Configuration spaces, props and wheel-free deformation quantization

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Abstract

The main theme of this thesis is higher algebraic structures that come from operads and props.

The first chapter is an introduction to the mathematical framework needed for the content of this thesis. The chapter does not contain any new results.

The second chapter is concerned with the construction of a configuration space model for a particular 2-colored differential graded operad encoding the structure of two A_∞ algebras with two A_∞ morphisms and a homotopy between the morphisms. The cohomology of this operad is shown to be the well-known 2-colored operad encoding the structure of two associative algebras and of an associative algebra morphism between them.

The third chapter is concerned with deformation quantization of (potentially) infinite dimensional (quasi-)Poisson manifolds. Our proof employs a variation on the *transcendental* methods pioneered by M. Kontsevich for the finite dimensional case. The first proof of the infinite dimensional case is due to B. Shoikhet. A key feature of the first proof is the construction of a universal L_∞ structure on formal polyvector fields. Our contribution is a simplification of B. Shoikhet proof by considering a more natural configuration space and a simpler choice of *propagator*. The result is also put into a natural context of the dg Lie algebras coming from graph complexes; the L_∞ structure is proved to come from a Maurer-Cartan element in the *oriented graph complex*.

The fourth chapter also deals with deformation quantization of (quasi-)Poisson structures in the infinite dimensional setting. Unlike the previous chapter, the methods used here are purely algebraic. Our main theorem is the possibility to deformation quantize quasi-Poisson structures by only using perturbative methods; in contrast to the transcendental methods employed in the previous chapter. We give two proofs of the theorem via the theory of dg operads, dg properads and dg props. We show that there is a dg prop morphism from a prop governing star-products to a dg prop(erad) governing (quasi-)Poisson structures. This morphism gives a theorem about the existence of a deformation quantization of (quasi-)Poisson structure. The proof proceeds by giving an explicit deformation quantization of *super-*

involutive Lie bialgebras and then lifting that to the dg properad governing quasi-Poisson structures. The prop governing star-products was first considered by S.A. Merkulov, but the properad governing quasi-Poisson structures is a new construction.

The second proof of the theorem employs the Merkulov-Willwacher polydifferential functor to transfer the problem of finding a morphism of dg props to that of finding a morphism of dg operads. We construct an extension of the well known operad of A_∞ algebras such that the representations of it in V are equivalent to an A_∞ structure on $V[[\hbar]]$. This new operad is also a minimal model of an operad that can be seen as the extension of the operad of associative algebras by a unary operation. We give an explicit map of operads from the extended associative operad to the operad we get when applying the Merkulov-Willwacher polydifferential functor to the properad of super-involutive Lie bialgebras. Lifting this map so as to go between their respective models gives a new proof of the main theorem.

Sammanfattning

Det gemensamma inslaget i den här avhandlingen är högre algebraiska strukturer som kommer från operader och propar.

Det första kapitlet är en introduktion till det matematiska ramverk som avhandlingen främst håller sig inom. Kapitlet innehåller inga nya resultat

Det andra kapitlet behandlar konstruktionen av en konfigurationsrumsmodel för den 2-färgade differentialgraderade operaden som beskriver strukturen av två A_∞ algebror med två A_∞ avbildningar och en homotopi mellan avbildningarna. Vi visar att kohomologin av denna operad är den välkända 2-färgade operaden som beskriver två associativa algebror och en avbildning mellan dem.

Det tredje kapitlet behandlar deformationskvantisering av (potentiellt) oändligtdimensionella (kvasi-)poissonmångfalder. Vårt bevis använder de slags *transcendentala* metoder som M. Kontsevich använde för att behandla det ändligtdimensionella fallet. Det första beviset för oändligtdimensionell deformationskvantisering gavs av B. Shoikhet. Ett viktigt inslag i beviset är konstruktionen av en universel L_∞ struktur på formella polyvektorfält. Vårt bidrag är en förenkling av B. Shoikhets bevis via användandet av en enklare *propagator*. Resultatet sätts även in i sammanhanget givit av differentialgraderade Lie algebror kommande från grafkomplex; L_∞ strukturen bevisas komma från ett Maurer-Cartan element i det *orienterade grafkomplexet*.

Det fjärde kapitlet behandlar också deformationskvantisering av kvasipoissonstrukturer i det oändligtdimensionella fallet. Till skillnad från det föregående kapitlet så är metoderna i detta kapitel rent algebraiska. Vårt huvudteorem är möjligheten att deformationskvantisera kvasipoissonstrukturer med hjälp av endast perturbativa metoder; i kontrast till de transcendentala metoder som användes i kapitlet innan.

Vi ger två bevis för teoremet med hjälp av teorin för dg operader, dg properader och dg propar. Vi visar att det finns en propavbildning från en prop som beskriver stjärnprodukter till en prop som beskriver kvasipoissonstrukturer. Denna avbildning ger ett teorem för existensen av en deformationskvantisering av kvasipoissonstrukturer. Beviset börjar med att beskriva en explicit deformationkvantisering av *superinvolutiva Lie bialgebror* och sen lyfts den associerade avbildningen till dg properaden som beskriver kvasipois-

sonstrukturer. Propen som beskriver stjärnprodukter konstruerades av S.A. Merkulov men properaden som beskriver kvasipoissonstrukterer är en ny konstruktion.

Det andra beviset av teoremet använder Merkulov-Willwachers polydifferentiella funktor för att överföra problemet att hitta en avbildning av dg propar till att hitta en avbildning av dg operader. Vi konstruerar en utvidning av operaden av A_∞ algebror. Representationerna av utvidgningen i ett vektorrum V är det samma som A_∞ strukturer i $A[[\hbar]]$. Denna operad är en minimal model för en utvidgning av operaden av associativa algebror genom att lägga till en unär operation. Vi beskriver en explicit avbildning från den utvidgade operaden av associativa algebror till den operad som bildas då man använder Merkulov-Willwachers polydifferentiella funktor på propen av superinvolutiva Lie bialgebror. Att lyfta denna avbildning till deras respektive modeler ger ett nytt bevis för teoremet.

Contents

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1. Introduction

1.1 Operads as a unifying language in mathematics

As this thesis practically revolves around the concept of an operad, I will give a little background on them and, by examples, try to convince the reader that they lead to interesting and beautiful mathematics.

1.1.1. General background and motivation Operads originated in the 1970s in the field of topology. The formal definition was given by P. May in [May], where he built upon the previous ideas of M. Bordmann, R. Vogt and J.D. Stasheff [BV1; BV2; St]. Since their invention, operads have come to have applications in several other areas of mathematics, most prominently in algebra, geometry and mathematical physics. Operads have been used to organize similar ideas under a common umbrella. To give an example from algebra. Commutative, associative and Lie algebras all have a cohomology theory of their own; the Harrison, Hochschild and Chevalley-Eilenberg cohomology, respectively. Within the framework of algebraic operads, each of these theories can be realized as an example of a general operad cohomology theory.

One can think of operads as a universal language in mathematics. It's often the case that the same operad will have an interesting incarnation in both algebra and geometry. In this way one can uncover deep and often unexpected connections. Let us discuss the example of the operad of little n-disks, denoted E_n. The first use of this operad was in topology to classify iterated and infinite loop spaces; a simply connected CW-complex X has the weak homotopy type of an n-fold (or infinite) loop space if it has the structure of an algebra over the operad E_n (or the E_∞ operad). Later, the associated *chains operad* on E_n, $\mathsf{Chains}(\mathsf{E}_n)$, was studied in a paper by F. Cohen [C]. In it he proved that the *homology operad* of E_2 coincides with the operad of *Gerstenhaber algebras*. This algebraic structure has important applications in algebra, geometry and mathematical physics. The operad $\mathsf{Chains}(\mathsf{E}_2)$ was proven to be formal by D. Tamarkin [Ta] and this was subsequently used to give a new proof of M. Kontsevich's formality theorem [Kon1]. This chains operad also plays a central role in the (now proven)

Deligne conjecture, which says that the Hochschild cochain complex of an associative algebra A (or more generally a A_∞ algebra) has the structure of an algebra over the operad $\mathsf{Chains}(\mathsf{E}_2)$. A generalization of this result states that the Hochschild cochain complex of a $\mathsf{Chains}(\mathsf{E}_n)$ algebra has the structure of a $\mathsf{Chains}(\mathsf{E}_{n+1})$ algebra [HKV]. Finally, the minimal model of this chain operad of little disks controls the Hertling-Manin system of non-linear partial differential equations which lie behind the notion of F-manifolds [Me7]. The operad of little disks demonstrates how operads can establish connections between very different mathematical structures.

1.1.2. Topological operads and configuration spaces Many important operads can be reinterpreted as operads of chains on a topological operad. We mentioned above the topological operad of little 2-disks, whose chain operad is equivalent to the operad controlling strongly homotopy Gerstenhaber algebras. Another example is given by the operad of little 1-disks, $D_1(\mathbb{R})$. The n-th part of this operad is (roughly speaking) given by the space of embeddings of n copies of $\mathbb{R}$ into $\mathbb{R}$ such that the image intervals are disjoint. The representations of $D_1(\mathbb{R})$ are the same thing as A_∞ spaces and the chains of this topological operad is a differential graded operad that is quasi-isomorphic to the operad Ass_∞. There is however another very useful way to connect the theory of A_∞ algebras to the theory of geometric operads.

Consider n points on the real line modulo the action of the affine group; $x \mapsto \lambda x + c$ where $\lambda \in \mathbb{R}^+$ and $c \in \mathbb{R}$. The space of such configurations of points is an $n-2$ dimensional manifold $C_n(\mathbb{R})$. This manifold $C_n(\mathbb{R})$ can be suitably compactified into a closed manifold with corners $\overline{C}_n(\mathbb{R})$ in a such a way that the whole family $\{\overline{C}_n(\mathbb{R})\}_{n\geq 2}$ gives us an operad in the category of smooth manifolds with corners. The associate operad of fundamental chains is identical to the operad of Ass_∞ algebras. Note that in this approach we get a geometric interpretation of the Ass_∞ operad in terms of manifolds, not just topological spaces! Therefore this approach gives us new mathematical tools when studying strongly homotopy algebras, as for example, manifolds with corners are always equipped with sheaves of differential forms which one can integrate and which obey the Stokes' theorem. Therefore such an interpretation of an algebraic operad in terms of an operad of configuration spaces opens up the possibility of obtaining transcendental results; results that cannot be achieved just through homological algebra and perturbative methods. There are two such famous transcendental results due to M. Kontsevich.

In the 90s M. Kontsevich made a ground breaking contribution to the field of mathematical physics by proving his *Formality conjecture* [Kon1].

The result gives an L_∞ quasi-isomorphism

$$\mathcal{K} : (\mathcal{T}_{poly}(\mathbb{R}^d), [-,-]_S, d=0) \longrightarrow (\mathcal{D}_{poly}(\mathcal{O}(\mathbb{R}^d)), [-,-]_G, d_H)$$

from the Lie algebra of polyvectorfields on $\mathbb{R}^d$ equipped with the Schouten-Nijenhuis bracket and the trivial differential to the Lie algebra of polydifferential operators on smooth functions on $\mathbb{R}^d$ equipped with the Gerstenhaber bracket and the Hochschild differential. The proof consists of an explicit construction. The map constructed by M. Kontsevich is given as a linear combination over a family of graphs. Each graph giving a polydifferential operator and a weight that is determined as the integral over a configuration space. The formality theorem can be formulated as a morphism of operads, i.e. as a morphism from the operad of fundamental chains of Kontsevich configuration spaces to the operad of Kontsevich graphs.

The second such transcendental result due to M. Kontsevich gives and explicit proof of the formality of the little disks operad [Kon2].

1.2 Outline of thesis

The thesis is mainly concerned with the higher algebraic structures coming from operads and props. The text is divided into four chapter.

The first chapter is an introduction to the mathematical framework in which the thesis is embedded. It does not contain any novel results and serves as a collection definitions, constructions and theorems that are required for the following chapters.

The second chapter investigates the notion of homotopies between (weak) morphisms of Ass_∞ algebras. The major achievement is a construction of a configurations space model for the 2-colored dg operad $\mathsf{Ho}(\mathsf{Ass})_\infty$ which controls a pair of A_∞ algebras, (V,μ^V), (W,μ^W), a pair of A_∞ morphism between them $f,g : (V,\mu^V) \to (W,\mu^W)$ and a homotopy between these morphism,

$$h : f \sim g.$$

Put another way a representation of our 2-colored operad is a diagram in the category of A_∞ algebras like this:

$$\begin{tikzcd} (V,\mu^V) \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & (W,\mu^W) \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd}$$

The configuration space model is given by considering a suitable compactification of the configuration space of points on the real line. In a very natural way this model generalizes previously known configuration space model that describe the 2-colored operad $\mathsf{Mor}(\mathsf{Ass})_\infty$. The 2-colored operad $\mathsf{Mor}(\mathsf{Ass})_\infty$ admits the configuration space model given by compactifications of configurations of n points on the line modulo the Lie group of translations. We also calculate the cohomology of the 2-colored dg operad $\mathsf{Ho}(\mathsf{Ass})_\infty$ and show that it is equal to $\mathsf{Mor}(\mathsf{Ass})$. Put another way, this result proves that $\mathsf{Ho}(\mathsf{Ass})_\infty$ is a non-minimal model of $\mathsf{Mor}(\mathsf{Ass})$. The result also implies, after some additional work, that $\mathsf{Ho}(\mathsf{Ass})_\infty$ is a non-minimal quasi-free model of the 2-colored dg operad $\mathsf{Ho}(\mathsf{Ass})$ which is defined as the operad encoding the structure of two dg associative algebras, two algebra morphisms between them and a homotopy between these two morphisms. Compared to the original presentation in loc. cit., the present presentation of the result is significantly modified with constructions simplified and exposition improved.

The third chapter is concerned with deformation quantization of (potentially) infinite dimensional formal (quasi-)Poisson structures. The proof in this chapter is a variation of M. Kontsevich's original proof for the finite dimensional case [Kon1]; We employ the *transcendental* methods pioneered in Loc. cit. and consider integrals over compactifications of configuration spaces. The first proof of the infinite dimensional case is due to B. Shoikhet [Sh]. An integral part of his proof is the construction of a universal Lie_∞ structure on the *polyvector fields* on a space V. Our contribution is a simplification of B. Shoikhet proof by considering a more natural configuration space and a simpler choice of *propagator*. The result is also put into a natural context of the dg Lie algebras coming from graph complexes. We show that the aforementioned Lie_∞ structure comes from a Maurer-Cartan element in the oriented graph complex defined by Willwacher [Wi2].

The fourth chapter is also concerned with the problem of deformation quantization in the infinite dimensional setting. Whereas the methods of chapter three are transcendental in nature — relying on integrals over configuration spaces — the methods of chapter four are purely algebraic. We give two proofs of deformation quantization of quasi-Poisson structures.

Following S.A. Merkulov [Me5; Me6] we reinterpret the problem of deformation quantization as that of finding a morphism of dg props. On one hand we have the dg prop $\mathsf{DefQ}^\hbar$ that can be characterized by having representations in a vector space V given by Maurer-Cartan elements in the full *polydifferential* Hochschild cochain complex $\mathrm{Hoch}^\bullet(\mathcal{O}_V[[\hbar]])$. These MC-elements correspond to (curved) Ass_∞ structures on $\mathcal{O}_V[[\hbar]]$. On the other hand we have the dg properad qPois that can be characterized by having representations in a vector space V given by Maurer-Cartan elements in the Kontsevich-Shoikhet Lie_∞ algebra $\mathcal{T}_{poly}(V)$ discussed in chapter three, i.e., quasi-Poisson structure in V. To prove a universal deformation quantization of quasi-Poisson structures it is enough to demonstrate a morphism of props $Q : \mathsf{DefQ}^\hbar \longrightarrow \mathsf{qPois}$. The proof uses that qPois is a cofibrant replacement of the much simpler properad $\mathsf{LieB}^\diamond_{odd}$, encoding odd Lie bialgebras with an extra relation corresponding to a higher notion of involutivity. It's a straightforward calculation to deformation quantize $\mathsf{LieB}^\diamond_{odd}$; to give a map of props $q : \mathsf{DefQ}^\hbar \longrightarrow \mathsf{LieB}^\diamond_{odd}$. By the cofibrancy of the two props $\mathsf{DefQ}^\hbar$ and qPois we can construct a lift of q such that the following diagram commutes

$$\begin{tikzcd} & \mathsf{qPois} \arrow[d, "\pi"] \\ \mathsf{DefQ}^\hbar \arrow[ur, dashed, "Q"] \arrow[r, "q"] & \mathsf{LieB}^\diamond_{odd} \end{tikzcd}$$

This gives us the necessary morphism to prove deformation quantization of quasi-Poisson structures. The prop $\mathsf{DefQ}^\hbar$ was first considered by S.A. Merkulov [Me5], but the properad qPois is a construction that is original to this thesis.

The second proof of the theorem employs the Merkulov-Willwacher polydifferential functor $\mathcal{O}$ ([MW1]) to transfer the problem of finding a morphism of dg props to that of finding a morphism of dg operads. We construct the operad $\mathsf{Ass}^\diamond_\infty$ which is an extensions of the Ass_∞ operad. The representations of $\mathsf{Ass}^\diamond_\infty$ in a $\mathbb{K}$-module V is equivalent to an Ass_∞ structure on the $\mathbb{K}[[\hbar]]$-module $V[[\hbar]]$. As the notation implies, the operad $\mathsf{Ass}^\diamond_\infty$ is a minimal resolution of an operad $\mathsf{Ass}^\diamond$. This operad is an extension of the classical Ass operad formed by adding a unary operation. The idea of the second proof is to use the functor $\mathcal{O}$ to transfer the map

$$p : \mathsf{LieB}^\diamond_{odd} \longrightarrow \mathsf{End}_V$$

giving a $\mathsf{LieB}^\diamond_{odd}$ structure in a vector space V to a map

$$\mathcal{O}p : \mathcal{O}\mathsf{LieB}^\diamond_{odd} \longrightarrow \mathsf{End}_{\odot^\bullet V}.$$

By constructing an explicit map

$$f : \mathsf{Ass}^\diamond \longrightarrow \mathcal{O}\mathsf{LieB}^\diamond_{odd}$$

we prove the existence of a universal deformation quantization by using the exactness of $\mathcal{O}$ and cofibrancy of qPois and $\mathsf{Ass}^\diamond_\infty$.

1.3 Admission

Some parts of this thesis were already published in the authors licentiate thesis [B]. To be precise, most of the content of the second chapter and parts of the exposition about monoidal categories and operads were taken from there.

1.4 Notations and conventions

Through out the thesis we let $\mathbb{K}$ be a field of characteristic zero. Vector spaces will always be over the field $\mathbb{K}$ unless specified otherwise. The set $\{1,2,\dots,n\}$ will be denoted by $[n]$. The group of permutations of $[n]$ will be denoted $\mathbb{S}_n$. Given a $\mathbb{Z}$-graded vector space $V=\bigoplus_{i\in\mathbb{Z}} V^{(i)}$, the *shifted* vector space $V[k]$ is defined to have the component $V[k]^{(i)}=V^{(i+k)}$ in degree i. Let s denote the degree 1 isomorphism $s: V\to V[1]$. We will often abbreviate differential graded as "dg".

1.5 General background

1.5.1. Differential graded Lie algebras and deformation theory Let g be a dg Lie algebra over the field $\mathbb{K}$ with Lie bracket $[-,-]$ and differential ∂. For a formal parameter $\hbar$ we can give the structure of a dg Lie algebra to $g\widehat{\otimes}\mathbb{K}[[\hbar]]=g[[\hbar]]$ by extending the bracket and differential to be $\mathbb{K}[[\hbar]]$-linear. Let $\mathfrak{m}=\hbar\mathbb{K}[[\hbar]]$ be the maximal ideal of $\mathbb{K}[[\hbar]]$.

Definition 1.5.1. *A Maurer-Cartan (or MC) element γ of the dg Lie algebra g is a formal power series $\gamma\in g^1\widehat{\otimes}\mathfrak{m}$ of degree 1 elements satisfying the equation*

$$\partial(\gamma)+\frac{1}{2}[\gamma,\gamma]=0.$$

Consider the Lie subalgebra $g^0\widehat{\otimes}\mathfrak{m}$ of formal power series of degree 0 elements in g without "constant term". This Lie algebra is the inverse limit of the system

$$g^0\widehat{\otimes}\mathfrak{m}=\varprojlim\left(\cdots\longrightarrow\frac{g^0\widehat{\otimes}\mathfrak{m}}{g^0\widehat{\otimes}\mathfrak{m}^{N+1}}\longrightarrow\frac{g^0\widehat{\otimes}\mathfrak{m}}{g^0\widehat{\otimes}\mathfrak{m}^{N}}\longrightarrow\frac{g^0\widehat{\otimes}\mathfrak{m}}{g^0\widehat{\otimes}\mathfrak{m}^{N-1}}\longrightarrow\cdots\right)$$

where the the maps are the natural projection maps.

The terms of the inverse system are all nilpotent Lie algebras, therefore a group can be defined by taking the exponential map

$$G=\exp\left(g^0\widehat{\otimes}\mathfrak{m}\right).$$

The group G acts on the MC elements of g according to the formula:

$$\exp(\xi)\gamma = \exp([-,\xi])\gamma + f([-,\xi])\partial\xi,$$

where f is the power series of the function

$$f(x) = \frac{e^x - 1}{x} = \sum_{n\geq 1} \frac{x^{n-1}}{n!}$$

about the point $x = 0$. We have the interpretation

$$\exp([-,\xi]) = \sum_{k\geq 0} \frac{1}{k!} \underbrace{[\ldots[[-,\xi],\xi],\ldots,\xi]}_{k} \in \mathbb{K}[[\hbar]] \,\widehat{\otimes}\, \mathrm{End}(g)^0$$

and in the same way we use the power series of $f(x)$ to define $f([-,\xi])$

From this we can define the *Deligne groupoid* $\mathrm{MC}(g)$ [GM]. The idea of the groupoid is to capture the formal deformation theory associated to the dg Lie algebra g. In this groupoid the objects are MC elements of g and morphisms between two MC elements γ_1 and γ_2 are elements of the group G which transform γ_1 to γ_2.

Let $\pi_0(\mathrm{MC}(g))$ denote the set of isomorphism classes of the groupoid $\mathrm{MC}(g)$.

Every morphism $\mu : g_1 \to g_2$ of dg Lie algebras gives us an explicit functor

$$\mu_* : \mathrm{MC}(g_1) \longrightarrow \mathrm{MC}(g_2)$$

between the corresponding Deligne groupoids.

For weakly equivalent dg Lie algebras there is a bijective correspondence of isomorphism classes of MC elements. According to [G], [GM] and [SS] we have the following theorem

Theorem 1.5.2. *If $\mu : L_1 \to L_2$ is a quasi-isomorphism of dg Lie algebras then μ_* induces a bijection between isomorphism classes of MC elements $\pi_0(\mathrm{MC}(L_1))$ and $\pi_0(\mathrm{MC}(L_2))$.*

Remark 1.5.3. *Theorem 1.5.2 is an essential ingredient in M. Kontsevich proof of deformation quantization for Poisson manifolds [Kon1]. Where for a finite dimensional manifold M the dg Lie algebras $(\mathcal{T}_{poly}(M), [\ ,\ ]_{SN}, 0)$ and $(\mathcal{D}_{poly}(M), [\ ,\ ]_G, d_H$ are found to be weakly equivalent - i.e. that they can be connected by a zig-zag of dg Lie algebra quasi-isomorphisms*

$$\mathcal{T}_{poly}(M) \to \bullet \leftarrow \bullet \to \ldots \leftarrow \bullet \to \mathcal{D}_{poly}(M)$$

or equivalently, which M. Kontsevich proves, that there is an Lie_∞ quasi-isomorphism

$$\mathcal{U} : \mathcal{T}_{poly}(M)[1] \longrightarrow \mathcal{D}_{poly}(M)[1].$$

It follows that there is a bijection between their sets of isomorphism classes of MC elements. The deformation quantization theorem follows from the fact that the isomorphism classes of MC elements of $\mathcal{T}_{poly}(M)$ are exactly Poisson structures on M and isomorphism classes of MC elements of $\mathcal{D}_{poly}(M)$ are exactly star-products on $C^{\infty}(M)$.

However, Theorem 1.5.2 has also been generalized and is a corollary to a result by E. Getzler in the paper [G]; Proposition 4.9, which is a more general result concerning the deformation theory of Lie_{∞} algebras.

1.5.2. Hochschild (co)homology and the HKR theorem(s) Let A be a $\mathbb{K}$-algebra and M an A-bimodule. For A and M we will define two related differential graded structures; The *Hochschild chain complex* $\mathrm{Hoch}_{\bullet}(A, M)$ and the the *Hochschild cochain complex* $\mathrm{Hoch}^{\bullet}(A, M)$.

The Hochschild chain complex is the differential graded vector space

$$\mathrm{Hoch}_{\bullet}(A, M) := \bigoplus_{k\geq 0} M \otimes A^{\otimes k}[k]$$

and degree -1 differential $d = \{d^{(n)}\}_{n\geq 0}$ defined by

$$\begin{gathered} M \otimes_{\mathbb{K}} A^{\otimes n} \xrightarrow{d^{(n)}} M \otimes_{\mathbb{K}} A^{\otimes(n-1)} \\ d^{(n)} := \textstyle\sum_{i=0}^{n}(-1)^i d_i^{(n)} \end{gathered}$$

where

$$\begin{aligned} d_0(m \otimes a_1 \otimes \ldots \otimes a_n) &= m \cdot a_1 \otimes a_2 \otimes \ldots \otimes a_n \\ d_i(m \otimes a_1 \otimes \ldots \otimes a_n) &= m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \quad 0 < i < n \\ d_n(m \otimes a_1 \otimes \ldots \otimes a_n) &= a_n \cdot m \otimes a_1 \otimes \ldots \otimes a_{n-1} \end{aligned}$$

The Hochschild cochain complex, $\mathrm{Hoch}^{\bullet}(A, M)$ is the differential graded vector space with graded components

$$\mathrm{Hoch}^n(A, M) := \mathrm{Hom}(A^{\otimes n}, M)[n]$$

and degree 1 differential

$$\begin{gathered} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{d^{(n)}} \mathrm{Hom}(A^{\otimes(n+1)}, M) \\ d^{(n)} = \textstyle\sum_{i=0}^{n}(-1)^i d_i^{(n)} \end{gathered}$$

where

$$d_0(f)(a_1 \otimes \ldots \otimes a_{n+1}) = a_1 \cdot f(a_2 \otimes \ldots \otimes a_{n+1})$$

$$d_i(f)(a_1 \otimes \ldots \otimes a_{n+1}) = f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \quad 0 < i < n$$
$$d_n(f)(a_1 \otimes \ldots \otimes a_{n+1}) = f(a_1 \otimes \ldots \otimes a_n) \cdot a_{n+1}$$

When $A = M$ we will simply write $\mathrm{Hoch}_\bullet(A)$ and $\mathrm{Hoch}^\bullet(A)$, respectively, for the Hochschild (co)chain complex of A as a bimodule over itself.

The homology of the complexes $(\mathrm{Hoch}_\bullet(A,M),d)$ and $(\mathrm{Hoch}^\bullet(A,M),d)$ are known as the Hochschild homology and Hochschild cohomology of M with coefficients in A. We denote this group by $\mathrm{HH}_\bullet(A,M)$ in the homology case and $\mathrm{HH}^\bullet(A,M)$ in the cohomology case. Again, we adopt the more compact notation of $\mathrm{HH}_\bullet(A)$ and $\mathrm{HH}^\bullet(A)$ when $A = M$.

In the particular case when $A = M$ the Hochschild cochain complex $\mathrm{Hoch}^\bullet(A)$ has the structure of a shifted dg Lie algebra with bracket $[-,-]_G$. Let $f \in \mathrm{Hoch}^n(A)$ and $g \in \mathrm{Hoch}^m(A)$, the degree -1 bracket in $\mathrm{Hoch}^\bullet(A)$ is defined as follows

$$\mathrm{Hoch}^n(A) \otimes \mathrm{Hoch}^m(A) \xrightarrow{[-,-]_G} \mathrm{Hoch}^{n+m-1}(A)$$
$$[f,g]_G := f \circ g - (-1)^{(|f|-1)(|g|-1)} g \circ f$$

where

$$f \circ g((x_1, x_2 \ldots\ldots x_{n+m-1})) =$$
$$\sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f(x_1, \ldots x_{i-1}, g(x_i, \ldots x_{i+m-1}), x_{i+m}, \ldots x_{n+m-1})$$

When A is a commutative algebra then the Hochschild cochain complex $\mathrm{Hoch}^\bullet(A)$ is equipped with a graded commutative product called the cup product. Let f and g be as above, then the cup product is defined as follows

$$\mathrm{Hoch}^n(A) \otimes \mathrm{Hoch}^m(A) \xrightarrow{-\cup-} \mathrm{Hoch}^{n+m}(A)$$
$$(x_1, \ldots, x_n) \otimes (y_1, \ldots y_m) \xmapsto{f \cup g} f(x_1, \ldots, x_n) \cdot g(y_1, \ldots, y_m)$$

The shifted Lie-bracket and the cup product descend to well-defined operations on the Hochschild cohomology $\mathrm{HH}^\bullet(A)$. On the cohomolgy the operations satisfy the coherence conditation that the adjoint action of the bracket is a derivation of the cup product. This structure of a shifted Lie bracket and graded commutative product for which the bracket acts as a derivation is known as a *Gerstenhaber algebra*.

In many situations it can be convenient to shift the Hochschild cochain complex so that the Lie bracket is of degree 0;

$$\mathrm{Hoch}^{\bullet+1}(A) = \bigoplus_{n \geq 0} \mathrm{Hom}(A^{\otimes n}, A)[n+1].$$

Kähler differentials and derivations

Let A be a commutative $\mathbb{K}$-algebra and let ϵ denote the multiplication map

$$\begin{aligned} A \otimes_{\mathbb{K}} A &\xrightarrow{\epsilon} A \\ \sum x_i \otimes y_i &\longmapsto \sum x_i y_i \end{aligned}$$

Let I be kernel of the map ϵ and consider the module I/I^2.

Proposition 1.5.4. *Let M be an A-module. There is an isomorphism between the module of derivations* $\operatorname{Der}_{\mathbb{K}}(A, M)$ *and the module of linear maps* $\operatorname{Hom}_{\mathbb{K}}(I/I^2, M)$.

The module I/I^2 is isomorphic to the module of *Kähler differentials* of A (as a $\mathbb{K}$-algebra); which we denote by $\Omega_{A/\mathbb{K}}$. The module $\Omega_{A/\mathbb{K}}$ is defined as the free A-module on the symbols da for $a \in A$ subject to the relations

- $d(a+b) = da + db$
- $d(ab) = adb + bda$
- $d\alpha = 0$ if $\alpha \in \mathbb{K}$

Explicitly, the isomorphism $\Omega_{A/\mathbb{K}} \longrightarrow I/I^2$ is given by $da \longmapsto 1 \otimes a - a \otimes 1$

Proposition 1.5.5. *Let A be commutative $\mathbb{K}$-algebra and let M be a symmetric A-bimodule, (i.e. $am = ma$.) Then*

$$\begin{aligned} \mathrm{HH}^1(A, M) &\cong \operatorname{Der}_{\mathbb{K}}(A, M) \\ \mathrm{HH}_1(A, M) &\cong M \otimes_A \Omega_{A/\mathbb{K}} \end{aligned}$$

specifically when $M = A$ we have that

$$\begin{aligned} \mathrm{HH}^1(A, A) &\cong \operatorname{Der}_{\mathbb{K}}(A) \\ \mathrm{HH}_1(A, A) &\cong \Omega_{A/\mathbb{K}} \end{aligned}$$

In the seminal paper by G. Hochschild, B. Kostant and A. Rosenberg [HKR], proposition 1.5.5 was given a generalization that has been come to be known as the *Hochschild-Kostant-Rosenberg Theorem.* The generalization works for certain well-behaved algebras. We remind the reader of some definitions before stating the theorem.

Definition 1.5.6. *Let A be a commutative $\mathbb{K}$-algebra. We say that A is essentially of finite type if it is a localization of a finitely generated $\mathbb{K}$-algebra.*

Definition 1.5.7. *Let A be a commutative $\mathbb{K}$-algebra. We say that A is smooth if it satisfies the following lifting property: Let C be commutative $\mathbb{K}$-algebra with square-zero ideal N with a morphism of $\mathbb{K}$-algebras $A \xrightarrow{e} C/N$. Then A is smooth if there is a lift $A \xrightarrow{f} C$ such that $p \circ f = e$ where p is the natural projection $C \xrightarrow{p} C/N$.*

Theorem 1.5.8. (Hochschild-Kostant-Rosenberg) *Let A be a smooth commutative $\mathbb{K}$-algebra of essentially finite type then there are two quasi-isomorphism of differential graded vector spaces*

$$\left(\bigwedge^\bullet \Omega_{A/\mathbb{K}}, 0\right) \xrightarrow{\simeq} (\mathrm{Hoch}_\bullet(A,A), d_H)$$
$$\left(\bigwedge^\bullet \mathrm{Der}_{\mathbb{K}}(A), 0\right) \xrightarrow{\simeq} \left(\mathrm{Hoch}^\bullet(A,A), d_H\right)$$

Proof. See e.g. [We] □

1.5.3. The polydifferential Hochschild (co)chain complex The Hochschild cochain complex $\mathrm{Hoch}^\bullet(A)$ is a huge object in the case where the algebra A is the algebra of smooth functions on a manifold M, $A := C^\infty(M)$. In this thesis we will mainly be interested in the subcomplex (in fact, the dg Lie subalgebra) consisting of the so called polydifferential operators, which are defined as follows.

Definition 1.5.9. *A polydifferential operator $D : C^\infty(M)^{\otimes n} \longrightarrow C^\infty(M)$, expressed in coordinates (x^i), is a map D of the form*

$$D : f_1 \otimes f_2 \otimes \ldots \otimes f_n \mapsto \sum_{(I_1,\ldots,I_n)} F^{(I_1,\ldots,I_n)}(x) \cdot \frac{\partial^{|I_1|}(f_1)}{(\partial x)^{I_1}} \cdot \frac{\partial^{|I_2|}(f_2)}{(\partial x)^{I_2}} \cdot \ldots \cdot \frac{\partial^{|I_n|}(f_n)}{(\partial x)^{I_n}}$$

where the I_i are multi-indices, $F^{I_1,\ldots,I_n}$ are smooth functions, $\cdot$ is the ordinary commutative product of functions and $\frac{\partial^{|I_i|}(f_i)}{(\partial x)^{I_i}}$ is the partial derivative of f_i that is associated to the multi-index I_i.

1.6 Deformation quantization

Deformation quantization is one approach to the procedure of quantizing a classical mechanical system to produce a quantum mechanical system-counterpart. In this formalization one considers the algebras of observables of the physical system; in the classical mechanical system one has the commutative algebra of smooth functions $C^\infty(M)$ and the aim is to find a deformation of the ordinary commutative product of functions to a non-commutative associative algebra structure on $C^\infty(M)[[\hbar]]$ satisfying some limit conditions. Such a product is called a star-product. We give a rigourous definition.

Definition 1.6.1. *Let A be the $\mathbb{R}$-algebra of smooth functions on a manifold M. A star-product on A is an associative $\mathbb{R}[[\hbar]]$-linear product on $A[[\hbar]]$;*

$$A[[\hbar]] \otimes_{\mathbb{R}[[\hbar]]} A[[\hbar]] \longrightarrow A[[\hbar]]$$

of the form

$$(f,g) \mapsto f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \dots$$

where B_i is bilinear and a differential operator in each argument.

We will consider star-products up to the equivalence of a particular group of automorphisms. Consider the set of $\mathbb{R}[[\hbar]]$-linear automorphisms

$$D : A[[\hbar]] \longrightarrow A[[\hbar]]$$

of the form

$$D(f) = f + \hbar D_1(f) + \hbar^2 D_2(f) + \dots \quad f \in A$$

where each D_i is a differential operator. For the group of these automorphism we can define an action on star-products $D : \star \mapsto \star'$;

$$f \star' g = D\big(D^{-1}(f) \star D^{-1}(g)\big) \quad f, g \in A[[\hbar]]$$

We say that two star-products are equivalent if they are related by such an automorphism.

The notion of star-product is closely connected to the deformation theory of algebras. We develop this connection in the rest of the section.

Let $\star$ be a star-product on A and suppose it has the explicit form

$$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \dots$$

We can show that the commutator of B_1 defines a Poisson bracket on the algebra A,

$$\{-,-\} : A \otimes A \to A$$
$$\{f,g\} := B_1(f,g) - B_1(g,f),$$

i.e. it's a Lie bracket which acts as a derivation of the associative product of A;

$$\{f, gh\} = \{f,g\}h + g\{f,h\}.$$

To see that this is the case we first notice that

$$[f,g] := \frac{1}{\hbar}(f \star g - g \star f)$$

defines a Lie bracket on $A[[\hbar]]$. If we reduce the commutator $[-,-]$ modulo $\hbar$ we still have a Lie bracket and this reduction exactly produces the commutator of B_1, i.e. $\{-,-\}$. Considering the associativity of the star-product $\star$ we find that the following relation holds for all $\alpha, \beta, \gamma \in A[[\hbar]]$

$$B_1(\alpha\beta,\gamma) - B_1(\alpha,\beta\gamma) - \alpha B_1(\beta,\gamma) + \gamma B_1(\alpha,\beta) = 0 \tag{1.1}$$

using equation 1.1 repeatedly we can demonstrate that $\{-,-\}$ is a Poisson bracket

$$\begin{aligned}
\{f, gh\} &= B_1(f, gh) - B_1(gh, f) \\
&= B_1(fg,h) - fB_1(g,h) + hB_1(f,g) - B_1(g,fh) + fB_1(g,h) - gB_1(h,f) \\
&= B_1(gf,h) - B_1(g,fh) + hB_1(f,g) - gB_1(h,f) \\
&= gB_1(f,h) + hB_1(f,g) - gB_1(h,f) - hB_1(g,f) \\
&= g\{f,h\} + h\{f,g\}
\end{aligned}$$

We also notice that equation 1.1 is exactly the criterion for B_1 to be a 2-cocycle in the Hochschild cochain complex of A. Furthermore, if the two star-products

$$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g)\dots$$

and

$$f \star' g = fg + \hbar B_1'(f,g) + \hbar^2 B_2'(f,g) + \dots$$

are equivalent through the automorphism $D = \mathrm{Id} + \hbar E$ then we can deduce that

$$B_1'(f,g) = B_1(f,g) + fE(g) + gE(f) - E(fg),$$

which means that B_1' and B_1 differ by a 2-coboundary. We summarize the discussion in a theorem.

Theorem 1.6.2. *The set of equivalence classes of first order deformations of an algebra A is bijective to the second Hochschild cohomology* $\mathrm{HH}^2(A)$.

Let A be an algebra and let $\star$ be star-product on A. We let μ denote the ordinary multiplication in A

$$\mu : A \otimes A \longrightarrow A$$

and $\mu_\hbar$ denote the multiplication coming from the star-products

$$\mu_\hbar : A[[\hbar]] \otimes A[[\hbar]] \longrightarrow A[[\hbar]]$$

The shifted Hochschild cochain complex of $\mathrm{Hoch}^{\bullet+1}(A)$ is a dg Lie algebra with the Hochschild differential d_H and Gerstenhaber bracket $[\ ,\ ]_G$. Let us apply the general idea of deformation theory and make some comments on its Deligne groupoid of Maurer-Cartan elements.

A Maurer-Cartan element for $\mathrm{Hoch}^{\bullet+1}(A)$ is an element $\gamma \in \mathrm{Hoch}^{1+1}(A)\widehat{\otimes}\mathfrak{m}$, (where $\mathfrak{m} = \hbar\mathbb{R}[[\hbar]]$ is the maximal ideal of $\mathbb{R}[[\hbar]]$) such that

$$d_H(\gamma) + \frac{1}{2}[\gamma,\gamma]_G = 0$$

We can think of $\mu_\hbar$ as an element of $\mathrm{Hoch}^{1+1}(A)[[\hbar]]$. We let the Lie bracket be extended by $\mathbb{R}[[\hbar]]$-linearity. Now the associativity of $\mu_\hbar$ can be phrased as

$$0 = 2(\mu_\hbar(\mu_\hbar(f,g),h) - \mu_\hbar(f,\mu_\hbar(f,g))) = [\mu_\hbar,\mu_\hbar]_G(f,g,h),$$

we conclude that

$$[\mu_\hbar,\mu_\hbar]_G = 0. \tag{1.2}$$

Let us consider the decomposition of $\mu_\hbar$ as a sum $\mu_\hbar = \mu + B$ where $B \in \mathrm{Hoch}^{1+1}(A)\widehat{\otimes}\mathfrak{m}$ then we can expand equation 1.2 as follows

$$[\mu_\hbar,\mu_\hbar]_G = [\mu,\mu]_G + 2[\mu,B] + [B,B]_G = 0$$

We notice two things:

- μ is associative and therefore $[\mu,\mu] = 0$
- the Hochschild differential can be understood as the adjoint action of μ;
$$d_H = \pm[\mu,-]$$

From this we can see that equation 1.2 is equivalent to the Maurer-Cartan equation for B;

$$d_H(B) + \frac{1}{2}[B,B]_G = 0$$

and that star-products on A are given by Maurer-Cartan elements of the dg Lie algebra $\left(\mathrm{Hoch}^{\bullet+1}(A), d_H, [-,-]_G\right)$.

1.7 Monoidal categories

The appropriate categorical setting for the higher algebraic structures like operads, properads and props is that of a symmetric monoidal category.

Definition 1.7.1. *A monoidal category is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object I together with three natural isomorphisms,*

i) the associator $\alpha_{A,B,C} : (A\otimes B)\otimes C \cong A\otimes(B\otimes C)$

ii) the left unitor $\rho_A : I \otimes A \cong A$

iii) the right unitor $\nu_A : A \otimes I \cong A$,

such that the following diagrams commute

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes D"] & (A \otimes B) \otimes (C \otimes D) \arrow[dd, "\alpha_{A,B,C\otimes D}"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A,B\otimes C,D}"] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[r, "A\otimes\alpha_{B,C,D}"] & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\nu_A\otimes B"] & & A \otimes (I \otimes B) \arrow[dl, "A\otimes\rho_B"] \\
& A \otimes B &
\end{tikzcd}$$

Furthermore, we say that monoidal category $\mathcal{C}$ *is symmetric if it's equipped with an isomorphism* $\sigma_{A,B} : A \otimes B \cong B \otimes A$ *such that the following diagrams commute:*

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\sigma_{A,I}"] \arrow[dr, "\nu_A"] & & I \otimes A \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\sigma_{A,B}\otimes C"] \arrow[d, "\alpha_{A,B,C}"] & (B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"] \\
A \otimes (B \otimes C) \arrow[d, "\sigma_{A,B\otimes C}"] & B \otimes (A \otimes C) \arrow[d, "B\otimes\sigma_{A,C}"] \\
(B \otimes C) \otimes A \arrow[r, "\alpha_{A,B,C}"] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
& B \otimes A \arrow[dr, "\sigma_{B,A}"] & \\
A \otimes B \arrow[ur, "\sigma_{A,B}"] \arrow[rr, equal, "A\otimes B"] & & A \otimes B
\end{tikzcd}$$

Example 1.7.2. *The category of sets with Cartesian products form a symmetric monoidal category with the one element set as a unit and the map* $\sigma : A \times B \to B \times A$ *being given on pairs* $(a, b) \mapsto (b, a)$.

Example 1.7.3. *The category of dg vector spaces together with the tensor product form a symmetric monoidal category with the ground field as the unit and the map* $\sigma : (A \otimes B) = \bigoplus_{i+j=n} A^i \otimes B^j \to \bigoplus_{i+j=n} B^j \otimes A^i = B \otimes A$ *given on homogeneous components as* $a \otimes b \mapsto (-)^{\deg a \deg b} b \otimes a$.

Example 1.7.4. *The category of topological spaces with Cartesian products form a symmetric monoidal category with the unit being the point and the map* $\sigma : A \times B \to B \times A$ *being given on pairs* $(a, b) \mapsto (b, a)$.

Definition 1.7.5. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *between monoidal categories* $(\mathcal{C}, \otimes, I, \alpha)$ *and* $(\mathcal{D}, \otimes, J, \beta)$ *together with a natural transformation*

$$\phi_{AB} : FA \otimes FB \to F(A \otimes B)$$

and a morphism $\psi : J \to FI$ *is called monoidal if the following diagrams commute*

$$\begin{tikzcd}
(FA \otimes FB) \otimes FC \arrow[r, "\beta"] \arrow[d, "\phi\otimes\mathrm{Id}"] & FA \otimes (FB \otimes FC) \arrow[d, "\mathrm{Id}\otimes\phi"] \\
F(A \otimes B) \otimes FC \arrow[d, "\phi"] & FA \otimes F(B \otimes C) \arrow[d, "\phi"] \\
F((A \otimes B) \otimes C) \arrow[r, "F\alpha"] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes J \arrow[r, "\mathrm{Id}\otimes\psi"] \arrow[d, "\nu_A"] & FA \otimes FI \arrow[d, "\phi"] \\
FA & F(A \otimes I) \arrow[l, "F\nu_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
J \otimes FA \arrow[r, "\psi\otimes\mathrm{Id}"] \arrow[d, "\rho_A"] & FI \otimes FA \arrow[d, "\phi"] \\
FA & F(I \otimes A) \arrow[l, "F\rho_A"']
\end{tikzcd}$$

Furthermore, the monoidal functor F *is called symmetric monoidal if it's a functor between symmetric monoidal categories and if the following diagram commutes*

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\sigma"] \arrow[d, "\phi"] & FB \otimes FA \arrow[d, "\phi"] \\
F(A \otimes B) & F(B \otimes A) \arrow[l, "F\sigma"']
\end{tikzcd}$$

Example 1.7.6. *Two central examples of symmetric monoidal functors are the chains functor on a topological space and the homology functor on complexes. That these functors are symmetric monoidal is essentially the content of the Eilenberg-Zilber theorem and the Künneth theorem, respectively.*

1.8 Operads

1.8.1. The classical definition of operads J.P. May gave the definition of an operad in a suitable category of topological spaces [May]. We will restate this definition with the generalization to objects of a symmetric monoidal category, as has been done in [MSS].

Definition 1.8.1. *Let G be a group and x an object in some category $\mathcal{C}$. A left action by G on x is a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(x,x)$, where $\mathrm{Aut}_{\mathcal{C}}(x,x)$ is the group of units in the monoid $\hom_{\mathcal{C}}(x,x)$. A right action by G on x is function $G \to \mathrm{Aut}_{\mathcal{C}}(x,x)$ such that it is a group homomorphism when composed with the inversion map $G \to G$*

Definition 1.8.2. *Let Σ be the category with objects the sets $[n] = \{1,\dots,n\}$ and morphisms the elements of the symmetric groups. A Σ-module in a category $\mathcal{C}$ is an element in $Fun(\Sigma^{op},\mathcal{C})$. Alternatively we could say that a E is a Σ-module if there are objects $E(n)$ (where it is understood that $E([n]) = E(n)$) for all $n \geq 0$ with a right action of $\mathbb{S}_n$.*

Definition 1.8.3. *A non-unital operad in a symmetric monoidal category $(\mathcal{C},\otimes,I)$ is a Σ-module $\{\mathsf{O}(n)\}_{n\geq 1}$ and a composition map*

$$\gamma : \mathsf{O}(k) \otimes \bigotimes_{r=1}^{k} \mathsf{O}(j_r) \to \mathsf{O}\left(\sum j_r\right)$$

such that the following diagrams commute:

1. *(associativity)*

$$\begin{tikzcd}
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}\mathsf{O}(j_r))\otimes(\bigotimes_{t=1}^{\sum j_r}\mathsf{O}(i_t)) \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "shuffle"] & \mathsf{O}(\sum_{r=1}^{k} j_r)\otimes(\bigotimes_{t=1}^{\sum j_r}\mathsf{O}(i_t)) \arrow[dd, "\gamma"] \\
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}(\mathsf{O}(j_r)\otimes(\bigotimes_{q=1+j_1+\ldots+j_{r-1}}^{j_1+\ldots+j_r}\mathsf{O}(i_q)) \arrow[d, "\mathrm{id}\otimes(\otimes_r\gamma)"] & \\
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}\mathsf{O}(\sum_{q=1}^{j_r} i_{j_1+\ldots+j_{r-1}+q})) \arrow[r, "\gamma"] & \mathsf{O}(\sum_{t=1}^{\sum j_r} i_t)
\end{tikzcd}$$

2. *(equivariance)*

$$\begin{tikzcd}
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}\mathsf{O}(j_r)) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\gamma"] & \mathsf{O}(k)\otimes\bigotimes_{r=1}^{k}\mathsf{O}(j_{\sigma(r)}) \arrow[d, "\gamma"] \\
\mathsf{O}(\sum_{r=1}^{k} j_r) \arrow[r, "\sigma(j_1,\ldots,j_k)"] & \mathsf{O}(\sum_{r=1}^{k} j_r)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}\mathsf{O}(j_r)) \arrow[r, "\mathrm{id}\otimes(\tau_1\otimes\ldots\otimes\tau_k)"] \arrow[d, "\gamma"] & \mathsf{O}(k)\otimes\bigotimes_{r=1}^{k}\mathsf{O}(j_r) \arrow[d, "\gamma"] \\
\mathsf{O}(\sum_{r=1}^{k} j_r) \arrow[r, "\tau_1\oplus\ldots\oplus\tau_k"] & \mathsf{O}(\sum_{r=1}^{k} j_r)
\end{tikzcd}$$

for $\sigma \in \mathbb{S}_k$ and $\tau_i \in \mathbb{S}_{j_i}$, where $\sigma(j_1,\ldots,j_k) \in \mathbb{S}_{\sum j_r}$ is the induced permutation action on the k blocks r_j and where $\tau_1 \oplus \ldots \oplus \tau_k \in \mathbb{S}_{\sum j_r}$ is the block sum permutation.

Definition 1.8.4. *A pseudo operad in a symmetric monoidal category $(\mathcal{C},\otimes,I)$ is a Σ-module $\{\mathsf{O}(n)\}_{n\geq 1}$ and with composition maps*

$$\circ_j : \mathsf{O}(n)\otimes\mathsf{O}(m) \to \mathsf{O}(n+m-1) \quad 1 \leq j \leq n$$

such that the following conditions are fulfilled

- *(associativity*

$$\circ_i(\circ_j\otimes\mathrm{id}) = \begin{cases} \circ_{j+p-1}(\circ_i\otimes\mathrm{id})(\mathrm{id}\otimes\tau) & \text{for } 1\leq i\leq j-1, \\ \circ_j(\mathrm{id}\otimes\circ_{i\,j+1}) & \text{for } j\leq i\leq j+n-1 \text{ and} \\ \circ_j(\circ_{i-n+1}\otimes\mathrm{id})(\mathrm{id}\otimes\tau) & \text{for } j+n\leq i, \end{cases}$$

  where τ is the transposition $\mathsf{O}(n)\otimes\mathsf{O}(m)\to\mathsf{O}(m)\otimes\mathsf{O}(n)$.

- *(equivariance)*

$$\circ_i(\sigma\otimes\rho) = (\sigma\circ_i\rho)\circ_{\sigma(i)}$$

  where $\sigma\in\mathbb{S}_n$, $\rho\in\mathbb{S}_m$ such that $\sigma\circ_i\rho\in\mathbb{S}_{m+n-1}$ with $\sigma\circ_i\rho = \sigma_{1,\ldots,1,m,1,\ldots,1}(1\times\cdots\times 1\times\rho\times 1\times\cdots\times 1)$, and where $\sigma_{1,\ldots,1,m,1,\ldots,1}$ is the block permutation on the n blocks $1,\ldots,1,m,1,\ldots,1$.

Definition 1.8.5. *An operad in a symmetric monoidal category $(\mathcal{C},\otimes,I)$ is a Σ-module $\{\mathsf{O}(n)\}_{n\geq 1}$, a unit map $\nu : I \to \mathsf{O}(1)$ and a composition map*

$$\gamma : \mathsf{O}(k)\otimes\bigotimes_{r=1}^{k}\mathsf{O}(j_r) \to \mathsf{O}\left(\sum j_r\right)$$

such that the following diagrams commute:

1. *(associativity)*

$$\begin{tikzcd}
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}\mathsf{O}(j_r))\otimes(\bigotimes_{t=1}^{\sum j_r}\mathsf{O}(i_t)) \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "shuffle"] & \mathsf{O}(\sum_{r=1}^{k} j_r)\otimes(\bigotimes_{t=1}^{\sum j_r}\mathsf{O}(i_t)) \arrow[dd, "\gamma"] \\
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}(\mathsf{O}(j_r)\otimes(\bigotimes_{q=1+j_1+\ldots+j_{r-1}}^{j_1+\ldots+j_r}\mathsf{O}(i_q)) \arrow[d, "\mathrm{id}\otimes(\otimes_r\gamma)"] & \\
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}\mathsf{O}(\sum_{q=1}^{j_r} i_{j_1+\ldots+j_{r-1}+q})) \arrow[r, "\gamma"] & \mathsf{O}(\sum_{t=1}^{\sum j_r} i_t)
\end{tikzcd}$$

2. *(unitality)*

$$\begin{tikzcd}
\mathsf{O}(k)\otimes(I)^{\otimes k} \arrow[d, "\mathrm{id}\otimes(\nu^{\otimes k})"'] \arrow[dr, "\cong"] & \\
\mathsf{O}(k)\otimes(\mathsf{O}(1)^{\otimes k}) \arrow[r, "\gamma"] & \mathsf{O}(k)
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes\mathsf{O}(k) \arrow[d, "\nu\otimes\mathrm{id}"] \arrow[dr, "\cong"] & \\
\mathsf{O}(1)\otimes\mathsf{O}(k) \arrow[r, "\gamma"] & \mathsf{O}(k)
\end{tikzcd}$$

3. *(equivariance)*

$$\begin{tikzcd}
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}\mathsf{O}(j_r)) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\gamma"] & \mathsf{O}(k)\otimes\bigotimes_{r=1}^{k}\mathsf{O}(j_{\sigma(r)}) \arrow[d, "\gamma"] \\
\mathsf{O}(\sum_{r=1}^{k} j_r) \arrow[r, "\sigma(j_1,\ldots,j_k)"] & \mathsf{O}(\sum_{r=1}^{k} j_r)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{O}(k)\otimes(\bigotimes_{r=1}^{k}\mathsf{O}(j_r)) \arrow[r, "\mathrm{id}\otimes(\tau_1\otimes\ldots\otimes\tau_k)"] \arrow[d, "\gamma"] & \mathsf{O}(k)\otimes\bigotimes_{r=1}^{k}\mathsf{O}(j_r) \arrow[d, "\gamma"] \\
\mathsf{O}(\sum_{r=1}^{k} j_r) \arrow[r, "\tau_1\oplus\ldots\oplus\tau_k"] & \mathsf{O}(\sum_{r=1}^{k} j_r)
\end{tikzcd}$$

for $\sigma\in\mathbb{S}_k$ and $\tau_i\in\mathbb{S}_{j_i}$, where $\sigma(j_1,\ldots,j_k)\in\mathbb{S}_{\sum j_r}$ is the induced permutation action on the k blocks r_j and where $\tau_1\oplus\ldots\oplus\tau_k\in\mathbb{S}_{\sum j_r}$ is the block sum permutation.

We can also give a partial definition of the operadic composition map.

Definition 1.8.6. *An operad in a symmetric monoidal category $(\mathcal{C},\otimes,I)$ is a Σ-module $\{\mathsf{O}(n)\}_{n\geq 1}$, a unit map $\nu: I\to\mathsf{O}(1)$ and n composition maps*

$$\circ_j:\mathsf{O}(n)\otimes\mathsf{O}(m)\to\mathsf{O}(n+m-1)\qquad 1\leq j\leq n$$

such that $\circ_j = \gamma\sigma\pi$ *where*

$$\begin{aligned}
\pi &: \mathsf{O}(n)\otimes\mathsf{O}(m) \cong \mathsf{O}(n)\otimes I^{j-1}\otimes\mathsf{O}(m)\otimes I^{n-j}\\
\sigma &: \mathsf{O}(n)\otimes I^{j-1}\otimes\mathsf{O}(m)\otimes I^{n-j} \to \mathsf{O}(n)\otimes\mathsf{O}(1)^{j-1}\otimes\mathsf{O}(m)\otimes\mathsf{O}(1)^{n-j}\\
\gamma &: \mathsf{O}(n)\otimes\mathsf{O}(1)^{j-1}\otimes\mathsf{O}(m)\otimes\mathsf{O}(1)^{n-j} \to \mathsf{O}(n+m-1).
\end{aligned}$$

where γ *satisfies the associativity, unitality and equivariance axiom of the preceding definition.*

Example 1.8.7. *Let* $\mathcal{C}$ *be a symmetric monoidal category with internal* hom-*functor* Hom *and let* X *be an object in* $\mathcal{C}$. *The endomorphism operad of* X, End_X, *is the objects* $\mathrm{Hom}(X^{\otimes k}, X)$ *with the operadic composition* γ *defined as a certain pre-composition scheme. Explicitly*

$$\gamma : \mathrm{Hom}(X^{\otimes n}, X)\otimes\mathrm{Hom}(X^{\otimes k_1}, X)\otimes\ldots\otimes\mathrm{Hom}(X^{\otimes k_n}, X) \longrightarrow \mathrm{Hom}(X^{\otimes(\sum k_i)}, X)$$

acting on $f\in\mathrm{Hom}(X^{\otimes n}, X)$ *and* $g_i\in\mathrm{Hom}(X^{\otimes k_i}, X)$ *such that*

$$\gamma(f, g_1,\ldots,g_n) = f(g_1(-),\ldots,g_n(-)) \in \mathrm{Hom}(X^{\otimes(\sum k_i)}, X).$$

Definition 1.8.8. *Let* $\mathsf{O} = \{\mathsf{O}(n)\}_{n\geq 1}$ *and* $\mathsf{P} = \{\mathsf{P}(n)\}_{n\geq 1}$ *be operads. A morphism* $\phi : \mathsf{O}\to\mathsf{P}$ *is a sequence of maps* $\phi(n) : \mathsf{O}(n)\to\mathsf{P}(n)$ *such that the following diagram commutes*

$$\begin{tikzcd}
\mathsf{O}(n)\otimes(\bigotimes_i \mathsf{O}(k_i)) \arrow[r, "\mathrm{id}\otimes(\bigotimes_i \phi(k_i))"] \arrow[d, "\gamma_{\mathsf{O}}"] & \mathsf{P}(n)\otimes(\bigotimes_i \mathsf{P}(k_i)) \arrow[d, "\gamma_{\mathsf{P}}"] \\
\mathsf{O}(\sum k_i) \arrow[r, "\phi(\sum k_i)"] & \mathsf{P}(\sum k_i)
\end{tikzcd}$$

where γ_{O} *is the composition map in* O *and* γ_{P} *is the composition map in* P.

Definition 1.8.9. *Let* O *be an operad in a symmetric monoidal category* $\mathcal{C}$. *An algebra* A *over* O *is an object from* $\mathcal{C}$ *and a morphism of operads* $\theta : \mathsf{O}\to\mathsf{End}_A$.

Definition 1.8.10. *An ideal* I *in an operad* O *is a collection of subobjects* $\mathsf{I}(n)\subset\mathsf{O}(n)$ *such that whenever* $i\in\mathsf{I}$ *then* $\gamma(\ldots,i,\ldots)\in\mathsf{I}$.

Given a family of elements, $(x_i)_{i\in I}$, from an operad O. The smallest ideal in O that contains all the x_i is *the ideal generated by the family* $(x_i)_{i\in I}$.

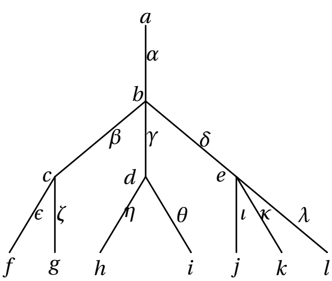
Definition 1.8.11. *The quotient of an operad* O *by an ideal* I *is the operad* $(\mathsf{O}/\mathsf{I})(n) := \mathsf{O}(n)/\mathsf{I}(n)$ *and with the induced composition map from* O.

1.8.2. An alternative definition of operads

Definition 1.8.12. *A rooted tree t is a genus 0 connected graph with one marked vertex called the root of t, $root(t)$. The root must be a vertex with only one edge connected to it. We call all other vertices with only one edge leaves and denote them with the set $leaves(t)$. The vertices with more than one edge are called internal vertices and we denote the set of them with $vert(t)$. Choosing a root in a tree endows the graph with a flow; for every edge at a vertex it is either leading toward or away from the root. Call all edges that lead away from the root at an internal vertex v input edges and denote the set of them with $in(v)$. Denote the set of edges of t with $edges(t)$. Call the edge connecting to the root the stem, $stem(t)$, the edges connecting to the leaves the branches, $branches(t)$ and call the union of these be the legs; $legs(t) = stem(t) \cup branches(t)$*

Example 1.8.13. *Consider the rooted tree t with vertices $a, b, \ldots, l$ and edges $\alpha, \beta, \ldots, \lambda$,*

$t:$



We designate the vertex a to be the root, $root(t) = \{a\}$. The set of internal vertices is given by $vert(t) = \{b, c, d, e\}$ and the set of leaves is given by the set $leaves(t) = \{f, g, h, i, j, k, l\}$. Input edges are as follows,

$$\begin{aligned} in(b) &= \{\beta, \gamma, \delta\} \\ in(c) &= \{\epsilon, \zeta\} \\ in(d) &= \{\eta, \theta\} \\ in(e) &= \{\iota, \kappa, \lambda\}. \end{aligned}$$

The legs are given as follow, $stem(t) = \{\alpha\}$ and $branch(t) = in(c) \cup in(d) \cup in(d) = \{\epsilon, \zeta, \ldots, \lambda\}$

Definition 1.8.14. *Let C be a non-empty set. A C-colored rooted tree is a rooted tree t together with a map $edges(t) \to C$. The set C is called the set of colors of t.*

Definition 1.8.15. *Let X be a finite set. Let $RT_C(X)$ be the set of C-colored rooted trees t with a bijection $leaves(t) \to X$.*

Definition 1.8.16. *Let* FinSet *denote the category of finite sets with bijections. A $\mathcal{C}$-valued S-module is a contravariant functor* O : FinSet $\to \mathcal{C}$, *where* $\mathcal{C}$ *is a symmetric monoidal category. Denote by* $\mathrm{Mod}_S(\mathcal{C})$ *the category of* $\mathcal{C}$*-valued* S*-modules with natural transformations as morphism.*

Definition 1.8.17. *Let* $\mathcal{C}$ *be symmetric monoidal category with finite colimits. Given a non-empty finite set* Y *with a bijection* $f : Y \to \{1,\ldots,n\}$ *and objects* A_y *in* $\mathcal{C}$ *for each* $y \in Y$*. Define the product*

$$\bigotimes_f A_f = A_{f^{-1}(1)} \otimes \ldots \otimes A_{f^{-1}(n)}.$$

There is a natural action of the symmetric group S_n *on this product*

$$\sigma^* : \bigotimes A_f \to \bigotimes A_{\sigma\circ f}$$

We define the unordered product over Y *as*

$$\bigotimes_{y\in Y} A_y = \mathrm{coequalizer}_{\sigma\in\mathbb{S}_n}\left\{\sigma^* : \coprod_{f:Y\cong\{1,\ldots,n\}} \bigotimes_{i=1}^{n} A_f \to \coprod_{f:Y\cong\{1,\ldots,n\}} \bigotimes_{i=1}^{n} A_f\right\}.$$

Let $\mathcal{C}$ be a symmetric monoidal category with small limits and colimits and with the property that the functor $A \otimes_{\mathcal{C}} -$ preserves colimits for any object A. Specifically this implies that $\mathcal{C}$ has an initial object 0. Let O be a S-module with $\mathsf{O}(\{\}) = 0$. We define the treewise tensor product as the unordered product

$$\mathsf{O}(t) := \bigotimes_{v\in vert(t)} \mathsf{O}(in(v)).$$

The treewise tensor product defines a functor $T :: \mathrm{Mod}_S(\mathcal{C}) \to \mathrm{Mod}_S(\mathcal{C})$ given by

$$T(\mathsf{O})(X) = \coprod_{t\in RT_C(X)} \mathsf{O}(t).$$

We have two transformations of functors:

1. $\iota : \mathrm{Id}_{S-\mathrm{mod}} \to T$, given on an S-module O and a set X as the map which takes $\mathsf{O}(X)$ to the coproduct of the treewise tensor product

   $$\iota_{\mathsf{O}}(X) : \mathsf{O}(X) \mapsto \coprod \mathsf{O}(cor)$$

   where cor is the graph with one vertex and $|X|$ branches and where the coproduct is taken over all ways to color the legs of the corolla. Clearly $\mathsf{O}(cor) = \mathsf{O}(X)$, regardless of coloring of cor.

2. $\alpha : T \circ T \to T$, given on an S-module O and a set X as the grafting of trees. The grafting works as follows: Given $t \in RT_C(X)$ and $v \in vert(t)$, then for every tree with an admissible[1] coloring of its legs, $\tau \in RT_C(in(v))$ we define the grafting of τ at the vertex v in t as the new tree t' where the vertex v has been replaced with the tree τ and the leaves of τ connected (according to the bijection $leaves(\tau) \to in(v)$) to the input edges of v. The grafting is performed in all possible ways on all vertices.

Lemma 1.8.18. *The tree functor T is a monad with composition map $\alpha : T \circ T \to T$ and unit $\iota : \mathrm{Id}_{\mathrm{Mod}_S(\mathcal{C})} \to T$.*

Proof. The grafting of trees as defined is naturally associative. Replacing a vertex with a corolla is the same as replacing a corolla with a vertex so the transformation ι is a unit. □

From this general framework the definition of the operad is easy to state.

Definition 1.8.19. *A C-colored operad in the symmetric monoidal category $\mathcal{C}$ is an algebra $(\mathsf{O}, \gamma : T(\mathsf{O}) \to \mathsf{O})$ for the monad (T, α, ι).*

For the most part it's not important to consider operads valued on a general finite set. For an operad O and the finite set $[n] = \{1, 2, \ldots, n\}$ we define $\mathsf{O}(n) := \mathsf{O}([n])$.

Definition 1.8.20. *The free C-colored operad on the S-module E is the algebra $T(E)$. We often denote the free operad by $\mathsf{Free}\langle E \rangle$.*

Definition 1.8.21. *A morphism of operads $f : (\mathsf{O}, \gamma) \to (\mathcal{P}, \nu)$ is map such that the following diagram commutes*

$$\begin{tikzcd} T(\mathsf{O}) \arrow[r, "T(f)"] \arrow[d, "\gamma"] & T(\mathsf{P}) \arrow[d, "\nu"] \\ \mathsf{O} \arrow[r, "f"] & \mathsf{P} \end{tikzcd}$$

Definition 1.8.22. *A differential graded operad or dg operad for short, is an operad in the symmetric monoidal category of differential graded vector spaces.*

Much of the operadic theory is concerning this class of operads. Examples includes the operads of dg associative, dg commutative and dg Lie algebras.

[1] The coloring is admissible if the bijection $leaves(\tau) \to in(v)$ preserves color and the non-input edge at v has the same color as the stem of τ.

Proposition 1.8.23. *Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor and let O be an operad in the symmetric monoidal category $\mathcal{C}$. The object $F(\mathsf{O})$ is an operad in the category $\mathcal{D}$.*

This important result has as a consequence that there is an associated operad of chains and homology coming from an operad of topological spaces.

Definition 1.8.24. *A C-colored operad is said to be of transformation type if the color set C is a union of two sets C_{in} and C_{out} such that input colors are always from the set C_{in} and the output color is always from the set C_{out}.*

1.8.3. Tensor products and shifts of algebraic operads Let the ambient symmetric monoidal category be that of dg vector spaces over $\mathbb{K}$. For this category a Σ-module, O, is exactly a series of differential graded vector spaces $\{\mathsf{O}(n)\}_{n\geq 1}$ such that $\mathsf{O}(n)$ has the structure of a module over the group ring $\mathbb{K}[\mathbb{S}_n]$. For this reason we shall refer to Σ-modules in this category as $\mathbb{S}$-modules.

Given operads O and P we can form the tensor product $\mathsf{O}\otimes\mathsf{P}$ defined by

$$(\mathsf{O}\otimes\mathsf{P})(n) := \mathsf{O}(n)\otimes\mathsf{P}(n).$$

One can show that $\mathsf{O}\otimes\mathsf{P}$ inherits the structure of an operad.

Consider the $\mathbb{S}$-module Λ

$$\Lambda(n) = \begin{cases} s^{1-n}\,\mathrm{sgn}_n & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases} \tag{1.3}$$

with sgn_n being the sign representation of $\mathbb{S}_n$. Let

$$\circ_i \;:\; \Lambda(n)\otimes\Lambda(k) \to \Lambda(n+k-1)$$

be the partial composition operations defined by

$$1_n \circ_i 1_k = (-1)^{(1-k)(n-i)} 1_{n+k-1}\,, \tag{1.4}$$

where 1_m denotes the generator $s^{1-m}1 \in s^{1-m}\,\mathrm{sgn}_m$. The obvious unit map $\iota = \mathrm{id} : \mathbb{K} \to \Lambda(1) \cong \mathbb{K}$ will equip the $\mathbb{S}$-module Λ with the structure of an operad. It's clear that representations of Λ in a vector space V are in bijection with representations of Com in the shifted space $V[1]$.

For an operad O we denote by $\mathsf{O}\{k\}$ the operad

$$\mathsf{O}\{k\} := \underbrace{\Lambda\otimes\ldots\otimes\Lambda}_{k}\otimes\mathsf{O}\,. \tag{1.5}$$

The representation of $\mathsf{O}\{k\}$ in dg space V are in bijection with representations of O in the shifted space $V[k]$.

Example 1.8.25. *Let* Lie *be the operad of Lie algebras. A representation of* Lie{1} *in a graded vector space V is equivalent to a binary operation:*

$$\{,\}: V \otimes V \to V$$

of degree -1 *satisfying the identities:*

$$\{v_1, v_2\} = (-1)^{|v_1||v_2|}\{v_2, v_1\},$$

$$\{\{v_1, v_2\}, v_3\} + (-1)^{|v_1|(|v_2|+|v_3|)}\{\{v_2, v_3\}, v_1\} + (-1)^{|v_3|(|v_1|+|v_2|)}\{\{v_3, v_1\}, v_2\} = 0\,.$$

1.9 Props and other generalizations of operads

For the theory of properads and props we follow B. Vallete [Va1; Va2].

1.9.1. Props and properads

Modules

We include here a short summary of the monoidal categories of $\mathbb{S}$-bimodules and their construction for vector spaces over the field $\mathbb{K}$. There is a natural generalization to the differential graded framework.

Definition 1.9.1. *An $\mathbb{S}$-bimodule* Q *is a collection of $\mathbb{K}$-modules* $\{\mathsf{Q}(m,n)\}_{m,n\geq 0}$ *such that each module* $\mathsf{Q}(m,n)$ *has an action of* $\mathbb{S}_m$ *on the left and an action of* $\mathbb{S}_n$ *on the right. The two actions are compatible in the sense that if* $F \in \mathsf{Q}(m,n)$ *and* $\sigma \in \mathbb{S}_m$, $\nu \in \mathbb{S}_n$ *then*

$$(\sigma(F))\nu = \sigma((F)\nu)).$$

Definition 1.9.2. *A morphisms of $\mathbb{S}$-bimodules* $f : \mathsf{Q} \longrightarrow \mathsf{P}$*, is a collection of morphisms* $f_{m,n} : \mathsf{Q}(m,n) \longrightarrow \mathsf{P}(m,n)$ *that are compatible with the action of* $\mathbb{S}_m$ *on the left and the action of* $\mathbb{S}_n$ *on the right.*

Remark 1.9.3. *The objects $\mathbb{S}$-bimodules with $\mathbb{S}$-bimodule morphism form the category* $\mathbb{S}-$biMod.

Definition 1.9.4. *Let* $\mathfrak{G}$ *be be set of non-planar directed graphs without loops or wheels. To each vertex the set of input/output edges are labeled by integers* $\{1, 2, \ldots, n\}$*. We allow for edges to only connect to a vertex at one end. Such edges are essentially input (or output depending on direction) edges for the graph as a whole. We will assume graphs of this type are drawn so that the directed edges always point downwards.*

Definition 1.9.5. *We say that an element of $\mathfrak{G}$ is a 2-level graph if the vertices can be distributed on two levels. We denote the set of 2-level graphs by $\mathfrak{G}^2$. For a 2-level graph we denote the vertices on the i-th by $\mathfrak{L}_i$*

Let $In(v)$ and $Out(v)$, respectively, denote the set of input and output edges of a vertex v.

Definition 1.9.6. *Composition product $\boxtimes$ for $\mathbb{S}$-bimodules. Let Q and P be $\mathbb{S}$-bimodules. We the define the composition product $\mathsf{Q} \boxtimes \mathsf{P}$ as the following $\mathbb{S}$-bimodule*

$$\mathsf{Q} \boxtimes \mathsf{P}(m,n) = \left(\bigoplus_{g \in \mathfrak{G}^2} \bigotimes_{v_2 \in \mathfrak{L}_2} \mathsf{Q}(|Out(v_2)|, |In(v_2)|) \otimes_{\mathbb{K}} \bigotimes_{v_1 \in \mathfrak{L}_1} \mathsf{P}(|Out(v_1)|, |In(v_1)|) \right) / \approx,$$

with the relation $\approx$ meaning that the action of symmetric groups on the input/output edges of vertex should be compatible with the action on the $\mathbb{S}$-bimodules.

Definition 1.9.7. *Concatenation product $\otimes$ for $\mathbb{S}$-bimodules. Let Q and P be $\mathbb{S}$-bimodules. We the define the concatenation product $\mathcal{Q} \otimes \mathcal{P}$ as the following $\mathbb{S}$-bimodule*

$$\mathsf{Q} \otimes \mathsf{P}(m,n) = \bigoplus_{\substack{m'+m''=m \\ n'+n''=n}} \mathbb{K}[\mathbb{S}_{m'+m''}] \otimes_{\mathbb{S}_{m'} \times \mathbb{S}_{m''}} \mathsf{Q}(m',n') \otimes_{\mathbb{K}} \mathsf{P}(m'',n'') \otimes_{\mathbb{S}_{n'} \times \mathbb{S}_{n''}} \mathbb{K}[\mathbb{S}_{n'+n''}]$$

Proposition 1.9.8. *The category $\mathbb{S}-$biMod with the concatenation product $\otimes$ and the unit object I;*

$$\mathsf{I}(m,n) = \begin{cases} \mathbb{K} & m = n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

is a monoidal category $(\mathbb{S}-\text{biMod}, \otimes, \mathsf{I})$.

Definition 1.9.9. *We say that a graph is connected if it's geometric realization is a connected as a topological space. We denote the set of connected graphs with $\mathfrak{G}_c$.*

Definition 1.9.10. *Connected composition product $\boxtimes_c$ for $\mathbb{S}$-bimodules. Let Q and P be $\mathbb{S}$-bimodules. We the define the connected composition product $\mathsf{Q} \boxtimes_c \mathsf{P}$ as the following $\mathbb{S}$-bimodule*

$$\mathsf{Q} \boxtimes_c \mathsf{P}(m,n) = \left(\bigoplus_{g \in \mathfrak{G}^2_c} \bigotimes_{v_2 \in \mathfrak{L}_2} \mathsf{Q}(|Out(v_2)|, |In(v_2)|) \otimes_{\mathbb{K}} \bigotimes_{v_1 \in \mathfrak{L}_1} \mathsf{P}(|Out(v_1)|, |In(v_1)|) \right) \Big/ \approx$$

Proposition 1.9.11. *The category $\mathbb{S}-$biMod with the connected composition product $\boxtimes_c$ and the unit object I;*

$$\mathsf{I}(m,n) = \begin{cases} \mathbb{K} & m = n = 1 \\ 0 & \textit{otherwise} \end{cases}$$

is a monoidal category $(\mathbb{S}-\text{biMod}, \boxtimes_c, \mathsf{I})$.

Definition 1.9.12. *The $\mathbb{S}$-bimodules $T_\otimes(\mathsf{P})$ and $\mathcal{S}(\mathsf{P})$. We denote the free monoid of P with respect to the monoidal product $\otimes$ as $T_\otimes(\mathsf{P})$. Explicitly, it's given by*

$$T_\otimes(\mathsf{P}) := \bigoplus_{n\geq 0} \mathsf{P}^{\otimes n}.$$

As the monoidal product $\otimes$ is symmetric we also define the truncated symmetric algebra $\mathcal{S}(\mathsf{P})$;

$$\mathcal{S}(\mathsf{P}) := \bigoplus_{n\geq 1} \left(\mathsf{P}^{\otimes n}\right)_{\mathbb{S}_n}.$$

The functorial assignment given by $\mathcal{S}$ express the relationship between $\boxtimes_c$ and $\boxtimes$.

Proposition 1.9.13. *For $\mathbb{S}$-bimodules P and Q the following relation holds*

$$\mathcal{S}(\mathsf{P} \boxtimes_c \mathsf{Q}) = \mathsf{P} \boxtimes \mathsf{Q}$$

Definition 1.9.14. *We say that an $\mathbb{S}$-bimodules Z is saturated if $\mathcal{S}(\mathsf{Z}) = \mathsf{Z}$.*

Example 1.9.15. *For any $\mathbb{S}$-bimodule Q we have that $\mathcal{S}(\mathsf{Q})$ is saturated.*

Let sat$-\mathbb{S}-$biMod denote the category of saturated $\mathbb{S}$-bimodules.

Remark 1.9.16. *Let I, as above, be defined as*

$$\mathsf{I}(m,n) = \begin{cases} \mathbb{K} & m = n = 1 \\ 0 & \textit{otherwise} \end{cases}$$

then $\mathcal{S}(\mathsf{I})$ is given explicitly as

$$\mathcal{S}(\mathsf{I}) = \begin{cases} 0 & m \neq n \\ \mathbb{K}[\mathbb{S}_m] & \textit{otherwise} \end{cases}$$

Proposition 1.9.17. *The category* sat$-\mathbb{S}-$biMod *with the composition product $\boxtimes$ and the unit object $\mathcal{S}(\mathsf{I})$ is a monoidal category* (sat$-\mathbb{S}-$biMod$, \boxtimes, \mathcal{S}(\mathsf{I}))$.

Props and properads

With the appropriate framework of monoidal categories developed we can state the definition of props and properads.

Definition 1.9.18. *A prop is a monoid (P, μ, ν) in the monoidal category* (sat$-\mathbb{S}-$biMod$, \boxtimes, \mathcal{S}(\mathsf{I}))$. *Which is equivalent to the following:*

- *The $\mathbb{S}$-bimodule P is closed under concatenation $\mathsf{P} \otimes \mathsf{P} \hookrightarrow \mathsf{P}$.*

- *The composition* $\mathsf{P}\boxtimes\mathsf{P}\xrightarrow{\mu}\mathsf{P}$ *is associative.*
- *The morphism* $\mathcal{S}(\mathsf{I})\xrightarrow{\nu}\mathsf{P}$ *is a unit.*

Properads are the specialization of props given by restricting to only consider connected compositions.

Definition 1.9.19. *A properad is a monoid* (P,μ,ν) *in the monoidal category* $(\mathbb{S}-\text{biMod},\boxtimes_c,\mathsf{I})$. *Which is equivalent to the following:*

- *The composition* $\mathsf{P}\boxtimes_c\mathsf{P}\xrightarrow{\mu}\mathsf{P}$ *is associative.*
- *The morphism* $\mathsf{I}\xrightarrow{\nu}\mathsf{P}$ *is a unit.*

Example 1.9.20. *To every vector space* V *we associate a canonical prop(erad); the endomorphism prop(erad)* End_V. *We make no symbolic distinction between the prop and properad case.*

- *The underlying* $\mathbb{S}$*-bimodule of the endomorphism prop is given as follows*
$$\mathsf{End}_V(n,m) := \text{Hom}(V^{\otimes m},V^{\otimes n})$$
- *The associative product* $\mathsf{End}_V\boxtimes\mathsf{End}_V\longrightarrow\mathsf{End}_V$ *is given by composition of functions.*
- *The unit morphism* $\mathcal{S}(\mathsf{I})\hookrightarrow\mathsf{End}_V$ *sends a permutation* $\sigma\in\Bbbk[\mathbb{S}_n]\in\mathcal{S}(\mathsf{I})(n,n)$ *to the map* $f_\sigma : V^{\otimes n}\to V^{\otimes n}$ *permutating the variables according to* σ .

If we restrict to connected compositions $\mathsf{End}_V\boxtimes_c\mathsf{End}_V\longrightarrow\mathsf{End}_V$ *we instead get the endomorphism properad.*

Definition 1.9.21. *Let* P *and* Q *be props . A morphism* $f:\mathsf{P}\to\mathsf{Q}$ *of* $\mathbb{S}$*-bimodules is a morphism of props if the following diagram commutes:*

$$\begin{tikzcd}
\mathsf{P}\boxtimes\mathsf{P} \arrow[r, "f\boxtimes f"] \arrow[d, "\mu_{\mathsf{P}}"] & \mathsf{Q}\boxtimes\mathsf{Q} \arrow[d, "\mu_{\mathsf{Q}}"] \\
\mathsf{O} \arrow[r, "f"] & \mathsf{P}
\end{tikzcd}$$

The definition of a morphism of properads is acquired if one, in the above definition, replaces the composition $\boxtimes$ with the connected composition $\boxtimes_c$.

Definition 1.9.22. *Let* V *be an* $\mathbb{S}$*-bimodule. The free properad on* V, *denoted by* $\mathsf{F}(V)$, *is defined as follows*

$$\mathsf{F}(V) := \left(\bigoplus_{g\in\mathfrak{G}_c}\bigotimes_{v\in\mathfrak{L}} V(|Out(v)|,|In(v)|)\right)\Big/\approx$$

Algebras over props and properads

We define the notion of an algebra over a properad and prop in complete analogy to how it's defined for operads.

Definition 1.9.23. *Let* P *be prop (or a properad). The structure of a* P*-algebra on the vector space* V *is a morphism of props*

$$\Xi : \mathsf{P} \longrightarrow \mathsf{End}_V$$

Example 1.9.24. *Many classical and well-known algebraic structures can be recovered as the algebra over some prop(erad). Examples include (co)associative algebras, (co)Lie algebras, bialgebras and Lie bialgebras.*

Differential graded analogue

One can mimic the above constructions for the category of differential graded $\mathbb{S}$-bimodules to generalize the framework to that of differential graded properads and differential graded props.

2. Configuration space model for A_∞ homotopies

2.1 Introduction

One of the first and most important operads is the topological Ass_∞ operad. It was introduced by J.D. Stasheff [St] with the help of the following motivating example.

2.1.1. A motivating topological example An H-space is a topological space H with a multiplication map

$$\mu : H \times H \to H$$

and a point $e \in H$ such that $\mu(-,e)$ and $\mu(e,-)$ are homotopic; μ has unit up to homotopy. We say that an H-space is topological monoid if the multiplication map is associative;

$$\mu \circ (\mathrm{Id} \times \mu) = \mu \circ (\mu \times \mathrm{Id}),$$

and the maps $\mu(-,e)$ and $\mu(e,-)$ are equal.

The space of maps $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = b$ where b is the base point of X is called the based loop space of X. Denote the (based) loop space on X with ΩX. We can define a multiplication on ΩX as the concatenation of loops;

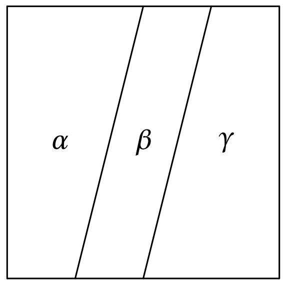
$$m_2 : \Omega X \times \Omega X \to \Omega X$$
$$(\gamma_1(t), \gamma_2(t)) \mapsto m_2(\gamma_1, \gamma_2)(t) := \begin{cases} \gamma_1(2t) & t \in \left[0, \frac{1}{2}\right] \\ \gamma_2(2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

The loop space ΩX with concatenation as multiplication and the constant map as unit is an H-space but not a topological monoid; the unit axiom is satisfied up to homotopy but the multiplication map fails to be associative. While multiplication in ΩX is non-associative, at least there is a homotopy between $m_2 \circ (\mathrm{Id} \times m_2)$ and $m_2 \circ (m_2 \times \mathrm{Id})$;

$$m_3 : [0,1] \times (\Omega X)^3 \to \Omega X.$$

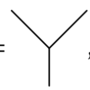
The homotopy can be explicitly defined in coordinates, but instead, we describe it with a picture as follows

$m_2(\alpha, m_2(\beta,\gamma))$

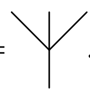


$m_2(m_2(\alpha,\beta),\gamma)$

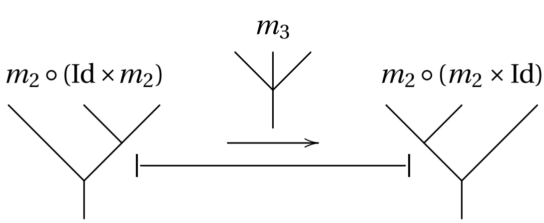
Where α, β and γ are appropriately variable-substituted so that on every horizontal line, each loop takes its full course before the next one starts. We represent m_2 with a corolla

$m_2 =$  ,

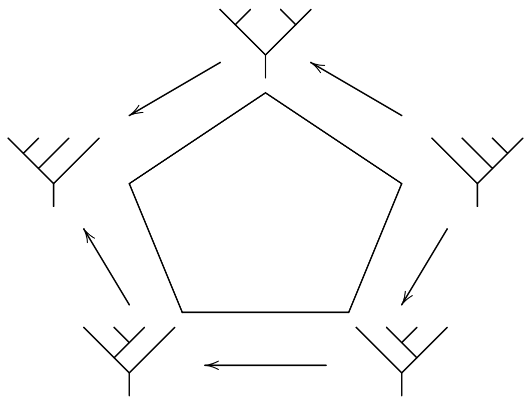
the two top edges are the inputs and the lower one is the output. Analogously, m_3 can be thought of as a map with three inputs and one output for every fixed $t \in [0,1]$;

$m_3 =$  .

It's natural to associate the interval to m_3 due to the parameter dependence. The following schematic picture of this would look like



Choosing to orientate the interval in the positive direction we see that $\partial m_3 := m_3(1) - m_3(0) = m_2(m_2 \times \mathrm{Id}) - m_2(\mathrm{Id} \times m_2)$. Along the same lines; four loops can be composed in five ways, and there are homotopies connecting them. The following picture describes the situation



The appearing pentagon is called $\mathcal{K}^4$. We define a homotopy

$$m_4 : \mathcal{K}^4 \times (\Omega X)^4 \to \Omega X,$$

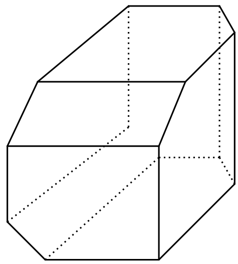
such that on the edges of $\mathcal{K}^4$ the map m_4 acts as the map corresponding to trees built from m_2 and m_3. On the corners m_4 act as trees built from m_2 alone. We orient $\mathcal{K}^4$ in the positive direction and we have that the induced orientation on the boundary will give signs to the formula for how m_4 acts on the boundary of $\mathcal{K}^4$. If the arrow along the edge points in the same direction as the orientation we have a plus sign, otherwise a minus sign. Explicitly

$$\begin{aligned}
\partial m_4 &= m_4|_{\partial(\mathcal{K}^4)} \\
&= \sum_{s \in \text{segments of } \partial\mathcal{K}^4} \pm m_4|_s \\
&= m_3 \circ (\mathrm{Id} \times \mathrm{Id} \times m_2) + m_3 \circ (m_2 \times \mathrm{Id} \times \mathrm{Id}) - \\
&\quad m_2 \circ (\mathrm{Id} \times m_3) - m_3 \circ (\mathrm{Id} \times m_2 \times \mathrm{Id}) - m_2 \times (m_3 \times \mathrm{Id}).
\end{aligned}$$

For five loops there are 14 binary trees built from m_2, one for each way to concatenate five loops. Letting each binary tree represent a vertex, adding an edge when there is a homotopy between two vertices and adding a face when two edges are homotopic we get polytope $\mathcal{K}^5$ describing a general homotopy

$$m_5 : \mathcal{K}^5 \times (\Omega X)^5 \to \Omega X.$$

The polytope $\mathcal{K}^5$ has two quadrilateral faces and six pentagonal faces. It is given a geometric representation in the following figure.



In general, for each $n \geq 2$ there is an $n-2$ dimensional polytope $\mathcal{K}^n$ where the vertices are labeled by binary rooted trees with n leaves. The spaces $\mathcal{K}^n$ are called *associahedrea* or *Stasheff's associahedrea.* For each associahedron $\mathcal{K}^n$, there exist a map $m_n : \mathcal{K}^n \times (\Omega X)^n \to \Omega X$. The map m_n is controlling the homotopies between the different ways to multiply together n loops, either by concatenation or by their induced higher multiplications. Note that $\mathcal{K}^2 = \{*\}$ is just a point and that $\mathcal{K}^3 = [0,1]$ is the unit interval.

A topological space X equipped with a series of coherent maps $m_n : \mathcal{K}^n \times X^n \to X$ subject to the relations which one reads of the loop space in the above manner is called an A_∞ space. The notion of an A_∞ space was first introduced in the thesis by Stasheff [St],which was devoted to the study of H-spaces. It explored the idea that a strictly associative multiplication can be weakened to a sequence of higher multiplications wherein the condition of associativity has been relaxed in a certain sense. From this perspective a topological monoid can be regarded as a special case in a larger class of spaces, the A_∞-spaces, where the associativity condition is demanded to hold up to coherent homotopy of higher multiplications.

It was proved that all A_∞ spaces X are of the same weak homotopy type as the loop space ΩY, for some space Y.

In fact the set of Stasheff's associahedra $\{\mathcal{K}^n\}$ give us an example of a non-symmetric topological operad.

2.1.2. Strongly homotopy algebras The algebraic analogue of A_∞ spaces are called A_∞ algebras and the examples include the singular chains on an A_∞ space. Morally one can say that A_∞ algebras are to associative algebras as A_∞ spaces are to topological monoids.

Having defined some algebraic structure it is natural to ask what the correct notion of morphism between these algebraic structures should be. In the case of associative algebras (A, m_A) and (B, m_B), where

$$m_A : A \otimes A \to A$$

and

$$m_B : B \otimes B \to B$$

are multiplication maps, we know that the correct notion of morphism is a linear map

$$f : A \to B,$$

such that

$$m_B \circ (f \otimes f) = f \circ m_A.$$

One can define a *2-colored* operad encoding the structure of an associative algebra morphism between a pair of associative algebras. We denote this operad by $\mathsf{Mor}(\mathsf{Ass})$. Due to the work of Van der Laan [VdL] there is a generalized theory of Koszul duality for colored operads that can be used to give models of operads like $\mathsf{Mor}(\mathsf{Ass})$. It was proved by M. Markl [Mar4] that the minimal model of $\mathsf{Mor}(\mathsf{Ass})$ is a 2-colored operad encoding a pair of A_∞ algebras and a series of maps between them, we denote this operad with $\mathsf{Mor}(\mathsf{Ass})_\infty$. The encoded maps are, not surprisingly, the correct notion of morphism between A_∞ algebras. Similarly one can describe a 2-colored operad describing the structure of a L_∞ morphism between two L_∞ algebras.

The structure of strongly homotopy algebras have also been exhibited in (closed) string field theory, see for example Zwiebach [Z] and Markl [Mar3].

2.2 Operads and A_∞ algebras

Definition 2.2.1. *The operad* Ass *is defined as the quotient*

$$\mathsf{Ass} = \mathsf{Free}\langle E\rangle / R$$

where the $\mathbb{S}$*-module* E *is given by*

$$E(n) = \begin{cases} \mathbb{K}[\mathbb{S}_2] & n = 2 \\ 0 & n \neq 2 \end{cases}$$

and the ideal R *is generated by a quadratic relation;* $R = \operatorname{span}\langle (\mathrm{id} \otimes \sigma) \circ \sigma - (\sigma \otimes \mathrm{id}) \circ \sigma\rangle_{\sigma \in \mathbb{S}_2} \subseteq \mathsf{Free}\langle E\rangle^{(\geq 2)}$.

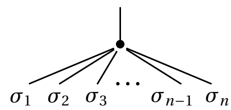
The operad Ass gives an operadic description of associative algebras; a dg associative algebra V is a map of operads $\mathsf{Ass} \longrightarrow \mathsf{End}_V$.

The operad Ass has a minimal model, which we call Ass_∞.

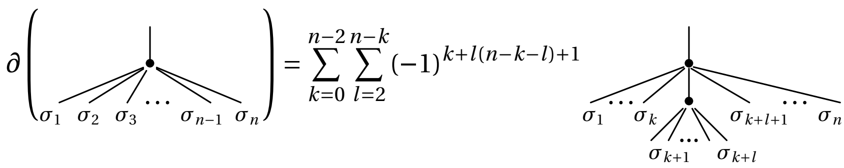
Definition 2.2.2. *The dg operad* $(\mathsf{Ass}_\infty, \partial)$ *is quasi-free on the* $\mathbb{S}$*-module* E *given by*

$$E(n) = \begin{cases} \mathbb{K}[\mathbb{S}_n][2-n] & n \geq 2 \\ 0 & n < 2 \end{cases}$$

Represent the generator $\sigma \in E(n)$ by the corolla



The differential is defined as



Definition 2.2.3. *The 2-colored operad* $\mathsf{Mor(Ass)}$ *is defined as the quotient*

$$\mathsf{Mor(Ass)} = \mathsf{Free}\langle E\rangle / R$$

where $E = E_0 \oplus E_1 \oplus E_f$ *with* $E_0 = E_1 = \mathbb{K}[\mathbb{S}_2]$ *and*

$$E_f(n) = \begin{cases} \text{span}\langle f\rangle & n = 1 \\ 0 & n \neq 1 \end{cases}$$

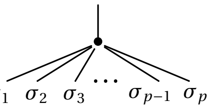
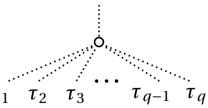
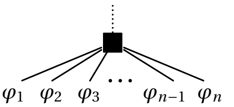
is the one-dimensional space spanned by the indeterminate f. The elements of E_i are monochrome with color i and the elements of E_f are two-colored, having input color 0 *and output color* 1*. The relations in this operad are given by $R = R_0 \oplus R_1 \oplus R_f$ where the spaces R_0 and R_1 correspond to the associative relation for the elements of E_0 and E_1, respectively;* $R_0 = \text{span}\langle(\text{id} \otimes \sigma_0) \circ \sigma_0 - (\sigma_0 \otimes \text{id}) \circ \sigma_0\rangle_{\sigma_0 \in \mathbb{S}_2}$ *and* $R_1 = \text{span}\langle(\text{id} \otimes \sigma_1) \circ \sigma_1 - (\sigma_1 \otimes \text{id}) \circ \sigma_1\rangle_{\sigma_1 \in \mathbb{S}_2}$*. Lastly, the space R_f is given as the space* $\text{span}\langle(f \otimes f) \circ \sigma_1 - \sigma_0 \circ f\rangle_{\sigma_0, \sigma_1 \in \mathbb{S}_2}$.

The operad $\mathsf{Mor(Ass)}$ gives an operadic description of dg (associative) algebra morphisms. The operad $\mathcal{M}or(As)$ has a minimal model, which we call $\mathsf{Mor(Ass)}_\infty$.

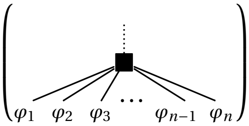
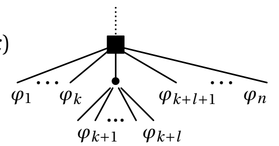
Definition 2.2.4. *The 2-colored dg operad* $(\mathsf{Mor(Ass)}_\infty, \partial)$ *is quasi-free on the $\mathbb{S}$-module $E = E_1 \oplus E_2 \oplus E_f$, where E_i is monochrome with color i and E_f is dichromatic with input color* 0 *and output color* 1*. The $\mathbb{S}$-modules E_1 and E_2 are copies (if we disregard the colors) of E appearing in Definition 2.2.2. The $\mathbb{S}$-modules E_f is defined as*

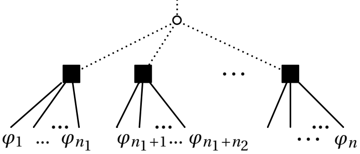
$$E_f(n) = \begin{cases} \mathbb{K}[\mathbb{S}_n][1-n] & n \geq 1 \\ 0 & n = 0 \end{cases}$$

Represent generators $\sigma \in E_0(p)$ and $\tau \in E_1(q)$ by monochrome corollas, either black edges or dashed edges and represent elements $\varphi \in E_f(n)$ by dichromatic corollas;

$\sigma \sim$  $\tau \sim$  $\varphi \sim$ 

The action of the differential ∂ on corollas σ and τ is identical to the one described in Definition 2.2.2. On the corolla representing φ the differential acts as follows:

$\partial\Big($  $\Big) = \sum_{l=2}^{n}\sum_{k=0}^{n-l} \epsilon(l,k)$ 

$+\sum_{k=2}^{n}\sum_{n=n_1+\ldots+n_k} \epsilon(k;n_1,\ldots,n_k)$ 

Where $\epsilon(k,l) = (-1)^{(k-1)(l-1)+n}$ and

$$\epsilon(k;n_1,\ldots,n_k) = (-1)^{(k-1)(n_1-1)+(k-2)(n_2-1)+\ldots+2(n_{k-2}-1)+n_{k-1}-1}$$

Definition 2.2.5. *The 2-colored dg operad* $(\mathsf{Ho}(\mathsf{Ass}), d)$ *is defined as the quotient*

$$\mathsf{Free}\langle E\rangle / R.$$

The free construction is taken on the space $E = E_0 \oplus E_1 \oplus E_f \oplus E_g \oplus E_h$ and the relations are given by the direct sum $R = R_0 \oplus R_1 \oplus R_f \oplus R_g \oplus R_h$. The spaces E_0, E_1, E_f are just like above, E_g is like E_f but spanned by the indeterminate g. The last generator E_h is defined as

$$E_h(n) = \begin{cases} \text{span}\langle h\rangle[-1] & n = 1 \\ 0 & n \neq 1 \end{cases},$$

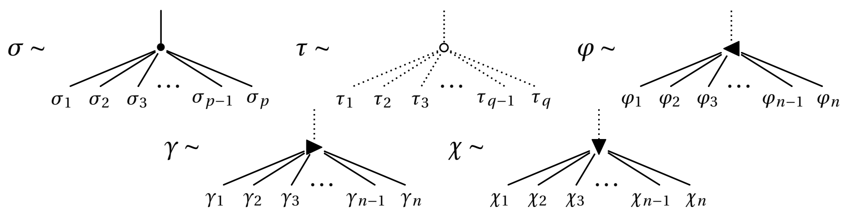
the one-dimensional space spanned by the indeterminate h and concentrated in degree 1. *The relation spaces R_0, R_1 and R_f are identical to the proceeding definition, and the space R_g is defined just like R_f with g in place of f. Lastly, the space R_h is given by the space* $\text{span}\langle (f\otimes h)\circ\sigma_1 + (h\otimes g)\circ\sigma_1 - \sigma_0\circ h\rangle_{\sigma_0,\sigma_1\in\mathbb{S}_2}$. *The differential d is defined to be trivial on all generators except h, for which it has the action $d(h) = f - g$.*

The operad $\mathrm{Ho(Ass)}$ gives an operadic description of a homotopy of dg algebra morphisms. There is a non-minimal model for $\mathrm{Ho(Ass)}$, which we call $\mathrm{Ho(Ass)}_\infty$.

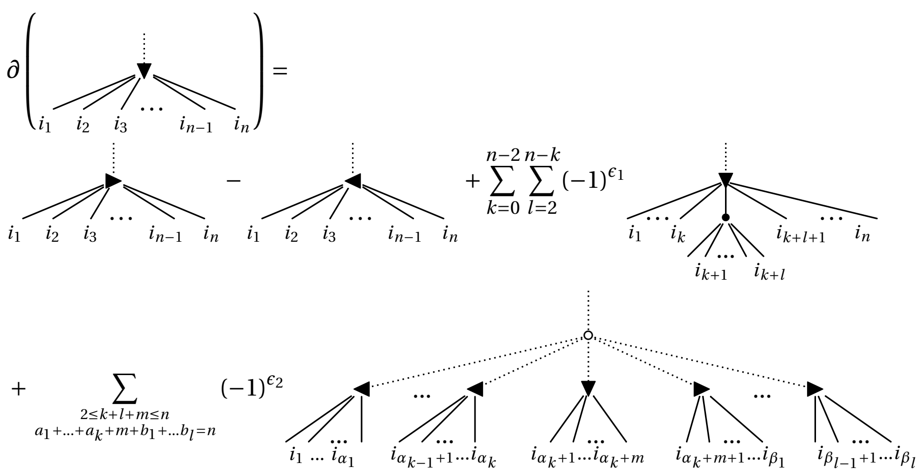
Definition 2.2.6. *The 2-colored dg operad* $(\mathrm{Ho(Ass)}_\infty, \partial)$ *is quasi-free on the* $\mathbb{S}$*-module* $E = E_0 \oplus E_1 \oplus E_f \oplus E_g \oplus E_h$*. The summands* E_0, E_1, E_f *are defined as in Definition 2.2.4,* E_g *is a copy of* E_f *and* E_h *is dichromatic with input color* 0 *and output color* 1 *and defined as*

$$E_h(n) = \begin{cases} \mathbb{K}[\mathbb{S}_n][-n] & n \geq 1 \\ 0 & n = 0 \end{cases}$$

Represent generators $\sigma \in E_0(p)$ *and* $\tau \in E_1(q)$ *by monochrome corollas, either black edges or dashed edges and represent elements* $\varphi \in E_f(n)$, $\gamma \in E_g(m)$ *and* $\chi \in E_h(l)$ *by dichromatic corollas;*



The action of the differential on the generators coming from E_0, E_1, E_f *and* E_g *is just as in Definition 2.2.2-2.2.4. The differential acts on the generator* $\chi \in E_h$ *as*



Where $1 \leq m \leq n-1$, $\alpha_i = a_1 + \ldots + a_i$ *and* $\beta_j = \alpha_k + m + b_1 + \ldots b_j$*. The factors* ϵ_1 *and* ϵ_2 *are signs;*

$$\epsilon_1 = (k-1)(l-1) + n + 1$$

$$\epsilon_2 = l + \sum_{1 \leq i \leq l} (1 - b_i) \left(n - \sum_{j \geq i} b_j \right) + m \sum_{1 \leq i \leq k} a_i + \sum_{2 \leq i \leq k} (1 - a_i) \left(\sum_{j < i} a_j \right)$$

From cofree coalgebras to A_∞ algebras

We state here the relation between two common non-operadic definition of strongly homotopy associative structures (algebras, morphisms and homotopies).

Definition 2.2.7. *Let V be a graded vector space. The tensor coalgebra T_cV is as a vector space the direct sum $\oplus_{k\geq 0} V^{\otimes i}$, where $V^{\otimes i}$ is the i-times iterated tensor product with itself,*

$$V^{\otimes i} = \underbrace{V \otimes \ldots \otimes V}_{i\text{-times}}.$$

T_cV can be given a coalgebra structure with the coproduct map

$$\Delta : T_cV \to T_cV \otimes T_cV$$

given on summand $T_c^n V = V^{\otimes n}$ as

$$\Delta : (v_1, \ldots, v_n) \to \sum_{i=0}^{n} (v_1, \ldots, v_i) \otimes (v_{i+}, \ldots, v_n),$$

where the term for $i = 0, n$ are $1 \otimes (v_1, \ldots, v_n)$ and $(v_1, \ldots, v_n) \otimes 1$ inside $V^{\otimes 0} \otimes V^{\otimes n}$ and $V^{\otimes n} \otimes V^{\otimes 0}$, respectively.

The reduced tensor coalgebra $\overline{T}_cV$ is as a vector space the direct sum $\oplus_{i\geq 1} V^{\otimes i}$, with coproduct

$$\Delta : \overline{T}_cV \to \overline{T}_cV \otimes \overline{T}_cV$$

given, as above, on summands as

$$\Delta : (v_1, \ldots, v_n) \to \sum_{i=1}^{n-1} (v_1 \ldots v_i) \otimes (v_{i+1}, \ldots, v_n).$$

Remark 2.2.8. *From the coproduct we define the partial coproducts:*

$$\Delta_{a+b}^{a,b} := V^{\otimes(a+b)} \hookrightarrow \overline{T}_cV \xrightarrow{\Delta} \overline{T}_cV \bigotimes \overline{T}_cV \to V^{\otimes a} \bigotimes V^{\otimes b}.$$

This will be a convenient short hand in many of the proofs of this section.

Proposition 2.2.9. *A map of vector spaces $b : \overline{T}_cV \to V$ can be lifted to a unique coderivation of coalgebras,*

$$B : \overline{T}_cV \to \overline{T}_cV,$$

such that $pr_1 \circ B = b$, when pr_1 is the natural projection $\overline{T}_cV \to V$. If B_n^m denotes the composition

$$B_n^m : V^{\otimes n} \hookrightarrow \overline{T}_cV \xrightarrow{B} \overline{T}_cV \to V^{\otimes m},$$

then the explicit formula for B_n^m is given by

$$B_n^m = \begin{cases} 0 \text{ if } n < m \\ \sum_{i+j=m-1} \mathrm{Id}^{\otimes i} \otimes b_{n+1-m} \otimes \mathrm{Id}^{\otimes j} \end{cases}$$

where $b_a := b|_{V^{\otimes a}}$. Furthermore, the map B is recovered as a product;

$$B = \prod_{n\geq 1} B^n \qquad B^n = \prod_{m\geq 1} B_m^n,$$

and note that $B_a^1 = b_a$

Proof. The proof is by induction. The case B_n^1 is clear from the projection property. Assume that for all $m < M$ we have that B_n^m is given by the formula. The equation

$$\Delta B = (\mathrm{Id} \bigotimes B + B \bigotimes \mathrm{Id})\Delta$$

is true. Specifically we can restrict its input to be $V^{\otimes n}$ and its output to be in $V^{\otimes (M-1)} \bigotimes V$, in which case the formula becomes:

$$\Delta_M^{M-1,1} B_m^M = (\mathrm{Id}^{\otimes (M-1)} \bigotimes B_{n+1-M}^1 + B_{n-1}^{M-1} \bigotimes \mathrm{Id})\Delta_n^{n-1,1}.$$

By the induction hypothesis we know that $B_{n-1}^{M-1} = \sum_{i+j=M-2} \mathrm{Id}^{\otimes i} \otimes b_{n+1-M} \otimes \mathrm{Id}^{\otimes j}$ so, as an M-fold tensor, the right hand side has the desired form; it's the formula given for B_n^M. □

Proposition 2.2.10. *A map of vector spaces $f : \overline{T}_c V \to W$ can uniquely be lifted to a morphism of coalgebras $F : \overline{T}_c V \to \overline{T}_c W$, such that $pr_1 \circ F = f$, when pr_1 is the natural projection $\overline{T}_c W \to W$. If we let F_n^m be a composition;*

$$F_n^m : V^{\otimes n} \hookrightarrow \overline{T}_c V \xrightarrow{F} \overline{T}_c W \to W^{\otimes m}$$

and let $f_k = f|_{V^{\otimes k}}$ then, explicitly, F_n^m will be of the form

$$F_n^m = \begin{cases} 0 \text{ if } n < m \\ \sum_{i_1+\cdots+i_m=n} f_{i_1} \otimes \ldots \otimes f_{i_m} \end{cases}$$

where F can be recovered as the product;

$$F = \prod_{m\geq 1} F^m \qquad F^m = \prod_{n\geq 1} F_n^m.$$

Proof. By the property of the projection, $pr_1 \circ F = f$, it follows that $F_n^1 = f_n$. We proceed by induction; assume that

$$F_n^m = \sum_{i_1+\cdots+i_m=n} f_{i_1} \otimes \ldots \otimes f_{i_m},$$

for all $m < M$. The equation

$$(F \bigotimes F) \circ \Delta = \Delta \circ F$$

can be restricted to taking the input $V^{\otimes n}$ and having the output $W^{\otimes(M-1)} \bigotimes W$, in which case it becomes

$$\Delta_M^{M-1,1} F_n^M = \sum_{i+j=n} (F_i^{M-1} \bigotimes F_j^1) \Delta_n^{i,j}.$$

Now we can expand F_i^{M-1} with the induction hypothesis and compare the two sides of the equation as an M-fold tensor. It follows that F_n^M is of the correct form. □

Proposition 2.2.11. *Let $F, G : \overline{T}_c V \to \overline{T}_c W$ be two morphisms of coalgebras and let $h : \overline{T}_c V \to W$ be a map of vector spaces then, there exist a unique map $H : \overline{T}_c V \to \overline{T}_c W$, such that $(H \bigotimes G + F \bigotimes H)\Delta = \Delta H$ and so that $pr_1 \circ H = h$ when pr_1 is the natural projection $\overline{T}_c W \to W$. Define H_n^m as the composition*

$$H_n^m : V^{\otimes n} \hookrightarrow \overline{T}_c V \xrightarrow{H} \overline{T}_c W \to W^{\otimes m}$$

Explicitly H_n^m is of the following form

$$H_n^m = \begin{cases} 0 \text{ if } m > n \\ \sum_{a+b=m-1} \sum_{i_1+\ldots+i_a+s+j_1+\ldots+j_b=n} F_{i_1}^1 \otimes \ldots \otimes F_{i_a}^1 \otimes h_s \otimes G_{j_1}^1 \otimes \ldots \otimes G_{j_b}^1, \end{cases}$$

where $a, b \geq 0$, $s > 0$ and, F_k^1 and G_l^1 are as in the previous proposition, and $h_i := h|_{V^{\otimes i}}$. From H_n^m we can recover H by taking the product;

$$H = \prod_{m \geq 1} H^m \qquad H^m = \prod_{n \geq 1} H_n^m.$$

Proof. We prove this with induction. When $m = 1$ this follows from the projection property; $H_n^1 = h_n$. Assume that

$$H_n^m = \sum_{a+b=m-1} \sum_{i_1+\ldots+i_a+s+j_1+\ldots+j_b=n} F_{i_1}^1 \otimes \ldots \otimes F_{i_a}^1 \otimes h_s \otimes G_{j_1}^1 \otimes \ldots \otimes G_{j_b}^1$$

for all $m < M$. Restrict the input to $V^{\otimes n}$ and consider the projection to the $(M-1, 1)$-th component in the equation $(H \bigotimes G + F \bigotimes H)\Delta = \Delta H$ to get:

$$\Delta_M^{M-1,1} \circ H_n^N = \sum_{i+j=n} (F_i^{M-1} \bigotimes H_j^1 + H_i^{M-1} \bigotimes G_j^1) \circ \Delta_n^{i,j},$$

if we expand H_i^{M-1} with the induction hypothesis and F_i^{M-1} with the help of Proposition 2.2.10 then we see that this, as an M-fold tensor, is precisely the formula that was given. □

Definition 2.2.12. *An A_∞ algebra is a graded vector space V equipped with the structure of a codifferential b_V on the associated reduced tensor coalgebra (of the shifted vector space);*

$$b_V : \overline{T_c}V[1] \to \overline{T_c}V[1].$$

A morphism of A_∞ algebras

$$f : (V, b_V) \to (W, b_W)$$

is a morphism of dg coalgebras

$$F : \overline{T_c}V[1] \to \overline{T_c}W1].$$

Let F and G be the morphism of dg coalgebras

$$F, G : (\overline{T_c}V[1], b_V) \to (\overline{T_c}W[1], b_W),$$

where b_V and b_W are codifferentials giving V and W the structure of A_∞ algebras. We say that a map

$$H : \overline{T_c}V[1] \to \overline{T_c}W[1]$$

is a homotopy of F and G if it satisfies two relations:

1. $(F \otimes H + H \otimes G) \circ \Delta_V = \Delta_W \circ H$
2. $F - G = b_W \circ H + H \circ b_V$.

By analyzing the relations it is possible reinterpret the definitions concerning A_∞ algebras without referencing the tensor coalgebra explicitly.

Theorem 2.2.13. *An A_∞ algebra structure on the graded vector space V is a sequence of maps $m_n : V^{\otimes n} \to V$ of degree $2 - n$ such that following equations are satisfied*

$$\begin{aligned}
m_1 \circ m_1 &= 0\\
-m_2 \circ (\mathrm{Id} \otimes m_1) - m_2 \circ (m_1 \otimes \mathrm{Id}) + m_1 \circ m_2 &= 0\\
-m_2 \circ (m_2 \otimes \mathrm{Id}) - m_2 \circ (\mathrm{Id} \otimes m_2) + m_3 \circ (m_1 \otimes \mathrm{Id}^{\otimes 2})&\\
+m_3 \circ (\mathrm{Id} \otimes m_1 \otimes \mathrm{Id}) + m_3 \circ (\mathrm{Id}^{\otimes 2} \otimes m_1) + m_1 \circ m_3 &= 0\\
&\vdots\\
\sum_{s+j+t=n} (-1)^{s+jt} m_{s+1+t} \circ (\mathrm{Id}^{\otimes s} \otimes m_j \otimes \mathrm{Id}^{\otimes t}) &= 0
\end{aligned}$$

Proof. The proof is a matter of expanding the expression $b_V \circ b_V = 0$ and recognizing that $m_n = s^{-1} \circ {b_V}_1^n \circ s^{\otimes n}$, where ${b_V}_1^n$ is the restriction of b_V to $(V[1])^{\otimes n}$ followed by the projection onto $V[1]$. The sign factor comes from applying the Koszul sign rule when shifts are reorganized. $\square$

We will occasionally denote an A_∞ algebra with the pair (V, m^V), where m^V is the system of maps given in the above theorem.

Theorem 2.2.14. *A morphism of A_∞ algebras $f : (V, m^V) \to (W, m^W)$ is a collection of maps $f_n : V^{\otimes n} \to W$ of degree $1-n$ such that*

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t} \circ (\mathrm{Id}^{\otimes r} \otimes m_s^V \otimes \mathrm{Id}^{\otimes t}) = \sum_{q=1}^{n} \sum_{i_1+\ldots+i_q=n} (-1)^p m_q^W \circ (f_{i_1} \otimes \ldots \otimes f_{i_q})$$

where $p = (q-1)(i_1-1) + (q-2)(i_2-1) + \ldots + 2(i_{q-2}-1) + (i_{q-1}-1)$.

Proof. Let $F : (\overline{T}_c V[1], B_V) \to (\overline{T}_c W[1], B_W)$ be a coalgebra morphism. Explicitly f_n is given as $s^{-1} \circ F_n^1 \circ s^{\otimes n}$, where F_n^1 is the restriction of F to the n:th component followed by the projection to the first; $F_n^1 : (V[1])^{\otimes n} \to W[1]$.

We start with the equation $B_W \circ F = F \circ B_V$. In it we restrict the input to $(V[1])^{\otimes n}$ and output to $W[1]$. The result is

$$\sum_{i=1}^{n} (B_W)_i^1 \circ F_n^i = \sum_{j=1}^{n} F_j^1 \circ (B_V)_n^j.$$

It can be determined that

$$B_n^m = \sum_{i+j=m-1} \mathrm{Id}^{\otimes i} \otimes B_s^1 \otimes \mathrm{Id}^{\otimes j}$$

and

$$F_n^m = \sum_{n_1+\ldots+n_m=n} F_{n_1}^1 \otimes \ldots \otimes F_{n_m}^1.$$

Using these explicit formulas we arrive at the expression

$$\sum_{i=1}^{n} (B_W)_i^1 \circ (\sum_{n_1+\ldots+n_i=n} F_{n_1}^1 \otimes \ldots \otimes F_{n_i}^1) = \sum_{i=1}^{n} F_i^1 \circ (\sum_{a+b=i-1} \mathrm{Id}^{\otimes a} (B_V)_{n+1-i}^1 \otimes \mathrm{Id}^b)$$

$\square$

Theorem 2.2.15. *Let (V, μ^V) and (W, μ^W) be two A_∞ algebras and let $f, g : (V, \mu^V) \to (W, \mu^W)$ be two A_∞ algebra morphisms given on the form of maps*

$$\begin{aligned} f_n &: V^{\otimes n} \to W \\ g_n &: V^{\otimes n} \to W \end{aligned}.$$

A system of maps of graded vector spaces $h_n : V^{\otimes n} \to W$ of degree $-n$ is a homotopy of f and g if

$$f_n - g_n = \sum_{m=1}^{n} \sum_{\substack{k+l=m-1 \\ i_1+\ldots+i_k+t+j_1+\ldots j_l=n}} (-1)^s \mu_m^W \circ \left(f_{i_1} \otimes \ldots \otimes f_{i_k} \otimes h_t \otimes g_{j_1} \otimes \ldots \otimes g_{j_l}\right)$$

$$+ \sum_{i+j+k=n} (-1)^{ij+k} h_{i+1+k} \circ \left(\mathrm{Id}^{\otimes i} \otimes \mu_j^V \otimes \mathrm{Id}^{\otimes k}\right),$$

$$s = l + \sum_{1 \le a \le l} (1-j_a)(n - \sum_{b \ge a} j_b) + t \sum_{1 \le a \le k} i_a + \sum_{2 \le a \le k} (1-i_a)(\sum_{b<a} i_b)$$

Proof. The proof is along the lines of the previous theorems. Use a careful analysis of the tensor coalgebra to lift h to a map H, f to F and g to G. In that setting you can apply the rule for homotopy, project the formula to the first component and lastly you recognize the sign that comes from taking into account the degree-shifts. □

2.3 The configuration spaces $Conf_n(\mathbb{R})$, $C_n(\mathbb{R})$ and $\mathfrak{C}_n(\mathbb{R})$

2.3.1. Families of uncompactified configuration spaces Given a set A we define the configuration space $Conf_A(\mathbb{R})$ as the set of injections of the set A into the real line;

$$Conf_A(\mathbb{R}) := \{A \hookrightarrow \mathbb{R}\}.$$

Sometimes we will consider the full set of maps $A \to \mathbb{R}$, and for it we introduce the notation

$$\widetilde{Conf_A}(\mathbb{R}) := \{A \to \mathbb{R}\}.$$

In the special case when $A = [n]$ we use the notation

$$Conf_n(\mathbb{R}) := Conf_{[n]}(\mathbb{R}).$$

The set $Conf_A(\mathbb{R})$ is a real oriented manifold of dimension $|A|$. As a space $Conf_A(\mathbb{R})$ is the union of $|A|!$ connected components, all isomorphic to

$$Conf^o_{|A|}(\mathbb{R}) := \{x_1 < x_2 < \ldots < x_{|A|}\}.$$

The orientation on $Conf^o_n(\mathbb{R})$ is given as the volume form $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$. The group S_n acts on $Conf_n(\mathbb{R})$ by permuting the elements of $[n]$. We assume that the action of S_n is orientation preserving on $Conf_n(\mathbb{R})$ and this fixes the orientation on all connected components of $Conf_n(\mathbb{R})$.

The 2-dimensional Lie group $G_{(2)} = \mathsf{Aff}(\mathbb{R})$ acts freely on $Conf_A(\mathbb{R})$ via the action

$$(x_{a_1}, \dots, x_{a_{|A|}}) \times (\lambda, \nu) = (\lambda x_{a_1} + \nu, \dots, \lambda x_{a_{|A|}} + \nu).$$

The quotient space from this action is an $(n-2)$-dimensional real oriented manifold. We define $C_A(\mathbb{R})$ and $C^o_A(\mathbb{R})$ as the quotient by the action of $G_{(2)}$

$$C_A(\mathbb{R}) := Conf_A(\mathbb{R})/G_{(2)}, \quad C^o_A(\mathbb{R}) := Conf^o_A(\mathbb{R})/G_{(2)}$$

The elements of $C^o_n(\mathbb{R}) := C^o_{[n]}(\mathbb{R})$ can be represented by the equivalence classes of the form $(0 = x_1 < x_2 \dots < x_{n-1} < x_n = 1)$. The orientation orientation on $C^o_n(\mathbb{R})$ is given by the form $dx_2 \wedge \dots \wedge dx_{n-1}$. Let us also remark that $C_n(\mathbb{R}) \cong S_n \times C^o_n(\mathbb{R})$.

Alternatively we can represent equivalence classes of $C_A(\mathbb{R})$ with elements $p = (x_1, \dots, x_n) \in Conf_A(R)$ subject to

$$x_c(p) = \frac{1}{|A|} \sum_{x \in A} x = 0 \qquad ||p|| = \sqrt{\sum_{x \in A} (x - x_c(p))^2} = 1.$$

We define

$$C^{st}_A(\mathbb{R}) := \{p \in Conf_A(\mathbb{R}) \mid x_c(p) = 0, ||p|| = 1\}, \quad C^{st}_n(\mathbb{R}) := C^{st}_{[n]}(\mathbb{R})$$

and

$$\widetilde{C}^{st}_A(\mathbb{R}) := \{p \in \widetilde{Conf_A}(\mathbb{R}) \mid x_c(p) = 0, ||p|| = 1\}, \quad \widetilde{C}^{st}_n(\mathbb{R}) = \widetilde{C}^{st}_{[n]}(\mathbb{R})$$

The 1-dimensional Lie group $G_{(1)} = \mathbb{R}$ acts freely on $Conf_n(\mathbb{R})$ by translation

$$(p, \nu) \mapsto p + \nu,$$

and we denote associated the quotient spaces

$$\mathfrak{C}_A(\mathbb{R}) := Conf_A(\mathbb{R})/G_{(1)}, \quad \mathfrak{C}_n(\mathbb{R}) := \mathfrak{C}_{[n]}(\mathbb{R}).$$

We also introduce the notation

$$\mathfrak{C}^{st}_A(\mathbb{R}) := \{p \in Conf_A(\mathbb{R}) \mid x_c(p) = 0\}, \quad \mathfrak{C}^{st}_n(\mathbb{R}) := \mathfrak{C}^{st}_{[n]}(\mathbb{R}).$$

We have three homeomorphisms associated to these configuration spaces.

1. The space $C_n(\mathbb{R})$ is naturally homeomorphic to $C^{st}_n(\mathbb{R})$

2. We have

$$\Psi_A : \mathfrak{C}_A(\mathbb{R}) \xrightarrow{\simeq} C_A^{st}(\mathbb{R}) \times (0,1)$$

given by

$$p \mapsto \left(\frac{p - x_c(p)}{||p||}, \frac{||p||}{1+||p||} \right).$$

3. We have

$$\Phi_n : Conf_n(\mathbb{R}) \xrightarrow{\simeq} (-1,1)^n \times C_n^{st}(\mathbb{R}) \times (0,1) \times (-1,1)$$

given by

$$p \mapsto \left(\frac{p - x_c(p)}{||p||}, \frac{||p||}{1+||p||}, \frac{x_c(p)}{1+|x_c(p)|} \right).$$

These homeomorphisms provide a starting point to determine compactifications of the configuration spaces in questions. The method employed is essentially a variation of the Fulton-MacPherson compactification for the configuration space of points in the complex plane as given by M. Kontsevich [Kon1].

2.4 The Fulton-MacPherson compactification of $C_n(\mathbb{R})$ and $\mathfrak{C}_n(\mathbb{R})$

In this subsection we give a short summary of a compactification procedure given in [**?**]. This paper attempts to build onto and expand the constructions of loc.cit., and therefore it's most natural for us to reiterate some of the constructions.

The compactification of $C_n(\mathbb{R})$

We introduce a topological compactification $\overline{C}_n(\mathbb{R})$ as the closure of the following injections

$$C_n(\mathbb{R}) \xrightarrow{\prod \pi_A} \prod_{\substack{|A|\subset[n],|A|\geq 2\\ A\ conn}} C_A(\mathbb{R}) \xrightarrow{\simeq} \prod_{\substack{|A|\subset[n],|A|\geq 2\\ A\ conn}} C_A^{st}(\mathbb{R}) \hookrightarrow \prod_{\substack{|A|\subset[n],|A|\geq 2\\ A\ conn}} \widetilde{C}_A^{st}(\mathbb{R})$$

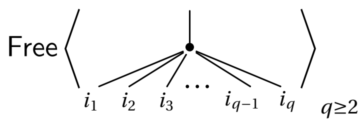
In the products A is a *connected* subsets of $[n]$. By a connected subset of $[n]$ we mean a set which contains all intermediate integers between any two integers that are included in it.

- The codimension one boundary strata of the configuration space $\overline{C}_n(\mathbb{R})$ is given by

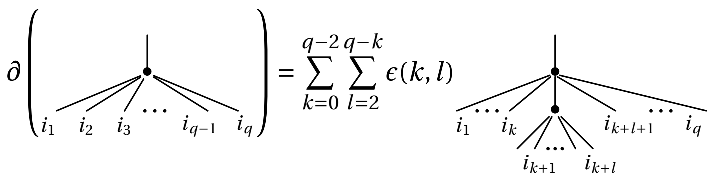
$$\partial\overline{C}_n(\mathbb{R}) = \bigcup_{A\subset[n]} \overline{C}_{n-|A|+1}(\mathbb{R}) \times \overline{C}_{|A|}(\mathbb{R}),$$

  where A is a connected proper subset of $[n]$ with two or more elements. By a connected subset of $[n]$ we mean a set which contains all intermediate integers between any two integers that are included in it.

- The face complex on $\overline{C}_\bullet(\mathbb{R})$ has the natural structure of a dg free operad;



  where the differential acts as follows



  The factor $\epsilon(k,l)$ is a sign that can be worked out to be $(-1)^{k+l(q-k-l)+1}$. Representations of this operad in differential graded vector space are given by A_∞ structures. Thus this is a description of the Ass_∞ operad.

The space $\overline{C}_n(\mathbb{R})$ as a smooth manifold with corners

Let $RT_{n,l}$ be the set of rooted trees with n legs and $l+1$ internal vertices. The set $RT_{n,l}$ parameterizes the codimension l boundary strata of $\overline{C}_n(\mathbb{R})$ in the following sense. Each tree $t \in RT_{n,l}$ describes a space $C_t(\mathbb{R})$ which is defined as the product

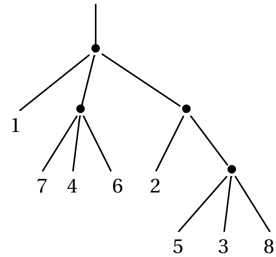
$$C_t(\mathbb{R}) := \prod_{v\in vert(t)} C_{|in(v)|}(\mathbb{R}),$$

where, like before, $vert(t)$ denote the set of internal vertices of t and $in(v)$ the set of input edges at the vertex v. From this one gets a description of $\overline{C}_n(\mathbb{R})$ as a stratified disjoint union of spaces

$$\overline{C}_n(\mathbb{R}) = \coprod_{l\geq 0} \prod_{t\in RT_{n,l}} C_t(\mathbb{R}).$$

To make the compactified configuration space $\overline{C}_n(\mathbb{R})$ into a smooth manifold with corners, we shall define coordinate charts U_t near the boundary stratum $C_t(\mathbb{R})$. We do this for a specific tree t but the general procedure should be clear from the given example. Let t be the tree

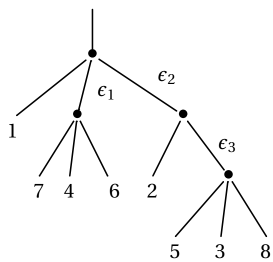
$t =$



We define the coordinate chart close to $C_t(\mathbb{R})$ in a three step procedure.

1. Associate to the tree t a *metric tree*, t_{metric} by endowing each internal edge with a bounded non-negative parameter ϵ;

   $t_{metric} =$

   with $\epsilon_1, \epsilon_2, \epsilon_3 \in [0, \epsilon)$.

2. Pick an S_n-equivariant section $\gamma : C_n(\mathbb{R}) \to Conf_n(\mathbb{R})$, of the natural projection $Conf_n(\mathbb{R}) \to C_n(\mathbb{R})$ and associate to the image of γ a smooth structure. The section could be either of the two description of $C_n(\mathbb{R})$ we mentioned above; $C_n^{st}(\mathbb{R})$ or the space of configurations where $x_1 = 0$ and $x_n = 1$.

3. The coordinate chart U_t can now be seen to be diffeomorphic to the smooth manifold with corners $[0,\epsilon)^{|E(t)|} \times \prod_{v \in vert(t)} C_{|in(v)}(\mathbb{R})$. The diffeomorphism is given by the map Φ_t,

   $$\Phi_t : [0,\epsilon)^{|E(t)|} \times \prod_{v \in vert(t)} C_{|in(v)|}(\mathbb{R}) \longrightarrow U_t.$$

   which we describe in the example of our tree t. Coordinate-wise it is defined as follows

   $$\begin{array}{ccccccccc} (0,\epsilon)^3 & \times & C_3^{st}(\mathbb{R}) & \times & C_3^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) & \times & C_3^{st}(\mathbb{R}) \\ (\epsilon_1, \epsilon_2, \epsilon_3) & \times & (x_1, x', x'') & \times & (x_7, x_4, x_6) & \times & (x_2, x''') & \times & (x_5, x_3, x_8) \end{array}$$

   $$\begin{array}{cc} \longrightarrow & C_8(\mathbb{R}) \\ & (y_1, y_7, y_4, y_6, y_2, y_5, y_3, y_8) \end{array}$$

according to

$$
\begin{aligned}
y_1 &= x_1\\
y_2 &= x'' + \epsilon_2 x_2\\
y_3 &= x'' + \epsilon_2(x''' + \epsilon_3 x_3)\\
y_4 &= x' + \epsilon_1 x_4\\
y_5 &= x'' + \epsilon_2(x''' + \epsilon_3 x_5)\\
y_6 &= x' + \epsilon_1 x_6\\
y_7 &= x' + \epsilon_1 x_4\\
y_8 &= x'' + \epsilon_2(x''' + \epsilon_3 x_5)
\end{aligned}
$$

In general the map Φ_t is given as the recursive ϵ-magnified substitution scheme. If the coordinates $x_i, \ldots, x_{i+k}$ lie in a corolla controlled by the internal edge associated to the coordinate x' and where the internal edge is parameterized by the factor ϵ, then the substitution give the new coordinates $x' + \epsilon x_i, \ldots, x' + \epsilon x_{i+k}$.

A compactification of $\widehat{\mathfrak{C}}_n(\mathbb{R})$

Define the compactification of $\mathfrak{C}_n(\mathbb{R})$ as the closure of the following inclusions

$$
\mathfrak{C}_n(\mathbb{R}) \xrightarrow{\prod \pi_A} \prod_{\substack{|A|\subset[n]\\|A|\geq 1\\ A\, conn}} \mathfrak{C}_A(\mathbb{R}) \xrightarrow{\prod \Psi_A} \prod_{\substack{|A|\subset[n]\\|A|\geq 1\\ A\, conn}} C_A^{st}(\mathbb{R}) \times (0,1) \hookrightarrow \prod_{\substack{|A|\subset[n]\\|A|\geq 1\\ A\, conn}} \widetilde{C}_A^{st}(\mathbb{R}) \times [0,1]
$$

- The codimension one boundary strata of the configuration space $\widehat{\mathfrak{C}}_n(\mathbb{R})$ is given by

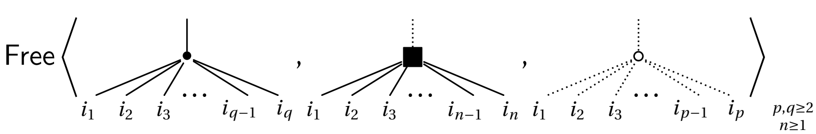
  $$\partial\widehat{\mathfrak{C}}_n(\mathbb{R}) = \bigcup \widehat{\mathfrak{C}}_{n-|A|+1}(\mathbb{R}) \times \overline{C}_{|A|}(\mathbb{R}) \cup \bigcup \overline{C}_k(\mathbb{R}) \times \widehat{\mathfrak{C}}_{|A_1|}(\mathbb{R}) \times \ldots \times \widehat{\mathfrak{C}}_{|A_k|}(\mathbb{R})$$

  where A is as above and where the A_i are connected disjoint subsets of $[n]$ such that $inf A_1 < \ldots < inf A_k$ and $\cup A_i = [n]$.

- The face complex of the disjoint union

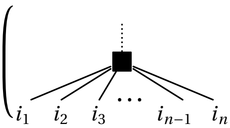
  $$\overline{C}_\bullet(\mathbb{R}) \sqcup \widehat{\mathfrak{C}}_\bullet(\mathbb{R}) \sqcup \overline{C}_\bullet(\mathbb{R})$$

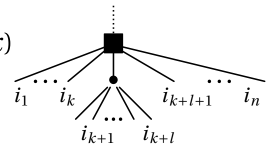
  has the natural structure of a dg free operad of transformation type;



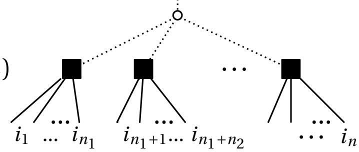
The differential has the following action

$$\partial\left(\quad\right) = \sum_{l=2}^{n}\sum_{k=0}^{n-l}\epsilon(l,k)$$





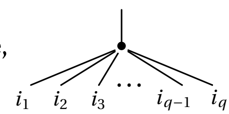
$$+\sum_{k=2}^{n}\sum_{n=n_1+\ldots+n_k}\epsilon(k;n_1,\ldots,n_k)$$



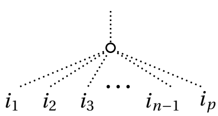
Where $\epsilon(k,l)=(-1)^{k+l+l(n-k)+1}$ and

$$\epsilon(k;n_1,\ldots,n_k)=(-1)^{(k-1)(n_1-1)+(k-2)(n_2-1)+\ldots+2(n_{k-2}-1)+n_{k-1}-1}$$

On the corollas corresponding to the A_∞ structure,



and



, the differential acts precisely like in the case $\overline{C}_n(\mathbb{R})$. Representations of this operad are given by three pieces of data: two A_∞ algebras, A and A', and a morphism of A_∞ algebras $A\to A'$. Thus this is the previously discussed operad $\mathsf{Mor}(\mathsf{Ass})_\infty$.

The space $\widehat{\mathfrak{C}}_n(\mathbb{R})$ as a smooth manifold with corners

We generalize the procedure for $\overline{C}_n(\mathbb{R})$ to $\widehat{\mathfrak{C}}_n(\mathbb{R})$. For every tree $t\in\mathsf{Mor}(\mathsf{Ass})_\infty$ we define the sets $vert_{\bullet,\circ}(t)$ and $vert_{\blacksquare}(t)$ as the vertices of t marked by $\{\bullet,\circ\}$ or $\blacksquare$. For the tree t we define $\mathfrak{C}_t(\mathbb{R})$ as a product;

$$\mathfrak{C}_t(\mathbb{R}) := \prod_{v\in vert_{\bullet,\circ}(t)} C_{|in(v)|}(\mathbb{R}) \times \prod_{v\in vert_{\blacksquare}(t)} \mathfrak{C}_{|in(v)|}(\mathbb{R}).$$

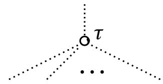
We can describe the space $\widehat{\mathfrak{C}}_n(\mathbb{R})$ as a stratified union of spaces;

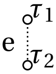
$$\widehat{\mathfrak{C}}_n(\mathbb{R}) = \prod_{t\in\mathcal{M}or(As)_\infty(n)} \mathfrak{C}_t(\mathbb{R}).$$

We shall define a coordinate chart U_t around every boundary stratum $\mathfrak{C}_t(\mathbb{R})$ with a metric tree. We associate to t the metric tree t_{metric} with for

1. every internal edge of the type  a small positive parameter ϵ;

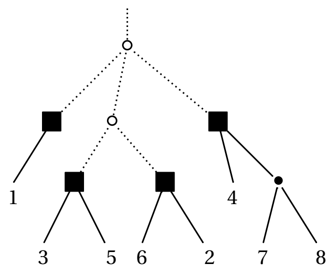
2. every vertex of a dashed corolla associate a large positive number τ,



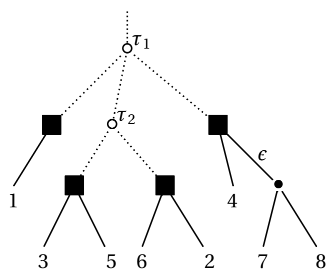
3. every subgraph of t_{metric} of the type 

an inequality $\tau_1 > \tau_2$.

Example 2.4.1. *As an example we consider a specific tree. The general method should be clear from this description. Let t be the following tree*



Then the associated metric tree, t_{metric}, is given by



Choose an equivariant section,

$$s : \mathfrak{C}_n(\mathbb{R}) \to Conf_n(\mathbb{R})$$

to the projection

$$Conf_n(\mathbb{R}) \to \mathfrak{C}_n(\mathbb{R})$$

and a smooth structure on the image of s. Define $\mathfrak{C}^{st}_n(\mathbb{R}) := s(\mathfrak{C}_n(\mathbb{R}))$, which is called the space of configurations in standard position. One possible choice of $\mathfrak{C}^{st}_n(\mathbb{R})$ is subspace of points in $Conf_n(\mathbb{R})$ where $\sum x_i = 0$.

The coordinate chart $U_t \subset \widehat{\mathfrak{C}}_n(\mathbb{R})$ is now defined to be diffeomorphic to the manifold with corners,

$$(l, +\infty]^{|vert_\circ|(t)} \times [0, s)^{|edge^{\blacksquare}_{\bullet}(t)|} \times \prod_{v \in vert_{\circ,\bullet}(t)} C^{st}_{|in(v)|}(\mathbb{R}) \times \prod_{v \in vert_{\blacksquare}(t)} \mathfrak{C}^{st}_{|in(t)|}(\mathbb{R})$$

where $vert_\circ$ denotes the set of vertices of type $\circ$, $vert_{\circ,\bullet}$ denotes the set of vertices of type $\circ$ or $\bullet$ and $edge^{\blacksquare}_{\bullet}$ denote the set of edges of type $\overset{\blacksquare}{\underset{\bullet}{|}}$. The diffeomorphism Φ_t between the coordinate chart U_t and the product above is read from metric tree. The map is given in coordinates, for the specific tree in the above example, as follows

$$
\begin{array}{ccccccccc}
(l,+\infty]^2 & \times & [0,s) & \times & C_3^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) \\
(\tau_1,\tau_2) & \times & \epsilon & \times & (x',x'',x''') & \times & (t',t'') & \times & (x_7,x_8) \\
\\
 & \times & \mathfrak{C}_1^{st}(\mathbb{R}) & \times & \mathfrak{C}_2^{st}(\mathbb{R}) & \times & \mathfrak{C}_2^{st}(\mathbb{R}) & \times & \mathfrak{C}_2^{st}(\mathbb{R}) \\
 & \times & x_1 & \times & (x_3,x_5) & \times & (x_6,x_2) & \times & (x_4,u)
\end{array}
\longrightarrow
\begin{array}{c}
\mathfrak{C}_8(\mathbb{R}) \\
(y_1,\dots,y_8)
\end{array}
$$

such that

$$
\begin{aligned}
y_1 &= \tau_1 x' + x_1 \\
y_2 &= \tau_1 x'' + \tau_2 t'' + x_2 \\
y_3 &= \tau_1 x'' + \tau_2 t' + x_3 \\
y_4 &= \tau_1 x''' + x_4 \\
y_5 &= \tau_1 x'' + \tau_2 t' + x_5 \\
y_6 &= \tau_1 x'' + \tau_2 t'' + x_6 \\
y_7 &= \tau_1 x''' + u + \epsilon x_7 \\
y_8 &= \tau_1 x''' + u + \epsilon x_8
\end{aligned}
$$

The boundary strata in U_t are given by allowing formally $\tau_1 = \infty, \tau_2 = \infty$ such that $\tau_1/\tau_2 = 0$ and $\epsilon = 0$.

2.5 The operad $\mathsf{Ho}(\mathsf{Ass})_\infty$

2.5.1. Compactification of the configuration space $Conf_\bullet(\mathbb{R})$ In this section we introduce our main result. We define the new compactification $\overline{Conf}_n(\mathbb{R})$ of the configuration space $Conf_n(\mathbb{R})$ as the closure of the following injections

$$
Conf_n(\mathbb{R}) \xrightarrow{\Phi\times\Psi_{[n]}} (-1,1)^n \times \mathfrak{C}_n \xrightarrow{\mathrm{id}\times\Psi_\bullet} [-1,1]^n \times \prod_{\substack{|A|\subset[n],|A|\geq 1\\ A\ conn}} \widetilde{C}_A^{st}(\mathbb{R}) \times [0,1]
$$

where

$$
(x_1,\dots,x_n) \overset{\Phi}{\longmapsto} \left(\frac{x_1}{1+|x_1|},\dots,\frac{x_n}{1+|x_n|}\right) \quad \text{and} \quad \Psi_\bullet = \prod_{\substack{|A|\subset[n],|A|\geq 1\\ A\ conn}} \Psi_A
$$

We extend the previous result for $\overline{C}_n$ and $\widehat{\mathfrak{C}}_n$ to the space $\overline{Conf}_n(\mathbb{R})$. The codimension one boundary strata of $\overline{Conf}_n(\mathbb{R})$ are given as

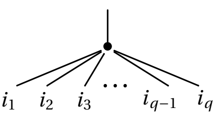
$$\partial\overline{Conf}_n(\mathbb{R}) = \bigcup \overline{Conf}_{n-|A|+1}(\mathbb{R}) \times \overline{C}_{|A|}(\mathbb{R}) \cup \widehat{\mathfrak{C}}_n(\mathbb{R}) \cup \widehat{\mathfrak{C}}_n(\mathbb{R})$$
$$\bigcup \overline{C}_{k+1+l}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{|A_1|}(\mathbb{R}) \times \ldots \times \widehat{\mathfrak{C}}_{|A_k|}(\mathbb{R}) \times \overline{Conf}_{|A|}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{|B_1|}(\mathbb{R}) \times \ldots \times \widehat{\mathfrak{C}}_{|B_l|}(\mathbb{R})$$

1. The first union runs over all connected subsets $A \subset [n]$ such that $|A| > 1$. The stratum correspond to the collapsing of the points of A into one point.

2. The stratum $\widehat{\mathfrak{C}}_n(\mathbb{R})$ appears when either all points go to plus or minus infinity but in such a manner that the distance between the points is finite.

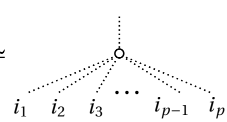
3. The second union runs over all partitions of $[n]$ into connected non-empty subsets $[n] = A_1 \cup \ldots \cup A_k \cup F \cup B_1 \cup \ldots \cup B_l$ with $|F| > 0$. These limit points correspond to when the points from $A_1, \ldots, A_k$ go to $-\infty$, the points from F stay in a finite position and the points from $B_1, \ldots, B_l$ go to ∞. The points do this such that each point in A_i and B_j remain a finite distance from each other; $||p_{A_i}||, ||p_{B_j}|| < \infty$.

By methods described in [**?**] we can consider the fundamental chains of $\{\overline{C}_\bullet(\mathbb{R}) \sqcup \widehat{\mathfrak{C}}_\bullet(\mathbb{R}) \sqcup \overline{Conf}_\bullet(\mathbb{R}) \sqcup \widehat{\mathfrak{C}}_\bullet(\mathbb{R}) \sqcup \overline{C}_\bullet(\mathbb{R})\}$ as a dg free operad with two colors. We identify the faces with corollas;

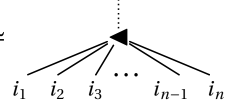
$\overline{C}_q(\mathbb{R}) \simeq$ 

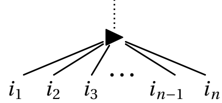
.

We need to illustrate two versions of this space as it appears either as collapsing or as controlling points at infinity. We distinguish between them by the color of their internal vertex and legs; drawn black or white/dashed.

$\overline{C}_p(\mathbb{R}) \simeq$ 

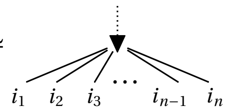
Points going to plus or minus infinity in a cluster are given a two-colored corolla:

$-\infty : \widehat{\mathfrak{C}}_n(\mathbb{R}) \simeq$ 

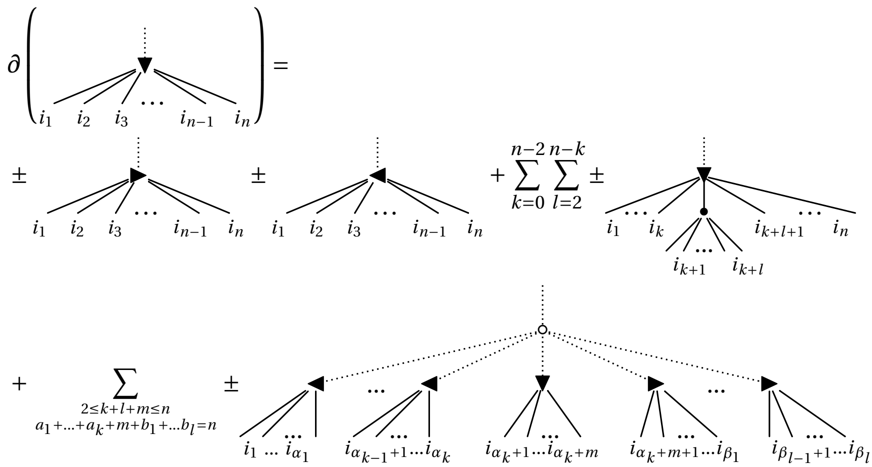
$\infty : \widehat{\mathfrak{C}}_n(\mathbb{R}) \simeq$ 

We represent points staying finite with a two-colored corolla as follows:

$$\overline{Conf}_n(\mathbb{R}) \simeq$$



In this graphical notation the differential has the following action:



Where $1 \leq m \leq n-1$, $\alpha_i = a_1 + \ldots + a_i$ and $\beta_j = \alpha_k + m + b_1 + \ldots b_j$ On the corollas corresponding to $\overline{C}_\bullet(\mathbb{R})$ and $\widehat{\mathfrak{C}}_\bullet(\mathbb{R})$ the differential acts identically to the differential in the $\mathsf{Mor}(\mathsf{Ass})_\infty$ operad.

Example 2.5.1. *To convince the reader we proceed to work out the codimension* 1 *boundary strata of* $\overline{Conf}_3(\mathbb{R})$:

$$\overline{Conf}_3(\mathbb{R}) \subset [-1,1]^3 \times \widetilde{C}^{st}_{12}(\mathbb{R}) \times [0,1] \times \widetilde{C}^{st}_{23}(\mathbb{R}) \times [0,1] \times \widetilde{C}^{st}_{123}(\mathbb{R}) \times [0,1]$$

the points of which can be written in coordinates

$$x = \left(\frac{x_1}{1+|x_1|}, \frac{x_2}{1+|x_2|}, \frac{x_3}{1+|x_3|}, *, \frac{||p_{12}||}{1+||p_{12}||}, *, \frac{||p_{23}||}{1+||p_{23}||}, \left(\frac{x_1-x_c}{||p||}, \frac{x_2-x_c}{||p||}, \frac{x_3-x_c}{||p||} \right), \frac{||p||}{1+||p||} \right)$$

where

$$||p_{12}|| = \frac{x_2-x_1}{\sqrt{2}}, \quad ||p_{23}|| = \frac{x_3-x_2}{\sqrt{2}} \quad x_c = \frac{x_1+x_2+x_3}{3}$$

and

$$||p|| = \sqrt{\left(\frac{2x_1-x_2-x_3}{3}\right)^2 + \left(\frac{-x_1+2x_2-x_3}{3}\right)^2 + \left(\frac{-x_1-x_2-2x_3}{3}\right)^2}$$

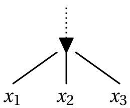
There is some redundant information in x. *The parameter* $||p||$ *is at a boundary value if and only if* $||p_{12}||$ *or* $||p_{23}||$ *is at a boundary value. The same is* *true for the coordinates of* $\widetilde{C}^{st}_{123}(\mathbb{R})$, *where a boundary point arises if* $||p_{12}||$ *or* $||p_{23}||$ *becomes zero.*

To analyze the codimension one boundary strata it is enough to consider the coordinates of the form $(y_1, y_2, y_3, s_{12}, s_{23})$ *with*

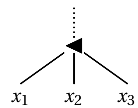
$$y_1 < y_2 < y_3 \in [-1,1] \text{ and } s_{12}, s_{23} \in [0,1]$$

We will see that the codimension one strata are given in twelve different ways:

1. *The configurations* $(y_1, y_2, y_3) = (-1, -1, -1)$ *with* s_{12} *and* s_{23} *being points of the closed unit interval. This can be achieved by* $x_1 = r - \sqrt{2}\lambda_1$, $x_2 = r$, $x_3 = r + \sqrt{2}\lambda_2$ *and then letting* $r \to -\infty$*. These limit points are scaling-invariant so we can identify them with a copy of* $\widehat{\mathfrak{C}}_{\{x_1,x_2,x_3\}}(\mathbb{R})$

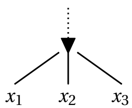


$$\xrightarrow[\substack{(y_1,y_2,y_3)\longrightarrow(-1,-1,-1)\\ s_{12}\longrightarrow\lambda_1\\ s_{23}\longrightarrow\lambda_2}]{}$$

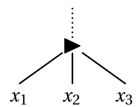


$$\cong \{(-1,-1,-1,s_{12},s_{23})\} \cong \widehat{\mathfrak{C}}_{\{x_1,x_2,x_3\}}(\mathbb{R})$$

2. *The configurations* $(y_1, y_2, y_3) = (1, 1, 1)$ *with* s_{12} *and* s_{23} *being points of the closed unit interval. This boundary strata can also be identified with a copy of* $\widehat{\mathfrak{C}}_{\{x_1,x_2,x_3\}}(\mathbb{R})$.

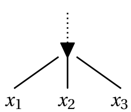


$$\xrightarrow[\substack{(y_1,y_2,y_3)\longrightarrow(1,1,1)\\ s_{12}\longrightarrow\lambda_1\\ s_{23}\longrightarrow\lambda_2}]{}$$

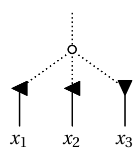


$$\cong \{(1,1,1,s_{12},s_{23})\} \cong \widehat{\mathfrak{C}}_{\{x_1,x_2,x_3\}}(\mathbb{R})$$

3. *The configurations* $(y_1, y_2, y_3) = (-1, -1, a)$ *with* $s_{12} = s_{23} = 1$*. Points of this type can be identified with elements of* $\overline{C}\{x_1, x_2, x_3\}R) \times \widehat{\mathfrak{C}}_{\{x_1\}} \times \widehat{\mathfrak{C}}_{\{x_2\}}(\mathbb{R}) \times \overline{Conf}_{\{x_3\}}(\mathbb{R})$.



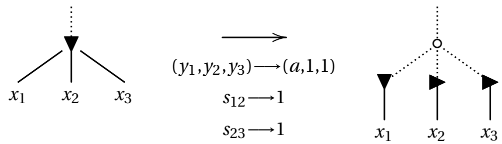
$$\xrightarrow[\substack{(y_1,y_2,y_3)\longrightarrow(-1,-1,a)\\ s_{12}\longrightarrow 1\\ s_{23}\longrightarrow 1}]{}$$



$$\cong \{(-1,-1,a,1,1)\} \cong \overline{C}_{\{x_1,x_2,x_3\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_1\}} \times \widehat{\mathfrak{C}}_{\{x_2\}}(\mathbb{R}) \times \overline{Conf}_{\{x_3\}}(\mathbb{R})$$

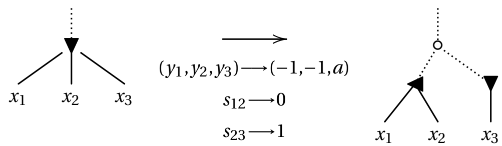
4. *The configurations* $(y_1, y_2, y_3) = (a, 1, 1)$ *with* $s_{12} = s_{23} = 1$*. Points of this type can be identified with elements of* $\overline{C}_{\{x_1,x_2,x_3\}}(\mathbb{R}) \times$

$\overline{Conf}_{\{x_1\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_2\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_3\}}(\mathbb{R})$.



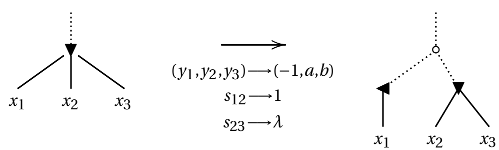
$\cong (a, 1, 1, 1, 1) \cong \overline{C}_{\{x_1,x_2,x_3\}}(\mathbb{R}) \times \overline{Conf}_{\{x_1\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_2\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_3\}}(\mathbb{R})$

5. *The configurations* $(y_1, y_2, y_3) = (-1, -1, a)$ *with* $s_{12} = 0$. *Points of this type can be identified with elements of* $\overline{C}_{\{x_2,x_3\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_1,x_2\}}(\mathbb{R}) \times \overline{Conf}_{\{x_3\}}(\mathbb{R})$.



$\cong (-1, -1, a, 0, 1) \cong \overline{C}_{\{x_2,x_3\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_1,x_2\}}(\mathbb{R}) \times \overline{Conf}_{\{x_3\}}(\mathbb{R})$

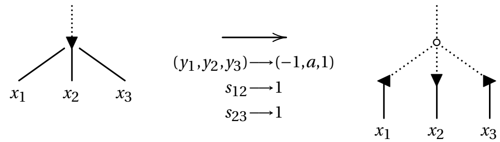
6. *The configurations* $(y_1, y_2, y_3) = (-1, a, b)$ *i.e.* $s_{12} = 1$. *Points of this type can be identified with elements of* $\overline{C}_{\{x_1,x_2\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_1\}}(\mathbb{R}) \times \overline{Conf}_{\{x_2,x_3\}}(\mathbb{R})$



$\cong (-1, a, b, 1, s_{23}) \cong \overline{C}_{\{x_1,x_2\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_1\}}(\mathbb{R}) \times \overline{Conf}_{\{x_2,x_3\}}(\mathbb{R})$

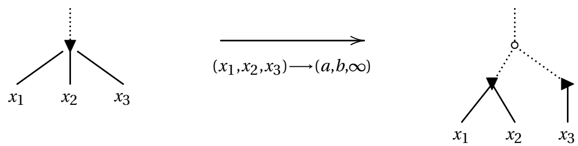
7. *The configurations* $(y_1, y_2, y_3) = (-1, a, 1)$ *i.e.* $s_{12} = s_{23} = 1$ *These points can be identified with* $\overline{C}_{\{x_1,x_2,x_3\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_1\}}(\mathbb{R}) \times$

$\overline{Conf}_{\{x_2\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_3\}}(\mathbb{R})$



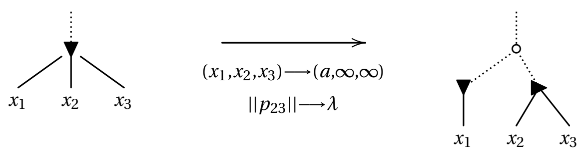
$\cong (-1, a, 1, 1, 1) \cong \overline{C}_{\{x_1,x_2,x_3\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_1\}}(\mathbb{R}) \times \overline{Conf}_{\{x_2\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_3\}}(\mathbb{R})$

8. *The configurations* $(y_1, y_2, y_3) = (a, b, 1)$ *i.e.* $s_{23} = 1$. *Points of this type can be identified with elements of* $\overline{C}_{\{x_2,x_3\}}(\mathbb{R}) \times \overline{Conf}_{\{x_1,x_2\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_3\}}(\mathbb{R})$.



$\cong (a, b, 1, s_{12}, 1) \cong \overline{C}_{\{x_1,x_2\}}(\mathbb{R}) \times \overline{Conf}_{\{x_1,x_2\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_3\}}(\mathbb{R})$

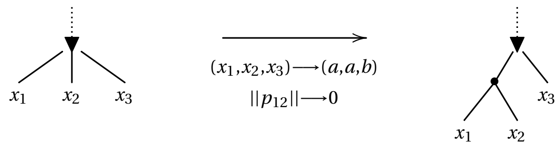
9. *The configurations* $(y_1, y_2, y_3) = (a, 1, 1)$ *with* $s_{23} = \lambda$ *being an arbitrary real number of the unit interval. Points of this type can be identified with elements of* $\overline{C}_{\{x_1,x_2\}}(\mathbb{R}) \times \overline{Conf}_{\{x_1\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_2,x_3\}}(\mathbb{R})$



$\cong (a, 1, 1, 1, 1) \cong \overline{C}_{\{x_1,x_2\}}(\mathbb{R}) \times \overline{Conf}_{\{x_1\}}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{\{x_2,x_3\}}(\mathbb{R})$

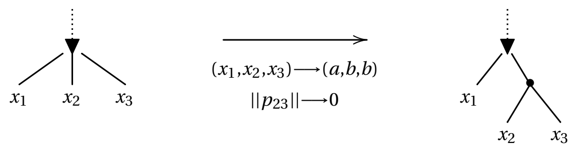
10. *The configurations* $(y_1, y_2, y_3) = (a, a, b)$ *i.e.* $s_{12} = 0$. *Points of this type can be identified with elements of* $\overline{Conf}_{\{x_2,x_3\}}(\mathbb{R}) \times$

$\overline{C}_{\{x_1,x_2\}}(\mathbb{R})$.



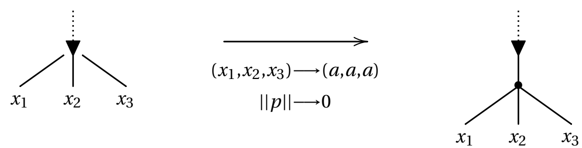
$\cong (a,a,b,0,s_{23}) \cong \overline{Conf}_{\{x_2,x_3\}}(\mathbb{R}) \times \overline{C}_{x_1,x_2}(\mathbb{R})$

11. *The configurations* $(y_1,y_2,y_3) = (a,b,b)$ *i.e.* $s_{23} = 0$. *Points of this type can be identified with elements of* $\overline{Conf}_{\{x_1,x_2\}}(\mathbb{R}) \times \overline{C}_{\{x_2,x_3\}}(\mathbb{R})$



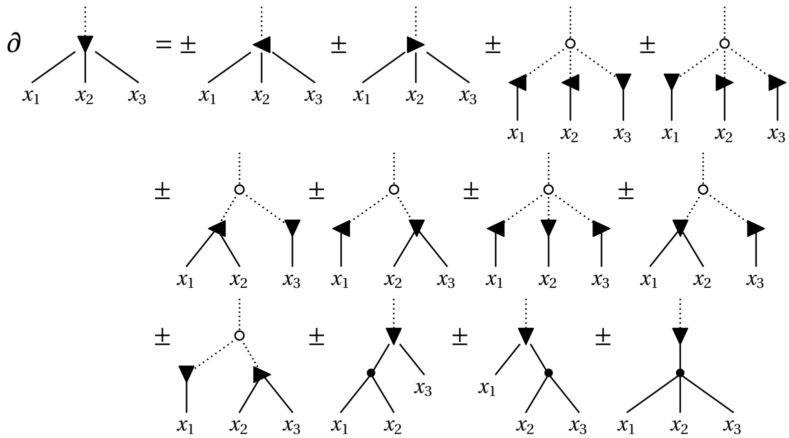
$\cong (a,b,b,s_{12,0}) \cong \overline{Conf}_{\{x_1,x_2\}}(\mathbb{R}) \times \overline{C}_{x_2,x_3}(\mathbb{R})$

12. *The configurations* $(y_1,y_2,y_3) = (a,a,a)$ *i.e.* $s_{12} = s_{23} = 0$. *Points of this type can be identified with elements of* $\overline{Conf}_{\{x_1\}}(\mathbb{R}) \times \overline{C}_{\{x_1,x_2,x_3\}}(\mathbb{R})$



$\cong (a,a,a,1,1) \cong \overline{Conf}_{\{x_1\}}(\mathbb{R}) \times \overline{C}_{\{x_1,x_2,x_3\}}(\mathbb{R})$

We summarize this in the formula:



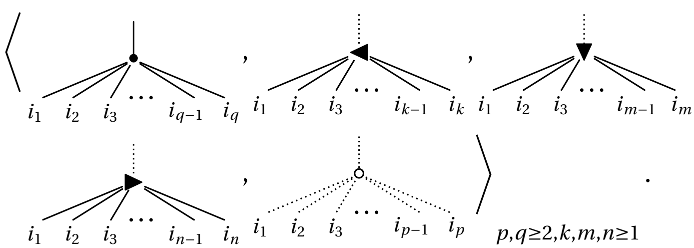
We summarize the result in our main theorem

Theorem 2.5.2. *The face complex on the disjoint union*

$$\overline{C}_\bullet(\mathbb{R}) \sqcup \widehat{\mathfrak{C}}_\bullet(\mathbb{R}) \sqcup \overline{Conf}_\bullet(\mathbb{R}) \sqcup \widehat{\mathfrak{C}}_\bullet(\mathbb{R}) \sqcup \overline{C}_\bullet(\mathbb{R})$$

is naturally a dg free operad of transformation type

$\mathsf{Ho}(\mathsf{Ass})_\infty = \mathsf{Free}$



Representation of this operad in a pair of vector spaces V^1 and V^2 is the structure of two A_∞ algebras, (V^1,μ^1) and (V^2,μ^2), two A_∞ morphisms, $f,g:(V^1,\mu^1)\to(V^2,\mu^2)$ and a homotopy h between the morphism $h: f \to g$. The action of the differential was described earlier.

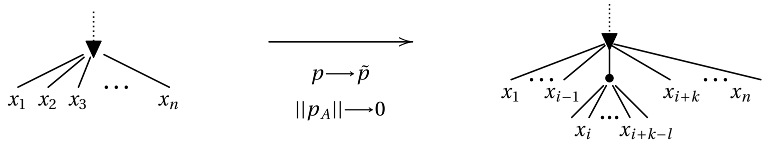
Proof. The proof is by inspection. We have worked out the cases of three points in detail and we can see that they correspond the to algebraic formulas of the previous section. The general case is treated in complete analogy.

Let p be a configuration of n points on the real line $p = (x_1 < x_2 < \ldots < x_n) \in Conf_n(\mathbb{R})$. The possible codimension 1 boundary strata can arise in three different ways.

1. A connected subset $A = (x_i < x_{i+1} < \ldots x_{i+k-1})$ of points collapsing into single point; A limit point

$$p \longrightarrow \tilde{p} = (a_1 < a_2 < \ldots a_{i-1} < a_i = a_{i+1} = \ldots = a_{i+k-1} < a_{i+k} < \ldots < a_n).$$

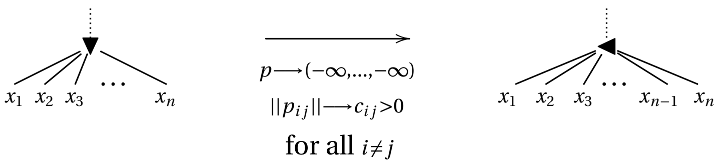
Points of this type can be identified with $\overline{Conf}_{n-k+1}(\mathbb{R}) \times \overline{C}_k(\mathbb{R})$.

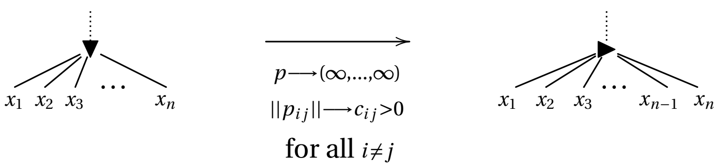


2. All n points moving in a cluster towards $\pm\infty$; A limit point

$$p \longrightarrow \pm(\infty, \infty, \ldots, \infty)$$

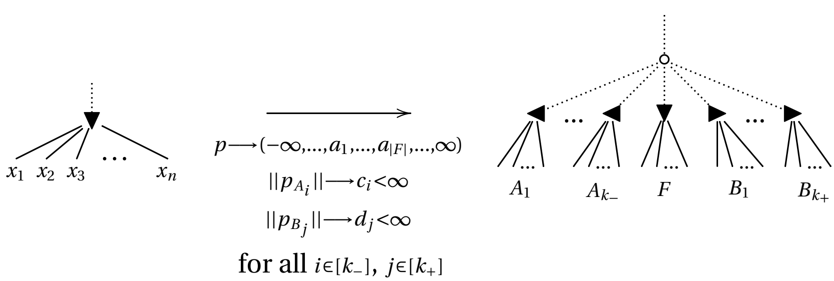
where the distance between points remain finite, e.g. it could look like $p = (t+\lambda_1, t+\lambda_2, \ldots, t+\lambda_n)$ with $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ and $t \longrightarrow \pm\infty$. Limit points of this type can be identified with $\widehat{\mathfrak{C}}_n(\mathbb{R})$.





3. For each $k \geq 2$ the n points congregate in $k = k_- + 1 + k_+$ clusters where k_- clusters move to $-\infty$, k_+ clusters move to $+\infty$ and one cluster for the points that stay finite. Within each of the $k_- + k_+$ clusters moving to $\pm\infty$ the distance between points remain finite, while the distance from any two points from different clusters tend to ∞. Every such configuration is determined by a disjoint union of connected subsets $A_1 \cup \ldots \cup A_{k_-} \cup F \cup B_1 \cup \ldots \cup B_{k_+} = [n]$ is with $\inf A_1 < \inf A_2 < \ldots < \inf A_{k_-} < \inf F < \inf B_1 < \inf B_2 < \ldots < \inf B_{k_+}$, and limit points of this type can then be identified with

$$\overline{C}_k(\mathbb{R}) \times \widehat{\mathfrak{C}}_{|A_1|}(\mathbb{R}) \times \ldots \times \widehat{\mathfrak{C}}_{|A_{k_-}|}(\mathbb{R}) \times \overline{Conf}_{|F|}(\mathbb{R}) \times \widehat{\mathfrak{C}}_{|B_1|}(\mathbb{R}) \times \ldots \times \widehat{\mathfrak{C}}_{|B_{k_+}|}(\mathbb{R}).$$



$\square$

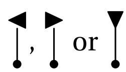
2.5.2. The space $\overline{Conf}_n(\mathbb{R})$ as a smooth manifold with corners. We shall endow the space $\overline{Conf}_n(\mathbb{R})$ with a manifold structure in an almost identical procedure to how the space $\widehat{\mathfrak{C}}_n(\mathbb{R})$ was treated. For every tree $t \in \mathsf{Ho(Ass)}_\infty$ we define the sets $vert_{\bullet,\circ}(t)$, $vert_{\blacktriangleleft,\blacktriangleright}(t)$ and $vert_{\blacktriangledown}$ as the vertices of t marked by $\{\bullet,\circ\}$, $\{\blacktriangleleft,\blacktriangleright\}$ or $\blacktriangledown$, respectively. For the tree t we define $Conf_t(\mathbb{R})$ as a product;

$$Conf_t(\mathbb{R}) := \prod_{v \in vert_{\bullet,\circ}(t)} C_{|in(v)|}(\mathbb{R}) \times \prod_{v \in vert_{\blacktriangleleft,\blacktriangleright}(t)} \mathfrak{C}_{|in(v)|}(\mathbb{R}) \times \prod_{v \in vert_{\blacktriangledown}} Conf_n(\mathbb{R}).$$

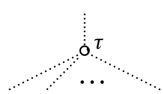
We can describe the space $\overline{Conf}_n(\mathbb{R})$ as a stratified union of spaces;

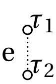
$$\overline{Conf}_n(\mathbb{R}) = \prod_{t \in \mathcal{H}o(As)_\infty(n)} Conf_t(\mathbb{R}).$$

We shall define a coordinate chart U_t around every boundary stratum $Conf_t(\mathbb{R})$ with a metric tree. We associate to t the metric tree t_{metric} with for

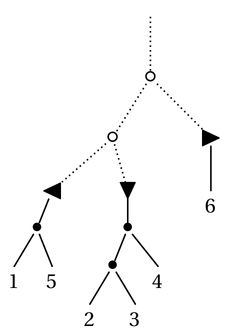
1. every internal edge of the types  a small positive parameter ϵ;

2. every vertex of a dashed corolla associate a large positive number τ,

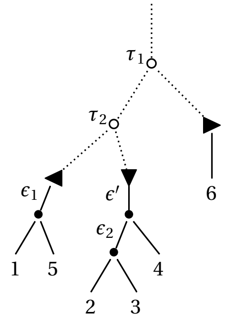


3. every subgraph of t_{metric} of the type  an inequality $\tau_1 > \tau_2$.

Example 2.5.3. *We consider a specific tree and associated the metric tree to it. The general method should be clear from this description. Let t be the following tree*



Then the associated metric tree, t_{metric}, is given by



The coordinate chart $U_t \subset \overline{Conf}_n(\mathbb{R})$ is now defined to be diffeomorphic to the manifold with corners,

$$(l,+\infty]^{|vert_\circ(t)|} \times [0,s)^{|edge_\bullet^{\blacktriangleleft,\blacktriangleright,\blacktriangledown}(t)|} \times \prod_{v\in vert_{\circ,\bullet}(t)} C^{st}_{|in(v)|}(\mathbb{R})$$
$$\times \prod_{v\in vert_{\blacktriangleleft,\blacktriangleright}(t)} \mathfrak{C}^{st}_{|in(t)|}(\mathbb{R}) \times \prod_{v\in vert_{\blacktriangledown}(t)} Conf_{|in(v)|}(\mathbb{R})$$

where $vert_\circ$ denotes the set of vertices of type $\circ$, $vert_{\circ,\bullet}$ denotes the set of vertices of type $\circ$ or $\bullet$ and so forth. The set $edge_\bullet^{\blacktriangleleft,\blacktriangleright,\blacktriangledown}$ is give set of edges of type $\blacktriangleleft$, $\blacktriangleright$ or $\blacktriangledown$. The diffeomorphism Φ_t between the coordinate chart U_t and the product above is read from the metric tree. The map is given in coordinates, for the specific tree in the above example, as follows

$$\begin{array}{ccccccccc}
(l,+\infty]^2 & \times & [0,s)^3 & \times & C_2^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) \\
(\tau_1,\tau_2) & \times & (\epsilon_1,\epsilon_2,\epsilon') & \times & (x_1'',x_2'') & \times & (x_1',x_2') & \times & (x_1,x_5) \\
\\
C_2^{st}(\mathbb{R}) & \times & C_2^{st}(\mathbb{R}) & \times & \mathfrak{C}_1^{st}(\mathbb{R}) & \times & \mathfrak{C}_1^{st}(\mathbb{R}) & \times & Conf_1(\mathbb{R}) \\
(x',x_4) & \times & (x_2,x_3) & \times & x_6 & \times & s & \times & u
\end{array}$$
$$\longrightarrow \begin{array}{c} Conf_6(\mathbb{R}) \\ (y_1,\dots,y_6) \end{array}$$

such that

$$\begin{aligned}
y_1 &= \tau_1 x_1'' + \tau_2 x_1 + t + \epsilon_1 x_1 \\
y_2 &= \tau_1 x'' + \tau_2 x' + u + \epsilon'(x' + \epsilon_2 x_2) \\
y_3 &= \tau_1 x'' + \tau_2 x' + u + \epsilon'(x' + \epsilon_2 x_3) \\
y_4 &= \tau_1 x'' + \tau_2 x' + u + \epsilon' x_4 \\
y_5 &= \tau_1 x_1'' + \tau_2 x_1 + t + \epsilon_1 x_5 \\
y_6 &= \tau_1 x'' + x_6
\end{aligned}$$

The boundary strata in U_t are given by allowing formally $\tau_1 = \infty, \tau_2 = \infty$ such that $\tau_1/\tau_2 = 0$ and $\epsilon_1 = 0, \epsilon_2 = 0, \epsilon' = 0$.

2.5.3. The cohomology of $\mathsf{Ho(Ass)}_\infty$ We will state two results without giving a proof, both of which we need in order to calculate the cohomology of the operad $\mathsf{Ho(Ass)}_\infty$.

Theorem 2.5.4. *[MeVa] Let* P *be a koszul operad. Define the 2-colored operad* $\mathsf{Mor(P)}$ *whose representations are two* P*-algebras and a* P*-algebra morphism between them. The operad* $\mathsf{Mor(P)}$ *has a minimal model given by the operad* $\mathsf{Mor(P)}_\infty$ *whose representations are two* P_∞*-algebras and a* P_∞*-morphism between them.*

Corollary 2.5.5. *The 2-colored operad* $\mathsf{Mor(Ass)}$ *has a minimal model given by the 2-colored dg operad* $\mathsf{Mor(Ass)}_\infty$.

Lemma 2.5.6. *Let* $f : B \to C$ *be a map of filtered complexes, where both* B *and* C *are complete and exhaustive. Fix* $r \geq 0$. *Suppose that* $f^r : E^r_{pq}(B) \cong E^r_{pq}(C)$ *for all* p *and* q. *Then* $f : \mathrm{H}(B) \to \mathrm{H}(C)$ *is an isomorphism.*

This result is known as the comparison lemma, and can be found in a textbook on homological algebra, e.g. [We].

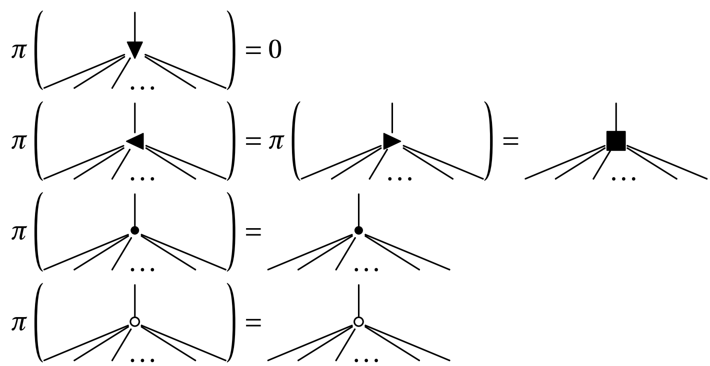
We can now state our result.

Theorem 2.5.7. *The natural projection of operads*

$$\pi : \mathsf{Ho(Ass)}_\infty \twoheadrightarrow \mathsf{Mor(Ass)}_\infty$$

is a quasi-isomorphism.

Proof. We describe the explicit action of the map π on corollas by using the presentation of $\mathcal{H}o(As)_\infty$ and $\mathcal{M}or(As)_\infty$ from theorem 2.5.2 and section 2.4, respectively;



The map π obviously respect the differentials of the operads.

We introduce a filtration on $\mathsf{Ho(Ass)}_\infty(n)$ and $\mathsf{Mor(Ass)}_\infty(n)$ on the number of internal vertices in a tree,

$$F_P\mathsf{Ho(Ass)}_\infty(n) = \{x \in \mathsf{Ho(Ass)}_\infty(n) | \text{number of internal vertices of } x \geq p\}$$

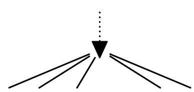
and

$$F_P\mathsf{Mor(Ass)}_\infty(n) = \{x \in \mathsf{Mor(Ass)}_\infty(n) | \text{number of internal vertices of } x \geq p\}.$$

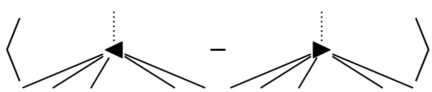
Clearly the differentials in $\mathsf{Ho(Ass)}_\infty$ and $\mathsf{Mor(Ass)}_\infty$ respect these filtrations as the number of vertices can only stay the same or increase when the differentials are applied. Note that the filtrations are both exhaustive and complete, this follows from that the objects in question are finite dimensional for any given n. The induced differential on $E^0_{pq}(\mathsf{Mor(Ass)}_\infty)$ will either map a corolla to zero or increase the number of vertices and therefore

$$E^1_{pq}(\mathsf{Mor(Ass)}_\infty) = \mathrm{H}(E^0_{pq}(\mathsf{Mor(Ass)}_\infty)) = E^0_{pq}(\mathsf{Mor(Ass)}_\infty).$$

On the other hand, in the case of $E^0_{pq}(\mathsf{Ho(Ass)}_\infty)$, we have that the differential will map all trees except those containing a corolla of type



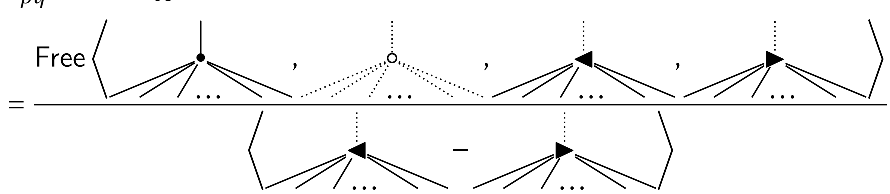
to zero. We get that the image of $\partial^0 : E^0_{pq}(\mathsf{Ho(Ass)}_\infty) \to E^0_{pq}(\mathsf{Ho(Ass)}_\infty)$ will consist of trees (operadically) generated by the difference of corollas;



$\subset \mathsf{Ho(Ass)}_\infty$.

The first page is then determined;

$E^1_{pq}(\mathsf{Ho(Ass)}_\infty)$



$\cong E^0_{pq}(\mathsf{Mor(Ass)}_\infty)$

$= E^1_{pq}(\mathsf{Mor(Ass)}_\infty)$

By Lemma 2.5.6 it follows that π is quasi-isomorphism. □

Corollary 2.5.8. *The 2-colored dg operad* $\mathsf{Ho(Ass)}_\infty$ *is a non-minimal quasi-free model of the 2-colored operad* $\mathsf{Ho(Ass)}$.

Proof. There is a natural projection of operads

$$p : \mathsf{Ho(Ass)}_\infty \twoheadrightarrow \mathsf{Ho(Ass)}$$

defined in the obvious fashion. We determine the cohomology of the operad $\mathsf{Ho(Ass)}$. Let $\mathrm{H}(\mathsf{Ho(Ass)}) = Z/B$, then, if μ^V and μ^W are the multiplications, $f, g : V \to W$ are the dg algebra morphisms and $h : f \sim g$ is the homotopy between them. We will have that $\partial f = \partial g = \partial \mu^V = \partial \mu^W = 0$, so these generators all constitute cycles. The boundaries are generated by $\partial h = f - g$. Hence it's easy to directly calculate the cohomology

$$\mathrm{H}(\mathsf{Ho(Ass)}) = Z/B = \langle f, g, \mu^V, \mu^W \rangle / (f-g) \cong \langle [f], \mu^V, \mu^W \rangle$$

and we see that the cohomology is equal to $\mathsf{Mor(Ass)}$.

It follows from the preceding theorem that

$$\mathrm{H}(\mathsf{Ho(Ass)}_\infty) \cong \mathsf{Mor(Ass)}$$

We will use this explicit description to prove that the projection is a quasi-isomorphism.

Let $\mu_\bullet^V$ and $\mu_\bullet^W$ be the parts of the operad corresponding to A_∞ structures in $\mathsf{Ho(Ass)}_\infty$. The projection p will send the binary parts to the associative products, μ^V and μ^W, and all higher multiplications to 0. The induced map on cohomology will then map the cohomology classes $[\mu_2^V]$ and $[\mu_2^W]$ to the cohomology classes μ_V and μ_W. Let $f_\bullet$ and $g_\bullet$ denote the operations corresponding to A_∞ morphism in $\mathsf{Ho(Ass)}_\infty$. The projection p will send their linear parts to f and g and all higher morphism to 0. The induced map on cohomology will then map the class $[f_1]\,(= [g_1])$ to $[f]\,(= [g])$. We conclude that p is a quasi-isomorphism of 2-colored dg operads. □

Corollary 2.5.9. *The 2-colored dg operad* $\mathsf{Ho(Ass)}_\infty$ *is a non-minimal quasi-free model of the 2-colored operad* $\mathsf{Mor(Ass)}$.

Proof. The model inducing quasi-isomorphism comes from the map

$$\tilde{p} : \mathsf{Ho(Ass)}_\infty \twoheadrightarrow \mathsf{Mor(Ass)},$$

which is given by post-composing the map π from Theorem 2.5.7 with the natural projection onto cohomology classes;

$$\mathsf{Ho(Ass)}_\infty \twoheadrightarrow \mathsf{Mor(Ass)}_\infty) \cong \mathsf{Mor(Ass)}.$$

□

3. Deformation quantization of quasi-Poisson structures

3.1 Operads of Graphs

3.1.1. The Lie algebra of Polyvector fields Let V be a d-dimensional graded vector space, and let $\mathcal{O}_V = \prod_{n\geq 0} \odot^n V^*$ be the commutative algebra of formal power series functions on V. Where we have defined the components of the graded dual V^* as $(V^*)^{(-n)} := \mathrm{Hom}(V^{(n)}, \mathbb{K})$.

Now let V_∞ be a infinite-dimensional graded vector space with infinite basis $\{e_1, e_2, \ldots\}$ and let V_k be the span of the k first basis-vectors,

$$V_k = \mathbb{K}\langle e_1, \ldots, e_k\rangle.$$

We define the graded-commutative algebra of power series on V_k as

$$\mathcal{O}_k := \prod_{n\geq 0} \odot^n V_k^*$$

There is a chain of injections of formal power series algebras,

$$\ldots \longrightarrow \mathcal{O}_n \longrightarrow \mathcal{O}_{n+1} \longrightarrow \mathcal{O}_{n+2} \longrightarrow \ldots$$

and we denote the associated *direct* limit by

$$\mathcal{O}_\infty := \lim_{n\to\infty} \mathcal{O}_n.$$

Suppose that $\{x^1, x^2, \ldots\}$ is the associated dual basis of V_∞^*, $|x^i| = -|e_i|$. The algebra $\mathcal{O}_\infty$ has the property that any given formal power series $f \in \mathcal{O}_\infty$ is expressible as a power series in a finite subset of $\{x^1, x^2, \ldots\}$, i.e. there exist some k such that $f \in \mathcal{O}_k$.

For V_∞, an infinite dimensional graded vector space, we define the polyvector fields $\mathcal{T}_{poly}(V_\infty)$ as the following product of spaces

$$\mathcal{T}_{poly}(V_\infty) := \prod_{m\geq 0} \mathrm{Hom}\left(\odot^m (V_\infty[-1]), \mathcal{O}_\infty\right)$$

Let ψ_i be associated basis in the shifted space $V_\infty[-1]$ with $|\psi_i|-|e_i| = |\psi_i|+|x^i| = 1$. We can explicitly describe $\mathcal{T}_{poly}(V_\infty)$ as the subset of

$$\mathbb{K}[[x^1, x^2, \ldots, \psi_1, \psi_2, \ldots]]$$

consisting of formal power series where the coefficient for each monomial in $\{\psi_1, \psi_2, \ldots\}$ is a power series in $\mathcal{O}_k$ for some k, i.e. a power series in a finite subset of $\{x^1, x^2, \ldots\}$. The space $\mathcal{T}_{poly}(V_\infty)$ is endowed with the *Schouten-Nijenhuis bracket*;

$$[f_1, f_2]_{SN} = \sum_{i=1}^{n} \frac{f_1 \overleftarrow{\partial}}{\partial \psi_i} \frac{\overrightarrow{\partial} f_2}{\partial x^i} - (-1)^{(|f_1|-1)(|f_2|-1)} \frac{f_2 \overleftarrow{\partial}}{\partial \psi_i} \frac{\overrightarrow{\partial} f_1}{\partial x^i}$$

This operation is of degree -1 and defines a Lie bracket on $\mathcal{T}_{poly}(V_\infty)[1]$.

3.1.2. Operads of graphs and their representations in $\mathcal{T}_{poly}(V)$

The operad $\mathsf{Gra}^{\circlearrowleft}$ and its representation in $\mathcal{T}_{poly}(V)$ when $\dim V < \infty$

We make some minor notational changes but otherwise follow the conventions in [Wi1; Wi2].

Let $\mathsf{gra}_{N,l}$ be the set of directed graphs G with N vertices, $Vert(G)$, ordered with $[N] = \{1, 2, \ldots, N\}$ and l directed edges , $Edge(G)$, ordered by $[l] = \{1, 2, \ldots, l\}$ The group $\mathbb{S}_l$ acts on a graph by reordering the edges. We use this set of graphs to generate an $\mathbb{S}$-module $\{\mathsf{Gra}_n^{\circlearrowleft}(N)\}_{N\geq 1}$,

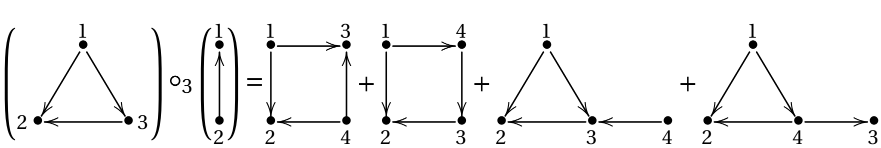
$$\mathsf{Gra}_n^{\circlearrowleft}(N) = \begin{cases} \bigoplus_{l\geq 0}(\mathbb{K}\langle \mathsf{gra}_{N,l}\rangle \otimes_{\mathbb{S}_l} \mathrm{Sgn}_l)[l(n-1)] & n \in 2\mathbb{Z} \\ \bigoplus_{l\geq 0}(\mathbb{K}\langle \mathsf{gra}_{N,l}\rangle \otimes_{\mathbb{S}_l} \mathrm{Sgn}_2^{\otimes l})[l(n-1)] & n \in 2\mathbb{Z}+1 \end{cases}$$

The $\mathbb{S}_N$-action on $\mathsf{Gra}_n^{\circlearrowleft}(N)$ is given by permuting the vertex labels.

The described $\mathbb{S}$-module can be given the structure of an operad; the *directed graphs operad*. The (partial) operadic composition in $\mathsf{Gra}_2^{\circlearrowleft}$ is given by a simple graph substitution scheme; The i-th composition $\Gamma_1 \circ_i \Gamma_2$ is defined to be the sum of graphs given by deleting the i-th vertex in Γ_1 and substituting in its place the graph Γ_2 and finally summing all the ways of reconnecting the edges indecent to the eliminated vertex to the graph Γ_2.

We present the following calculation of the operadic composition as an example of how it works.

Example 3.1.1. *A simple example of a partial composition:*



For a finite dimensional V we have a representation of the operad $\mathsf{Gra}_2^{\circlearrowleft}$ in $\mathcal{T}_{poly}(V)$,

$$\rho : \mathsf{Gra}_2^{\circlearrowleft} \longrightarrow \mathsf{End}_{\mathcal{T}_{poly}(V)}.$$

To each $\Gamma \in \mathsf{Gra}_2^{\circlearrowleft}(N)$ we associate a map

$$\rho(\Gamma) := \Phi_\Gamma \in \mathsf{End}_{\mathcal{T}_{poly}(V)}(N) = \mathrm{Hom}(\mathcal{T}_{poly}(V)^{\otimes N}, \mathcal{T}_{poly}(V)).$$

The map Φ_Γ is defined as the composition of two maps, $\mu \circ \phi$, where μ is just the regular multiplication map in the graded commutative algebra and where $\phi = \prod_{e \in Edge(\Gamma)} \Delta_e$, the product is taken over the edges in their associated ordering. The map Δ_e is defined on an edge $e = \overset{i}{\bullet}\!\longrightarrow\!\overset{j}{\bullet}$ as follows

$$\Delta_e = \sum_\alpha \mathrm{id}^{\otimes i-1} \otimes \frac{\partial}{\partial x^\alpha} \otimes \mathrm{id}^{\otimes j-i-1} \otimes \frac{\partial}{\partial \psi_\alpha} \otimes \mathrm{id}^{\otimes n-j} + \\ \mathrm{id}^{\otimes i-1} \otimes \frac{\partial}{\partial \psi_\alpha} \otimes \mathrm{id}^{\otimes j-i-1} \otimes \frac{\partial}{\partial x^\alpha} \otimes \mathrm{id}^{\otimes n-j}$$

The operad $\mathsf{Gra}_2^{\uparrow}$ and its representation in $\mathcal{T}_{poly}(V)$ for V any dimension

For each $N \geq 1$ we define the sub-$\mathbb{S}_N$-module $\mathsf{Gra}_n^{\uparrow}(N) \subset \mathsf{Gra}_n^{\circlearrowleft}(N)$ as the subspace spanned by the set of graphs which don't contain oriented cycles of directed edges. We denote the set of such graphs that have V vertices and E edges by $\mathsf{gra}_{V,E}^{\uparrow}$. As in the case of the directed graphs, the action of the symmetric groups on $\mathsf{Gra}_n^{\uparrow}$ is given by permutation of the vertex-labels. In a completely analogously manner, the $\mathbb{S}$-module $\mathsf{Gra}_2^{\uparrow}$ can be given the structure of an operad called the *oriented graphs operad*. The composition in $\mathsf{Gra}_2^{\uparrow}$ is the same one defined for $\mathsf{Gra}_2^{\circlearrowleft}$ but restricted to graphs without oriented cycles of directed edges, and as the composition preserves this property, we have a well-defined suboperad.

For an arbitrary vector space V (not necessarily finite dimensional), we can define a representation of $\mathsf{Gra}_2^{\uparrow}$ in $\mathcal{T}_{poly}(V)$. The representation

$$\rho^{\uparrow} : \mathsf{Gra}_2^{\uparrow} \longrightarrow \mathsf{End}_{\mathcal{T}_{poly}(V)}$$

is given in much the same manner as in the directed case with a finite dimensional V; to each $\Gamma \in \mathsf{Gra}_2^{\uparrow}(N)$ we associate the map

$$\rho^{\uparrow}(\Gamma) := \Phi_\Gamma^{\uparrow} \in \mathsf{End}_{\mathcal{T}_{poly}(V)}(N) = \mathrm{Hom}(\mathcal{T}_{poly}(V)^{\otimes N}, \mathcal{T}_{poly}(V)).$$

Where we define the map $\Phi_\Gamma^{\uparrow}$ as the composition of two maps, $\mu \circ \phi$, where μ is just the regular multiplication map in the graded commutative algebra and where $\phi^{\uparrow} = \prod_{e\in Edge(\Gamma)} \Delta_e^{\uparrow}$, the product is taken over the edges in their associated ordering. The map $\Delta_e^{\uparrow}$ is defined on an edge $e = \overset{i}{\bullet}\!\!\longrightarrow\!\!\overset{j}{\bullet}$ as follows

$$\Delta_e^{\uparrow} = \sum_{\alpha} \mathrm{id}^{\otimes i-1} \otimes \frac{\partial}{\partial x^{\alpha}} \otimes \mathrm{id}^{\otimes j-i-1} \otimes \frac{\partial}{\partial \psi_{\alpha}} \otimes \mathrm{id}^{\otimes n-j}.$$

The difference between Δ_e and $\Delta_e^{\uparrow}$ is that we don't symmetrize to forget the direction of the edge e.

The undirected graphs operad Gra

For completeness and for the purpose of stating some known results, we also want to consider an operad of undirected graphs Gra_n. It is defined along the same lines as $\mathsf{Gra}_n^{\circlearrowleft}$. The generating set of graphs $\mathsf{gra}_{N,l}$ is replaced by set of undirected graphs $\mathsf{ugra}_{N,l}$. To be precise, $\mathsf{ugra}_{N,l}$ is the set of *undirected* graphs G with N vertices, $Vert(G)$, ordered with $[N] = \{1,2,\dots,N\}$ and l edges , $Edge(G)$, ordered by $[l] = \{1,2,\dots,l\}$.

The operadic composition is defined in the exact same way as for $\mathsf{Gra}_n^{\circlearrowleft}$ and $\mathsf{Gra}_n^{\uparrow}$; the lack of directed edges is of no consequence.

Remark 3.1.2. *There is map of operads* $\mathsf{Gra}_n \longrightarrow \mathsf{Gra}_n^{\circlearrowleft}$ *defined on a graph G by sending it to the sum of all the ways to add directions on the edges of G.*

3.1.3. The Kontsevich graph complex and its oriented version. Let

$$f^{\uparrow} : \mathsf{Lie}\{n-1\} \longrightarrow \mathsf{Gra}_n^{\uparrow}$$

be the map of operads given by

$$f^{\uparrow}\left(\begin{smallmatrix} & | & \\ & \bullet & \\ / & & \backslash \end{smallmatrix} \right) = \overset{1}{\bullet}\!\!\longrightarrow\!\!\overset{2}{\bullet} + \overset{1}{\bullet}\!\!\longleftarrow\!\!\overset{2}{\bullet}$$

From this we define the map $f^{\circlearrowleft} := i \circ f^{\uparrow}$ where i is the natural inclusion $\mathsf{Gra}_n^{\uparrow} \hookrightarrow \mathsf{Gra}_n^{\circlearrowleft}$. For the undirected graphs operad we define the map $f : \mathsf{Lie}\{n-1\} \longrightarrow \mathsf{Gra}_n$ as

$$f\left(\begin{smallmatrix} & | & \\ & \bullet & \\ / & & \backslash \end{smallmatrix} \right) = \overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}$$

We shall define particular dg Lie algebras, called *graph complexes*, by considering the associated deformation complex to $f^{\circlearrowleft}$, $f^{\uparrow}$ and f. A, for this text, relevant dictionary follows:

- The *full directed graph complex*: $\mathsf{fGC}_n^{\circlearrowleft} := \mathsf{Def}(\mathsf{Lie}\{n-1\} \xrightarrow{f} \mathsf{Gra}_n^{\circlearrowleft})$.

- The *full oriented graph complex*: $\mathsf{fGC}_n^{\uparrow} := \mathsf{Def}(\mathsf{Lie}\{n-1\} \xrightarrow{f^{\uparrow}} \mathsf{Gra}_n^{\uparrow})$.
- The *full undirected graph complex*: $\mathsf{fGC}_n := \mathsf{Def}(\mathsf{Lie}\{n-1\} \xrightarrow{f} \mathsf{Gra}_n)$, This chain complex is also known as the (full) Kontsevich graph complex.

Relative to these we define the following subcomplexes

- $\mathsf{fcGC}_n^{\circlearrowleft}$ which is spanned by directed graphs that are connected.
- $\mathsf{cGC}_n^{\circlearrowleft}$ which is spanned by directed and connected graphs that contain at least one trivalent vertex and all other vertices are at least bivalent.
- $\mathsf{GC}_n^{\circlearrowleft}$ which is spanned by directed graphs that are connected and at least bivalent.
- $\mathsf{fcGC}_n^{\uparrow}$ which is spanned by directed graphs that are connected and does not contain any oriented cycles of edges.
- $\mathsf{GC}_n^{\uparrow}$ which is spanned by directed graphs that are connected, at least bivalent and does not contain any oriented cycles of edges.
- fcGC_n which is spanned by undirected graphs that are connected.
- $\mathsf{GC}_n^{\uparrow}$ which is spanned by undirected graphs that are connected and at least trivalent.

Proposition 3.1.3. *There is a natural identification of complexes coming from splitting a graph into a product of its connected components*

$$\mathsf{fGC}_n^{\circlearrowleft} = S^+(\mathsf{fcGC}_n^{\circlearrowleft}[-n])[n]$$
$$\mathsf{fGC}_n^{\uparrow} = S^+(\mathsf{fcGC}_n^{\uparrow}[-n])[n]$$
$$\mathsf{fGC}_n = S^+(\mathsf{fcGC}_n[-n])[n]$$

where S^+ denotes the non-unital symmetric algebra functor.

For the case of the undirected graph complex, M. Kontsevich and T. Willwacher has shown how its cohomology is related to that of the connected version.

Proposition 3.1.4 ([Wil]). GC_n *is a sub-dg Lie algebra. The cohomology satisfies*

$$\mathrm{H}^\bullet(\mathsf{fcGC}_n) \cong \mathrm{H}^\bullet(\mathsf{GC}_n) \oplus \bigoplus_{\substack{j\geq 1 \\ j \underset{4}{\equiv} 2n+1}} \mathbb{K}[n-j].$$

The class $\mathbb{K}(n-j]$ is represented by a loop with j edges.

The directed graph complex is closely related to the undirected graph complex as the following proposition demonstrates.

Proposition 3.1.5 ([Wi1])**.** *There is an explicit quasi-isomorphism*

$$q : \mathsf{GC}_n \longrightarrow \mathsf{cGC}_n^{\circlearrowleft}$$

induced by the map sending each graph to the sum of directed graphs where each edge is taken in both directions.

For $n = 2$ the directed and undirected graph complexes have had their zeroth cohomology calculated by T. Willwacher. The result is very interesting from several points of view.

Theorem 3.1.6 ([Wi1])**.** *The zeroth cohomology of* $\mathsf{cGC}_2^{\circlearrowleft}$ *is the Grothendieck-Teichmüller Lie algebra,*

$$\mathrm{H}^0(\mathsf{cGC}_2^{\circlearrowleft}) \cong \mathfrak{grt}_1 \cong \mathrm{H}^0(\mathsf{GC}_2).$$

It has also been discovered by T. Willwacher that the undirected graph complex is a related to the oriented graph complex but with a shifted degree;

Theorem 3.1.7 ([Wi2])**.** *The cohomology of the oriented graph complex* $\mathsf{GC}_n^{\uparrow}$ *is isomorphic to the cohomology of the connected undirected graph complex* fcGC_{n-1}

$$\mathrm{H}^\bullet(\mathsf{GC}_n^{\uparrow}) \cong \mathrm{H}^\bullet(\mathsf{fcGC}_{n-1}) = \mathrm{H}^\bullet(\mathsf{GC}_{n-1}) \oplus \bigoplus_{\substack{j\geq 1\\ j \equiv_4 2n+1}} \mathbb{K}[n-j].$$

An important open problem related to the graph complex GC_2 is to determine its first cohomology group. The conjecture, named the Drinfeld-Kontsevich conjecture by T. Willwacher, is that the first cohomology group is zero.

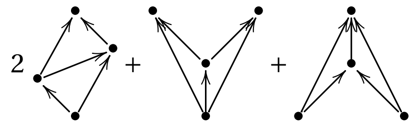
Conjecture 3.1.8 (**Drinfeld-Kontsevich**)**.** *The cohomology group* $\mathrm{H}^1(\mathsf{GC}_2)$ *is trivial.*

In contrast to the open problem mentioned above we have the following result.

Theorem 3.1.9 ([Wi2])**.** *The first cohomology group of* $\mathsf{GC}_2^{\uparrow}$ *is one-dimensional*

$$\mathrm{H}^1(\mathsf{GC}_2^{\uparrow}) \cong \mathrm{H}^1(\mathsf{fcGC}_1)$$

Remark 3.1.10. *The class that spans* $\mathrm{H}^1(\mathsf{GC}_2^{\uparrow})$ *in* $\mathsf{GC}_2^{\uparrow}$ *is given by the following linear combination discovered by B. Shoikhet [Sh]*



Remark 3.1.11. *In [Wi2] T. Willwacher notes that the zeroth cohomology of the oriented graph complex for* $n = 2$ *is trivial;*

$$\mathrm{H}^0(\mathsf{fcGC}_2^{\uparrow}) = 0.$$

3.1.4. Universal deformations of Schouten-Nijenhuis bracket For any Lie algebra $(\mathfrak{g}, [,]) = \xi : \mathsf{Lie} \longrightarrow \mathsf{End}_{\mathfrak{g}}$, the deformation complex of $\mathfrak{g}$ is given by the operadic deformation complex of the morphism ρ;

$$\mathsf{Def}\left(\mathsf{Lie} \xrightarrow{\xi} \mathsf{End}_{\mathfrak{g}}\right) = \prod_{n\geq 2} \mathrm{Hom}(\wedge^{\bullet}\mathfrak{g}, \mathfrak{g})[1-n].$$

The Maurer-Cartan elements of this Lie algebra determine Lie_{∞}-structures on $\mathfrak{g}$. If we, instead, assume that the bracket has degree -1 then the structure is determined by a map

$$\mathsf{Lie}\{1\} \longrightarrow \mathsf{End}_{\mathfrak{g}}.$$

We define the dg Lie algebra

$$\mathsf{CE}^{\bullet}(\mathfrak{g}, \mathfrak{g}) = \mathsf{Def}(\mathsf{Lie}\{1\} \longrightarrow \mathsf{End}_{\mathfrak{g}}) = \prod_{n\geq 2} \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})[3-2n].$$

Then the Maurer-Cartan elements of $\mathsf{CE}^{\bullet}(\mathfrak{g}, \mathfrak{g})$ that have degree 2 are exactly $(\mathsf{Lie}\{1\})_{\infty}$ structures on $\mathfrak{g}$.

Consider the special case $\mathfrak{g} = \mathcal{T}_{poly}(V)$ of polyvector fields on a vector space V. The representation $\rho : \mathsf{Gra}^{\uparrow} \longrightarrow \mathsf{End}_{\mathcal{T}_{poly}(V)}$ can be precomposed with the morphism $f^{\uparrow}$ to yield a map

$$\xi := \rho \circ f^{\uparrow} : \mathsf{Lie}\{1\} \longrightarrow \mathsf{End}_{\mathcal{T}_{poly}(V)},$$

determining a $\mathsf{Lie}\{1\}$ structure on $\mathcal{T}_{poly}(V)$ which is exactly the Shouten-Nijenhuis-bracket. The deformation complex of the morphism $\xi = \rho \circ f^{\uparrow}$ is, thus, the deformation complex for this bracket;

$$\mathsf{Def}\left(\mathsf{Lie}\{1\} \xrightarrow{\xi} \mathsf{End}_{\mathcal{T}_{poly}(V)}\right) = \mathsf{CE}^{\bullet}\left(\mathcal{T}_{poly}(V), \mathcal{T}_{poly}(V)\right).$$

Precomposition with $f^{\uparrow}$

$$-\circ f^{\uparrow} : \mathrm{Hom}_{\mathsf{dgOper}}\left(\mathsf{Gra}_2^{\uparrow}, \mathsf{End}_{\mathcal{T}_{poly}(V)}\right) \xrightarrow{-\circ f^{\uparrow}} \mathrm{Hom}_{\mathsf{dgOper}}\left(\mathsf{Lie}\{1\}, \mathsf{End}_{\mathcal{T}_{poly}(V)}\right),$$

is functorial and induces a map of dg Lie algebras of the associated deformation complexes

$$\mathsf{Def}\left(\mathsf{Lie}\{1\} \xrightarrow{f^{\uparrow}} \mathsf{Gra}_2^{\uparrow}\right) \longrightarrow \mathsf{Def}\left(\mathsf{Lie}\{1\} \stackrel{\xi=\rho\circ f^{\uparrow}}{\longrightarrow} \mathsf{End}_{\mathcal{T}_{poly}(V)}\right).$$

Or expressed in the terminology of graph complexes, there is a map

$$\mathsf{fGC}_2^{\uparrow} \longrightarrow \mathsf{CE}^{\bullet}(\mathcal{T}_{poly}(V), \mathcal{T}_{poly}(V)).$$

In a completely analogously manner it can be demonstrated that there is a canonical map

$$\mathsf{fGC}_2^{\circlearrowleft} \longrightarrow \mathsf{CE}^{\bullet}(\mathcal{T}_{poly}(V), \mathcal{T}_{poly}(V))$$

for finite dimensional V.

We can understand $\mathsf{fGC}_2^{\circlearrowleft}$ as a *universal* deformation complex of the Schouten Lie algebra of polyvector fields on vector spaces V, when V is finite dimensional. And likewise, we can understand $\mathsf{fGC}_2^{\uparrow}$ as a *universal* deformation complex of the Schouten Lie algebra of polyvector fields on vector spaces of arbitrary dimension.

We make two remarks, one for the finite dimensional case and one for the arbitrary dimensional case.

- The cohomology group $\mathrm{H}^1(\mathsf{fGC}_2^{\circlearrowleft})$ determines all homotopy non-trivial infinitesimal deformations of the Schouten bracket when V is finite dimensional. The Drinfeld-Kontsevich conjecture would then imply that the Schouten bracket on $\mathcal{T}_{poly}(V)$ is rigid i.e. there are no non-trivial deformations in finite dimension.

- The cohomology group $\mathrm{H}^1(\mathsf{fGC}_2^{\uparrow})$ determines all homotopy non-trivial infinitesimal deformations of the Schouten bracket when V is of arbitrary dimension. By Theorem 3.1.9 the cohomology $\mathrm{H}^1(\mathsf{GC}_2^{\uparrow})$ is one-dimensional and thus there is a unique infinitesimal non-trivial deformation of the Schouten bracket. The full deformation was constructed by Kontsevich-Shoikhet and will be described in the next section.

3.2 Configuration spaces and transcendental methods

3.2.1. The Kontsevich-Shoikhet L_∞ structure on $\mathcal{T}_{poly}(V)$ [1] We repeat the construction of two configuration spaces due to B. Shoikhet [Sh]. The spaces are generalizations of the configuration space C_n which M. Kontsevich used to prove his famous Formality theorem [Kon1].

Definition 3.2.1. *Let $G(n,E)$ be the set of directed and connected graphs Γ with n vertices and E edges such that:*

- *The graph Γ has no oriented cycles of edges.*
- *The vertices of Γ by labeled by $\{1,\dots,n\}$ and let $l(v)$ be the label corresponding to the vertex $v \in V(\Gamma)$.*
- *for every directed edge $v_1 \xrightarrow{e} v_2$ the label increases along the direction of the edge $l(v_2) > l(v_1)$.*

Remark 3.2.2. *The set $G(n,E)$ is a subset of* $\mathsf{gra}_{\mathsf{n,E}}{}^{\uparrow}$.

Let $Conf_n$ denote the configuration space of n distinct points in the complex plane $\mathbb{C}$. Let Call the quotient space of this action C_n.

Let $Conf_A$ stand for the space of injections, $A \hookrightarrow \mathbb{C}$, of a finite non-empty set A into the complex plane and $\widetilde{Conf}_A$ for the space of all possible maps. Let G be the three dimensional Lie group of transformations $\{z \mapsto az+w | a \in \mathbb{R}^+, w \in \mathbb{C}\}$. The group G acts freely on $Conf_A$. We define $C_A = Conf_A/G$ and, for a configuration $p = \{z_i\}_{i\in A} \in Conf_A$, we set,

$$z_c(p) := \frac{1}{\#A}\sum_{i\in A} z_i, \quad |p - z_c(p)| := \sqrt{\sum_{i\in A} |z_i - z_c(p)|^2}.$$

There is a section $s : C_A \longrightarrow Conf_A$ given by on a configuration $p = \{z_i\}_{i\in A}$ as

$$s : p \mapsto \frac{p - z_c(p)}{|p - z_c(p)|}$$

The image of s is denoted by C_A^{st};

$$C_A^{st} = \{p \in Conf_n \mid z_c(p) = 0,\ |p - z_c(p)| = 1\}$$

Similarly we can construct the space $\widetilde{C}_A^{st}$;

$$\widetilde{C}_A^{st} = \{p \in \widetilde{Conf}_A(\mathbb{C}) \mid z_c(p) = 0,\ |p - z_c(p)| = 1\},$$

which is a compact $(2\#A-3)$-dimensional manifold with boundary. The compactification $\overline{C}_\bullet$ can be defined as the closure of an embedding,

$$C_n \xrightarrow{\prod \pi_A} \prod_{\substack{A\subseteq[n]\\ \#A\geq 2}} C_A \xrightarrow{\simeq} \prod_{\substack{A\subseteq[n]\\ \#A\geq 2}} C_A^{st} \hookrightarrow \prod_{\substack{A\subseteq[n]\\ \#A\geq 2}} \widetilde{C}_A^{st}$$

where the product is taken over all possible subsets A of $[n]$ with $\#A \geq 2$, and

$$\begin{array}{cccc} \pi_A : & C_n & \longrightarrow & C_A \\ & p = \{z_i\}_{i\in[n]} & \longmapsto & p_A := \{z_i\}_{i\in A} \end{array}$$

is the natural forgetful map.

Remark 3.2.3. *The face complex of the compactified configuration space $\overline{C}_n$ assemble to the operad* $\mathsf{Lie}\{1\}_\infty$*, as was determined by E. Getzler and J.D.S. Jones in [GJ].*

Relative to an admissible graph we can define a particular subspace of C_n. Let $\Gamma \in G(n,E)$ and define the configuration space of n complex points associated to Γ;

$$C_{n,\Gamma} := \{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid \mathrm{Im}(z_{l(u)} - z_{l(v)}) < 0 \text{ when } (v,u) \text{ is an edge of } \Gamma\}/G$$

where G is the three-dimensional group of transformations mentioned above. It should be clear how to mimic the compactification of $C_\bullet$ to the configuration space $C_{\bullet,\Gamma}$.

For a subgraph $\Gamma' \subset \Gamma$ there is a natural forgetful projection map

$$p_{\Gamma,\Gamma'} : \overline{C}_{n,\Gamma} \longrightarrow \overline{C}_{n',\Gamma'}$$

such that all coordinates and relations not associated to Γ' are forgotten.

Let Γ be an admissible graph and e an edge of Γ with endpoint vertices labeled by i and j. Suppose that the edge is directed from the j-vertex (to the i-vertex). Let Γ_e be the subgraph given by the edge e. To every edge e we associate the 1-form

$$\phi_e = p^*_{\Gamma,\Gamma_e}(d\mathrm{Arg}(z_j - z_i))$$

The labels on the vertices induce an enumeration on the edges; first enumerate the edges outgoing from the vertex labeled by 1 in the order of the end-vertices, then continue to the vertex labeled by 2 and so on.

We define the weight c_Γ as an integral over the configuration space $\overline{C}_{n,\Gamma}$;

$$c_\Gamma = \frac{1}{\pi^{2n-3}} \int_{\overline{C}_{n,\Gamma}} \bigwedge_{e\in E(\Gamma)} \phi_e.$$

We also denote

$$\Omega_\Gamma := \bigwedge_{e\in E(\Gamma)} \phi_e$$

fore future use.

With these definitions and constructions we can describe the deformation of the Schouten bracket that was mentioned in the end of subsection 3.1.4 and also give the $\mathsf{Lie}\{1\}_\infty$ structure it implies the existence of.

3.2.2. Explicit description of the Kontsevich-Shoikhet Lie_∞ structure on $\mathcal{T}_{poly}(V)$ Let us first assume that V is a finite dimensional vector space.

Given a graph $\Gamma \in G(n,E)$ let $in(v)$ denote the set of directed edges ending in v and let $star(v)$ denote the set of directed edges that start in v

Let Γ be graph in the set $G(n,E)$. Let V be a finite dimensional vector space. We define the multilinear map

$$\mathcal{L}_\Gamma : \wedge^n \mathcal{T}_{poly}(V)[1] \longrightarrow \mathcal{T}_{poly}(V)[3-n]$$

by the formula

$$\mathcal{L}_\Gamma(\gamma_1,\dots,\gamma_n) = \sum_{I:E\to\{1,2,\dots,\dim V\}} \mathcal{L}^I_\Gamma(\gamma_1,\dots\gamma_n)$$

where

$$\mathcal{L}^I_\Gamma(\gamma_1,\dots,\gamma_n) = \bigwedge_{v\in V(\Gamma)} \Psi^I_v$$

and the wedge product is taken in the order the vertices of Γ are labeled, Ψ^I_v is defined as the sum

$$\Psi^I_v = \left(\prod_{e\in in(v)} \frac{\partial}{\partial x^{I(e)}}\right)(\gamma_{l(v)}(\wedge_{e\in star(v)} dx^{I(e)}))$$

A result by B. Shoikhet states this construction can be extended to the infinite dimensional case.

Lemma 3.2.4 ([Sh])**.** *Let Γ be a graph without oriented cycles of directed edges. The map $\mathcal{L}_\Gamma$ is well-defined for an infinite-dimensional V.*

Theorem 3.2.5 ([Sh])**.** *The Kontsevich-Shoikhet Lie_∞ structure is defined on polyvectors $\gamma_1,\dots,\gamma_n \in \mathcal{T}_{poly}(V)$ as*

$$\mu_n(\gamma_1,\dots,\gamma_n) = \sum_{\Gamma\in G(n,2n-3)} \sum_{\sigma\in\mathbb{S}_n} (-1)^\sigma c_\Gamma \mathcal{L}_\Gamma(\gamma_{\sigma(1)},\dots,\gamma_{\sigma(n)})$$

where the sign $(-1)^\sigma$ is determined by the rule that $(-1)^{(\gamma_i+1)(\gamma_j+1)}$ appears when γ_i switches place with γ_j.

Remark 3.2.6. *Let $\{\mu_n : \wedge^n \mathcal{T}_{poly}(V)[1] \to \mathcal{T}_{poly}(V)[3-n]\}_{n\geq 1}$ be the Lie_∞ structure defined by γ^{KS}. B. Shoikhet proved that $\mu_{2n+1} = 0$.*

Theorem 3.2.5 can be reinterpreted as the existence of a Maurer-Cartan element in the oriented graph complex of section 3.1.4.

Theorem 3.2.7. *The sum of graphs*

$$\gamma^{KS} = \sum_{n\geq 2} \sum_{\Gamma\in \mathsf{gra}^{\uparrow}_{n,2n-3}} c_\Gamma \cdot \Gamma$$

is a Maurer-Cartan element of $\mathsf{fGC}_2^{\uparrow}$. *Thus* γ^{KS} *determine a* Lie_∞ *structure on* $\mathcal{T}_{poly}(V)[1]$ *which is a deformation of the Schouten bracket.*

Proof. Given an oriented directed graph Γ with n vertices and $2n-4$ edges, i.e.e an element $\Gamma \in \mathsf{gra}^{\uparrow}_{n,2n-4}$. A subset of vertices $A \subset Vert(\Gamma) \simeq [n]$ is called *admissible* if $2 \leq \#A \leq n-1$ and the associated subgraph Γ_A (which is by definition has vertices A together with all edges between them inherited from Γ) belongs to $\mathsf{gra}^{\uparrow}_{\#A,2\#A-3}$. Note that in this case the quotient graph Γ/Γ_A, that is the graph obtained from Γ by shrinking all vertices and edges of the subgraph Γ_A into a single new vertex, belongs to $\mathsf{gra}^{\uparrow}_{n-\#A+1,2(n-\#A+1)-3}$.

By the Stokes theorem, for any $\Gamma \in \mathsf{gra}^{\uparrow}_{n,2n-4}$, we have using the fact that the differential forms Ω_Γ are closed,

$$\begin{aligned}
0 &= \int_{\overline{C}_{n,\Gamma}} d\Omega_\Gamma \\
&= \int_{\partial\overline{C}_{n,\Gamma}} \Omega_\Gamma \\
&= \sum_{\substack{A\subsetneq[n]\\ \#A\geq 2}} (-1)^{\sigma_A} \int_{\overline{C}_{n-\#A+1,\Gamma/\Gamma_A}} \Omega_{\Gamma/\Gamma_A} \int_{\overline{C}_{\#A,\Gamma_A}} \Omega_{\Gamma_A} \\
&= \sum_{\substack{A\subset V(\Gamma)\\ A\, is\, admissible}} (-1)^{\sigma_A} c_{\Gamma_A} c_{\Gamma/\Gamma_A}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
[\gamma^{KS},\gamma^{KS}] &= \sum_{n_1,n_2\geq 2} \sum_{\Gamma_1\in \mathsf{gra}^{\uparrow}_{n_1,2n_1-3}} \sum_{\Gamma_2\in \mathsf{gra}^{\uparrow}_{n_2,2n_2-3}} c_{\Gamma_1} c_{\Gamma_2} \cdot [\Gamma_1,\Gamma_2] \\
&= \sum_{n_1,n_2\geq 2} \sum_{\Gamma_1\in \mathsf{gra}^{\uparrow}_{n_1,2n_1-3}} \sum_{\Gamma_2\in \mathsf{gra}^{\uparrow}_{n_2,2n_2-3}} c_{\Gamma_1} c_{\Gamma_2} \cdot \sum_{v\in Vert(\Gamma_1)} \Gamma_1 \bullet_v \Gamma_2
\end{aligned}$$

where $\bullet_v$ means operadic substitution of the graph Γ_2 into the vertex v of the graph Γ_1. This substitution gives a graph with n_1+n_2-1 vertices and $2(n_1+n_2-1)-4$ edges, i.e. an element of $\mathsf{gra}^{\uparrow}_{N,2N-4}$ for $N = n_1+n_2-1$. Hence we can rewrite the above expression as follows,

$$[\gamma^{KS},\gamma^{KS}] = \sum_{N\geq 3} \sum_{\Gamma\in \mathsf{gra}^{\uparrow}_{N,2N-4}} \left(\sum_{\substack{A\subset Vert(\Gamma)\\ A\, is\, admissible}} (-1)^{\sigma_A} c_{\Gamma_A} c_{\Gamma/\Gamma_A} \right) \Gamma$$

$$= \quad 0,$$

which proves the claim.

□

3.2.3. Formal quasi-Poisson structures Let us begin by making some remarks about Maurer-Cartan elements. The Kontsevich-Shoikhet Lie_∞ structure in $\mathcal{T}_{poly}(V)[1]$ involves infinitely many operations so that in order to define a Maurer-Cartan element of the latter one has to introduce formal parameters to assure convergence. Consider a formal power series extension, $\mathcal{T}_{poly}(V)[1][[\lambda]]$, of the space of polyvector fields, λ being a formal parameter (of homological degree zero); the Kontsevich-Shoikhet Lie_∞ operations μ_n extend by $\mathbb{R}[[\lambda]]$ linearity to $\mathcal{T}_{poly}(V)[1][[\lambda]]$ and hence make the latter into a topological Lie_∞ algebra. A degree 1 element $\hat{\pi} \in \mathcal{T}_{poly}(V)[[\lambda]]$ of the form

$$\hat{\pi} = \lambda\pi = \lambda\left(\pi_0 + \lambda\pi_1 + \lambda^2\pi_2 + \ldots\right)$$

for some $\pi \in \mathcal{T}_{poly}(V)[[\lambda]]$ which satisfies an equation

$$0 = \frac{1}{2}\mu_2(\hat{\pi},\hat{\pi}) + \frac{1}{4!}\mu_4(\hat{\pi},\hat{\pi},\hat{\pi},\hat{\pi}) + \ldots + \frac{1}{(2k)!}\mu_{2k}(\hat{\pi},\ldots,\hat{\pi}) + \ldots$$

is called a Maurer-Cartan element of the Kontsevich-Shoikhet Lie_∞ algebra. Notice that this equation is well-define, and can be rewritten in terms of π as follows

$$0 = \frac{1}{2}\mu_2(\pi,\pi) + \frac{\lambda^2}{4!}\mu_4(\pi,\pi,\pi,\pi) + \ldots + \frac{\lambda^{2k-2}}{(2k)!}\mu_{2k}(\pi,\ldots,\pi) + \ldots$$

The equation is clearly invariant under the transformation $\lambda \to -\lambda$ so that it makes sense to look for solutions π which are also invariant under such a transformation, i.e. which are formal power series in λ^2. This class of Maurer-Cartan elements of the the Kontsevich-Shoikhet Lie_∞ algebra are of special interest to us in this work, and we call them *quasi-Poisson* structures. Replacing $\lambda^2 \to \hbar$ we arrive at the following definition.

Definition 3.2.8. *A quasi-Poisson structure on the vector space V is an element $\pi \in \hbar\,\mathcal{T}_{poly}(V)[[\hbar]]$ such that*

$$0 = \frac{1}{2}\mu_2(\pi,\pi) + \frac{\hbar}{4!}\mu_4(\pi,\pi,\pi,\pi) + \ldots + \frac{\hbar^{k-1}}{(2k)!}\mu_{2k}(\pi,\ldots,\pi) + \ldots$$

i.e. an MC-element for the Lie_∞ *algebra* $(\mathcal{T}_{poly}(V)[1],\mu_\bullet)\otimes\mathbb{K}[[\hbar]]$

Remark 3.2.9. *It was proven by Merkulov and Willwacher [MW2] that, for any* finite*-dimensional vector space V, the Kontsevich-Shoikhet* Lie_∞ *algebra* $(\mathcal{T}_{poly}(V)[1], \mu_\bullet)$ *is* Lie_∞ *isomorphic to the standard Schouten algebra* $(\mathcal{T}_{poly}(V)[1], [\ ,\ ]_{SN})$*, but the isomorphism is highly non-trivial and depends on the choice of associator. This result implies that, in finite dimensions, there is a 1-1 correspondence (up to gauge equivalence) between the ordinary Poisson structures and the above quasi-Poisson structures. In* infinite *dimensions, however, these notions become very different. We show in Chapter 4 below, that (finite or infinite-dimensional) quasi-Poisson structures can be quantized by an inductive procedure without using associators.*

Next we recall B. Shoikhet's explicit formulae for the universal deformation quantization of quasi-Poisson structures which use the Kontsevich propagator, and then we offer some new and much simpler formulae which use the standard homogeneous volume form on the circle S^1. Both sets of formulae rely on transcendental methods.

Definition 3.2.10. *A directed graph* Γ *is admissible if satisfies the following:*

- *The graph* Γ *has no oriented cycles of edges.*
- *The vertices of* Γ *are of two types. The first type labeled by* $\{1, 2, \ldots, n\}$ *and the second type labeled by* $\{\bar{1}, \bar{2}, \ldots, \bar{m}\}$ *with* $2n + m \geq 2$.
- *If* $v_1 \xrightarrow{e} v_2$ *is an edge of* Γ *with* v_1 *and* v_2 *being of the first type then* $l(v_1) < l(v_2)$*, where* $l(v)$ *denotes the label of the vertex* v.

Let $G(n, m, E)$ denote the set of admissible graphs with n vertices of the first type, m vertices of the second type and E edges.

Definition 3.2.11. *Let* z_1 *and* z_2 *be two points in the upper half-plane. We define the binary relation* $z_2 \leq_P z_1$ *if the the point* z_2 *is contained in the half-circle* C *with diameter on the real axis and "highest" point at* z_1.

Given an admissible graph Γ we shall define a configuration space of points in the upper half-plane.

Definition 3.2.12. *Let* $Conf_{n,m}$ *be the set of* $n+m$ *distinct points in the upper half-plane with* m *points being on the real line;*

$$Conf_{n,m} = \{(z_1, \ldots, z_n) \in \mathbb{H}^n,\ (x_{\bar{1}}, \ldots, x_{\bar{m}}) \in \mathbb{R}^m |\ z_i \neq z_j, x_{\bar{i}} \neq x_{\bar{j}}\}.$$

Let $\Gamma \in G(n, m, E)$ *be an admissible graph and define the subset* $Conf_{n,m,\Gamma} \subset Conf_{n,m}$ *as the subset of configurations coherent with* Γ;

$$Conf_{n,m,\Gamma} =$$

$$\left\{ (z_1, \ldots, z_n, x_{\bar{1}}, \ldots, x_{\bar{m}}) \in \mathbb{H}^n \times \mathbb{R}^m \ : \ \begin{array}{ll} x_{\bar{1}} < \ldots < x_{\bar{m}} & \textit{always} \\ z_{l(v_1)} \leq_P z_{l(v_2)} & \textit{if there is an edge } v_2 \xrightarrow{e} v_1 \\ x_{l(u_1)} \leq_P z_{l(u_2)} & \textit{if there is an edge } u_2 \xrightarrow{e'} u_1 \end{array} \right\}$$

where u_2, v_2, v_1 are vertices of the first type and u_1 a vertex of the second type. The 2 dimensional group of real affine transformations $G^2 = \{z \mapsto az + b \mid a \in \mathbb{R}_+,\ b \in \mathbb{R}\}$ act on $Conf_{n,m}$. We define $C_{n,m}$ as the quotient $Conf_{n,m}/G^2$. The group action is well defined on the subspace $Conf_{n,m,\Gamma}$ and we can also define $C_{n,m,\Gamma}$ which is the quotient $Conf_{n,m,\Gamma}/G^2$.

In [Kon1] M. Kontsevich defined for the space $C_{n,m}$ a compactification $\overline{C}_{n,m}$ as the closure of the image of the following inclusions

$$\begin{array}{ccccc} C_{n,m} & \longrightarrow & C_{2n+m} & \hookrightarrow & \overline{C}_{2n+m} \\ \{z_1,\ldots,z_n;x_1,\ldots,x_m\} & \longmapsto & \{z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n;x_1,\ldots,x_m\} & & \end{array}$$

For an admissible graph Γ, the configuration space $C_{n,m,\Gamma}$ can be compactified in analogy with the construction above. Let $\overline{C}_{n,m,\Gamma}$ denote the compactification of $C_{n,m,\Gamma}$.n

Definition 3.2.13 ([Kon1; Sh])**.** *Let Γ_0 denote the 2-vertex graph with one directed edge; $\Gamma_0 : 1 \xrightarrow{e} 2$. A modified angle function is a map θ of $\overline{C}_{2,0,\Gamma_0}$ to a unit circle S^1 such that θ is the angle varying from the $-\pi$ to 0 on the upper half-circle, and θ contracts the two other upper boundary components to a point $0 \in S^1$.*

Example 3.2.14. *An example of a modified angle function is given by the doubled Konsevich's harmonic angle:*

$$f(z,w) = \frac{1}{i}\mathrm{Log}\frac{(z-w)(z-\bar{w})}{(\bar{z}-w)(\bar{z}-\bar{w})}$$

Given a subgraph $\Gamma' \subset \Gamma$ there is a natural projection map

$$p_{\Gamma,\Gamma'} : \overline{C}_{n,m,\Gamma} \longrightarrow \overline{C}_{n',m',\Gamma'}.$$

Definition 3.2.15. *Let e be an edge of an admissible graph Γ. Let θ be a modified angle function. Define the 1-form ϕ_e as the pullback of $d\theta$ along p_{Γ,Γ_0};*

$$\phi_e = p^*_{\Gamma,\Gamma_0}(d\theta)$$

The weight W_Γ associated to the space $\overline{C}_{n,m,\Gamma}$ is given by the integral

$$W_\Gamma = \frac{1}{\pi^{2n+m-2}}\int_{\overline{C}_{n,m,\Gamma}} \bigwedge_{e\in E(\Gamma)} \phi_e$$

3.2.4. Description of boundary strata of $\overline{C}_{n,\Gamma}$ The only boundary strata that will be of consequence is those of codimension 1. The codimension 1 boundary strata of $\overline{C}_{n,\Gamma}$ fall into two types.

- Type 1: some proper subset of points $S = \{p_{i_1}, \ldots, p_{i_s}\},\ s \geq 2$ approach each other. Let Γ_1 be the restriction of the graph Γ to the vertices corresponding to the set S and let Γ_2 be the graph obtained from contracting the subgraph Γ_1 to a vertex. The boundary strata of this configuration is then given as the product $C_{n-S+1,\Gamma_2} \times C_{s,\Gamma_1}$.

- Type 2: some point q with an edge $p \longrightarrow q$ approaches the horizontal line through p.

3.2.5. Description of boundary strata of $\overline{C}_{n,m,\Gamma}$ with hyperbolic height order The codimension 1 boundary strata of $\overline{C}_{n,m,\Gamma}$ comes in three types:

- Type 1: Some points $p_{i_1}, p_{i_2}, \ldots, p_{i_s} \in \mathbb{H}$ $s \geq 2$ approach each other in such a way that they stay inside the appropriate geodesic half-circles. This corresponds to a boundary strata isomorphic to the product

  $$C_{n-s+1,m,\Gamma_2} \times C_{s,\Gamma_1},$$

  where Γ_1 is the subgraph given by restricting Γ to the subset of vertices labeled by $\{i_1, \ldots i_s\}$ and Γ_2 is the graph given by contracting the subgraph Γ_1 into a new vertex.

- Type 2: Some points $p_{i_1}, \ldots, p_{i_s} \in \mathbb{H}$ and some points $q_{j_1}, \ldots, q_{j_r} \in \mathbb{R}$ such that $2m+n-1 \geq 2s+r \geq 2$ approach each other and a point on the real line in such a that the points stay inside the appropriate geodesic half-circles. This boundary strata is isomorphic to the product $C_{s,r,\Gamma_1} \times C_{n-r+1,m-r+1,\Gamma_2}$ where Γ_1 is the subgraph induced by the restriction to the $i_1, \ldots i_s$ labeled vertices and Γ_2 is the graph attained by contracting the subgraph Γ_1 into a new vertex of the second type.

- some point p with an incident edge $q \longrightarrow p$ approaches the geodesic half-circle of a point q.

3.2.6. The Lie_∞ morphism F. We use this subsection to remind the reader of the definition of the map

$$\mathcal{U}_\Gamma : \mathcal{T}_{poly}(V)^{\otimes |Vert(\Gamma)|} \longrightarrow \mathcal{D}_{poly}(V)$$

which was given by M. Kontsevich [Kon1].

For any admissible graph Γ with n vertices of the first type, m vertices of the second type and $2n+m-2+l$ edges $l \in \mathbb{Z}$, we define the linear map

$$\mathcal{U}_\Gamma : \mathcal{T}_{poly}(V)^{\otimes n} \to \mathcal{D}_{poly}(V)[1+l-n].$$

This map has only one non-zero graded component $\mathcal{U}_{\Gamma_{(k_1,\dots,k_n)}}$ where k_i+1 is the number of outgoing edges from the i-labeled vertex in Γ. For $l=0$ the map $\mathcal{U}_\Gamma$ becomes a pre-Lie_∞-morphism after anti-symmetrization. Let $\gamma_1,\dots,\gamma_n$ be polyvector fields of degree $k_1+1,\dots,k_n+1$ respectively and let $f_1,\dots,f_m$ be functions on V. We define the polydifferential operator

$$(f_1,\dots,f_m) \mapsto \Phi,$$

where the formula for Φ is given as follows

$$\Phi := (\mathcal{U}_\Gamma(\gamma_1,\dots,\gamma_n))(f_1,\dots,f_m)$$

where Φ is sum over all configurations of indices running from 1 to $\dim V$ labeled by $Edge(\Gamma)$;

$$\sum_{I:Edge(\Gamma)\to\{1,\dots,\dim V\}} \Phi_I$$

The term Φ_I is the product over all $n+m$ vertices of Γ of certain partial derivatives of functions g_j and of coefficients of the polyvectors γ_i :

- For each vertex $i \in \{1,2,\dots,n\}$ of the first type we associate the function $\psi_i \in V^*$ which is a coefficient of the polyvector field γ_i :

  $$\psi_i = \langle \gamma_1, dx^{I(e_i^1)} \otimes \dots \otimes dx^{I(e_i^{k_i+1})} \rangle$$

  Here we use the identification of polyvector fields and skew-symmetric tensor fields;

  $$\xi_1 \wedge \dots \wedge \xi_{k+1} \mapsto \sum_{\sigma\in\mathbb{S}_{k+1}} \operatorname{sgn}(\sigma)\ \xi_{\sigma_i} \otimes \dots \otimes \xi_{\sigma_{k+1}}$$

- For each vertex $\overline{j} \in \{\overline{1},\dots,\overline{m}\}$ of the second type the associated function $\psi_{\overline{j}}$ is the function f_j.

At each vertex of Γ we "put" a function ψ_i or $\psi_{\overline{j}}$. At each edge $e \in Edge(\Gamma)$, there correspond coordinates $I(e)$ in V. In the next step we put into each vertex v a certain partial derivative of the function ψ_v;

$$\left(\prod_{e\in In(v)\subset Edge(\Gamma)} \frac{\partial}{\partial x^{I(e)}}\right)\psi_v.$$

The product of these functions running over the set of vertices of v defines Φ_I.

The complete formality morphism constructed by M. Kontsevich is linear combination of $\mathcal{U}_\Gamma$ with coefficients given by the weights c_Γ;

$$\mathcal{U}_n : \mathcal{T}_{poly}(V)^{\otimes n} \longrightarrow \mathcal{D}_{poly}(V)$$

where

$$U_n = \sum_{m\geq 0} \sum_{\Gamma\in G_{n,m}} c_\Gamma \cdot \mathcal{U}_\Gamma$$

and the second sum is taken over all admissible graphs with n vertices of the first type m vertices of the second type and $2n+m-2$ edges.

3.3 Transcendental quantization formula from dArg-propagator

We will give a new proof of B.Shoikets theorem for infinite dimensional deformation quantization by using the dArg-propagator and the order on $\mathbb{C}$ given by imaginary part.

Let Γ be an admissible graph with n vertices of type I, m vertices of type II and edge set E. Define the configuration space $C_{n,m,\Gamma}$ as the subset of configurations $(z_1,\dots,z_n;x_1,\dots,x_m) \in C_{n,m}$ such that $\mathrm{Im}\ z_i > \mathrm{Im}\ z_j$, whenever there is an edge $e = \overset{j}{\bullet}\!\longrightarrow\!\overset{i}{\bullet} \in E$.

Remark 3.3.1. *This definition is similar to the one given by B. Shoikhet [Sh]. Our definition is a slight simplification as we don't have to consider any intricacies of hyperbolic geometry.*

Recall the definition of the weights c_Γ

$$c_\Gamma = \int_{\overline{C}_{p,\Gamma}} \bigwedge_{e\in E(\Gamma)} \phi_e$$

where $\phi_e = \frac{d\mathrm{Arg}(z_i - z_j)}{\pi}$ for an edge $e = \overset{j}{\bullet}\!\longrightarrow\!\overset{i}{\bullet} \in E(\Gamma)$.

We define another weight coming from the configuration space $\overline{C}_{n,m,\Gamma}$;

$$w_\Gamma = \int_{\overline{C}_{n,m,\Gamma}} \bigwedge_{e\in E(\Gamma)} \phi_e$$

with $\phi_e = \frac{d\mathrm{Arg}(z_i - z_j)}{\pi}$ as above.

3.3.1. Description of boundary strata of $\overline{C}_{n,m,\Gamma}$ with height order by imaginary part The codimension 1 boundary strata of $\overline{C}_{n,m,\Gamma}$ comes in three types:

- Type 1: Some points $p_{i_1}, p_{i_2}, \ldots, p_{i_s} \in \mathbb{H}$ with $s \geq 2$ approach each other in such a way that the relation $\operatorname{Im} p_{i_a} > \operatorname{Im} p_{i_b}$ for every edge $\overset{i_a}{\bullet} \longrightarrow \overset{i_b}{\bullet}$. This corresponds to a boundary strata isomorphic to the product

$$C_{n-s+1,m,\Gamma_2} \times C_{s,\Gamma_1},$$

  where Γ_1 is the subgraph given by restricting Γ to the subset of vertices labeled by $\{i_1, \ldots i_s\}$ and Γ_2 is the graph given by contracting the subgraph Γ_1 into a new vertex.

- Type 2: Some points $p_{i_1}, \ldots, p_{i_s} \in \mathbb{H}$ and some points $q_{j_1}, \ldots, q_{j_r} \in \mathbb{R}$ such that $2m + n - 1 \geq 2s + r \geq 2$ approach each other at point on the real line in such that the points in the upper half-plane stay at their relative heights as determined by the directed edges. This boundary strata is isomorphic to the product $C_{s,r,\Gamma_1} \times C_{n-r+1,m-r+1,\Gamma_2}$ where Γ_1 is the subgraph induced by the restriction to the $i_1, \ldots i_s$ labeled vertices and Γ_2 is the graph attained by contracting the subgraph Γ_1 into a new vertex of the second type.

- Type 3: Some point p with an incident edge $q \longrightarrow p$ approaches the line $\{z \in \mathbb{H} \mid \operatorname{Im} z = \operatorname{Im} q\}$.

3.3.2. Infinite dimensional deformation quantization To produce a formality morphism in the infinite dimensional setting B. Shoikhet applies Kontsevich's $\mathcal{U}_\Gamma$ to define the Lie_∞ morphism

$$\mathcal{F}_\bullet : \mathcal{T}_{poly}(V)[1] \longrightarrow \mathcal{D}_{poly}(V)[1],$$

explicitly it's given by a formula which is completely analogous to the one given by M. Kontsevich;

$$\mathcal{F}_n = \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} \omega_\Gamma \cdot \mathcal{U}_\Gamma.$$

Theorem 3.3.2. *[Sh] Let V be a $\mathbb{Z}_{\geq 0}$ graded vector space over $\mathbb{C}$ with finite-dimensional graded-components V^i. Then the maps F_n constitute a Lie_∞ quasi-isomorphism*

$$(\mathcal{T}_{poly}(V)[1], \mu_\bullet) \longrightarrow (\mathcal{D}_{poly}(V)[1], d_H, [-,-]_G).$$

Where the first component is the Hochschild-Kostant-Rosenberg map, $\mu_\bullet$ is the Kontsevich-Shoikhet Lie_∞ structure, d_H Hochschild differential and $[-,-]_G$ the Lie bracket on the Hochschilld complex.

Proof. Let the $S(a,b)$ denote the subgroup of (a,b) shuffles. To show that F is a Lie_∞ morphism we have to demonstrate that the following equation holds for all $n \geq 2$ (The case $n = 1$ is the Hochschild-Kostant-Rosenberg Theorem)

$$\begin{aligned}0 = {} & d_H(F_n(\gamma_1, \dots, \gamma_n)) \\ & + \sum_{\substack{a+b=n \\ a,b\geq 1}} \sum_{\sigma\in S(a,b)} \pm \left[F_a(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(a)}), F_b(\gamma_{\sigma(a+1)}, \dots, \gamma_{\sigma(a+b)})\right]_G \\ & + \sum_{\substack{a+b=n \\ a,b\geq 1}} \sum_{\sigma\in S(a,b)} \pm F_{a+1}(\mu_k(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(a)}), \gamma_{\sigma(a+1)}, \dots, \gamma_{\sigma(a+b)})\end{aligned}$$

We employ the same trick as M. Kontsevich (cf. [Kon1]) and consider $F_0 : \mathcal{T}_{poly}(V)^{\otimes 0} \to \mathcal{D}_{poly}(V)$ defined by $F_0 : 1 \mapsto m_V$ where $m_V \in \mathrm{Hom}(\mathcal{O}_V^{\otimes 2}, \mathcal{O}_V)$ is the ordinary multiplication. We can see F_0 in terms of the polydifferential operator coming from a graph Γ_0 which has 0 vertices of the first type, 2 vertices of the second type and 0 edges. The associated weight ω_{Γ_0} is equal to 1. The above equation can now be given the following equivalent form

$$\begin{aligned}0 = {} & \sum_{\substack{a+b=n \\ a,b\geq 0}} \sum_{\sigma\in S(a,b)} \pm \left[F_a(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(a)}), F_b(\gamma_{\sigma(a+1)}, \dots, \gamma_{\sigma(a+b)})\right]_G \\ & + \sum_{\substack{a+b=n \\ a,b\geq 1}} \sum_{\sigma\in S(a,b)} \pm F_{a+1}(\mu_k(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(a)}), \gamma_{\sigma(a+1)}, \dots, \gamma_{\sigma(a+b)})\end{aligned}$$

We substitute $F_\bullet$ and $\mu_\bullet$ for their definitions as sums of operators given by graphs

$$\begin{aligned} & \sum_{\substack{a+b=n \\ a,b\geq 0 \\ \sigma\in S(a,b)}} \pm \left[\sum_{\substack{\Gamma_1\in G_{a,m} \\ m\geq 0}} \omega_{\Gamma_1} \cdot \mathcal{U}_{\Gamma_1}(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(a)}), \sum_{\substack{\Gamma_2\in G_{b,m} \\ m\geq 0}} \omega_{\Gamma_2} \cdot \mathcal{U}_{\Gamma_2}(\gamma_{\sigma(a+1)}, \dots, \gamma_{\sigma(a+b)}) \right]_G \\ & + \sum_{\substack{a+b=n \\ a,b\geq 1 \\ \sigma\in S(a,b)}} \pm \sum_{\substack{\Gamma_3\in G_{b+1,m} \\ m\geq 0}} \omega_{\Gamma_3} \cdot \mathcal{U}_{\Gamma_3} \left(\sum_{\substack{\Gamma_4\in \mathsf{gra}^{\uparrow}_{a,2a-3} \\ \nu\in\mathbb{S}_k}} \pm c_{\Gamma_4} \cdot \mathcal{L}_{\Gamma_4}(\gamma_{\nu(\sigma(1))} \dots), \gamma_{\sigma(a+1)}, \dots, \gamma_{\sigma(a+b)} \right) \\ = {} & \\ & \sum_{\substack{a+b=n \\ a,b\geq 0 \\ \sigma\in S(a,b)}} \sum_{\substack{\Gamma_1\in G_{a,m} \\ m\geq 0}} \sum_{\substack{\Gamma_2\in G_{b,m} \\ m\geq 0}} \pm \omega_{\Gamma_1} \cdot \omega_{\Gamma_2} \left[\mathcal{U}_{\Gamma_1}(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(a)}), \mathcal{U}_{\Gamma_2}(\gamma_{\sigma(a+1)}, \dots, \gamma_{\sigma(a+b)})\right]_G \\ & + \sum_{\substack{a+b=n \\ a,b\geq 1 \\ \sigma\in S(a,b)}} \sum_{\substack{\Gamma_3\in G_{b+1,m} \\ m\geq 0}} \sum_{\substack{\Gamma_4\in \mathsf{gra}^{\uparrow}_{a,2a-3} \\ \nu\in\mathbb{S}_k}} \pm \omega_{\Gamma_3} \cdot c_{\Gamma_4} \cdot \mathcal{U}_{\Gamma_3} \left(\mathcal{L}_{\Gamma_4}(\gamma_{\nu(\sigma(1))} \dots), \gamma_{\sigma(a+1)}, \dots, \gamma_{\sigma(a+b)}\right)\end{aligned}$$

For a fixed m, this expression is of the form $\sum_\Gamma \alpha_\Gamma \mathcal{U}_\Gamma$ where α_Γ are coefficients coming from the products of weights and where Γ are admissible graphs with n vertices of the first type, m vertices of the second type and $2n+m-3$ edges. Specifically the weights are given as

$$\sum \omega_{\Gamma_1} \cdot \omega_{\Gamma_2} + \sum \omega_{\Gamma_3} \cdot c_{\Gamma_4}$$

The first sum is over pairs of graphs (Γ_1, Γ_2) such that they together have n vertices of type 1 and $m+1$ vertices of type 2. The second sum is over pairs of graphs (Γ_3, Γ_4) such that they together have $n+1$ vertices of type 1 and Γ_3 alone has m vertices type 2 while Γ_4 has no vertices of type 2. For all four graphs the relation $2\#|\text{vertices of type } 1| + \#|\text{vertices of type } 2| - 2 = |\text{edges}|$ is satisfied. The pair of graphs (Γ_1, Γ_2) and (Γ_3, Γ_4) that determine the coefficient for $\mathcal{U}_\Gamma$ are the one that appear in the process of determining the codimension 1 boundary strata for $\overline{C}_{n,m,\Gamma}$. This fact is why the weight α_Γ vanishes. We consider the integral of a closed differential form over the compactified configuration space $\overline{C}_{n,m,\Gamma}$.

By closedness of the form and Stokes' theorem we have that

$$0 = \int_{\overline{C}_{n,m,\Gamma}} d\left(\bigwedge_{e \in E(\Gamma)} \phi_e\right) = \int_{\partial(\overline{C}_{n,m,\Gamma})} \bigwedge_{e \in E(\Gamma)} \phi_e$$

A decomposition of $\partial(\overline{C}_{n,m,\Gamma})$ was given in Subsection 3.3.1, we find that

$$\int_{\partial(\overline{C}_{n,m,\Gamma})} \bigwedge_{e \in E(\Gamma)} \phi_e = \int_{\partial T1} \bigwedge_{e \in E(\Gamma)} \phi_e + \int_{\partial T2} \bigwedge_{e \in E(\Gamma)} \phi_e + \int_{\partial T3} \bigwedge_{e \in E(\Gamma)} \phi_e$$

In fact

$$\int_{\partial T3} \bigwedge_{e \in E(\Gamma)} \phi_e = 0$$

We conclude the calculation

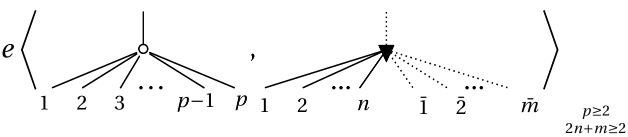
$$\begin{aligned}
0 &= \int_{\overline{C}_{n,m,\Gamma}} d\left(\bigwedge_{e \in E(\Gamma)} \phi_e\right) \\
&\dots \\
&= \int_{\partial T1} \bigwedge_{e \in E(\Gamma)} \phi_e + \int_{\partial T2} \bigwedge_{e \in E(\Gamma)} \phi_e \\
&= \sum_{s=2}^{n} \sum_{(\Gamma_1,\Gamma_2) \in G_1(s,\Gamma)} \int_{C_{n-s+1,m,\Gamma_2} \times C_{s,\Gamma_1}} \bigwedge_{e \in E(\Gamma)} \phi_e \\
&\quad + \sum_{I=2}^{2m+n-1} \sum_{2s+r=I} \sum_{(\Gamma_1,\Gamma_2) \in G_2(s,r,\Gamma)} \int_{C_{s,r,\Gamma_1} \times C_{n-r+1,m-r+1,\Gamma_2}} \bigwedge_{e \in E(\Gamma)} \phi_e \\
&= \sum_{s=2}^{n} \sum_{(\Gamma_1,\Gamma_2) \in G_1(s,\Gamma)} \left(\int_{C_{n-s+1,m,\Gamma_2}} \bigwedge_{e \in E(\Gamma_1)} \phi_e\right) \cdot \left(\int_{C_{s,\Gamma_1}} \bigwedge_{e \in E(\Gamma_2)} \phi_e\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{I=2}^{2m+n-1}\sum_{2s+r=I}\sum_{(\Gamma_1,\Gamma_2)\in G_2(s,r,\Gamma)}\left(\int_{C_{s,r,\Gamma_1}}\bigwedge_{e\in E(\Gamma_1)}\phi_e\right)\cdot\left(\int_{C_{n-r+1,m-r+1,\Gamma_2}}\bigwedge_{e\in E(\Gamma_2)}\phi_e\right)\\
&=\sum_{s=2}^{n}\sum_{(\Gamma_1,\Gamma_2)\in G_1(s,\Gamma)} w_{\Gamma_1}\cdot c_{\Gamma_2}+\sum_{I=2}^{2m+n-1}\sum_{2s+r=I}\sum_{(\Gamma_1,\Gamma_2)\in G_2(s,r,\Gamma)} w_{\Gamma_1}\cdot w_{\Gamma_2}\\
&=\alpha_\Gamma
\end{aligned}
$$

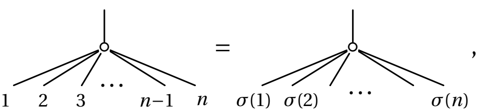
□

3.4 Formality as a map of operads $F:\mathsf{OC}_\infty\longrightarrow\mathsf{KGra}$

3.4.1. Open-closed homotopy algebras The operad OC_∞ is a quasi-free 2-colored dg operad [KS] generated by two types of corollas, of degree $3-2n$ and degree $2-2n-m$, respectively;

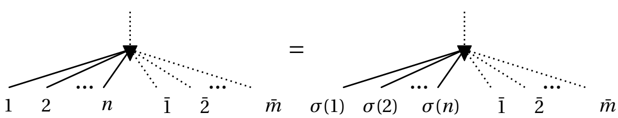
$\mathsf{OC}_\infty = \mathcal{F}ree$ 

The first class of corollas are subject to the relation



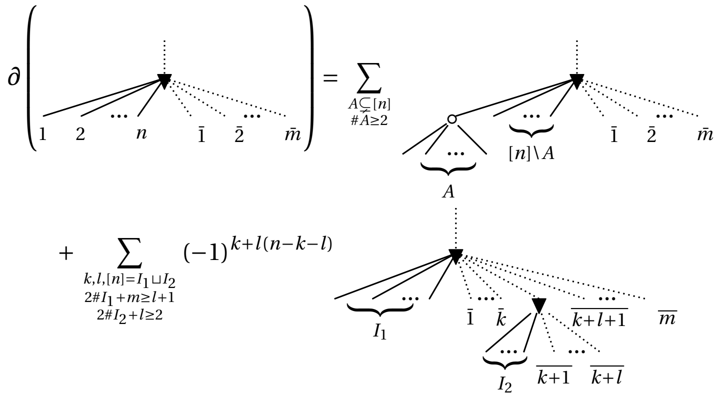
$$, \quad \forall\sigma\in\mathbb{S}_n,\ n\geq 2 \qquad (3.1)$$

with differential defined by the second class of corollas are subject to



$$, \quad 2n+m\geq 2, \forall\ \sigma\in\mathbb{S}_n$$

The action of the differential on these corollas is given by the following formula

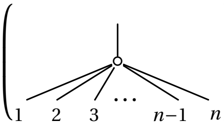


A representation of OC_∞ in a pair of dg vector spaces (X_c, X_o) have been called *open-closed homotopy algebras* or OCHA for short.

It was shown in [H] that representations of OC_∞ are equivalent to degree one codifferentials in the tensor product, $\odot^\bullet(X_c[2]) \bigotimes \otimes^\bullet(X_o[1])$, of the free graded cocommutative coalgebra cogenerated by $X_c[2]$ and the free coalgebra cogenerated by $X_o[1]$.

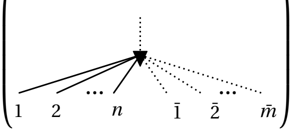
Being a free operad, OC_∞ has the property that any representation, ρ, is uniquely determined by the values on the corollas,

$$\nu_n := \rho\left(\quad \right) \in \mathrm{Hom}(\odot^n X_c, X_c[3-2n]), \quad n \geq 2,$$



and

$$\mu_{n,m} := \rho\left(\quad \right) \in \mathrm{Hom}\left(\bigodot^{n} X_c \otimes \bigotimes^{m} X_o, X_o\right)[2-2n-m],$$



$$2n + m \geq 2$$

which satisfy quadratic relations presented above for the differential ∂.

Let $\mathrm{Coder}(\otimes^\bullet(X_o[1]), [\ ,\ ])$ be the Lie algebra of coderivations of the free coalgebra, $\otimes^\bullet(X_o[1])$, cogenerated by $X_o[1]$. The coderivations are not required to preserve the co-unit so that MC elements in this Lie algebra describe, in general, non-flat A_∞-structures on X_o. Recall that we have an isomorphism of vector spaces,

$$\mathrm{Coder}(\otimes^\bullet(X_o[1])) = \bigoplus_{m \geq 0} \mathrm{Hom}(\otimes^m X_o, X_o)[1-m]$$

The structure of an Open-closed homotopy algebras was reinterpreted by S.A Merkulov [Me3].

Proposition 3.4.1. *An* OC_∞ *structure on a pair of vector spaces* (X_c, X_o);

$$\mathsf{OC}_\infty \longrightarrow \mathsf{End}_{(X_c, X_o)},$$

is equivalent to

1. *A* $\mathsf{Lie}\{1\}_\infty$ *structure* ν *on* X_c; $\{\nu_k : \odot^k X_c \to X_c[3-2k]\}_{k \geq 0}$. *Stated otherwise, it's a* Lie_∞ *structure on* $X_c[1]$.

2. *An* Ass_∞ *structure* μ *on* X_o; $\{\mu_k : \otimes^k X_o \to X_o[2-k]\}$. *The associated MC-element of the Lie algebra* $\mathrm{Coder}(\otimes^\bullet X_o[1])$ *corresponding to* μ *gives a differential* $d_\mu = [\mu, -]$.

3. *A* Lie_∞ *morphism* $F : (X_c[1], \nu) \to (\mathrm{Coder}(\otimes^\bullet X_o[1]), [-,-], d_\mu)$;

$$\left\{F_k : \odot^k X_c \longrightarrow \mathrm{Coder}(\otimes^\bullet X_o[1])[1-2k]\right\}_{k\geq 1}$$

such that the composition

$$\odot^k X_c \xrightarrow{F_k} \mathrm{Coder}(\otimes^\bullet (X_o[1]))[1-2k] \xrightarrow{proj} \mathrm{Hom}(\otimes^m X_o, X_o)[1-m-2k]$$

coincides with $\mu_{n,m}$ *for* $n \geq 1, m \geq 0$.

Remark 3.4.2. *The face complex of Kontsevich's compactified configuration space* $\overline{C}_{n,m}$ *was considered in [Me3]. This face complex can be given the structure of an operad in the category of smooth manifolds, and the associated operad of fundamental chains is isomorphic to the operad* OC_∞.

3.4.2. The operad of Kontsevich graphs. In the framework *stable formality morphisms* developed by Rossi-Willwacher the existence of a formality map is reinterpreted as a morphism of certain 2-colored operads;

$$\mathsf{OC}_\infty \to \mathsf{KGra}$$

We will start with a definition of the 2-colored operad of Kontsevich graphs KGra. The operad is so called because it consists of the type of graphs considered by M. Kontsevich in his original proof of the Formality theorem. The operad KGra is intimately connected to the monochromatic operad $\mathsf{Gra}_2^{\circlearrowleft}$ consisting of directed graphs.

Recall from Definition 3.2.10 that an admissible graph has vertices of two types, and the edge-set is subject to the restriction that no edge may begin in a vertex of the second type. Let the colors of KGra be denoted by o and c (standing for open and closed, respectively). We denote the subspace of operations with n inputs of color c and m inputs with color o and with output in color o or c by $\mathsf{KGra}_c(n,m)$ or $\mathsf{KGra}_o(n,m)$ respectively. Define these subspaces as follows

$$\mathsf{KGra}_c(n,m) = \begin{cases} \mathsf{Gra}_2^{\circlearrowleft}(n) & m = 0 \\ \{0\} & m \geq 1 \end{cases}$$

and

$$\mathsf{KGra}_o(n,m) = \bigoplus_{k\geq 0} (\mathbb{K}\langle G(n,m,k)\rangle \otimes_{\mathbb{S}_n} \mathrm{Sgn}_k)[k]$$

In order to promote KGra to an operad we will define its operadic composition by means of partial composition. As for the previous operads of graphs the compositions are defined by substitution of a graph into a vertex and summing over all the ways to reconnect the edges. The partial composition of two graphs from KGra_c is defined exactly as for $\mathsf{Gra}_2^{\circlearrowleft}$. The partial composition of $\mathsf{KGra}_o \otimes \mathsf{KGra}_c$ to KGra_o at either a vertex of color c or o and there are partial compositions $\mathsf{KGra}_o \otimes \mathsf{KGra}_o$ to KGra_o at a vertex of color o. Just like for the other operads of graphs the partial composition includes a re-labeling of vertices and an induced total order on edges.

As was implicit in M. Kontsevich definition of the individual components $\mathcal{U}_\Gamma$ of the Formality morphism, the pair $(\mathcal{T}_{poly}(X), \mathcal{O}_X)$, $X = \mathbb{K}^d$ form an algebra over the 2-colored operad KGra.

Recall that we can think of $\mathcal{T}_{poly}(X)$ as being the graded commutative algebra generated by $\{x^a\}$, coordinates of X and $\{\psi_a\}$ the associated vector fields ($\psi_a = \frac{\partial}{\partial x^a}$). Let Γ be an admissible graph with n vertices of type I and m vertices of type II and let E denote the set of edges. For every edge $e = \overset{i}{\bullet}\!\longrightarrow\!\overset{j}{\bullet}$ we set

$$\Delta_e^{\uparrow} = \sum_\alpha \mathrm{id}^{\otimes i-1} \otimes \frac{\partial}{\partial x^\alpha} \otimes \mathrm{id}^{\otimes j-i-1} \otimes \frac{\partial}{\partial \psi_\alpha} \otimes \mathrm{id}^{\otimes n-j}.$$

A map $\Phi_\Gamma^{\uparrow} \in \mathrm{Hom}(\mathcal{T}_{poly}(X)^{\otimes(n+m)}, \mathcal{T}_{poly}(X))$ of degree $|E|$ can now be defined as follows. The map $\Phi_\Gamma^{\uparrow}$ is the composition of two maps, $\mu \circ \phi^{\uparrow}$, where μ is just the regular associative multiplication map in the graded commutative algebra and where $\phi^{\uparrow} = \prod_{e \in Edge(\Gamma)} \Delta_e^{\uparrow}$, the product is taken over the edges in their associated ordering. By the natural inclusion $\iota : \mathcal{O}_X \longrightarrow \mathcal{T}_{poly}(X)$ and the natural projection $\pi : \mathcal{T}_{poly} \to \mathcal{O}_X$ we can define a map

$$\Pi_\Gamma : \mathcal{T}_{poly}(X)^{\otimes n} \otimes \mathcal{O}_X^{\otimes m} \longrightarrow \mathcal{O}_X$$

as the composition

$$\Pi_\Gamma : \mathcal{T}_{poly}(X)^{\otimes n} \otimes \mathcal{O}_X^{\otimes m} \xrightarrow{\iota^{\otimes m}} \mathcal{T}_{poly}(X)^{\otimes(n+m)} \xrightarrow{\Phi_\Gamma^{\uparrow}} \mathcal{T}_{poly}(X) \xrightarrow{\pi} \mathcal{O}_X$$

The association $\Gamma \mapsto \Pi_\Gamma$ gives the map of operads mentioned above

$$\Pi : \mathsf{KGra} \longrightarrow \mathsf{End}_{(\mathcal{T}_{poly}(X), \mathcal{O}_X)}$$

3.4.3. Stable formality maps

Definition 3.4.3. *A stable formality morphism is map of operads*

$$\Xi : \mathsf{OC}_\infty \longrightarrow \mathsf{KGra}$$

such that the induced OC_∞ structure on $(\mathcal{T}_{poly}(X), \mathcal{O}_X)$ coincides with the Kontsevich-Shoikhet $\mathsf{Lie}\{1\}_\infty$ structure on $\mathcal{T}_{poly}(X)$, the standard graded commutative A_∞ structure on $\mathcal{O}_X$ and such that the one black vertex-part of the Lie_∞ morphism coincides with the Hochschild-Kostant-Rosenberg quasi-isomorphism.

Theorem 3.4.4. *There exist a stable formality morphism* $\Xi : \mathsf{OC}_\infty \longrightarrow \mathsf{KGra}$ *given on generators as follows.*

i) The $\mathsf{Lie}\{1\}_\infty$ *generators:*

$$\Xi(\nu_p) = \sum_{\Gamma \in G(p,0,2p-3)} c_\Gamma \Gamma$$

$$c_\Gamma = \int_{\overline{C}_{p,\Gamma}} \bigwedge_{e \in E(\Gamma)} \phi_e$$

where $\phi_e = \frac{d\mathrm{Arg}(z_i - z_j)}{\pi}$ *for an edge* $e = \overset{j}{\bullet} \!\longrightarrow\! \overset{i}{\bullet} \in E(\Gamma)$.

ii) The A_∞ *generators*

$$\Xi(\mu_{0,m}) = \begin{cases} \Gamma_{\circ\circ} & m = 2 \\ 0 & m \geq 3 \end{cases}$$

Where the graph $\Gamma_{\circ\circ}$ *is the graph ith two vertices of type II (and no edges);*

$$\Gamma_{\circ\circ} = \circ \quad \circ$$

iii) The Lie_∞ *morphism generators*

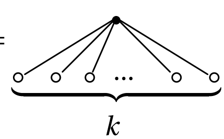
$$\Xi(\mu_{n,m}) = \sum_{\Gamma \in G(n,m,E)} w_\Gamma \Gamma$$

with $n \geq 1$ *and* $E = 2n + m - 2$ *and weights*

$$w_\Gamma = \int_{\overline{C}_{n,m,\Gamma}} \bigwedge_{e \in E(\Gamma)} \phi_e$$

with $\phi_e = \frac{d\mathrm{Arg}(z_i - z_j)}{\pi}$ *as above.*

Proof. The content of *i)* is exactly Theorem 3.2.7. It's an obvious consequence of how the representation $\mathsf{KGra} \to \mathsf{End}_{(\mathcal{T}_{poly}(X),\mathcal{O}_X)}$ is defined that the graph $\Gamma_{\circ\circ}$ produces the standard graded commutative multiplication on $\mathcal{O}_X$ and therefore *ii)* is satisfied. The graphs of the form

$\Gamma_k^\bullet =$ 

have weight $\frac{1}{k!}$ and thus give us the *Hochschild-Kostant-Rosenberg* quasi-isomorphism;

$$\Pi \circ \Xi(\mu_{1,k}) = \Pi\left(\omega_{\Gamma_k^\bullet} \Gamma_k^\bullet\right) = \mathrm{HKR}_k : \mathcal{T}_{poly}(X)^{(k)} \longrightarrow \mathrm{Hom}(\mathcal{O}_X^{\otimes k}, \mathcal{O}_X)[1-k]$$

The proof of *iii)* is the content of Theorem 3.3.2. $\square$

4. Wheel-free deformation quantization

4.1 A propic approach to deformation quantization

4.1.1. The dg Lie algebra of polydifferential operators Let $\mathcal{O}_V = \odot^\bullet V^*$ be the free graded commutative algebra generated by V^*, where we as usual mean that the k-th graded component of V^* is the linear dual of the k-th graded component of V; $(V^*)^{(k)} = (V^{(k)})^*$. It is well known that the graded space of linear maps $\bigoplus_{k\geq 0}\mathrm{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V)[1-k]$ can be given the structure of a dg Lie algebra with the Gerstenhaber bracket, $[-,-]_G$, and the Hochschild differential, d_H. This dg Lie algebra is known as the *Hochschild cochain complex* of $\mathcal{O}_V$;

$$\mathrm{Hoch}^\bullet(\mathcal{O}_V) := \bigoplus_{k\geq 0}\mathrm{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V)[1-k].$$

The Hochschild cochain complex has an interesting dg Lie subalgebra, $\mathcal{D}_V \subset \mathrm{Hoch}^\bullet(\mathcal{O}_V)$, consisting of the operators in $\mathrm{Hoch}^\bullet(\mathcal{O}_V)$ that, for $k \geq 1$, vanish on an element $f_1 \otimes f_2 \otimes \ldots \otimes f_k \in \mathcal{O}_V^{\otimes k}$ if at least one of the polynomials f_i is a constant. Any degree one element, $\Gamma \in \mathcal{D}_V$ can be decomposed into a sum, $\Gamma = \sum_{k\geq 0}\Gamma_k$, where each Γ_k is a "polydifferential" map $\mathcal{O}_V^{\otimes k} \longrightarrow \mathcal{O}_V$ of degree $2-k$. If we fix s system of local coordinates $\{x^\alpha\}$ that span V, then Γ can be represented as a power series (of *finite* total order as a polydifferential operator)

$$\Gamma = \sum_{k\geq 0}\sum_{I_1,\ldots,I_k,J}\Gamma_J^{I_1,\ldots,I_k}x^J\partial_{I_1}\otimes\ldots\otimes\partial_{I_k}.$$

In this series the $\Gamma_J^{I_1,\ldots,I_k}$ are a scalars from our base field and the $I_1,\ldots,I_k,J$ are multi-indices; If $I = a_1a_2\ldots a_{|I|}$ then

$$x^I := x^{a_1}x^{a_2}\ldots x^{a_{|I|}} \quad \text{and} \quad \partial_I := \frac{\partial^{|I|}}{\partial x^{a_1}\ldots\partial x^{a_{|I|}}}$$

We say that an element $\Gamma \in \mathcal{D}_V$ is a Maurer-Cartan element in the Hochschild cochain complex if it satisfies the equation

$$d_H\Gamma + \frac{1}{2}[\Gamma,\Gamma]_G = 0.$$

A Maurer-Cartan element in $\mathcal{D}_V$ (or more generally, in $\mathrm{Hoch}^\bullet(\mathcal{O}_V)$) determines a deformation of the associative (graded commutative) algebra $\mathcal{O}_V$.

4.1.2. Prop profile of deformations In [Me4] S.A Merkulov gives a very interesting alternative description of the Maurer-Cartan elements in the Hochschild complex. In loc. cit. the author constructs a *d*g prop, DefQ, such that the representations of it is exactly the structure of a Maurer-Cartan element in the Hochschild cochain complex.

We remind the reader about the definition of the prop DefQ and its perturbative version $\mathsf{DefQ}^\hbar$.

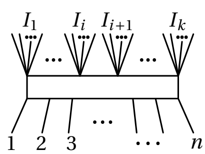
Definition 4.1.1. *Define the* $\mathbb{S}$*-bimodule* $\mathsf{D} = \{\mathsf{D}(m,n)\}_{m,n\geq 0}$ *where we set*

$$\begin{aligned}\mathsf{D}(m,n) &= \mathsf{D}(m)\otimes 1_n[m-2],\\ \mathsf{D}(0) &= \mathbb{K}[-2]\\ \mathsf{D}(m\geq 1) &= \bigoplus_{k\geq 1}\bigoplus_{[m]=I_1\sqcup\ldots\sqcup I_k}\mathrm{Ind}^{\mathbb{S}_m}_{\mathbb{S}_{|I_1|}\times\ldots\times\mathbb{S}_{|I_k|}} 1_{|I_1|}\otimes\ldots\otimes 1_{|I_k|}[k-2]\end{aligned}$$

The prop DefQ *is now defined as the free prop generated by the* D*;*

$$\mathsf{DefQ} := \mathsf{Free}\langle\mathsf{D}\rangle.$$

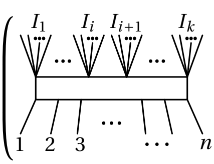
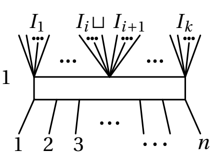
Let the generators be graphically represented by planar corollas of degree $2-k$



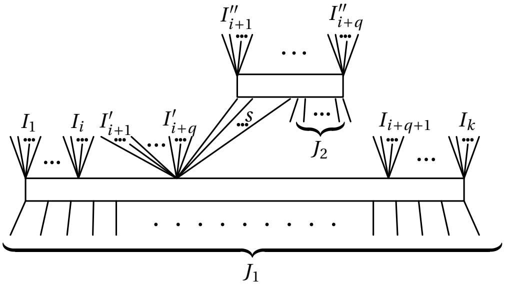
(4.1)

Where the edges labeled $1,\ldots,n$ correspond to the input. The input edges are symmetric; the order in which they are written down does not matter. The edges labeled by elements from $I_1\sqcup\ldots\sqcup I_k = \{1,\ldots,m\}$ correspond to the output. Within each group labeled by a I_i, the edges are symmetric.

The differential on DefQ is defined by its action on the generators

$\delta\Bigg($  $\Bigg) = \sum_{i=1}^{k}(-1)^{i+1}$ 

$$+\sum_{\substack{p+q=k+1\\ p\geq 1, q\geq 0}}\sum_{i=0}^{p-1}\sum_{\substack{I_{i+1}=I'_{i+1}\sqcup I''_{i+1}\\ \cdots\cdots\cdots\\ I_{i+q}=I'_{i+q}\sqcup I''_{i+q}}}\sum_{[n]=J_1\sqcup J_2}\sum_{s\geq 0}(-1)^{(p+1)q+i(q-1)}$$

$$\frac{1}{s!}$$


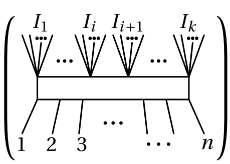
The proof that δ squares to zero was given in [Me4].

Remark 4.1.2. *Notice the limits on the parameter s. As s may be zero, the graph created by the differential can be disconnected; the disjoint union of two corollas. The number s is also not bounded above so we shall actually consider* DefQ *to be completed on the sum of the number of vertices and the genus.*

The above formula for the differential shows that the free prop DefQ makes sense as a *differential* prop only if it is considered as genus completed as the r.h.s. of the above formula involves infinitely many graphs of increasing genus. This immediately raises a question: what can be a representation of the completed prop DefQ in an arbitrary vector space V? A morphism of dg props

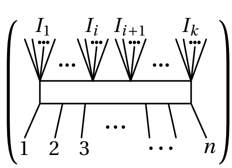
$$\rho : \mathsf{DefQ} \longrightarrow \mathsf{End}_V$$

is uniquely specified by its values on the generators,

ρ


$\in \mathrm{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V) \subset \mathsf{End}_V$

which can be arbitrary. Hence an arbitrary representation ρ gives us an infinite sum of polydifferential operators with order tending, in general, to infinity. This is definitely not the object we are interested in as the Gerstenhaber brackets $[\ ,\ ]_G$ can diverge on such infinite sums of polydifferential operators. Therefore we introduce a new notion of an *admissible* representation ρ_{adm} which, by definition, satisfies the condition that the values

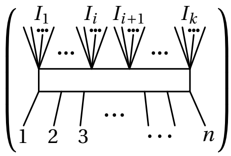
ρ_{adm}


vanish for *sufficiently for sufficiently large values of the cardinalities* $|I_i|$. We give a rigourous definition

Let N be a positive integer. We say that a representation

$$\rho : \mathsf{DefQ} \longrightarrow \mathsf{End}_V$$

is N-admissible if

ρ  $= 0$

whenever $|I_i| > N$ for some $i = 1, 2, \dots k$.

Let $\mathrm{Rep}_N^{\mathsf{DefQ}}(V)$ denote the set of N-admissible representation of DefQ in V. Clearly there is a chain of inclusions

$$\cdots \longrightarrow \mathrm{Rep}_{N-1}^{\mathsf{DefQ}}(V) \longrightarrow \mathrm{Rep}_{N}^{\mathsf{DefQ}}(V) \longrightarrow \mathrm{Rep}_{N+1}^{\mathsf{DefQ}}(V) \longrightarrow \cdot$$

we say that a representation is *admissible* if it is an element of the direct limit

$$\lim_{N\to\infty} \mathrm{Rep}_N^{\mathsf{DefQ}}(V)$$

of the above diagram.

This condition assures that we get polydifferential operators of finite order. There is a one-to-one correspondence between the set of *admissible* representations and the Hochschild polydifferential complex

$$\mathcal{D}_V \subset \mathrm{Hoch}^\bullet(\mathcal{O}_V) := \bigoplus_{k\geq 0} \mathrm{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V)[1-k].$$

Moreover, the following stronger result holds true.

Proposition 4.1.3 ([Me4])**.** *There is a one-to-one correspondence between admissible representations*

$$\rho : (\mathsf{DefQ}, \delta) \to (\mathsf{End}_V, d)$$

and Maurer-Cartan elements, Γ*, in the Hochschild dg Lie algebra of polydifferential operators,* $\mathcal{D}_V$*, that is, degree one polydifferential operators on* $\mathcal{O}_V$ *satisfying the equation* $d_H\Gamma + \frac{1}{2}[\Gamma,\Gamma]_G = 0$.

4.1.3. Prop profile of perturbative deformations There is a perturbative analogue of the polydifferential operators on $\mathcal{O}_V$, $\mathcal{D}_V^\hbar$, in which Maurer-Carten elements correspond to perturbative deformations of the ordinary product in $\mathcal{O}_V$. Formally we define $\mathcal{D}_V^\hbar := \mathcal{D}_V \otimes \hbar\mathbb{K}[[\hbar]]$. This makes $\mathcal{D}_V^\hbar$ into a dg Lie algebra of polydifferential operators on $\mathcal{O}_V[[\hbar]]$ which vanish at $\hbar = 0$. The elements $\Gamma \in \mathcal{D}_V^\hbar$ that solve the associated Maurer-Cartan equation correspond to generalized A_∞ structures on $\mathcal{O}_V[[\hbar]]$. These Maurer-Cartan elements can also be described as the representation of a dg prop.

Definition 4.1.4. *Let* $\mathsf{DefQ}^\hbar$ *be the free prop generated by* $\mathsf{D}^\hbar$;

$$\mathsf{DefQ}^\hbar := \mathsf{Free}\langle \mathsf{D}^\hbar \rangle.$$

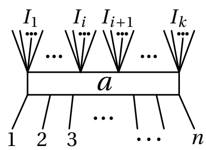
Where $\mathsf{D}^\hbar = \{\mathsf{D}^\hbar(m,n)\}$ *is defined by the following*

$$\mathsf{D}^a(m,n) := \mathsf{D}(m) \otimes 1_n[m-2], \textit{ where } a+1, m, n \in \mathbb{Z}_{\geq 0}$$

$$\mathsf{D}^\hbar(m,n) := \bigoplus_{a=1}^{\infty} \mathsf{D}^a(m,n)$$

where $\mathsf{D}(m)$ *is the same module introduced in the non-perturbative case.*

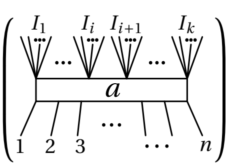
Analogously to DefQ, the generators of $\mathsf{DefQ}^\hbar$ can be identified with corollas,

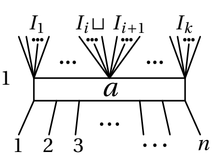


, (4.2)

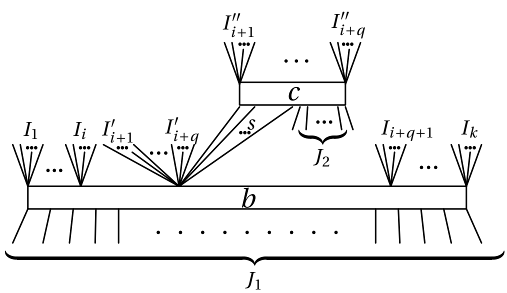
which have the same properties as those defined in the previous section but additionally they also carry a numerical label $a \in \mathbb{Z}_{>0}$.

The differential in $\mathsf{DefQ}^\hbar$ is defined on generators by the following graphical expression:

$\delta\Big($ 

$\Big) = \sum_{i=1}^{k} (-1)^{i+1}$ 

$$+ \sum_{\substack{b+c=a \\ b,c\geq 1}} \sum_{\substack{p+q=k+1 \\ p\geq 1, q\geq 0}} \sum_{i=0}^{p-1} \sum_{\substack{I_{i+1}=I'_{i+1}\sqcup I''_{i+1} \\ \cdots \\ I_{i+q}=I'_{i+q}\sqcup I''_{i+q}}} \sum_{[n]=J_1\sqcup J_2} \sum_{s\geq 0} (-1)^{(p+1)q+i(q-1)}$$

$\frac{1}{s!}$ 

One defines the notion of an *admissible* representation of $\mathsf{DefQ}^\hbar$ in exactly the same way as in the case of DefQ to assure that we do work with operators of infinite order. For the prop $\mathsf{DefQ}^\hbar$ the following proposition is analogous to Proposition 4.1.3 from the previous section.

Proposition 4.1.5 ([Me4]). *There is a one-to-one correspondence between admissible representations*

$$\rho : (\mathsf{DefQ}^\hbar, \delta) \to (\mathsf{End}_{V[[\hbar]]}, d)$$

of $\mathsf{DefQ}^\hbar$ *in a* $\mathbb{K}[[\hbar]]$ *extension of the dg vector space* (V, d) *and curved* A_∞ *structures in* $\mathcal{O}_V[[\hbar]]$*, i.e. Maurer-Cartan elements,* Γ*, in* $\mathcal{D}_V[[\hbar]]$ *satisfying the equations* $d_H\Gamma + \frac{1}{2}[\Gamma,\Gamma]_G = 0$ *and* $\Gamma|_{\hbar=0} = d$.

4.1.4. Propic formulation of deformation quantization Suppose we have a mathematical structure ϕ on a dg vector space (V, d) that can be defined as the representation of a dg properad (P, ∂);

$$\rho_\phi : (\mathsf{P}, \partial) \longrightarrow (\mathsf{End}_V, d).$$

Following [Me4] we understand the problem of deformation quantization of this structure as finding a morphism of dg props

$$q : (\mathsf{DefQ}, \delta) \longrightarrow (\mathsf{P}, \partial)$$

that satisfy some boundary condition to guarantee non-triviality. The composition $q \circ \rho_\phi$ gives us an explicit representation of (DefQ, δ) in (V, d) and by the propositions 4.1.3 this determines a star-product on $\mathcal{O}_V$.

This procedure of deformation quantization for P-algebras has the corollary that if $\mathsf{C}(\mathsf{P})$ is a cofibrant replacement of P then the cofibrancy of DefQ implies the existence of a lift for the map q;

$$\widehat{q} : \mathsf{DefQ} \longrightarrow \mathsf{C}(\mathsf{P}).$$

Alternatively, if we wish to prove that a certain *formal* dg properad (Q, d) can be deformation quantized then it is enough to find a morphism $(\mathsf{DefQ}, \delta) \longrightarrow (\mathrm{H}^\bullet(\mathsf{Q}), 0)$.

The main result of [Me4] was the proof for deformation quantization of a *formal graded Poisson structure* in a dg vector space (V, d).

4.1.5. Properad of quasi-Poisson structures In [Sh] B. Shoikhet determines a universal Lie_∞ structure on $\mathcal{T}_{poly}(V)[1]$ and goes on to define the notion of quasi-Poisson structure as a Maurer-Cartan element in this Lie_∞-algebra.

We shall see that the notion of quasi-Poisson structure has a natural properadic interpretation.

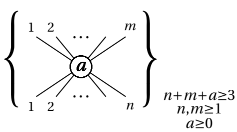
Consider the Kontsevich-Shoiket Maurer-Cartan element as an infinte sum of graphs

$$\gamma^{KS} = \Gamma_2 + \Gamma_4 + \Gamma_6 + \ldots \in \mathsf{GC}_2^{\uparrow}, \tag{4.3}$$

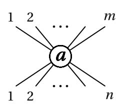
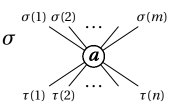
where Γ_n is the sum of graphs with n vertices. This element has a natural perturbative analogue

$$\hat{\gamma}^{KS} = \Gamma_2 + \hbar\Gamma_4 + \ldots + \hbar^{k-1}\Gamma_{2k} + \ldots \in \mathsf{GC}_2^{\uparrow}[[\hbar]] := \mathsf{GC}_2^{\uparrow} \otimes \mathbb{K}[[\hbar]].$$

We define a free properad, qPois on the following generators

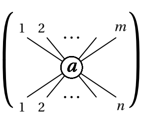
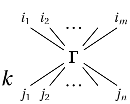


with the relation

 $= (-1)^{\sigma}$  $\quad \forall\sigma \in \mathbb{S}_m, \forall\tau \in \mathbb{S}_n.$

We will consider qPois to be completed with respect to the number of vertices.

The dg Lie algebra $\mathsf{GC}_2^{\uparrow}[[\hbar]]$ acts on qPois. by derivation

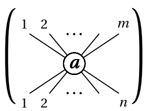
$\hbar^k\Gamma$  $= \displaystyle\sum_{\substack{grafting\\ inputs/outputs}} \; \sum_{a_1+\ldots+a_{|V(\Gamma)|}=a-k}$ 

The first sum is over all the ways to graft the output and input edges to the vertices of Γ. The second sum is over all the ways to decorate the vertices in Γ by non-negative weights a_i such that the sum of the weights total to $a-k$.

From this action it follows that $\hat{\gamma}^{KS}$ defines a differential ∂ on qPois.

Proposition 4.1.6. *Representations* qPois $\longrightarrow \mathsf{End}_V$ *are in one-to-one correspondence with quasi-Poisson structures on* (V,d)*, i.e. Maurer-Cartan elements in the Kontsevich-Shoikhet* $\mathsf{Lie}\{1\}_\infty$ *algebra* $(\mathcal{T}_{poly}(V)[[\hbar]], \mu_\bullet)$.

Proof. Let ρ : qPois $\longrightarrow \mathsf{End}_V$ be a map of dg props. Each corolla



gives rise to a map $\wedge^n V \to \odot^m V$, or equivalently, the image of every such corolla is an element $\pi^{(a)}(m,n) \in \wedge^n V \otimes \odot^m V^*$ and therefore it can be regarded as polyvector field on V. From the image of these corollas we define the following power series

$$\pi^{\diamond} = \sum_{m,n\geq 1} \sum_{a\geq 0} \hbar^a \pi^{(a)}(m,n)$$

where we make the addition that $\pi^{(0)}(1,1) := d$. An inspection of the condition $\rho \circ \partial = d_{\mathsf{End_V}} \circ \rho$ reveals that this implies that $\pi^{\diamond}$ satisfies the Maurer-Cartan equation

$$\frac{1}{2}\mu_2(\pi^{\diamond},\pi^{\diamond}) + \frac{\hbar}{4!}\mu_4(\pi^{\diamond},\pi^{\diamond},\pi^{\diamond},\pi^{\diamond}) + \ldots = 0$$

for the Kontsevich-Shoikhet $\mathsf{Lie}\{1\}_\infty$ structure $\mu_\bullet$ on $\mathcal{T}_{poly}(V)$. □

4.2 Wheel-free deformation quantization

4.2.1. Odd Lie bialgebras and the cohomology of qPois We will give the "classical" algebraic definition of odd Lie bialgebras and then the properadic definition.

Definition 4.2.1. *The structure of an odd Lie bialgebra in a graded vector space V is a given by a degree 1 skew-symmetric bilinear map*

$$[-,-] : V \wedge V \to V[1]$$

such that the pair $(V[-1],[-,-])$ is a graded Lie algebra and a map

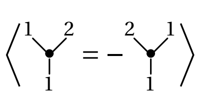
$$\Delta : V \to V \wedge V$$

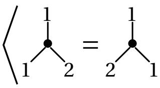
such that the pair (V,Δ) is graded Lie coalgebra. Furthermore, we require that the following compatibility rule is satisfied

$$\begin{aligned}\Delta \circ [-,-] =& ([-,-]\otimes \mathrm{id})\circ(\mathrm{id}\otimes\Delta) + \tau\circ([-,-]\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta) - \\ &([-,-]\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\circ\tau - \tau\circ([-,-]\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\circ\tau\end{aligned}$$

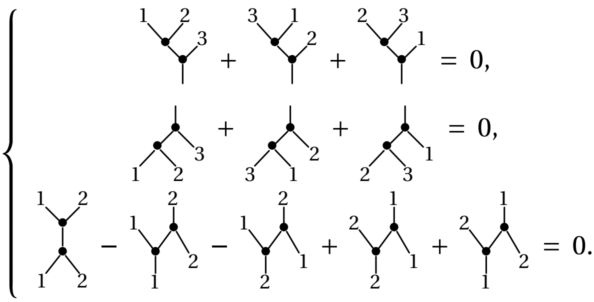
where τ denotes the twist map $\tau : x\otimes y \to y\otimes x$.

Let us now consider the properad of odd Lie bialgebra, denoted LieB_{odd}. This properad can be represented as a quotient of a free properd F modulo an ideal of relations R. The free properad F is generated by the $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geq 1}$ with all $E(m,n) = 0$ except

$E(2,1) := \mathrm{id}_1 \otimes \mathrm{sgn}_2 = \mathrm{span}$ 

$$E(1,2) := \mathrm{id}_2[-1] \otimes \mathrm{id}_1 = \mathrm{span}\left\langle \right.$$

$$\left.\right\rangle$$

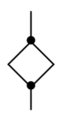
The ideal R is generated by the following relations

$R:$

(4.4)

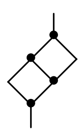
Remark 4.2.2. *For odd Lie bialgebras, due to the cocommutativity of the cobracket and the skew-commutativity of the bracket, the standard involutive relation is automatically satisfied,*

$$[-,-] \circ \Delta = 0,$$

or in terms of graphs


$= 0.$

Let $\mathsf{LieB}^{\diamond}_{odd}$ be the properad of odd Lie bialgebras modulo the additional relation


$= 0$

Remark 4.2.3. *We call these Lie bialgebras* super-involutive*, because they don't only satisfy the ordinary notion of involutivity but also a higher notion of involutivity corresponding to the genus 2 graphs given above.*

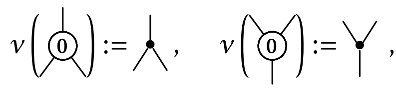
It follows immediately from the definition of the differential in qPois that there is an epimorphism of properads

$$g : \mathsf{LieB}^{\diamond}_{odd} \longrightarrow \mathrm{H}(\mathsf{qPois})$$

and existence of such a map is enough to prove the Main Theorem below. After this chapter was completed, it was proven in [KMW] that the cohomology of qPois is precisely the properad $\mathsf{LieB}^{\diamond}_{odd}$, i.e. that the map g is an isomorphism. We shall use this latest result below though it is worth emphasizing that it is *not* required for the purposes of this result. The natural “forgetful" map

$$\nu : \mathsf{qPois} \longrightarrow \mathsf{LieB}^{\diamond}_{odd}$$

which vanishes on all generators except the following ones,



(4.5)

is therefore a *quasi-isomorphism.*

4.2.2. Super-involutive Lie bialgebras Our next purpose is to construct a non-trivial morphism of props

$$\rho_0 : (\mathsf{DefQ}^{\hbar}, \delta) \longrightarrow (\mathsf{LieB}^{\diamond}_{odd}, 0).$$

If $r : \mathsf{LieB}^{\diamond}_{odd} \to \mathsf{End}_V$ is a representation of $\mathsf{LieB}^{\diamond}_{odd}$ in a graded vector space V, that is a quasi-Poisson structure in V, the above morphism ρ_0 induces a representation $r : \mathsf{DefQ}^{\hbar} \to \mathsf{End}_V$, that is, a Maurer-Cartan element in $\mathcal{D}_V[[\hbar]]$ satisfying certain conditions, see Proposition 4.1.5 above.

Any prop can be understood as a totality of its representations in all possible graded vector spaces V. Therefore, we can read of the morphism ρ_0 via a universal construction of Maurer-Cartan elements Γ in $\mathcal{D}_V[[\hbar]]$ from an arbitrary quasi-Poisson structure in V (universal in the sense that the construction does not depend on a particular choice of V). Moreover, it is enough to work with a sufficiently large family of graded vector spaces V which satisfy the condition that any non-zero element in $\mathsf{LieB}^{\diamond}_{odd}$ or in $\mathsf{DefQ}^{\hbar}$ can be represented by an element in End_V that does not vanishing identically for sufficiently generic V in our family. To simplify the rules of sign we choose to work (following the standard trick in mathematical and theoretical physics) with all possible representations of the above props in the category of free modules V over graded commutative $\mathbb{K}$-algebras Λ which satisfy the condition that V (and hence V^*) admits a set of generators $\{x^1, \dots, x^n\}$ (over Λ) of homological degree zero for some (arbitrary large) $n \in \mathbb{N}$. For sufficiently large n and generic Λ the non-vanishing condition can be obviously satisfied so that we loose no information about our props (cf. [MeVa]) while working in this family of representations. We can view every such Λ-module V as a Λ-supermanifold (cf. [R]) and hence talk about formal polyvector fields on V,

$$\sum C^{b_1,\dots,b_l}_{a_1,\dots,a_k} x^{a_1} \cdots x^{a_k} \frac{\partial}{\partial x^{b_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{b_l}}$$

whose coefficients are elements of the graded commutative ring Λ. This is a standard "coordinate" approach to supermanifolds in mathematical physics.

In this context a representation $r : \mathsf{LieB}^{\diamond}_{odd} \to \mathsf{End}_V$ is the same as the following pair of polyvector fields

$$\nu = \Phi^{ab}_{c} x^c \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b} \in \wedge^2 T_{V^*}$$
$$\xi = C^c_{ab} x^a x^b \frac{\partial}{\partial x^c} \in \mathcal{T}_{V^*}$$

where the structure constants

$$\Phi^{ab}_{c} = -\Phi^{ba}_{c}$$

are degree zero elements in Λ, and

$$C^c_{ab} = C^c_{ba}$$

are degree 1 elements in Λ. More elementary, ν is a degree 0 co-Lie bracket on V and ξ is a degree 1 Lie bracket on V. If one wants to employ a geometric intuition, with respect to the above interpretation as polyvector fields, then they are given as a linear Poisson structure and a quadratic degree 1 vector field.

In the same local coordinates, the differential d_V on V will determine a degree 1 linear vector field of the following form

$$d_V = L^b_a x^a \frac{\partial}{\partial x^b} \in \mathcal{T}_{V^*}$$

The relations of $\mathsf{LieB}^{\diamond}_{odd}$ imply that the element $\Lambda = d_V + \nu + \xi$ is a graded Poisson structure on V^* subject to the super-involutive relation. Explicitly the element Λ will satisfy the Maurer-Cartan equation for the classical Schouten bracket on $\mathcal{T}_{poly}(V^*)$,

$$\frac{1}{2}[\Lambda, \Lambda]_{SN} = 0,$$

and the equation

$$C^a_{bc} C^b_{de} \Phi^{ec}_{f} \Phi^{ef}_{g} x^g \frac{\partial}{\partial x^a} = 0 \tag{4.6}$$

which corresponds to the super-involutive relation.

4.2.3. Structure constants The fact that ν and ξ make up an odd Lie bialgebra on V^* implies that certain relations on structure constant for ν and ξ hold. We will state some of them explicitly for the sake of future calculations.

The structure constants of ν satisfy the following relations

$$\Phi_a^{bc} + \Phi_a^{cb} = 0$$
$$\Phi_i^{ab}\Phi_d^{ic} + \Phi_i^{bc}\Phi_d^{ia} + \Phi_i^{ca}\Phi_d^{ib} = 0$$

Analogously one has formulas for ξ. The structure constants of ξ satisfy the following relations

$$C_{ab}^{\alpha} - C_{ba}^{\alpha} = 0$$
$$C_{ic}^{\alpha}C_{ab}^{i} + C_{ia}^{\alpha}C_{bc}^{i} + C_{ib}^{\alpha}C_{ca}^{i} = 0$$

The Lie bialgebra compatibility between the bracket and co-bracket gives the following relation on structure constants.

$$C_{kj}^{a}\Phi_i^{jb} + C_{ij}^{a}\Phi_k^{jb} - C_{kj}^{b}\Phi_i^{ja} - C_{ij}^{b}\Phi_k^{ja} - \Phi_j^{ab}C_{ki}^{j} = 0$$

4.2.4. Quantization procedure To deformation quantize the pair (ν, ξ) is to construct from it a degree 2 function $\Gamma_0 \in \mathrm{Hom}_2(\mathbb{K}, \mathcal{O}_V)[[\hbar]]$, a differential operator $\Gamma_1 \in \mathrm{Hom}_1(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]]$, and a bi-differential operator $\Gamma_2 \in \mathrm{Hom}_0(\mathcal{O}_V^{\otimes 2}, \mathcal{O}_V)[[\hbar]]$ such that the following equations are satisfied

$$\begin{aligned} \Gamma_1\Gamma_0 &= 0 \\ \Gamma_1^2 + [\Gamma_0, \Gamma_2]_G &= 0 \\ d_H\Gamma_1 + [\Gamma_1, \Gamma_2]_G &= 0 \\ d_H\Gamma_2 + \frac{1}{2}[\Gamma_2, \Gamma_2]_G &= 0 \end{aligned} \tag{4.7}$$

The bi-differential operator Γ_2

As ν determines a linear Poisson structure on V^* we can deformation quantize it to produce Γ_2 with the Poincare-Birkhoff-Witt isomorphism. The construction is well known and we restate it here for completeness. Let $[-,-]$ denote the Lie bracket on V^* coming from ν and form the homogenous universal enveloping algebra $\mathcal{U}_\hbar$ defined as the quotient

$$\mathcal{U}_\hbar := \frac{\widehat{\otimes}^{\bullet} V^*[[\hbar]]}{J}$$

where the ideal J is generated by all expressions of the form $(x \otimes y - y \otimes x - \hbar[x, y])$ with $x, y \in V^*$. Let $- \circ - : \mathcal{U}_\hbar^{\otimes 2} \longrightarrow \mathcal{U}_\hbar$ denote the associative product in the enveloping algebra. The Poincaré-Birkhoff-Witt theorem gives an isomorphism

$$s : \mathcal{O}_V[[\hbar]] \longrightarrow \mathcal{U}_\hbar$$

explicitly defined as

$$s(x^{i_1} x^{i_2} \ldots x^{i_k}) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} x^{\sigma(i_1)} \circ x^{\sigma(i_2)} \circ \ldots \circ x^{\sigma(i_k)}$$

We define the star-product through the isomorphism s;

$$f \star_\hbar g := s^{-1}(s(f) \circ s(g))$$

and from it define the bi-differential operator Γ_2,

$$\Gamma_2(f \otimes g) := f \star_\hbar g - fg.$$

That $\star_\hbar$ is an (even degree) associative product on $\mathcal{O}_V[[\hbar]]$ is equivalent to that $\frac{1}{2}[\star_\hbar, \star_\hbar]_G = 0$. We have that $\star_\hbar = \mu + \Gamma_2$, where μ denotes the ordinary product of functions. That this choice of Γ_2 satisfied $d_H \Gamma_2 + \frac{1}{2}[\Gamma_2, \Gamma_2]_G = 0$ can now be seen by expanding the expression $[\star_\hbar, \star_\hbar]_G$;

$$0 = \frac{1}{2}[\star_\hbar, \star_\hbar]_G = \frac{1}{2}[\mu, \mu] + [\mu, \Gamma_2] + \frac{1}{2}[\Gamma_2, \Gamma_2]$$

the ordinary multiplication is associative and therefore $[\mu, \mu]_G = 0$.

The differential operator Γ_1

To find Γ_1 we need to solve the equation $d_H \Gamma_1 + [\Gamma_1, \Gamma_2]$, which is equivalent to that Γ_1 is a derivation of the star-product;

$$\Gamma_1(f \star_\hbar g) = \Gamma_1(f) \star_\hbar g + f \star_\hbar \Gamma_1(g)$$

To start we form the map $\theta : V^*[[\hbar]] \longrightarrow \widehat{\bigotimes}^\bullet V^*[[\hbar]]$ by using the structure constants of ξ and d_V;

$$\theta(x^a) := \frac{1}{2} \hbar C^a_{ij} x^i \otimes x^j + \hbar L^a_i x^i$$

Let $\hat{\theta}$ be the extension of θ to a derivation of the tensor algebra $\widehat{\bigotimes}^\bullet V^*[[\hbar]]$. By using the relations for ξ, ν and d_V implied by the $\mathsf{LieB}^\diamond_{odd}$ representation we can see that this derivation preserves the ideal

$$J = \left\langle x^a \otimes x^b - x^b \otimes x^a - \hbar \Phi^{ab}_k x^k \right\rangle$$

and so it also becomes a derivation of the universal enveloping algebra $\mathcal{U}_\hbar$.

Before proceeding we will establish a technical lemma

Lemma 4.2.4. *The following relations are exhibited*

$$
\begin{aligned}
x^a \otimes x^b \otimes x^c =& x^c \otimes x^a \otimes x^b + \hbar\Phi^{ac}_j x^j \otimes x^b + \hbar\Phi^{bc}_i x^a \otimes x^i \\
x^a \star_\hbar x^b \star_\hbar x^c =& x^a \otimes x^b \otimes x^c - \frac{\hbar}{2}\left(\Phi^{bc}_i x^a \otimes x^i + \Phi^{ac}_i x^b \otimes x^i + \Phi^{ab}_i x^c \otimes x^i\right) \\
& - \frac{\hbar^2}{3}\left(\Phi^{ab}_i \Phi^{ic}_j x^j + \Phi^{ac}_i \Phi^{ib}_j x^j\right)
\end{aligned}
$$

Proof. We derive the first expression from

$$x^\alpha \otimes x^\beta - x^\beta \otimes x^\alpha - \hbar\Phi^{\alpha\beta}_k x^k = 0;$$

$$
\begin{aligned}
x^a \otimes x^b \otimes x^c &= x^a \otimes \left(x^c \otimes x^b + \hbar\Phi^{bc}_i x^i\right) \\
&= x^a \otimes x^c \otimes x^b + \hbar\Phi^{bc}_i x^a \otimes x^i \\
&= \left(x^c \otimes x^a + \hbar\Phi^{ac}_j x^j\right) \otimes x^b + \hbar\Phi^{bc}_i x^a \otimes x^i \\
&= x^c \otimes x^a \otimes x^b + \hbar\Phi^{ac}_j x^j \otimes x^b + \hbar\Phi^{bc}_i x^a \otimes x^i
\end{aligned}
$$

The second expression is derived in a process of repeatedly using the quadratic relation

$$x^\alpha \star_\hbar x^\beta = x^\alpha \otimes x^\beta - \frac{\hbar}{2}\Phi^{\alpha\beta}_i x^i$$

and we omit the proof □

Lemma 4.2.5. *The map $s^{-1} \circ \hat{\theta} \circ s$ is a derivation of the star-product $\star_\hbar$.*

Proof.

$$
\begin{aligned}
&\hat{\theta}(x^a \otimes x^b - x^b \otimes x^a - \hbar\Phi^{ab}_i x^i) \\
&= \hat{\theta}(x^a) \otimes x^b + x^a \otimes \hat{\theta}(x^b) - \hat{\theta}(x^b) \otimes x^a - x^b \otimes \hat{\theta}(x^a) - \hbar\Phi^{ab}_c \hat{\theta}(x^c) \\
&= \left(\frac{1}{2}\hbar C^a_{ij} x^i \otimes x^j + \hbar L^a_i x^i\right) \otimes x^b + x^a \otimes \left(\frac{1}{2}\hbar C^b_{ij} x^i \otimes x^j + \hbar L^b_i x^i\right) - \left(\frac{1}{2}\hbar C^b_{ij} x^i \otimes x^j + \hbar L^b_i x^i\right) \otimes x^a \\
&- x^b \otimes \left(\frac{1}{2}\hbar C^a_{ij} x^i \otimes x^j + \hbar L^a_i x^i\right) - \hbar\Phi^{ab}_i \left(\frac{1}{2}\hbar C^i_{kj} x^k \otimes x^j + \hbar L^i_k x^k\right) \\
&= \hbar\left[\frac{1}{2} C^a_{ij} x^i \otimes x^j \otimes x^b + \frac{1}{2} C^b_{ij} x^a \otimes x^i \otimes x^j - \frac{1}{2} C^b_{ij} x^i \otimes x^j \otimes x^a - \frac{1}{2} C^a_{ij} x^b \otimes x^i \otimes x^j\right. \\
&\left.+ L^a_i x^i \otimes x^b + L^b_i x^a \otimes x^i - L^b_i x^i \otimes x^a - L^a_i x^b \otimes x^i\right] - \hbar^2\left[\frac{1}{2}\Phi^{ab}_i C^i_{kj} x^k \otimes x^j + \Phi^{ab}_i L^i_k x^k\right] \\
&= \hbar\left[\frac{C^a_{ij}}{2}\left(-x^b \otimes x^i \otimes x^j + x^i \otimes x^j \otimes x^b\right) + \frac{C^b_{ij}}{2}\left(x^a \otimes x^i \otimes x^j - x^i \otimes x^j \otimes x^a\right)\right. \\
&\left.+ L^a_i\left(x^i \otimes x^b - x^b \otimes x^i\right) + L^b_i\left(x^a \otimes x^i - x^i \otimes x^a\right)\right] - \hbar^2\left[\frac{1}{2}\Phi^{ab}_i C^i_{kj} x^k \otimes x^j + \Phi^{ab}_i L^i_k x^k\right]
\end{aligned}
$$

[using the relation implied by the ideal J directly and Lemma 4.2.4]

$$\underset{(\mathrm{mod}\ J)}{\equiv} \hbar^2\left[\frac{C^a_{ij}}{2}\left(\Phi^{ib}_k x^k \otimes x^j + \Phi^{jb}_k x^i \otimes x^k\right) - \frac{C^b_{ij}}{2}\left(\Phi^{ia}_k x^k \otimes x^j + \Phi^{ja}_k x^i \otimes x^k\right) - \frac{1}{2}\Phi^{ab}_i C^i_{kj} x^k \otimes x^j\right.$$

$$+L^a_i\Phi^{ib}_k x^k + L^b_i\Phi^{ai}_k x^k - \Phi^{ab}_i L^i_k x^k\Big]$$

$[d_V$ is a derivation of $v]$

$$\underset{(\mathrm{mod}\ J)}{\equiv} \hbar^2\left[\frac{C^a_{ij}}{2}\left(\Phi^{ib}_k x^k\otimes x^j + \Phi^{jb}_k x^i\otimes x^k\right) - \frac{C^b_{ij}}{2}\left(\Phi^{ia}_k x^k\otimes x^j + \Phi^{ja}_k x^i\otimes x^k\right) - \frac{1}{2}\Phi^{ab}_i C^i_{kj} x^k\otimes x^j\right]$$

[by relabeling indices]

$$\underset{(\mathrm{mod}\ J)}{\equiv} \frac{\hbar^2}{2}\left(C^a_{kj}\Phi^{jb}_i + C^a_{ij}\Phi^{jb}_k - C^b_{kj}\Phi^{ja}_i - C^b_{ij}\Phi^{ja}_k - \Phi^{ab}_j C^j_{ki}\right)x^k\otimes x^i$$

[by the Lie bialgebra compatibility between bracket and co-bracket]

$$\underset{(\mathrm{mod}\ J)}{\equiv} 0$$

□

This choice of derivation also squares to zero;

Lemma 4.2.6. *The map $s^{-1}\circ\hat{\theta}\circ s$ is a differential.*

Proof. It's enough to prove that $\hat{\theta}^2(x^a) = 0$ for an arbitrary basis element x^a.

$$\begin{aligned}
\hat{\theta}^2(x^a) =&\hat{\theta}\left(\frac{1}{2}\hbar C^a_{bc}x^b\star_\hbar x^c + \hbar L^a_b x^b\right)\\
=&\frac{1}{2}\hbar C^a_{bc}\hat{\theta}(x^b)\star_\hbar x^c + \frac{1}{2}\hbar C^a_{bc}x^b\star_\hbar\hat{\theta}(x^c) + \hbar L^a_b\hat{\theta}(x^b)\\
=&\frac{1}{2}\hbar C^a_{bc}\left(\frac{1}{2}\hbar C^b_{de}x^d\star_\hbar x^e + \hbar L^b_d x^d\right)\star_\hbar x^c + \frac{1}{2}\hbar C^a_{bc}x^b\star_\hbar\left(\frac{1}{2}\hbar C^c_{de}x^d\star_\hbar x^e + \hbar L^c_d x^d\right)\\
&+\hbar L^a_b\left(\frac{1}{2}\hbar C^b_{cd}x^c\star_\hbar x^d + \hbar L^b_c x^c\right)\\
=&\frac{1}{4}h^2 C^a_{bc}C^b_{de}x^d\star_\hbar x^e\star_\hbar x^c + \frac{1}{4}\hbar^2 C^a_{bc}C^c_{de}x^b\star_\hbar x^d\star_\hbar x^e + \frac{1}{2}\hbar^2 C^a_{bc}L^c_d x^b\star_\hbar x^d\\
&+\frac{1}{2}\hbar^2 C^a_{bc}L^b_d x^d\star_\hbar x^c + \frac{1}{2}\hbar^2 L^a_b C^b_{cd}x^c\star_\hbar x^d + \hbar^2 L^a_b L^b_c x^c\\
&\left[\text{relabeling indices and cancelling the linear term by using that } L^a_b L^b_c = 0\right]\\
=&\frac{1}{4}h^2 C^a_{id}C^i_{bc}x^b\star_\hbar x^c\star_\hbar x^d + \frac{1}{4}\hbar^2 C^a_{bi}C^i_{cd}x^b\star_\hbar x^c\star_\hbar x^d + \frac{1}{2}\hbar^2 C^a_{bi}L^i_c x^b\star_\hbar x^c\\
&+\frac{1}{2}\hbar^2 C^a_{ic}L^i_b x^b\star_\hbar x^c + \frac{1}{2}\hbar^2 L^a_i C^i_{bc}x^b\star_\hbar x^c\\
=&\frac{h^2}{4}\left(C^a_{id}C^i_{bc} + C^a_{bi}C^i_{cd}\right)x^b\star_\hbar x^c\star_\hbar x^d + \frac{\hbar^2}{2}\left(C^a_{bi}L^i_c + C^a_{ic}L^i_b + L^a_i C^i_{bc}\right)x^b\star_\hbar x^c\\
&[\text{by the Jacobi identity and since } d_V \text{ is a derivation of } \xi]\\
=&-\frac{h^2}{4}C^a_{ic}C^i_{db}x^b\star_\hbar x^c\star_\hbar x^d\\
&[\text{by Lemma 4.2.4}]\\
=&-\frac{h^2}{4}C^a_{ic}C^i_{db}\left(x^b\otimes x^c\otimes x^d - \frac{\hbar}{2}\left(\Phi^{cd}_i x^b\otimes x^i + \Phi^{bd}_i x^c\otimes x^i + \Phi^{bc}_i x^d\otimes x^i\right)\right.\\
&\left.-\frac{\hbar^2}{3}\left(\Phi^{bc}_i\Phi^{id}_j x^j + \Phi^{bd}_i\Phi^{ic}_j x^j\right)\right)\\
=&-\frac{h^2}{4}C^a_{ic}C^i_{db}x^b\otimes x^c\otimes x^d + \frac{h^3}{8}C^a_{ic}C^i_{db}\left(\Phi^{cd}_i x^b\otimes x^i + \Phi^{bd}_i x^c\otimes x^i + \Phi^{bc}_i x^d\otimes x^i\right)\\
&+\frac{h^4}{12}C^a_{ic}C^i_{db}\left(\Phi^{bc}_i\Phi^{id}_j x^j + \Phi^{bd}_i\Phi^{ic}_j x^j\right)
\end{aligned}$$

$\left[\text{using that } C^{-}_{\alpha\beta}\Phi^{\alpha\beta}_{-}=0\text{, the commutativity of } \xi \text{ and skew-commutativity of } \nu\right]$

$$
\begin{aligned}
&= -\frac{h^2}{4}C^a_{ic}C^i_{db}\,x^b\otimes x^c\otimes x^d + \frac{h^4}{12}C^a_{ic}C^i_{db}\,\Phi^{bc}_j\Phi^{jd}_k\,x^k \\
&\quad [\text{by the super-involutive relation}] \\
&= -\frac{h^2}{12}\left(C^a_{ic}C^i_{db}\,x^b\otimes x^c\otimes x^d + C^a_{id}C^i_{bc}\,x^d\otimes x^b\otimes x^c + C^a_{ib}C^i_{cd}\,x^c\otimes x^d\otimes x^b\right) \\
&\quad \left[\text{by } x^a\otimes x^b - x^b\otimes x^a - \hbar\Phi^{ab}_k x^k = 0\,\right] \\
&= -\frac{h^2}{12}\left(C^a_{ic}C^i_{db} + C^a_{id}C^i_{bc} + C^a_{ib}C^i_{cd}\right)x^b\otimes x^c\otimes x^d - \frac{\hbar^3}{12}C^a_{id}C^i_{bc}\Phi^{db}_j\,x^j\otimes x^c \\
&\quad -\frac{\hbar^3}{12}C^a_{id}C^i_{bc}\Phi^{dc}_j\,x^b\otimes x^j - \frac{\hbar^3}{12}C^a_{ib}C^i_{cd}\Phi^{db}_j\,x^c\otimes x^j - \frac{\hbar^3}{12}C^a_{ib}C^i_{cd}\Phi^{cb}_j\,x^j\otimes x^d \\
&\quad = 0 \\
&\quad [\text{using the Jacobi identity and the skew-commutativity of } \nu]
\end{aligned}
$$

□

We have constructed Γ_1 which solves $d_H\Gamma_1 + [\Gamma_1,\Gamma_2]_G = 0$ and satisfies $\Gamma_1^2 = 0$, thus the last two equations for the star-product can be solved for by setting $\Gamma_0 = 0$.

4.2.5. Main Theorem *There is a universal perturbative quantization of (possibly infinite-dimensional) quasi-Poisson structures which does not use Drinfeld's associators.*

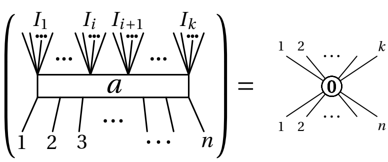
This theorem is an immediate corollary of the following proposition

Proposition

There is a morphism of dg props

$$\rho : \mathsf{DefQ}^{\hbar} \longrightarrow \mathsf{qPois}$$

such that

$\pi_1 \circ \rho$ 

for $a = n + k - 2$ and $|I_1| = \ldots = |I_k| = 1$ (4.8)

where $\pi_1 : \mathsf{qPois} \to (\mathsf{qPois})_1$ is the projection of the free prop qPois to the subspace, $(\mathsf{qPois.})_1$, consisting of graphs with precisely 1 internal vertex.

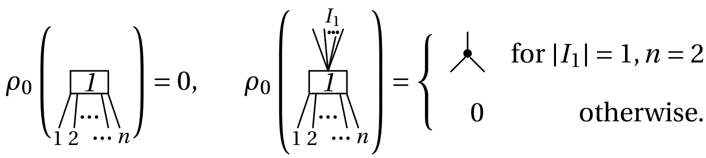
Proof. The above formulae show that there is a morphism of dg props

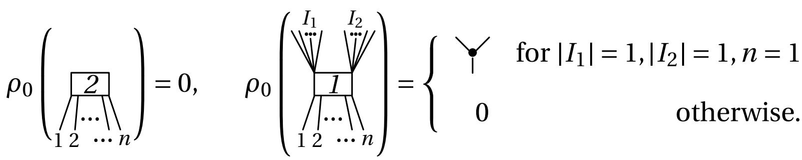
$$\rho_0 : (\mathsf{DefQ}^{\hbar}, \delta) \longrightarrow (\mathsf{LieB}^{\diamond}_{odd}, 0)$$

such that

- ρ_0 vanishes on all generators (4.2) with $k = 0$ and $k \geq 3$,
- ρ_0 sends the generators (4.2) with $k = 1$ and weight a into a graph incarnation of the $\hbar^a$-summand of the above explicit solution Γ_1 of the equations (4.7)
- ρ_0 sends the generators (4.2) with $k = 2$ and weight a into a graph incarnation of the $\hbar^a$-summand of the above explicit solution Γ_2 of the equations (4.7).

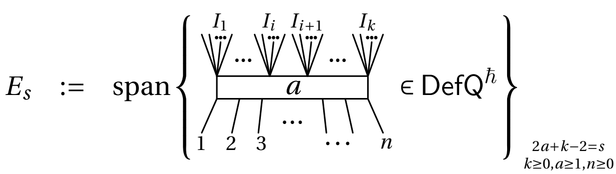
In particular one has the following formulae (cf. [Me5])



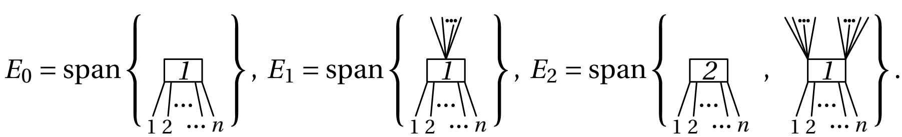


and hence can use a standard lifting argument and a cofibrant structure of $\mathsf{DefQ}^\hbar$ first observed in [Me5] to finish the proof of the Main Theorem.

Define E_s to be zero for negative s and, for $s \geq 0$,



For example,



Let $\mathsf{DefQ}^\hbar_s \subset \mathsf{DefQ}^\hbar$ be the free prop generated by $\oplus_{i=0}^s E_i$. Thereby we get an increasing filtration, $0 \subset \mathsf{DefQ}^\hbar_0 \subset \ldots \subset \mathsf{DefQ}^\hbar_s \subset \mathsf{DefQ}^\hbar_{s+1} \ldots$ with

$$\lim_{s\to\infty} \mathsf{DefQ}^\hbar_s = \mathsf{DefQ}^\hbar.$$

A straightforward inspection of the formula for the differential δ in $\mathsf{DefQ}^{\hbar}$ implies

$$\delta E_{s+1} \subset \mathsf{DefQ}^{\hbar}_s,$$

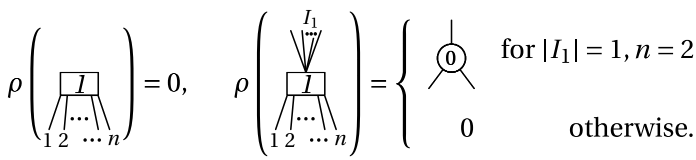
i.e. that the dg prop $(\mathsf{DefQ}^{\hbar}, \delta)$ has a cofibrant structure (even an elemental cofibrant structure in the sense of Definition 17 in [Mar4] formulated for props rather than for operads). As the natural projection

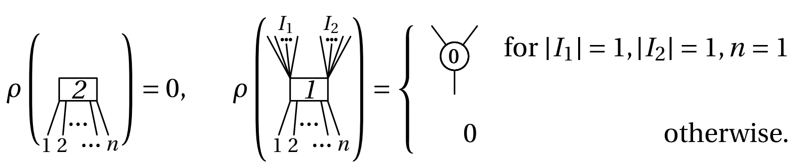
$$\pi : \mathsf{qPois} \longrightarrow \mathsf{LieB}^{\diamond}_{odd}$$

is a quasi-isomorphism, one can use an inductive argument to construct a lift of the morphism ρ_0 to the required morphism ρ making the following diagram commutative,

$$\begin{tikzcd} & \mathsf{qPois} \arrow[d, "\pi"] \\ \mathsf{DefQ}^{\hbar} \arrow[ur, "\rho"] \arrow[r, "\rho_0"'] & \mathsf{LieB}^{\diamond}_{odd} \end{tikzcd}$$

Indeed, we start our induction by defining the values of ρ on the subspace $E_0 \oplus E_1 \oplus E_2$ as follows





On this subset of generators the equations $\pi \circ \rho = \rho_0$ and (4.8) hold obviously true

Let us assume next that we have a morphism

$$\rho_s : \mathsf{DefQ}^{\hbar}_s \longrightarrow \mathsf{qPois}$$

satisfying the condition $\pi \circ \rho_s = \rho_0$ for some $s \geq 3$. Let us show that we can extend ρ_s to a morphism of dg props,

$$\begin{array}{rccc} \rho_{s+1} : & \mathsf{DefQ}^{\hbar}_{s+1} & \longrightarrow & \mathsf{qPois} \\ & a & \longrightarrow & \rho_{s+1}(a) \end{array}$$

such that $\pi \circ \rho_{s+1} = \rho_0$ and the condition (4.8) hold true. Let a' be a lift of $\rho_0(a)$ along the surjection π. Then $\rho_s(\delta a) - \partial a'$ is a cycle in qPois. which the projection π sends to zero. As π is a quasi-isomorphism, this element must be exact, $\rho_s(\delta a) - \partial a' = \partial a''$, for some $a'' \in \mathsf{qPois}$. We set $\rho_{s+1}(a) := a' + a''$ completing thereby the inductive construction of ρ as a morphism of dg props. The condition $\pi \circ \rho = \rho_0$ is satisfied by construction. The second condition (4.8) is also satisfied by induction due to exactly the same observation as the one used in (cf. [Me4]) — the Gerstenhaber brackets, $[\gamma, \gamma]_G$, of a degree 1 polyvector field γ (viewed as a polydifferential operator) contain the Schouten-Nijenhuis bracket, $[\gamma, \gamma]_{SN}$, as one of the the summands. □

Proof of the Main Theorem

Given a quasi-Poisson structure on a (possibly infinite-dimensional) graded vector space V. This is the same as a representation

$$\nu : \mathsf{qPois} \longrightarrow \mathsf{End}_V$$

of the prop qPois. in V. Composing ν with the above morphism ρ we obtain a representation

$$\nu^\hbar : \mathsf{DefQ}^\hbar \xrightarrow{\rho} \mathsf{qPois} \xrightarrow{\nu} \mathsf{End}_V$$

of the prop $\mathsf{DefQ}^\hbar$ in V which gives us the required deformation quantization of ν. At no stage of this construction are Drinfeld' associators utilized. The proof is completed.

Remark

The map $\rho : \mathsf{DefQ}^\hbar \longrightarrow \mathsf{qPois}$ takes values in the ordinary prop, not in the wheeled completion of the minimal resolution of the prop LieB_{odd} as in [Me4]. Therefore the map ρ sends to zero all generators (4.2) of $\mathsf{DefQ}^\hbar$ with $k = 0$.

Let I be the differential ideal of $\mathsf{DefQ}^\hbar$ generated by corollas (4.2) with $k = 0$; this ideal is clearly differential so that it makes sense to consider the quotient dg prop

$$\mathsf{DefQ}_0^\hbar := \mathsf{DefQ}^\hbar.$$

Our morphism ρ factors through this prop, i.e. it can be written as the composition

$$\rho : \mathsf{DefQ}^\hbar \longrightarrow \mathsf{DefQ}_0^\hbar \xrightarrow{\rho_0} \mathsf{qPois}$$

where the first arrow is the canonical projection.

The prop $\mathsf{DefQ}_0^\hbar$ has a nice algebraic meaning. If admissible representations of $\mathsf{DefQ}^\hbar$ in a dg vector space (V, d) give us *curved* A_∞ structures in

$\mathcal{O}_V[[\hbar]]$ (see Proposition 4.1.5, representations of $\mathsf{DefQ}_0^{\hbar}$ in V give us *ordinary or flat* A_∞ structures in V.

We conclude therefore that contrary to the case of universal quantizations of ordinary graded Poisson structures which gives rise to *curved* A_∞ structures in $\mathcal{O}_V[[\hbar]]$, our universal deformation quantization of quasi-Poisson structures always gives us flat A_∞-structures, that is, Maurer-Cartan elements Γ in the reduced polydifferential Hochschild complex,

$$d_H\Gamma + \frac{1}{2}[\Gamma,\Gamma]_G = 0, \quad \Gamma|_{\hbar=0} = d,$$

with

$$\Gamma \in \mathcal{D}_V^{\geq 1}[[\hbar]] \subset \left(\bigoplus_{k\geq 1} \mathrm{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V)[1-k] \right)[[\hbar]].$$

Put another way, our universal quantization of quasi-Poisson structures give rise to morphisms of dg operads

$$\varepsilon_\hbar : \mathsf{Ass}_\infty \longrightarrow \mathsf{End}_{\odot^\bullet V}[[\hbar]]$$

such that $\varepsilon_0 := \varepsilon_\hbar|_{\hbar=0}$ equals to the standard morphism

$$\varepsilon_0 : \mathsf{Ass}_\infty \longrightarrow \mathsf{End}_{\odot^\bullet V}$$

which factors through the composition

$$\varepsilon_0 : \mathsf{Ass}_\infty \longrightarrow \mathsf{Ass} \longrightarrow \mathsf{End}_{\odot^\bullet V}$$

and sends the generator of Ass into the standard graded commutative product in the symmetric tensor algebra $\odot^\bullet V$. We shall use this observation heavily below when showing a second “more conceptual" proof of the Main Theorem.

4.3 A second proof of the Main Theorem

Merkulov-Willwacher polydifferential functor

The authors of [MW1] constructed an exact functor from the category of dg (augmented) props to the category of operads which has the property that for any prop $\mathsf{P} = \{\mathsf{P}(m,n)\}_{m,n\geq 1}$ and its representation

$$\rho : \mathsf{P} \longrightarrow \mathsf{End}_V$$

in a dg vector space V, the associated operad $\mathcal{O}\mathsf{P} = \{\mathcal{O}\mathsf{P}(k)\}_{k\geq 1}$ admits an associated representation,

$$\rho^{poly} : \mathcal{O}\mathsf{P} \longrightarrow \mathsf{End}_{\odot^\bullet V}$$

in the graded commutative algebra $\odot^\bullet V$ on which elements $p \in \mathsf{P}$ act as polydifferential operators.

Let P be an augmented prop. As an $\mathbb{S}$-module the operad $\mathcal{O}\mathsf{P} = \{\mathcal{O}\mathsf{P}(k)\}_{k\geq 1}$ is defined as follows [MW1],

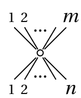
$$\mathcal{O}\mathsf{P}(k) := \mathsf{Com}(k) \oplus \prod_{m,n\geq 1} \bigoplus_{\substack{[m]=J_1\sqcup\ldots\sqcup J_k \\ \#J_1,\ldots,\#J_k\geq 0}} \mathcal{O}\mathsf{P}^n_{J_1,\ldots,J_k}$$

where

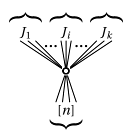
$$\mathcal{O}\mathsf{P}^n_{J_1,\ldots,J_k} := \mathrm{id}_{\mathbb{S}_{J_1}} \otimes \ldots \otimes \mathrm{id}_{\mathbb{S}_{J_k}} \otimes_{\mathbb{S}_{J_1}\times\ldots\times\mathbb{S}_{J_k}} \otimes \bar{\mathsf{P}}(m,n) \otimes_{\mathbb{S}_n^{op}} \mathrm{id}_n$$

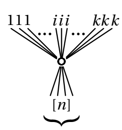
where id_I stands for the trivial one-dimensional representation of the permutation group $\mathbb{S}_I$, and $\mathsf{Com} = \{\mathsf{Com}(k)\}_{k\geq 1}$ for the operad of commutative associative algebras. Thus an element of the summand $\mathcal{O}\mathsf{P}^n_{J_1,\ldots,J_k} \subset \mathcal{O}\mathsf{P}(k)$ is an element of $\mathsf{P}(\#J_1+\ldots+\#J_k, n)$ with all its n inputs symmetrized and all its outputs in each bunch $J_s \subset [m]$, $s \in [k]$, also symmetrized. We assume from now on that all legs in each bunch J_s are labelled by the same integer s; this defines an action of the group $\mathbb{S}_k$ on $\mathcal{O}\mathsf{P}(k)$.

It is often useful to represent elements p of the (non-unital) prop $\bar{\mathsf{P}}$ as (decorated) corollas,

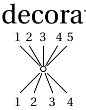
$p \ \sim$  $\in \bar{\mathsf{P}}(m,n)$

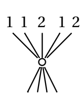
The image of such an element under the projection $\pi^n_{J_1,\ldots,J_k}$ is represented pictorially as the same corolla whose output legs are decorated by the same symbol 1 (which is omitted in the pictures) and the input legs decorated with possibly *coinciding* indices as in the following picture

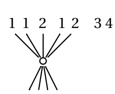


or


$1 \leq i \leq k.$

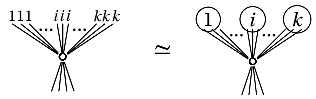
Note that some of the sets J_i can be empty so that some of the numbers decorating inputs can have no legs attached! For example, an element $q =$  $\in \bar{\mathsf{P}}(5,4)$ can generate several different elements in $\mathcal{O}\mathsf{P}$,



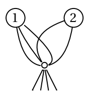
$\in \mathcal{O}\mathsf{P}(2)$,


$\in \mathcal{O}\mathsf{P}(4)$, *etc.* (4.9)

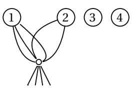
Often (but not always) it is useful to represent elements of $\mathcal{O}\mathsf{P}$ not as corollas decorated by elements from $\mathcal{P}$ whose legs are labelled by possibly coinciding natural numbers, but as graphs having two types of vertices: the small one (with is decorated by an element of $\bar{\mathcal{P}}$) and new big ones corresponding to inputs of $\mathcal{O}\mathsf{P}$ and having a “non-coinciding" numerical labelling



In this notation elements (4.9) gets represented, respectively, as



and



while generators of $\mathcal{C}om(n) \subset \mathcal{O}\mathsf{P}(n)$ as ① ② $\cdots$ ⓝ .

Using this notation it is easy to define an operadic composition,

$$\circ_i : \mathcal{O}\mathsf{P}(m) \otimes \mathcal{O}\mathsf{P}(n) \longrightarrow \mathcal{O}\mathsf{P}(m+n-1)$$

for any $i \in [m]$: take any elements $\Gamma_1 \in \mathcal{O}\mathsf{P}(m)$ and $\Gamma_2 \in \mathcal{O}\mathsf{P}(n)$, then $\Gamma_1 \circ_1 \Gamma_2$ is sum a graphs obtained by substituting the graph Γ_2 into the i-th white vertex of Γ_1 and then taking a sum over all possible ways of attaching half-edges of the i-th vertex to the output legs and/or white vertices of Γ_2.

It is important to notice that the functor $\mathcal{O}$ is exact. For example, if

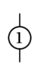
$$\nu : \mathsf{qPois} \longrightarrow \mathsf{LieB}^{\diamond}_{odd}$$

is a quasi-isomorphism of props, then

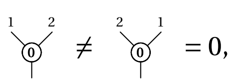
$$\mathcal{O}\nu : \mathcal{O}\mathsf{qPois} \longrightarrow \mathcal{O}\mathsf{LieB}^{\diamond}_{odd}$$

is a quasi-isomorphism of the associated polydifferential operads.

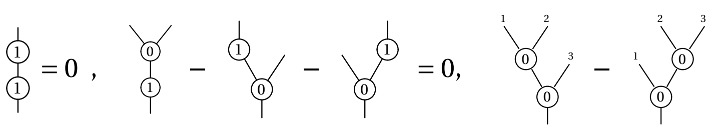
4.3.1. An extended version of the operad Ass Let $\mathsf{Ass}^{\diamond}$ be an operad generated by degree 1 corolla



and degree zero corollas



subject to the following relations,

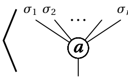


Thus $\mathsf{Ass}^{\diamond}$ is an extension of the operad of associative algebras Ass by one extra degree 1 generator of arity $(1,1)$ subject to quadratic relations.

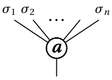
We want to construct a minimal resolution of $\mathsf{Ass}^{\diamond}$.

An extended version of the operad Ass_∞

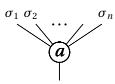
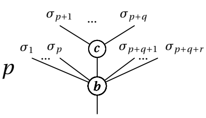
Let $\mathsf{Ass}_\infty^\diamond$ be the free operad generated by the following $\mathbb{S}$-module

$$E^\diamond(n) = \mathrm{span}\left\langle \right.$$

$$\left.\right\rangle_{\substack{\sigma\in\mathbb{S}_n \\ n+a\geq 2,\ a\geq 0}} \qquad (4.10)$$

the degree of the generators is given in the same manner as for the regular Ass_∞ operad;


$$= 2-n.$$

The standard differential in Ass_∞ extends naturally to $\mathsf{Ass}^\diamond$ by the following formula (which is analogous to the one in the operad "$\mathcal{L}ie_\infty^\diamond$" introduced in [CMW])

$$d^\diamond\left(\right.$$

$$\left.\right) = \sum_{\substack{p+q+r=n\\ a=b+c}} (-1)^{qr+p}$$


Theorem 4.3.1. *The dg operad* $(\mathsf{Ass}_\infty^\diamond, d^\diamond)$ *is a minimal resolution of* $\mathsf{Ass}^\diamond$.

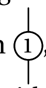
Proof. We shall use the *exact* functor F introduced in Step 1 of the proof of Theorem 2.7.1 in [CMW],

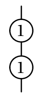
$$F : \text{category of dg } \frac{1}{2}\text{-props} \longrightarrow \text{category of dg properads},$$

where, for a $\frac{1}{2}$-prop P,

$$F(\mathsf{P})(m,n) := \bigoplus_{\Gamma\in\overline{\mathrm{Gr}}(m,n)} \left(\bigotimes_{v\in v(\Gamma)} \mathsf{P}(Out(v), In(v)) \otimes \odot^\bullet H^1(\Gamma,\partial\Gamma)\right)_{Aut(\Gamma)},$$

with $\overline{\mathrm{Gr}}(m,n)$ representing the set of all (isomorphism classes of) oriented graphs with n output legs and m input legs an such that they do not have internal edges which might correspond to $\frac{1}{2}$-prop compositions. Here $H^1(\Gamma,\partial\Gamma)$ is the relative cohomology of the connected graph Γ viewed as a 1-dimensional CW complex; the space $H^1(\Gamma,\partial\Gamma)$ is assumed to live in cohomological degree 1. In particular the symmetric tensor algebra $\odot^\bullet H^1(\Gamma,\partial\Gamma)$ is finite dimensional, and the square of any relative cohomology class vanishes. The differential acts trivially on the $H^1(\Gamma,\partial\Gamma)$ part.

For any graph Γ, $H^1(\Gamma,\partial\Gamma)$ may be identified with the space of formal linear combinations of edges of Γ, modulo the relations that the sum of incoming edges at any vertex equals the sum of outgoing edges. Then $\odot^p H^1(\Gamma,\partial\Gamma)$ may be identified with formal linear combinations of p-fold wedge products of edges, modulo similar relations. Such a product of p edges may be represented combinatorially by putting a marking of the form , on these p edges. As for any edge e the wedge product $e \wedge e$ vanishes identically, we have a relation



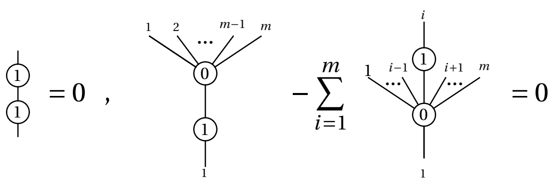
$$= 0$$

for these markings.

Operads form a special case of $\frac{1}{2}$-props, and the functor F restricted to dg operads gives us an exact endofunctor

$$F : \text{category of dg operads} \longrightarrow \text{category of dg operads}.$$

Let us apply this functor to dg operads Ass_∞ and Ass. The operad $F(\mathsf{Ass})$ is precisely the operad $\mathsf{Ass}^\diamond$. The dg operad $F(\mathsf{Ass}_\infty)$ is generated by corollas (4.10) with either $a = m = n = 1$ or $a = 0$ and $m+n \geq 3$ subject to the relations



The differential in $F(\mathsf{Ass}_\infty)$ acts non-trivially only on corollas with weight zero, and is given by the formula identical to the case of Ass_∞.

As the functor F is exact, we have a canonical projection

$$F(\mathsf{Ass}_\infty) \longrightarrow F(\mathsf{Ass}) \equiv \mathsf{Ass}^\diamond$$

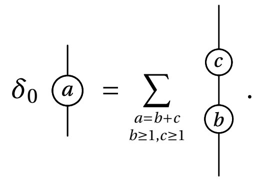
which is a quasi-isomorphism.

We also have an epimorphism of dg operads

$$p : \mathsf{Ass}_\infty^\diamond \longrightarrow F(\mathsf{Ass}_\infty)$$

which sends to zero all generators (4.10) which do not satisfy the condition $a = m = n = 1$ or the condition $a = 0$ and $m + n \geq 3$ (it is easy to check that this projection respects the differentials). To complete the proof of the theorem we have to show that the morphism p is a quasi-isomorphism. Consider a filtration of $\mathsf{Ass}_\infty^\diamond$ given, for any graph Γ, by the difference $a(\Gamma) - n(\Gamma)$, where $a(\Gamma)$ is the sum of all decorations of non-bivalent vertices and $n(\Gamma)$ is the sum of valences of non-bivalent vertices (cf.[CMW]). On the 0-th page

$E_0^{pq}\mathsf{Ass}_\infty^\diamond$ of this spectral sequence the induced differential δ_0 acts only on bivalent vertices by splitting them as follows



By Proposition 2.6.1 from [CMW], the associated δ_0-cohomology (i.e. the first page of this spectral sequence)

$$H(E_0^{pq}\mathsf{Ass}_\infty^\diamond, \delta_0) = E_1^{pq}\mathsf{Ass}_\infty^\diamond$$

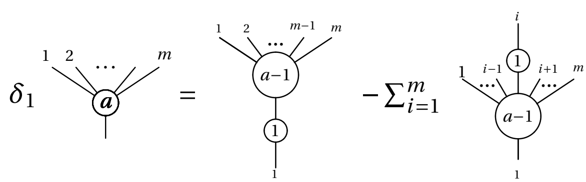
consists of graphs which

i) have no bivalent vertices,

ii) have each non-bivalent vertex assigned weight $a \in \mathbb{Z}^{\geq 0}$ and

iii) have every edge either decorated by the symbol ① or not decorated at all.

The induced differential δ_1 in $E_1^{pq}\mathsf{Ass}_\infty^\diamond$ is given by



Now comes the main observation that each complex $(E_1^{pq}\mathsf{Ass}_\infty^\diamond(m), \delta_1)$, $m \geq 2$, is identical to the following one

$$\hat{F}(\mathsf{Ass}_\infty)(m) = \bigoplus_{\Gamma \in \overline{\mathsf{G}}(m,1)} \left(\bigotimes_{v \in v(\Gamma)} \mathsf{Ass}_\infty(Out(v)) \otimes \odot^\bullet C(\Gamma, \partial\Gamma) \right)_{Aut(\Gamma)},$$

where $\overline{\mathsf{G}}(m,1)$ is the set of planar trees with one input leg and m output legs, and $C(\Gamma, \partial\Gamma)$ is the relative chain complex of the graph Γ viewed as the 1-dimensional CW complex; the differential in $\hat{F}(\mathsf{Ass}_\infty)$ is induced from the standard chain differential in $C(\Gamma, \partial\Gamma)$. Indeed, $C(\Gamma, \partial\Gamma)$ is a graded vector space

$$C(\Gamma, \partial\Gamma) = \mathrm{span}\langle V(\Gamma)\rangle[0] \ \oplus \ \mathrm{span}\langle E(\Gamma)\rangle[1]$$

concentrated in degree zero and degree one; the degree zero part is spanned over a field $\mathbb{K}$ by the (internal) vertices of the graph Γ, and the degree one part is spanned over $\mathbb{K}$ by the set of its internal edges and legs. The differential ∂ in $C(\Gamma, \partial\Gamma)$ is trivial on edges, and is given on an arbitrary vertex $v \in V(\Gamma)$ by the formula

$$dv = e_{in(v)} - \sum_{e_v \in Out(v)} e_v,$$

where $Out(v)$ is the set of edges outgoing from v and $e_{in(v)}$ is the unique edge incoming to the vertex v. Therefore the symmetric tensor algebra

$$\bigodot^{\bullet} C(\Gamma, \partial\Gamma)$$

is spanned over $\mathbb{K}$ by monomials of the form

$$\prod_{v \in V(\Gamma)} v^{a_v} \otimes \prod_{e \in E(\Gamma)} e^{\alpha_e}$$

where a_v is some non-negative natural number, and α_v takes values 0 or 1. An element of

$$\bigoplus_{\Gamma \in \overline{\mathsf{G}}(m,1)} \left(\bigotimes_{v \in v(\Gamma)} \mathsf{Ass}_\infty(Out(v)) \prod_{v \in V(\Gamma)} v^{a_v} \otimes \prod_{e \in E(\Gamma)} e^{\alpha_e} \otimes \right)_{Aut(\Gamma)}$$

can be understood as an element of Ass_∞ whose vertices v are decorated by non-negative numbers a_v and whose edges e are either decorated with symbol 1 (in the case $\alpha_e = 1$) or not decorated at all (in the case $\alpha_e = 0$). The differential can then be identified with δ_1 above. As $\mathrm{H}^0(\Gamma, \partial\Gamma) = 0$, we conclude immediately that

$$\mathrm{H}(E_1^{pq} \mathsf{Ass}_\infty^\diamond, \delta_1) \simeq F(\mathsf{Ass}_\infty)$$

proving thereby that the map p is a quasi-isomorphism.

The proof of the theorem is completed.

$\square$

4.3.2. Representations of the operad $\mathsf{Ass}_\infty^\diamond$ The representations of $\mathsf{Ass}_\infty^\diamond$ in a $\mathbb{K}$-module V is a series of maps

$$\mu_n^{(a)} : V^{\otimes n} \longrightarrow V$$

subject to the compatibility condition C_n for each $n \geq 1$

$$C_n: \quad \sum_{\substack{p+q+r=n \\ a=b+c}} (-1)^{qr+p} \mu_{p+1+r}^{(b)} \circ \left(\mathrm{id}^{\otimes p} \otimes \mu_q^{(c)} \otimes \mathrm{id}^r \right) = 0$$

We can think of the collection $\mu_n^{(0)}, \mu_n^{(1)}, \dots$ as homogenous $\hbar$ components of a continuous morphism of $\mathbb{K}[[\hbar]]$

$$\mu_n : V^{\otimes n}[[\hbar]] \longrightarrow V[[\hbar]],$$

which if extended to a completed tensor product of $\mathbb{K}[[\hbar]]$-modules can be interpreted as

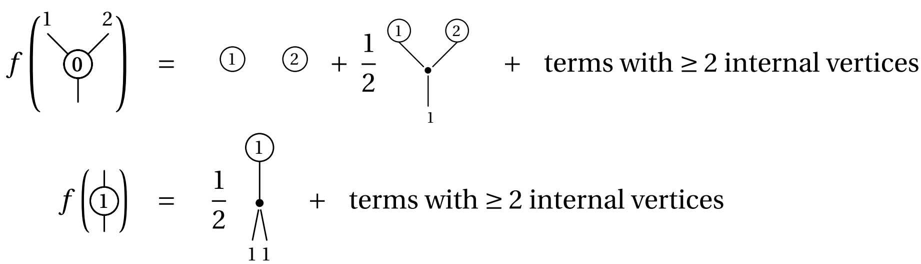
$$\mu_n^{\hbar} : (V[[\hbar]])^{\otimes n} \longrightarrow V[[\hbar]].$$

One can check that the system of maps $\mu_\bullet^{\hbar}$ constitute an Ass_∞ structure on the $\mathbb{K}[[\hbar]]$-module $V[[\hbar]]$.

4.3.3. Lemma There is a morphism of operads

$$f : \mathsf{Ass}^\diamond \longrightarrow \mathcal{O}\mathsf{LieB}^\diamond_{odd}$$

such that



Proof. Given any $\mathsf{LieB}^\diamond_{odd}$ structure in an arbitrary graded vector space V, we constructed above in section 4.2.4 a universal Maurer-Cartan element (Γ_2, Γ_1) of the Hochschild dg Lie algebra of $\mathcal{O}_V$, i.e. the structure of a dg associative algebra structure in $\mathcal{O}_V$ which deforms the standard graded commutative multiplication in $\mathcal{O}_V$. Put another way, we get a universal representation of $\mathsf{Ass}^\diamond$ in $\mathcal{O}_V$ which sends the multiplication generator of $\mathsf{Ass}^\diamond$ into Γ_2 and the $(1,1)$ generator of $\mathsf{Ass}^\diamond$ into Γ_1. Restating this result in terms of graphs, we obtain the above claim.

□

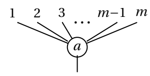
4.3.4. Theorem There exists a morphism of dg props

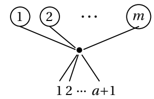
$$F : \mathsf{Ass}^\diamond_\infty \longrightarrow \mathcal{O}\mathsf{qPois}$$

which makes the following diagram commutative,

$$\begin{tikzcd} \mathsf{Ass}^\diamond_\infty \arrow[r, "F"] \arrow[d, "p"'] & \mathcal{O}\mathsf{qPois} \arrow[d, "\pi"] \\ \mathsf{Ass}^\diamond \arrow[r, "f"'] & \mathcal{O}\mathsf{LieB}^\diamond_{odd} \end{tikzcd}$$

and satisfies the condition

$$\pi_1 \circ F \left(\vphantom{\Big|} \right.$$


$$\left. \vphantom{\Big|} \right) = \lambda_m \tag{4.11}$$


where λ_m is a non-zero constant of unspecified (and inconsequential) value and where $\pi_1 : \mathsf{qPois} \to (\mathsf{qPois})_1$ is the projection of the free prop qPois to the subspace, $(\mathsf{qPois})_1$, consisting of graphs with precisely 1 internal vertex of weight zero.

Proof. The existence of F follows immediately from existence of the morphism f and the fact that vertical lines represent cofibrant replacements of dg operads. The second condition of $\pi_1 \circ F$ follows from the standard induction (cf. [Me4]).

□

4.3.5. Second proof of the Main Theorem Given a quasi-Poisson structure in a dg space V, i.e., given a representation of dg properads

$$\rho : \mathsf{qPois} \longrightarrow \mathsf{End}_V.$$

By the very definition of the Merkulov-Willwacher polydifferential functor $\mathcal{O}$, there is an associated representation of dg operads

$$\mathcal{O}\rho : \mathcal{O}\mathsf{qPois} \longrightarrow \mathsf{End}_{\odot^\bullet V}$$

in the symmetric tensor algebra $\odot^\bullet V$. Composing it with the canonical morphism F from the above Theorem we obtain a morphism of dg operads,

$$\mathsf{Ass}_\infty^\diamond \longrightarrow \mathsf{End}_{\odot^\bullet V}$$

which gives us the required universal deformation quantization of the given quasi-Poisson structure.

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