Students' understandings of multiplication

Kerstin Larsson

Stockholm University

ISBN 978-91-7649-515-5 (print)
ISBN 978-91-7649-516-2 (PDF)

Printed in Sweden by Holmbergs, Malmö 2016
Distributor: Department of Mathematics and Science Education

List of papers

1. Larsson, K. (2016). Finding Erik and Alva: Uncovering students who reason additively when multiplying. *Nordic Studies in Mathematics Education, 21*(2), 69-88.

2. Larsson, K. (2015). Sixth grade students' explanations and justification of distributivity. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 295-301). Prague: Charles University in Prague, Faculty of Education and ERME.

3. Larsson, K., Pettersson, K., & Andrews, P. (resubmitted). The ambiguous role of equal groups in students' understanding of distributivity.

4. Larsson, K., Pettersson, K., & Andrews, P. (under review). Students' conceptualisation of multiplication as repeated addition or equal groups in relation to multi-digit and decimal numbers.

Abstract

Multiplicative reasoning permeates many mathematical topics, for example fractions and functions. Hence there is consensus on the importance of acquiring multiplicative reasoning. Multiplication is typically introduced as repeated addition, but when it is extended to include multi-digits and decimals a more general view of multiplication is required.

There are conflicting reports in previous research concerning students' understandings of multiplication. For example, repeated addition has been suggested both to support students' understanding of calculations and as a hindrance to students' conceptualisation of the two-dimensionality of multiplication. The relative difficulty of commutativity and distributivity is also debated, and there is a possible conflict in how multiplicative reasoning is described and assessed. These inconsistencies are addressed in a study with the aim of understanding more about students' understandings of multiplication when it is expanded to comprise multi-digits and decimals.

Understanding is perceived as connections between representations of different types of knowledge, linked together by reasoning. Especially connections between three components of multiplication were investigated; models for multiplication, calculations and arithmetical properties. Explicit reasoning made the connections observable and externalised mental representations.

Twenty-two students were recurrently interviewed during five semesters in grades five to seven to find answers to the overarching research question: What do students' responses to different forms of multiplicative tasks in the domain of multi-digits and decimals reveal about their understandings of multiplication? The students were invited to solve different forms of tasks during clinical interviews, both individually and in pairs. The tasks involved story telling to given multiplications, explicit explanations of multiplication, calculation problems including explanations and justifications for the calculations and evaluation of suggested calculation strategies. Additionally the students were given written word problems to solve.

The students' understandings of multiplication were robustly rooted in repeated addition or equally sized groups. This was beneficial for their understandings of calculations and distributivity, but hindered them from fluent use of commutativity and to conceptualise decimal multiplication. The robustness of their views might be explained by the introduction to multiplication, which typically is by repeated addition and modelled by equally sized groups. The robustness is discussed in relation to previous research and the dilemma that more general models for multiplication, such as rectangular area, are harder to conceptualise than models that are only susceptible to natural numbers.

The study indicated that to evaluate and explain others' calculation strategies elicited more reasoning and deeper mathematical thinking compared to evaluating and explaining calculations conducted by the students themselves. Furthermore, the different forms of tasks revealed various lines of reasoning and to get a richly composed picture of students' multiplicative reasoning and understandings of multiplication, a wide variety of forms of tasks is suggested.

Keywords: Multiplication; students' understanding; connections; multiplicative reasoning; models for multiplication; calculations; arithmetical properties

Acknowledgements

The common style for this section is to start by comparing the doctoral project to a journey and thank those who have accompanied and helped you during that journey, typically ending with recognising the informants. However, I choose to start with my deepest thanks to all you wonderful students who have shared your reasoning with me. Without your contribution this thesis would have been impossible to conduct. I am also grateful to the anonymous teachers, who so generously opened your classrooms for me.

I have had the best possible supervisors. They have supported the project by wise comments, suggestions, encouragement and discouragement when needed. Kerstin Pettersson, I did not make your job easy. For example, I have suggested a number of side-tracks, that you had to talk me out of, but you never made me feel that I was wasting your time. You have, keeping your door open, always been ready to discuss small and big issues of my project, giving productive critique at exactly the right time and always, always in a way that made me leave your room filled with new energy. What I feel is best expressed by paraphrasing En dum en in Dunderklumpen: Lycklig den doktorand som får dig! Paul Andrews, I have given you a hard time too, being completely ignorant about narratives. You have spent endless hours struggling to teach me how to create a narrative. It might be a never-ending story; I still do not know how to do it. Furthermore, you have given me new perspectives of how to frame my research and invaluable linguistic support. Thank you! Astrid Pettersson, I thank you for giving me the opportunity to become a doctoral student at MND, for good advice with respect to what type of tasks to use in the study and for suggesting to go to Åre. You also warned me about the ups and downs that I could expect to experience as a doctoral student.

On the up side of being a doctoral student there were some key moments, in which academics have supported the doctoral project by reading, commenting and discussing my work. I thank the 90 % readers, Wim Van Dooren and Lovisa Sumpter, the 50 % readers, Cecilia Kilhamn and Lisa Björklund Boistrup, and the 10 % readers, Inger Wistedt and Torbjörn Tambour. Another significant moment on the up side was the visit in Leuven, for which I am grateful to Lieven Verschaffel; you invited me for a fruitful stay with your research group at Katholieke Universiteit. During my stay, you gave me productive feedback of my work and arranged for meetings with several members of the staff, who generously spent time discussing mathematics education research. I am most grateful to all of you at the IPT centre.

Moving from the ups to the downs of being a doctoral student, I now turn to a key moment of misery. In the summer of 2013 I was ready to give up, when a number of scholars engaged themselves in long discussions about my project and mathematics education research. Thank you Dina Tirosh, Joanne Mulligan, Jenny Young-Loveridge and Rose Griffiths! You convinced me to go on by assuring me that students' understanding of mathematics is still an important branch of mathematics education research.

Now I turn to some groups, in which the persons are too numerous to be mentioned by names, that have supported my doctoral studies in different ways. The members of the Mathematics Education Research Group at my department have read and commented on various texts and I am grateful for your time and efforts. The various researchers and doctoral students that I have met during courses, conferences and other types of meetings, have contributed to my persistence to clarify what I mean. Thank you for good questions and good fun. My colleagues at MND, I want you to know that I highly appreciate that you have ever so often expressed that you believe in me and showed interest in my project. Thank you!

In the real world, outside of Academia, my friends and my family have contributed to the completion of this thesis by sharing my noisy laughter. I especially thank; A-laget for helping me to keep my feet on the ground and to remember what is important, Smaklökarna for good wines and food, my best friend Eva for positive energy and all other friends and family for your encouraging comments.

Last, but not least, there are three special persons that have helped me go through these years by just being themselves, my children and my partner in crime. I specifically thank Martin for involving me in intriguing conversations about science and for suggesting the perfect writing music. Sofia, I particularly thank you for marvellous massages and for the cheering when I needed it the most. Niclas, I thank you for, challenging conversations, jokes and puns, good food and drinks, diverse vacations, productive criticism and questions, comforting hugs, your brutally straightforward critique, and your sharp eye for inconsistencies down to every single comma, which I, since I am a big fan of interposed subordinate clauses, tend to use several of in my writing. I did not thrust my computer into the wall, not even close, but I am happy that you endured my shifting temper and still wanted to spend time with me. I love you all.

Contents

1 Introduction ... 1

2 Literature review ... 4
2.1 Multiplicative reasoning ... 4
2.2 Models for multiplication ... 6
2.2.1 Categorisations of models for multiplication ... 7
2.2.2 Properties of models for multiplication ... 10
2.2.3 Students’ understandings of models for multiplication ... 11
2.3 Multiplication and arithmetical properties ... 16
2.3.1 The commutative property ... 16
2.3.2 The distributive property ... 17
2.3.3 The associative property ... 17
2.4 Calculations in multiplication ... 19
2.4.1 Repeated addition ... 19
2.4.2 Overgeneralisation of addition strategies ... 20
2.4.3 Multiplicative strategies for multi-digits ... 20

3 Aim and research questions ... 23

4 Students’ understandings ... 25
4.1 Understanding of mathematical concepts ... 25
4.1.1 Representations ... 26
4.1.2 Connections ... 27
4.2 Understandings of multiplication ... 29
4.2.1 Analytical tool ... 31

5 Method ... 33
5.1 Participants ... 33
5.1.1 Teaching context ... 34
5.2 Data collections and tasks ... 35
5.2.1 Clinical individual interviews ... 36
5.2.2 Paired interviews ... 39
5.2.3 Written tests ... 40
5.2.4 Additional material ... 41
5.3 Analyses ... 42
5.3.1 Transcriptions ... 42
5.3.2 Analyses for paper 1 ... 43
5.3.3 Analyses for paper 2 ... 43

5.3.4 Analyses for paper 3 44
5.3.5 Analyses for paper 4 45
5.4 Ethical considerations 50

6 Summary of papers 52
6.1 Paper 1 – Finding Erik and Alva: Uncovering students who reason additively when multiplying....................52
6.2 Paper 2 – Sixth grade students' explanations and justifications of distributivity 54
6.3 Paper 3 – The ambiguous role of equal groups in students' understandings of distributivity....................55
6.4 Paper 4 – Students' conceptualisation of multiplication as repeated addition or equal groups in relation to multi-digit and decimal numbers .57

7 Discussion 59
7.1 Results....................59
7.1.1 Students' understandings of multiplication 59
7.1.2 Different forms of tasks....................63
7.2 The project 65
7.3 Implications for instruction....................69
7.4 Contribution and future research 70

Sammanfattning 72

References 76

1 Introduction

Multiplicative reasoning permeates many mathematical topics, for example fractions, ratio and functions (Vergnaud, 1994) and it forms a foundation for proportional reasoning, which is "the capstone of children's elementary school arithmetic and the cornerstone of all that is to follow" (Lesh, Post, & Behr, 1988, p. 94). This quote emphasises the importance of multiplicative reasoning for success in mathematics and warrants investigations of students' development of multiplicative reasoning in many studies including this thesis.

Multiplication is typically introduced as repeated addition of equally sized groups (Izsák, 2004; Watanabe, 2003), which is regarded as a natural way to introduce students to multiplication (Fischbein, Deir, Nello, & Marino, 1985). There are reports of students successfully exploiting this notion of multiplication to invent sophisticated calculation strategies underpinned by the three arithmetical properties, commutativity, distributivity and associativity (Ambrose, Baek, & Carpenter, 2003; Carpenter, Franke, & Levi, 2003; Lampert, 1986). However, the notion of multiplication as repeated addition of equally sized groups is insufficient, since it is not applicable to multiplication beyond natural numbers; it is hard to conceptualise to add something for example exactly 3.17 times (Greer, 1992; Verschaffel, Greer, & De Corte, 2007). Furthermore, is it obscuring why commutativity is valid and it reduces multiplication to a one-dimensional operation, which might hinder the development of multiplicative reasoning (Barmby, Harries, Higgins, & Suggate, 2009; Confrey & Smith, 1995). The role of equal groups is an example of tensions in the research literature with respect to students' understandings of multiplication. This and other tensions concerning students' understandings of multiplication, at the stage when multiplication is extended to multi-digit and decimal numbers, are addressed in this thesis.

Understanding of multiplication is multi-faceted, for example, it involves not only the ability to reason multiplicatively, but also calculations, justifying calculation strategies, explaining connections to other operations, employing arithmetical properties and knowing when to multiply and when to not (Carpenter et al., 2003; Clark & Kamii, 1996; Greer, 1992; Kilpatrick, Swafford, & Findell, 2001; Nunes et al., 2009; Park & Nunes, 2001; Siemon, Bleckly, & Neal, 2012; Van Dooren, De Bock, & Verschaffel, 2010; Verschaffel et al., 2007; Young-Loveridge & Mills, 2009).

The aim for this thesis is to understand more about students' understandings of multiplication when the operation is expanded to comprise multi-digits and decimals, specified as an overarching research question: What do students' responses to different forms of multiplicative tasks in the domain of multi-digits and decimals reveal about their understandings of multiplication?

Understanding of mathematics is often described as connections between pieces of knowledge, procedural knowledge, and conceptual knowledge (Baroody, Feil, & Johnson, 2007; Berthold, Eysink, & Renkl, 2009; Hiebert & Carpenter, 1992; Hiebert & Wearne, 1992). Mental representations of these concepts, or pieces of knowledge, are suggested to be connected by reasoning to form a person's understanding of mathematical concepts (Barmby et al., 2009). Explicit reasoning and externalised representations are observable and it is suggested that understanding can be inferred from them.

When this notion of understanding is applied to understanding of multiplication, it can be viewed as connections between three components and includes both procedural and conceptual knowledge. The three components are models for multiplication, such as equally sized groups; calculations, such as repeated addition; and arithmetical properties, such as commutativity. The connections can become discernable by reasoning, such as to reason that $3 \cdot 5$ must equal $5 \cdot 3$ since you can see that it is the same amount of cookies on a tray with three rows of cookies with five in each row even if you turn the tray 90°. This example shows how reasoning can connect commutativity, an arithmetical property, to a model for multiplication, rectangular array.

To investigate students' understandings of multiplication by their responses to different forms of tasks, twenty-two students were recurrently interviewed, both individually and in pairs, during five semesters in grades five to seven. The first interviews took place the first semester in the fifth grade, which was before multi-digit and decimal multiplication was introduced. During the interviews the students were given different forms of tasks, for example calculation tasks followed by prompts to explain how and why the calculation worked and story telling tasks, similar to problem posing to given multiplications. Additionally, the students were given written tests of multiplicative word problems during regular mathematics lessons.

The results from the study, reported in this thesis, are organised in four papers, investigating four issues related to the overarching question and what the literature review had revealed as incongruent reports of students' multiplicative understandings. The first paper investigates the possible incongruence of descriptions of multiplicative reasoning as closely related to calculative acts (Clark & Kamii, 1996; Sowder et al., 1998; Steffe, 1994; Tzur et al., 2013) and testing it by interpretations of multiplicative comparison problems (Clark & Kamii, 1996; Fernandez, Llinares, Van Dooren, De Bock, &

Verschaffel, 2012; Van Dooren, De Bock, & Verschaffel, 2010; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010). The second paper investigates what students' evaluations of erroneous calculations of multi-digit multiplications can reveal about their understandings of distributivity. Distributivity has been reported both as and troublesome to understand (Carpenter, Levi, Franke, & Koehler, 2005; Ding & Li, 2014) and as something young students successfully can learn (Ambrose et al., 2003; Barmby et al., 2009; Carpenter et al., 2003; Izsák, 2004; Lampert, 1986; Young-Loveridge & Mills, 2009). The third paper investigates the role of equal groups in relation to students' understandings of distributivity, which is debated, as mentioned earlier. In the fourth paper, two students' understandings of multiplication is investigated through what connections they demonstrate over the five semesters of the study, thus allowing a description of how their understandings vary over time.

After this brief introduction, relevant literature is reviewed in chapter 2, to clarify what is known about students' understandings of multiplication. In chapter 3, aim and research questions for this thesis are presented as a consequence of the literature review. In chapter 4, there is a review of how students' understanding of mathematics can be construed and investigated. In chapter 5, I present methods and methodological choices, including participants, data collections, tasks, analyses and ethical considerations with respect to what is presented in this thesis. Chapter 6 consists of summaries of the four papers, which constitute the results. Finally, in chapter 7, I discuss the results, methodological choices for the study, implications for instruction and the contribution to the field including suggestions for further research.

2 Literature review

In this chapter, I first review literature concerning multiplicative reasoning with focus on the transition from additive reasoning to multiplicative reasoning. When the multiplicative reasoning is extended to comprise multi-digits and decimals, three domains of special interest for students' understandings of multiplication has emerged, models for multiplication, the arithmetical properties and calculations, each domain is reviewed in its own section.

2.1 Multiplicative reasoning

There is consensus on the pivotal role of multiplicative reasoning for mathematics in many domains (Barmby et al., 2009; Chandler & Kamii, 2009; Clark & Kamii, 1996; Confrey & Smith, 1995; Nunes et al., 2009; Park & Nunes, 2001; Sowder et al., 1998; Steffe, 1994; P. Thompson & Saldanha, 2003; Van Dooren, De Bock, & Verschaffel, 2010). The ability to reason multiplicatively is considered as a significant conceptual leap of students' understanding (Simon & Blume, 1994; Tzur et al., 2013) and it is a foundation for all mathematics after primary school, hence essential to develop (Lesh et al., 1988). For example, multiplicative reasoning is the foundation for our place value system, fractions, ratio, proportionality and functions (Vergnaud, 1988, 1994), which warrants various types of studies in order to gain better understanding of how students understand multiplication.

The transition from additive to multiplicative reasoning is a process stretched over several years (Clark & Kamii, 1996; Simon & Blume, 1994; P. Thompson & Saldanha, 2003) and supposedly requires instruction (Sowder et al., 1998). Multiplicative reasoning involves the ability to handle nested units on more levels of abstraction compared to additive reasoning (Clark & Kamii, 1996). This is also described as coordination of composite units (Sowder et al., 1998; Steffe, 1994; Tzur et al., 2013). To explain the notion of coordination of composite numbers, I use the example of $3 \cdot 4$. Both 3 and 4 are composite numbers, meaning that they both consist of more than one unit. If we think of $3 \cdot 4$ as representing 3 boxes with 4 toy cars in each, the coordination for a very young student can be to count four (one-two-three-four) toy cars for each of the three (one-two-three) boxes to find the answer of how many toy cars in total. Thus composite numbers are coordinated at three levels; the total composite number (12 toy cars) is comprised of composite numbers (3 boxes) of composite numbers (4 in each box) (Clark & Kamii, 1996; Steffe, 1994). Even though 4 refers to toy cars per box, most young students think of 4 to represent toy cars, or only think of the number and ignore the quantity (Vergnaud, 1983). To keep track of the simultaneous count of two composite units and the resulting composite unit as well as the transformation of quantities, is considered to be a sign of multiplicative reasoning (Sowder et al., 1998; Steffe, 1994; Tzur et al., 2013). Another way to describe the coordination of composite units, as an indicator of multiplicative reasoning, is to perceive one bag of twelve marbles as representing both 1 and 12 simultaneously (Steffe, 1994). This ability underpins understanding of our enumeration system, where one hundred is one hundred, ten tens and one hundred ones at the same time (Chandler & Kamii, 2009; Nunes et al., 2009), which also can be described as one-to-many correspondence (Anghileri, 2006; Bakker, Van den Heuvel-Panhuizen, & Robitzsch, 2014; Nunes & Bryant, 2010; Nunes, Bryant, Evans, & Barros, 2015; Park & Nunes, 2001). Additionally, it has been suggested that multiplicative reasoning involves multiplication with expanded factors, such as 20 + 6 (Izsák, 2004), which is regarded as more advanced, especially if two factors are expanded (Ambrose et al., 2003).

Furthermore, multiplicative reasoning, in contrast to additive, involves coordination of dimensions and quantities (Schwartz, 1988; Simon & Blume, 1994). Multiplication transforms quantities, for example, multiplication of lengths is two-dimensional, and resulting in area, while addition of lengths is a linear, one-dimensional, operation resulting in length. Another example is that the number of cookies multiplied by the cost per cookie, is transformed to the cost for all cookies (Vergnaud, 1983, 1988). In the latter example the intensive quantity, cost per cookie, (Schwartz, 1988) is distinctly different from the extensive quantities that are applicable to addition and subtraction (Piaget, 1952). Mass is an example of extensive quantity; the total weight of two boxes can be calculated by addition. Intensive quantities are most often a combination of two extensive quantities and their unit measure includes the word "per", for example kronor per kg (Schwartz, 1988). Thus, an intensive quantity forms a multiplicative relation, a ratio, between its two parts, which make intensive quantities more problematic to understand than extensive (Howe, Nunes, & Bryant, 2010; Nunes & Bryant, 2010; Simon & Placa, 2012).

One way to investigate students' multiplicative reasoning is to invite them to solve multiplicative comparison problems, such as 3 times as many as 5 (see e.g. Clark & Kamii, 1996). Students in lower primary grades tend to reason additively to this type of problems, hence answering 8 after adding 3 and 5. When students grow older they slowly learn to use multiplicative reasoning. Multiplicative comparison of two quantities involves ratio, which might explain the slow process of transition to multiplicative reasoning (Lamon, 1993, 2007; Sowder et al., 1998). However, in the process of learning multiplicative reasoning there is evidence that students overuse it on additive problems (Fernandez et al., 2012; Van Dooren, De Bock, Janssens, & Verschaffel, 2008; Van Dooren, De Bock, & Verschaffel, 2010). In order to develop a robust ability to reason multiplicatively, one must be able to distinguish when multiplicative reasoning is appropriate and when not (Clark & Kamii, 1996; Confrey & Smith, 1995; Simon & Blume, 1994; Van Dooren et al., 2008; Van Dooren, De Bock, & Verschaffel, 2010; Van Dooren, De Bock, Vleugels, et al., 2010).

In sum, literature has described multiplicative reasoning as ability to coordinate composite numbers at several levels of abstraction and researchers have investigated students' multiplicative reasoning by their calculations (e.g. Steffe, 1994; Tzur et al., 2013). Others have tested students' ability to reason multiplicatively by their ability to interpret proportional or multiplicative comparison problems (Clark & Kamii, 1996; Van Dooren, De Bock, & Verschaffel, 2010). However, it is not clear if the ability to reason multiplicatively is equally measured by these two forms of tasks.

2.2 Models for multiplication

Multiplicative situations occurring in real life can be described verbally. For example, if I buy 3 boxes of cookies with 6 cookies in each, it can be formulated as a problem asking for how many cookies I bought. Such problems form a genre in school mathematics referred to as word problems, irrespective of the degree of realistic contextual content. A line of research has shown that students can engage in solving less realistic word problems even though they realise that that the solutions are unrealistic, which is referred to as playing the game of school mathematics (Verschaffel, Greer, & De Corte, 2000, 2002). Even though this line of research is interesting and important for the overall understanding of students' accomplishment, it is not further reviewed here, since it is not directly connected to the study of students' understanding of multiplication, rather to socio-mathematical norms (Cobb & Yackel, 1998). Multiplicative word problems can be categorised as various models for multiplication, the cookie problem above is distinctly different from a question of how many different meals can be combined from 6 main courses and 3 side orders, even though both can be solved by the same multiplication.

Research concerning students' understanding can distinguish between models *of* and models *for* a concept (Van den Heuvel-Panhuizen, 2003; Verschaffel et al., 2007). When students solve word problems, they might construct a model of the problem, which typically is an illustration of the specific situation in the problem. To solve a word problem of how many cookies there are in 3 boxes with 6 cookies in each, a young student might draw a picture of 3 boxes with 6 cookies. The cookie drawing might not be useful to a similar problem of bags with marbles, since the illustration is very close to the problem and hence the cookies are not exchangeable to marbles. A model *for* multiplication is more general and can be used as a thinking tool for a class of problems; therefore it is considered as a sign of a deeper understanding compared to a model *of* multiplication (Van den Heuvel-Panhuizen, 2003). Models for multiplication do not need to be drawn; they can for example be mental representations or verbal stories (Greer, 1992; Van den Heuvel-Panhuizen, 2003). Hereafter I use the term model for multiplication both when I refer to the mathematically and psychologically derived models for multiplication presented below and the models for multiplication that students have been found to use as thinking tools and explanations for multiplicative problems. In the following I review categorisations of models for multiplication, what properties various models have, and students' understandings of models for multiplication.

2.2.1 Categorisations of models for multiplication

The formulation of word problems determines what situation is described. Hence some researchers (e.g. Mulligan, 1992; Mulligan & Mitchelmore, 1997; Nesher, 1988) refer to models for multiplication as semantic structures. Even though there is less research concerning multiplicative models compared to additive models (Greer, 1992; Verschaffel et al., 2007), scholars have categorised real world situations from different perspectives resulting in different taxonomies of models for multiplication. In order to understand the significance of different models for multiplication the origin of classifications is reviewed.

From a mathematical point of view, using dimension analysis, Vergnaud (1983, 1988, 1994) and Schwartz (1988) have both defined three multiplicative situations, but their definitions differ. Vergnaud's categories are isomorphism of measures, product of measures and multiple proportions. Multiplication is viewed as a relation between four measures, whereof one can be implicit. In Schwartz' categories, multiplication is viewed as a relation between three measures, and based on the quantities involved.

In Vergnaud's (1983, 1988, 1994) classification, isomorphism of measures is a relation between two measure spaces, for instance the number of cookies and the cost of cookies. In a problem to find the cost for 12 cookies when 4 cookies cost 20 kronor the problem consists of two measure spaces, the number of cookies (4 and 12) and the cost (20 and x). In many multiplicative word problems there are only two numbers, with the number 1 implicit. For example, to ask how much 6 cookies cost if the price is 5 kronor per cookie includes an implicit 1, which is hidden in the cost, since 1 cookie costs 5 kronor. Product of measures involves a transformation of dimensions, where two dimensions are transformed into a third; for example the length and width of a rectangle are transformed into the area when multiplied. Another example of product of measures is the combination of six main courses and three side orders resulting in eighteen meals, which often is denoted Cartesian product (Greer, 1992) and constitutes a basis for combinatorics. Finally, multiple proportional problems deal with (at least) three measure spaces, in which one is proportional to both the other measure spaces, which are independent from each other. For example, the consumption of water is proportional both to the number of persons and the number of days. This type of problem can be broken down into two or more steps, for example one step of the water problem is to find out the consumption for one person for a number of days and the next step to multiply the product with the number of persons. Thus multiple proportional problems can be transformed to isomorphism of measures, or product of measures, in two or more steps.

Schwartz' (1988) dimensional analysis is based on whether the quantities are extensive or intensive, suggesting that extensive quantity times intensive quantity forms one category, extensive times extensive forms another, and intensive times intensive a third. An example of a problem with an extensive quantity multiplied by an intensive quantity is the 12 cookies (extensive quantity) at the cost of 5 kronor/cookie (intensive quantity), which corresponds directly to isomorphism of measures. Problems in which extensive quantities are multiplied by extensive quantities are referent transforming and can be exemplified by rectangular area that corresponds to product of measures. Problems in which intensive quantities are multiplied by intensive quantities are more complex and mostly employed in science, but an example given by Nesher (1988) is to multiply the average speed in km/h with number of hour/day to find the new intensive quantity of km/day.

Other categorisations are based on the perspective of the learners' conceptualisation of different situations, real or in the format of a word problem (Carpenter, Fennema, & Franke, 1996; Greer, 1992). These categorisations typically involve more categories than the mathematically derived categorisations described above. In Greer's review of categorisations of models for multiplication a list of ten different models is presented. Seven of these correspond to isomorphism of measures in Vergnaud's (1983) classification and the three other correspond to product of measures. For example, equal measures and equal groups, which both correspond to isomorphism of measures, are considered as different categories, since equal measures deals with continuous quantities, such as how many pieces of 3.5 metres can be cut off from 21 metres of fabric, while equal groups deals with discrete objects (Greer, 1992). Continuous and discrete measures are conceptualised differently by young learners. The remaining five categories corresponding to isomorphism of measures are multiplicative comparison, rate, measure conversion, part/whole and multiplicative change. The latter two are most often included in multiplicative comparison and not further described. Measure conversion, as for example how many centimetres are 4 inches, can be viewed as a special case of rate (Bell, Greer, Grimison, & Mangan, 1989), which as well as multiplicative comparison is further explained below.

Rate is commonly described as an invariant relationship between two measures, for example, the price of 5 kronor/cookie, in the problem to find the total price for 4 cookies at the price of 5 kronor each (Bell et al., 1989; Mulligan & Mitchelmore, 1997). For younger students the distinction between countable objects and objects with a value warrants the need to categorise the word problems into different categories (Mulligan & Mitchelmore, 1997). Rate problems can be subcategorised into for example price-problems, as the cookie problem, speed-problems, as how far Martin walked if he walked at the average speed of 5 km/h for 3 hours, and conversion-problems, such as how much is 4 euro worth in Swedish kronor if 1 euro is worth 10 kronor (Bell, Fischbein, & Greer, 1984; Bell et al., 1989). Even though rate problems are more abstract than equal groups, since they do not describe actual groups of discrete objects, students might solve them in the same way (Carpenter, Fennema, Franke, Levi, & Empson, 1999), for example, by perceiving Martin's 3 hours of walking as 3 groups and the speed of 5 km/h as the number of kilometres in each group. The rate model in price-problems can also be thought of as multiplicative comparison (Greer, 1992). Multiplicative comparison involves two different groups being compared to each other, if Sofia has 10 books and Martin 40, he has 4 times as many, but can also involve price problems; if I buy 4 items I have to pay 4 times as much as for 1 item. In the literature on proportional reasoning, rate, ratio and proportional problems are central, and all three can be considered as types of multiplicative comparison of different kinds of quantities with different units (Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998; Shield & Dole, 2013).

The three models in Greer's (1992) list that correspond to Vergnaud's (1988, 1994) product of measures are rectangular area, Cartesian product and product of measure. Rectangular area and Cartesian product are already explained and product of measure in this list is a generalisation of Cartesian product, combining pairs from two disjoint sets so that each object in one set is combined with each object in the other. Cartesian product transforms quantities; in the example above dishes are transformed to meals.

As this review of categorisations demonstrates, the categorisation of models for multiplication can be conducted in different ways, leading to different taxonomies and "the categories can be extended, collapsed, or refined depending on the purpose of the investigation" (Mulligan & Mitchelmore, 1997, p. 310). In the end of Greer's (1992) review of categorisations, he suggests combining the ten categories into four different models for multiplication which often are used in research in the middle school years (Greer, 1992; Izsák, 2004; Mulligan & Mitchelmore, 1997): equal groups (including equal measures and rate), multiplicative comparison, rectangular array/area, and Cartesian product.

However, rectangular array involves discrete objects and hence might be considered as a separate category rather than unified with rectangular area, which involves continuous quantities. Cartesian product is rare in Swedish multiplicative instruction until combinatorics is introduced. Rate can both be viewed as equal groups and multiplicative comparison and the expression has no equivalent in Swedish. For the stage when multiplication is expanded to multi-digits and decimals, it seems adequate to include price problems in equal groups, and measure conversions and speed problems in multiplicative comparisons. Thus the following four categories of models for multiplication became important to my study:

- Equal groups, including rate such as price per item
- Rectangular array
- Rectangular area
- Multiplicative comparison, including rate such as measure conversion and speed

It is worth noticing that these categories of models for multiplication are not distinctly different. For example, when equal groups of objects are placed in orthogonal rows and columns, the situation can simultaneously be perceived as a rectangular array and as equal groups. Similarly, a rectangle divided in area unit squares may be viewed both as array (of the squares) and as area (by the measures of the sides).

2.2.2 Properties of models for multiplication

Rectangular array and area are symmetrical; the factors have the same role, while equal groups and multiplicative comparison are asymmetrical (Bell et al., 1989; Carpenter et al., 1999; Greer, 1992). In asymmetrical models, for example equal groups, the multiplier (the number of groups) has a different role from the multiplicand (the number of objects in each group). This aspect makes a model more or less appropriate to demonstrate multiplicative commutativity. That $a \cdot b$ equals $b \cdot a$ is visually perceivable in symmetrical models, which makes symmetrical models psychologically commutative (Verschaffel et al., 2007). A rectangular array, such as a box of 4 rows of 7 soda cans, makes it observable that the total amount of soda cans has not changed when rotated 90°. It is not immediately obvious that 7 bags of 4 marbles must be same amount of marbles as 4 bags of 7 marbles (Bell et al., 1989; Carpenter et al., 1999; Lo, Grant, & Flowers, 2008).

Another property of a model for multiplication is what number types, such as decimals or integers, it is applicable to. The number type has been reported to affect students' reasoning more in multiplicative operations than in additive, and therefore affects the categories of models (Greer, 1992).

Models with discrete objects, such as equal groups and rectangular array, are not suitable for decimals, which rectangular area and multiplicative comparison are. For example, 3.8 bags of 4.9 marbles in each or 3.8 rows and 4.9 columns of soda cans are hard to imagine. A room of 3.8 metres in width and 4.9 metres in length or that Sofia has 3.8 times as much money as her brother, who has 4.9 euro, is possible to conceptualise. However, equal groups can deal with continuous quantities and hence decimal numbers in the multiplicand, but the multiplier needs to be a positive integer. For example, 3 books with the width of 2.7 cm would need $3 \cdot 2.7$ cm of space in the bookshelf. Also the rate-model for price problems (here included in equal groups) can make sense of decimal multiplication. It is possible to buy 3.8 kg of potatoes for 4.9 kronor/kg. The potato-situation can also be viewed as a multiplicative comparison; each kg of potatoes has the value of 4.9 kronor. The amount of potatoes and the cost in kronor form two quantities that are multiplicatively connected; the quantity of potatoes (in kg) can be measured by its value (in kronor/kg) and it is a proportional relation (P. Thompson & Saldanha, 2003).

Multiplicative comparison can be considered conceptually different from the other three models, since comparisons typically involve two sets; for example, if Sofia has 10 books and Martin 40, he has four times as many, since $4 \cdot 10$ is 40, but in all there are 50 books. One set is (theoretically) divided into subsets of the other set's magnitude, Martins 40 books are divided into 4 groups of 10 books. This can also be viewed as one set is used as a unit to measure the other set and thus the two sets are compared multiplicatively (Lamon, 1993; Sowder et al., 1998; P. Thompson, 1994). Therefore multiplicative comparison is considered key to proportional reasoning (Lesh et al., 1988; P. Thompson & Saldanha, 2003).

2.2.3 Students' understandings of models for multiplication

There is evidence from studies concerning prospective teachers' understandings of mathematical concepts that the same or similar misconceptions and lines of reasoning exist for them as for younger students (Lo et al., 2008; McClain, 2003). Hence, the review does not only relate to students in compulsory school; also prospective teachers' understandings of the models are included.

To investigate the relative difficulty of different models for multiplication, students have been invited to solve word problems reflecting different models for multiplication and to pose word problems to match given calculations. Based on students' success rates of solving word problems, a number of studies have demonstrated that equal groups and rectangular array and area problems are easier than multiplicative comparison and hardest are Cartesian product (Greer, 1992; Mulligan, 1992). These reports give a hint of what is easier even though students might solve a word problem of a certain model for multiplication by means of another (Greer, 1992; Mulligan & Mitchelmore, 1997). For example, a student might solve a word problem presented as rectangular area as equal groups of area units.

The prevalent explanation in the literature to larger success rates for some models is the frequent occurrence of these types of problems in instruction (Verschaffel et al., 2007); equal groups problems are frequent in primary school and a common model in the introduction of multiplication (Fischbein et al., 1985; Izsák, 2004; Watanabe, 2003). Another explanation is that the intuitive model for calculating multiplication is suggested to be repeated addition (Fischbein et al., 1985), which is easy to connect to equal groups and rectangular array (De Corte & Verschaffel, 1996; Simon & Blume, 1994). The relative success of rectangular area problems might be a consequence of rote learning of the area formula, which is activated by the use of key words like 'area' (De Corte, Verschaffel, & Van Coillie, 1988; Simon & Blume, 1994).

Coincidently, studies have shown that to connect multiplication to rectangular area and conceptually grasp the relation of multiplication and area have proved to be an obstacle both for students and prospective teachers (Izsák, 2005; Lo et al., 2008; Simon & Blume, 1994; Verschaffel et al., 2007). If multiplication is conceptualised as a linear, one-dimensional operation, such as repeated addition, skip-counting or jumps on a number line, a rectangular area might not be regarded as a representation of multiplication. It is possible to learn the area formula, $A = l \cdot w$, without connecting the multiplication to area (Simon & Blume, 1994). Similarly, an array of squares can be counted or viewed as equal groups without connection to multiplication and knowledge that area is found by multiplication (Izsák, 2005).

Problem posing is directly linked to problem solving (Silver & Cai, 1996), which recently has attracted more attention in mathematics education research (Cai, Hwang, Jiang, & Silber, 2015; Silver, 2013). Problem posing can be useful for uncovering how students conceptualise multiplication (De Corte & Verschaffel, 1996; Prediger, 2008). An individually construed meaning of a concept or operation is seen as "the key to analyse understanding" (Prediger, 2008, p. 6). The type of problems students pose can reflect the instruction they have met as well as their conceptual understandings or misunderstandings (Cai et al., 2015; Tichá & Hošpesová, 2013).

A variety of activities are viewed as problem posing, and in one suggested classification, five categories of problem posing prompts have been identified: a) free situations to pose a problem without constrictions, b) to pose a problem for a given answer, c) to use given information to pose a problem, d) to pose problem for a given situation, and e) to pose a problem for a given calculation (Pittalis, Christou, Mousoulides, & Pitta-Panzai, 2004). In this thesis problem posing refers solely to the last category, to pose a problem for a given multiplicative calculation, such as $23 \cdot 39$. Hence literature in this line of problem posing is reviewed, but not the larger field of problem posing.

The most frequent form of word problems that is generated among prospective teachers and different ages of students is equal groups problems, including equal measures (De Corte & Verschaffel, 1996; Nesher, 1988). For example, more than 80 % of the correct word problems were equal groups in De Corte and Verschaffel's study when the multiplications included at least one integer. When both factors were decimal numbers, the most common model was rectangular area. However, there are cultural differences to what types of problems that are typically posed. Israeli 10–12 year old students posed 41 % multiplicative comparison problems and 34 % equal groups problems in a similar study as De Corte and Verschaffel's, possibly since there is a short and often used Hebrew expression with the meaning of 'times as many' (Nesher, 1988). The expression 'P-five' has the meaning of 'five times as many as' (there is also an analogous short way to express 'five more' in Hebrew, 'Bae-five').

The high frequency of equal groups problems can be explained by the intuitive models theory, which suggests that one model for each operation is more natural and hence easier (Fischbein et al., 1985). According to Fischbein et al., repeated addition is the most natural and easy model for multiplication and it is closely related to equal groups (De Corte & Verschaffel, 1996; Simon & Blume, 1994). Note that in Fischbein et al.'s (1985) study, as well as in some other studies (e.g. Mulligan & Mitchelmore, 1997) the word 'model' is used to denote what calculation strategy students use rather than the structural level of a model for multiplication as reflecting a real world situation.

The preference for equal groups can be explained by influence of instruction and everyday experiences (De Corte & Verschaffel, 1996; English, 1998; Fischbein et al., 1985; Kinda, 2013). In initial instruction, students generally meet equal groups problems (and in Israel multiplicative comparison problems) (Fischbein et al., 1985; Izsák, 2004; Watanabe, 2003). In later instruction, multiplication is not explicitly connected to any model or real world situation, leaving the students with only the initially taught model for multiplication (De Corte & Verschaffel, 1996). This impact of instruction is partly what the intuitive model theory (Fischbein et al., 1985) predicts, even though De Corte and Verschaffel (1996) argue that the instructional implications in combination with "a tendency to reduce cognitive load" (p. 240), is a more plausible explanation than an innate inclination to use repeated addition, which the intuitive model theory suggests (Fischbein et al., 1985).

Decimal numbers are known to create problems for students in many ways (Lamon, 2007). With respect to multiplication, decimals affect both problem solving and posing. Indeed, the number types are reported to have larger impact than the model for multiplication in relation to how difficult students find the task (De Corte & Verschaffel, 1996; Greer, 1992;

Verschaffel et al., 2007). One aspect of the difficulties is denoted the multiplier effect, since a decimal as multiplier makes the problem harder, while a decimal as the multiplicand has no or very small effect on the difficulty (Bell et al., 1989; De Corte & Verschaffel, 1996). Another aspect is whether the decimal number is less than or larger than 1, with the former violating the belief that multiplication makes bigger (Bell et al., 1989; Bell, Swan, & Taylor, 1981; Fischbein et al., 1985; Greer, 1992). These two obstacles for decimal multiplication are well established from a number of studies concerning problem posing and solving (Greer, 1992).

The multiplier effect challenge repeated addition as a calculation procedure; it is not possible to add for example 3.8 exactly 4.6 times (Fischbein et al., 1985; Greer, 1992; P. Thompson & Saldanha, 2003; Verschaffel et al., 2007). Also in problem posing to given multiplications, this effect is reported to cause a decrease of appropriate problems (De Corte & Verschaffel, 1996). However, De Corte and Verschaffel suggest that the multiplier effect should not be overgeneralised, since they found that decimals less than 1 had a greater impact on the number of posed word problems. For example, both $5.3 \cdot 0.6$ and $7.4 \cdot 3.8$ violate the multiplier effect, but $5.3 \cdot 0.6$ also violates the multiplication makes bigger belief. Since there was a significant difference in the number of appropriate posed problems between these two given calculations, but no significant difference between $5.3 \cdot 0.6$ and $0.7 \cdot 0.2$, the presence of decimals less than 1 are suggested as more influential than the multiplier effect.

Other effects of the multiplication makes bigger belief, and its analogous division makes smaller, have been reported to cause erroneous solutions to word problems such as to find the cost for 0.45 kg of potatoes at 6.90 kronor/kg. Students typically suggest to divide 6.9 by 0.45, believing that division produces smaller and multiplication larger results (Bell et al., 1989; Bell et al., 1981). Similar findings were recently reported from a Chinese study, in which students were invited to solve and pose word problems to given calculations (Chen, Van Dooren, Jing, & Verschaffel, 2015). The tasks involved the same numbers and varied by decimals larger than and less than 1 in one or both factors. In a post test, the students had to judge which operation would produce a smaller or larger result, for example, they were asked to choose whether multiplication or division by 0.7 would be less than 0.7. The post test also included to put smaller than or larger than signs in number sentences, such as if $0.6 \cdot 0.49$ were larger than or smaller than 0.6. The number of correct solutions was affected by decimals less than 1 in all types of problems, suggesting that decimals less than 1 affect students' reasoning according to the belief that multiplication makes bigger.

Solving a problem by its inverse operation, following the multiplication makes bigger belief, has been observed both when students choose from a list of expressions and when answering by making the calculation, even though students' own solutions elicit a larger amount of correct answers than choosing from a list (De Corte et al., 1988). This is probably depending on the opportunity to use informal strategies in combination with estimation of the magnitude of the answer.

It is suggested that the root of the multiplication makes bigger rule lies in the 'natural number bias', meaning that rules and properties of natural numbers inappropriately is applied to rational numbers (Ni & Zhou, 2005; Vamvakoussi, Van Dooren, & Verschaffel, 2012). Since we initially learn arithmetic in the set of natural numbers, our experiences make us believe that addition and multiplication makes the result bigger and subtraction and division makes it smaller (Greer, 1992; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). To overcome the problems concerning decimal multiplication, it is suggested that students need greater meta-cognitive awareness of how multiplication affects decimals, supported by models for multiplication that are susceptible to decimals (Greer, 1992; Verschaffel & De Corte, 1997). In a recent study among secondary students testing the natural number bias, the students were given algebraic statements concerning the effect of operations that they should declare true or false (Van Hoof et al., 2015). The statements were either congruent to natural numbers, meaning that if the variable was exchanged by a natural number it led to a correct conclusion, or incongruent, exchanging the variable with a natural number led to an erroneous conclusion. The results indicate that the natural number bias is stronger for multiplication and division compared to addition and subtraction. In another study, it was suggested that secondary students had overcome the natural number bias as measured by correct answers to evaluate relative size of fractions (Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). However, by using reaction time measurement it was found that they used significantly longer time to give correct answer to tasks that were incongruent to natural numbers compared to tasks that were congruent. In sum, the natural number bias affects both primary and secondary students, and the effect of the natural number bias can be observed even after students have learnt to bypass it. The multiplicative operations are more sensible to the effect than the additive. This has led to recommendations to be aware of undesirable effects of introducing multiplication by repeated addition since it may cause "later conceptual difficulties" (Van Hoof et al., 2015, p. 37).

There seems to be a pedagogical dilemma in the literature concerning students' understandings of models for multiplication and recommendations for teaching. On the one hand, there is consensus on the equal groups model as one of the most accessible models for young students. On the other hand, the introduction and extensive use of the equal groups model is reported to reduce multiplication to repeated addition, which is not supporting multiplicative reasoning. Hence it has been suggested that students should learn to use rectangular area and array, which emphasise the two-dimensionality of multiplication (Barmby et al., 2009; Fuson, 2003; Izsák, 2004; Young-Loveridge & Mills, 2009). Coincidently, rectangular area and array have been reported as hard to conceptualise as models for multiplication (Izsák, 2005; Lo et al., 2008; Simon & Blume, 1994; Verschaffel et al., 2007). With respect to decimal multiplication, there are well-established findings concerning the multiplier effect and the multiplication makes bigger belief, explained by the natural number bias. However, there is a plea for more research of how "multiplicative concepts develop beyond the domain of positive integers" (Verschaffel et al., 2007, p. 588).

2.3 Multiplication and arithmetical properties

Three arithmetical properties, the commutative property, the distributive property and the associative property, underpin calculations and hence understanding of multiplication as well as forming a foundation for understanding algebra in later school years (Baek, 2008; Carpenter et al., 2003; Carpenter et al., 2005; Ding & Li, 2014; Schifter, Monk, Russel, & Bastable, 2008). In this section I review what earlier research report concerning students' understandings and use of each of the properties in relation to multiplication.

2.3.1 The commutative property

Commutativity is valid for addition and multiplication and states that the two numbers can change order: $a + b = b + a$ and $a \cdot b = b \cdot a$. Most children discover the commutative property for addition without instruction (Canobi, Reeve, & Pattison, 2002), but not for multiplication (Ambrose et al., 2003; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998), even though some students discover commutativity when they memorise the number facts in the multiplication table (Anghileri, 2006; Baroody, 1999; Fuson, 2003).

One explanation for the need of instruction is that the model of equal groups and to calculate by repeated addition do not emphasise commutativity (Greer, 1992; Schliemann et al., 1998). This was for example demonstrated in a Brazilian study comparing unschooled street sellers to schoolchildren. The street sellers more seldom solved a task such as to find the total cost of 60 items of 4 cruzeiros by exploiting commutativity and solved it as if it was 4 items of 60 cruzeiros (Schliemann et al., 1998). The school children had learnt to use commutativity and were not depending on repeated addition. In another study, English fourth grade students seemed to understand commutativity quite well (Squire, Davies, & Bryant, 2004). The students took a multiple-choice test with tasks concerning commutativity and distributivity in combination with cues formulated as three different models for multiplication: equal groups, rectangular area and Cartesian product. An example of a problem given with a cue with equal groups is "Christopher has 33 bags of coins, each with 18 coins in them. Altogether he has 594 coins. James has 18 bags, each with 33 coins in them. How many coins does James have?" (Squire et al., 2004, p. 520). The students' results suggest that they understood the commutative property, irrespective of model for multiplication. Even though commutativity is thought not to develop spontaneously this study asserts that the English nine- and ten-year old students had learnt to use commutativity irrespective of the model for multiplication.

2.3.2 The distributive property

Distributivity is defined for multiplication over addition and states that $a \cdot (b + c) = a \cdot b + a \cdot c$, which underpins most multiplication algorithms. The distributive property is powerful, fundamental for understanding multiplication and forms the basis for algebra and generalisations, but considered problematic for students to learn (Carpenter et al., 2005; Ding & Li, 2014). Hence it is advocated that students should be given opportunity to learn distributivity "in the context of whole-number multiplication" (Izsák, 2004, p. 38).

The difficulty of learning distributivity is supported by Squire et al. (2004), who assert that the students had "very poor understanding of distributivity" (p. 515). The students were more successful in distributivity when the cue reflected equal groups compared to rectangular area or Cartesian product, which led Squire et al. to suggest that instruction about distributivity would benefit from use of the equal groups model. This suggestion is in line with a report of how American fourth grade students were successfully introduced to distributivity by stories and drawings of equal groups (Lampert, 1986). Other studies have shown that students employ distributivity when they develop repeated addition for multi-digit calculations and that implicit understanding of distributivity does not need instruction (Ambrose et al., 2003; Baek, 2008; Schifter et al., 2008; Schliemann et al., 1998), indicating that distributivity is not difficult to learn, which is in contrast to what Squire et al. (2004) reported.

Yet other studies, in which students were invited to calculate by using an array, have shown that the rectangular array model can support instruction about distributivity as measured by an increase of students' correct calculations (Barmby et al., 2009; Izsák, 2004; Young-Loveridge & Mills, 2009). Hence, it is not yet clear whether equal groups or rectangular array serves best to support students' understanding of distributivity or how difficult distributivity is for students to learn.

2.3.3 The associative property

Associativity is valid for addition and multiplication and states, when there are more than two numbers, the order of operations can be changed:

$(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. This property can be used in flexible mental arithmetic such as to transform $16 \cdot 25$ to $4 \cdot 4 \cdot 25$ by a factorisation and then apply the associative property to change the order of calculations, $(4 \cdot 4) \cdot 25$, to $4 \cdot (4 \cdot 25) = 4 \cdot 100$.

Compared to commutativity and distributivity, there is not much research concerning students' understandings of associativity (Ding, Li, & Capraro, 2013; Robinson, Ninowski, & Gray, 2006). According to Schifter et al. (2008), this is not surprising, at least not in primary grades, since students seldom meet more than two factors in multiplication. Nevertheless, they report of students making sense of doubling-halving strategies by means of both equal groups and rectangular area, using concrete materials. Doubling and halving strategies are underpinned by associativity, for example $12 \cdot 3 = (6 \cdot 2) \cdot 3 = 6 \cdot (2 \cdot 3) = 6 \cdot 6$. Students may reason that if the number of groups is twice as many, the number of objects must be halved in each group, and that the area of a rectangle is not changed if it is half as long and twice as wide. In a Canadian study, it was asserted that sixth and eighth grade students did not apply associativity frequently when presented as a three term problem in the format $a \cdot b/c$, thus not only testing associativity but also inverse relation of multiplication and division (Robinson et al., 2006). A problem connected to associativity noticed in a study among prospective primary teachers is that associativity and commutativity are confused (Ding et al., 2013), probably since both are expressed as changing the order (Zaslavsky & Peled, 1996). The difference, that commutativity allows change of the order of the numbers and associativity the order of the operations, might not be clarified in instruction. Ding et al. (2013) explain the prospective teachers' confusion by their own learning experiences from primary and secondary school.

To summarise, in spite of the rare number of studies concerning the arithmetical properties, the literature review established the importance of understanding and using the properties both for arithmetic calculations and future studies in algebra. One possible reason for the rare number of studies concerning students' understandings of the properties is the problematic in designing studies to investigate understanding of the properties (J. Torbeyns, personal communication, February 17, 2016). All studies in the review regarding students' understandings of the arithmetical properties have made inference of understandings of the properties from tasks involving calculations. The literature suggests that distributivity is difficult to learn (Carpenter et al., 2005; Ding & Li, 2014) and harder to understand than commutativity (Squire et al., 2004). At the same time, other scholars suggest that commutativity is harder to understand than distributivity (e.g. Schliemann et al., 1998). It also suggests that well-organised instruction on distributivity underpinned by models of multiplication is fruitful, and that both equal groups and rectangular arrays and area can be used as models of multiplication (Barmby et al., 2009; Izsák, 2004; Lampert, 1986; Young-Loveridge & Mills, 2009).

2.4 Calculations in multiplication

In this section I review literature concerning multiplicative calculations of multi-digits. Compared to the first experiences with single-digit multiplication, this topic has attracted less research (Fuson, 2003; Izsák, 2004; Verschaffel et al., 2007). In the review I describe what types of problems the transition from single-digit multiplication of integers to multi-digits can entail. It is organised in three themes: the ambiguous role of repeated addition, overgeneralisations of addition strategies leading to erroneous reasoning and multiplicative calculations for multi-digits.

Studies on recall of number facts in the multiplication table are omitted from this thesis, since they typically are conducted in the domain of single-digits. This does not imply that I find procedural knowledge of calculations and number fact fluency as unimportant or uninteresting, on the contrary, but they are not within the scope of this thesis.

2.4.1 Repeated addition

As already mentioned, repeated addition is questioned with respect to multiplicative reasoning. Multiplicative reasoning involves calculative acts coordinating composite units (Sowder et al., 1998), including expanded factors, such as 20 + 6 (Izsák, 2004), and is based on one-to-many correspondence, while addition has its roots in part-whole relations (Nunes et al., 2015; Park & Nunes, 2001; Vergnaud, 1983). Repeated addition is therefore sometimes described as linear or one-dimensional, in contrast to the two-dimensionality of multiplication (Barmby et al., 2009; Confrey & Smith, 1995). At the same time, there is evidence of repeated addition as a successful intermediate strategy among students who invent their own strategies (Ambrose et al., 2003; Baek, 2008). Fischbein et al. (1985) suggest that repeated addition is the intuitive model, and strategy, for multiplication. Some researchers question this and argue that one-to-many correspondence (Nunes & Bryant, 2010) or splitting (Confrey, 1994; Confrey & Smith, 1995) is separating multiplicative reasoning from additive, and therefore the operations have different roots. Irrespective of the origin of multiplication, repeated addition transforms multiplication into addition by ignoring the quantity transformation (P. Thompson & Saldanha, 2003).

However, in the initial learning of multiplication, repeated addition is thought of as an intermediate stage; young students start by counting all objects, and move over repeated addition to multiplicative calculations (Fuson, 2003; Mulligan & Mitchelmore, 1997). In this initial stage, repeated addition is considered more sophisticated than to count all, but when the multi-digit multiplication is introduced it becomes a cumbersome strategy and other multiplicative strategies are needed (Anghileri, 1999; Lo et al., 2008; Mulligan & Watson, 1998; Verschaffel et al., 2007; Young-Loveridge & Mills, 2009). Repeated addition is closely associated with the equal groups model (De Corte & Verschaffel, 1996; Simon & Blume, 1994), preserving the one-dimensionality of addition (Anghileri, 2000; Barmby et al., 2009). On the other hand, repeated addition has proved to support students' use of distributivity and associativity (Ambrose et al., 2003; Baek, 2008), thus repeated addition to multi-digit multiplication seems to be both disadvantageous and beneficial.

2.4.2 Overgeneralisation of addition strategies

Influence of additive reasoning, manifested in overgeneralisation of additive strategies to multiplication, can form a structural hindrance (Simon & Blume, 1994). For example, to calculate $19 + 26$ as $(10 + 20) + (9 + 6)$ is a sensible addition strategy (Beishuizen, 1993; Fuson et al., 1997), but to multiply only within ones and tens separately and then add the two products, such as to calculate $19 \cdot 26$ as $(10 \cdot 20) + (9 \cdot 6)$ is not (Foxman & Beishuizen, 2002; Lo et al., 2008; Young-Loveridge & Mills, 2009). This mistake has been found both in mental arithmetic and in vertical algorithms, both among students and prospective teachers (Foxman & Beishuizen, 2002; Lo et al., 2008; Young-Loveridge & Mills, 2009). Some prospective teachers claimed that it should be valid to use addition strategies to multiplication, since multiplication is repeated addition (Lo et al., 2008).

Another influence of addition is to assume that an increase (or decrease) of 1 to one factor will increase (or decrease) the product by 1 (Squire et al., 2004). For example, to believe that $5 \cdot 19$ can be calculated as $5 \cdot 20 - 1$, since 1 was added to 19, 1 should be subtracted. This type of error was most frequent to distributive tasks in their study; if the cue stated that $26 \cdot 21 = 546$ and the problem was $27 \cdot 21$, most wrong answers were 547. Another way to explain these erroneous strategies for multiplication, is to refer them to misunderstandings of distributivity (Squire et al., 2004).

2.4.3 Multiplicative strategies for multi-digits

Multiplicative strategies for multi-digit multiplication are underpinned by commutativity, distributivity and associativity. Students' understandings of the properties can be implicit, as in the reports of student-invented strategies starting by repeated addition and finding ways to simplify the additions (Ambrose et al., 2003; Baek, 2008; Lampert, 1986).

An example of a multiplicative strategy using associativity for $16 \cdot 25$ is to group the 25s in four groups of 4, by dividing 16 by 4 thus transforming the task to $4 \cdot (4 \cdot 25) = 4 \cdot 100$. An intermediate strategy is to successively double the multiplicand (Baek, 2008), for example to write 25 sixteen times make eight groups of 50, four groups of 100, two groups of 200 and finally one group of 400, see figure 1.

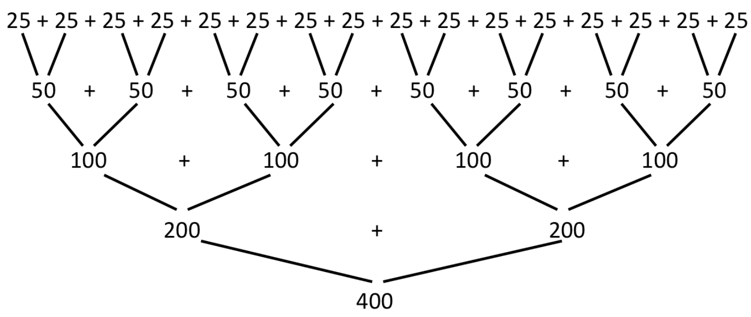

Figure 1. Successive doubling strategy

This strategy illustrates the transition from repeated addition to multiplicative strategies, and the successive doubling can be regarded as a mix of repeated addition and multiplicative reasoning. A more sophisticated strategy would be to directly group the 25s in four groups as described earlier.

Another multiplicative strategy using distributivity is to partition the multiplier in chunks that are easy to calculate; for example, to exploit that 10 and 5 are easy to use as multipliers, $16 \cdot 25$ can be calculated as $10 \cdot 25 + 5 \cdot 25 + 1 \cdot 25$ (Ambrose et al., 2003).

A number of findings concerning students' use of distributivity in their calculations are related to classroom studies imposing models for multiplication, both equal groups (Lampert, 1986) and rectangular array and area (Barmby et al., 2009; Izsák, 2004; Young-Loveridge & Mills, 2009) have been used. In Lampert's study, American fourth grade students learnt to calculate multi-digit multiplication by distributivity through a series of lessons based on the model of equal groups and repeated addition represented by word problems and drawings. She claims, that by connecting the number sentences to stories and drawings, the students could avoid common mistakes, such as those described in section 2.4.2, since they kept in mind what the story was about.

The three classroom studies, in which rectangular array was imposed as a thinking and calculation tool to foster multiplicative thinking based on distributivity, were conducted in England (Barmby et al., 2009), the USA (Izsák, 2004) and New Zealand (Young-Loveridge & Mills, 2009) with students between 9 and 13 years old. The English study presented an array of dots on a computer screen, the New Zealand used dots on paper and the American used a series of material starting with an array of unit squares, which gradually were transformed into rectangular area over rectangles of 100s, 10s and 1s to illustrate the magnitudes. Both the American and the New Zealand study encouraged students to connect the array and distributivity by connections to numerical calculations, for example, by using the grid method. The English study emphasised to partitioning the factors in chunks that are easy to calculate. For example, the array of dots was organised in four $5 \cdot 5$ areas that were closer together forming visible 100s as well as 25s. Irrespective of these differences, they all reported that the majority of students gained from the instruction material as measured by performance on multi-digit calculations. However, in all three studies there was a small group of students who did not make any connection between the array and the calculation. For example, students made the calculation first and then struggled to find a way to show the result in the array. Similar findings are reported for prospective teachers trying to use rectangular area as a representation for distributive calculations and struggling to connect their calculations to the picture (Lo et al., 2008). The difficulty to connect rectangular area and multiplication by distributivity might be explained by absence of connections between area and multiplication, rather than problems concerning distributivity. To know the area formula does not necessarily imply that multiplication is viewed two-dimensionally and hence to connect the area to the operation is not self-evident if multiplication is viewed one-dimensionally as in repeated addition (Izsák, 2005; Simon & Blume, 1994).

Other instructional attempts for connecting distributive calculations to conceptual knowledge can be found in textbooks. In an analysis of Chinese textbooks it was reported that distributivity was typically explained by word problems in combination with two separate calculations (Ding & Li, 2014). For example, 5 jackets for 65 yuan and 5 trousers for 45 yuan, combined with two ways to calculate ($(5 \cdot 65) + (5 \cdot 45)$ and $5 \cdot (65 + 45)$). This is another instance of using an asymmetrical model for multiplication, as in Lampert's (1986) study, and also an explicit way to show that it does not matter in which order the calculations are performed.

3 Aim and research questions

To sum up the literature review, three components, in addition to multiplicative reasoning, have been identified in literature as important building blocks for understanding of multiplication in the stage when multiplication is extended to multi-digits and decimals: models for multiplication, the arithmetical properties and calculations.

Multiplicative reasoning is essential for mathematics after primary school and it has proved to be challenging for students to master. Multiplicative reasoning involves coordination of composite numbers at several levels, including expanded numbers such as 20 + 6, and to interpret and understand multiplicative comparison. However, most studies seem to use multiplicative comparison to investigate students' multiplicative reasoning, not calculations. Therefore one question for further investigation is the relation between students' understandings of multiplicative comparison problems and their calculations.

Models for multiplication has been suggested as an important tool for development of conceptual understanding of properties of operations, but the theoretical suggestions are not always consistent with empirical findings. Especially the relations between understanding of distributivity and the models of equal groups, rectangular array and area have inconsistencies. The equal groups model has been suggested as less suitable than rectangular array to make sense of distributivity, but students have successfully employed the equal group model to learn distributivity. The array model has also proved to be useful for distributivity, even though some students had problems connecting the model to the multiplicative calculation. Rectangular area is suggested to support conceptualisation of decimal multiplication, but empirical studies have shown that area can be unconnected to multiplication, and then the model loses its explanatory function. There are also inconsistent claims in the literature concerning how difficult distributivity is to learn, both in comparison to commutativity and to calculations. These findings, in combination with the strong emphasis on how important distributivity is for calculations and algebra, raise the need to better understand the process of how students make sense of distributivity.

The aim for this thesis is to understand more about students' understandings of multiplication when it is expanded to comprise multi-digits and decimals, specified as an overarching research question:

> What do students' responses to different forms of multiplicative tasks in the domain of multi-digits and decimals reveal about their understandings of multiplication?

This question is operationalised in the following research questions, which are addressed in the four papers:

1. What is the sufficiency of multiplicative comparison problems for uncovering students' multiplicative reasoning? (paper 1)
2. What do students' evaluations of erroneous calculations of multi-digit multiplicative problems reveal about their understandings of distributivity? (paper 2)
3. What is the role the of the equal groups model for multiplication for students' understandings of the distributive property? (paper 3)
4. How do students' connections between models for multiplication, calculations and the arithmetical properties vary over time? (paper 4)

This thesis is problem-driven, meaning that questions were formulated before a suitable methodology was chosen. The overarching question, regarding what students' responses to different forms of multiplicative tasks in the domain of multi-digits and decimals reveal about their understandings of multiplication, places this thesis in the cognitive field of mathematics education and signals that it is a descriptive study.

4 Students' understandings

The notion of *students' understandings* is central in this thesis. In this chapter I review literature about students' understandings and how this notion can be investigated.

4.1 Understanding of mathematical concepts

Mathematical understanding has often been described as comprising both procedural and conceptual knowledge (Barmby et al., 2009; Baroody et al., 2007; Kilpatrick et al., 2001; Rittle-Johnson, Schneider, & Star, 2015; Star, 2005). According to Star and Stylianides (2013) the terms procedural and conceptual knowledge are a continuation of Skemp's (1976/2006) distinction of instrumental and relational understanding and have been used since the 1980s. However, there is a discrepancy in what they are employed to denote; some literature describes procedural and conceptual knowledge as different types of knowledge, and some as different quality of knowledge (Maciejewski & Star, 2016).

Procedural knowledge is defined as step-by-step knowledge, such as to perform a calculation in an algorithm (Maciejewski & Star, 2016; Rittle-Johnson et al., 2015), while conceptual knowledge often is described as connected to other "units of knowledge" (Hiebert & Carpenter, 1992, p. 78) or as a "connected web of knowledge" (Hiebert & Lefevre, 1986, p. 3). This has led to a view of procedural knowledge as unconnected and hence of less quality. However, procedural knowledge can be richly connected both to other procedures and to conceptual knowledge (Baroody et al., 2007; Maciejewski & Star, 2016; Star, 2005). The quality of the knowledge would rather lie in more, stronger and well-organised connections (Baroody et al., 2007; Star, 2005).

Definitions of conceptual knowledge in the literature often refer to the definition from Hiebert and Lefevre above, describing it as well connected (Crooks & Alibali, 2014). The definition of conceptual knowledge and how it can be measured were investigated in Crooks and Alibali's study and they report of three commonly used categories of definitions: general principled knowledge, knowledge of principles underlying procedures and connection knowledge. General principled knowledge is to know what principles regulate a domain, for example that division is the inversion to multiplication, and might be abstract and implicit, thus not necessarily verbalisable (Crooks & Alibali, 2014; Rittle-Johnson et al., 2015). Knowledge of principles underlying procedures is similar as general principled knowledge, but explicitly explains and justifies a procedure. In contrast, connection knowledge links together knowledge, both conceptual and procedural, thus it is another type of knowledge (Crooks & Alibali, 2014). In the view of Star and his colleagues (Maciejewski & Star, 2016; Rittle-Johnson et al., 2015; Star, 2005; Star & Stylianides, 2013), conceptual knowledge can be shallow, sparsely connected, or deep, well connected. An example of shallow conceptual knowledge can be to recite a definition, such as the formula for rectangular area, which Hiebert and Carpenter (1992) denote as learnt without conceptual connections. The quality of the knowledge would rather lie in more, stronger and well-organised connections (Baroody et al., 2007; Star, 2005).

Several researchers argue that conceptual and procedural knowledge are complementary (Maciejewski & Star, 2016) and that there is a bidirectional relation between procedural and conceptual knowledge in mathematics (Rittle-Johnson et al., 2015). The bidirectional relation implies that procedural and conceptual knowledge are dependent on each other to grow deeper. For example, by learning procedures, such as an algorithm, the underlying principles are strengthened, and reciprocally, by learning the underlying principles the procedural knowledge grows. Consequently, well-connected conceptual and procedural knowledge, compared to less connected, is described as a sign of deeper understanding in a number of frameworks for students' understanding in mathematics (Baroody et al., 2007; Star, 2005). Indeed, connectedness is suggested as central to a deep and strong understanding (Barmby et al., 2009; Baroody et al., 2007; Gray & Tall, 1994; Richland, Stigler, & Holyoak, 2012).

This implies that to investigate understandings of multiplication, both procedural and conceptual knowledge, as well as connections between them, needs to be studied. Procedural knowledge may become observable in procedures, but conceptual knowledge and the connections need to be externally represented to become observable.

4.1.1 Representations

There are both theoretical assumptions and empirical evidence that representations, both external and internal, have an important function in the construction of conceptual understanding and in the communication about concepts and procedures (Barmby, Bolden, Raine, & Thompson, 2013; Hiebert & Carpenter, 1992). Internal representations are described as hidden within a persons mind, "as the way in which concepts are stored mentally" (Bolden, Barmby, Raine, & Gardner, 2014, p. 60) while external representations are what other persons can see and hear (Goldin & Shteingold, 2001).

To investigate external representations and infer understandings of a concept builds on the idea of "correspondence between the form of external representations with which students interact and the internal representations they create" (Hiebert & Carpenter, 1992, p. 90), which has been a basis for several studies (Berthold et al., 2009; Crooks & Alibali, 2014; Hiebert & Wearne, 1992; Rittle-Johnson & Alibali, 1999).

External representations can be categorised according to mode: verbal, visual, symbolic or numerical (Selling, 2016). For example, a model for multiplication can be represented verbally, as a word problem, or visually, as a drawing. Representations in one mode, for example visual, can be subcategorised, for example as diagrams, concrete materials or drawings. Concrete objects, such as manipulatives placed in rows and columns or sketched drawings of a chocolate bar, are examples of representations of the model of rectangular array (Barmby et al., 2009). External representations can serve as thinking tools for the abstract mathematics they represent (Greeno & Hall, 1997; Izsák, 2005; Pape & Tchoshanov, 2001; Selling, 2016).

There are at least two roles of representations; as means to communicate, both with one self and others, and as means to construct understanding. Communication and understanding are not exclusive, but overlapping; the act of using representations as thinking tools is a way of communicating and it affects the conceptual understanding (Hiebert & Carpenter, 1992; Pape & Tchoshanov, 2001; Selling, 2016).

In order to function as a representation that aids communication the representation needs to have a meaning that the communicators agree on (Greeno & Hall, 1997; Hall, 1998), which might need negotiating in the social setting of a classroom or a research interview (diSessa, 2004; Pape & Tchoshanov, 2001). To communicate by means of external representations may include creating representations (Greeno & Hall, 1997). In the creative act of inventing representations students might externalise their internal representations (Pape & Tchoshanov, 2001), which draws on the theoretical assumption of similarity of internal and external representations (Barmby et al., 2013; Hiebert & Carpenter, 1992).

Representations as building blocks for construction of understanding are thought to be central in the learning of mathematics (Acevedo Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009; Goldin & Shteingold, 2001; Greeno & Hall, 1997; Hiebert & Carpenter, 1992; Izsák, 2005; Panasuk & Beyranevand, 2010). Furthermore, external representations can be used to "facilitate an argument and to support conclusions" (Pape & Tchoshanov, 2001, p. 125).

4.1.2 Connections

Understanding can be viewed as connections between representations of different types of knowledge, and the nature of connections has been suggested to be reasoning (Barmby et al., 2009). Reasoning is a large research field in mathematics education and is described in several ways, for example, as making generalisations and constructing arguments for generalisations being true or false (Stylianides, Stylianides, & Shilling-Traina, 2013) or as “the line of thought adopted to produce assertions and reach conclusions in task solving” (Lithner, 2008, p. 257). In a broad sense, reasoning can be defined as “the process of drawing conclusions”, as a line of thought independent of specific tasks (Leighton, 2003). I choose to use this broad and unspecified definition of reasoning and accept reasoning as what constitutes the connections as suggested by Barmby et al. (2009). This leads to the opportunity to view utterances, supported by drawings, gestures and other external representations, as reasoning connecting different representations of different types of knowledge. An example of reasoning connecting two representations in multiplication is to reason that adding sixteen 25s is the same as adding eight 50s since the result is the same. This reasoning is connecting the representation of multiplication as repeated addition to implicit knowledge of associativity. The same reasoning could be considered as stronger if it also links to a representation of the sixteen 25s as becoming half as many 25s but twice as large, since the 25s are put together two and two to 50s. This example demonstrates that connections, seen as reasoning, can be of different quality.

In several studies external representations and connections between them are employed to infer students’ understanding. Two examples of studies are first graders understandings of place value (Hiebert & Wearne, 1992) and high school students’ understandings about probability (Berthold et al., 2009), which illustrate the wide range of age groups it has been applied to. In this thesis I adhere to the assumption that the internal and the external representation for a concept can be similar, but not necessarily the same (Hiebert & Carpenter, 1992), and that it is possible to infer understanding by observing connections, in the form of reasoning, between these representations (Barmby et al., 2009).

To have parallel representations, and connections between them, which enables to switch representation to what best fits the problem at hand, is viewed to indicate deeper understanding than having a single representation for a mathematical concept (Acevedo Nistal et al., 2009; Dreyfus, 1991; Panasuk & Beyranevand, 2010). I also agree with the body of research that suggests that the more and stronger connections between representations and between forms of knowledge, as well as the more structured the connections are, the deeper is the understanding (Barmby et al., 2009; Baroody et al., 2007; Crooks & Alibali, 2014; Gray & Tall, 1994; Hiebert & Carpenter, 1992; Hiebert & Wearne, 1992; Izsák, 2005; Maciejewski & Star, 2016; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2015; Star, 2005; Star & Stylianides, 2013).

In this thesis a model for students' understandings of multiplication has been drawn from the reviewed literature above, placing connections in the centre. The connections link procedural and conceptual knowledge, both viewed as representations of multiplication. The connections are seen as reasoning.

4.2 Understandings of multiplication

In this section I draw from the reviewed literature in section 4.1 to present a model for students' understandings of multiplication. In chapter 2 central components for multiplication in the stage when multiplication is expanded to multi-digits and decimals were identified. In this section I put these components together in a model for mapping connections between the components in a simplified diagram in figure 2.

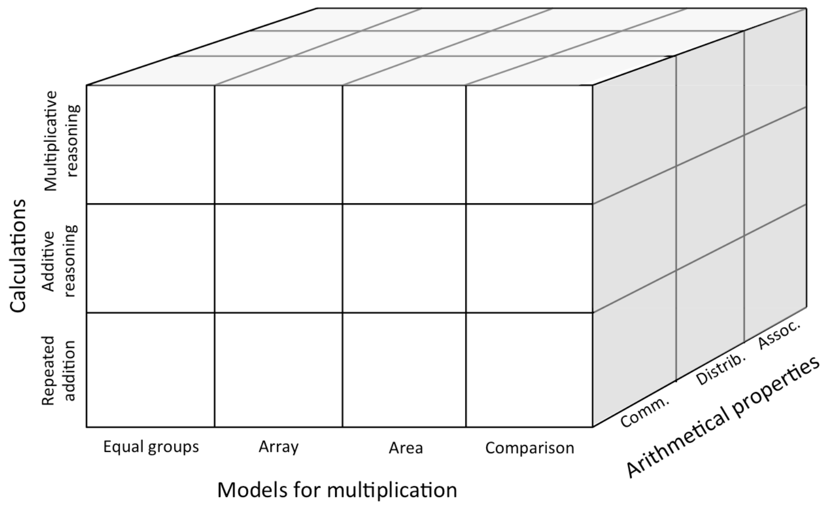

Figure 2. The multiplicative cuboid

The width reflects the dimension of models for multiplication, the height reflects the dimension of calculations and the depth the dimension of the arithmetical properties. The width can be divided into as many models of multiplication as one wishes to investigate, for example symmetrical and asymmetrical or finer grained by dividing it into more models, as in figure 2. The height can be divided in the number of distinctly different calculation strategies that are of interest, here repeated addition, addition influenced strategies and multiplicative strategies are chosen, and the depth in the arithmetical properties, commutativity, distributivity and associativity, which all are applicable to multiplication and underpin calculation strategies.

The components are not of the same type and they are illustrated as different dimensions of a cuboid.

There is no clarified hierarchy of importance of different models, although some models are assumed to be easier than other. More importantly, they have different properties and hence different feasibility to various properties and number sets, as was described in section 2.2. The arithmetical properties are mathematical laws, and hence not hierarchically ordered, even though commutativity and distributivity are suggested to be more central than associativity in the stage of expanding multiplication from single-digits (Schifter et al., 2008), and their importance for multiplicative reasoning was described in section 2.3. However, among the calculations there is a hierarchy. Repeated addition is a less sophisticated strategy than multiplicative strategies underpinned by the arithmetical properties. Between the repeated addition and multiplicative strategies are strategies reflecting overgeneralisation of addition, which implies that they generally yield erroneous answers as described in section 2.4. It could be argued that the erroneous answers should place additive reasoning strategies below repeated addition. But the effort of trying to use a more sophisticated strategy warrants its placement above repeated addition.

The small cubes of the cuboid represent three-way connections. An example of a such a connection is to calculate $16 \cdot 25$ by use of the distributive property, as $10 \cdot 25 + 6 \cdot 25$, and connecting it to the model of equal groups by explaining that one can first calculate 10 of the groups and then the remaining 6 groups, irrespective whether knowledge of distributivity is implicit or explicit. This example of connections would fit in the cube where all three dimensions meet. To calculate in the same manner without connecting it to any model for multiplication would be in the two-dimensional square on the surface of the cuboid where the arithmetical properties and calculations meet. An example of unconnected understanding is to calculate $16 \cdot 25$, for example by an algorithm, without explanation why it works neither from implicit knowledge of distributivity nor any model for multiplication. This example would be placed on the one-dimensional edge of the calculation dimension.

This model does not imply that to perform calculations without any reference to models of multiplication or knowledge of what properties are used is a sign of less connected components and thus more shallow understanding. When a calculation strategy has become familiar and is frequently used, there is no need to make connections to models or properties. But when asked to explain *why* the procedure works, the connections to properties and models can be a sign of deeper understanding.

The cuboid gives opportunity to map a person's demonstrated connections. The external representations, which a person creates by verbal utterances, drawings and calculations that are related to any of the three components, can, potentially, comprise a connection to one of or both the other components, thus fit into one of the squares or cubes.

4.2.1 Analytical tool

To use the multiplicative cuboid for fine-grained analyses it needs to be transformed into two-dimensional shape. Since there are three dimensions of the cuboid, three matrices can be created, see figures 3–5. The matrices make it possible to record connections between two of the dimensions at a time and were presented and discussed in a slightly different layout at an international conference (Larsson, 2015).

To demonstrate what can be recorded in the cells I show an example of a connection that a student, here called Erik, made between repeated addition and commutativity by calculating 5 · 19 as repeated addition by adding both five 19s and nineteen 5s, see figure 3.

	Commutative property	Distributive property	Associative property
Repeated addition	Erik calculated 5 · 19 as 19 + 19 +… and as 5 + 5 +…		
Addition influenced strategies			
Multiplicative strategies			

Figure 3. Matrix for connections between arithmetical properties and calculations

In figure 4 the connections between models for multiplication and the arithmetical properties are focussed. An example of a possible connection can be to explain distributivity by referring to divide equal groups into two parts.

	Commutative property	Distributive property	Associative property
Equal groups	questionable		
Rectangular area			
Rectangular array			
Multiplicative comparison	questionable		

Figure 4. Matrix for connections between arithmetical properties and models for multiplication

Note that it says “questionable” in two of the cells in figure 4; to connect commutativity to asymmetrical models of multiplication is questionable since asymmetrical models are thought to conceal commutativity (Bell et al., 1989; Greer, 1992; Lo et al., 2008; Verschaffel et al., 2007). However, there are reports of students making this connection (Carpenter et al., 2003). In figure 5 connections between models for multiplication and calculations can be noted.

	Equal groups	Rectangular area	Rectangular array	Multiplicative comparison
Repeated addition				
Addition influenced strategies				
Multiplicative strategies				

Figure 5. Matrix for connections between models for multiplication and calculations

The multiplicative cuboid, and the matrices derived from it, enables analyses of how connections can be more or less frequent as well as what connections different students demonstrate, possibly creating different patterns of connections.

Note that it is not necessarily a sign of deep understanding to have many connections, to fill most cells; the structure and quality of what is connected and the connections are also important (Barmby et al., 2009; Baroody et al., 2007; Hiebert & Carpenter, 1992; Hiebert & Wearne, 1992; Izsák, 2005; Star, 2005). For example, to have connections to repeated addition and commutativity is not necessarily strengthening understanding of multiplication. The example in figure 3 is not as sophisticated as to connect commutativity to a symmetrical model for multiplication to make arguments of generalised commutativity.

The multiplicative cuboid could, in theory, reveal compartmentalised understandings by no or few cells containing utterances. However, this would draw on the assumption that all connections are made observable by explicit reasoning, which is unlikely to happen. Despite efforts to offer a variety of tasks to elicit reasoning and representations for multiplication, there would still be connections and representations that are abstract and not verbalisable (Crooks & Alibali, 2014).

5 Method

In this chapter, I present methodological choices in relation to my research questions. I start by describing how the participants were chosen followed by data collection activities, including tasks and rationales for these activities and tasks. This is followed by a description of analyses and finally I present what ethical considerations were made.

5.1 Participants

The participating students comprise a convenience sample (Bryman, 2008) with no pretension of being representative. However, I chose the school for its diverse profile of students considering both the parents' socio-economic status and the proportion of immigrant students. The results in mathematics are known to vary in relation to these factors (Hansson, 2012). This particular school increased the possibilities for a diverse group of students in order to enable various demonstrations of students' understandings. Inviting students from schools with different profiles could have accomplished the diversity, but would have made the data collections more time consuming. In addition to the school's diverse profile, it included three parallel classes of the grade in question to ensure enough students.

After getting agreement for the study from the school's principle, I contacted the three teachers that were teaching mathematics to grade four in spring 2012. Two of the teachers were planned to teach the same classes in mathematics in grades five and six and thus suitable for the study. Since the study was investigating the extension of multiplication to multi-digits and decimals, I wanted to start the study just before these domains were introduced, which is the second semester of grade five. Hence the study should take place in grades five and six, when the students would be 10–13 years old. The decision to include the first semester of the seventh grade was made when the students were in sixth grade. Both teachers were well-qualified and experienced, they had 14 and over 25 years of experience respectively. I described the research project to the teachers, both written and orally, and provided opportunity to ask further questions. Both teachers agreed to participate.

Next step was to get the students' and their parents' informed consent. A written description of the study, including contact information to both my supervisors and me, was handed out to the students to take home. The aim for the research and how the collected data would be used were explained in the letter. The participants were also informed of their right to withdraw from the study at any time and that they would be kept anonymous in all presentations of the study. Both the student and a parent needed to sign a document where they explicitly agreed either to participate fully, including video and audio recordings, or to participate partially, which was to participate in everything except recordings. Before handing out the written information, all students were orally informed about the study, its rationales and its aims. I spent three half days in each of the classes to enable the students to get accustomed to me and ask more questions about the study in a more informal way. In addition I could learn all the students' names prior to the study.

In the first semester twenty-seven students and their parents agreed to participate fully and nineteen students chose to participate partially, without recordings. In the following years some students left the school and a few newly arrived were invited to participate. A stable group of twenty-two students participated fully through the whole study. The last semester of the study, in the seventh grade, only eight of the students were interviewed. These eight were chosen as representatives for different learning trajectories discovered by tentative analyses of the data from earlier semesters.

5.1.1 Teaching context

Both teachers taught the students through fourth to sixth grade and one also the seventh grade. They used the same textbook series, but according to their own claims and my observations when visiting their classes, there were differences in their instruction. One of the teachers, who taught eighteen of the twenty-two students and followed them to seventh grade, followed the textbook closely, and the students worked individually, solving the tasks in the textbooks most of the time, which is common practice in Swedish mathematics instruction (Skolverket, 2009). The other teacher spent more time on whole class teaching, mathematical games and other activities and hence less time was dedicated to individual work in the textbooks.

The textbook series consists of one book for each semester and typically presents a problem type and how it can be solved on the top of a page and a number of problems, to which the solution method is applicable, on the rest of the page. There are also pages with mixed problems, summaries and challenges. The books cover different mathematical content areas, such as large numbers, geometry and decimal numbers, in separate chapters.

In a simple analysis of the multiplication problems in the six textbooks for grades four to six, it was found that more than half of the problems (64 %) were uncontextualised; such as 'calculate $40 \cdot 20$'. The contextualised problems were predominantly equal groups problems (78 %) given as word problems. In the first book for grade four, there were pictures of rectangular arrays and of equal groups, in the five subsequent textbooks pictures for multiplicative situations were rare, except for images of money.

The textbook series promotes both horizontal and vertical multiplication calculations by partitioning the numbers by place value and explanations were provided by comparing ones to 1 kronor coins, tens to 10 kronor coins and hundreds to 100 kronor notes. Both the vertical and the horizontal calculations were presented by procedural step-by-step instruction. Once in the book for the second semester of the sixth grade, an alternative calculation was presented to simplify decimal multiplication by doubling and halving in order to transfer the problem into integers, for example $3.5 \cdot 6 = 7 \cdot 3$.

The second semester in grade five, multi-digit and decimal multiplication were introduced. Both were introduced by procedural step-by-step instruction for how to calculate both vertically and horizontally, and all the decimals were larger than 1. In the first semester of the sixth grade, decimals less than 1 were included together with tasks involving to reflect over the product not being larger than the factors. Percentage was introduced the second semester in grade six, as a way to find *x* % of a number through finding 1 % (or 10 %) and then multiply.

5.2 Data collections and tasks

In order to get different types of data, a number of different forms of tasks were used, which is considered as essential in studies aiming at investigating what connections students make (Barmby et al., 2009; Bisanz, Watchorn, Piatt, & Sherman, 2009; Hiebert & Carpenter, 1992). Individual clinical interviews give opportunity to follow the students' line of reasoning by following up questions (Ginsburg, 1997), thus both procedural and conceptual knowledge can be investigated (Crooks & Alibali, 2014). Paired interviews can capture students' reasoning while it takes place when students solve problems together (Izsák, 2004; Schoenfeld, 1985). Written tests allow each student to think about the problems without the potential stress of a researcher watching and additionally have the advantage to be time efficient. As a complement, I also collected the students' solutions to the written national tests in grade six as well as interviewed the teachers once in the beginning of the study. In the following I describe data collection methods and tasks that were used in the four parts of my study reported in this thesis.

5.2.1 Clinical individual interviews

To investigate the students' understandings of multiplication, as manifested by their reasoning, clinical individual interviews were the main method to collect data. Each individual clinical interview involved four forms of tasks; multi-digit calculations, explanation for calculations, story telling to given multiplications and questions to explicitly explain multiplication. The questions were adapted to what the students said; that is I followed their lines of thinking in order to understand how they perceived multiplication.

The interviews were conducted during the school day in a small room close to the classroom. Each interview started by a reminder that the reason for my research was curiosity of how students really think when they work with operations, that the interview was recorded and that the student could withdraw from the research at any point if he or she wanted, without having to give any explanation. The curiosity of students' thinking is described as respecting the student by showing a real interest in his or her thinking, typical of clinical interviews (Ginsburg, 1997). The reminder of my curiosity also included that I explicitly stated that I was not interested in the correctness of their answers, but how they reached them.

A separate audio recorder and a smartpen were used for recordings. The smartpen records both audio and what is written (or drawn) during the interview. Both written and audio recordings can be replayed as it occurred, enabling a detailed transcription of the combination of the written and oral data.

The explicit explanations for multiplication made it possible to compare the students' answers over time as well as to compare what models for multiplication the stories they told reflected. Furthermore, explicit explanations have been reported as a means to directly measure students' conceptual knowledge and via the calculations students could demonstrate procedural knowledge, while explanations or justifications might demonstrate underlying conceptual knowledge (Crooks & Alibali, 2014). The numbers in the calculation tasks, which are presented in table 1 on next page, together with the number expressions for story telling, were chosen to elicit strategies underpinned by the arithmetical properties, predominantly distributivity. From the second semester, the calculation tasks were chosen after a preliminary analysis had been made of the students' solutions the previous semester to make use of experiences and adapt the tasks to challenge the students' reasoning.

Table 1. Calculation and story telling tasks during individual interviews

Time	Calculation	Story telling
Grade 5 1st semester	$5 \cdot 19$ $16 \cdot 25$	Any number
Grade 5 2nd semester	$15 \cdot 24$	Any number
Grade 6 3rd semester	$19 \cdot 42$	$19 \cdot 42$ $4.6 \cdot 3.9$
Grade 6 4th semester	$39 \cdot 23$ finish: $37 \cdot 21 = 20 \cdot 37 + ?$ $45 \cdot 19 = 20 \cdot 45 - ?$ mimic for $29 \cdot 42$	$39 \cdot 23$ $3.8 \cdot 4.9$
Grade 7 5th semester	$16 \cdot 25$ $2 \cdot 0.8$ $0.7 \cdot 4$ $0.2 \cdot 0.9$	$16 \cdot 25$ $2 \cdot 0.8$ $0.2 \cdot 0.9$

Single- by multi-digit numbers, such as $5 \cdot 19$, had been part of instruction at the time of the first interview, while two multi-digit numbers, such as $16 \cdot 25$, had not. By starting with a type of task that they had met in instruction, I hypothesised that students would feel more at ease, hence the first task was $5 \cdot 19$. Multiplication of two two-digit numbers was what the literature had indicated as problematic, hence what was focussed on throughout the study. All numbers in calculation tasks were chosen to elicit reasoning by distributivity and in one case, $16 \cdot 25$, also associativity. For example, 16 can be factorised into $4 \cdot 4$ and the task can then be solved as $(4 \cdot 4) \cdot 25 = 4 \cdot (4 \cdot 25)$, simplifying the calculation. Both tasks, $5 \cdot 19$ and $16 \cdot 25$, came from other studies (Foxman & Beishuizen, 2002; Ruthven, 1998), in which they had proved to elicit different types of reasoning, for example $5 \cdot 19$ can be solved by finding half of $10 \cdot 19$ or by $5 \cdot 20 - 5$. Since $16 \cdot 25$ was productive the first semester by eliciting distributivity among the students in this study, a similar task was chosen for the second semester, $15 \cdot 24$. Both have one number ending with a five, which had elicited reasoning about fives as easy to think with and the proximity of 24 to 25 might be productive for rounding up. Numbers ending with a nine are prone to compensation strategies, such as to round up to the next ten and adjust the answer by subtracting, such as to solve $19 \cdot 42$ as $20 \cdot 42 - 42$ (Heirdsfield, Cooper, Mulligan, & Irons, 1999). Hence both calculation tasks in grade six had one factor ending with nine. For the last interview in grade seven, $16 \cdot 25$ was reused to investigate students' development of strategies.

The two unfinished calculations and the mimic task were chosen to challenge students who had not yet tried any calculation strategy involving distributivity. These tasks can also be construed as measuring the underlying principles of procedures for those students that had used distributivity earlier

(Crooks & Alibali, 2014). The unfinished calculations were presented as they are written in table 1, including a use of commutativity, such as $37 \cdot 21 = 20 \cdot 37 + ?$ The reason for this choice was to not bias the task towards one of the two types of students with a firm view of which factor denoted the multiplier. Among the students, within the same class, there were some students with a firm view of the first factor denoting the multiplier and some students with a firm view of the second factor denoting the multiplier. In the mimic task, 29 and 42 were chosen to create opportunity to use both of the unfinished strategies, that is to split 42 in 40 + 2 and to round up 29 to 30 and compensate.

The two tasks with an integer multiplied by a decimal number used in grade seven, forced students to abandon repeated addition procedures, or to treat the natural number as the multiplier and hence use commutativity to one of the tasks. Since some students demonstrated a firm view of the first factor being the multiplier, while other students showed an equally firm view of the second factor being the multiplier, two versions were offered to all students. The multiplication of two numbers less than 1 violates the 'multiplication makes bigger' belief (Bell et al., 1981). The number 0.9 was chosen since it is close to 1, which might support ways of reasoning connected to rounding up and compensate.

In addition to calculations, the students were invited to story telling, similar to problem posing. To pose a problem to a numerical expression, such as $15 \cdot 24$, might expose students' conceptualisation of the operation (De Corte & Verschaffel, 1996; Prediger, 2008). To tell a story is similar to posing a problem, but the story does not necessarily need to comprise a question. For example "Sofia has three times as many pens as Martin, and since Martin has five pens, Sofia must have fifteen", is an example of a story matching the multiplication $3 \cdot 5 = 15$. The stories that students tell can give an image of what the students perceive as adequate multiplication by which model for multiplication they choose (De Corte & Verschaffel, 1996), but also as a reflection of what they have experienced in mathematics instruction (Verschaffel et al., 2007). By inviting students to tell stories to multi-digit and decimal multiplication, it was hypothesised that the stories would give an indication of how they conceptualised multiplication (Prediger, 2008).

During the first year of the study, grade five, the students could tell a story matching any multiplicative expression. From the second semester of the study, the student was prompted to find more stories for the same expression that they themselves thought of as different, thus given opportunity to demonstrate knowledge of more models for multiplication and what the student viewed as significant differences within multiplication.
From grade six they were prompted to tell a story to the same multi-digit task as they had first calculated, see table 1, in order to create opportunities for connecting the calculation strategy to a story. Almost all stories the students told during fifth grade had reflected equal groups. To challenge their

choice of model, they were also invited to tell matching stories to decimals from grade six. The decimal multiplications in grade six, $4.6 \cdot 3.9$ and $3.8 \cdot 4.9$, were chosen to be applicable both to the measures for a room, in metres, and for the cost of candy at 3.90 kronor/hg or 4.90 kronor/hg, which is a common every-day context in Sweden.

5.2.2 Paired interviews

It is hard to investigate how students think and reason when solving problems (Schoenfeld, 1985). To ask the student after the solution might result in a description of the successful decisions that led to the solution and the mistakes are omitted. This would lead to less data supporting my understanding of how the students were reasoning while they were working. Some researchers have tried to ask for simultaneous reporting of the solutions, talking out loud. This is however hard for some persons to do, especially young students (Schoenfeld, 1985). To put the student with a peer might get more detailed access to the thinking as they were working (Izsák, 2004). A drawback can be that one of the students dominates the problem solving and hence the reasoning of the less active student is not captured, but on the other hand it might give more detailed information of the reasoning process for the first student, or in best case, both students (Schoenfeld, 1985). The choice of putting students in pairs rather than larger groups, even though some interviews were in groups of three for practical reasons, was based on a pilot study, in which larger groups worked less well, typically by one student taking charge of the discussion, compared to pairs, in which both more often reasoned together.

The paired interviews were conducted in the same manner as the individual interviews (see section 5.2.1) with two exceptions; they were also video recorded and I did not interfere with the students' work unless to clarify something or if I could not hear what they said. Video recordings could ensure that the students' voices were correctly identified. In addition, a pilot study had revealed that when students were discussing and explaining a problem together the gestures and pointing increased, which contributed vital information of what their utterances were referring to.

The evaluation tasks were followed by prompts to explain and justify their reasoning, if they did not do that spontaneously. In this thesis only the paired interview from last semester in grade six is reported.

The students were invited to evaluate three invalid calculation strategies for the multi-digit multiplication, $26 \cdot 19$. All three suggested strategies originate from students in the study, suggested or conducted during individual interviews to multi-digit multiplication. They were presented on separate cards as if fictitious students were suggesting them, see figure 6. The suggested strategies reflect errors with respect to distributivity and influence of additive reasoning or overgeneralisation of addition calculations. They can also be regarded as tasks aiming for conceptual knowledge defined as principles underlying procedures (Crooks & Alibali, 2014; Rittle-Johnson & Alibali, 1999).

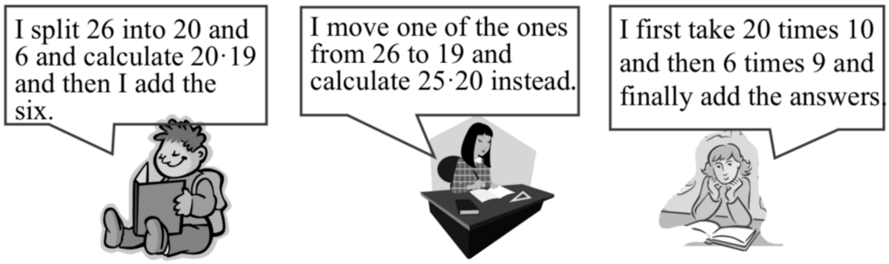

Figure 6. Three erroneous strategies

The first suggested strategy, to split 26 into 20 and 6, multiply 20 by 19 and then add the 6, reflects influence of additive reasoning: since 20 and 19 are already used in the multiplication, there is not anything else to do with the 6 that is 'left'. The two other suggested strategies have analogous strategies for addition. The second suggested strategy, to move 1 from 26 to 19 and calculate $25 \cdot 20$, is analogous to a compensating strategy for addition, $26 + 19 = (25 + 1) + 19 = 25 + (1 + 19) = 25 + 20$, and underpinned by associativity. The third suggested strategy, to partition both factors by place value and multiply ones and tens separately is also analogous to addition strategies and was described in the section about overgeneralisation of addition (section 2.4.2) since it has been reported in prior studies. During the paired interview the students were invited to evaluate if the suggested strategy was valid or not and explain why.

5.2.3 Written tests

Written tests with word problems were used to give information of students' passive knowledge of different multiplicative models, as a complement to their active knowledge when telling stories. In this thesis, only three of the ten word problems from the first semester were employed, namely the three problems reflecting multiplicative comparison and these are presented in table 2 on next page. The multiplicative comparison problems were used to measure students' ability to reason multiplicatively. This approach to evaluating multiplicative reasoning is in line with other studies (e.g. Clark & Kamii, 1996; Van Dooren, De Bock, & Verschaffel, 2010) and was contrasted to the students' calculations of $5 \cdot 19$ and $16 \cdot 25$ the same semester.

The written test were given during ordinary mathematics lessons to the whole classes, with oral information that the solution was more important than the answer to ensure that the students would write an expression how they calculated the answer. The test consisted of ten word problems and was divided into two sets of five problems, in order to avoid test fatigue, that were given one week apart.

Table 2. The multiplicative comparison word problems from the first semester (translated to English)

Question number	The word problems
3	Sofia has 50 kronor. Martin has 3 times as much money as Sofia. How much money has Martin got?
8	Sofia has 50 kronor. Martin has 150 kronor. How many times as much money has Martin?
10	Max has 150 kronor. This is 3 times as much money as Mollie has. How much money has Mollie?

The word problems were written in simple language resembling textbook tasks, with no superfluous or missing information. The problems were posed as multiplication and division problems, both partitive and quotitive. All numbers were ‘easy’, such as multiples of 50 or within the multiplication table. With too easy numbers there is a risk that students might use a superficial technique to only look at the numbers to decide what operation will solve the problem (Greer, 1992). This was balanced by the opportunity that easy numbers might help the students to keep their attention on the structure of the problems (Tzur et al., 2013).

5.2.4 Additional material

In addition to tasks generated explicitly for this project, some other data were collected: I interviewed the teachers during the first semester of the study and collected copies of the students’ solutions to the written parts of the national test in mathematics for grade six. Both the interviews and the written test material could contribute to a thicker description of students’ mathematical experiences and achievement.

The interviews with the teachers were conducted to get background information, such as their education and experience and what textbook series they used, but I mainly wanted to hear how they themselves described their instruction.

The written test materials were collected for all students who had accepted full as well as partial participation. Swedish students do national tests in the end of grade three, six and nine. The tests in mathematics for grade six in 2014 consisted of an oral part and four written parts. The test items are under secrecy, and I was given permission from the schools principle to collect the written tests for the students in the study under the condition that I do not reveal any of the test items. I copied the participating students’ four written test parts and marked them according to the guidelines for marking, provided from the Swedish National Agency for Education. These markings can be compared to the national results for the test, which is available on the National Agency's web pages. This implies that I can compare the students' overall mathematical achievements in the end of grade six to the national level.

5.3 Analyses

Analyses have been conducted by various methods, depending on the aim of each paper. In this section I first describe how transcriptions were conducted, since that was common for all papers. This is followed by a description of the analyses in relation to the four research questions, addressed in each of the papers.

5.3.1 Transcriptions

The analyses of the interviews started with transcription, transforming spoken language to written (Bryman, 2008). Transcriptions were primarily made from the audio recordings and later complemented by information from the smartpen, and in the paired interviews also from the video recordings.

The transcriptions were verbatim, but in written language, including sounds like "mm" and similar. Examples of changes to written language were to write "jag" [I] instead of "ja" and "är det" [is it] instead of "ere". The meaning of what the students think can be clearer in written language (Szabo, 2013) and serves my purpose better. When students were quiet longer than approximately three seconds, I also included how long pauses they made. Overlapping talk, strong emphases and laughter were noted, but no other audio-related information.

The smartpen allowed me to complement what students said with their writing and drawings exactly when it happened. This contributed to very precise information of what the students said and wrote simultaneously. The written materials were inserted in the transcriptions when they contributed to understanding what was said. In a similar fashion I complemented the transcription with descriptions from the video recordings. I only included gestures that I judged as significant to understand what the students meant. Examples of gestures always included were when students said *this*, *here*, *there* and similar while pointing at something. Example of other gestures noted in the transcripts were hand movements that students made, when they explained what impact different speed had on the distance between two swimmers, reported of at a conference (Larsson & Pettersson, 2015). In the final phase I complemented the transcripts with punctuation marks to enhance readability. The transcripts are in Swedish, and analyses of them were conducted in Swedish. Only excerpts that are presented are translated into English, and these are sometimes slightly changed into more formal language out of respect for the students.

5.3.2 Analyses for paper 1

The first paper investigates the sufficiency of multiplicative comparison problems for uncovering students' multiplicative reasoning. To find answers to this, the whole cohort of fully participating students' solutions to the three multiplicative comparison problems presented in section 5.2.3 were analysed. This was compared to how they had calculated in the interview the same period during first semester in grade five.

The answers to the word problems were analysed as described by Van Dooren et al. (2010), coding the answers as additive, multiplicative or other/undefined. An additive solution to 3 times as much as 50 would be 53 ($50 + 3$) and a multiplicative 150 ($3 \cdot 50$). If a student had answered all three items by multiplicative reasoning he or she was considered to reason multiplicatively. If a student demonstrated two or more additive answers he or she was considered to reason mainly additively to the word problems. The remaining students were considered to demonstrate mixed reasoning.

For the calculation tasks, the transcripts of the students' calculations of $5 \cdot 19$ and $16 \cdot 25$ were analysed, categorising the calculations as additive, multiplicative or other/undefined. Additive calculations were calculation strategies influenced by addition, such as to calculate the ones separately from the tens, typically yielding incorrect answers. In the calculation task, repeated addition was categorised as an additive calculation since the students' calculations were focussed, not only their answers. To repeatedly add multi-digit numbers is not considered as conceptualised multiplication (P. Thompson & Saldanha, 2003), even though it can lead to a correct answer. The multiplicative calculations were underpinned by the distributive or associative property. Most common was to partition one of the factors and for example solve $16 \cdot 25$ as $10 \cdot 25 + 6 \cdot 25$. Finally, some students' calculations could not be categorised according to reasoning and were categorised as other/undefined Students were then categorised as showing additive or multiplicative reasoning to both the calculations or as demonstrating mixed reasoning if not both calculations were additive or multiplicative.

5.3.3 Analyses for paper 2

The second paper answers to what students' evaluations of erroneous calculations of multi-digit multiplicative problems can reveal about their understandings of distributivity. Hence, the data to analyse were the transcripts from the paired interviews the second semester in grade six. In this interview the students were presented with three different erroneous strategies to calculate $26 \cdot 19$ (see figure 6 in section 5.2.2) and the students were invited to evaluate whether each suggested strategy was valid or not and explain why.

The analyses of these transcripts were conducted by a bottom-up procedure (Andrews, 2009), in order to investigate how the cohort of students reasoned to each of the three strategies. The data were iteratively read and clustered in groups of similar reasoning, re-read and checked for overlaps, possible refining or unitising of groups. The groups were labelled according to the overall idea that the reasoning reflected, which yielded seven categories labelled as general justification, equal groups, counterexample, check the answer, experience, additive reasoning and other/no answer. For example, the category equal groups covered reasoning in which students contextualised the numbers to the equal groups model for multiplication. Each pair could employ more categories which all were noted. See paper 2 for more examples of the reasoning categories.

This categorisation was presented and discussed at an international conference, which resulted in a second analysis of the categories. In the second analysis, properties of the reasoning of the categories were identified and compared, also as a bottom-up process. This resulted in five categories at a more general level: investigative reasoning on a meta-level, multiplicative reasoning by the distributive property, procedural reasoning, descriptive reasoning and not showing multiplicative reasoning.

5.3.4 Analyses for paper 3

In the third paper, the role of equal groups for students' understandings of distributivity was investigated. To investigate this the transcript from two students' paired interview explaining erroneous strategies was analysed. The two students were chosen since I knew from the analyses conducted for paper 2 that they had employed equal groups several times during their reasoning, in which they explained distributivity. Hence it was hypothesised that their reasoning might provide insights to how the equal groups model was exploited to support distributive understanding.

The emphasis was on the students' understandings and a framework for such investigations was employed, the representational-reasoning framework, described by Barmby et al. (2009). Briefly described, this framework views understanding as mental representations of a concept and connections between these representations. The connections consist of reasoning, which is broadly defined as the process to come to a conclusion. Reasoning can be observed and allows students' understandings to be inferred. In our case we, one co-author and me, inferred understandings related to equal groups and distributivity, since we investigated reasoning that connected representations of these.

In the transcript we identified students' explanations that drew on any model for multiplication, and how the model was exploited in relation to distributivity. This was done separately and then compared. We had identified the same excerpts and the same model, equal groups, in each of these excerpts. The reasoning in those excerpts was mainly verbal, but some visual information, such as drawings, written calculations and gestures were included in the excerpts. Each instance of reasoning was analysed independently in a process of iterative readings, to control if and how the reasoning was linking equal groups to distributivity. Some minor differences between how we had perceived the reasoning were discussed and in some cases the recordings were revisited to also include tone of voice, in order to decide which of possible interpretations of utterances that made more sense.

5.3.5 Analyses for paper 4

In the fourth paper, students' connections between models for multiplication, calculations and the arithmetical properties were investigated in relation to how they varied over time. The analytical process in this study started by identifying two students who could exemplify students with different understandings of multiplications over the five semesters of the study. I found two students who had demonstrated different understandings during the first semester in fifth grade, while they both achieved the highest possible grade in the national test in the end of grade six. Next step of the analyses was to decide which parts of the collected data that would best serve the purpose of this study. A tentative analysis was therefore conducted for all collected data regarding the two students. The paired interviews were excluded, since too much of their reasoning was convoluted in the discussions with their peers. The written work did not reveal much of their reasoning, leading to the decision to only use the transcripts from the individual clinical interviews from all five semesters.

The analysis of each transcript started by reading the whole interview session to get an overview. If necessary for straightening out ambiguity, this was complemented by listening to the audio and/or looking at the animation of the smartpen. In the next reading, I noted the parts where any of the components were employed, that is whether the student was calculating or was using or reasoning about an arithmetical property or a model for multiplication. The noted parts formed units of various lengths that could be meaningfully analysed. Below is an example of how such a unit was distinguished from the surrounding transcript. The part in italics was identified as a unit separated from the parts above and below. Note that I have changed the transcript into more formal written language out of respect for Ida and to not make her reasoning obscured by spoken idioms. This was not made prior to my analyses. In this unit, Ida used a multiplicative calculation and distributivity, but no model for multiplication.

Kerstin:	I am very curious how you think when you are going to calculate five times nineteen.
Ida:	I have never calculated that.
Kerstin:	No, I didn't think so. It isn't like a multiplication table task.
Ida:	*No. [pause 4 sec] First I would take five times ten, because that is fifty. Yes! What a good way! Then I take five times ten again, and then it will be one hundred. And then minus five, so it becomes ninety-five*
Kerstin:	*When you said it is fifty, did you think that you had //had times I think//*[1]
Ida:	*//Yes, well//, that I, had taken five times ten and then five times ten again.*
Kerstin:	Do you have other ways to calculate?

The entire units were copied into matrices for connections (see figures 3–5 in section 4.2.1). If the same unit demonstrated connections in more than one cell, it was placed everywhere it fitted. The matrices ensured that the data were treated systematically.

When all units were copied into the matrices, there were still some units left not matching any of the cells even though there were instances of one of the components. An example is the following excerpt:

Kerstin:	Zero point seven times four?
Emil:	Seven times four equals twenty-eight, and divided by ten is two point eight.

Emil performed a calculation, but it was not connected to any other component by reasoning, since it was a procedure he had learnt from instruction, which he did not explain or justify. This was the case with some calculations and they were put in a one-dimensional matrix representing only the calculation component, see figure 7.

Calculations	
Repeated addition	
Addition influenced strategies	
Multiplicative strategies	Kerstin: Zero point seven times four? Emil: Seven times four equals twenty-eight, and divided by ten is two point eight.

Figure 7. One-dimensional matrix for calculations

A similar one-dimensional matrix was needed to categorise the students' multiplicative stories in the story telling task, since they were not connected to calculations or properties due to the design of the task. In the matrices for models for multiplication all stories from the story-telling task were noted together with the explanations of what multiplication is.

[1] Text written between pairs of // shows overlapping speech with what is written between the next pairs of //.

The excerpts in the matrices were coded to get an overview over the data material, see figure 8. For example, Emil stated that multiplication is repeated addition once in the second semester of the study, and in the fourth semester he explained that multiplication is applicable to percentages. The remaining notations are equal groups stories that Emil told during his interviews or gave as explanations to what multiplication is. Each notation represents an excerpt, thus one can see that Emil told two equal group stories during the interview the first semester.

Emil	Story telling and explanations of multiplication
Semester 1	- Equal groups - Equal groups
Semester 2	- Repeated addition - Equal groups - Equal groups
Semester 3	- Equal groups - Equal groups - Equal groups
Semester 4	- Percentages - Equal groups - Equal groups - Equal groups
Semester 5	- Equal groups - Equal groups

Figure 8. Compilation of models for multiplication all semesters

The compilation of models in figure 8 was combined with the two-dimensional matrices in figures 3–5 (see section 4.2.1) for all semesters into a matrix shown in figure 9 on next page. From this matrix I could discern patterns of how students connected, or not connected, the three components. In figure 9 one can see which connections Emil made between all three components all semesters. Normal text style in the cells indicates that he had shown the connection during one interview, and bold text style indicates that he had shown the connection during more than one interview.

The first two columns, labelled calculations and properties, show what three-way connections Emil made between models for multiplication and calculations and properties. The connections that he demonstrated between properties and calculations (from the matrix in figure 3) are written in the cells. These were also connected to the equal groups model, thus in this combination matrix written on the equal groups row. Consequently, it is noted that he connected all three components in three ways: equal groups, additive calculation and distributivity; equal groups, multiplicative calculations and distributivity; and finally, equal groups, multiplicative calculation and associativity.

<table>
<tr><th>Emil</th><th>Calculations</th><th>Properties</th><th>Stories</th></tr>
<tr><td rowspan="2">Equal groups</td><td>Additive:
- Distributivity</td><td>Distributivity:
- Multiplicative
- Additive</td><td>natural × natural
natural × decimal
decimal × decimal</td></tr>
<tr><td>Multiplicative:
- Distributivity
- Associativity</td><td>Associativity:
- Multiplicative</td><td></td></tr>
<tr><td>Array</td><td></td><td></td><td></td></tr>
<tr><td>Area</td><td></td><td></td><td></td></tr>
<tr><td>Comparison</td><td></td><td></td><td></td></tr>
</table>

Figure 9. Matrix for models in relation to calculations, properties and stories.

In addition, the third column, labelled stories, shows an abbreviation of the one-dimensional matrix in figure 8, for what stories Emil told during the whole study, and what number types the stories were told to. Natural × natural means that both factors were natural numbers and natural × decimal that one factor was a natural number and the other a decimal number. The same matrix for Ida, see figure 10, showed a significantly different pattern, which demonstrates the usefulness of the matrix to find different patterns for different students.

Ida	Calculations	Properties	Stories
Equal groups			**natural × natural** natural × decimal
Array			natural × natural
Area			decimal × decimal
Comparison			decimal × decimal

Figure 10. Matrix for models in relation to calculations, properties and stories.

All Ida's calculations were unconnected to any model for multiplication. She demonstrated connections between calculations and properties, which were noted in the matrix for that type of two-way connections (see figure 3 in section 4.2.1). Since she never connected her calculations and properties to any model for multiplication, they were not put into this matrix, and therefore the cells are empty. She had, however, told stories of different models for multiplication, which is displayed in the third column.

To illustrate demonstrated connections between calculations and properties with respect to variation over time, the matrices in figure 9 and 10 were restructured, see figure 11.

Calculations	Properties	Stories/what is
REPEATED ADDITION S3: - Repeated addition S5: - Repeated addition (boring)	ASSOCIATIVITY S5: - 16 · 25 - 16 · 25	S1: - Equal groups - Equal groups S2: - Repeated addition - Equal groups - Equal groups S3: - Equal groups - Equal groups - Equal groups S4: - Percentages - Equal groups - Equal groups - Equal groups S5 - Equal groups - Equal groups
ADDITIVE S5: - Distributivity	COMMUTATIVITY Knows, see notes	
MULTIPLICATIVE S1: - Distributivity - Distributivity - Distributivity S2: - Distributivity - Distributivity - Distributivity S3: - Distributivity - Distributivity S4: - Distributivity - Distributivity - Distributivity - Distributivity - Distributivity - Distributivity S5: - Distributivity - Distributivity - Distributivity - Associativity - Associativity	DISTRIBUTIVITY S1: - 5 · 19 - 16 · 25 - 9:ans S2: - 15 · 24 - 15 · 24 - 15 · 24 S3: - 19 · 42 - 19 · 42 S4: - 39 · 23 - 39 · 23 - Finish - Finish - Mimic - Mimic S5: - 16 · 25 - 16 · 25 - 16 · 25	

Figure 11. The combined matrix for Emil.

In figure 11, Emil's calculations for all five semesters are presented as an example of such a restructured matrix. In the first two columns his demonstrated connections between calculations and properties are displayed by writing each instance in both columns. These two columns are organised in three sections, one for each type of calculation and property respectively. The third column includes the stories and explicit explanations to what multiplication is, from the matrix in figure 8. All sections include information of which semester the connection was demonstrated, S1 means semester 1 and so on. For example, during the second semester (S2), Emil demonstrated three different ways for calculating $15 \cdot 24$, all by distributivity. This can be read both as "distributivity" in the in the multiplicative section of the calculation column and as "$15 \cdot 24$" in the distributivity section of the properties column. Under commutativity I have entered “Knows, see notes”, since there was too much information to make a short summary. The grey colour over “distributivity” and “mimic” denotes that he made errors in the two calculations when he tried to mimic a distributive strategy

The iterative reading (and sometimes listening) of the students’ reasoning and the different matrices with different degrees of details guided me to use the matrices of connections in combination with one-dimensional matrices to map each student’s understanding of multiplication. This enhanced the possibility of making a fair and useful description of both the students’ understandings of multiplication and how it varied over time. I combined the matrices shown in figures 9 and 11 for Emil, and the corresponding matrices for Ida, in the analyses for paper 4. This combination served to demonstrate the complexity of students’ multiplicative reasoning without giving too many details, which might entail losing sight of a pattern, and at the same time not ignore too many details, thus losing significant data.

5.4 Ethical considerations

The ethical considerations concern both the participants and the quality of research. There are several different systems of criteria for qualitative research and an overarching quality measure suggested to be suitable for qualitative research is denoted trustworthiness (Bryman, 2008). I have construed trustworthiness as being as clear and open about methods, data and choices as possible. However, to protect the participants and at the same time be open to ensure validity forms a dilemma. To be totally open would imply that the participants’ identities would be revealed, for example by sharing the video recordings in order to increase trustworthiness in relation to what the students said. To partly address this dilemma, I have, for example, engaged a small number of researchers in the research group at Stockholm University to analyse small parts of the data to ensure inter-code reliability.

As my descriptions of data collections activities indicate, I have followed the guidelines from the Swedish Research Council ([Vetenskapsrådet]), making sure that both the students and their legal guardians were informed of the study, and agreed to participate by a written consent. I have also ensured that the data have been kept securely from other persons to ensure the students’ and their teachers’ anonymity. The pseudonyms were chosen from a list of common names given to children in Sweden the same year as the students were born. The names and the students were randomly paired, giving the students names according to their gender. In all conversations and presentations I have used the pseudonyms.

I have considered how to present data from a small-scale study in an ethically proper fashion, in relation to both the participants and research. Even though I refrain from presenting the real names and what school I have visit-ed, the students themselves, their teachers, parents and friends know that they participated. I need to present what they have said and done to give justice to the actual situations, to be true to the data. This must however be conducted in such a way that the students' anonymity is not jeopardised as well as ensure treating the students' respectfully. For example, I have transformed the transcriptions into more formal written language before presenting them and most often the presentations are given in English, which further aid to protect the students from being recognised. A verbatim transcription including spoken language might give a negative image of the speaker (Kvale & Brinkmann, 2009) and would not be respectful.

If any student can identify him- or herself is an open question. Do they remember what they said two to four years ago? If any of them do, I hope that my presentations of their reasoning and explanations make them feel proud of their interesting and enlightening explanations, irrespective of its correctness. As I repeated to them every time I interviewed them: I am not interested in the correctness. I am not here to evaluate you, your answers or your thinking. I am interested in how you think when working and explaining. It is equally interesting if the reasoning is correct or not, what matters is to help us mathematics teachers to understand how you students actually think.

6 Summary of papers

In this chapter each paper is summarised with focus on the results and discussions. The full papers are printed in the end of this thesis.

6.1 Paper 1 – Finding Erik and Alva: Uncovering students who reason additively when multiplying

The first paper reports a study from the first semester of grade five, in which the sufficiency of multiplicative comparison problems for uncovering students' multiplicative reasoning was investigated. Twenty-two students' answers to three multiplicative comparison problems from a written test were compared to their calculations of two multiplications, $5 \cdot 19$ and $16 \cdot 25$.

Multiplicative comparison problems have proved valuable to investigate whether a student has yet developed multiplicative reasoning or not. To be considered to reason multiplicatively is based on the interpretation of expressions such as "three times as much as 50", understanding that 150 is the correct answer, not 53, which is considered to reflect additive reasoning. However, in the literature the definition of multiplicative reasoning often refers to reasoning closely connected to calculations, such as to coordinate composite numbers and for example realise that an increase of 1 in one factor is equal to an increase of the other factors' magnitude. In a calculation of $5 \cdot 19$ this explains why $5 \cdot 20$ is 5 more and not 1 more, even though 1 was added to 19.

The students' answers to each multiplicative comparison problem were categorised as demonstrating additive or multiplicative reasoning in line with earlier research, assessing the answers. The students were then categorised into additive reasoning, multiplicative reasoning or mixed reasoning, depending on how many of the problems were answered additively and multiplicatively, respectively. Similarly, the students' calculations were categorised as additive or multiplicative. The students were then categorised into three groups of additive, multiplicative or mixed reasoning as in the word problem task, depending on how many calculations were multiplicative.

The results from both forms of tasks were combined and showed that ten of the twenty-two students had demonstrated the same type of reasoning to both forms of tasks. The remaining twelve students had demonstrated different types of reasoning when interpreting multiplicative comparison problems and when calculating. The mixed results for seven students can be accounted for by the transitional stage they are expected to demonstrate, the path from additive to multiplicative reasoning is not straightforward and students may demonstrate many different strategies and ways of reasoning simultaneously.

The remaining five students are more problematic. All of them demonstrated additive reasoning to the calculations, even though one of them showed multiplicative reasoning and four demonstrated mixed reasoning to the word problems. That is, all these five students correctly interpreted at least two multiplicative comparison problems, thus implying that they were reasoning multiplicatively or on their way to multiplicative reasoning. This is problematic; since the evaluation of their calculations demonstrated that they should be offered instruction to support their transition to multiplicative reasoning, since they demonstrated influence of additive reasoning. The multiplicative comparison problems are typically employed to investigate students' transition from additive to multiplicative reasoning, irrespective of the calculation.

Two students, Erik and Alva, were chosen as representatives for the group of five students that could correctly interpret the multiplicative comparison problems but demonstrated additive reasoning in their calculations. Both Erik and Alva stated that they had no other calculation strategy than repeated addition, which is not sufficient as numbers get larger and decimals are introduced. In order to identify students like Erik and Alva, I suggest that evaluating whether they can understand multiplicative comparison problems is insufficient; the calculations also play a significant role for the development of multiplicative reasoning.

It was also discussed whether the way students chose to write their solution to the multiplicative comparison problems could be used in the evaluation of their reasoning. To write the solution as $50 + 50 + 50$ or as $3 \cdot 50$ could possibly reflect how the student has reasoned while solving the problem, even though students might do one thing when calculating and then write it in another way. However, the written solutions for these twenty-two students seemed to actually reflect their preferred way of calculating; students who preferred to use repeated addition in the clinical interviews had written solutions by repeated addition, which not was the case for those who preferred to use multiplicative calculations during interviews.

The conclusion from paper 1 is that the method to identify students as multiplicative reasoners by their answers to multiplicative comparison problems is not sufficient; the calculations also play a significant role for the development of multiplicative reasoning.

6.2 Paper 2 – Sixth grade students' explanations and justifications of distributivity

In the second paper, students' explanations of and justifications for distributivity were investigated through paired interviews at the end of grade six, the fourth semester of the study. The research question was: What do students' evaluations of erroneous calculations of multi-digit multiplicative problems reveal about their understandings of multiplication?

There were nineteen students from the cohort that participated in paired interviews (one was working alone) evaluating three suggested calculation strategies. All suggested strategies were connected to distributivity and reflected mistakes that were influenced by additive reasoning. Students from this cohort had earlier demonstrated these erroneous strategies during individual interviews. In the paired interview, the students were invited to evaluate each strategy's validity and explain why the strategy was valid or invalid. The suggested strategies were presented as a fictitious student was saying it (see figure 6 in section 5.2.2). All strategies were suggestions to calculate $26 \cdot 19$ in an easier way: splitting 26 into 20 and 6 and multiply 20 by 19 and then add 6; moving one from 26 to 19 and then calculate $20 \cdot 25$; and to split both numbers into tens and ones, multiply them separately and add the results, $20 \cdot 10 + 6 \cdot 9$. By inviting the students to evaluate and explain erroneous calculation strategies, it was possible to infer understandings of the distributive property from their reasoning.

The explanations that the students gave to the suggested strategies were categorised into five groups: a) demonstrating investigative reasoning on meta-level; b) demonstrating reasoning by distributivity; c) demonstrating procedural reasoning; d) demonstrating descriptive reasoning; e) not demonstrating multiplicative reasoning. The first four (a–d) reflected multiplicative reasoning at different levels of sophistication.

Reasoning belonging to the first three categories (a–c) correctly found the strategies invalid. Investigative reasoning was employed to investigate under which circumstances the strategy was valid, making general justifications. Reasoning by distributivity was demonstrated by contextualising the number expression to the equal groups model or by giving counterexamples. Procedural reasoning was shown by suggesting calculating the result and seeing if the correct answer was reached, thus not focussing how the strategy worked, but whether it produced the correct answer. Descriptive reasoning denotes that the students described the strategy based on their experiences. This was used both as a validation, "I know that it works, because that is how I do", and as refutation, "I know that does not work because I used to do like that", and hence on a lower level of sophistication than procedural reasoning. Students who did not demonstrate multiplicative reasoning either displayed additive reasoning or could not give any explanations.

Signs of the students' understandings of distributivity became discernable in the reasoning, since the reasoning both demonstrated what they understood and what was unclear to them. To contextualise the strategy by the equal groups model was successful, it allowed students to explain the validity of the distributive property, thus suggesting that it is a robust model for distributivity. For example, a story of twenty-six piles with nineteen sticks in each supported the students' understanding that taking away one pile was equivalent to taking away nineteen sticks. At the same time the equal groups stories had a constraining effect on flexible use of commutativity. Even though students knew commutativity to be valid in multiplication, equal groups models caused some misunderstandings between students who perceived different factors to denote the multiplier.

The conclusion was that the equal groups model for multiplication was a powerful model for communication, explaining distributivity and justifying why erroneous strategies were invalid. Coincidently, some pairs were partly constrained by the model in relation to fluent use of commutativity. Furthermore, to evaluate and explain erroneous strategies had potential to elicit reasoning about distributivity, and hence to make it possible to infer parts of the students' understandings of multiplication.

Alternative explanations to the fact that some students did not demonstrate multiplicative reasoning were discussed, suggesting that it could depend on difficulties to express their thinking as well as perceiving the issue as self-evident instead of a more shallow understanding of distributivity. To perceive the error as self-evident was suggested with respect to the first suggested strategy; in which 20 were multiplied by 19 and the remaining 6 from 26 was added. Students might have found it obvious that 6 also needed to be multiplied by 19. The other two suggested strategies were evoking longer and more elaborated reasoning, and hence were suggested to be more productive with respect to elicit students' understandings of distributivity.

6.3 Paper 3 – The ambiguous role of equal groups in students' understandings of distributivity

In the third paper, the role of equal groups for students' understandings of distributivity was investigated by means of a case study of two students in the end of the sixth grade. We, one co-author and me, used data from the same tasks as for paper 2, the paired interview in which the students were invited to evaluate and explain erroneous strategies. Here we focussed on two students' paired reasoning and contrasted it to their individual reasoning two weeks earlier when they calculated similar tasks during interviews, and what we know about the instruction they took part in.

From the analyses conducted for paper 2 we knew that the students, with pseudonyms Anton and Lucas, had succeeded to explain distributivity by contextualising the problem as equal groups. Hence, we conjectured that their reasoning might provide insights to how the equal groups model was exploited to support understanding of distributivity.

Distributivity is considered as fundamental for understanding of multiplication, but the literature is inconsistent with respect to how difficult distributivity is and whether equal groups is a suitable model to understand distributivity. Furthermore, there are reports of overgeneralisations of addition strategies causing errors violating distributivity, for example to solve $26 \cdot 19$ as $20 \cdot 10 + 6 \cdot 9$. This erroneous strategy was part of the suggested strategies, and both Anton and Lucas had used it two weeks earlier when calculating $39 \cdot 23$. Another example of overgeneralisation of addition that were suggested during this interview, is to move a part of one factor to the other, such as to move 1 from 23 to 39 and thus calculate $40 \cdot 22$ instead. Lucas suggested this strategy during his individual interview two weeks earlier.

When Anton and Lucas evaluated and explained the suggested strategies, they connected the calculations to the equal groups model by contextualising the calculations to heaps of things and bags with coins. Their reasoning was supported by the asymmetry of equal groups, for example to denote 26 as the multiplier helped them to infer that to take away 1 from 26, meant that one heap or bag was removed, hence 19 things. Thus they coordinated the numbers multiplicatively. When they explained why cross-multiplication (to multiply the tens with the ones, such as $20 \cdot 9$ and $6 \cdot 10$ in $26 \cdot 19$) was necessary, their contextualisation of bags with coins helped them to construct more general reasoning concerning multi-digit multiplication and distributivity.

The difference between their reasoning when evaluating and explaining compared to when calculating might be due to the nature of the tasks, to evaluate and explain may imply that deeper mathematical thinking is evoked compared to conducting calculations. It is also possible that their reasoning gained from working together with a peer, thus being forced to formulate their thinking out loud.

The role of equal groups with respect to multiplicative reasoning has been questioned, since equal groups does not emphasise the two-dimensionality of multiplication, but rather the one-dimensionality of repeated addition. Our study shows that this is not necessarily the case; Anton and Lucas reasoned multiplicatively by contextualisation of multi-digit multiplication to equal groups. However, the sustainability of equal groups is questionable, since it cannot explain decimal multiplication, hence other models are needed too.

6.4 Paper 4 – Students' conceptualisation of multiplication as repeated addition or equal groups in relation to multi-digit and decimal numbers

In the fourth paper, we investigated how students connect the three components: models for multiplication, calculations, and arithmetical properties as well as how these connections changed as multiplication was extended from single-digit to multi-digit and decimal numbers. This is accomplished by a case study of two students, here called Emil and Ida, whom we interviewed over five semesters in grades five to seven.

The data for this study came from clinical individual interviews with the two students. The choice of which students to report was guided by the demonstrated understanding of multiplication from the first semester (see Larsson, 2013) and their results to the national tests in grade six. Emil and Ida both had reached the highest performance level on the national test and we knew that they had demonstrated different understandings of multiplication in the first semester, which led us to hypothesise that they would demonstrate different connections over the five semesters of the study. In the interviews, the students were invited to explain what multiplication is, to calculate multiplicative tasks and to tell stories to match multiplications as described in section 5.2.1.

The analyses, which are thoroughly described in section 5.3.5, revealed that Emil constantly connected equal groups to multiplication. During all interviews, he told, or tried to tell, stories of equal groups. When both numbers were decimals, Emil told unrealistic stories in grade six, but in grade seven he concluded that it was impossible to tell such stories to decimals. He stated "here it is two things that are not whole, so you cannot have zero point nine candy, you can have that, but you cannot have that in zero point two jars", thus pinpointing the unfeasibility of the equal groups model for decimals. Emil's multi-digit calculations were, through the entire study, manifested in strategies underpinned by distributivity. His calculations were connected to the model of equal groups, which he employed to justify his calculation strategies. However, he did not consider distributive strategies susceptible to decimal numbers, which he stated to be "a completely different system for how it works". When Emil discussed his calculation strategies he sometimes reminded himself by saying out loud that "you could calculate in the other direction" or similar before applying commutativity, indicating that commutativity was not as self-evident as distributivity to Emil.

The analyses of Ida's interviews demonstrated that she was calculating by repeated addition in the start of the study and gradually changed to distributive strategies, parallel to overgeneralisations of addition. By the end of the study she demonstrated that distributivity was applicable to decimal numbers and employed associativity to multi-digits. She told stories of equal groups, rectangular array, multiplicative comparison and rectangular area during these five semesters, the latter two to decimal multiplication, thus telling realistic stories. Ida did not employ commutativity fluently, she reminded herself that “it doesn’t matter what direction” when applying it. Ida did not demonstrate any explicit connections to her calculations; they were all numerical, talking about numbers as entities.

Emil and Ida were both high achieving students according to the national test in grade six and demonstrated similarities as well as differences in how they connected models for multiplication, calculations and arithmetical properties, and how these connections changed over time and in relation to multi-digit and decimal numbers.

They demonstrated similarities in their firm view of asymmetrical multiplication, Emil by the model of equal groups, and Ida by the procedure of repeated addition, even though Ida reasoned partly additive and Emil was fluently using distributivity in the beginning of the study. This firm connection to asymmetrical multiplication probably constrained their fluent use of commutativity, and, in the case of Emil, to make sense of decimal multiplication. Ida demonstrated a significant development during the five semesters of the study. She had active access to multiple models for multiplication. This study cannot connect those two observations as a causal correspondence, but suggests that Emil’s shortcomings to apply multiplication to decimals had its roots in the single model of equal groups.

Two alternative explanations to both students’ firm views of multiplication as asymmetrical are discussed: the intuitive models theory and conceptual change theory. Both theories offer explanations to why students hold on to what they have learnt early, which raises questions about how the students were introduced to multiplication. In their instruction through fourth to sixth grade, multiplication was predominantly presented asymmetrically in their textbooks, which might be part of the explanation. The theories lead to different suggestions concerning both early and later instruction to support students to extend their understandings of multiplication and a proposal is that we need to introduce parallel models in instruction. Symmetrical models might be used parallel to asymmetrical, highlighting different aspects of multiplication.

7 Discussion

In this chapter, I discuss the results, the project, which includes methodological choices, implications, the contribution and future research.

7.1 Results

The result discussion is focussed on results concerning the students' understandings of multiplication and what different forms of tasks can reveal, which reflects the overarching research question:

> What do students' responses to different forms of multiplicative tasks in the domain of multi-digits and decimals reveal about their understandings of multiplication?

7.1.1 Students' understandings of multiplication

The most salient outcome from this study is the ambiguous role of equal groups and repeated addition for students' understandings of multiplication. The students in this study perceived multiplication as the procedure of repeated addition, as the model of equal groups, or as a combination of them. I do not view repeated addition as a model for multiplication; I view it as a calculation procedure. However, I do recognise that repeated addition and the equal groups model are congruent (De Corte & Verschaffel, 1996) and that they can play a similar role, as they share asymmetric properties (Greer, 1992; Park & Nunes, 2001). A body of research, including this thesis, views procedural and conceptual knowledge as intertwined and as supporting the construct of each other (Baroody et al., 2007; Maciejewski & Star, 2016; Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2015; Star, 2005). In the case of additive and multiplicative reasoning this is ambiguous; even though "there is a conceptual discontinuity between multiplication and addition, there is a procedural connection" (Park & Nunes, 2001, p. 764). Therefore, I discuss repeated addition and equal groups together when I focus on the asymmetry and calculations, but not when I focus on the students' conceptualisation of multiplication.

The strong role of equal groups and repeated addition could be predicted. For example, the multiplier effect is based on a view of multiplication as repeated addition and that students prefer to pose equal groups problems to match multiplicative expressions is well known (e.g. De Corte & Verschaffel, 1996). However, the robustness of the students' asymmetrical view entails details concerning advantages and disadvantages not reported for students in traditional instruction at the stage when multi-digit and decimal multiplication is introduced.

The persistence of equal groups and repeated addition can be explained by different theories. The intuitive model theory suggests that repeated addition is deeply rooted and resistant to change for two reasons, it is the initially taught procedure for multiplication and "correspond to features of human mental behavior that are primary, natural, and basic" (Fischbein et al., 1985, p. 15). The intuitive model theory predicts that the reasoning will be influenced by repeated addition long after more generalised models and calculations have been incorporated in the students' repertoire. The long lasting effect of initial instruction is generally agreed upon, but there are different views of the roots of multiplicative reasoning. For example, it is suggested that the intuitive and informal idea of multiplication children have before instruction is embedded in a one-to-many correspondence (Nunes & Bryant, 2010) or as splitting (Confrey, 1994; Confrey & Smith, 1995), rather than as repeated addition, which makes multiplication conceptually different from addition.

Conceptual change theory offers complementary explanations to the resistance to change initially taught models and procedures. The term change can be misleading, since it is not a sudden change; it is a gradual process that stretches over several years and may include parallel conceptualisations (Vamvakoussi & Vosniadou, 2010; Vosniadou & Verschaffel, 2004). To conceptualise multiplication as rectangular array and area parallel to repeated addition and equal groups could thus be described as a conceptual change. In the process of expanding and changing conceptualisations, synthetic concepts can occur. Synthetic concepts are intermediate and erroneous concepts that may evolve as a result of experiences (Vamvakoussi & Vosniadou, 2010) and they can coexist with correct concepts (Durkin & Rittle-Johnson, 2015). For example, the idea that multiplication makes bigger can be explained by the experiences to multiply in the set of natural numbers and be labelled as a synthetic concept. Similarly, overgeneralisations of addition calculation strategies to multiplication can be viewed as synthetic concepts.

Both conceptual change theory and the intuitive model theory suggest that it is a slow and cumbersome process to change one's conceptualisations. Therefore it can be argued that to construct sustainable models from the start is easier than to remedy synthetic concepts (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004). Coincidently, it is generally agreed that a sound basis for mathematics instruction is to build on students' informal strategies (Nunes & Bryant, 2010; Selter, 1998; Verschaffel et al., 2007).

This may form a pedagogical dilemma. If the informal strategy for multiplication is repeated addition, it might form obstacles to the development of multiplicative reasoning (Nunes & Bryant, 2010; Selter, 1998).

The positive influence that the repeated addition and/or equal groups view of multiplication can have on students' reasoning in relation to distributivity is noteworthy. For example, to conclude that the strategy of moving 1 from 26 to 19 is equivalent to a subtraction of 19 and an addition of 25 as Hugo did (in paper 2) was guided by repeated addition. He suggested that the fictitious student could continue by compensating what was wrong:

Hugo:	She has multiplied twenty times and then she must take away what stands for one time, that is twenty-five. She has to take away twenty-five. […] Then she gets that one times nineteen, so she has plus nineteen.

Similarly, Lucas and Anton (in paper 3) sorted out the same problem by constructing a verbal representation drawing on equal groups:

Anton:	Yes, but if you think that if one has twenty-six heaps and take away one heap, then one…then one gets…
Lucas:	There are nineteen in each. There are nineteen things in each heap.
Anton:	There are nineteen things in twenty-six heaps.
Lucas:	Yes. If one takes away one heap, then one takes away nineteen. And that is supposed to be handed out in twenty-five heaps. It doesn't work. One can't add. It will be nineteen coins in twenty-five heaps. It will be too little.
Anton:	Yes, exactly.
Lucas:	So it doesn't work.

It is far from trivial to explain this particular strategy. The coordination of expanded factors, such as (20 + 6), is described as advanced in the literature, especially when both factors are expanded (Ambrose et al., 2003; Izsák, 2004). To simultaneously increase and decrease the factors, strains the multiplicative coordination of numbers. Numerically we can write the move of 1 from 26 to 19 as

$$(26 - 1) \cdot (19 + 1) = 26 \cdot 19 + 26 - 19 - 1$$

and algebraically the move of 1 as

$$(a - 1) \cdot (b + 1) = ab + a - b - 1$$

The magnitude of the error of moving 1 from one factor to the other, is 1 less than the difference between the factors, as can be seen in the last part of the equations above[2]. In the case of $26 \cdot 19$ the result of $25 \cdot 20$ is $26 - 19 - 1$ less than of $26 \cdot 19$, that is 1 less than the difference of 26 and 19; the erroneous strategy will yield an answer that is 6 more than the correct answer.

[2] Under the condition that $a > b$

Or as Lucas said, there will be too little, only nineteen, to be handed out in twenty-five heaps.

Emil and Ida (in paper 4) gave similar explanations as Hugo and Lucas for their calculations, referring to repeated addition and equal groups. These findings of understanding of distributivity, by use of equal groups and repeated addition, are in line with a number of studies (Ambrose et al., 2003; Carpenter et al., 2005; Lampert, 1986; Schifter et al., 2008). However, the other studies are reports from interventions in which the students are encouraged to reason relationally and to justify their reasoning by mathematical arguments. The students I report with respect to distributivity take part in a traditional Swedish instruction, typically by individual work in the textbook according to the teacher.

The students' use of verbal representations of the equal groups model can be viewed as a tool for thinking and communicating (Greeno & Hall, 1997; Izsák, 2005; Selling, 2016; Yackel, 2001), thus lessening the burden on the working memory (Pape & Tchoshanov, 2001). By imagining, for example heaps of sticks, the meaning of multiplication became clear and possibly helped the students to construct more and stronger connections between the calculation and the model as suggested in research (Barmby et al., 2009; Baroody et al., 2007; Hiebert & Carpenter, 1992; Richland et al., 2012). Verbal representations of the equal groups model were powerful in relation to thinking, communicating and understanding distributivity, thus confirming the role of representations for conceptual understanding, suggested in literature (Goldin & Shteingold, 2001; Greeno & Hall, 1997; Hiebert & Carpenter, 1992; Izsák, 2005; Panasuk & Beyranevand, 2010).

However, there are disadvantages in relating multiplication to repeated addition and equal groups. For example, in relation to the asymmetry, Ida and Emil (in paper 4) had to explicitly remind themselves that commutativity is valid in multiplication before using it, and a number of student pairs in paper 2 had to overcome different views of which factor denoted the multiplier, by explicit reminders of commutativity. This is analogous to reports of secondary students using significantly longer time to solve problems that are incongruent to the natural number structure, compared to problems following the natural number structure (Van Hoof et al., 2013). Event though the secondary students solved the incongruent problems correctly, they needed more time to solve them; even though Emil and Ida knew and correctly used commutativity, they stopped and said that it was all right to swap the numbers. Furthermore, as was seen in the work of Erik and Alva (in paper 1), the procedure of repeated addition is cumbersome and prone to errors in the domain of multi-digits.

Repeated addition becomes even more problematic in the domain of decimals, as has been discussed by a number of scholars (Greer, 1992; P. Thompson & Saldanha, 2003; Verschaffel et al., 2007). Decimals were included in the story telling tasks in sixth and seventh grade, and among the students reported of in this thesis there was only one, Ida (in paper 4), who told appropriate stories for decimal multiplication. The other students gave up, said it was impossible or told what they called 'weird' stories such as broken trees with partially eaten pears, as Anton (in paper 3). Emil (in paper 4) explained why it was impossible:

> Emil: One should calculate something, as for example that one has four candies in four jars, how many together. But here it is that one has two things that are not whole, so one cannot have zero point nine candy, one can have that, but one cannot have that in zero point two jars.

Other models are needed to overcome the problem that Emil pinpoints. There is an on-going debate concerning what role equal groups and repeated addition should have in instruction, especially regarding distributivity. This study contributes to the debate by detailed descriptions of students' reasoning about multi-digit multiplication. The equal groups model and repeated addition helped them to draw sophisticated conclusions from calculations where both factors were changed and to explain distributivity. This cannot be ignored, but neither can the disadvantages of repeated addition and equal groups with respect to commutativity and decimal numbers be ignored, which this study also provides detailed information of. The question of how multiplication, especially initially, best can be modelled remains.

7.1.2 Different forms of tasks

With respect to different forms of tasks, it seems that to evaluate and explain a suggested calculation, as in paper 2 and 3, evoked reasoning that draws on deeper mathematical thinking compared to explain one's own calculation. This can be viewed as analogous to findings concerning problem solving, in which students who categorised problems became better problem solvers (Van Dooren, De Bock, Vleugels, et al., 2010). Van Dooren and his colleagues suggest that the act of categorisation promoted deeper and more mathematical thinking, presumably because the categorisation task requires the student to take a step back and think about the structure of the problems. Similarly, the tasks to evaluate and explain might have promoted more focus on conceptual knowledge, such as principled knowledge and knowledge of principles underlying procedures (Crooks & Alibali, 2014; Rittle-Johnson & Alibali, 1999). This suggestion is in line with arguments concerning early algebra through generalised arithmetic. To foster students to explain and make mathematical arguments for their calculations, rather than focussing on the calculations per se, can enhance both arithmetical calculation skills and conceptual knowledge of arithmetical properties, such as distributivity (Bastable & Schifter, 2008; Carpenter et al., 2003; Carpenter et al., 2005; Schifter et al., 2008).

To reason about and justify calculation strategies might stimulate the use of models for multiplication as tools to think and communicate (Selling, 2016; Yackel, 2001). For example, at the time of the paired interview both Lucas and Anton knew that cross-multiplication was necessary, but not until they connected it to a model could they explain why. The act of making this connection can support the understanding of multiplication (Barmby et al., 2009; Dreyfus, 1991).

The three suggested calculation strategies did not provide the same amount of information concerning the students' understandings of distributivity. One suggested strategy, to solve $26 \cdot 19$ as $20 \cdot 19 + 6$, generated less reasoning than the other two. This indicates that it was not only to evaluate and explain that were important to elicit students' reasoning, but also the properties of the suggested strategies. The strategy that evoked less information might have been too easy to evaluate as erroneous, while the other two suggested strategies (to simplify by 'moving' 1 and to only multiply within place value) were more challenging to evaluate as seen by the students' longer discussions. Both are direct applications of valid addition strategies. Overgeneralisation of the latter strategy is reported in the literature both for young students and prospective teachers (Foxman & Beishuizen, 2002; Lo et al., 2008; Young-Loveridge & Mills, 2009) and was used in this group of twenty-two students during all semesters of the study. The students' instruction can provide an explanation to the frequent presence of the place value mistake. To partition by place value is the standard calculation that their textbooks promote, both for addition and multiplication. If the instruction emphasises step-by-step procedures, as these students' textbooks do, it is well known that 'buggy algorithms' might occur (I. Thompson, 1999; Verschaffel et al., 2007). The strategy to move a part from one factor to the other was used less frequently even though several students suggested it during the first two years of the study. I have not found anything in the literature concerning this specific type of overgeneralisation and it was not an addition strategy promoted by the textbooks for grades four to six; hence it is harder to find explanations for its occurrence. It is possible that the strategy had been part of instruction in the classroom or in earlier grades. In both cases of overgeneralisation of addition strategies it is possible that the students reasoned as the prospective teachers that Lo et al. (2008) report; that addition strategies should apply to multiplication since multiplication is repeated addition.

Two different forms of tasks, multiplicative comparison word problems and multi-digit calculations, were employed in an assessment of students' multiplicative reasoning in paper 1. For a group of five students, these two different forms of tasks generated different results with respect to their reasoning being assessed as multiplicative or not. One explanation to this result might be the choice to categorise repeated addition as multiplicative in the multiplicative comparison problem, but as additive in the calculation task.

This can be viewed as a stricter categorisation in the calculation task compared to the word problem task. It can also be viewed as reflecting the two different types of descriptions of multiplicative reasoning reported in the literature. To transform multiplication into repeated addition implies that the coordination of composite units is sidestepped (Sowder et al., 1998; Tzur et al., 2013), hence some researchers describe repeated addition as reflecting additive reasoning (Bakker et al., 2014; Nunes et al., 2015; P. Thompson & Saldanha, 2003; Vergnaud, 1983). However, other researchers describe multiplicative reasoning as understanding the multiplicative relationship in multiplicative comparison and proportion problems, and evaluate it by distinguishing additive comparison from multiplicative (Clark & Kamii, 1996; Fernandez et al., 2012; Van Dooren et al., 2008; Van Dooren, De Bock, & Verschaffel, 2010). I view the abilities to coordinate numbers multiplicatively in calculations and to recognise, distinguish and solve multiplicative problems as equally important and as an example of connecting several aspects of multiplication, thus forming more and stronger connections, which is in line with what for example Barmby et al. (2009) and Baroody et al. (2007) suggest.

7.2 The project

Students' understandings of multiplication were investigated in this project. To study someone's understanding directly is impossible, but it might be inferred from something that can be directly observed, such as external representations (Goldin & Shteingold, 2001) and reasoning (Barmby et al., 2009). For example, it is problematic to directly investigate students' understandings of arithmetical properties (J. Torbeyns, personal communication, February 17, 2016), but through the students' calculations and explanatory reasoning and use of representations of models for multiplication I could infer how they understood the properties. These inferences were in line with the literature about conceptual understanding and external representations (Barmby et al., 2009; Berthold et al., 2009; Crooks & Alibali, 2014; Hiebert & Carpenter, 1992; Hiebert & Wearne, 1992; Rittle-Johnson & Alibali, 1999). The students' understandings might be different from what I inferred. It is possible, or rather plausible, that they understood more than they shared with me, but I could only study what was demonstrated (Goldin & Shteingold, 2001). The results can be construed in the light of earlier studies. For example, the theory of 'multiplication makes bigger' (Bell et al., 1989; Bell et al., 1981) served both to explain and predict students' reasoning by comparing and contrasting their reasoning (Cobb, 2007).

Understanding of an arithmetical operation involves solving and posing problems, calculations and reasoning. To investigate understanding, I drew on multiple sources of data, combined methods and used different forms of tasks, as recommended in literature (Barmby et al., 2009; Bisanz et al., 2009; Hiebert & Carpenter, 1992). The literature review convinced me that both large cross-sectional studies and small-scale case studies were feasible in order to extend the knowledge of students' understandings of multiplication, since both have been conducted to investigate students' understandings of mathematical concepts (e.g. Tzur et al., 2013; Van Dooren, De Bock, & Verschaffel, 2010). The opportunity to say something more general increases with the number of cases, but the level of details decreases (Battista et al., 2009). It is possible that other tasks and methods could have elicited more and other reasoning among the students. For example, solving word problems in the individual interviews might have complemented the written tests with more information on the students' reasoning with respect to various models for multiplication as well as number types. However, the interviews were time consuming, both for the students and me, and therefore the amount of tasks was kept relatively small.

Other possible weaknesses concerning the approach are that little data concerning the students' instruction were included and the relatively small number of students. The findings would plausibly have been more enlightening if they had been compared to the instruction of the students to a greater extent. The textbooks that were used extensively in the case of eighteen of the twenty-two students could be included as data, but to observe instruction was not feasible in this project. The methodology was partly chosen from pragmatic considerations; the doctoral project must be possible to conduct within the given time frame. The number of students was too large to conduct detailed analyses of all data within the project. However, I had two reasons to not exclude any of the students; I did not want to risk having too few students in the end of the project due to dropouts and I did not want to disappoint any student who wanted to participate.

To be interviewed both individually and in pairs was popular. The students showed that they enjoyed having full attention from a person who made efforts to understand them and who demonstrated a genuine interest and curiosity about their thinking, as was proposed by Ginsburg (1997). Most of the students who were involved looked forward to being interviewed, as shown by their asking if it was their turn to come to me soon. During the interviews I experienced that the students appeared relaxed. At every interview, before starting the recordings, I explicitly asked the student if he or she was feeling at ease being recorded and reminded the student that he or she was free to ask me to stop interviewing at any point. In one single case the student said no to audio recording, but still wanted to be interviewed, hence I did not record him and just took notes. Another student asked me to stop recording in one interview when she got stuck on a calculation, but she still wanted to go on being interviewed both on that occasion and in later semesters. I trust that the students' positive attitude to participate in the study reflects that they really enjoyed being listened to, and that they experienced that I treated them respectfully, which is a hallmark of clinical interviews (Ginsburg, 1997).

To conduct clinical interviews is not easy and I had no prior experiences when the study started. When reading the transcripts from different semesters this became obvious. In the first semester I sometimes posed leading questions and did not always follow up students' statements. In the later semesters I had improved my interviewing skills and made fewer mistakes.

A framework can be employed to distinguish what is investigated from other information (Larson, 2014). In this study the multiplicative cuboid served as such a framework, supporting the distinction of students' reasoning, with respect to the three components and connections between them, from other information. However, the cuboid and the two-way matrices derived from it, were insufficient as analytical tools, since only connected data were framed. The complementing one-dimensional matrices were needed to give justice to the students' reasoning, but most important, the complexity of the data required a combination of matrices of different types.

One might argue that to include overgeneralisations of addition in the framework is questionable, since they reflect erroneous strategies. However, it can be argued that they not only reflect additive reasoning, but students' initial attempts to employ multiplicative reasoning by distributivity. The first suggested strategy in papers 2 and 3, to split 26 into 20 and 6 and multiply 20 by 19 and add the 6, can be viewed as such an attempt. Another example is Ida's reasoning when she calculated $15 \cdot 24$ and started by taking $10 \cdot 10$ and then suggested that since she had already taken 10 from both numbers she must go on and multiply 5 by 14. The students in this group commonly demonstrated such reasoning.

A limitation of the multiplicative cuboid, as presented in this thesis, is its suitability to decimal multiplication; it was more appropriate to multi-digit than to decimal numbers. The nature of numbers in multiplication is reported as more influential than other dimensions, such as model for multiplication, for students' conceptualisation of multiplication problems (Greer, 1992; Verschaffel et al., 2007). To be useful for analyses of decimal multiplication one of the dimensions of the cuboid might be divided into categories of number types, such as combinations of integers, decimals larger than 1 and decimals less than 1. The number type could replace the arithmetical properties dimension of the cuboid. The arithmetical properties could be placed as subcategories in the multiplicative strategies along the calculation dimension, since the multiplicative calculation strategies were underpinned by the arithmetical properties.

Despite these drawbacks, I think that variations of the multiplicative cuboid, including the one-dimensional complementing matrices, can be useful in further studies aiming at mapping students' understandings of multiplication. A multiplicative cuboid for analyses of younger students than this study would need slightly different categories in the calculation dimension and for older students, for example when expanding multiplication to polynomials, it might need other dimensions. In such cases, the model's three components might be extended or collapsed into other categories of components analogous to how categories of models for multiplication can be refined or unitised (Mulligan & Mitchelmore, 1997).

The analytical process of connections was extended over a long time, refined and changed and discussed several times among a small group within the mathematics education research group at Stockholm University. Parts of the matrices have also been discussed and scrutinised by international researchers at a conference (Larsson, 2015) and other meetings. I am grateful for all these discussions and critical examination of the matrices, it has improved the analytical work and the versions presented here. I do not want to label these versions as final, since there remain opportunities for improvement. As for every model, the 'multiplicative cuboid' is a simplification. It does not tell the full story of what constitutes understanding of multiplication. There are other dimensions, which affect students' development of multiplicative reasoning, for example, affective factors, the interaction with other students, teachers and parents, and individual cognitive disposition.

Initially I had decided to exclusively investigate the transition from single-digits to multiple-digits and stay in the domain of natural numbers. From the early analyses I found that decimals were required to challenge the students who only used the equal groups model or repeated addition procedure and thus they were included. This also made sense since the introduction of decimal multiplication took place during the same semester as for multi-digits. The unfinished calculations and mimicking task have the same background; tasks were needed to challenge students who did not use distributivity, to get more information about their reasoning. These choices were possible to make since the study run over several years. At the same time, these choices can be regarded to weaken the study. I had new types of tasks included in different semesters, which made a developmental perspective of the study weak. I have, for example, no way of knowing students' understandings of decimal multiplication before grade six.

In hindsight, I can identify some problematic choices in the methodology that were not discussed in paper 1, the comparison of two forms of tasks to assess students' multiplicative reasoning. Firstly, the multiplicative comparison problems were mixed with other multiplicative problems. If they had been mixed with additive word problems it might have caused students to reason differently to the problems. Secondly, the distinction for what was categorised as additive or multiplicative reasoning was well defined for each answer, but not for the whole task. To be considered reasoning multiplicatively, all three word problems needed to be correctly solved, while two additive answers were enough to be considered to reason mainly additively. This choice might seem arbitrary, but it does not change the result much, all five students that were identified as reasoning additively when calculating would still be the same demonstrating the same additive reasoning while calculating. Thirdly, the choice to use easy numbers might have affected the outcome in two ways: to help the students to focus the structure as was intended and suggested by Tzur et al. (2013), or to increase superficial strategies as Greer (1992) proposed. Easy numbers might also elicit more multiplicative reasoning compared to, for example, decimal ratios (Van Dooren, De Bock, & Verschaffel, 2010). Both possible drawbacks of using easy numbers might have skewed the results to show more answers reflecting multiplicative reasoning. Furthermore, in problems with too easy numbers students might 'see' the answer without calculating, and the possibility that the students actually wrote their thinking might have increased if more challenging numbers had been used (Sowder et al., 1998). On the other hand, too difficult numbers might have led more students to ignore the structure of the problems and apply additive reasoning since the numbers were too problematic to use multiplicatively. I think the issue of choosing numbers deserves more attention in research as it might influence students' reasoning.

Finally, I have an over-all reflection with respect to the role of different forms of tasks for research. It was very informative to employ a number of different forms of tasks. In paper 4, in which Emil and Ida were recurrently interviewed, four forms of tasks were used and they contributed to a more nuanced picture of the students' reasoning. For example, Ida's reasoning was solely numerical throughout the study except when prompted to tell stories and Ida told stories reflecting four different models. In this case the story telling task was productive to elicit more, or another aspect, of Ida's understanding of multiplication to become observable. Even though I found the mix of various forms of tasks productive, there is probably room for further variation of forms of tasks. This makes me wonder what forms of tasks that could have been even more productive to create situations in which more students would have shown more of their understandings.

7.3 Implications for instruction

The results from this study demonstrate that students' understandings of multiplication can be robustly rooted in repeated addition and equal groups. This is both advantageous and disadvantageous. Considering the view of mathematical understanding as connections, there is much that points at a need for multiple models for multiplication and several calculation strategies. Connections refer to connections between the three components; calculations, models for multiplication and arithmetical properties, but also to connections between conceptual and procedural knowledge. Additionally, it could entail connections between multiplication and other operations, connections between multiplication and other domains of mathematics, and connections to everyday experiences. I cannot rule out that instruction in

Swedish classrooms offers multiple models for multiplication and multiple calculation strategies to students. However, these students' reasoning and the textbook series for grades four to six demonstrated a strong emphasis on equal groups and repeated addition.

To include more models for multiplication can enhance the understanding of multiplication, as has been suggested in the literature (Acevedo Nistal et al., 2009; Dreyfus, 1991; Greer, 1992; Panasuk & Beyranevand, 2010). Equal groups and repeated addition are insufficient in the domain of decimals. It has been advocated that students should not have to learn both about decimals and new models simultaneously, hence other models should be offered in the domain of natural numbers (Izsák, 2004). But the literature also points out that 'more' is not enough, it is also the quality of the structure of more models or representations that matters (Barmby et al., 2013; Hiebert & Carpenter, 1992). To compare and contrast different models' properties might be a way to provide that quality. The rectangular area model is suggested as a more general model, susceptible to decimals and polynomials, but also challenging to understand (Greer, 1992; Izsák, 2005; Simon & Blume, 1994; Verschaffel et al., 2007). Therefore, I suggest that models' properties are compared, investigated and discussed in instruction.

Based on the results from paper 3, in which Anton and Lucas demonstrated such advanced reasoning by explaining erroneous strategies exploiting equal groups, I suggest to give more opportunities for students to explore and discuss suggested strategies and explanations. The very act of explaining and justifying has been reported to support students' understandings (Berthold et al., 2009). To use misconceptions, as well as correct and creative strategies, from the same group of students, might help the students to draw conclusions that are close to their own held beliefs with respect to the operation.

7.4 Contribution and future research

This doctoral project contributes to the literature on students' understandings of multiplication by detailed descriptions of how students can reason on different forms of tasks and how their reasoning can change (or not change) during the time period in which multiplication is extended beyond single-digit numbers. As far as what I am aware, previous studies concerning understanding of multiplication, that have contributed with detailed descriptions of students' reasoning, are either intervention studies or focussed on earlier stages, such as the introduction of multiplication. The students in this project participated in traditional Swedish mathematics instruction. The detailed descriptions of how these students reason involve a combination of their calculations, their rationales for how they calculate and interpretations of what multiplication is, and what is allowed to do in multiplication. Thus, the descriptions combine different aspects of multiplication. This follows a line of research conducted in the initial years of instruction, which has produced much knowledge of how young children reason to various arithmetic problems with natural numbers (e.g. Bastable & Schifter, 2008; Carpenter et al., 1999). Here it is extended to middle school students and the challenges to extend understanding of multiplication beyond single-digits, in a setting of traditional instruction.

The overgeneralisation of an addition strategy, to move a part of one factor to the other, was found among several students in this study and explored through the eyes of the same group of students. As far as I know, this type of overgeneralisation is not described in previous research.

The understanding of the three arithmetical properties is not a well-researched area. All three properties are considered important for flexible calculations and as a foundation for understanding algebra (Carpenter et al., 2005; Ding & Li, 2014). The literature review gives anecdotal and ambiguous evidence of students' understandings of the arithmetical properties, and hence more research is needed. For example, to investigate what type of instruction that promotes understanding and flexible use of the arithmetical properties would contribute to our knowledge.

The predominant result from the project was details concerning the ambiguous role of equal groups and repeated addition; how students were supported and constrained by the same model and procedure. The constraints were prevalent in the reasoning even for high achieving students, thus indicating that it is not a problem connected to weak mathematical achievement. This needs to be addressed in future research. I suggest that investigations if and how it is possible to reduce the negative effects of equal groups and repeated addition with respect to decimals and commutativity, without jeopardising the positive effects to distributivity, is conducted. This could for example be investigated through interventions in which explicit comparisons of various models' features are addressed. Such a study would preferably be longitudinal, to map effects of introduction in a longer perspective, including the extension of multiplication beyond single-digits, to include multi-digits and decimals.

Sammanfattning

Multiplikation genomsyrar stora delar av matematiken, till exempel bygger vårt positionssystem och tal i bråkform på multiplikativa relationer och det råder enighet i tidigare forskning om att det är viktigt att tillägna sig förmåga att resonera multiplikativt. Multiplikativt resonemang innebär bland annat att inse skillnaden mellan multiplikativ och additiv jämförelse, det vill säga att 3 gånger mer än 5 är 15 och inte 8, samt att simultant kunna fokusera både helhet och delar samt hur dessa är relaterade. Det kan till exempel innebära att kunna se 60 kulor som är fördelade i 4 påsar med 15 i varje som en helhet *samtidigt* som man ser de fyra delarna och att varje påse representerar 15 kulor. Processen att utveckla förmåga att resonera multiplikativt beskrivs allmänt som besvärlig och tidskrävande. Denna process omfattar bland annat att lära sig räkneoperationen multiplikation. Multiplikation introduceras ofta som en upprepad addition och förknippas med lika stora grupper av föremål. När multiplikation utvidgas till att omfatta rationella tal, utmanas föreställningen av multiplikation som upprepad addition. Det är svårt att föreställa sig hur man kan addera till exempel 3,8 exakt 4,9 gånger. Redan vid multiplikation inom de naturliga talen kan upprepad addition vara otillräcklig, till exempel när talen som ska multipliceras är stora. När multiplikation utvidgas till att omfatta både flersiffriga tal och tal i decimalform, behöver därför synen på multiplikation som upprepad addition av lika stora grupper förändras eller utvidgas, vilket har visat sig vara problematiskt för många elever.

Förmågan att resonera multiplikativt har i tidigare forskning beskrivits som att simultant fokusera både helhet och delar samt hur dessa är relaterade. Detta är nära förknippat med hur beräkningar utförs, till exempel innebär det att inse att $a \cdot (b + 1)$ inte är lika med $ab + 1$, utan att det korrekta är att det är lika med $ab + a$, eftersom en ökning med 1 av den ena faktorn innebär en ökning av produkten som är lika stor som den andra faktorn. Många studier har undersökt elevers förmåga att resonera multiplikativt genom att de fått lösa textuppgifter som reflekterar multiplikativ jämförelse, det vill säga proportionella samband. Då testas elevernas förmåga att tolka en multiplikationsmodell uttryckt som en textuppgift snarare än deras förmåga att samtidigt hantera helhet och delar i en multiplikativ beräkning. Att testa förmåga att resonera multiplikativt enbart genom att bedöma elevers svar på textuppgifter utan hänsyn till hur beräkningarna har gjorts, är ett exempel på diskrepans i tidigare forskning inom elevers förståelse för multiplikation. Ett annat exempel där tidigare forskning inte är entydig, är vilken roll upprepad addition och lika stora grupper har, och borde ha, i relation till distributivitet. Det finns forskning som visar att elever lyckas väl med att utveckla förståelse för distributivitet med hjälp av beräkningar som bygger på upprepad addition och multiplikationsmodellen lika stora grupper. Samtidigt pekar en del forskning ut rektangelformationer, som hur ägg ligger i en äggkartong, och rektangelarea som lämpligare modeller än lika stora grupper, för att förstå och använda den distributiva lagen. Vidare beskrivs distributivitet som svårare för eleverna att lära sig än kommutativitet i en del studier och vice versa i andra. Däremot verkar det vara stor enighet om vikten av att kunna koppla samman multiplikation med olika modeller, som till exempel rektangelformation och lika stora grupper, samt att tillägna sig förståelse för de aritmetiska egenskaperna; kommutativitet, distributivitet och associativitet.

Grunden för den studie som beskrivs i avhandlingen utgörs av de ovan beskrivna motsättningarna i litteraturen i kombination med samstämmigheten i vikten av förståelse för multiplikation och de aritmetiska egenskaperna. Syftet med studien är att få en utökad kunskap om elevers förståelser av multiplikation, när den utvidgas till att omfatta flersiffriga tal och tal i decimalform. Den övergripande forskningsfrågan var: Vad visar elevers lösningar till olika typer av multiplikativa uppgifter, inom multiplikation med flersiffriga tal och decimaltal, av deras förståelse av multiplikation?

Studien genomfördes genom att följa tjugotvå elever under fem terminer i årskurs fem till sju. Eleverna arbetade med olika typer av uppgifter under återkommande kliniska intervjuer som genomfördes både individuellt och i par. Under intervjuerna fick eleverna konstruera räknehändelser till givna multiplikationer, ge explicita förklaringar av vad multiplikation är, utföra beräkningar följda av uppmaningar att förklara och motivera hur och varför strategin de använt fungerar, samt utvärdera beräkningsstrategier presenterade som förslag från fiktiva elever. Dessutom gav eleverna skriftliga svar till textuppgifter under ordinarie matematiklektioner.

Att förstå ett matematiskt begrepp har i forskning ofta beskrivits som att ha gjort kopplingar mellan begrepps- och procedurkunskap, vilket för multiplikation kan innebära att man vet både vad $3 \cdot 5$ innebär och hur man räknar ut det. Vidare kan förståelse innebära kopplingar mellan kunskapstyperna och olika representationer för begreppet, till exempel genom att man kan koppla ihop multiplikationen med en bild eller en räknehändelse. Kopplingen mellan de olika kunskapstyperna och representationerna kan ses som resonemang. Ett exempel på ett resonemang är att $3 \cdot 5$ måste vara lika mycket som $5 \cdot 3$, eftersom om det finns tre rader med fem bullar på en bakplåt så är det lika många bullar även om man vrider på plåten så att det är fem rader med tre bullar. Detta resonemang binder samman en aritmetisk lag med en multiplikationsmodell, nämligen kommutativa lagen med modellen rektangelformation.

Tre komponenter av multiplikationskunskaper har lyfts fram i tidigare forskning som särskilt betydelsefulla för utvidgningen av multiplikation till flersiffriga tal och decimaltal: multiplikationsmodeller, beräkningar och de aritmetiska lagarna. Att till exempel utföra beräkningen av $16 \cdot 25$ som $10 \cdot 25 + 6 \cdot 25$ innebär att den distributiva lagen används, vilket kan tolkas som att det finns en koppling mellan komponenterna beräkning och aritmetisk lag. Att förklara beräkningen genom att beskriva $16 \cdot 25$ som sexton påsar med tjugofem kulor i varje påse och att man först räknar ut hur många kulor det är i tio av påsarna och därefter hur många det är i de andra sex och slutligen summerar resultaten, kan tolkas som en koppling av beräkningen till en multiplikationsmodell. I föreliggande studie granskas kopplingar mellan de tre komponenterna multiplikationsmodeller, beräkningar och aritmetiska egenskaper. Detta görs genom att studera elevernas resonemang, som ses som kopplingar, i syfte att beskriva deras förståelser av multiplikation.

Resultaten av studien är uppdelade i fyra fristående rapporter (paper). Den första rapporten beskriver en undersökning om den eventuella motsättningen mellan att lösa multiplikativa jämförelseproblem och att utföra multiplikationsberäkningar, genom att jämföra elevernas resultat från dessa två typer av uppgifter. Det visade sig att en grupp elever som kunde förstå och lösa multiplikativa jämförelseproblem utförde beräkningar med hjälp av additiva resonemang. Resultatet indikerar att det inte tillräckligt att enbart se till elevernas tolkning av multiplikativ jämförelse för att utvärdera elevers förmåga att resonera multiplikativt, även hur de utför beräkningarna bör undersökas. Den andra rapporten beskriver vad elevers utvärderingar av felaktiga beräkningsstrategier kan berätta om elevers förståelse för distributivitet. Där framkom att eleverna resonerade på många olika sätt och att de elever som kopplade samman beräkningsstrategierna med multiplikationsmodellen lika stora grupper kom längre i sina resonemang än de elever som enbart resonerade numeriskt. Samtidigt som kopplingen till modellen lika stora grupper verkade bidra till mer utvecklade resonemang, verkade kopplingen till modellen ibland hindra eleverna från att utnyttja kommutativitet. Dessutom visade det sig att de felaktiga beräkningsstrategier som eleverna fick undersöka var olika utmanande och ledde till olika typer av resonemang, vilket understryker vikten av att explicit beskriva de uppgifter som används i en studie för att andra ska kunna tolka och värdera resultaten. Den tredje rapporten beskriver vilken roll multiplikationsmodellen lika stora grupper kan ha för elevers förståelse för distributivitet genom en fallstudie av två elevers resonemang kring de felaktiga strategierna från den andra av de fyra rapporterna. Dessa båda elever utnyttjade kontextualiseringen av beräkningarna för att argumentera kring och förklara distributivitet. Samtidigt verkade modellen lika stora grupper hindra dem att förstå vad multiplikation av decimaltal kan innebära. Den fjärde rapporten beskriver hur elevers förståelse för multiplikation kan manifesteras genom de kopplingar de gör och hur denna förståelse kan variera över tid. En dubbel fallstudie med två elever visade två olika lärostigar under studiens fem terminer.

Resultaten från de fyra rapporterna diskuteras gemensamt för att svara på den övergripande frågan om hur elevers lösningar till olika former av multiplikativa uppgifter kan visa deras förståelse för multiplikation.
Sammantaget visade sig elevernas förståelse av multiplikation vara djupt rotad i upprepad addition och modellen lika stora grupper. Att koppla samman multiplikation med lika stora grupper var fördelaktigt för deras förståelse för distributivitet, men samtidigt begränsande för tillämpning av kommutativitet och förståelse för multiplikation av decimaltal. En förklaring till den djupt rotade bilden av multiplikation som upprepad addition och lika grupper kan vara hur multiplikation introduceras, vilket diskuteras i relation till tidigare forskning. Diskussionen behandlar även det allmängiltiga dilemmat om hur mer generella modeller för multiplikation (t.ex. rektangelarea) är svårare att konceptualisera än de mer konkreta modellerna (t.ex. lika stora grupper), vilka i sin tur inte räcker till då talområdet utvidgas från de naturliga talen till att omfatta även rationella tal.

I relation till olika uppgiftstyper indikerar studien att en del uppgiftstyper var mer produktiva för att generera varierande typer av resonemang än andra. Till exempel verkade uppgiften att utvärdera en fiktiv elevs felaktiga beräkningsstrategi skapa större möjligheter för eleverna att resonera multiplikativt jämfört med uppgiften att motivera och förklara sin egen beräkningsstrategi. I framtida forskning som syftar till utökad kunskap om elevers förståelse för multiplikation föreslås därför att en variation av uppgiftstyper används.

References

Acevedo Nistal, A., Van Dooren, W., Clarebout, G., Elen, J., & Verschaffel, L. (2009). Conceptualising, investigating and stimulating representational flexibility in mathematical problem solving and learning: A critical review. *ZDM, 41*(5), 627-636. doi:10.1007/s11858-009-0189-1

Ambrose, R., Baek, J. M., & Carpenter, T. (2003). Children's invention of multidigit multiplication and division algorithms. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 305-336). Mahwah, NJ: Lawrence Erlbaum.

Andrews, P. (2009). Comparative studies of mathematics teachers' observable learning objectives: Validating low inference codes. *Educational Studies in Mathematics, 71*(2), 97-122. doi:10.1007/s10649-008-9165-x

Anghileri, J. (1999). Issues in teaching multiplication and division. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 184-194). Buckingham: Open University Press.

Anghileri, J. (2000). *Teaching number sense*. New York: Continuum.

Anghileri, J. (2006). *Teaching number sense*. London; New York: Continuum.

Baek, J. M. (2008). Developing algebraic thinking through exploration in multiplication. In C. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics, seventieth yearbook* (pp. 141-154). Reston, VA.: National Council of Teachers of Mathematics.

Bakker, M., Van den Heuvel-Panhuizen, M., & Robitzsch, A. (2014). First-graders' knowledge of multiplicative reasoning before formal instruction in this domain. *Contemporary Educational Psychology, 39*(1), 59-73. doi:10.1016/j.cedpsych.2013.11.001

Barmby, P., Bolden, D., Raine, S., & Thompson, L. (2013). Developing the use of diagrammatic representations in primary mathematics through professional development. *Educational Research, 55*(3), 263-290. doi:10.1080/00131881.2013.825164

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics, 70*(3), 217-241. doi:10.1007/s10649-008-9145-1

Baroody, A. (1999). The roles of estimation and the commutativity principle in the development of third graders' mental multiplication. *Journal*

of Experimental Child Psychology, 74(3), 157-193. doi:10.1006/jecp.1999.2524
Baroody, A., Feil, Y., & Johnson, A. (2007). An alternative reconceptualization of procedural and conceptual knowledge. *Journal for Research in Mathematics Education, 38*(2), 115-131. doi:10.2307/30034952
Bastable, V., & Schifter, D. (2008). Classroom stories: Examples of elementary students engaged in early algebra. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 165-184). New York: Lawrence Erlbaum Associates/National Council of Teachers of Mathematics.
Battista, M., Smith, M., Boerst, T., Sutton, J., Confrey, J., White, D., . . . Quander, J. (2009). Research in mathematics education: Multiple methods for multiple uses. *Journal for Research in Mathematics Education, 40*(3), 216-240.
Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education, 24*(4), 294-323. doi:10.2307/749464
Bell, A., Fischbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure and context. *Educational Studies in Mathematics, 15*(2), 129-147. doi:10.1007/BF00305893
Bell, A., Greer, B., Grimison, L., & Mangan, C. (1989). Children's performance on multiplicative word problems: Elements of a descriptive theory. *Journal for Research in Mathematics Education, 20*(5), 434-449. doi:10.2307/749419
Bell, A., Swan, M., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational Studies in Mathematics, 12*(4), 399-420. doi:10.1007/BF00308139
Ben-Chaim, D., Fey, J., Fitzgerald, W., Benedetto, C., & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics, 36*(3), 247-273. doi:10.1023/A:1003235712092
Berthold, K., Eysink, T., & Renkl, A. (2009). Assisting self-explanation prompts are more effective than open prompts when learning with multiple representations. *Instructional Science, 37*(4), 345-363. doi:10.1007/s11251-008-9051-z
Bisanz, J., Watchorn, R., Piatt, C., & Sherman, J. (2009). On "understanding" children's developing use of inversion. *Mathematical Thinking and Learning, 11*(1-2), 10-24. doi:10.1080/10986060802583907
Bolden, D., Barmby, P., Raine, S., & Gardner, M. (2014). How young children view mathematical representations: A study using eye-tracking technology. *Educational Research, 57*(1), 59-79. doi:10.1080/00131881.2014.983718

Bryman, A. (2008). *Social research methods*. Oxford: Oxford University Press.

Cai, J., Hwang, S., Jiang, C., & Silber, S. (2015). Problem-posing research in mathematics education: Some answered and unanswered questions. In F. M. Singer, N. Ellerton, & J. Cai (Eds.), *Mathematical problem posing: From research to effective practice* (pp. 3-34). New York: Springer Science Business Media. Retrieved from http://dx.doi.org/10.1007/978-1-4614-6258-3. doi:10.1007/978-1-4614-6258-3_1

Canobi, K., Reeve, R., & Pattison, P. (2002). Young children's understanding of addition concepts. *Educational Psychology, 22*(5), 513-532. doi:10.1080/0144341022000023608

Carpenter, T., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20. doi:10.2307/1001789

Carpenter, T., Fennema, E., Franke, M. L., Levi, L., & Empson, S. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Carpenter, T., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carpenter, T., Levi, L., Franke, M. L., & Koehler, J. (2005). Algebra in elementary school: Developing relational thinking. *ZDM, 37*(1), 53-59. doi:10.1007/BF02655897

Chandler, C., & Kamii, C. (2009). Giving change when payment is made with a dime: The difficulty of tens and ones. *Journal for Research in Mathematics Education, 40*(2), 97-118.

Chen, L., Van Dooren, W., Jing, M., & Verschaffel, L. (2015). Effect of learning context on students' understanding of the multiplication and division rule for rational numbers. In X. Sun, B. Kaur, & J. Novotná (Eds.), *Proceedings of the International Commission of Mathematical Instruction (ICMI) Study 23 Conference on the Primary Mathematics Study on Whole Numbers* (pp. 526-533). Macau, China: University of Macau.

Clark, F., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41-51. doi:10.2307/749196

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspective. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics.* (Vol. 1, pp. 3-38). Charlotte, NC: Information Age Pub.

Cobb, P., & Yackel, E. (1998). A constructivist perspective on the culture of the mathematics classroom. In F. Seeger, J. Voigt, & U. Waschescio (Eds.), *The culture of the mathematics classroom* (pp. 158-190). Cambridge: Cambridge University Press.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291-330). Albany: State Univ. of New York Press.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86. doi:10.2307/749228

Crooks, N., & Alibali, M. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review, 34*(4), 344-377. doi:10.1016/j.dr.2014.10.001

De Corte, E., & Verschaffel, L. (1996). An empirical test of the impact of primitive intuitive models of operations on solving word problems with a multiplicative structure. *Learning and Instruction, 6*(3), 219-242. doi:10.1016/0959-4752(96)00004-7

De Corte, E., Verschaffel, L., & Van Coillie, V. (1988). Influence of number size, problem structure and response mode on children's solutions of multiplication word problems. *Journal of Mathematical Behavior, 7*, 197-216.

Ding, M., & Li, X. (2014). Transition from concrete to abstract representations: The distributive property in a Chinese textbook series. *Educational Studies in Mathematics, 87*(1), 103-121. doi:10.1007/s10649-014-9558-y

Ding, M., Li, X., & Capraro, M. (2013). Preservice elementary teachers' knowledge for teaching the associative property of multiplication: A preliminary analysis. *The Journal of Mathematical Behavior, 32*(1), 36-52. doi:10.1016/j.jmathb.2012.09.002

diSessa, A. (2004). Metarepresentation: Native competence and targets for instruction. *Cognition and Instruction, 22*(3), 293-331. doi:10.1207/s1532690xci2203_2

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht: Kluwer.

Durkin, K., & Rittle-Johnson, B. (2015). Diagnosing misconceptions: Revealing changing decimal fraction knowledge. *Learning and Instruction, 37*, 21-29. doi:10.1016/j.learninstruc.2014.08.003

English, L. (1998). Children's problem posing within formal and informal contexts. *Journal for Research in Mathematics Education, 29*(1), 83-106. doi:10.2307/749719

Fernandez, C., Llinares, S., Van Dooren, W., De Bock, D., & Verschaffel, L. (2012). The development of students' use of additive and proportional methods along primary and secondary school. *European Journal of Psychology of Education, 27*(3), 421-438. doi:10.1007/s10212-011-0087-0

Fischbein, E., Deir, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3-17.

Foxman, D., & Beishuizen, M. (2002). Mental calculation methods used by 11-year-olds in different attainment bands: A reanalysis of data from the 1987 APU survey in the UK. *Educational Studies in Mathematics, 51*(1-2), 41-69. doi:10.1023/A:1022416021640

Fuson, K. (2003). Developing mathematical power in whole number operations. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 68-94). Reston, VA: National Council of Teachers of Mathematics.

Fuson, K., Wearne, D., Hiebert, J., Murray, H., Human, P., Olivier, A., . . . Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130-162. doi:10.2307/749759

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice* Retrieved from http://dx.doi.org/10.1017/CBO9780511527777 doi:10.1017/CB09780511527777

Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 1-23). Reston, Va.: National Council of Teachers of Mathematics.

Gray, E., & Tall, D. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116-140. doi:10.2307/749505

Greeno, J., & Hall, R. (1997). Practicing representation. *Phi Delta Kappan, 78*(5), 361-367.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Hall, N. (1998). Concrete representations and the procedural analogy theory. *The Journal of Mathematical Behavior, 17*(1), 33-51. doi:10.1016/S0732-3123(99)80060-1

Hansson, Å. (2012). The meaning of mathematics instruction in multilingual classrooms: Analyzing the importance of responsibility for learning. *Educational Studies in Mathematics, 81*(1), 103-125. doi:10.1007/s10649-012-9385-y

Heirdsfield, A., Cooper, T., Mulligan, J., & Irons, C. (1999). Children's mental multiplication and division strategies. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Psychology of Mathematics Education Conference* (pp. 89-96). Haifa, Israel.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook of research on* *mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, N.J.: Erlbaum.

Hiebert, J., & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for Research in Mathematics Education, 23*(2), 98-122. doi:10.2307/749496

Howe, C., Nunes, T., & Bryant, P. (2010). Intensive quantities: Why they matter to developmental research. *British Journal of Developmental Psychology, 28*(2), 307-329. doi:10.1348/026151009X410362

Izsák, A. (2004). Teaching and learning two-digit multiplication: Coordinating analyses of classroom practices and individual student learning. *Mathematical Thinking & Learning, 6*(1), 37-79. doi:10.1207/s15327833mtl0601_3

Izsák, A. (2005). "You have to count the squares": Applying knowledge in pieces to learning rectangular area. *Journal of the Learning Sciences, 14*(3), 361-403. doi:10.1207/s15327809jls1403_2

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington D.C.: National Academy Press.

Kinda, S. (2013). Generating scenarios of division as sharing and grouping: A study of Japanese elementary and university students. *Mathematical Thinking and Learning, 15*(3), 190-200. doi:10.1080/10986065.2013.794256

Kvale, S., & Brinkmann, S. (2009). *Den kvalitativa forskningsintervjun*. Lund: Studentlitteratur.

Lamon, S. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24*(1), 41-61. doi:10.2307/749385

Lamon, S. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics* (pp. 629-668). Charlotte, NC: Informatin Age Publishing Inc.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction, 3*(4), 305-342. doi:10.1207/s1532690xci0304_1

Larson, N. (2014). *Matematikämnet och stadiebytet mellan grundskolan och gymnasieskolan: En enkät- och klassrumsstudie [Mathematics and the transition from lower to upper secondary school: A survey and classroom study] (Doctoral thesis)*: Linköping University, Department of Behavioural Sciences and Learning, Linköping, Sweden.

Larsson, K. (2013). Multiplicative thinking in relation to commutativity and forms of representation. In J. Novotná & H. Moraová (Eds.),

International Symposium Elementary Maths Teaching, SEMT'13: Tasks and tools in elementary mathematics (pp. 179-187). Prague: Charles University, Faculty of Education.
Larsson, K. (2015). Connections for learning mulitplication. In J. Novotná & H. Moraová (Eds.), *Proceedings from Symposium Elementary Mathematics Education 15, (SEMT´15) Developing mathematical language and reasoning* (pp. 202-211). Prague: Charles University, Faculty of Education.
Larsson, K., & Pettersson, K. (2015). Discerning multiplicative and additive reasoning in co-variation problems. In X. Sun, B. Kaur, & J. Novotná (Eds.), *Proceedings of the International Commission of Mathematical Instruction (ICMI) Study 23 Conference on the Primary Mathematics Study on Whole Numbers* (pp. 559-566). Macau, China: University of Macau.
Leighton, J. (2003). Describing and defining reason. In J. Leighton & R. Sternberg (Eds.), *The nature of reasoning* (pp. 3-11). Cambridge: Cambridge University Press. Retrieved from http://dx.doi.org/10.1017/CBO9780511818714. doi:10.1017/CBO9780511818714
Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Hillsdale, N.J.: Lawrence Erlbaum Associates.
Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67*(3), 255-276. doi:10.1007/s10649-007-9104-2
Lo, J.-J., Grant, T., & Flowers, J. (2008). Challenges in deepening prospective teachers' understanding of multiplication through justification. *Journal of Mathematics Teacher Education, 11*(1), 5-22. doi:10.1007/s10857-007-9056-6
Maciejewski, W., & Star, J. (2016). Developing flexible procedural knowledge in undergraduate calculus. *Research in Mathematics Education*, 1-18. doi:10.1080/14794802.2016.1148626
McClain, K. (2003). Supporting preservice teachers' understanding of place value and multidigit arithmetic. *Mathematical Thinking and Learning, 5*(4), 281-306. doi:10.1207/S15327833MTL0504_03
Mulligan, J. (1992). Children’s solutions to multiplication and division word problems: A longitudinal study. *Mathematics Education Research Journal, 4*(1), 24-41. doi:10.1007/BF03217230
Mulligan, J., & Mitchelmore, M. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28*(3), 309-330. doi:10.2307/749783
Mulligan, J., & Watson, J. (1998). A developmental multimodal model for multiplication and division. *Mathematics Education Research Journal, 10*(2), 61-86. doi:10.1007/BF03217343
Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In J. Hiebert & M. J. Behr

(Eds.), *Number concepts and operations in the middle grades* (pp. 19-40). Hillsdale, N.J.: Lawrence Erlbaum Associates.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27-52. doi:10.1207/s15326985ep4001_3

Nunes, T., & Bryant, P. (2010). Paper 4: Understanding relations and their graphical representation. In T. Nunes, P. Bryant, & A. Watson (Eds.), *Key understandings in mathematics learning*. Retrieved from http://www.nuffieldfoundation.org/key-understandings-mathematics-learning.

Nunes, T., Bryant, P., Burman, D., Bell, D., Evans, D., & Hallett, D. (2009). Deaf children's informal knowledge of multiplicative reasoning. *Journal of Deaf Studies and Deaf Education, 14*(2), 260-277. doi:10.1093/deafed/enn040

Nunes, T., Bryant, P., Evans, D., & Barros, R. (2015). Assessing quantitative reasoning in young children. *Mathematical Thinking and Learning: An International Journal, 17*(2-3), 178-196. doi:10.1080/10986065.2015.1016815

Panasuk, R., & Beyranevand, M. (2010). Algebra students' ability to recognize multiple representations and achievement. *International Journal for Mathematics Teaching and Learning, 22*, 1-21. Retrieved from http://www.cimt.plymouth.ac.uk/journal/default.htm

Pape, S., & Tchoshanov, M. (2001). The role of representation(s) in developing mathematical understanding. *Theory into Practice, 40*(2), 118-127. doi:10.1207/s15430421tip4002_6

Park, J.-H., & Nunes, T. (2001). The development of the concept of multiplication. *Cognitive Development, 16*(3), 763-773. doi:10.1016/S0885-2014(01)00058-2

Piaget, J. (1952). *The child's conception of number*. London: Routledge & Kegan Paul.

Pittalis, M., Christou, C., Mousoulides, N., & Pitta-Panzai, D. (2004). A structural model for problem posing. In M. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Annual Meeting of the International Group for the Psychology of Mathematics Education* (pp. 49-56). Bergen, Norway: Bergen University College.

Prediger, S. (2008). The relevance of didactic categories for analysing obstacles in conceptual change: Revisiting the case of multiplication of fractions. *Learning and Instruction, 18*(1), 3-17. doi:10.1016/j.learninstruc.2006.08.001

Richland, L., Stigler, J., & Holyoak, K. (2012). Teaching the conceptual structure of mathematics. *Educational Psychologist, 47*(3), 189-203. doi:10.1080/00461520.2012.667065

Rittle-Johnson, B., & Alibali, M. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*(1), 175-189.

Rittle-Johnson, B., Schneider, M., & Star, J. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review, 27*(4), 587-597. doi:10.1007/s10648-015-9302-x

Robinson, K., Ninowski, J., & Gray, M. (2006). Children's understanding of the arithmetic concepts of inversion and associativity. *Journal of Experimental Child Psychology, 94*(4), 349-362. doi:10.1016/j.jecp.2006.03.004

Ruthven, K. (1998). The use of mental, written and calculator strategies of numerical computation by upper primary pupils within a 'calculator-aware' number curriculum. *British Educational Research Journal, 24*(1), 21-42. doi:10.1080/0141192980240103

Schifter, D., Monk, S., Russel, S., & Bastable, V. (2008). Early algebra: What does understanding the laws of arithmetic mean in the elementary grades. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 413-447). New York: Lawrence Erlbaum Associates/National Council of Teachers of Mathematics.

Schliemann, A., Araujo, C., Cassundé, M. A., Macedo, S., & Nicéas, L. (1998). Use of multiplicative commutativity by school children and street sellers. *Journal for Research in Mathemtics Education, 29*(4), 422-435. doi:10.2307/749859

Schoenfeld, A. (1985). Making sense of "Out Loud" problem-solvning protocols. *Journal of Mathematical Behavior, 4*(2), 171-191.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Hillsdale, N.J.: Lawrence Erlbaum Associates.

Selling, S. K. (2016). Learning to represent, representing to learn. *Journal of Mathematical Behavior, 41*, 191-209. doi:10.1016/j.jmathb.2015.10.003

Selter, C. (1998). Building on children's mathematics: A teaching experiment in grade three. *Educational Studies in Mathematics, 36*(1), 1-27. doi:10.1023/A:1003111425076

Shield, M., & Dole, S. (2013). Assessing the potential of mathematics textbooks to promote deep learning. *Educational Studies in Mathematics, 82*(2), 183-199. doi:10.1007/s10649-012-9415-9

Siemon, D., Bleckly, J., & Neal, D. (2012). Working with the big ideas in number and the Australian curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen, & D. Siemon (Eds.), *Engaging the Australian curriculum mathematics: Perspectives from the field* (pp. 19-45).

Silver, E. (2013). Problem-posing research in mathematics education: Looking back, looking around, and looking ahead. *Educational Studies in Mathematics, 83*(1), 157-162. doi:10.1007/s10649-013-9477-3

Silver, E., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*(5), 521-539. doi:10.2307/749846

Simon, M., & Blume, G. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*(5), 472-494. doi:10.2307/749486

Simon, M., & Placa, N. (2012). Reasoning about intensive quantities in whole-number multiplication? A possible basis for ratio understanding. *For the Learning of Mathematics, 32*(2), 35-41.

Skemp, R. R. (1976/2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School, 12*(2), 88-95.

Skolverket. (2009). *Vad påverkar resultaten i svensk grundskola? Kunskapsöversikt om betydelsen av olika faktorer [What affects the results in the Swedish compulsory school? Systematic review of the importance of various factors]*. Stockholm: Skolverket.

Sowder, J., Armstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education, 1*(2), 127-155. doi:10.1023/A:1009980419975

Squire, S., Davies, C., & Bryant, P. (2004). Does the cue help? Children's understanding of multiplicative concepts in different problem contexts. *British Journal of Educational Psychology, 74*(4), 515-532. doi:10.1348/0007099042376364

Star, J. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36*(5), 404-411.

Star, J., & Stylianides, G. J. (2013). Procedural and conceptual knowledge: Exploring the gap between knowledge type and knowledge quality. *Canadian Journal of Science, Mathematics and Technology Education, 13*(2), 169-181. doi:10.1080/14926156.2013.784828

Steffe, L. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany: State University of New York Press.

Stylianides, G., Stylianides, A., & Shilling-Traina, L. (2013). Prospective teachers' challenges in teaching reasoning-and-proving. *International Journal of Science and Mathematics Education, 11*(6), 1463-1490. doi:10.1007/s10763-013-9409-9

Szabo, A. (2013). *Matematiska förmågors interaktion och det matematiska minnets roll vid lösning av matematiska problem. Licentiatavhandling [The interaction of mathematical abilities and the role of mathematical memory in the solution of mathematical problems] (Licentiate thesis).* Stockholm University, Stockholm, Sweden.

Thompson, I. (1999). Written methods of calculation. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 169-183). Buckingham: Open University Press.

Thompson, P. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-236). Albany: State Univ. of New York Press.

Thompson, P., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *A research companion to Principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Tichá, M., & Hošpesová, A. (2013). Developing teachers' subject didactic competence through problem posing. *Educational Studies in Mathematics, 83*(1), 133-143. doi:10.1007/s10649-012-9455-1

Tzur, R., Johnson, H., McClintock, E., Kenney, R., Xin, Y., Luo, S., . . . Xianyan, J. (2013). Distinguishing schemes and tasks in children's development of multiplicative reasoning /Distinción de esquemas y tareas en el desarrollo del razonamiento multiplicativo de los niños. *PNA, 7*(3), 85-101.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *Journal of Mathematical Behavior, 31*(3), 344-355. doi:10.1016/j.jmathb.2012.02.001

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*(2), 181-209. doi:10.1080/07370001003676603

Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*(1), 9-35. doi:10.1023/B:EDUC.0000005212.03219.dc

Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2004). Remedying secondary school students' illusion of linearity: a teaching experiment aiming at conceptual change. *Learning and Instruction, 14*(5), 485-501. doi:10.1016/j.learninstruc.2004.06.019

Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative: An inventory and conceptual analysis of students' overuse of linearity. *Journal for Research in Mathematics Education, 39*(3), 311-342.

Van Dooren, W., De Bock, D., & Verschaffel, L. (2010). From addition to multiplication ... and back: The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction, 28*(3), 360-381. doi:10.1080/07370008.2010.488306

Van Dooren, W., De Bock, D., Vleugels, K., & Verschaffel, L. (2010). Just answering ... or thinking? Contrasting pupils' solutions and classifications of missing-value word problems. *Mathematical*

Thinking and Learning, 12(1), 20-35. doi:10.1080/10986060903465806

Van Hoof, J., Lijnen, T., Verschaffel, L., & Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research in Mathematics Education, 15*(2), 154-164. doi:10.1080/14794802.2013.797747

Van Hoof, J., Vandewalle, J., Verschaffel, L., & Van Dooren, W. (2015). In search for the natural number bias in secondary school students' interpretation of the effect of arithmetical operations. *Learning and Instruction, 37*, 30-38. doi:10.1016/j.learninstruc.2014.03.004

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). Orlando, Fla.: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, Virginia: The National Council of Teachers of Mathematics.

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41-59). Albany: State University of New York.

Verschaffel, L., & De Corte, E. (1997). Word problems: A vehicle for promoting authentic mathematical understanding and problem solving in the primary school. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 69-97). Hove, East Sussex: Psychology Press.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Lisse, Netherlands: Swets & Zeitlinger Publishers.

Verschaffel, L., Greer, B., & De Corte, E. (2002). Everyday knowledge and mathematical modeling of school word problems. In K. Gravemeijer, R. Lehrer, B. Van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 257-276). Dordrecht: Kluwer.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 557-628). Charlotte, NC: Information Age Pub.

Vosniadou, S., & Verschaffel, L. (2004). Extending the conceptual change approach to mathematics learning and teaching. *Learning and Instruction, 14*(5), 445-451. doi:10.1016/j.learninstruc.2004.06.014

Watanabe, T. (2003). Teaching multiplication: An analysis of elementary school mathematics teachers' manuals from Japan and the United States. *The Elementary School Journal, 104*(2), 111-125.

Yackel, E. (2001). Perspectives on arithmetic from classroom-based research in the United States of America. In J. Anghileri (Ed.), *Principles and* *practices in arithmetic teaching: Innovative approaches for the primary classroom* (pp. 15-31). Buckingham: Open University Press.

Young-Loveridge, J., & Mills, J. (2009). Supporting multiplicative thinking: Multi-digit multiplication using array-based materials. *Findings from the New Zealand Numeracy Development Projects 2008* (pp. 48-64). Wellington: Learning Media.

Zaslavsky, O., & Peled, I. (1996). Inhibiting factors in generating examples by mathematics teachers and student teachers: The case of binary operation. *Journal for Research in Mathematics Education, 27*(1), 67-78. doi:10.2307/749198

[Vetenskapsrådet], S. R. C. CODEX, Rules and guidelines for research. Retrieved from http://www.codex.vr.se/forskninghumsam.shtml