ON STRONGLY PRIME IDEALS AND STRONGLY ZERO-DIMENSIONAL RINGS

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Abstract

A prime ideal $\mathfrak{p}$ is said to be strongly prime if whenever $\mathfrak{p}$ contains an intersection of ideals, $\mathfrak{p}$ contains one of the ideals in the intersection. A commutative ring with this property for every prime ideal is called strongly zero-dimensional. Some equivalent conditions are given and it is proved that a zero-dimensional ring is strongly zero dimensional if and only if the ring is quasi semi-local. A ring is called strongly n-regular if in each ideal $\mathfrak{a}$ there is an element a such that $x = ax$ for all $x \in \mathfrak{a}^n$. Connections between the concepts strongly zero-dimensional and strongly n-regular are considered.

1. Introduction.

A prime ideal $\mathfrak{p}$ containing a finite intersection $\bigcap_{i=1}^{n} \mathfrak{a}_i$ of ideals, obviously contains one of the ideals in the intersection. But for infinite intersections the corresponding result may not hold, as elementary examples show. For example the intersection of the non-zero ideals in the ring of integers is the zero ideal. Thus the following question seems rather apt. When does the implication $\mathfrak{p} \supseteq \bigcap_{i \in \mathfrak{I}} \mathfrak{a}_i \Rightarrow \mathfrak{p} \supseteq \mathfrak{a}_i$ *for some* $i \in \mathfrak{I}$ hold, and when is this not the case?

The work on this problem was initiated by C. Jayaram, K.H. Oral & Ü Tekir in [2], where they introduced the concepts of a strongly prime ideal and a strongly zero-dimensional ring, defined as follows.

Definitions 1.1 (Jayaram, Oral, Tekir). A prime ideal $\mathfrak{p}$ of R is said to be *strongly prime*, if whenever $\mathfrak{p}$ contains an intersection of ideals, it contains one of the ideals in the intersection, i.e. $\mathfrak{p} \supseteq \bigcap_{i \in \mathfrak{I}} \mathfrak{a}_i \Rightarrow \mathfrak{p} \supseteq \mathfrak{a}_i$ for some $i \in \mathfrak{I}$. R is said to be a *strongly zero-dimensional* ring if every prime ideal of R is strongly prime.

Among other things Jayaram, Oral & Tekir proved that strongly zero-dimensional rings have Krull dimension 0, they gave a list of equivalent conditions, and they investigated connections with von-Neumann regular rings. In this paper we shall consider some other aspects of the theory, and introduce a few related concepts. All rings are assumed to be commutative with a unit element, and R always denotes such a ring. It should be noted also, that the concept has been extended to the concept of strongly zero-dimensional modules. Cf [4]. In this paper, however, we study rings only.

The two main results, in this paper, on strongly zero-dimensional rings are Theorems 1.8 and 2.4. In Theorem 1.8 we show that R is strongly zero-dimensional if and only if R sastisfies the following two conditions. No maximal ideal contains the intersection of the other maximal ideals and each chain of the form $(x) \supseteq (x^2) \supseteq (x^3) \supseteq \cdots$ stops. In Theorem 2.4 this result is improved and we prove that among the zero-dimensional rings, the strongly zero-dimensional rings are precisely the rings with only a finite number of maximal ideals.

We adopt the modern convention and call rings with only a finite number of maximal ideals quasi-semi-local rings (the term semi-local is restricted to the Noetherian case). Similarly a ring with only one maximal ideal is said to be quasi-local. A key result preceding Theorem 2.4 is, that in a ring with an infinite number of maximal ideals, there is always a maximal ideal which contains the intersection of the other maximal ideals. However, it is quite possible for *some* maximal ideal to be strongly prime even if the ring is not quasi semi-local.

During the investigation we will also encounter some other problems, which are related to the questions on when a ring is strongly zero-dimensional or when a prime ideal is strongly prime. When do infinite intersections commute with localizing at a prime ideal? When do infinite intersections commute with radicals?

In the final section we introduce, what I have chosen to call strongly n-regular rings. The rings in this class, which includes the commutative von Neumann regular rings, are all strongly zero-dimensional. The converse is not true, and we provide an example of a ring which is strongly zero-dimensional but not strongly n-regular for any n.

The most trivial examples of strongly zero-dimensional rings are the zero-dimensional quasi-local rings. Indeed in such a ring all ideals are contained in the one and only prime ideal. Another easy example is afforded by the Artinian rings, since in an Artinian ring all intersections are effectively finite. Moreover, since strongly-zero dimensional rings are zero dimensional ([2, Theorem 3.9]) it follows that the Artinian rings are the only Noetherian rings, which are strongly zero-dimensional.

The fact that strongly zero-dimensional rings are zero dimensional has the following counterpart for prime ideals. This is essentially [2, Lemma 2.5], but we repeat the short argument.

Proposition 1.2. *Let $\mathfrak{p}$ be a strongly prime ideal. Then $\mathfrak{p}$ is a maximal ideal.*

Proof. Obviously $\mathfrak{p}/\mathfrak{a}$ is also strongly prime in $R/\mathfrak{a}$ for any ideal $\mathfrak{a} \subseteq \mathfrak{p}$ of R. Thus we can reduce to the case, where R is an integral domain with strongly prime zero ideal, and we shall prove that R is a field. In this case the intersection of all non-zero ideals must be a non-zero ideal, and hence there is in R a unique smallest non-zero ideal, say Rr. Now take any $a \neq 0$ in R. Then $ar \neq 0$ and hence $r = arb$ some b. It follows that $1 = ab$, and we have proved that R is a field.

In [2, Theorem 2.9] there is provided a list of conditions on a ring equivalent to the condition that the ring is strongly zero-dimensional. We shall presently extend this list, but first we consider prime ideals.

Proposition 1.3 *A prime ideal $\mathfrak{p}$ is strongly prime if and only if for every indexed set $\mathfrak{a}_i$, $i \in \mathfrak{I}$ of ideals*

$$\Big(\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i\Big)_{\mathfrak{p}} = \bigcap_{i\in\mathfrak{I}} (\mathfrak{a}_i)_{\mathfrak{p}}$$

i.e., localizing at $\mathfrak{p}$ commutes with the formation of intersections.

Proof. Note that finite intersections always commute with localization. Also note that the inclusion $\big(\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i\big)_{\mathfrak{p}} \subseteq \bigcap_{i\in\mathfrak{I}} (a_i)_{\mathfrak{p}}$ holds in general. For take $\frac{x}{s} \in (\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i)_{\mathfrak{p}}$. Then $xt \in \bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i$ for some $t \notin \mathfrak{p}$, and hence $xt \in \mathfrak{a}_i$ for all i, which means that $\frac{x}{s} \in (\mathfrak{a}_i)_{\mathfrak{p}}$ for all i, and hence $\frac{x}{s} \in \bigcap_{i\in\mathfrak{I}} (\mathfrak{a}_i)_{\mathfrak{p}}$. It now remains to prove that, given $\mathfrak{p}$, the reverse inclusion holds for all sets of $\mathfrak{a}_i$, if and only if $\mathfrak{p}$ is strongly prime. First suppose that $\mathfrak{p}$ is strongly prime and take $x \in \bigcap_{i\in\mathfrak{I}} (\mathfrak{a}_i)_{\mathfrak{p}}$. Then $(\mathfrak{a}_i : x) \nsubseteq \mathfrak{p}$ holds for all i, and hence (since $\mathfrak{p}$ is strongly prime) $\bigcap_{i\in\mathfrak{I}} (\mathfrak{a}_i : x) \nsubseteq \mathfrak{p}$. But $(\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i : x) = \bigcap_{i\in\mathfrak{I}} (\mathfrak{a}_i : x)$, so $(\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i : x) \nsubseteq \mathfrak{p}$, and hence $x \in (\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i)_{\mathfrak{p}}$. For the converse assume that localizing at $\mathfrak{p}$ commutes with intersections, and suppose $\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i \subseteq \mathfrak{p}$. Then $(\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i)_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}}$, and hence $\bigcap_{i\in\mathfrak{I}} (\mathfrak{a}_i)_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}}$. But $\mathfrak{p}R_{\mathfrak{p}}$ is the maximal ideal in a quasi-local ring, so we must have $(\mathfrak{a}_i)_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}}$ for some i, and this implies that $\mathfrak{a}_i \subseteq \mathfrak{p}$.

For rings we obtain a corollary.

Corollary 1.4. *R is strongly zero-dimensional if and only if for every prime ideal $\mathfrak{p}$ and set of ideals $\mathfrak{a}_i$, $i \in \mathfrak{I}$, the equality $(\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i)_{\mathfrak{p}} = \bigcap_{i\in\mathfrak{I}} (\mathfrak{a}_i)_{\mathfrak{p}}$ holds.*

An Artinian-like concept bearing some relevance to our study is the following.

Definition 1.5. We shall say that R satisfies the *descending chain condition* (DCC for short) *on principal powers* if every chain $(x) \supseteq (x^2) \supseteq (x^3) \supseteq \cdots$ stops, i.e., $(x^n) = (x^{n+1})$ for some n.

The referee has pointed out that a ring R (not necessarily commutative) is said to be strongly π-regular if there for every $x \in R$ is an integer n and a $y \in R$ such that $x^n = x^{n+1}y$. Thus the rings with DCC on principal powers are exactly the commutative strongly π-regular rings. The concept of strongly π-regular rings was introduced by I. Kaplansky in [3]. A more recent paper is [1].

We do not assume, in the definition of DCC on principal powers, that n can be chosen uniformly, i.e., that there is an integer n which works for all $x \in R$. In Section 3 we shall encounter a similar, but stronger, condition. If $n = 1$ works for all x, then R is, by definition, a commutative von-Neumann regular ring.

We give two equivalent conditions.

Proposition 1.6. *The following conditions on R are equivalent.*

(i) *R satisfies the DCC on principal powers*

(ii) *For every $x \in R$ the relation $(x) + \operatorname{Ann}(x^n) = R$ holds for some n*

(iii) *radicals commute with intersections, i.e., $\sqrt{\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i} = \bigcap_{i\in\mathfrak{I}} \sqrt{\mathfrak{a}_i}$ for any set of ideals $\mathfrak{a}_i$.*

Proof. The equivalence of (i) and (ii) is easily seen. If $(x^n) = (x^{n+1})$, then $x^n = yx^{n+1}$ for some y, and hence $x^n(1 - xy) = 0$, i.e., $1 - xy \in \operatorname{Ann}(x^n)$, and so $(x) + \operatorname{Ann}(x^n) = (1)$. Conversely, if $(x) + \operatorname{Ann}(x^n) = (1)$, we have, say, $1 = ax + y$ where $y \in \operatorname{Ann}(x^n)$. This yields $(1 - ax)x^n = 0$, and hence $x^n \in (x^{n+1})$. In order to prove the equivalence of (i) and (iii), first supose (i) and consider a set of ideals $\mathfrak{a}_i$, $i \in \mathfrak{I}$. Since $\sqrt{\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i} \subseteq \bigcap_{i\in\mathfrak{I}} \sqrt{\mathfrak{a}_i}$ holds in general it is enough to show that $\sqrt{\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i} \supseteq \bigcap_{i\in\mathfrak{I}} \sqrt{\mathfrak{a}_i}$. Take an $x \in \bigcap_{i\in\mathfrak{I}} \sqrt{\mathfrak{a}_i}$ and, say, $(x^n) = (x^{n+1})$. Then $x^n \in \mathfrak{a}_i$ for all $i \in \mathfrak{I}$, and hence $x \in \sqrt{\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i}$. For the converse, suppose that (i) does not hold, and take an $x \in R$ such that $(x) \supset (x^2) \supset (x^3) \supset \cdots$ never stops. Clearly $x \in \sqrt{(x^n)}$ for all n, i.e., $x \in \bigcap_{n=1}^{\infty} \sqrt{(x^n)}$. But $x \notin \sqrt{\bigcap_{n=1}^{\infty} (x^n)}$, and this shows that (iii) does not hold.

These rings are clearly zero-dimensional.

Proposition 1.7. *If R satisfies the DCC on principal powers, then the dimension of R is zero.*

Proof. Suppose that R is not zero-dimensional, and let $\mathfrak{p}_1 \supset \mathfrak{p}_2$ be prime ideals of R. Take an $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$. Then $x^n \notin \mathfrak{p}_2$ for all n, and hence $\operatorname{Ann}(x^n) \subseteq \mathfrak{p}_2$. Thus $(x) + \operatorname{Ann}(x^n) \neq R$ for all n, which shows that R does not satisfy the DCC on principal powers.

We are now able to give another characterization of strongly zero-dimensional rings.

Theorem 1.8. *R is strongly zero-dimensional if and only if the following two conditions hold.*

(i) *No maximal ideal of R contains the intersection of the other maximal ideals*

and

(ii) *R satisfies the DCC on principal powers.*

Proof. First suppose that R is strongly zero-dimensional. The truth of (i) is immediate. To show (ii), let $\mathfrak{a}_i$, $i \in \mathfrak{I}$ be a set of ideals and let $\mathfrak{p}$ be any prime ideal containing $\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i$. Then, for some i, we have $\mathfrak{p} \supseteq \mathfrak{a}_i$ and hence $\mathfrak{p} \supseteq \sqrt{\mathfrak{a}_i}$. Thus $\mathfrak{p} \supseteq \bigcap_{i\in\mathfrak{I}} \sqrt{\mathfrak{a}_i}$. This shows that $\sqrt{\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i} \supseteq \bigcap_{i\in\mathfrak{I}} \sqrt{\mathfrak{a}_i}$, and since the opposite inclusion holds in general, we have proved (ii) using condition (iii) of Proposition 1.6. Next suppose R satisfies (i) and (ii). Then we know from Proposition 1.7 that R is zero-dimensional, so in proving that R is strongly zero-dimensional we only need consider the situation $\mathfrak{m} \supseteq \bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i$, where $\mathfrak{m}$ is a maximal ideal. Now we obtain $\mathfrak{m} \supseteq \sqrt{\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i}$, and hence by condition (ii), $\mathfrak{m} \supseteq \bigcap_{i\in\mathfrak{I}} \sqrt{\mathfrak{a}_i}$. But the $\sqrt{\mathfrak{a}_i}$ are intersections of maximal ideals, and it follows from condition (i) that $\mathfrak{m}$ must occur among these maximal ideals. Thus $\mathfrak{m} \supseteq \mathfrak{a}_i$ for some i and we have proved that R is strongly zero-dimensional.

Condition (i) of Theorem 1.8 is never fulfilled in a ring with an infinite number of maximal ideals, as we prove next.

Proposition 1.9. *Suppose* $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots$ *is an infinite set of distinct maximal ideals of R. Then there is in R a maximal ideal which contains the intersection of the other maximal ideals of R.*

Proof. If for some k we have $\mathfrak{m}_k \supseteq \bigcap_{i\neq k, i=1}^{\infty} \mathfrak{m}_i$ we are done, so suppose this is not the case. Let $\mathfrak{a} = \{x \in R; x \notin \mathfrak{m}_i$ only for a finite number of $i\}$. Then $\mathfrak{a}$ is readily seen to be a proper ideal of R. Moreover, for each k, $\mathfrak{a} \supseteq \bigcap_{i\neq k, i=1}^{\infty} \mathfrak{m}_i$ and hence $\mathfrak{m}_k \not\supseteq \mathfrak{a}$. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a}$. Then $\mathfrak{m} \supseteq \bigcap_{i=1}^{\infty} \mathfrak{m}_i$ but $\mathfrak{m} \neq \mathfrak{m}_i$ for all i. Thus $\mathfrak{m}$ contains the intersection of all the other maximal ideals.

Corollary 1.10. *Every strongly zero-dimensional ring is quasi semi-local.*

We shall see this again, by other means in Section 3. So in looking for strongly zero-dimensional rings we should concentrate on zero-dimensional quasi-semi-local rings. And, in fact, we shall prove that all such rings are strongly zero-dimensional.

However a maximal ideal in a ring which is not quasi semi-local might be strongly prime. This is illustrated by the following proposition and the example which follows.

Proposition 1.11. *Suppose R is a ring with DCC on principal powers. Then a maximal ideal* $\mathfrak{m}$ *in R is strongly prime if and only if* $\mathfrak{m}$ *does not contain the intersection of the other maximal ideals.*

Proof. The "only if"-part of the proposition is obvious, so we only need consider the "if"-part. To this end, suppose $\mathfrak{m} \supseteq \bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i$. Then $\mathfrak{m} \supseteq \sqrt{\bigcap_{i\in\mathfrak{I}} \mathfrak{a}_i}$ and hence, by Proposition 1.6 $\mathfrak{m} \supseteq \bigcap_{i\in\mathfrak{I}} \sqrt{\mathfrak{a}_i}$. But the $\sqrt{\mathfrak{a}_i}$ are all intersections of maximal ideals (remember that R

is zero-dimensional), so assuming that $\mathfrak{m}$ does not contain the intersection of the other maximal ideals, we must, for some i, have $\mathfrak{m} \supseteq \sqrt{\mathfrak{a}_i}$ and hence $\mathfrak{m} \supseteq \mathfrak{a}_i$. Thus $\mathfrak{m} \supseteq \bigcap_{i \in \mathfrak{I}} \mathfrak{a}_i$ implies $\mathfrak{m} \supseteq \mathfrak{a}_i$ for some i, which proves that $\mathfrak{m}$ is strongly zero-dimensional.

Example 1.12. Let k be a field and consider the ring $R = k \times k \times k \times \cdots$. For every $x \in R$ it is a fact easily seen that $(x) = (x^2)$. Thus R satisfies the DCC on principal powers. But R is not quasi semi-local, and hence there is in R a maximal ideal containing the intersection of the other maximal ideals. As such a maximal ideal we can take any maximal ideal, say $\mathfrak{m}$, containing the proper ideal $k \oplus k \oplus k \oplus \cdots$. Let, namely, $\mathfrak{m}_1 = 0 \times k \times k \times \cdots$, $\mathfrak{m}_2 = k \times 0 \times k \times k \cdots$, and so on. Then $\bigcap_{i=1}^{\infty} \mathfrak{m}_i = (0)$, but $\mathfrak{m} \neq \mathfrak{m}_i$ for every i. This shows that $\mathfrak{m}$ is not strongly prime. However we claim that the $\mathfrak{m}_i$ are strongly prime. For simplicity of notation we let $i = 1$ (the proof is obviously analogous for other i). According to Proposition 1.11, we only have to show the impossibility of $\mathfrak{m}_1 \supseteq \bigcap \mathfrak{n}_j$, where $\mathfrak{n}_j$ runs over the set of maximal ideals different from $\mathfrak{m}_1$. To each $\mathfrak{n}_j$ associate the set $\mathfrak{n}'_j = \{a \in k; (a, 0, 0, 0, \ldots) \in \mathfrak{n}_j\}$. This is an ideal of k, and hence 0 or k. But $\mathfrak{n}'_j = 0$ for a certain j, would imply that $\mathfrak{n}_j \subseteq \mathfrak{m}_1$, so this can not be the case. Hence all $\mathfrak{n}'_j = k$. Thus $(1, 0, 0, \ldots) \in \bigcap \mathfrak{n}_j$. But $(1, 0, 0, \ldots) \notin \mathfrak{m}_1$ and hence $\mathfrak{m}_1 \not\supseteq \bigcap \mathfrak{n}_j$.

2. Quasi-semi-local rings.

Proposition 2.1. *Let $R_1, R_2, \ldots R_n$ be strongly zero-dimensional rings. Then their direct product $R_1 \times R_2 \times \cdots \times R_n$ is also strongly zero-dimensional.*

Proof. Just note that all ideals of R are of the form $\mathfrak{a}_1 \times \mathfrak{a}_2 \times \cdots \times \mathfrak{a}_n$, where $\mathfrak{a}_i$ is an ideal of R_i and this ideal is prime (and maximal) if and only one of the $\mathfrak{a}_i$ is maximal and $\mathfrak{a}_k = R_k$ for $k \neq i$. For convenience of notation, suppose $i = 1$, and write the maximal ideal as $\mathfrak{m} \times R_2 \times \cdots \times R_n$. Suppose $\mathfrak{m} \times R_2 \times \cdots \times R_n \supseteq \bigcap_{i \in \mathfrak{I}} \mathfrak{a}_i$, where $\mathfrak{a}_i = \mathfrak{a}_{1,i} \times \cdots \times \mathfrak{a}_{n,i}$. Then $\mathfrak{m} \supseteq \bigcap_{i \in \mathfrak{I}} \mathfrak{a}_{1,i}$, and hence, since R_1 is strongly zero-dimensional, $\mathfrak{m} \supseteq \mathfrak{a}_{1,i}$ for some $i \in \mathfrak{I}$. Thus $\mathfrak{m} \times R_2 \times \cdots \times R_n \supseteq \mathfrak{a}_{1,i} \times \cdots \times \mathfrak{a}_{n,i} = \mathfrak{a}_i$.

Corollary 2.2. *Any finite direct product of zero-dimensional quasi-local rings is strongly zero-dimensional.*

We shall presently use this to prove that, in fact, every quasi-semi-local zero-dimensional ring is strongly zero-dimensional. For this we will need the following proposition. It is probably well known, but since I know of no reference, I give a proof.

Proposition 2.3 *In a zero-dimensional quasi-semi-local ring every ideal has a primary decomposition.*

Proof. It is clearly enough to consider the zero ideal. Let $S_{\mathfrak{m}}(0)$ denote the contraction of (0) under the homomorphism $R \to R_{\mathfrak{m}}$. Then it is a general fact that $(0) = \bigcap_{\mathfrak{m}} S_{\mathfrak{m}}(0)$, where $\mathfrak{m}$ runs over the set of maximal ideals (this does not require the ring to be quasi semi-local). In order to show that (0) has a primary decomposition it is therefore enough to show that $S_{\mathfrak{m}}(0)$ is primary for $\mathfrak{m}$, for every maximal ideal $\mathfrak{m}$. But this is clear since in $R_{\mathfrak{m}}$ we have $\sqrt{0} = \mathfrak{m}_{\mathfrak{m}}$, so $\sqrt{0}$ is primary for $\mathfrak{m}_{\mathfrak{m}}$ and the contraction of a primary ideal is primary.

Theorem 2.4. *Suppose that R is zero dimensional. Then the following conditions are equivalent.*

(i) *R is strongly zero-dimensional*

(ii) *R is quasi semi-local.*

(iii) $R = R_1 \times R_2 \times \cdots \times R_n$ for some zero-dimensional quasi-local rings $R_1, R_2, \ldots, R_n$

Proof. We already know that strongly zero-dimensional rings are quasi semi-local and hence that (i) $\Rightarrow$ (ii). Next suppose (ii) that R is quasi-semi-local and let $(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$ be a primary decomposition, where $\mathfrak{q}_i$ is primary for $\mathfrak{m}_i$. Then, since the $\mathfrak{q}_i$ are pairwise coprime (note $\sqrt{\mathfrak{q}_i + \mathfrak{q}_j} \supseteq \sqrt{\mathfrak{q}_i} + \sqrt{\mathfrak{q}_j} = \mathfrak{m}_i + \mathfrak{m}_j = R$, and hence $\mathfrak{q}_i + \mathfrak{q}_j = R$, if $i \neq j$), there is an isomorphism of rings $R \cong R/\mathfrak{q}_1 \times R/\mathfrak{q}_2 \times \cdots \times R/\mathfrak{q}_n$. Here each $R/\mathfrak{q}_i$ is zero dimensional and quasi local, and we have proved that (ii)$\Rightarrow$(iii). Finally, suppose (iii). Then each R_i is obviously strongly zero-dimensional and hence (i) follows from Corollary 2.2.

We give an example.

Example 2.5. Let $A = k[x_1, x_2, x_3, \ldots]$, where k is a field, let $\mathfrak{m}_1 = (x_1, x_2, \ldots)$, $\mathfrak{m}_2 = (x_1 - 1, x_2 - 1, x_3 - 1, \ldots)$, and let $\mathfrak{q}_1 = (x_1, x_2^2, x_3^3, \ldots)$ and $\mathfrak{q}_2 = (x_1 - 1, (x_2 - 1)^2, (x_3 - 1)^3, \ldots)$. Consider the ring $R = A/(\mathfrak{q}_1 \cap \mathfrak{q}_2)$. Then $\overline{\mathfrak{m}}_1$ and $\overline{\mathfrak{m}}_2$ are the only maximal ideals of R. Obviously R is zero dimensional, and a primary decompostion of (0) is already given $(0) = \overline{\mathfrak{q}}_1 \cap \overline{\mathfrak{q}}_2$. Now $\overline{x_1 - 1}, \overline{x_2 - 1}, \ldots \notin \overline{\mathfrak{m}}_1$, and hence, since R is strongly zero-dimensional, $\bigcap_{i=1}^{\infty} (\overline{x_i - 1}) \not\subseteq \overline{\mathfrak{m}}_1$. This may seem surprising at first sight, since in A we have $\bigcap_{i=1}^{\infty} (x_i - 1) = 0$. But note that $\overline{x_1 - 1} = \overline{(x_1 - 1)(1 - x_k^k)} + \overline{(x_1 - 1)x_k^k} = \overline{(x_1 - 1)(1 - x_k)(1 + x_k + \cdots + x_k^{k-1})} \in (\overline{x_k - 1})$ for all k. Thus actually $\bigcap_{i=1}^{\infty} (\overline{x_i - 1}) = (x_1 - 1)$.

3. The strongly n-regular rings.

This is a concept, which is similar to the concept of DCC on principal powers, but much stronger.

Definition 3.1. We shall say that R is *strongly n-regular*, for a positive integer n, if in each ideal $\mathfrak{a}$ of R, there is an element a, such that $x = ax$ holds for every $x \in \mathfrak{a}^n$.

Note that this yields $\mathfrak{a}^n = \mathfrak{a}^{n+1} = \cdots$ and also $\mathfrak{a}^n = (a^n) = (a^{n+1}) = \cdots$ and hence $\mathfrak{a}^n$ is generated by the idempotent element a^n. We immediately obtain the following alternative characterization (cf Proposition 1.6). Also note that strongly n-regular rings are zero-dimensional as follows from Proposition 17.

Lemma 3.2. *R is strongly n-regular if and only if every ideal $\mathfrak{a}$ satisfies $\mathfrak{a} + \operatorname{Ann} \mathfrak{a}^n = R$.*

Proposition 3.3. *Let R be a strongly n-regular ring. Then $\mathfrak{N}^n = 0$, where $\mathfrak{N}$ denotes the nilradical of R.*

Proof. There is an $a \in \mathfrak{N}$ such that $\mathfrak{N}^n = (a^n)$ and $a^n = a^{n+1} = \cdots$. But a is nilpotent. Hence $a^n = 0$, and $\mathfrak{N}^n = 0$ follows.

The property of being strongly n-regular is cleary inherited by factor rings, so the relation $(\sqrt{\mathfrak{a}})^n \subseteq \mathfrak{a}$ holds for any ideal $\mathfrak{a}$ in a strongly n-regular ring. The next proposition shows a connection with strongly zero-dimensional rings. Together with Proposition 3.3 this shows that, for a strongly zero-dimensional ring, the condition that $\mathfrak{N}^n = 0$ is equivalent to the condition that R be strongly n-regular.

Proposition 3.4. *Let R be a strongly zero-dimensional ring with nilradical $\mathfrak{N}$ such that $\mathfrak{N}^n = 0$. Then R is strongly n-regular.*

Proof. Let $\mathfrak{a}$ be an ideal and let $\mathfrak{m}_i$, $i \in \mathfrak{I}$ be the maximal ideals containing $\mathfrak{a}$, and $\mathfrak{m}_j$, $j \in \mathfrak{I}'$ the maximal ideals not containing $\mathfrak{a}$. Put $\mathfrak{b} = \bigcap_{j \in \mathfrak{I}'} \mathfrak{m}_j$. Since R is strongly zero-dimensional, we obtain $\mathfrak{m}_i \not\supseteq \mathfrak{b}$ for $i \in \mathfrak{I}$. It follows that $\mathfrak{a}+\mathfrak{b} \not\subseteq \mathfrak{m}$ is true for every maximal ideal $\mathfrak{m}$, and hence that $\mathfrak{a} + \mathfrak{b} = R$. But then also $\mathfrak{a} + \mathfrak{b}^n = R$, say $1 = a + b$, where $a \in \mathfrak{a}$ and $b \in \mathfrak{b}^n$. Take any $x \in \mathfrak{a}^n$. Then $xb \in \mathfrak{a}^n \mathfrak{b}^n \subseteq (\bigcap_{i \in \mathfrak{I}} \mathfrak{m}_i)^n (\bigcap_{j \in \mathfrak{I}'} \mathfrak{m}_j)^n = (\bigcap_{i \in \mathfrak{I}} \mathfrak{m}_i \cdot \bigcap_{j \in \mathfrak{I}'} \mathfrak{m}_j)^n \subseteq (\bigcap_{i \in \mathfrak{I}} \mathfrak{m}_i \cap \bigcap_{j \in \mathfrak{I}'} \mathfrak{m}_j)^n = \mathfrak{N}^n = 0$. Thus $xb = 0$ and hence $x = x(a + b) = xa$. This completes the proof.

Note that strongly 1-regular rings are principal ideal rings, such that every ideal is generated by an idempotent. We therefore obtain the following.

Corollary 3.5. *Let R be a strongly zero-dimensional ring and $\mathfrak{N}$ its nilradical. Then $R/\mathfrak{N}$ is an Artinian principal ideal ring.*

Remark. This was also shown in [2 Theorem 2.10].

We already know (Corollary 1.10) that strongly zero-dimensional rings are quasi semi-local, but here is another argument for this.

Corollary 3.6. *Every strongly zero-dimensional ring is quasi semi-local, i.e., possesses only a finite number of maximal ideals.*

Proof. Since $R/\mathfrak{N}$ is quasi semi-local, this is also true for R.

Proposition 3.7. *In a strongly n-regular ring no maximal ideal can contain the intersection of the other maximal ideals.*

Proof. Let the maximal ideals be $\mathfrak{m}_i$, $i \in \mathfrak{I}$ and take one of them, say $\mathfrak{m}_0$. We know from Lemma 3.2 that $\mathfrak{m}_0 + \operatorname{Ann} \mathfrak{m}_0^n = R$, so there is an $s \notin \mathfrak{m}_0$ with $s\mathfrak{m}_0^n = 0$. But $\mathfrak{m}_0^n \not\subseteq \mathfrak{m}_i$ for $i \neq 0$. Hence $s \in \mathfrak{m}_i$ for all $i \neq 0$. Thus $s \in \bigcap_{i \in \mathfrak{I}, i \neq 0} \mathfrak{m}_i$ and hence $\bigcap_{i \in \mathfrak{I}, i \neq 0} \mathfrak{m}_i \not\subseteq \mathfrak{m}_0$.

Example 3.8. Return to the strongly zero-dimensional ring of Example 2.5. Here the nilradical of R is not nilpotent, so R is not strongly n-regular for any n.

Corollary 3.9. *Every strongly n-regular ring is strongly zero-dimensional and hence quasi semi-local.*

Proof. We only have to apply Theorem 1.8.

References

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