

CONNECTIVITY THROUGH BOUNDS FOR THE CASTELNUOVO-MUMFORD REGULARITY

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ABSTRACT. In this note we generalize and unify two results on connectivity of graphs: one by Balinsky and Barnette, one by Athanasiadis. This is done through a simple proof using commutative algebra tools. In particular we use bounds for the Castelnuovo–Mumford regularity of their Stanley–Reisner rings. As a result, if Δ is a simplicial d -pseudomanifold and s is the largest integer such that Δ has a missing face of size s , then the 1-skeleton of Δ is $\lceil \frac{(s+1)d}{s} \rceil$ -connected. We also show that this value is tight.

1. INTRODUCTION

We say that a graph G having more than m vertices is m -connected whenever it is impossible to disconnect it by removing fewer than m vertices together with their incident edges. Interesting results on the connectivity of G can be found if G is the 1-skeleton of a pure polyhedral complex. The first step in this direction was taken in 1922 by Steinz [12] who characterized the 1-skeleta of 3-dimensional polytopes as the planar 3-connected graphs. Later Balinsky [3] proved that the 1-skeleton of a $(d+1)$ -dimensional convex polytope is $(d+1)$ -connected. This result has been extended to polyhedral d -pseudomanifolds by Barnette [4].

Theorem 1 (Balinsky, Barnette). *The 1-skeleton of a d -dimensional polyhedral pseudomanifold is $(d+1)$ -connected.*

In the simplicial case, Athanasiadis [2] proved a stronger result for the 1-skeleton of a flag (i.e. coinciding with the clique complex of its 1-skeleton) simplicial d -pseudomanifold.

Theorem 2 (Athanasiadis). *The 1-skeleton of a flag simplicial d -pseudomanifold is $2d$ -connected.*

This topic has recently attracted interest among commutative algebraists. Björner and Vorwerk [6] extended Athanasiadis' result interpolating it with Barnette's one, thanks to a generalization of flag complexes. Recently, Adiprasito, Goodarzi and Varbaro [1] provided an algebraic method that allowed them to obtain a more general result.

In this note we also present an algebraic approach, which allows us to bridge the gap between Theorem 1 and Theorem 2 in a different direction than the one taken by [6] and [1]. This can be done under even weaker hypotheses.

The strategy involves finding upper bounds for the Castelnuovo–Mumford regularity of the Stanley–Reisner ring of a simplicial complex Δ which are expressed in terms of the number of vertices of Δ . Such bounds are common in literature, as the Castelnuovo–Mumford regularity itself acts as an upper bound for a wide variety of algebraic invariants (such as maximum degree for which the local cohomology modules vanish, the integer from which the Hilbert function behaves polynomially, or the maximum degree of the syzygies).

Furthermore, we will require these bounds to behave well up to taking restrictions of the simplicial complex (we call such bounds *suitable*). Then we focus on a set T of vertices of Δ which have to be removed from the 1-skeleton G of Δ in order to disconnect it. By setting some hypotheses on Δ , we can bound the regularity of the Stanley–Reisner ring of the restriction $\Delta|_T$ from below. Such hypotheses are weaker than requiring Δ to be a pseudomanifold. By reversing and applying a suitable bound for

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the regularity, we find a lower bound for the cardinality of the subset, therefore we can estimate the connectivity of G .

2. PRELIMINARIES

Let Δ be a simplicial complex on the vertex set $[n] = \{1, \dots, n\}$. We call the faces of dimension 0 and 1 *vertices* and *edges*, respectively. We call a face of Δ which is maximal with respect to inclusion *facet*, and we say that Δ is *pure* if its facets have the same dimension. We define the 1-skeleton of a simplicial complex Δ as the set of all the faces of Δ of dimension lower than or equal to 1. For our purposes we will deal with undirected simple graphs, therefore we define a graph to be the 1-skeleton of some simplicial complex Δ . Given a subset $T \subset [n]$ we denote by $\Delta|_T$ the *restriction of Δ to T* , i.e. all the faces σ of Δ such that $\sigma \subseteq T$. A subcomplex of Δ is called *induced* if it is a restriction of Δ to some set $T \subseteq [n]$.

Let \mathbb{k} be an arbitrary field and $S = \mathbb{k}[x_1, \dots, x_n]$ the polynomial ring on n variables. The *Stanley–Reisner ring* of the complex Δ (with respect to the field \mathbb{k}) is the graded ring $\mathbb{k}[\Delta] = S/I_\Delta$ where the *Stanley–Reisner ideal* I_Δ is the ideal generated by all the squarefree monomials $x_{i_1} \cdots x_{i_r} \in S$ such that $\{i_1, \dots, i_r\} \notin \Delta$.

If $F \subseteq [n]$ and $F \notin \Delta$, but all of its proper subsets are in Δ , then we say that F is a *missing face* of Δ of size $|F|$. Note that I_Δ is generated by all the monomials $x_{i_1} \cdots x_{i_r} \in S$ such that $\{i_1, \dots, i_r\}$ is a missing face of Δ .

Let M be a finitely generated graded S -module. Recall that the Hilbert’s Syzygy Theorem grants the existence of a *minimal graded free resolution* of M , i.e. a chain complex \mathbf{F} of graduated free modules of minimal rank with degree-preserving maps such that \mathbf{F} has an exact augmentation $\mathbf{F} \rightarrow M \rightarrow 0$. Let

$$0 \rightarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{s,j}} \xrightarrow{\phi_s} \dots \xrightarrow{\phi_2} \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{1,j}} \xrightarrow{\phi_1} \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{0,j}} \rightarrow 0$$

be a minimal graded free resolution of M , where the shifting numbers $-j$ are chosen in order to let the maps ϕ_i be degree-preserving. We call the exponents $\beta_{i,j} = \beta_{i,j}(M)$ *graded Betti numbers*. Furthermore we define the *Castelnuovo–Mumford regularity* of M as $\text{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}$.

We denote by $\tilde{H}_i(\Delta; \mathbb{k})$ the i^{th} *reduced (simplicial) homology* of Δ over the field \mathbb{k} . Hochster’s formula [9] relates the Betti numbers of the Stanley–Reisner ring $\mathbb{k}[\Delta]$ to the reduced homology of restrictions of Δ as follows

$$\beta_{i,j}(\mathbb{k}[\Delta]) = \sum_{\substack{T \subseteq [n] \\ |T|=j}} \dim_{\mathbb{k}} \tilde{H}_{j-i-1}(\Delta|_T; \mathbb{k}).$$

It immediately follows that if some restriction $\Delta|_T$ of Δ has nonzero homology in homological degree k then

$$\text{reg}(\mathbb{k}[\Delta]) \geq k + 1; \tag{1}$$

note that the equality holds whenever k is the maximum integer such that some restriction of Δ has nonzero homology in homological degree k .

3. SUITABLE BOUNDS

We now introduce a new family of simplicial complexes to which we will extend the results on connectivity.

Definition 3. We say that the simplicial complex Δ is a *vertex minimal k -cycle* if, for some field \mathbb{k} , $\tilde{H}_k(\Delta|_T; \mathbb{k}) \neq 0$ if and only if $T = [n]$.

Recall that a simplicial complex Δ is *strongly connected* if for every couple of facets τ and σ of Δ , there exist a sequence $\tau_0, \tau_1, \dots, \tau_m$ of facets of Δ with $\tau = \tau_0$ and $\sigma = \tau_m$ such that $\tau_{i-1} \cap \tau_i$ is a codimensional 1 face of both τ_{i-1} and τ_i , for $1 \leq i \leq m$. A *simplicial d -pseudomanifold* is a strongly connected simplicial complex each of whose $(d-1)$ -dimensional face is contained in exactly two facets. As a consequence of being strongly connected, a pseudomanifold is also pure. Note that a d -dimensional simplicial pseudomanifold is a vertex minimal d -cycle, but is not necessary that a vertex minimal d -cycle is pure or that each of its $(d-1)$ -dimensional faces is contained in exactly two facets.

In the following example we construct a d -dimensional vertex minimal 2-cycle for an arbitrary $d \geq 2$. For $d = 3$, it is pure, but not strongly connected, and its $(d - 1)$ -dimensional faces are contained in just one facet; for $d \geq 4$, Δ is not even pure.

Example 4. Let Q be a polygon in \mathbb{R}^2 with $d + 1$ vertices, for some $d \geq 2$. Let V be the set of the $2d + 2$ vertices of the prism $P = \text{conv}(Q \times \{0\}, Q \times \{1\}) \subset \mathbb{R}^3$. We define the d -dimensional simplicial complex Δ on the vertex set V as the simplicial complex whose faces are all the subsets of V whose elements lie on a common face of (the boundary of) P .

The following theorem relates the connectivity of a vertex minimal k -cycle Δ to the regularity of the Stanley–Reisner ring of specific restrictions of Δ . It allows us to prove the main results of this note in Section 4.

Theorem 5. *Let Δ be a vertex minimal k -cycle and let T be a set of vertices of Δ such that $\Delta|_T$ is disconnected. Then $\text{reg}(\mathbb{k}[\Delta|_{[n]\setminus T}]) \geq k$.*

Proof. Let U_1 be a subset of vertices of T such that $\Delta|_{U_1}$ is one of the connected components of $\Delta|_T$, and let $U_2 = T \setminus U_1$. Let $\Gamma_1 = \Delta|_{[n]\setminus U_2}$ and $\Gamma_2 = \Delta|_{[n]\setminus U_1}$. Note that $\Gamma_1 \cup \Gamma_2 = \Delta$ and $\Gamma_1 \cap \Gamma_2 = \Delta|_{[n]\setminus T}$. By applying the Mayer–Vietoris sequence for the reduced homology of simplicial complexes to Γ_1 and Γ_2 we obtain the following exact sequence

$$0 \rightarrow \tilde{H}_k(\Delta; \mathbb{k}) \rightarrow \tilde{H}_{k-1}(\Delta|_{[n]\setminus T}; \mathbb{k}) \rightarrow \cdots,$$

where the first zero comes from the hypothesis that $[n] \setminus U_1$ and $[n] \setminus U_2$ are proper subsets of $[n]$ and therefore $\tilde{H}_k(\Gamma_1; \mathbb{k}) = \tilde{H}_k(\Gamma_2; \mathbb{k}) = 0$, because Δ is a vertex minimal k -cycle. Then, since $\tilde{H}_k(\Delta; \mathbb{k}) \neq 0$, $\tilde{H}_{k-1}(\Delta|_{[n]\setminus T}; \mathbb{k}) \neq 0$.

We conclude by inequality (1). □

Note that an upper bound for the regularity of the Stanley–Reisner ring $\mathbb{k}[\Delta]$ of a simplicial complex Δ given in terms of the number of vertices of Δ can be reversed and applied to Theorem 5. In this way it is possible to give a lower bound for the number of vertices one needs to remove from Δ in order to disconnect it, provided that the bound can be applied to restrictions of Δ .

More specifically, we will say that an upper bound for $\text{reg}(\mathbb{k}[\Delta])$ in terms of n is *suitable* for the purposes of this note, if the same bound holds true for $\text{reg}(\mathbb{k}[\Delta|_T])$ for each proper and nonempty subset $T \subset [n]$ by substituting n by $|T|$ in the bound itself.

A family of suitable bounds can be achieved thanks to the Taylor resolution ([13], see also [5]). This bound is well known, but for the convenience of the reader we provide here a quick argument.

Let I be a monomial ideal in S . The degree j part of the Taylor resolution \mathbf{F} of S/I in homological degree i must have rank at least $\beta_{i,j}(S/I)$. If moreover I is squarefree, let s be the maximum degree of one of its minimal generators m_1, \dots, m_r . In this case, the multigrade vector of each generator of the degree j part of \mathbf{F} in homological degree i is an element of $\{0, 1\}^n$. Since the number of nonzero entries of this vector cannot exceed si , we conclude that $\beta_{i,j}(S/I) \neq 0$ only when $j \leq si$.

We can now obtain a bound for the Castelnuovo–Mumford regularity of S/I in terms of s and the number of indeterminates n . Indeed there must be a nonzero Betti number $\beta_{i,j}(S/I)$ such that $j - i = \text{reg}(S/I)$. As observed before $j \leq si = s(j - \text{reg}(S/I))$, therefore, since I squarefree also implies that $j \leq n$, we obtain

$$\text{reg}(S/I) \leq \frac{n(s-1)}{s}. \quad (2)$$

This bound is suitable as the maximum degree s does not increase up to restrictions of Δ .

Improved bounds can be obtained by strengthening the hypotheses. We report a result proved by Dao, Huneke, Schweig [7, Theorem 4.9], where the hypotheses have been rewritten thanks to the characterization for the q -step linearity given by Eisenbud, Green, Hulek and Popescu [8, Theorem 2.1].

Theorem 6 ([7]). *Let Δ be a flag simplicial complex and q a positive integer such that Δ contains no induced m -cycles for $4 \leq m \leq q + 3$. Then*

$$\text{reg}(\mathbb{k}[\Delta]) \leq \min \left\{ \log_{\frac{q+4}{2}} \left(\frac{n-1}{q+1} \right) + 2, \log_{\frac{q+4}{2}} \left(\frac{(n-1) \ln(\frac{q+4}{2})}{q+1} + \frac{2}{q+4} \right) + 2 \right\}.$$

The first term of the right hand side is tighter than the second one whenever $q \geq 2$. Since no new induced subcycles are formed through restriction, also this bound is suitable.

4. GENERALIZATION OF RESULTS ON CONNECTIVITY

Recall that a graph G is said to be m -connected, if it has more than m vertices and any subgraph obtained from G by deleting fewer than m vertices and their incident edges is connected (necessarily with at least one edge). In the language of simplicial complexes, the previous definition can be restated as follows.

Definition 7. Let G be 1-dimensional simplicial complex on a vertex set V . Then G is m -connected if $|V| > m$ and $G|_{V \setminus T}$ is connected for any subset $T \subseteq V$ with $|T| < m$.

Note that by setting $|V| > m$, we exclude the trivial case of the complete graph K_m on m -vertices.

The following corollary generalizes and interpolates Theorems 1 and 2.

Corollary 8. Let G be the 1-skeleton of a vertex minimal k -cycle Δ , and let s be the largest integer such that Δ has a missing face of size s . Then G is $\left\lceil \frac{sk}{s-1} \right\rceil$ -connected.

Proof. Note that s is the maximum degree of the minimal generators of the Stanley–Reisner ideal I_Δ . Then apply the suitable bound (2) to Theorem 5. \square

Note that Balinsky–Barnette’s result is obtained by looking at $\left\lceil \frac{sk}{s-1} \right\rceil$ for $s \gg 0$, while Athanasiadis’ one is obtained by setting $s = 2$. Furthermore the class of vertex minimal cycles is ampler than the one of minimal cycles.

We now present a family of simplicial complexes for which the previous bound on the connectivity is tight. We thank Eran Nevo for suggesting to build the following example in the same manner as the family of homology spheres he introduced in [11], for different purposes. In our case, for each $s \geq 2$ and each $k \geq s - 1$ we build a vertex minimal k -cycle Δ whose Stanley–Reisner ideal I_Δ is generated by monomials of degree not exceeding s such that it is possible to disconnect the 1-skeleton of Δ by removing exactly $\left\lceil \frac{sk}{s-1} \right\rceil$ vertices. Note that the condition $k \geq s - 1$ is not restrictive as $k = s - 2$ holds true only if the vertex minimal k -cycle Δ is the boundary of an $(s - 1)$ -simplex.

Example 9. Let $s \geq 2$ and $k \geq s - 1$ be two integers. Let $sk = (s - 1)q' + r'$ the Euclidean division of sk by $s - 1$, for a proper $q' \geq 0$ and a remainder $0 \leq r' \leq s - 2$. Let moreover $\left\lceil \frac{sk}{s-1} \right\rceil = sq + r$ be the Euclidean division of $\left\lceil \frac{sk}{s-1} \right\rceil$ by s , for a proper $q \geq 0$ and a remainder $0 \leq r \leq s - 1$. Note that $r' = 0$ if and only if $r = 0$, as the second Euclidean division has no remainder if and only if $\left\lceil \frac{sk}{s-1} \right\rceil = \frac{sk}{s-1}$.

We first note that $r \neq 1$. Suppose otherwise, then $r' \neq 0$ and $\left\lceil \frac{sk}{s-1} \right\rceil = q' + 1$. The second Euclidean division can be rewritten as

$$q' + 1 = sq + 1$$

and therefore, rewriting q' as $\frac{sk - r'}{s - 1}$, we get

$$sk = (s - 1)sq + r'.$$

So

$$s(k - sq + q) = r',$$

which is impossible, as $s \geq 2$ and $0 \leq r' \leq s - 2$.

So the remainder r must either equal 0 or satisfy $2 \leq r \leq s - 1$. In both cases we build Δ explicitly.

Recall that the *simplicial join* $\Delta_1 * \Delta_2$ of two simplicial complexes Δ_1 and Δ_2 on two disjoint vertex sets is the simplicial complex whose faces can be written as the union of a face of Δ_1 with a face of Δ_2 . Moreover, recall that the simplicial join of two spheres of dimension d_1 and d_2 is a $(d_1 + d_2 + 1)$ -sphere.

If $r = r' = 0$, we define $\Delta = \partial\sigma^1 * \partial\sigma^{s-1} * \dots * \partial\sigma^{s-1}$, where $\partial\sigma^i$ denotes the boundary of an i -dimensional simplex, and $\partial\sigma^{s-1}$ appears q times in the join. In this way, Δ is a sphere of dimension $q(s - 1)$. We now prove that $q(s - 1) = k$; by definition of q we get

$$q(s - 1) = \frac{sk}{s(s - 1)}(s - 1) = k.$$

Conversely, let $2 \leq r \leq s - 1$. As observed before, $r' \neq 0$. In this case we define Δ as the join $\partial\sigma^1 * \partial\sigma^{s-1} * \dots * \partial\sigma^{s-1} * \partial\sigma^{r-1}$, where $\partial\sigma^{s-1}$ appears q times. In this way, Δ is a $(q(s-1) + r - 1)$ -sphere. We now prove that $q(s-1) + r - 1 = k$. Indeed,

$$q(s-1) + r - 1 = \frac{q' + 1 - r}{s}(s-1) + r - 1 = k - \frac{r - r' - 1}{s}.$$

The quantity $\frac{r-r'-1}{s}$ has to be an integer, and since $0 \leq r \leq s - 1$ and $0 \leq r' \leq s - 2$, the only integer value it can equal is 0.

In both the cases Δ is an k -sphere on $\left\lceil \frac{sk}{s-1} \right\rceil + 2$ vertices, indeed the second Euclidean division counts exactly the number of vertices of the join except for the two vertices of $\partial\sigma^1$. Moreover, in both the cases, the largest i such that $\partial\sigma^i$ is in Δ is $s - 1$, and therefore the Stanley–Reisner ideal I_Δ is generated by monomials of degree at most s . So Δ satisfies the hypothesis of Corollary 8.

If we remove from the 1-skeleton of Δ all the vertices but the two belonging to $\partial\sigma^1$, we disconnect it. Note that we are removing $n - 2 = \left\lceil \frac{sk}{s-1} \right\rceil$ vertices, therefore the 1-skeleton of Δ can not be more than $\left\lceil \frac{sk}{s-1} \right\rceil$ -connected.

If we have sufficient hypotheses to apply the bound for the regularity given by Dao, Huneke, Schweig (see Theorem 6) we can obtain the following result in which the connectivity of the simplicial complex grows exponentially on k .

Corollary 10. *Let G be the 1-skeleton of a vertex minimal k -cycle Δ which is flag and without induced q -cycles for $q \geq 4$. Then G is M -connected, where*

$$M = \max \left\{ \left\lceil (q+1) \left(\frac{q+4}{2} \right)^{k-2} + 1 \right\rceil, \left\lceil \frac{q+1}{\ln\left(\frac{q+4}{2}\right)} \left(\left(\frac{q+4}{2} \right)^{k-2} - \frac{2}{q+4} \right) + 1 \right\rceil \right\}.$$

Proof. Apply Theorem 6 to Theorem 5. □

For the sake of readability, we emphasize that from the corollary it follows that G results at least $\left\lceil \left(\frac{q}{2} \right)^{k-1} \right\rceil$ -connected.

A family of simplicial pseudomanifolds of arbitrary dimension which satisfy the hypotheses of the previous corollary has been built by Januszkiewicz and Świątkowski in [10].

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