FINITE UNIONS OF OVERRINGS OF AN INTEGRAL DOMAIN

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Abstract

Let R be an integral domain, and let $A, A_1, A_2, \ldots, A_s$ be overrings of R, where A is of the form $S^{-1}R$, where $S = R \setminus \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$ for some prime ideals $\mathfrak{p}_i$, and where each A_i, $i \geq 2$, is of the form $S_i^{-1}R$ for some multiplicatively closed subset S_i of R. It is shown that if $A \subseteq A_1 \cup \cdots \cup A_s$, then $A \subseteq A_i$ for some i.

This investigation was initiated and inspired by a theorem of A. Azarang, proved as Corollary 3.10 in [1]. The theorem, by Azarang properly called the *valuation avoidance lemma*, states that if $V, V_1, V_2, \ldots, V_s$ are valuation rings with a common field of fractions K, and if $V \subseteq V_1 \cup \cdots \cup V_s$, then in fact $V \subseteq V_i$ for some i. In other words, if V avoids every V_i (i.e. $V \nsubseteq V_i$ for every i), then V avoids the union of the V_i:s (i.e., $V \nsubseteq V_1 \cup V_2 \cup \cdots \cup V_s$).

We have previously studied avoidances of ideals and modules over commutative rings in a few papers, the most recent being [2], where a list of further references is given. One of the earliest papers, and an excellent source of inspiration, on finite unions of ideals and modules is the paper by N. McCoy [5].

By developing the techniques used in [2] a bit further, we shall be able to generalize Azarang's result to the following situation. R is an integral domain. A and $A_1, A_2, \ldots,$

A_s are overrings of R, i.e. rings between R and the field of fractions K of R. For $s \geq 2$ we suppose that A_i is a ring of fractions of R, say $A_i = S_i^{-1}R$. Also we suppose that $A = R_{\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n}$ where the $\mathfrak{p}_i$ are prime ideals, i.e., that $A = S^{-1}R$ where S is the complement of $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. Then if $A \subseteq A_1 \cup A_2 \cup \cdots \cup A_s$, in fact $A \subseteq A_i$ for some i. This is Theorem 6 below.

It is an advantage to formulate some of our results for modules and then apply these results to rings, so we begin by turning our attention to modules. In the sequel M, N and so on will always denote modules over a commutative ring R.

Speaking informally one might say that there are two kinds of avoidances of modules, each with corresponding avoidance-lemmas, namely the following:

(i) the fact that a single module M avoids each module in a set $M_1, M_2, \ldots, M_n$ of modules, does under certain conditions imply that M avoids the union $M_1 \cup \cdots \cup M_n$ of the modules, i.e., $M \nsubseteq M_1 \cup \cdots \cup M_n$

(ii) the fact that each module in a set $M_1, M_2, \ldots, M_n$ of modules avoids a single module M, does under certain conditions imply that the intersection $M_1 \cap \cdots \cap M_n$ avoids M, i.e., $M_1 \cap \cdots \cap M_n \nsubseteq M$.

An example of a theorem of the first kind, is the usual prime avoidance lemma: If an ideal $\mathfrak{a} \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, where the $\mathfrak{p}_i$ are prime, then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i. Another example is of course Azarang's result. Note also that for $n = 2$ we have avoidance of the first kind without any assumptions on M_1 and M_2.

An example of a theorem of the second kind, is the still more elementary fact, that if the intersection of ideals $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n \subseteq \mathfrak{p}$, where $\mathfrak{p}$ is prime, then some $\mathfrak{a}_i \subseteq \mathfrak{p}$.

The two kinds of avoidances are related, as can be seen from the usual proof of the prime avoidance lemma. This is also illustrated by the following lemma, which is [2, Corollary 4], and which will be used in the proof of Proposition 3 below.

Lemma 1. *Suppose $s \geq 2$ and $N \subseteq N_1 \cup \cdots \cup N_s$ but $N \nsubseteq N_2 \cup \cdots \cup N_s$. Then $N \cap N_2 \cap N_3 \cap \cdots \cap N_s \subseteq N_1$.*

In [2], following the terminology of Heinzer-Ratliff-Rush in [3], a module N over a commutative ring was said to be *strongly irreducible* if $N \supseteq N_1 \cap N_2$ implies that $N \supseteq N_1$ or $N \supseteq N_2$. By induction this extends to finite intersections. Note that this property relates to avoidance of the second kind. The most obvious example of strongly irreducible modules are prime ideals. Modules, which are finite intersections of strongly irreducible modules, were in [2] said to be *pseudo-radical*, in analogy with radical ideals, which by definition are intersections of prime ideals.

We will extend the notions of strongly irreducible and pseudo-radical, and prove some general results. These will then be applied in the special case, where R is an integral domain, and the modules are overrings of R.

Our main result in [2] was that if $N \subseteq N_1 \cup \cdots \cup N_s$, where all N_i except possibly two are pseudo-radical, then $N \subseteq N_i$ for some i. Informally speaking: from a condition related to avoidance of the second kind follows an avoidance of the first kind.

For the purpose of this paper, to generalize the valuation avoidance lemma of Azarang, we need two new concepts very similar to strongly irreducible and pseudo-radical, but in a more restricted sense, as follows.

Definition 2. *Let $\mathcal{F}$ be a set of R-modules. Then the R-module N is said to be strongly irreducible relative $\mathcal{F}$, if the following implication holds*

$$N \supseteq N_1 \cap \cdots \cap N_s, \text{ where } N_i \in \mathcal{F} \text{ for } i = 1, \ldots, s \Rightarrow N \supseteq N_i \text{ for some } i$$

An R-module L which is an intersection of (possibly infinitely many) strongly irreducible modules relative $\mathcal{F}$, is said to be pseudo-radical relative $\mathcal{F}$.

That the two avoidance-situations are related is again established by the proposition which follows.

Proposition 3. *Suppose $N \subseteq N_1 \cup \cdots \cup N_s$, where N_1 is pseudo-radical relative the set $\{N, N_2, N_3, \ldots, N_s\}$. Then either $N \subseteq N_1$ or $N \subseteq N_2 \cup N_3 \cup \cdots \cup N_s$.*

Proof. Let $\mathcal{F} = \{N, N_2, N_3, \ldots, N_s\}$ and suppose $N \not\subseteq N_2 \cup N_3 \cup \cdots \cup N_s$. We have, say, $N_1 = \bigcap L_i$, where the L_i are strongly irreducible relative $\mathcal{F}$. Fix an i and consider the covering $N \subseteq L_i \cup N_2 \cup N_3 \cup \cdots \cup N_s$, and remove from the covering any superflous N_i. It is a consequence of Lemma 1 that $N \cap N_2 \cap N_3 \cap \cdots \cap N_s \subseteq L_i$, and hence $N \subseteq L_i$ (since L_i is strongly irreducible relative $\mathcal{F}$). Since this holds for every L_i, we have $N \subseteq N_1$.

Remark. As an immediate application, suppose $N \subseteq N_1 \cup \cdots \cup N_s$, where N_i is pseudo-radical relative $\{N, N_j; j > i\}$, for $i = 1, 2, \ldots, s-2$. Then $N \subseteq N_i$ for some i. This can be seen as follows. If $N \not\subseteq N_i$ for $i \leq s-2$, we can successively delete these N_i, and obtain $N \subseteq N_{s-1} \cup N_s$, and hence $N \subseteq N_{s-1}$ or $N \subseteq N_s$.

It is true, that this does not really make sense, unless we have an example of a module being strongly irreducible relative some $\mathcal{F}$, without being strongly irreducible in the more general sense. Before providing such an example (it will appear in Proposition 5), we state without proof a few basic and well-known facts on overrings of an integral domain R.

(i) Let A be any overring of R. Then $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$, the intersection taken over all maximal ideals of A.

(ii) Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ be prime ideals of R, no two being comparable. Let $B = R_{\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n}$ and let $\mathfrak{q}_i$ be the extensions of $\mathfrak{p}_i$, $i = 1, \ldots, n$ to B. Then $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ are the maximal ideals of B, $R_{\mathfrak{p}_i} = B_{\mathfrak{q}_i}$ for each i, and $B = B_{\mathfrak{q}_1} \cap \cdots \cap B_{\mathfrak{q}_n} = R_{\mathfrak{p}_1} \cap \cdots \cap R_{\mathfrak{p}_n}$.

When N is strongly irreducible or pseudo-radical relative a certain $\mathcal{F}$, then this $\mathcal{F}$ can be assumed to be closed under the formation of finite intersections, or in other words we have the following.

Lemma 4. *Suppose N is strongly irreducible relative a class $\mathcal{F}$ of R-modules. Let $\mathcal{F}'$ be the class of finite intersections of modules from $\mathcal{F}$. Then N is strongly irreducible relative $\mathcal{F}'$. The same is true with pseudo-radical in place of strongly irreducible.*

Proof. This is rather obvious. Suppose N is strongly irreducible relative $\mathcal{F}$ and suppose $N \supseteq N_1 \cap \cdots \cap N_s$, where each $N_i \in \mathcal{F}'$, and say $N_i = N_{i,1} \cap \cdots \cap N_{i,k_i}$. Then $N \supseteq N_{i,j}$ some i, j, and hence $N \supseteq N_i$. The result for pseudo-radical N follows as an immediate consequence.

We follow the modern terminology and call a ring with only one maximal ideal a quasi-local ring, and we show next that every quasi-local overring is strongly irreducible relative the class of rings, which are obtained by localizing R at a finite union of prime ideals.

Proposition 5. *Let R be an integral domain and let $\mathcal{F} = \{R_{\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n}; \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ prime ideals of $R\}$. Then every quasi-local overring of R is strongly irreducible relative $\mathcal{F}$, and hence every overring of R is pseudo-radical relative $\mathcal{F}$.*

Proof. The second statement follows from the first, since every domain A is the intersection of its localizations $A_{\mathfrak{m}}$. Now, a typical ring in $\mathcal{F}$, is a finite intersection of rings of the form $R_{\mathfrak{p}}$. Thus, by Lemma 4, it is enough to prove that every quasi-local overring A of R is strongly irreducible relative the smaller class $\{R_{\mathfrak{p}}; \mathfrak{p}$ prime in $R\}$. Now suppose that A is quasi-local and that $A \supseteq R_{\mathfrak{p}_1} \cap \cdots \cap R_{\mathfrak{p}_n}$. Here we may assume that no two $R_{\mathfrak{p}_i}$ are comparable. Then no two of the $\mathfrak{p}_i$ are comparable and hence $R_{\mathfrak{p}_1} \cap \cdots \cap R_{\mathfrak{p}_n} = B_{\mathfrak{q}_1} \cap \cdots \cap B_{\mathfrak{q}_n}$, where $B = R_{\mathfrak{p}_1} \cap \cdots \cap R_{\mathfrak{p}_n}$, and where $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ are the maximal ideals of B. Let $\mathfrak{m}$ be the maximal ideal of A, and put $\mathfrak{p} = B \cap \mathfrak{m}$. Then $B_{\mathfrak{p}} \subseteq A$, but $\mathfrak{p} \subseteq \mathfrak{q}_i$, some i, and therefore $B_{\mathfrak{p}} \supseteq B_{\mathfrak{q}_i} = R_{\mathfrak{p}_i}$, and hence $A \supseteq R_{\mathfrak{p}_i}$.

We have now all we need to prove a theorem on the avoidance of certain overrings of a domain.

Theorem 6. *Suppose $A \subseteq A_1 \cup \cdots \cup A_s$, where $A, A_1, A_2, \ldots, A_s$ are overrings of an integral domain R, where A is of the form $R_{\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n}$ for prime ideals $\mathfrak{p}_i$ and where each A_i, $i \geq 2$ is of the form $S_i^{-1}R$ for some multiplicatively closed subset S_i of R. Then $A \subseteq A_i$ for some i.*

Remark. Note that we suppose that A is R localized at a finite union of prime ideals, whereas the A_i may be R localized at a possibly infinite union of prime ideals. On A_1, there is no restriction at all.

Proof. We begin by proving the theorem in the special case where $A_i = R_{\mathfrak{p}_i}$, $i \geq 2$. Then according to Proposition 5, A_1 is pseudo-radical relative $A, A_2, A_3, \ldots, A_s$. Hence, if $A \not\subseteq A_1$, we can, by Proposition 3, delete A_1 from the relation $A \subseteq A_1 \cup \cdots \cup A_s$. Next if also $A \not\subseteq A_2$, then A_2 may be deleted, and proceeding like this we find $A \subseteq A_i$ for some i. This completes the proof in the special case. In the general case, suppose (in order to derive a contradiction) that $A \not\subseteq A_i$ for all i. Then there are, for $i = 2, 3, \ldots, s$, maximal ideals $\mathfrak{m}_i$ of A_i, such that $A \not\subseteq (A_i)_{\mathfrak{m}_i}$. Note that $(A_i)_{\mathfrak{m}_i} = R_{\mathfrak{n}_i}$, where $\mathfrak{n}_i = \mathfrak{m}_i \cap R$. We have $A \subseteq A_1 \cup (A_1)_{\mathfrak{m}_2} \cup \cdots \cup (A_s)_{\mathfrak{m}_s}$, and hence, by the special case just proved, $A \subseteq (A_i)_{\mathfrak{m}_i}$ for some i, and there is a contradiction.

Remark. The valuation avoidance lemma is a special case of this, since if $V \subseteq V_1 \cup \cdots \cup V_n$, where $V, V_1, V_2, \ldots, V_n$ are all valuation overrings of $R = V_1 \cap \cdots \cap V_n$, then every V_i is of the form $R_{\mathfrak{p}_i}$. We refer to [4, Theorem 107] for this.

Note from the proof, that we only used Proposition 5, with $\mathcal{F}$ being the smaller class of the overrings $R_{\mathfrak{p}}$. Note also that from the theorem proved in the special case where $A_i = R_{\mathfrak{p}_i}$, $i \geq 2$, we deduced the full theorem. Considering this, a question which comes naturally to ones mind, is whether $\mathcal{F}$ in Proposition 5 could be extended to include every overring of the form $S^{-1}R$. But this is not so, as the following example shows.

Example 7. Let $R = k[x, y]$, where k is a field. Let $A = R_{(y)}$, which is a quasi-local overring of R. Next let $S = \{x^n y^n; n \geq 0\}$ and $T = \{(x+1)^n y^n; n \geq 0\}$. Then $S^{-1}R \not\subseteq R_{(y)}$, $T^{-1} \not\subseteq R_{(y)}$, but $S^{-1}R \cap T^{-1}R = R$, so $S^{-1}R \cap T^{-1}R \subseteq R_{(y)}$. This shows that $R_{(y)}$ is not strongly irreducible relative the class of overrings of the form $S^{-1}R$.

For Bézout domains, Theorem 6 becomes more attractive.

Corollary 8. *Let R be a Bézout domain and $A = R_{\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n}$ for prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$. Further let $A_1, A_2, \ldots, A_s$ be any overrings of R. Then if $A \subseteq A_1 \cup \cdots \cup A_s$, in fact $A \subseteq A_i$ for some i.*

Proof. This follows from the fact, which can be readily proved, that every overring of a Bézout domain R is of the form $S^{-1}R$.

References

1. A. Azarang, The space of maximal subrings of a commutative ring, *Comm. Algebra* 43 (2015), 795-811.
2. C. Gottlieb, Finite unions of submodules, *Comm. Algebra* 43 (2015) no. 2, 847-855.
3. W. Heinzer, L. Ratliff, D. Rush, Strongly irreducible ideals of a commutative rings., *J. Pure Appl. Algebra* 166 (2002), no. 3, 267-275.
4. I. Kaplansky, *Commutative rings*, The University of Chicago Press, 1970, ISBN 0-226-42454-5.
5. N. McCoy, A note on finite unions of ideals and subgroups, *Proc. Amer. Math. Soc.* 8 (1957), 633-637.