Mathematical abilities and mathematical memory during problem solving and some aspects of mathematics education for gifted pupils

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Abstract

This thesis reports on two different investigations.

The first is a systematic review of pedagogical and organizational practices associated with gifted pupils' education in mathematics, and on the empirical basis for those practices. The review shows that certain practices – for example, enrichment programs and differentiated instructions in heterogeneous classrooms or acceleration programs and ability groupings outside those classrooms – may be beneficial for the development of gifted pupils. Also, motivational characteristics of and gender differences between mathematically gifted pupils are discussed. Around 60% of analysed papers report on empirical studies, while remaining articles are based on literature reviews, theoretical discourses and the authors' personal experiences – acceleration programs and ability groupings are supported by more empirical data than practices aimed for the heterogeneous classroom. Further, the analyses indicate that successful acceleration programs and ability groupings should fulfil some important criteria; pupils' participation should be voluntary, the teaching should be adapted to the capacity of participants, introduced tasks should be challenging, by offering more depth and less breadth within a certain topic, and teachers engaged in these practices should be prepared for the characteristics of gifted pupils.

The second investigation reports on the interaction of mathematical abilities and the role of mathematical memory in the context of non-routine problems. In this respect, six Swedish high-achieving students from upper secondary school were observed individually on two occasions approximately one year apart. For these studies, an analytical framework, based on the mathematical ability defined by Krutetskii (1976), was developed. Concerning the interaction of mathematical abilities, it was found that every problem-solving activity started with an orientation phase, which was followed by a phase of processing mathematical information and every activity ended with a checking phase, when the correctness of obtained results was controlled. Further, mathematical memory was observed in close interaction with the ability to obtain and formalize mathematical information, for relatively small amounts of the total time dedicated to problem solving. Participants selected problem-solving methods at the orientation phase and found it difficult to abandon or modify those methods. In addition, when solving problems one year apart, even when not recalling the previously solved problem, participants approached both problems with methods that were identical at the individual level. The analyses show that participants who applied algebraic methods were more successful than participants who applied particular methods. Thus, by demonstrating that the success of participants' problem-solving activities is dependent on applied methods, it is suggested that mathematical memory, despite its relatively modest presence, has a pivotal role in participants' problem-solving activities. Finally, it is indicated that participants who applied particular methods were not able to generalize mathematical relations and operations – a mathematical ability considered an important prerequisite for the development of mathematical memory – at appropriate levels.

Keywords: *mathematical abilities, mathematical memory, high-achieving students, problem solving, mathematics education for gifted pupils.*

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Introduction

Mathematical abilities and mathematics education for gifted pupils, despite being central to this thesis, are not unproblematic topics, particularly from the perspective of identifying giftedness.

To illustrate this problem, I begin with two historical anecdotes. The first concerns a child prodigy. He was born into a poor family; a strict father who saw no purpose in education and an encouraging but illiterate mother. At the age of ten his teacher asked his class to calculate the sum of the first one hundred integers (Bell, 1953), a task expected to challenge even much older pupils. However, this particular boy, instantly recognising a fundamental property of the integers, solved it within moments by means of an approach that is still applied to such series today (Bell, 1953; Hayes, 2006). By continuing this pattern of extraordinary mathematical achievement – for example, proposing a version of the prime number theorem at the age of 15 – Carl Friedrich Gauss (1777-1855) became one of the most outstanding mathematicians of all time. Hence, it might be reasonable to assume that today the school-boy Gauss would have been described as both mathematically gifted and extremely high-achieving.

The second story concerns a child who, after losing his father before his birth, was initially raised by his maternal grandmother and who, after his mother's remarriage, exhibited such hostility towards his mother and stepfather that at one point he threatened "to burn them and the house over them" (Cohen, 1970, p. 43). He performed modestly in school, found social interactions problematic and, at the age of 17, dropped out of school. However, in an attempt to circumvent his mother's desire that he take over the family farm, he returned to school and eventually gained admission to Trinity College in Cambridge. Within a few years, marked by independent study and untiring perseverance, he presented ground-breaking theories that, among other cornerstones of contemporary science, led to the development of mechanics and modern calculus. Isaac Newton (1643-1727) is still considered one of the greatest mathematicians of all time but at school was barely ordinary.

Of course, not all pupils who perform poorly in school will become great scholars or all child prodigies outstanding mathematicians. My purpose, therefore, is to problematize not only the intuitive assumption that giftedness automatically implies excellent school performance but also the identification of giftedness. My aim is to nuance in the very opening of this thesis the often-discussed relationship between mathematical abilities and school achievement.

Consequently, the first objective of the research presented in this thesis was to investigate the research concerning pedagogical and organizational practices concerning gifted pupils' education in mathematics and, in so doing, to determine which of those practices are supported by empirical evidence. To fulfil this objective, I undertook a systematic review of the literature related to mathematics education for gifted pupils (e.g. Grant & Booth, 2009; Lerman, 2014; Rethlefsen, Murad, & Livingston, 2014; Wu, Aylward, Roberts, & Evans, 2012; Ziegler & Raul, 2000). During this process, it became clear that the concepts *giftedness*, *high ability* and *high achievement* are often used interchangeably within the field. Consequently, it became relevant to consider whether mathematically gifted pupils are high-achievers or if high achievement indicates mathematical giftedness.

In these respects, it should be emphasized that even though both gifted and high-achieving students display appropriate mathematical abilities (Krutetskii, 1976, pp. 67–70) and that gifted students in general excel in mathematics (e.g. Leikin, 2010; Rotiger & Fello, 2005; Usiskin, 2000), it has also been observed that methods for the identification of giftedness ignore those gifted learners who achieve below their potential (e.g. Davis & Rimm, 2004; Gagné, 1985). This dilemma might be explained by the proposition that both giftedness and mathematical giftedness are culturally, socially and politically conditioned (e.g. Borland, 2005; Gyarmathy, 2013; Freeman, 2004; Karp, 2017; Mönks & Katzko, 2005; Persson, 2010; Rotiger & Fello, 2005; Ziegler & Raul, 2000). Indeed, the anecdotes from the lives of Newton and Gauss seem to further problematize the association between school achievements and future performances in mathematics, which leads me to the reflections that prompted the second broad objective of my research.

It is reasonable to assume that readers of this thesis, at some point of their mathematics education will have experienced problems that they felt they could not immediately solve. I remember my own experiences, where, typically after a brief period of thinking, I would be able to determine whether or not I could solve a problem. And in those cases where I could not remember how to solve it, I resolved to study harder and solve even more problems prior to upcoming tests. Later, when teaching mathematics in upper secondary school, I would discuss with my students the difficulties that confronted them and, throughout these discussions, students habitually mentioned that success or failure was dependent on whether they remembered how to solve a particular problem.

Thus, how do we act when we do not know how a given problem should be solved? Shall we just trust our memory, or do we have other abilities that may support us? What if, for example, there are characteristic activities of those who are good at problem solving from which we might learn something? And the subsequent question could be, is school mathematics a subject where previous knowledge and memory are attributed essential importance or is mathematics a subject where pupils are able to construct their own knowledge, in ways similar to other subjects, for example, English and history? Consequently, the second objective of this thesis is to examine the interaction of mathematical abilities and the role of mathematical memory (Krutetskii, 1976) – which is a specific memory function related to mathematical activities – in the context of non-routine mathematical problems.

Research on mathematical abilities and mathematical thinking has its origins in the late nineteenth century (Calkins, 1894). Since then, researchers have continually attempted to delimit the nature of those abilities associated with mathematics (e.g. Hadamard, 1945; Krutetskii, 1976; Leikin, 2014; Vilkomir & O'Donoghue, 2009). After being dominated by the psychometric paradigm during the first half of the 20th century, whereby mathematical abilities were considered as innate and developable only within limited restraints, the research field was transformed by Krutetskii's (1976) longitudinal study and its resultant model of mathematical ability. Krutetskii describes mathematical ability as a complex dynamic phenomenon constituted by following abilities:

- the ability to obtain mathematical information (i.e. formalized perception of mathematical material),
- the ability to process mathematical information (i.e. logical thought, generalization of mathematical objects, relations and operations, the ability to curtail the process of mathematical reasoning and flexibility in mental processes),
- retaining mathematical information (i.e. mathematical memory, which is a generalized memory for mathematical relationships, type characteristics and methods of problem-solving) and
- a general synthetic component, described as a "mathematical cast of mind" (Krutetskii, 1976, pp. 350–351).

Moreover, while studies of mathematically talented students have continued to build successfully on Krutetskii's work (e.g. Deal & Wismer, 2010; Garofalo, 1993; Heinze, 2005; Leikin, 2010; Sriraman, 2003; Vilkomir & O´Donoghue, 2009), recent delimitations of mathematical intelligence or of the abilities of mathematically promising students resonate sufficiently closely with Krutetskii's model (e.g. Juter & Sriraman, 2011; Sheffield, 2003) for me to argue that it remains a trustworthy and significant analytical framework for studies of mathematical ability.

Another important factor in introducing this thesis is to highlight the fact that all empirical work was carried out in Sweden. With respect to notions of giftedness, principled egalitarianism and inclusion have hindered an acceptance of any need to identify and then develop gifted students in the Swedish school system (Dodillet, 2017; Persson, 2010). Thus, based on indications that older high-achieving students are able to display appropriate mathematical abilities and, importantly, intelligible forms of mathematical memory (Krutetskii, 1976), I selected participants among upper secondary school students, who, based on their performances, were extremely high-achievers in mathematics.

In addition, I would like to underline, as indicated above, that my intention was to investigate the interaction of mathematical abilities and mathematical memory in the context of problem solving. That is, my primary objective was not to investigate the distinctive characteristics of gifted and high-achieving students. However, as it will be demonstrated in this thesis, the special attributes of focused mathematical abilities and of mathematical memory entailed the participation of high-achieving students in these studies.

Finally, it might be helpful to mention my own background. I was educated in a socialist school system, a meritocracy in which education was the only way to succeed in life. However, despite its many shortcomings and omnipresent ideological influence, this educational system offered a form of mathematics education based on the considerable successes of the Hungarian and Romanian mathematics traditions. Throughout my experiences of school mathematics, where algebra and geometry were taught as separate subjects, problem solving was both the tool and the objective of mathematics learning. Every lesson contained problem solving performed by pupils at the blackboard, and every topic was introduced through problem solving by the teacher in the manner described by Andrews (2003) and Szalontai (2000). Thus, one might say that I was educated within a problem-solving oriented mathematics tradition, which, apart from its elitist view on school performance, provided me with many authentic experiences of problem solving. In addition, parallel to my studies in upper secondary school and some years afterwards, I played professional basketball. During these years, I played in some outstanding teams and was able to observe on an everyday basis how talent, supported by relentless practice, developed into (or failed to lead to) excellent performances. Later, after ending my career in sports, alongside my work as a teacher of mathematics in Swedish upper secondary schools and provider of university courses in mathematics for excelling secondary school students, I co-authored several series of mathematics textbooks for upper secondary school based on the Swedish curricula. Thus, it seems that some of the main topics of this thesis are inextricably linked to my personal background and experiences.

Lastly, it should be noted that, due to unforeseen health problems, my time as a PhD student involved four different supervisors at different stages of my work. Each of these, after managing the sensitive task of becoming familiar with the main ideas of this thesis at substantially different stages of the project, guided my work in exceptional ways. And it should also be mentioned that, despite the unanticipated time losses caused by the above disclosed circumstances, this thesis was completed according to its initial time schedule.

Theoretical background

In this section, I will examine the literature associated with those subjects that may be viewed as pivotal for this thesis.

Firstly, based on the observation that giftedness and high achievement in school are closely interrelated concepts in the educational context (e.g. Csíkszentmihalyi & Robinson, 1986; Sriraman & Leikin, 2017; Sternberg & Davidson, 2005; Stoeger, 2009), I will present a literature review on general giftedness. Secondly, based on the main topics of this thesis, I will present a review of the research associated with mathematical abilities and with mathematics education for gifted pupils. Thirdly, according to the circumstance that mathematical memory (Krutetskii, 1976) is attributed an essential role in this thesis, I will present some aspects of mathematical memory from psychological, educational, cognitive and neuroscientific perspectives. And fourthly, since mathematical abilities are basically associated with and observed through problem-solving activities (e.g. Blum & Niss, 1991; Cai & Lester, 2005; Carlson & Bloom, 2005; Garofalo, 1993; Halmos, 1980; Kilpatrick, 2016; Lester & Kehle, 2003; Mason, 2016; Mason, Burton, Stacey, 1982; Nunokawa, 2005; Pólya, 1966; Schoenfeld, 1985; Singer & Voica, 2017), I will present some aspects of the research on mathematical problem solving.

General giftedness

In this section, I will first offer a general view on giftedness which also involves a discussion about giftedness and exceptional intellectual achievements. Secondly, according to the widely agreed view that giftedness is a social, cultural and political construction (e.g. Borland, 2005; Callahan & Miller, 2005; Freeman, 2004; Jeltova & Grigorenko, 2005; Karp, 2017; Mönks & Katzko, 2005) I will mention giftedness in the Swedish educational context.

General views on giftedness

Most terms describing exceptional intellectual achievement originate from everyday language and are often used interchangeably and synonymously, for example, *gifted*, *talented* or *able* (e.g. Csíkszentmihalyi & Robinson, 1986; Sriraman & Leikin, 2017; Stoeger, 2009; Ziegler, 2005).

From being labelled as "heavenly children" in both Plato's Greece and in Confucius' China, early descriptions of gifted children indicate that giftedness was understood as a gift from the gods and not a characteristic which could be developed by the individual (Stoeger, 2009). This view is also emphasized by the Bible when stating that each of us has "gifts differing according to the grace that was given to us" (Romans 12:6, World English Bible). However, during the middle ages the comprehension of giftedness shifted towards more logical and meaningful descriptions. For example, in 1537, the philosopher Paracelsus pioneered the use of the term "talent" when describing exceptional potential and achievement, thereby indicating that those who possess such aptitudes were no longer viewed as supernatural, but as humans who could use their potential in ways that are beneficial for both themselves and for society (Passow, Mönks, & Heller, 1993). The ability to think logically, critically and autonomously was highly valued during the Enlightenment and many of the descriptions and attributes associated with talent and giftedness became more rational and less emotional (Cassirer, 1998).

During the industrial revolution, *intelligence*, *talent* and *giftedness* became recognized as important human resources, and researchers mobilized considerable efforts to establish empirical methods to identify and develop giftedness. However, despite the substantial research in the field of giftedness and intelligence, there remains a lack of consensus as to which attributes or characteristics should be included in the definition of giftedness and which methods are optimal in order to identify, rationalize or measure giftedness and intelligence (e.g. Carman, 2013; Pitta-Pantazi, 2017; Stoeger, 2009). Moreover, despite the introduction of scientific descriptions during the last century, exceptional intelligence remains subject to explanation by irrational beliefs (Heller, Mönks, Sternberg, & Subotnik, 2000).

The first genuinely scientific attempts to identify human intelligence were rooted in the traditions of psychometrics; according to psychometric models, mental abilities are to some extent innate, can be measured at young ages and are only slightly affected during the lifetime of the individual. Thus, the field of psychometrics agrees that mental abilities can be determined through intelligence tests which result in an individual intelligence quotient (IQ). The first IQ tests, constructed more than a hundred years ago by Binet, were developed into the Stanford-Binet Intelligence Scales and became widely used during the following decades.

Further developments of early IQ tests achieved relatively high levels of acceptance during the major part of the 20th century. Nevertheless, during its last decades, psychometric approaches have been questioned in their role as predictors of intellectual performances (e.g. Stoeger, 2009). A substantial part of the criticism was grounded in the observations that IQ is not constant during the lifetime of the individual (e.g. Weinert, 1998), that improved learning opportunities of the second half of the twentieth century precipitated consistent rises in average IQ scores (Flynn, 2009) and in studies showing that socioeconomic factors are almost as strong predictors for future performances as psychometric tests (e.g. Stoeger, 2009; Ziegler, 2005). In this context, it should also be noted that psychometric models are fundamentally hierarchical by situating general giftedness – a direct outcome of logical and reversible thinking – at the top of the hierarchy, while considering contextual and operational abilities and aptitudes of more specific nature as less fundamental components.

Subsequently, psychometric models linking giftedness and talent to a limited psychological concept have been reformed and replaced by *multifactorial* approaches. A significant step in that direction was taken by Marland (1972), who, in a report to the U.S. Congress proposed that the model of giftedness, beside general intellectual ability (IQ), should be complemented with a specific academic aptitude (displayed in mathematics, science and language), creative or productive thinking, leadership ability, talent in visual and performing arts and psychomotor ability – thereby suggesting that high-performers in any of the mentioned areas, should be regarded as gifted and talented.

In the following decade, in a significant attempt to delimit a multifactorial model, Renzulli's (1978) – influenced by biographies of successful individuals, who were not excelling in school but contributed essentially to their fields, and inspired by experiences from everyday life (Mitchell, 2010) – presented the *three-ring conception of giftedness* theory. According to the three-ring conception, giftedness may be treated as a behaviour of certain people during certain circumstances and is expressed in the intersection of above-average intellectual abilities, creativity and profound task commitment (Renzulli, 1978). Later, Sternberg (1997, 1998), by drawing on early multifactorial cognitive models and indicating that componential, experiential and practical abilities are crucial components of human intelligence, presented his *triarchic theory of intelligence* – the subsequent widespread agreement on Sternberg's model represented another important positioning of the research field towards more cognitive and practical approaches at the expense of psychometric theories. And even though a deeper discussion of these models is beyond the aim of this thesis, in the given context it should be mentioned that the criticism of Sternberg's model depends largely on its unempirical character (e.g. Gottfredson, 2003).

Further developments of multifactorial models resulted, for example, in the *multifactor* model of giftedness (Mönks & Katzko, 2005) and *the actiotope* model of giftedness (Ziegler, 2005). The multifactor model (Mönks & Katzko, 2005) is grounded in developmental psychology, while the actiotope model (Ziegler, 2005) is drawn on the hypothesis that the development of the individuals' actions repertoire is beneficial for, and could be integrated into, larger systems. These models are considering giftedness intrinsically as a potential which, in its development towards excellence, depends on sociocultural factors. For example, by describing giftedness as "an individual potential for exceptional achievements in one or more domains" (Mönks & Katzko, 2005, p.191) Mönks and Katzko suggest that learning environment, family, peers and motivation are pivotal in the development of giftedness. Further, Ziegler (2005) proposes that the quality of actions displayed at young ages is an excellent predictor of future exceptional performances.

However, when concerning the identification of giftedness, even though it may be understood as a paradox, it seems that it is not unproblematic to find well-justified alternatives to IQ tests, and that IQ tests are still widely used in the identification process (e.g. Feldman, 2003; Leikin, Leikin, & Waisman, 2017; Silverman, 2009).

Moreover, in a review of empirical studies on giftedness (Ziegler & Raul, 2000), the identification models for giftedness are criticized partly for their incoherence and partly because

> gifted research is conducted within the framework of a fragmented research community where studies are performed under various methodological viewpoints, which are often unsatisfactory. (Ziegler & Raul, 2000, p. 113)

As observed, the literature indicates that neither multifactorial models of intelligence nor the extension of psychometric models with additional attributes are able to provide more than a limited explanation of the complex phenomenon of giftedness and talent (e.g. Davis & Rimm, 2004; Ziegler & Raul, 2000; Ziegler, 2005) and, importantly, that the distinction between giftedness and talent is not clarified. Nevertheless, it should be mentioned that there are some theoretical models that, based on the view that general giftedness should be measured with IQ tests, differentiate these two concepts (e.g. Gagné, 2005; Feldhusen, 1995). For example, Gagné (2005) considers giftedness as a set of natural abilities which, during optimal circumstances, may lead to systematically developed skills, that are defined as talent. However, a vast majority of researchers either fail to make a sharp distinction between these concepts (e.g. Borland, 2005; Callahan & Miller, 2005; Csíkszentmihalyi & Robinson, 1986; Freeman, 2004; Mönks & Katzko, 2005; Pitta-Pantazi & Leikin, in press; Plucker & Barab, 2005 Sriraman & Leikin, 2017; Stoeger, 2009) or object to the above mentioned theoretical models (e.g. Robinson, 2005; Ziegler, 2005).

In conclusion, three main aspects of general giftedness were highlighted in the above review of the research literature. Firstly, that there is a lack of a generally agreed definition of giftedness in the research field (e.g. Carman, 2013; Pitta-Pantazi, 2017). Secondly, that the majority of models of giftedness include a psychometric component, that is, the IQ of the individual (e.g. Gagné, 2005; Feldhusen, 1995; Robinson, 2005; Stoeger, 2009). And, thirdly, that terms associated with excellent intellectual abilities, for example the notions gifted, talented, able, highly-able and intelligent, are frequently used in synonymous and interchangeable ways in the research field (e.g. Csíkszentmihalyi & Robinson, 1986; Sriraman & Leikin, 2017; Stoeger, 2009).

The Swedish educational context

Important for the framing of this thesis, the concepts of giftedness and excellence should be discussed in the Swedish cultural and educational context, not least because for the past four decades, the Swedish school system, characterized as egalitarian and inclusive, has demonstrated a low interest for the identification and education of intellectually gifted pupils (e.g. Dodillet, 2017; Persson, 2010). Consequently, prior to the period – between 2010 and 2014 – when the data for this thesis were collected, gifted and talented students were given little attention in Sweden (e.g. Dodillet, 2017). One of the reasons for this is that the main objective of the Swedish school system was to provide support for all students, in order to achieve a basic level – often interpreted as a level necessary for the lowest passing scores – of knowledge and competence (e.g. Dodillet, 2017; Persson, 2010). Moreover, in the given context, Persson, based on research carried out by Englund (2005) and Persson, Balogh and Joswig (2000) suggests that:

> It is likely that nowhere is resistance to assist gifted students in school stronger than in the Scandinavian countries for historical, cultural, and political reasons, particularly in Sweden (Persson, 2010, pp. 536–537)

Nevertheless, in 2009, as a pioneering project in the Swedish educational system, 20 upper secondary schools – 10 for mathematics and natural sciences and 10 for social sciences – were selected for an experimental educational program for excelling students, that was intended to be implemented in so called "spetsklasser" that can be translated as "peak performance classes" or "top-classes" (e.g. Dodillet, 2017). This experimental program for excelling students was initially planned to run between 2009 and 2014, but was later extended until 2022 and currently includes 26 schools.

However, an analysis of the implementation and the outcome of these programs, performed by Dodillet (2017), uncovers a number of problematic issues associated with the education of gifted and excelling students in Sweden. One of these problems concerns the fact that even though the intentions of the government associated with top-classes (described in parliamentary documents) were in concordance with other countries' excellence initiatives, the Swedish National Agency for Education (NAE) – the organisation responsible for the implementation of the program in Sweden – based on the country's egalitarian tradition, did not implement the project according to the parliamentary directives (Dodillet, 2017). Moreover, Dodillet, emphasizes that the NAE

> did not just deviate from the official regulations by implementing elite education as a project for interested rather than gifted or high-performing students. Treating the top-class reform as a consistent project, it also opposed the official aim of running a school experiment in which different approaches to the

> implementation of special classes for top-students could be evaluated to find innovative concepts. (Dodillet, 2017, p. 13)

Another problematic issue identified by the analysis is that the evaluations performed by NAE regarding top-classes reinforced the impression that NAE "lacks the elitist perspective that characterized the political initiative and is embedded in the reform's formal structure" (Dodillet, 2017, p. 8). Further, Dodillet states, that

> Given the government's understanding of top-classes, one might expect that the official evaluations by the NAE would touch on top-class students' prior knowledge, giftedness, or talent, and mention whether the schools managed to recruit "the best". However, these issues are ignored in the evaluations, which instead focus on the students' gender, socio-economic status, and ethnic background, and problematize a "distorted recruitment" arising from the fact that "the top students are recruited from a homogeneous group" (Skolverket, 2013, 2014). (Dodillet, 2017, p. 8)

Thus, as indicated above, regardless of considerable governmental efforts to implement special programs for gifted learners, at the same time that the studies included in this thesis were conducted, the Swedish educational system failed to interpret these programs as intended.

However, one year after the data collection for this thesis was completed, the NAE (Skolverket, 2015) provided a general support material for gifted pupils within the compulsory school system. Yet, in an attempt to emphasize its egalitarian view – namely, that all learners in the Swedish school system are considered as gifted – the pupils in the focus for the mentioned guidance are labelled "särskilt begåvade" which means "particularly gifted". Thus, it is reasonable to assume that the NAE, through this statement, continues to assume that intellectually gifted individuals are not the norm for the educational system; that is, that excellent performances are viewed as particular phenomena and not something achievable for all learners.

Accordingly, it seems that at the time the data collection for this thesis was carried out, even though considerable reforms – intended to meet the needs of gifted and excelling learners – were on their way to be implemented, the concepts of giftedness and excellence were neither well enough understood or accepted within the Swedish educational system.

Implications for this thesis

As indicated above, the literature is not convergent in defining and identifying general giftedness or in the terminology for describing talented, able, highly-able or intelligent learners. Thus, in my ambition to, as far as possible, avoid confounding factors connected to the terminology, in my studies included in this thesis, I elected to use the term *gifted* – and consequently the corresponding term *begåvad* in the Swedish language paper – when referring to pupils with exceptional intellectual abilities.

As mentioned in the literature review, the research field shifted focus from psychometric approaches towards multifactorial models for identification of giftedness, thereby delimiting the concept as a combination of several different abilities displayed on certain occasions. Further, it is also indicated that because of the fragmentation of the research field and its unsatisfactory methodological considerations, it is difficult to delimit methods which in a reliable way can identify giftedness. In addition, during the time period for the data collection associated with this thesis, the Swedish school system demonstrated a low acceptance for the identification and education of the gifted. Consequently, given the methodological difficulties related to the identification of giftedness and the Swedish educational system's resistance towards the assistance of gifted learners, I considered that identifying gifted pupils was not an appropriate practice when selecting participants for the studies included in this thesis.

Thus, when framing this thesis, based partly on the above discussed absence of reliable methods for identifying giftedness in young individuals and partly on the problematic situation that gifted and excelling pupils were facing in the Swedish educational system, I decided not to use psychometric or multifactorial methods to identify participants' intellectual abilities. Instead, I decided to pay particular attention to the participants' mathematical actions and behaviour exhibited during the different phases of data collection reported in this thesis.

Mathematical abilities and mathematics education for gifted pupils

Even though neither mathematics nor mathematical thinking are axiomatically defined concepts (e.g. Sternberg, 1998), studies have shown that we are born with an ability to estimate amounts and quantities – known as the approximate number system (Dehaene, 1997) – and that an active engagement with mathematics may, under favourable circumstances, lead to the development of complex and well-organized mathematical abilities (e.g. Henningsen & Stein, 1997; Krutetskii, 1976). In this section, I will first offer an overview of the literature on mathematical abilities, secondly, I will discuss the conditions of mathematics education for gifted pupils.

Mathematical abilities

Empirical attempts to delimit students' mathematical abilities are generally situated in the context of problem solving and has engaged researchers since the last decades of the 1800s (e.g. Krutetskii, 1976; Vilkomir &

O'Donoghue, 2009). For example, Mary Calkins (1894), in her studies of Harvard students from different disciplines with the aim to identify the mental operations associated with mathematics, observed that the memories of mathematicians were more concrete than verbal and that there was no difference between the ease of memorizing between mathematicians and other students. Moreover, she noticed that students who preferred geometry to algebra seemed to be more mathematically inclined, that classification and reasoning were more strongly developed in mathematicians than in other students and that, when engaged in mathematical activities, there were no important distinctions between men and women (Calkins, 1894).

Nevertheless, during the first half of the 20th century, research on mathematical abilities – in concordance with the research on general giftedness – was strongly influenced by psychometric approaches (Krutetskii, 1976). Yet, the psychometric paradigm, considering mental abilities as innate and static, was not only unable to identify the characteristics of those abilities that are viewed as mathematical (e.g. Krutetskii, 1976; Leikin, 2014; Vilkomir & O'Donoghue, 2009) but also contributed to more negative connotations to the concept of ability (Adey, Csapó, Demetriou, Hautamäki, & Shayer, 2007). Another problematic hypothesis associated with early psychometric approaches considered mathematical abilities only marginally developable through the teaching and the learning of the subject.

In a divergence from the psychometric standards of the time, the French mathematician Hadamard (1945) – by sending mail surveys to prominent mathematicians and physicists of the early 1900s – made a significant effort to expose the nature of mathematical thinking. Hence, even though Hadamard's main focus was not the delimitation of mathematical abilities, he identified four significant mental phases associated with problem solving; *preparation* (i.e. a profound understanding of the given problem), *incubation* (i.e. a period during which the mind subconsciously and almost independently works with the problem), *illumination* (i.e. when the mind generates a novel and useful idea after the incubation phase), and *verification* (i.e. when the mind assess the newly generated idea). Further, during the second half of the 20th century, in veins similar to conceptions of general giftedness, the research on mathematical abilities continued its migration from psychometrical ideas towards sociocultural representations, thereby assuming that mathematical abilities are neither static nor innate, but instead intellectual characteristics which can be assimilated and developed in flexible ways.

Nevertheless, a moment of great significance occurred in 1968, when the Soviet psychologist Vadim Krutetskii (1976) published his seminal work on mathematical abilities, thereby underpinning research for the following five decades on those who are gifted and high-achieving in mathematics (e.g. Leikin, 2010; Sriraman, 2003; Vilkomir & O´Donoghue, 2009). Therefore, the longitudinal study conducted by Krutetskii (1976), observing around 200 pupils between 1955 and 1966, is of pivotal importance for this thesis.

In order to uncover the structure of mathematical ability, Krutetskii's research team observed pupils at qualitatively different levels; that is, "groups of mathematically capable, average, and relatively incapable pupils were singled out" (Krutetskii, 1976, p. 176). By focusing the pupils' problem-solving activities, the research team developed a framework that describes mathematical ability as a complex phenomenon, with the following components:

1) The ability to obtain mathematical information (i.e. formalized perception of mathematical material),
2) The ability to process mathematical information (i.e. logical thought, rapid and broad generalization of mathematical objects, relations and operations, the ability to curtail the process of mathematical reasoning, flexibility in mental processes, striving for clarity and simplicity of solutions),
3) Retaining mathematical information (i.e. mathematical memory, which is a generalized memory for mathematical relationships, type characteristics, schemes of arguments and proofs and methods of problem-solving) and
4) A general synthetic component, described as a "mathematical cast of mind". (Krutetskii, 1976, pp. 350–351)

In Krutetskii's model, these four components establish a dynamic system, where each component can be developed through appropriate mathematical activities. Moreover, due to the dynamic aspect of the system, it has been indicated that mathematically able students develop their respective mathematical abilities to different extents, thereby permitting a wide interaction of the mentioned components in which modestly developed abilities are compensated by better developed abilities.

Conversely, some abilities, traditionally viewed as mathematical and undoubtedly useful in mathematics, are not considered as necessary components of the mathematical ability in schoolchildren; for example, "swiftness of mental processes", "abilities for rapid and precise calculation" often performed in head, "a memory for symbols, numbers and formulas", an "ability for spatial concepts" and the "ability to visualize abstract mathematical relationships" (Krutetskii, 1976, p. 351).

In addition, the research team differentiated three different mathematical casts of minds among capable pupils, namely,

- an *analytic* type, characterized by abstract patterns of thoughts and well-developed verbal-logical skills combined with less well-developed abilities to visualize the subject,
- a *geometric* type, with well-developed abilities to visualize the subject, that complement less well-developed verbal-logical skills, and

- a *harmonic* type, which is a combination of the analytic and geometric types. (Krutetskii, 1976, pp. 317–329)

Thus, in the context of this thesis, some aspects of Krutetskii's model should be highlighted.

Firstly, when discussing differences between components of the complex mathematical ability, Krutetskii emphasises that the *general synthetic component* is not a mental characteristic that is observable during problem solving, mainly because it refers to a general mindset of mathematically dedicated pupils. That is, by using a metaphor, one can say that mathematically gifted pupils observe the world thorough mathematical lenses; for example, when learning astronomy, they may construct tables with calculated moon-phases and distances between planets and stars, or, if studying geography, their tables will contain data about population density, length of main rivers, heights of mountains, different kinds of graphs, cartographic projections and diagrams (Krutetskii, 1976, p. 303). Additionally, Krutetskii's general synthetic component may be also interpreted from the perspective of Sternberg's (1997) triarchic model of intelligence, where those who are *synthetically gifted*, do not necessarily achieve the highest scores on IQ tests. According to Sternberg (1997), synthetic giftedness – even though it is not measurable by psychometric tests – is interpreted in terms of creativity and intuition, attributes that are particularly useful in the development of new ideas or creative solutions to given problems.

The second aspect concerns *mathematical memory*. By differentiating it from the mechanical recall of numbers, multiplication tables or algorithms, and by drawing on essential properties of the ability to *generalize mathematical information*, mathematical memory is described as a "memory for generalized, curtailed and flexible systems" (Krutetskii, 1976, p. 352). In other words, gifted pupils often remember the methods applied to the solution of a problem several months ago, but are usually unable to recall its contextual details. However, since mathematical memory is a key topic in my studies, I will discuss the concept in a more detailed manner in the upcoming section.

Thirdly, when characterizing mathematical giftedness, special attention was accorded to the *ability to generalize* mathematical objects, relations and operations (Krutetskii, 1976). By observing that "the ability to generalize is by its nature a general ability and usually characterizes the general property of teachability" (Krutetskii, 1976, p. 353) it is also specified that this is "a matter not of the ability to generalize, but the ability *to generalize numerical and spatial relations, expressed in number or letter symbolism*" (Krutetskii, 1976, p. 353). Thereby, it seems that the ability to generalize is closely related to mathematical memory.

Fourthly, the critical role of the teaching in mathematics is emphasized by noting that "the specific content of the structure of abilities largely depends on teaching methods, since it is formed during instruction" (Krutetskii, 1976, p. 351), and, importantly, that mathematical giftedness is an "aggregate of mathematical abilities that opens up for successful performance in mathematical activity" (Krutetskii, 1976, p. 77). Further, it should be mentioned that, although studies (e.g. Brandl, 2011; Leikin, 2014; Öystein, 2011) indicate that high-achieving pupils are not necessarily mathematically gifted, Krutetskii claims that abilities for learning the subject should be viewed as an appropriate mathematical ability (Krutetskii, 1976, pp. 67–70)

> Thus, we believe that the question whether abilities for *learning* mathematics can be regarded as a manifestation of mathematical ability in the proper sense of the word should be answered affirmatively. (Krutetskii, 1976, p. 69)

Finally, considering that Krutetskii's research was concluded in the 1960's, it is natural to ask, five decades on, whether his framework retains its currency in the research field. In this respect, it is useful to note that researchers continue to draw on Krutetskii's ideas and extend his work (e.g. Deal & Wismer, 2010; Garofalo, 1993; Heinze, 2005; Leikin, 2010; Sriraman, 2003; Vilkomir & O′Donoghue, 2009). Further, recent research on the characteristics of mathematically gifted students (e.g. Juter & Sriraman, 2011; Sheffield, 2003) displays essential similarities with Krutetskii's main concepts. For example, Sheffield (2003) concludes that, during problem solving, mathematically promising students demonstrate

a) a *mathematical frame of mind* (e.g. the individual enjoys exploring patterns and sees mathematics in a variety of everyday situations);
b) *mathematical formalization and generalization* (e.g. generalizing the structure of a problem often from only a few examples, thinking logically and developing proofs and arguments);
c) *mathematical creativity* (e.g. processing information flexibly, exhibiting original approaches to problem-solving, reversing processes and striving for mathematical elegance and clarity) and
d) *mathematical curiosity and perseverance* (e.g. curiosity about mathematical relationships, asking "why" and "what if" and showing persistence when solving difficult problems). (Sheffield, 2003, pp. 3–4)

Accordingly, it can be observed that the *mathematical frame of mind* in the above framework is similar to Krutetskii's (1976) *general synthetic component*, that *mathematical formalization and generalization* in the above framework (Sheffield, 2003) displays similarities with both the *formalized perception of mathematical material* and the *generalization of mathematical objects, relations and operations* in Krutetskii's (1976) model. Further, that components of *mathematical creativity*, that is, *processing information flexibly*, *reversing processes* and *striving for mathematical elegance and clarity* in Sheffield's (2003) framework correspond to the components of the *ability to process mathematical information* delimited by Krutetskii (1976), that is, to the *ability to curtail the process of mathematical reasoning, flexibility in mental processes, striving for clarity and simplicity of solutions*. Thus, despite some differences, it seems that there are also significant similarities between the frameworks of Krutetskii (1976) and Sheffield (2003).

Moreover, Juter and Sriraman (2011), when summarizing the main theoretical approaches to mathematical giftedness and creativity, show that cognitive studies delimit mathematical intelligence with important connections to Krutetskii's framework. Thus, according to cognitive studies, mathematical intelligence is composed by:

> 1) the ability to abstract, generalize and discern mathematical structures; 2) data management; 3) ability to master principles of logical thinking and inference; 4) analogical, heuristic thinking and posing related problems; 5) flexibility and reversibility of mathematical operations; 6) an intuitive awareness of mathematical proof; 7) ability to independently discover mathematical principles; 8) decision making abilities in problem solving situations; 9) the ability to visualize problems and/or relations and 10) distinguish between empirical and theoretical principles. (Juter & Sriraman, 2011, pp. 49–50)

As seen above, the *ability to abstract, generalize and discern mathematical structures* display substantial connections to Krutetskii's (1976) *ability to generalize mathematical objects, relations and operations* and to the components of *the ability to retain mathematical information*, described by Krutetskii (1976) as *generalized mathematical relationships, type characteristics, schemes of arguments and proofs and methods of problem-solving*. Further, the *ability to master principles of logical thinking* and the *flexibility and reversibility of mathematical operations* in the model of mathematical intelligence (Juter & Sriraman, 2011) are similar to components of the *ability to process mathematical information* in Krutetskii's (1976) model, for example, to *logical thought* and *flexibility in mental processes*. In addition, the *ability to visualize problems and/or relations* (Juter & Sriraman, 2011) resemble the *ability to obtain mathematical information*, described as *formalized perception of mathematical material* by Krutetskii (1976). Thus, it seems that there are some important similarities between Krutetskii's (1976) model and the mathematical intelligence framework discussed by Juter and Sriraman (2011).

In the context, it should also be mentioned that in a vein different from the above presented frameworks, some models for mathematical talent consider high mathematical performance in middle school as an outcome of some well-structured innate abilities – such as physical constitution, brain structure, character traits, number sense, competencies of spatial perception and linguistic and general cognitive potentials – that have been developed through significant inter- and intrapersonal catalysts, as well as by personality traits associated with talent (Benölken, 2015; Fritzlar, Rodeck, & Käpnick, 2006). However, in contrast to previously described frameworks for mathematical ability (e.g. Krutetskii, 1976; Sheffield, 2003) which are essentially based on operationalized components, these models for mathematical talent connect mathematical performances more directly to innate abilities and, importantly, unlike Krutetskii's (1976) framework, do not pay particular attention to memory functions in a mathematical context.

Further, when discussing Krutetskii's framework from the perspective of mathematical problem solving, there does not seem to exist notable discordances between his delimitation of mathematical abilities (Krutetskii, 1976) and those characteristics that are typically associated with mathematical problem solving. For example, in a synthesis of the typically sociological problem-solving related research performed by Schoenfeld and the basically psychological research conducted by De Corte and Verschaffel, Andrews and Xenofontos (2015) identified four sets of necessary problem-solver characteristics that display essential similarities with the abilities presented in Krutetskii's (1976) model. These are: a well-organized and flexible subject knowledge, that also include guidelines supporting mathematical argumentation; an appropriate set of retrievable problem-solving strategies; the ability to monitor and control problem-solving attempts, in ways that demonstrate an understanding of the applied methods; and finally, a productive mathematical disposition associated with the belief that mathematical problems are worth solving (Andrews & Xenofontos, 2015).

However, a notable distinction between the above listed general characteristics of problem-solvers and Krutetskii's mathematical abilities is that the former are attributable to all learners, while the latter are typically associated with able pupils. Thus, at least with respect to problem solving, it seems that research drawn on philosophically disjoint fields does not indicate critical variance between the abilities of able pupils and the goals for all learners.

In conclusion, by observing that a clear majority of the models for describing mathematically promising or able learners, as well as the mentioned characteristics of mathematical problem-solvers, offer relatively small deviations from Krutetskii's (1976) main concepts, it is reasonable to assume that his framework still offers a relevant, trustworthy and versatile understanding of students' mathematical abilities.

Mathematics education for gifted pupils

The mathematics education of gifted pupils – as well as their overall education (e.g. Persson, 2014) – represents specific challenges for the educational system (e.g. Freiman & Rejali, 2011; Mattsson, 2013; Rogers, 2007; Pettersson, 2011; Pitta-Pantazi, 2017). Further, it is also indicated that mathematics teachers rarely meet the needs of gifted pupils in the most commonly occurring form of school organization, that is, in heterogeneous (mixed ability) classes (e.g. Leikin & Stanger, 2011; Shayshon, Gal, Tesler, & Ko, 2014).

Additionally, the relatively large number of pedagogical, didactical and organizational practices recommended by the research field, in order to develop gifted pupils' mathematical performances and abilities, create a substantial level of uncertainty among mathematics teachers when deciding which approaches are best suited for their own practice (e.g. Leikin, 2010; Lester & Schroeder, 1983; Reed, 2004; Rogers, 2007; Rotiger & Fello, 2005). Moreover, teachers' uncertainty concerning pedagogical practices may be accentuated by the observation that a large number of practices, for example practices concerning mathematical problem solving, are recommended for all pupils, regardless of their talent for mathematics and thereby often coincide with recommendations for gifted pupils (e.g. Ambrus & Barczi-Veres, 2016; Blum & Niss, 1991; Cai & Lester, 2005; Kilpatrick, 2016; Leikin, 2010; Lithner, 2008).

In addition, a substantial part of the provision for the gifted – for example, enrichment, acceleration or "curriculum compacting" – has been subjected to criticism because of only meeting the needs of those who display giftedness, and thereby ignoring those gifted learners who achieve below their potential (e.g. Davis & Rimm, 2004; Gagné, 1985). And it has also been shown that even though the absence of appropriate teaching methods has the consequence that many gifted pupils perform much below their potential, if teaching is adapted to their needs and capabilities, they will develop according to their potential (e.g. Persson, 2010).

As indicated, the mathematical abilities of gifted pupils – regardless of the methodological considerations connected to the identification of mathematical giftedness – constitute a dynamic system, in which modestly developed components are compensated by well-developed ones (e.g. Juter & Sriraman, 2011; Krutetskii, 1976; Pitta-Pantazi & Leikin, in press; Sheffield, 2003). Thus, the complexity of and the individual differences between gifted pupils' mathematical abilities – and the recommendation that abilities should primarily be regarded as a potential and not as a warrant for exceptional performances (e.g. Brandl, 2011; Leikin, 2010; Usiskin, 2000) – suggests that mathematically gifted pupils constitute a divergent group, which cannot be identified by simple methods or stimulated within uncertain practices (e.g. Hoeflinger, 1998; Pitta-Pantazi, 2017).

Hence, even though it has been suggested that the identification of gifted pupils is a necessary stage prior to the application of pedagogical and organizational practices intended to meet their needs, it has been emphasized that the identification process has both cultural, social and political characteristics and is essentially dependent of the methods of identification (e.g. Gyarmathy, 2013; Karp, 2017; Karp & Bengmark, 2011; Leikin, 2014; Rotiger & Fello, 2005; Ziegler & Raul, 2000). Accordingly, there is an obvious risk that some gifted pupils will remain unidentified and thereby fail to benefit from developmental opportunities (e.g. Rotiger & Fello, 2005; Shayshon et al., 2014; Ziegler & Raul, 2000). Thus, when identifying mathematically gifted pupils, not only mathematical performances or IQ tests should be taken in consideration, because, even if a pupil is a top performer in the subject, further investigations are necessary in order to determinate mathematical giftedness (e.g. Leikin, Waisman, & Leikin, 2013; R. Leikin et al., 2017; Lupkowski-Shoplik, Sayler, & Assouline, 1994; Rotiger & Fello, 2005). In that respect, consultation with teachers and parents should be complemented with an investigation of the pupil's social and emotional development.

Accordingly, the identification process, beside evaluating mathematical performances, should also include observation of pupils' mathematical abilities and interest (e.g. Coleman, 2003; Pitta-Pantazi, 2017) and should take into consideration that gender, ethnicity and socioeconomic status might also influence the process (Birch, 1984). Nevertheless, despite a recommended early identification, which is considered beneficial for the development of mathematical abilities (e.g. Pitta-Pantazi, 2017), there is a lack of a conclusive connection between mathematical giftedness and accomplishments in the subject in a longer time perspective; that is, early performances are, regardless of their nature, not strong enough predictors for later achievements in mathematics (Juter & Sriraman, 2011).

Moreover, regardless of the increased attention on mathematically gifted pupils (e.g. Leikin, Paz-Baruch, & Leikin, 2013, 2014; Pitta-Pantazi & Leikin, in press; Sriraman & Leikin, 2017), for reasons of social justice and equality, research has typically focused on low achieving pupils (Swanson & Jerman, 2006) and relatively few studies have investigated the development of the mathematically gifted (e.g. Leikin, 2010, 2011; Sowell, 1993; Usiskin, 2000).

Even though the majority of gifted pupils study in heterogeneous classes in regular schools – and given that the integration of gifted pupils in the heterogeneous classroom is a challenging task (Renzulli, 2008) – the situation of these pupils in heterogeneous classes has been an overlooked area within the research field (e.g. Shayshon et al., 2014). Additionally, and importantly, most investigations on the efficiency of classroom-based teaching methods focus on pupils who are generally gifted (e.g. Davis & Rimm, 2004) rather than specifically gifted in mathematics and the majority of reviews of the research field are published at least a decade ago (e.g. Hoeflinger, 1998; Lester & Schroeder, 1983; Rogers, 2007; Rotiger & Fello, 2005; Sowell, 1993).

One consequence of the above described situation is that, in the spring of 2014, when framing one of the studies included in this thesis, it was difficult to find an up-to-date review of the research on pedagogical and organizational practices aimed to nurture and develop gifted pupils in the mathematics classroom.

In addition, even though the literature indicates a fair convergence regarding theories and frameworks on mathematical abilities and giftedness, it should be highlighted, that there is a lack of consensus on the terminology which describe pupils with exceptional mathematical abilities. Thus, in an analogous development to the research on general giftedness, the concepts *mathematically gifted*, *mathematically talented*, *mathematically able*, *mathematically highly-able* and *mathematically promising* are used synonymously in the literature (e.g. Pitta-Pantazi & Leikin, in press; Sowell, 1993; Sriraman & Leikin, 2017).

Finally, the presented literature review suggests that teachers receive insufficient guidance with regard to the practices they would like to apply in their encounters with gifted pupils in the mathematical classroom, and, importantly, that it is not unproblematic for teachers to evaluate the efficiency of recommended practices (e.g. Leikin, 2010; Lester & Schroeder, 1983; Reed, 2004; Rogers, 2007; Rotiger & Fello, 2005). Consequently, it is not unreasonable to assume that teachers would benefit from a review and categorization of practices, focused on developing gifted pupils in the mathematics classroom, that are recommended by researchers in the field.

Implications for this thesis

As indicated in the literature review, Krutetskii's (1976) framework still represents a solid research-model for identifying and analysing the individual structure of gifted pupils' mathematical abilities. Moreover, this framework contains a well-defined ability associated with the recalling of mathematical relationships and problem-solving methods, defined as *mathematical memory*. Thus, given the conditions that memory associated with mathematical activity is in the focus of my studies and that Krutetskii's framework is still used extensively in the research field, in framing this thesis, I decided to construct an analytical framework based on Krutetskii's model of mathematical ability.

In addition, with respect to Krutetskii's model, it is indicated that the *general synthetic* component of the mathematical ability is not observable during problem solving – thus, this ability was not included in the analytical framework construed to analyse students' problem-solving activities. Conversely, based on the observation that mathematical memory is discernible during problem-solving activities of gifted and high-achieving students (Krutetskii, 1976), mathematical memory was given a particular attention.

In a different vein, the described frameworks for mathematical talent (e.g. Fritzlar et al., 2006), connect high mathematical performances to a well-defined set of abilities determined by birth and do not offer a detailed explanation of memory functions in mathematics. Thus, due to the earlier discussed problematic view on innate giftedness in the Swedish educational context (e.g. Dodillet, 2017; Persson, 2010) and, importantly, because memory functions associated to mathematics have a central position in my studies, these frameworks for mathematical talent were not included in the analytical framework of this thesis.

Having said that, based on the suggestion that the development of mathematical abilities is influenced by the teaching of the subject (e.g. Krutetskii, 1976; Usiskin, 2000), and, importantly, that high-achievers are able to display appropriate mathematical abilities during problem solving (Krutetskii, 1976), it was reasonable to assume that Krutetskii's framework applies also for the analysis of the abilities of high-achieving students. Moreover, as indicated, the identification of generally or mathematically gifted pupils is not only a challenging and intricate process, but also a practice with low acceptance in the Swedish educational system. Consequently, in the framing of this thesis, I decided to observe *high-achieving* pupils, a group which could be identified in a reliable manner and without negative connotations associated with the concept of giftedness in the Swedish school context.

It has also been suggested that the terminology associated with exceptional mathematical performances is by some means convergent, and, that notions that describe those performances are used in interchangeable ways. Thus, in this thesis, I decided to primarily use the terms *mathematically gifted* (mainly because of its exhibited analogy to general giftedness) when addressing gifted, talented, able or highly-able students in mathematics and *mathematically high-achievers*, when describing pupils with excellent performances in school mathematics. In this manner, my aim was to avoid the use of several terms which are often understood synonymously and used interchangeable in the research community. However, when framing these decisions, I was aware that mathematically gifted and mathematically high-achieving pupils represent sets that considerably intersect, rather than completely intersect, each other.

Finally, as shown, many mathematics teachers do not meet the needs of gifted pupils and display considerable levels of uncertainty concerning the pedagogical and organizational practices directed towards these pupils. Also, several pedagogical practices which are recommended for gifted learners omit the needs of those who do not display their abilities in terms of school performances, and, furthermore, correspond with those practices which are suggested for all students in heterogeneous classes. In addition, it seems that a significant number of recommendations concerning the development of gifted pupils in mathematics are based on studies which were conducted at least a decade ago. In conclusion, I decided to undertake a systematic analysis of the research literature concerning gifted pupils' learning in mathematics and the school situation of mathematically gifted pupils.

Mathematical memory

Even though numbers and algorithms are fundamental in mathematics, the ability defined as mathematical memory (Krutetskii, 1976) does not include the mechanical recalling of numbers, algorithms or procedures. Addressing different aspects of mathematics, mathematical memory concerns knowledge of a more general character; for example, mathematical relationships, schemes of arguments and methods of problem-solving (Krutetskii, 1976, p. 300). In this section, I will try to present different aspects of the mathematical memory. That is, after displaying the differences between the notion of remembering numbers and mathematical memory, as defined by Krutetskii, the concept of mathematical memory will be discussed in the context of cognitive theories and standpoints from the field of cognitive neuroscience.

Remembering numbers

Considering that knowledge deliberately recalled from memory is a direct outcome of learning, it is widely agreed that memory functions are essential both in learning mathematics and in mathematical problem solving (e.g. M. Leikin et al., 2013, 2014; Raghubar, Barnes, & Hecht, 2010). Further, from a similar perspective, the cognitive scientist Gärdenfors proposes that "learning and memory are two sides of the same coin. Without memory, our learning it would not be worth that much. And vice versa, without learning we would not need any memory" (Gärdenfors, 2010, pp. 46–47; my translation). Thus, when considering mathematics, it seems that the "crucial question is not whether memory plays a role in understanding mathematics but what it is that is remembered and how it is remembered by those who understand it" (Byers & Erlwanger, 1985, p. 261).

From a historical perspective, it should be noted that research that deals with memory functions started more than 130 years ago, but during the first eight decades the topic was almost exclusively studied through quantitative approaches. A foundation stone of the research on memory functions was laid by Ebbinghaus in 1885, when he investigated the ways in which meaningless information, such as nonsense syllables, for example, DEB, GIK and WAF, are remembered (Byers & Erlwanger, 1985).

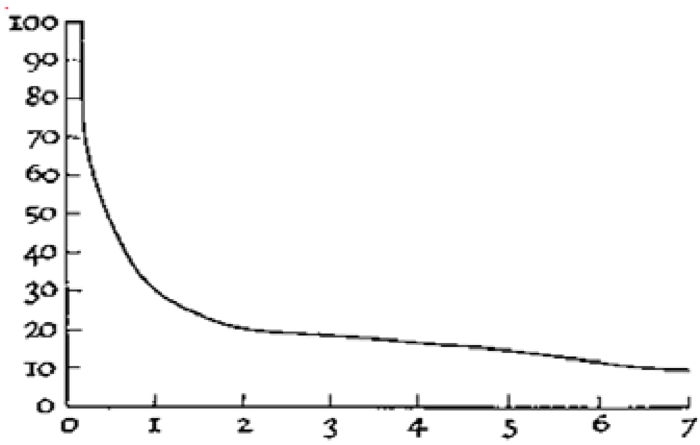


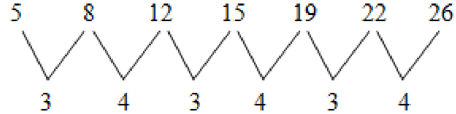
Figure 1. Ebbinghaus' Forgetting Curve. The horizontal axis shows elapsed days after the experiment, the vertical axis shows remembered syllables, in percentage (Rohracher, 1947, p. 247).

Figure 1 shows that one day after the experiment more than 70 percent of the meaningless information is forgotten and, importantly, that the amount of remembered nonsense syllables decreases in a relatively slow exponential pattern after the first couple of days. Moreover, quantitative approaches showed that the proportion of an arbitrary academic course that is forgotten after a year is similar to the proportion of insignificant information forgotten after a single day (Hilgard & Atkinson, 1957). And further studies in the quantitative paradigm emphasized that learning is supported essentially by repetition and meaning (Morgan & Deese, 1957).

However, a different approach to the existing paradigm, expressed in basically qualitative terms, was initiated in the 1940s. Accordingly, Katona (1940) – when analysing individual's strategies for memorizing numbers – observed that large numbers are easier to memorize if one can discern a comprehensible pattern among the composing digits, instead of mechanically memorizing the numbers. By asking randomly divided individuals in three groups to memorize the following sequence

5 8 1 2 1 5 1 9 2 2 2 6

Katona demonstrated qualitative differences in the use of memory among the different studied groups (Marton, Dahlgren, Svensson, & Säljö, 1999). For example, one group had to memorize the sequence by discerning sets of three digit numbers (e.g. 581, 215, 192 etc.), a second group had to observe the sequence in a narrative context of federal expenses, while a third group was given three minutes in order to find an understandable pattern among the digits. Some weeks later, when asking the participants to recall the sequence, the group which tried to find patterns among the digits outperformed the other two groups; that is, one fourth of the individuals in the third group could recall the correct sequence while no participants from the other two groups could (Marton et al., 1999). An explanation to the substantial difference between the groups was that some individuals in the third group observed that a re-grouping of the digits displayed a systematic addition with 3 and 4, as shown below:



Thereby, Katona (1940) demonstrated that remembering numbers is facilitated by relatively simple mathematical rules, interpreted as a qualitative aspect of the stored information.

Two decades later, in a different approach, Bruner (1962) demonstrated that detailed knowledge stored in memory may be recalled effectively by using simple interrelated representations. And further studies (e.g. Bruner, 1973; Davis, 1978) pointed out that mathematical understanding depends on the type of the actual knowledge, that is, there are qualitative differences between remembering numbers and procedures or mathematical concepts and generalizations. Moreover, when discussing the teaching of the subject, Byers and Erlwanger (1985) claim that memory is evident in every mathematical activity and therefore, the memory's role in understanding mathematics should be highlighted.

Finally, it should be noted that recent studies (e.g. M. Leikin et al., 2013, 2014; R. Leikin et al., 2017) have demonstrated significant interactions between memory functions, giftedness and exceptional performances in mathematics. Moreover, a relatively recent review of developmental and cognitive approaches on working memory and its connections to mathematics suggests that there are some particular memory processes which may facilitate learning, that certain aspects of working memory are specific to early mathematical learning and, importantly, that it is difficult to construct a comprehensive model over the impact of working memory on mathematical disabilities (Raghubar et al., 2010).

Mathematical memory according to Krutetskii

Although numbers are fundamental in mathematics, Krutetskii (1976) underlines that neither remembering numbers and multiplication tables nor recalling algorithms can be equated with mathematical memory. When observing the mathematical memory, he noted that able students memorized contextual information of a problem basically only during the problem-solving process and forgot it mostly straight afterwards; nevertheless, able students could, even several months later, recall the *general method* they used when solving the given problem (Krutetskii, 1976, pp. 295–296).

In addition, Krutetskii's (1976) research team remarked that, when solving mathematical problems, able students exhibited so-called *periodic associations*, that is, associations to the context of a problem which were applied only over a limited time period. Furthermore, it should also be mentioned that low-achievers could often recall numbers or superfluous data from the context of a problem, but very seldom the method which was necessary for solving it. For example, Krutetskii's team introduced a problem that incorporated details about the different parts of a fish and some relationships between them before inviting pupils to calculate the weight of the whole fish. When recalling the problem, low-achieving pupils stated that it included "something about a fish weighing 2 poods" (Krutetskii, 1976, p. 299) but did not remember the methods applied to solve the problem (which they had difficulties with). Typically, able students and high-achievers did not recall specific data about the fish but described the solution method associated with it:

> "I did a problem on different combinations of the parts of a whole – about a fish whose tail and head weigh so much, whose head and body weigh so much, and whose tail and body weigh so much" (Krutetskii, 1976, p. 299)

Interestingly, when able pupils solved a problem of a certain type and three months later were confronted with different problems of the same type, they often stated that they have solved the very same problem previously. According to Krutetskii, this event occurred because able pupils retain both the problem type and the generalized method for solving it; therefore, different problems of the same type seem to be extremely familiar to them (Krutetskii, 1976, p. 296).

Consequently, mathematical memory, as defined by Krutetskii, is a memory for mathematical relationships, schemes of arguments, proofs and methods of problem-solving (Krutetskii, 1976, p. 300). In that sense, it is not surprising that mathematical memory is more developed and easier to observe in mathematically able students than in low-achievers; moreover, Krutetskii indicates that it is a result of how information is remembered during mathematical activities:

> ... the memory of a mathematically able pupil is markedly selective: the brain retains not all of the mathematical information that enters it, but primarily that which is "refined" of concrete data and which represents generalized and curtailed structures. This is the most convenient and economical method of retaining mathematical information. (Krutetskii, 1976, p. 300)

Additionally, the quality of mathematical generalizations has very little to do with pupils' general memory abilities and "may be detected only in operations with mathematical material" (Krutetskii, 1976, p. 300) – thus, according to Krutetskii, mathematical memory is shaped and primarily actualized within mathematical activities.

In an attempt to bring a deeper understanding to the notion of mathematical memory, it may also be beneficial to elucidate it in the light of different studies conducted close to the time period of Krutetskii's research. As mentioned in the previous section, in contrast to the intuitive assumption that remembering numbers can be equated to understanding mathematics, after the 1960s, several studies demonstrated significant qualitative differences between *understanding* and *knowledge*. For example, Gagné (1970) described understanding as a set of intellectual abilities and emphasized that it is not meaningful to evaluate the process of learning without studying the ability to memorize information. Moreover, he stressed that there are significant qualitative differences between the ability to remember images, the ability to remember stories, and the ability to re-actualize intellectual aptitudes, because the differences between such abilities are also shown by the differences in how related memory-units are organized in human memory (Gagné, 1970).

In a different approach, Skemp (1976) differentiated between *instrumental* and *relational understanding* – thereby indicating that the nature of understanding is closely associated with the mathematical context in which it is applied. Hence, instrumental understanding is described as a lower level of understanding, needed for recognizing a particular type of task and for the correct application of algorithms and rules when solving the given task. Conversely, relational understanding emerges from an assessment of the mathematical area that a given problem belongs to, and thereby based on the perception of how the problem is related to an appropriate scheme, or how methods and formulas are related to each other within the respective mathematical area (Skemp, 1976). Therefore, by using an analogy to the above forms of understanding, it should be helpful to observe that the description of mathematical memory – associated with mathematical relationships and methods of problem-solving (Krutetskii, 1976) – exhibits significant parallels to the higher levels of understanding introduced by Gagné (1970) and Skemp (1976), respectively.

Subsequently, it is reasonable to assume that mathematical memory draws on a deeper understanding of the subject and thereby may also be viewed as a non-controversial characteristic of gifted and high-achieving students. Moreover, it seems that mathematical memory is essentially selective. That is, it focuses on those aspects which are meaningful and necessary for problem solving – such as methods and generalized relations within a specific mathematical area – and ignores numbers and contextual information which are not important for problem solving.

Having in mind that the delimitation of mathematical memory is not only a result of a twelve-year study of around 200 pupils but also a consequence of a comparative assessment of gifted and less able pupils (Krutetskii, 1976), it might be beneficial to highlight some additional aspects of the concept.

Thus, firstly it should be noted that, even though Krutetskii indicates that both gifted and high-achieving students are able to display and articulate appropriate mathematical abilities (Krutetskii, 1976, pp. 67–70), he underlines that mathematical memory is better developed in gifted and high-achieving pupils than in average performers and low-achievers (Krutetskii, 1976, pp. 295–301).

Secondly, when seeking connections between the age of pupils and mathematical memory, the research team could not observe accurate manifestations of mathematical memory in young children:

> In the primary grades we observed no manifestations of mathematical memory proper in his developed forms (when only generalizations and mental patterns would be remembered). At this age, able pupils … usually remembered both concrete data and relationships equally well. The general and the particular, the relevant and the irrelevant, the necessary and the unnecessary are retained side by side in their memories. (Krutetskii, 1976, p. 339)

Hence, in contrast to primary grade pupils, gifted students from secondary school demonstrated appropriate structures of mathematical memory. In an attempt to explain that, Krutetskii notes:

> The ability to curtail reasoning, the generalizing memory and the striving for economy and rationality in solutions were formed at later stages. There is reason to believe that these components of mathematical abilities are formed on the basis of the initial ability to generalize mathematical material. (Krutetskii, 1976, p. 341)

Accordingly, it seems that mathematical memory is not only formed at later stages in the child's development, but is also based on and developed as a consequence of the ability to generalize mathematical content.

And thirdly, it should be mentioned that – drawing on the observation that mathematical memory is mostly exhibited at the early stages of problem-solving when the individual discovers the formal structure of a problem – mathematical memory is closely related to and extremely difficult to discern from the ability to obtain mathematical information (Krutetskii, 1976, pp. 350–353).

Finally, it should also be mentioned that when mathematicians solve challenging problems, they attribute fundamental roles to *intuition* and *insight* (Hadamard, 1945; Mann, Chamberlin, & Graefe, 2017; Pólya, 1966). In this case, the insight or intuition refers to the situation when the individual construes the idea which will lead to the solution of the problem, and the time period prior to the insight is described as extremely challenging but crucial for the success of the process (e.g. Mann et al., 2017). Similarly, Krutetskii's team noted that gifted pupils became considerably stressed before offering an insight which solved the given problem, but also, that the insight was often exhibited after unsuccessful approaches or without a comprehensible connection to previous problem-solving attempts. However, contextual interviews with participants, conducted after problem solving, showed that the insights were often an outcome of pupils' previous generalizations and based on mathematical memory – thereby the research team highlighted the importance of insight or inspiration during problem solving but also that the researchers faced considerable difficulties when observing mathematical memory in pupils' activities during problem solving (Krutetskii, 1976)..

Mathematical memory in the context of cognitive theories and neuroscientific perspectives

In an attempt to situate mathematical memory in a broader perspective, in this section I describe some associations between mathematical memory, as described by Krutetskii (1976), and memory functions of the human brain, as perceived through the lenses of cognitive studies and approaches from the field of cognitive neuroscience.

A pioneering study conducted more than sixty years ago (Furlong, 1951) suggested that memory is dependent on the type of information which is remembered. Furlong described the difference between the ways a person who is not involved in mathematics remembers seeing someone post a letter and recalling the square root of a number, as it follows:

> In the former case the mind looks back to a past event: we recollect, reminisce, retrospect; there is imagery. In the latter case this looking back is absent, and there is little or no imagery. We have retained a piece of infor-

> mation; that is all. There is retentiveness but not retrospection. (Furlong, 1951, p. 6)

Hence, even though mathematicians or mathematic teachers will argue against the manner in which the square root of a number is remembered, for example, claiming that recalling the square root of 2 is connected to several images associated in their memory (Davis, 1996), Furlong points out an essential difference between different ways of remembering information. Specifically, that different types of information are recalled by different memory systems – as in the above example, the images of the posted letter, perceived as a previous event, were associated with *episodic memory* while the square root of a number, a fact without imagery, was associated with *semantic memory* (Nyberg & Bäckman, 2009). That is, when the person tried to remember information about the posting of the letter, he could visualize an earlier and comparable episodic experience. Conversely, the recalling of a value associated with the square root of a random number was stored only as a fact in his memory, without connections to previously experienced events.

When initiating a discussion about cognitive frameworks for memory functions, it might be beneficial to notice that the elements of these explanatory models are derived from everyday language and thereby limited in their attempt to offer accurate descriptions of the focused topic (Squire, 2004). However, cognitive studies and studies from the field of neuroscience indicate that a basic function of human memory is not to remember events and facts, but to forget unnecessary and superfluous information (e.g. Olson et al., 2009). In the perspective of these studies, a generally agreed model of human memory systems is shown in Figure 2 (e.g. Moscovitch, 1992; Olson et al., 2009; Squire, 2004).

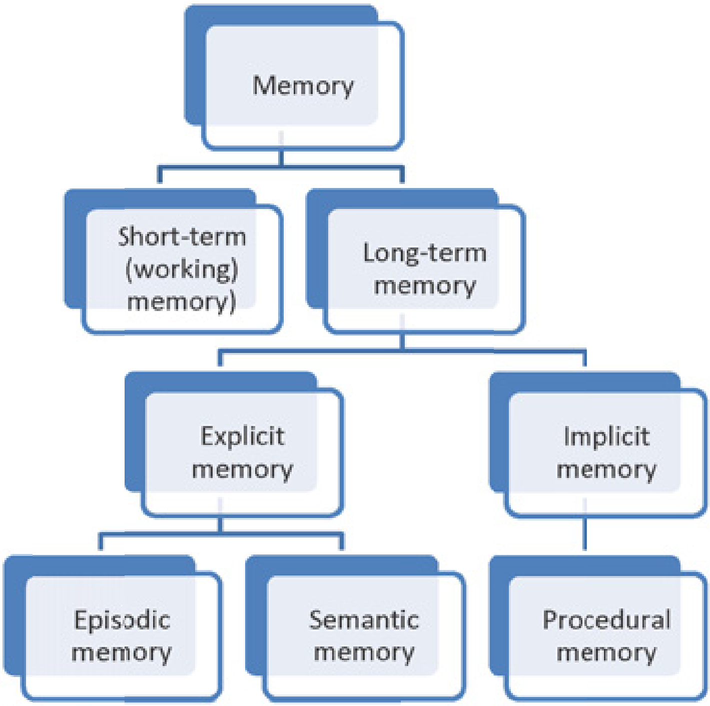


Figure 2. Human memory systems.

As indicated above, these systems are essentially categorized by the length of the time that information is stored in different partitions, and the most important distinction is between *short-term* and *long-term memory*. In the context, it should also be underlined that a substantial part of short-term memory consists of *working memory*. Nevertheless, studies in cognitive neuroscience indicate that a main function of the human brain is to relate all new information to previous knowledge and experience (e.g. Buckner & Wheeler, 2001; Ingvar, 2009; Shipp, 2007). Thus, according to the model, working memory hosts a continuous and cyclical process – for example, mathematical problem solving – in order to discover and mirror new information and, if needed, to update and reform existing knowledge.

However, the capacity of working memory is fundamentally limited by two factors: by the time period that information can be maintained in the system and by the number of items (information-units) that can be kept simultaneously in it (e.g. Buckner & Wheeler, 2001; Ingvar, 2009; Nyberg et al., 2003). Consequently, given the fact that information processed in working memory is typically lost after 30 seconds, it must be repeated in order to keep it in working memory (e.g. Buckner & Wheeler, 2001; Nyberg et al., 2003; Olson et al., 2009; Sternberg & Sternberg, 2012). In addition, in order to balance the limitations concerning the restricted number of information-units that can be kept simultaneously in working memory, a part of the processed information is brought into working memory by means of *automaticity* (Ingvar, 2009). Automaticity can be described as an extremely effective, but very inflexible, process which does not affect further memory functions in working memory. Moreover, from a neuroscientific point of view, an essential function of the cerebral cortex is to develop highly effective automated processes, because this is the most efficient way to use the memory system. Thus, it seems that "all learning leads to automation" (Ingvar, 2009, p. 36; my translation). Accordingly, when relating mathematical problem solving to the cognitive model, it would be reasonable to assume that information is processed (the problem is solved) in the working memory, while mathematical procedures and problem-solving methods are stored in long-term memory.

It is generally agreed that most of the information retained in human memory is stored in long-term memory (e.g. Olson et al., 2009; Sternberg & Sternberg, 2012). In the corresponding model (Figure 2), long-term memory has two subsystems, depending on the type of information stored in respective systems: the *implicit* memory and the *explicit* memory.

The implicit memory stores information about procedures and patterns of movement – for example, how to cycle, to write or to swim, or how to use a specific algorithm – that can be activated by certain events. Hence, in a mathematical context, *procedural* memory is a significant part of the system because it consists of information on *how* to do something, in contrast to *why* to do it (e.g. Nyberg & Bäckman, 2009; Sternberg & Sternberg, 2012; Squire, 2004). Thus, it seems that processes that retrieve information from procedural memory in a mathematical context – for example, the recalling of algorithms and multiplication tables – can be considered automated processes (e.g. Ingvar, 2009; Squire, 2004). Further, the implicit nature of procedural knowledge was also demonstrated by studies reporting that many pupils face difficulties when asked to explain how algorithms should be applied when not currently working with them (Davis, Hill, & Smith 2000).

According to different principles, explicit memory stores information about experiences and facts which can be consciously recalled and explained; consequently, the previously mentioned *semantic* and *episodic* systems are located in this memory partition (e.g. Davis et al., 2000; Nyberg & Bäckman, 2009). As indicated, episodic memories are associated with some sorts of imagery and thereby linked to the circumstances in which they were acquired, for example, remembering what we ate for lunch some days ago, or the profession of someone we met at a Saturday dinner (Nyberg & Bäckman, 2009).

Drawn on a somehow different structure, but still in an explicit manner, semantic memory denotes knowledge about various subjects and comprises facts, pictures and theories which are retained without concrete information about the circumstances in which they were acquired; for example, remembering the name of the capital city of a country which one has never visited or has no connections to, or associating the notion of chair to the word table

(Nyberg & Bäckman, 2009). Moreover, semantic memory allows us to structure and categorize our knowledge and to understand how the structures and categories are related to each other (Gärdenfors, 2010). However, given the fact that a great proportion of human knowledge belongs to the explicit system – thereby containing both episodic and semantic attributes – it is not unproblematic to differentiate semantic memories from episodic ones (e.g. Davis et al., 2000; Gärdenfors, 2010). Especially, in the mathematical context, it has been indicated that procedural skills may generate episodic memories in order to facilitate the consolidation of semantic memories, that is, factual knowledge in mathematics (Davis et al., 2000). Thus, from a mathematical point of view, it is reasonable to assume that explicit memory is associated with the ability to create and use mental schemas for problem-solving or to understand the main ideas of a particular mathematical area (e.g. Davis et al., 2000; Gärdenfors, 2010).

In conclusion, I will try to summarize the above described human memory system in the perspective of mathematical problem solving. According to the cognitive model, information is processed in the working memory and stored in long-term memory. However, the processed information can be retained in working memory only for a short period of time and the number of information-units that can be processed simultaneously in working memory is extremely limited (e.g. Buckner & Wheeler, 2001; Ingvar, 2009). Therefore, information is stored in the respective subsystems of long-time memory, that is, in implicit and explicit memory (e.g. Olson et al., 2009; Sternberg & Sternberg, 2012). Consequently, in order to balance the limitations of the system, information is brought into working memory through repetitive and cyclical processes (e.g. Buckner & Wheeler, 2001; Ingvar, 2009). Hence, it might be reasonable to assume that procedural knowledge, for example, algorithms and multiplication tables, is transferred from procedural memory to working memory by automaticity during problem solving (Ingvar, 2009).

Finally, it is important to highlight that Krutetskii's (1976) definition of mathematical memory excludes the recalling of numbers, algorithms and table skills, which, from a perspective of cognitive neuroscience, are automated processes. Thus, it seems that mathematical memory – with its generalized methods for problem-solving and abbreviated mathematical relationships – belongs to the explicit memory. Krutetskii's observation, that mathematical memory is not observable in young children, is supported by recent studies, indicating that young children possess few mechanisms that connect implicit memory to explicit memory (Murphy, McKone, & Slee, 2003). Therefore, it is reasonable to assume, that – because information from explicit memory can be consciously recalled and explained – mathematically high-achieving older pupils should be able to explain why they are using generalized methods during problem solving (e.g. Davis et al., 2000; Krutetskii, 1976; Nyberg & Bäckman, 2009).

Implications for this thesis

The presented literature review situates memory in mathematics and *mathematical memory* within the domains of psychological and educational studies, and addresses cognitive and neuroscientific standpoints associated with memory functions in school mathematics.

Accordingly, it seems that mathematical memory is formed during mathematical activities, based on the ability to *generalize mathematical relationships*. Consequently, in this thesis, I elected to investigate whether the ability to generalize mathematical relationships is observable during students' problem-solving process.

Additionally, as specified, mathematical memory is not observable in young children and in low-achievers; that is, young gifted pupils remember both concrete data and relationships in equal terms and do not discern the relevant information from the irrelevant, while low-achievers are more sensitive to the contextual data of a problem than to the applied problem-solving methods. Consequently, I decided to observe *high-achieving* students from upper secondary school.

Cognitive studies on the human memory system, as well as studies from the field of cognitive neuroscience, indicate that mathematical memory is most probably located in the *explicit memory* and thereby possible to be articulated by gifted students in a problem-solving context. Therefore, I decided to pay attention to students' problem-solving related behaviours in order to differentiate their use of *implicit knowledge* – such as algorithms or multiplication tables – from the recalling of generalized mathematical relationships and problem-solving methods, as manifestations of *explicit knowledge*. In this respect, I assumed that students will typically not be able to explain their implicit knowledge, while they may be able to describe and motivate applied problem-solving methods and mathematical relations.

Further, as indicated, during periods when students are quiet and inactive, it might be difficult to differentiate the ability to *obtain mathematical information* from *mathematical memory*. Subsequently, I decided to use a technology which digitises the problem-solving process in terms of short time-intervals. Additionally, in order to offer additional opportunities for participants to expose *semantic* and *explicit* memories, I decided to conduct individual reflective interviews after every problem-solving activity. My assumption was that the opportunities during the reflective interviews would facilitate the display and articulation of those abilities, which were not observable during problem solving.

Mathematical problem solving

A defining work of modern research on mathematical problem solving is George Pólya's *How to solve it* (Pólya, 1966), which is also considered a foundation stone of educational research on problem solving in school mathematics (e.g. Blum & Niss, 1991; Cai & Lester, 2005; Carlson & Bloom, 2005; Felmer, Pehkonen, & Kilpatrick., 2016; Kapa, 2001; Kilpatrick, 2016; Mason et al., 1982; Nunokawa, 2005; Schoenfeld, 1985; Singer & Voica, 2013).

However, despite its importance for the teaching and learning of mathematics, it seems that problem solving has a cyclical focus in many mathematics curricula of the western world – for example, in the USA, where cycles of teaching problem solving alternate with cycles of teaching the basics of the subject (e.g. Mason, 2016; Zimmermann, 2016). As a consequence of this, problem solving is placed once in every decade in the nucleus of the mathematics curricula (Lesh, 2006; Mason, 2016; Schoenfeld, 1992).

In this section, I will first introduce the notion of mathematical problems. Secondly, I will examine the literature on problem solving in mathematics, and thirdly, I will discuss some aspects of the research on the teaching of problem solving in mathematics.

Mathematical problems

When discussing mathematical problems, we should remind ourselves that even though mathematics is a subject consisting of several necessary components, for example, axioms, theorems, proofs, concepts, formulas, methods and definitions, problem solving is considered a central component of mathematics and mathematics education (e.g. Ambrus & Barczi-Veres, 2016; Halmos, 1980; Kilpatrick, 2016; Mason, 2016; Pólya, 1966, Schoenfeld, 1985). In this respect, Halmos expresses the significance of problem solving in school mathematics as it follows:

> The major part of every meaningful life is the solution of problems … it is the duty of all teachers, and all teachers of mathematics in particular, to expose their students to problems much more than to facts. (Halmos, 1980, p. 523)

Given its pivotal role in mathematics, I will try to expose some of the main characteristics of mathematical problems. In this aspect, it is generally agreed that what qualifies a *mathematical task* to be regarded as a *mathematical problem* is "not so much a function of various task variables as it is of the characteristics of the problem solver" (Lester, 1994, p. 664). Thus, it seems that the *relationship* between the solver and the proposed task in combination with the *challenge* that the solver faces when solving the task are of great importance when determining mathematical problems (Carlson & Bloom, 2005). Another way to describe mathematical problems is to consider them as tasks which pose some challenging questions to those individuals who are not in immediate possession of the methods or algorithms that are sufficient to answer the question and thereby to solve the task (e.g. Blum & Niss, 1991). Also, a task which is considered to represent a mathematical problem for a solver at a certain time, may not represent a problem at additional occasions later on; that is, if the solver becomes familiar with certain kinds of task, then those tasks will be considered *routine tasks* by the solver (Arcavi & Friedlander, 2007). Pólya (1966) addressed the difference between tasks and mathematical problems by highlighting the cognitive actions connected to the process of problem-solving, and by describing *routine* and *non-routine problems* from a teacher's point of view:

> There are problems and problems, and all sorts of differences between problems. Yet the difference which is the most important for the teacher is that between 'routine' and 'nonroutine' problems. The nonroutine problem demands some degree of creativity and originality from the student, the routine problem does not. ... I shall not explain what is a nonroutine mathematical problem: If you have never solved one, if you have never experienced the tension and triumph of discovery, and if, after some years of teaching, you have not yet observed such tension and triumph in one of your students, look for another job and stop teaching mathematics. (Pólya, 1966, pp. 126–127)

Thus, it is reasonable to assume that two basic characteristics of mathematical problems are the novelty and the challenge they represent to the individual who solve them. And, in the context of this thesis, it might be appropriate to mention that in a study of seventh grade students, Garofalo (1993) showed that successful, meaning-oriented, students prefer to solve *non-routine* problems, while less successful, number-oriented, students prefer *routine* problems. Hence, by relating his findings to Skemp (1976), it is indicated that the two different types of problem-solving can be considered as two different subjects that are taught under the common notion of mathematics.

Also, Krutetskii's (1976) team, in their ambition to as far as possible avoid the influence of students' previous experience and knowledge chose to use problems which can be characterized as non-routine:

> … we are studying ability, not knowledge, habits and skills … It is hard to isolate the factor of ability in this intricate complex of causes. Apparently one should select problems in such way that only ability will primarily influence their solutions. It is clearly impossible to completely eliminate the influence of past experience, knowledge and habits. (Krutetskii, 1976, p. 94)

Another way to characterize mathematical problems can be made according to the context of the problems. In that way, mathematical problems can be described as *pure mathematical* or *applied* problems (e.g. Blum & Niss, 1991; Haylock & Cockburn 2008). With that respect, applied problems are situated in a perceptible context and contain questions that are connected to the real world, that is, the world outside of pure mathematics, such as other school subjects or everyday life (e.g. Andrews & Xenofontos, 2015; Blum & Niss, 1991). Conversely, pure mathematical problems are entirely integrated in the field of mathematics, and even those derived from the real world, once they have been converted into an abstract mathematical context, cannot be considered as applied problems (e.g. Blum & Niss, 1991).

Problem solving in mathematics

In the research literature, the process of problem-solving is mainly characterized from the perspective of above presented descriptions of mathematical problems (e.g. Andrews & Xenofontos, 2015; Cai & Lester, 2005; Mason, 2016; Nunokawa, 2005). For example, by starting with the essential contribution of Pólya (1966), a substantial number of attempts to delimit problem solving in mathematics (e.g. Felmer et al., 2016; Garofalo, 1993; Halmos, 1980; Lester & Kehle, 2003) can be considered convergent in their nature and reflect a view of problem solving as "an activity requiring the individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine" (Cai & Lester, 2005, p. 221).

Additionally, it should also be noticed that problem solving is occasionally described as a process when the solver communicates her or his thinking to others through some intelligible representations of the problem (e.g. Cai & Lester, 2005; Lester & Kehle, 2003). Hence, in an attempt to broaden the notion, by pointing out that most definitions of problem solving omit the central role of representations – for example, facts, figures and symbols – and associated patterns of inference in the problem-solving process. In this respect, Lester and Kehle (2003) propose a more complex definition:

> Successful problem solving in mathematics involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve the tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity. (Lester & Kehle, 2003, p. 510)

Further, a more recent explanation has been offered by Kilpatrick (2016) from a slightly different and to some extent holistic approach. That is, by building on the ideas of the gestalt psychologist Duncker (1945) and the work of Pólya (1966), Kilpatrick (2016) describes mathematical problem solving as the process of reducing the formulation of a given problem into more trivial tasks; that is, as "getting from where you are to where you want to be through successive reformulations of the problem until it becomes something you can manage" while replacing "unsuccessful efforts by successful ones through a heuristic inquiry process" (Kilpatrick, 2016, p. 69).

After the presented approaches to define mathematical problem solving, with respect to the research carried out in this thesis, I will introduce some ideas which are considered significant for the process of problem-solving in mathematics.

A substantial part of the research attempting to delimit the problem-solving process has drawn on Pólya's (1966) four phase model, which can be described as *understanding* the problem, *developing* a plan, *carrying out* the plan and *looking back* at the completed solution. For example, Mason, Burton and Stacey (1982) define a three-phase framework, containing *entry*, *attack* and *review*, while Kapa (2001) delimits the following six phases of the process: *identifying and defining* the problem; *mental representation* of the problem; *planning* how to proceed; *executing* the solution according to the plan; *evaluation* of what you know about your performance and *reaction* to feedback. Moreover, Nunokawa (2005), by extending Pólya's work, suggests that problem solving has two fundamental phases; that is, *applying mathematical knowledge* through exploring the problem situation and *obtaining pieces of information* about the problem situation. In addition, Nunokawa (2005) emphasizes that if the tension between the two above mentioned phases is not resolved successfully, then further exploration of the problem situation will take place by a re-evaluation of the obtained information. More recently, Singer and Voica (2013), considering Pólya's (1966) model being significantly dependent of teachers' didactical actions, proposed a framework with a greater accent on the solver's cognitive process. The proposed model consists of four stages, so called operational categories: *decoding* the information; *representing and understanding* the problem via a mental model; *processing*, that is, using mental configurations suggested by the problem and personal mathematical competence in order to identify a mathematical model that can be associated with the given problem; and *implementing* the techniques of the associated model in order to solve the problem (Singer & Voica, 2013).

However, despite the pivotal role of Pólya's work in the field (Hensberry & Jacobbe, 2012) it should also be highlighted that his model has been subjected to occasional criticism, primarily because of his seeming to have described mathematical problem solving as an essentially *linear* and *cognitive* process rather than a *cyclical* and *metacognitive* process (e.g. Garofalo & Lester, 1985; Nunokawa, 1994). With respect to the distinction between linear and cyclical processes, there is nothing in Pólya's work to suggest he saw the process as anything but cyclical. Indeed, when observing mathematicians during problem solving in more recent studies, Carlson and Bloom (2005) noticed that mathematicians rarely solve problems in a linear way, typically fluctuating between the different phases of the process in cyclical approaches. Consequently, they recommend a more modern framework drawn "on the large body of research, as grounded by and modified in response to our close observations of these mathematicians" (Carlson & Bloom, 2005, p. 45) with the four phases of *orientation*, *planning*, *executing*, and *checking* (Carlson & Bloom, 2005). However, it is difficult to see how these four phases, other than in their choice of words, differ from those of Pólya, even though it is not unreasonable to assume that future frameworks will highlight additionally the cyclical nature of problem solving in mathematics (Felmer et al., 2016).

By stressing the difference between cognition and metacognition, it might be important to mention that the latter represents a comprehensive way of thinking, that is, *thinking about thinking* or, in other words, *cognition about cognition* (Flavell, 1979). Thus, for example, forty years after Pólya, whose book first appeared in 1945, Garofalo and Lester (1985) approached mathematical problem solving in terms of metacognitive actions, by describing it according to the phases of *orientation*, *organization*, *execution*, and *verification* – and when characterizing the process, they also accentuated that the shifts between the phases occurred when "metacognitive decisions resulted in some form of cognitive action" (Carlson & Bloom, 2005, p. 47). Accordingly, from this perspective, Pólya's (1966) model was criticized by Garofalo and Lester (1985) as representing a cognitive attempt that fails to address metacognitive attributes. However, in a recent article, Kilpatrick (2016) argues against this criticism, by noticing that, even though Pólya "did not use the term *metacognition* in his writings on problem solving, he certainly considered questions of regulation and control of one's cognitive efforts" (Kilpatrick, 2016, p. 76). Additionally, from the perspective of understanding Pólya's work, it might be beneficial to mention that metacognition, as a notion, was not coined by Flavell until the 1970s (Flavell, 1979).

The teaching of problem solving in mathematics

Finally, another aspect of the research on mathematical problem solving is how problem solving is integrated into school mathematics (e.g. Felmer et al., 2016; Schoenfeld, 1985; Zimmermann, 2016). Indeed, while both teachers and educational researchers view problem solving as a complex and challenging issue (e.g. Zimmermann, 2016), textbooks mostly focus on routine tasks and are particularly weak in stimulating students' problem-solving abilities (e.g. Brehmer, Ryve, & Van Steenbrugge, 2016; Cai, Jiang, Hwang, Nie, & Hu, 2016). Besides, the teaching of problem solving requires teachers who are able to direct pupils' attention to those generalities that are contained by the tasks that pupils are working with (Watson & Mason, 2006).

When initiating a discussion about how problem solving is integrated in school mathematics, it should be highlighted that the position of problem solving in the teaching of the subject and the ways the topic is taught are essentially dependent on the characteristics of the particular educational system. Specifically, significant cultural differences between school mathematics in different geographical areas influence the manner in which problem solving is approached by teachers, and it is not unlikely that these cultural differences have important consequences for students' behaviour and success in problem solving (e.g. Andrews, 2007; Andrews & Xenofontos, 2015; Arcavi & Friedlander, 2007; Cai & Lester, 2005).

However, when addressing problem solving in a didactical context, research offers a variety of perspectives. As with studies (e.g. Cai & Lester, 2005) that emphasize the importance of solution representations used by students and pedagogical representations used by teachers and highlight problem solving in the context of applications and modelling (e.g. Blum & Niss, 1991). Moreover, some researchers (e.g. Ambrus & Barczi-Veres, 2016) suggest that problem solving should be placed in the centre of school mathematics, as seen in the pentagonal curriculum model used in Singapore (Leinwand & Ginsburg, 2007), while others (Lester & Cai, 2016) debate if problem solving should be taught as a separate topic. In addition, some studies emphasize the benefits of the mathematical struggle that students face when constructing their own solutions to problems (Jonsson, Norqvist, Liljekvist, & Lithner, 2014), although others stress that formulating mathematical problems should be regarded as "both a goal and a means of mathematics teaching" (Kilpatrick, 2016, p. 73).

Nevertheless, even though the variety of perspectives on problem-solving demonstrates substantial divergence within the research field, the recommendations may be summarized in four broad categories.

Firstly, in order to help students becoming successful problem solvers, problem solving should be integrated in the teaching of school mathematics, that is, it should not be taught as a separate subject (e.g. Ambrus & Barczi-Veres, 2016; Leinwand & Ginsburg, 2007; Lester & Cai, 2016). And, when integrating it, students should be encouraged to use generalized, algebraic representations, rather than relying on concrete ones, because it is suggested that students who use algebraic methods are more successful in their problem solving (e.g. Cai & Lester, 2005). Secondly, collaborative problem-solving should be given an essential role in school mathematics (e.g. Ambrus & Barczi-Veres, 2016; Kilpatrick, 2016), because students learn a great deal of mathematics when cooperating with their peers (e.g. Ambrus & Barczi-Veres, 2016). Thirdly, that appropriately designed mathematical problems have the potential to develop students' mathematical reasoning, thereby also maintaining students' interest for the subject and facilitating their conceptual understanding of mathematics (Jonsson et al., 2014; Lithner, 2008). And fourthly, that students should be encouraged to reason inductively by encountering problems without given strategies (e.g. Ambrus & Barczi-Veres, 2016; Cai & Lester, 2005; Lithner, 2008) but also to be aware of the problematic nature of analogical reasoning when applying previously encountered relations at atypical situations (e.g. Degrande, Verschaffel, & Van Dooren, 2016).

Implications for this thesis

The above review indicates that problem solving is a highly-efficient activity for exposing and differentiating gifted and high-achieving students' mathematical abilities. Thus, in this thesis – in my attempt to analyse and expose their mathematical abilities – I decided to observe students *during mathematical problem solving*.

Moreover, the review shows not only that there are essential differences between *mathematical tasks* and *mathematical problems*, but also that, for a more accurate display of their mathematical abilities, students should be confronted with *non-routine problems*. Thus, the selection of problems seemed to be pivotal in the framing of this thesis, and therefore, I decided to use non-routine mathematical problems when observing the participants. In addition, based on Krutetskii's (1976) observations of gifted and high-achieving students' mathematical memory, when investigating possible displays of mathematical memory over a larger period of time, I decided to offer problems that, besides being non-routine, could be solved with *methods* that are *relatively similar*.

In addition, based on the indication that the use of generalized algebraic representations is a more efficient method during problem solving than the application of concrete representations (e.g. Cai & Lester, 2005), I decided to select problems that can be approached with both generalized and concrete representations. In that way, I assumed that it might be possible to discern some qualitative differences between the observed problem-solving activities.

Finally, the literature review indicates that mathematical problem solving may be viewed in *cognitive* or *metacognitive* terms, and that it may also be perceived as series of actions in *linear* or *cyclical* patterns, respectively. Therefore, I decided that, when observing participants' during problem solving and contextual interviews, I will be aware of possible associations between the participants' actions and utterances and the components of above discussed problem-solving frameworks. In that way, I assumed that the participants' problem-solving process could offer perspectives that are complementary to those indicated by the interaction of exhibited mathematical abilities.

Research questions

Prior to introducing the research questions that this thesis is built on, I will present a short summary of those theoretical reflections that have motivated and influenced the development of the research questions.

Giftedness

The literature review indicates that there is lack of a comprehensive definition of giftedness in the research field, and, that there is a substantial divergence among methods for identification of general giftedness. It has also been indicated that even though the frameworks aimed to discern and analyse individual mathematical abilities are relatively convergent with respect to their basic elements, the identification of gifted pupils is a challenging process, which has low acceptance in the Swedish egalitarian and inclusive school system. Further, it has been shown, that notions describing individuals with high-level intellectual abilities or learners displaying exceptional mathematical performances are used synonymously and in compatible ways in the research field.

Mathematical abilities

With respect to mathematical abilities, it has been indicated that Krutetskii's (1976) framework represents an appropriate and widely accepted model for analysing the individual mathematical ability in high-achieving and gifted students. However, drawing on the observation that mathematical ability is a complex and dynamic phenomenon, where modestly developed abilities are compensated by well-developed ones, some particular aspects of the mathematical ability should be emphasized in the framing of this thesis. Namely, that mathematical abilities are formed and developed during mathematical activities and thereby dependent of the teaching of the subject, that the general synthetic component of the mathematical ability is not observable during problem solving, that mathematical memory is extremely difficult to differentiate from the ability to obtain mathematical information – but observable during problem solving – and, importantly, that the development of mathematical memory is based on the ability to generalize mathematical material. In addition, the seemingly close relationship between the ability to learn school mathematics and displayed exceptional mathematical abilities indicates that high-achieving students are able to display and articulate their mathematical abilities during activities within the subject.

Mathematical memory

By emphasizing that information associated with mathematical memory represents a higher level of understanding than the mechanical recalling of numbers and algorithms, it has been indicated that there are significant qualitative differences between mechanical memory, that is, a memory for numbers, multiplication tables and algorithms, and mathematical memory (Krutetskii, 1976), that is, a memory for generalized mathematical relationships and problem-solving methods. Moreover, studies conducted within the respective fields of psychology and educational research suggest that mathematical memory is associated with relational understanding of the subject which is a more profound way of construing mathematics than the instrumental understanding of how rules and algorithms should be used in mathematics (Skemp, 1976). Besides, from the horizons of cognitive and neuroscientific studies, the basic elements of mathematical memory are associated with the explicit memory system, thereby representing information which can be consciously recalled and explained by gifted students. Moreover, studies indicate that mathematical memory is not observable in young pupils or in low-achievers because these groups are typically not able to discern knowledge based on mathematical generalizations – for example, methods for problem solving – from data associated with the context of a particular mathematical problem.

Problem solving in mathematics

With respect to mathematical problem solving, the review suggests that it is an appropriate activity for displaying individual mathematical abilities of gifted or high-achieving students. Additionally, in the context it has been emphasized that there are significant differences between tasks and problems and that the structure of a mathematical problem comprehends those abilities which are needed in order to solve the given problem. Moreover, it has also been indicated that the impact of a mathematical problem is highly dependent on the individual experiences of those who are solving it; that is, when analysing students' mathematical abilities, non-routine problems or problems without given strategies should be preferred instead of routine tasks. And it has also been mentioned that the process of problem-solving may be understood in both cognitive and metacognitive perspectives, and, that students' activities during problem solving may be viewed as linear or cyclical, depending on the shifting between the respective phases of the problem-solving process. However, regardless of their composing elements, it has been emphasized that the discussed frameworks display important similarities and are essentially drawn on Pólya's (1966) model, introduced for almost seventy years ago.

Mathematics education for gifted pupils

When discussing gifted pupils' mathematics education, the theoretical background suggests that a majority of mathematics teachers exhibit considerable levels of uncertainty in the choice of pedagogical and organizational practices designed to develop gifted students. Besides, it has been also indicated that a substantial part of those pedagogical and organizational practices which are recommended for all learners in heterogeneous classrooms, coincide with those practices which are suggested in the context of gifted pupils. Moreover, a considerable number of the practices recommended for gifted pupils fail to consider that not all gifted learners are exposing their abilities in the everyday school environment, and thereby not all of them will be able to benefit from offered developmental opportunities. Further, it was also exposed that a substantial part of the research field's recommendations associated with the development of gifted students in mathematics is grounded in literature reviews which might be perceived as not up-to-date and that only a few suggestions are grounded in recent empirical studies.

The research questions

Based on the above presented summary of the theoretical background, the following four topics are examined in this thesis:

- RQ1: Which pedagogical and organizational practices associated with gifted pupils' education are recommended by the research community and which of those practices have an empirical basis?
- RQ2: How do mathematical abilities, as defined by Krutetskii, interact during mathematical problem solving and what is the role of mathematical memory during problem solving?
- RQ3: What is the relationship between students' mathematical abilities and their problem-solving performance?
- RQ4: What is the impact of mathematical memory on students' problem-solving activities over a larger period of time?

These research questions were addressed in three research phases, summarized in the four papers included in this thesis. To facilitate the reader's comprehension of the context of these four papers, I will describe briefly the research from which they emerged.

The first phase of data collection, associated with RQ1, addresses pedagogical and organizational practices recommended in the research field in order to develop gifted pupils in the mathematics classroom. During this phase, a systematic review of research was undertaken. A detailed description of the methodological considerations associated with this phase of data collection and the subsequent analysis of the collected data are presented in the next section. The results that emerged from the first phase of data collection are shown in Paper I, "Mathematics education for gifted pupils – a survey of research" (in Swedish "Matematikundervisning för begåvade elever – en forskningsöversikt").

The second phase of data collection was defined by the research questions RQ2 and RQ3, concerning the interaction of mathematical abilities and the role of mathematical memory during mathematical problem solving, respectively the relationship between students' mathematical abilities and their problem-solving performances. In order to address RQ2 and RQ3, individual problem-solving activities of high-achieving students from Swedish upper secondary school were scrutinized. The observations of and interviews with participants were carried out during a single day. The methodological considerations associated with the data collection and the subsequent analyses are described in detail in the respective sections of this thesis. The papers referring to this phase are Paper II, "Examining the interaction of mathematical abilities and mathematical memory: A study of problem-solving activity of high-achieving Swedish upper secondary students" and Paper III, "Uncovering the relationship between mathematical ability and problem-solving performance of Swedish upper secondary school students".

The third phase of data collection, determined by RQ4, concerns the impact of mathematical memory on students' problem-solving performances over a larger period of time. In order to address RQ4, individual problem-solving activities of high-achieving students from Swedish upper secondary school were observed during two different occasions that occurred approximately one year apart. The participants were identical with the participants associated with the second phase of data collection, which occurred approximately one year prior to the third phase of data collection. The methodological considerations connected to the data collection and analysis for the third phase are presented in the respective sections of this thesis. The results of this phase are displayed in Paper IV, "Mathematical memory revisited: mathematical problem solving by high achieving students".

Methodology

Acknowledging the different research questions, the three phases of data collection drew on two methodological traditions. Three of the papers reported on investigations of the problem-solving behaviour of students, while one scrutinized the research literature associated with the education of gifted pupils, and it is to this that I turn first.

Methodology associated with the first phase of data collection

The first phase of data collection had a perspective different from the studies associated with gifted pupils' problem-solving activities and resulted in Paper I, which is a *review article*. One purpose of this phase of data collection was to expose those pedagogical and organizational practices concerning gifted pupils' development in mathematics, which are recommended by the research community. Another purpose was to expose the ways the recommended methods are supported by empirical evidence.

Primarily, it is important to mention that a main objective of a review article is to summarize various aspects of the current research on a specific topic (e.g. Grant & Booth, 2009; Rethlefsen et al., 2014; Ziegler & Raul, 2000). One critical difference between scientific papers reporting on empirical studies and papers reviewing research on a specific topic is that papers reporting on individual research occasionally lead to particular results which may be difficult to evaluate in the overall perspective of the research field, while review articles typically aim to provide information that facilitates the understanding of a given topic from the perspective of several research articles.

Moreover, it seems that the quality of review articles is predominantly determined by *the methods* used for data collection, *the analysis* of the collected data and *the competence* of those who perform the analysis of the selected papers (e.g. Grant & Booth, 2009; Wu et al., 2012). However, even though these criteria have been emphasized in the research field over the past three decades, and review articles are among the most frequently retrieved from research databases, an analysis of fourteen review articles shows that only a small part of them present methodologies which are both established and explicit (e.g. Grant & Booth, 2009). In addition, it should be mentioned that after examining the research on gifted pupils' mathematics education I also share the view that only a small part of review articles presents conventional and explicit methodologies. Finally, the divergence among the methods used in those investigations that review articles are based on entails that this type of articles is often labelled with synonymous concepts, for example, *systematic review*, *overview*, *review article*, *review of evidence*, *literature review* or *comprehensive review* (e.g. Felmer et al., 2016; Grant & Booth, 2009; Lerman, 2014; Wu et al., 2012; Ziegler & Raul, 2000). Consequently, in this thesis, I will use the term *review article* in order to differentiate Paper I, included in this thesis, from the literature reviews associated with the main topics in the theoretical background.

Accordingly, when framing the first phase of data collection, a main concern was to find a method that allowed me to collect data which, based on the research questions, could potentially represent a comprehensive understanding of the focused topic. Consequently, an appropriate method seemed to be a *systematic review* of published research articles on mathematics education for gifted pupils (e.g. Grant & Booth, 2009; Wu et al., 2012). However, even though a complete investigation of a particular research field may not be possible, it is suggested that data collection is performed by searching electronic databases for research articles. Moreover, drawing on the observation that several factors – for example, indexing of articles or algorithms used by search engines – may affect the extent to which relevant papers are retrieved and thereby also the results of a systematic review, it is suggested that database-searching is performed with particular attention for search strategies and search terms (e.g. Wu et al., 2012; Ziegler & Raul, 2000). In addition, it is recommended that the author of a systematic review relies on his or her experience of both theory and practice from the research field when examining published literature, unpublished material or other complementary sources (e.g. Grant & Booth, 2009). However, for this author, being relatively new to the field of mathematics education for gifted pupils, the inclusion of additional material, was restricted to some frequently referred volumes (e.g. Johnsen & Kendrick, 2005; Sriraman & Lee, 2011) during the spring of 2014, when the data was collected.

Thus, drawing on the relatively limited synthesis performed in the research field (e.g. Hoeflinger, 1998; Leikin, 2010: Lerman, 2014; Lester & Schroeder, 1983; Rogers, 2007; Sowell, 1993; Sriraman & Lee, 2011), I decided to use the electronic search engine provided by Stockholm University, and delivered by EBSCO Discovery Services (EDS), a leading resource for scholarly research, which performs searches in around 16 900 academic journals, periodicals, reports and books. In my attempt to collect appropriate articles, but also to reduce the risk of missing relevant papers which may not be retrieved from research databases, the following methods for data collection were applied.

- By using EDS, I searched for articles about mathematics education for gifted pupils in six English-language journals within the research on education of gifted pupils. The journals were selected based on review articles within the field of gifted education (e.g. Ziegler & Raul, 2000). The journals under consideration were *Gifted Child Quarterly*, *Gifted Education International*, *High Ability Studies*, *Journal for the Education of the Gifted*, *Roeper Review* and *The Journal of Secondary Gifted Education*. To these journals, the following search terms were applied: “mathematics”, “mathematics education”, “math education” and “mathematical problem solving”. The data collection was restricted to articles published between 1953 and 2014 and the retrieved results were sorted according to the criterion “Relevance” in EDS.

- By using EDS, I examined the whole database of academic journals for *peer reviewed* articles published between 1966 and 2014 about mathematically gifted pupils. This examination was performed by using the search terms “mathematically gifted” and “mathematically talented” and the retrieved results were sorted according to the criterion “Relevance” in EDS.
- I have also examined three frequently referred anthologies on the education on mathematically gifted pupils, published the decade prior to 2014, namely the volumes edited by Johnsen and Kendrick (2005), by Sriraman and Lee (2011) and by Lerman (2014).

In order to keep the data collection and the subsequent analysis on a reasonable level, and given that I was going to perform the systematic review alone and during a restricted amount of time, I decided to select the first 80 retrieved articles of every search to the further process of analysis. The analysis of the selected articles is described in detail in the corresponding sections.

Methodology associated with the second and third phases of data collection

The second and third phases of data collection resulted in three papers, namely, in Paper II, Paper III and Paper IV. The methodology associated with the second and third phases is predominantly based on research questions RQ2, RQ3 and RQ4, which are described in the previous section.

As indicated in the theoretical background, an analysis of a given problem, regardless of the mathematical domain it belongs to, displays the structure of the mathematical reasoning needed for its solution (e.g. Halmos, 1980; Krutetskii, 1976; M. Leikin et al., 2014; R. Leikin et al., 2017; Pólya, 1966; Silver, 1994; Singer & Voica, 2017; Sriraman, 2003, 2004b; Tan & Sriraman, 2017). Further, it is also indicated that analysing gifted and high-achieving students’ problem-solving activities is an effective method for displaying the individual structure of their mathematical abilities (e.g. Krutetskii, 1976; M. Leikin et al., 2014; R. Leikin et al., 2017; Öystein, 2011; Pitta-Pantazi, 2017; Singer & Voica, 2017; Tan & Sriraman, 2017).

The empirical studies included in this thesis were framed by some complementary methods. Accordingly, I observed six high-achieving students in the context of non-routine mathematical problems and conducted reflective interviews with them after every completed observation.

The applied methods were predominantly qualitative, with assumptions and empirical data as basic components. By noticing that applied methods may influence the outcome of qualitative studies, in a meta-analysis of qualitative attempts from different research fields, Larsson (2005) emphasizes the importance of an internal harmony between the research questions, the data collection and the methods of analysis. Further, Howe and Eisenhart (1990) point out that the internal harmony should determine the study in a hierarchic manner; that is, methods for observation and data collection should be determined by the research questions, instead of framing them in the reverse order.

Thus, in this section I will present those assumptions which concluded in the methods for observation and data collection associated with RQ2, RQ3 and RQ4.

Case studies

Inspired by the indication that the analysis of a given problem displays the structure of the mathematical reasoning required for its solution (e.g. Halmos, 1980; Krutetskii, 1976; M. Leikin et al., 2014; R. Leikin et al., 2017; Pólya, 1966; Silver, 1994; Singer & Voica, 2017; Sriraman, 2003, 2004b; Tan & Sriraman, 2017) and encouraged by studies that use problem-solving activities in order to display students' mathematical abilities (e.g. Krutetskii, 1976; M. Leikin et al., 2014; R. Leikin et al., 2017; Öystein, 2011; Pitta-Pantazi, 2017; Singer & Voica, 2017; Tan & Sriraman, 2017), I decided to observe students during mathematical problem solving. In order to maintain a suitable level of trustworthiness during observations and data collection, all participants solved the same problems. A more detailed description of those criteria that lead to the selection of participants and mathematical problems is presented in the upcoming sections.

Studies indicate that classroom interaction may affect pupils' thought processes and that this interaction does not have to be restricted to oral communication, also gestures, pictures or other physical modalities may have impact on pupils' thinking (e.g. Kress & van Leeuwen, 2001; Norris, 2002; van Leeuwen & Jewitt, 2001). Besides, Krutetskii (1976) argues that a correct delimitation and analysis of mathematical abilities might not have been possible during his studies if pupils have been observed in classroom situations, because the interaction with their peers would have affected their thinking and behaviour.

Consequently, in order to minimize the influence of disturbing factors and the interaction among participants, the participants in these studies were observed individually. That is, I conducted *case studies* (e.g. Bassey, 1999; Bennett, 2015; George & Bennett, 2005; Gerring, 2004; Orum, 2015; Yin, 1981, 2015) in order to uncover the structure of the participants' mathematical abilities and the role of mathematical memory during problem solving. Moreover, the case studies carried out during the second and third phases of data collection were designed because they were "expected in some way to be typical of something more general" (Bassey, 1999, p. 62).

Thus, after introducing it, the notion of case study might deserve some additional attention in the methodological framework of this thesis.

According to Yin (1981, 2015) a case study is an "empirical inquiry that closely examines a contemporary phenomenon (the case) within its *real-world context*" (Yin, 2015, p. 194), while Orum (2015) defines a case study as "an in-depth study of a single phenomenon whose boundaries and content can be made conceptually and empirically clear" (Orum, 2015, p. 202). However, by adopting a somehow wider perspective Woodside (2010) and Bennett (2015) consider that also investigations of historical events may be viewed as case studies.

Beyond the above description, there are some additional characteristics that contribute to the categorization of case studies. Yin (2015) shows that case studies may involve multiple sources of evidence about a case, for example, direct observations, extensive interviews and, if it is motivated, the analysis of relevant documents. Furthermore, he suggests that a *quantitative sub-analysis* of different aspects of a given case study might be carried out and that *multiple cases* should be used in order to compare or contrast a given theoretical proposition (Yin, 2015). Similarly, Bassey (1999) indicates that a *collective case study* – involving several cases – should be carried out if the purpose of the researcher is to reach a deeper understanding of a more general phenomenon.

In an attempt to differentiate those cases that are selected for observations, Orum (2015) proposes three types of cases: *the typical case*, a case that is believed to be highly representative according to certain criteria and thereby offering the possibility of results that might be generalizable for a larger population; *the prototypical case*, where a prototype represents a non-average case, that have the presumed potential to predict how a particular phenomenon may develop in the future; and, the *deviant case*, that represents a particular case which, with its particular qualities, has the potential to lead to assumptions that may be beneficial for a larger population (Orum, 2015, pp. 202–204).

Further, it is also highlighted that "accounting for the context creates an extra burden on a case study" (Yin, 2015, p. 195), that is, in the design of a case study, it may be difficult to maintain clear restrains or distinct boundaries between the studied case and its context. And already in his early work, Yin (1981) recommends that – in order to avoid findings that emerge in lengthy narratives without a distinct structure – to build case studies on clear conceptual frameworks.

In addition, when discussing the epistemological orientation of a case study, Yin (2015) suggests that a *qualitative case study* should take into consideration the difference between the *relativist* and the *realist* orientation when framing the findings of the study, which, according to the study's orientation, may be interpreted as *observer dependent* or *observer independent*. Moreover, according the mentioned epistemological standpoints associated with case studies, Yin (2009) differentiates case studies for being *exploratory*, *explanatory* or *descriptive*, respectively. In a different attempt of categorization, Bassey (1999) delimits several categories of cases studies, such as *theory-seeking*, *theory-testing*, *story-telling*, *picture-drawing*, and *evaluative* case studies. Additionally, important for this thesis, Bassey notes that theory-testing case studies can be viewed as basically instrumental (Stake, 1995) and explanatory (Yin, 2009) while theory-seeking case studies may be regarded as instrumental and exploratory (Yin, 2009).

In conclusion, based on the above presented theoretical background associated with the concept, the case studies referred in this thesis will be further categorised after presenting the methods for observation and data collection related to the second and third phases of data collection.

Framing the observation method

In order to investigate the interaction of mathematical abilities and the role of mathematical memory, I tried to find a framework, which in a reliable manner delimits and explains mathematical ability and mathematical memory. Beside describing essential components and indicating the complex dynamics between those components, Krutetskii's (1976) framework defines mathematical memory as a memory for generalized mathematical relationships, type characteristics, schemes of arguments and proofs and problem-solving methods. The mentioned definition of mathematical memory is validated by psychological studies on understanding (e.g. Gagné, 1970; Skemp, 1976) and by relatively recent educational studies on mathematical abilities (e.g. Vilkomir & O´Donoghue, 2009). In a somehow different vein, cognitive theories on memory functions locate mathematical memory within the explicit memory system by characterizing it as information which may be consciously recalled and explained (e.g. Davis et al., 2000; Nyberg & Bäckman, 2009; Sternberg & Sternberg, 2012).

In addition, studies with the aim to delimit mathematical abilities, indicate fundamental similarities with Krutetskii's framework (e.g. Juter & Sriraman, 2011; Sheffield, 2003) and, importantly, Krutetskii's framework has a substantial acceptance in the research field and is frequently applied to the investigation or description of pupils' mathematical abilities (e.g. Deal & Wismer, 2010; Garofalo, 1993; Heinze, 2005; Leikin, 2010; Sriraman, 2003; Vilkomir & O´Donoghue, 2009).

Accordingly, it is reasonable to assume that Krutetskii's (1976) framework, in combination with introduced cognitive and neuroscientific models for memory functions (e.g. Davis et al., 2000; Nyberg & Bäckman, 2009; Sternberg & Sternberg, 2012), represents a solid ground for observing and understanding mathematical abilities and mathematical memory in the context of mathematical problem solving.

Further, drawing on the observation that mathematical ability is a direct outcome of the learning the subject (Krutetskii, 1976, pp. 67–70), the literature review indicates that high-achieving pupils – in contrast to their low-achieving peers – are able to express appropriate mathematical abilities (e.g. Krutetskii, 1976; Leikin, 2014; Pitta-Pantazi, 2017).

When concerning mathematical memory, the theoretical background shows that gifted students and high-achievers are able to recall problem-solving methods several months after solving a problem (Krutetskii, 1976, pp. 295–301). Additionally, it is also indicated that young pupils do not possess well-developed mechanisms – for discerning between explicit and implicit memories – in order to differentiate facts and numbers from generalized mathematical relationships, and, that mathematical memory is difficult to observe in young pupils and in low-achievers (e.g. Krutetskii, 1976; Raghubar et al., 2010; Murphy et al., 2003). Thus, it seems reasonable to assume that high-achievers from upper secondary school, are able to express appropriate mathematical abilities and structures of mathematical memory that are observable during problem solving (Krutetskii, 1976).

Additionally, the literature review exposes at least three uncertainties connected to the concept of giftedness. Firstly, that not only is there a lack of a generally agreed definition of giftedness in the research field, but also that, regardless of applied definitions, IQ remains an important indicator of giftedness (Carman, 2013; Stoeger, 2009). Secondly, that identifying mathematically gifted pupils is an intricate and demanding process (e.g. Birch, 1984; Coleman, 2003; M. Leikin et al., 2013; R. Leikin et al., 2017; Lupkowski-Shoplik et al., 1994; Pitta-Pantazi, 2017; Rotiger & Fello, 2005). Thirdly, that the identification of gifted pupils and the use of IQ tests has low acceptance in the Swedish school system (e.g. Dodillet, 2017; Persson, 2010).

Thus, it should be emphasized that – based on the identified divergence concerning frameworks for the identification of giftedness, in combination with the fact that identifying gifted pupils was a controversial topic in the Swedish educational system when the data collection was carried out – the identification of gifted pupils is not the purpose of the analytical framework related to the second and third phases of data collection. The mentioned analytical framework is presented in the analysis section of this thesis.

From another perspective, the literature review indicates that mathematical abilities are essentially developed and refined during mathematical activities (e.g. Krutetskii, 1976; Leikin, 2014; Pitta-Pantazi, 2017). Further, by drawing on basic distinctions between routine and non-routine problems (e.g. Blum & Niss, 1991; Carlson & Bloom, 2005; Garofalo, 1993; Lester, 1994; Pólya, 1966; Schoenfeld, 1985), it is recommended that non-routine problems are applied in order to observe and investigate students' mathematical abilities (e.g. Krutetskii, 1976; M. Leikin et al., 2014; R. Leikin et al., 2017; Öystein, 2011; Singer & Voica, 2017; Tan & Sriraman, 2017). And even though it seems that previous experiences and knowledge may not be entirely eliminated when investigating mathematical abilities, Krutetskii emphasizes that:

> Apparently one should select problems in such way that only ability will primarily influence their solutions. It is clearly impossible to completely eliminate the influence of past experience, knowledge and habits. (Krutetskii, 1976, p. 94)

Thus, even if it is impossible to completely discern mathematical abilities from the subject knowledge, it seems rational that solving non-routine problems is an appropriate method for observing and investigating mathematical abilities in gifted or high-achieving students. And given the low acceptance of identification and development of giftedness in the Swedish educational system (e.g. Dodillet, 2017; Persson, 2010), selecting students who, based on their performances in school mathematics, are high-achievers, seems an appropriate way to select participants for the second and third phases of data collection.

In conclusion, after taking all above assumptions and indications in consideration, I decided to observe high-achieving Swedish upper secondary school students in the context of non-routine mathematical problems. Further, in order to understand the participants' actions and to examine the role of mathematical memory, I decided to use an analytical framework based on Krutetskii's (1976) model of mathematical ability.

Framing the data collection

Studies observing students' mathematical problem solving indicate that the data collection should not be restricted to written solutions; that is, students should be offered opportunities to complement their written solutions by communicating their thoughts and mathematical reasoning during the problem-solving process (e.g. Krutetskii, 1976; Lupkowski-Shoplik et al., 1994; Öystein, 2011; Schoenfeld, 1992; Singer & Voica, 2017). Consequently, beside written solutions, participants were offered complementary opportunities to express their mathematical reasoning associated with the problem-solving process.

However, because students are not familiar with solving problems through oral presentations (e.g. Bentley 2003; Krutetskii, 1976), they often perceive that oral presentations should include explanations about how the problem was solved, or, that they should describe their mental processes associated with the given problem (Krutetskii, 1976). Thus, in order to facilitate for participants to communicate their thoughts during problem solving, they have encouraged not only to write down every step of their solutions, but also to "think out aloud" when solving the problems. Moreover, when a student did not write or said anything for a while, drawn on Krutetskii's (1976) observations, some of the following questions were posed:

- What is bothering you?
- Why do you do that?
- What do you want to do and why?
- What are you thinking about?

As mentioned, the case studies referred in this thesis were carried out individually. In order to make the participants feel more familiar during the studies, every observation took place in a private room at their school and participants were given as much time as they needed to solve each task. And, to prevent the negative effects of result-related stress, they were informed that the studies would have no impact on their marks in school mathematics. In order to avoid as far as possible that the participants will discuss about the proposed problems, all observations were carried out during single days and the participants were asked to not talk with each other about the proposed problems.

According to the literature review, the cognitive paradigm displays significant qualitative differences between information and knowledge connected to different parts of the human memory system (e.g. Moscovitch, 1992; Olson et al., 2009; Sternberg & Sternberg, 2012; Squire, 2004). Consequently, from a mathematical perspective, there are qualitative differences between the use of algorithms and formulas – that is, knowledge retrieved from implicit memory to working memory by automaticity and thereby representing knowledge which cannot be explained by students when not engaged in the mentioned tasks (e.g. Davis et al., 2000; Ingvar, 2009; Nyberg & Bäckman, 2009) – and mathematical memory, consisting of methods for problem solving and mathematical relationships which are located in the explicit memory (e.g. Davis et al., 2000; Nyberg & Bäckman, 2009). In this respect, mathematical memory is constituted of generalized knowledge that might be consciously recalled and described by gifted students in a problem-solving context (e.g. Davis et al., 2000; Krutetskii, 1976; Nyberg & Bäckman, 2009).

Students are not used to communicate and verbalize their actions during problem solving; accordingly, to minimize the risk that some of their actions during problem solving would remain unexplained, it is recommended that *reflective interviews* are carried out directly after completed problem-solving activities (e.g. Ginsburg, 1981; Krutetskii, 1976; Lupkowski-Shoplik et al., 1994; Nogueira de Lima & Tall, 2008; Öystein, 2011; Singer & Voica, 2017). Thus, after completing the respective problem-solving activity, each student participated in a reflective interview. The purpose of these interviews was not only to discover and specify those cognitive processes which had not been observed during problem solving but also to evaluate the individual competence-level of each participant in the displayed actions (Ginsburg, 1981).

In addition, it has been indicated that the retrieval of implicit and automated processes to working memory is performed at very high speeds (e.g. Buckner & Wheeler, 2001; Nyberg et al., 2003; Ingvar, 2009) and, that it can be problematic to discern semantic memories – for example, elements of mathematic memory – from episodic memories (e.g. Davis et al., 2000; Nyberg & Bäckman, 2009; Sternberg & Sternberg, 2012). Also, Krutetskii's team reported several difficulties when trying to differentiate the ability to obtain mathematical information from mathematical memory, in situations that pupils were quiet, without writing or drawing anything (Krutetskii, 1976, pp. 350–353). Krutetskii states this dilemma as it follows:

> ... the study of problem-solving is greatly complicated because the process is not always expressed objectively enough; many links in the mental process of solving a problem escape the investigator. (Krutetskii, 1976, p. 92)

Several studies emphasize the above mentioned difficulties (e.g. Bloom & Sosniak, 1985; Öystein, 2011; Singer & Voica, 2017; Sriraman, 2003) when reporting on students' written solutions or about group discussions and contextual interviews performed with students. Thus, it seems that within the field of qualitative research there is a lack of reliable methods for an accurate display of mental processes associated with mathematics (e.g. Leikin, 2014; M. Leikin et al., 2013; Singer & Voica, 2017).

Accordingly, in order to increase the accuracy of the observations and assuming that during the observations it will not be possible to completely uncover participants' mental processes, multiple technologies for data collection were applied. Every observation and reflective interview was recorded with the use of a technology that enables a digital recording of writings, drawings and other actions connected to verbal utterances. That is, the participants used a digital pen which converted every problem-solving activity to files, which may be described as cartoons exposing the linear occurrence of every drawing, writing and verbal utterance. In addition, for safety reasons, every problem-solving activity and subsequent reflective interview was recorded with an audio recorder.

Consequently, the participants were observed individually and were given as much time as they needed, when solving mathematical problems. During problem solving they were encouraged to "think out loud", to write down every step of the process and if they did not write or say anything for a while, structured questions were posed in order to articulate their mathematical thinking. Every problem-solving activity was followed by a reflective interview, which offered additional opportunities for participants to explain their actions. Observations and reflective interviews were carried out at single days, during the second and third phases of data collection. In addition, the participants were asked not to discuss with each other about the problems they solved.

In conclusion, the case studies associated with the second and third phases of data collection may be categorized in the light of the presented methodological considerations. In that respect, the fact that a limited number of participants were examined individually – by using both direct observations and reflective interviews – in the context of solving the same mathematical problems, might be characterized as a study of multiple cases, or as a collective case study, which "closely examines a contemporary phenomenon (the case) within its *real-world context*" (Yin, 2015, p. 194). The circumstance that the interaction of mathematical abilities and the role of mathematical memory were examined by adapting a generally agreed framework for mathematical abilities (Krutetskii, 1976) would categorize these studies as theory-testing (Bassey, 1999), instrumental (Stake, 1995) and explanatory (Yin, 2009) case studies (Andrews, 2016). The particular framework – drawn on Krutetskii's (1976) work – applied for the analysis of participants' mathematical activities is described in detail in the analysis section of this thesis. Finally, the selection of participants from a group of extremely high-achieving students who felt motivated to participate in these studies, leads to the conclusion that the examined cases might be characterized as *deviant cases* (Orum, 2015), that, based on participants' excellent school performances, have the potential to generate assumptions about mathematical problem solving that might be beneficial for a larger group of students.

Participants

As indicated, an accurate exhibition of higher order components of the complex mathematical ability is difficult to observe in young children and low-achievers. However, high-achieving students from secondary school should be able to display appropriate mathematical abilities and, importantly, to use their mathematical memory in intelligible ways when solving problems (e.g. Krutetskii, 1976). Consequently, the participants were selected from a class of 16-17-year-old students from a Swedish upper secondary school, attending an advanced mathematics programme. All students from the class were top performers in an acceleration programme in compulsory school, and, importantly, during their last year of compulsory school (in ninth grade) they have completed the first mathematics course of upper secondary school with the highest result. Thereby these students were mathematically situated within the top 5 percent of students nationwide.

Prior to the empirical study reported in Paper II, in order to get familiarised with the students and to offer them opportunities to accept me as a mathematical colleague rather than as a teacher or an examiner, I spent 30 hours, over a period of four months, as a participant observer in their mathematics classroom. During this period, we discussed mathematics and school in general and some mathematical theories and problem solving in particular. Interestingly, in the beginning of this period some students tested my mathematical knowledge and posed intricate questions about problem solving. However, after some weeks, when feeling more confident with my mathematical skills and after realizing that I was one of the authors of their textbooks, they accepted me as a mathematical peer and, importantly, declared that they felt comfortable when solving problems with me.

Participation in the studies was optional and, when asked, 10-12 students wished to participate. However, because of the relatively complex conditions of the study, after consulting their teacher, six students, three boys and three girls, were invited to participate. In this context, it should be mentioned that gender parity was not in the focus for the second and third phases of data collection.

In addition, in order to maintain the established cordial relation with the students, I continued to interact with them during their mathematics classes also throughout the year between the second and the third phase of data collection.

Tasks

Given the different objectives of the second and third phases of data collection, observations were carried out during two single days, approximately a year apart. At each occasion, the participants solved two different mathematical problems that were framed by the conditions of respective studies.

As mentioned in the section referring to the framing of data collection, the intention of the studies carried out during the second and third phases of data collection was not to identify mathematical giftedness or high achievement in the participants. The aim of the studies was to observe the structure of mathematical abilities and the role of the mathematical memory in the context of non-routine mathematical problems.

Thus, in order to avoid, as far as possible, the influence of previous problem-solving experiences and subject knowledge (e.g. Blum & Niss, 1991; Carlson & Bloom, 2005; Garofalo, 1993; Lester, 1994; Pólya, 1966; Schoenfeld, 1985) and to minimize the impact of analogical reasoning (e.g. Antonietti, Ignazi, & Perego, 2000; Degrande et al., 2016; Nogueira de Lima & Tall, 2008), the main criteria were to select problems that were perceived as non-routine and accessible by the participants. In addition, in order to offer the participants opportunities to express a broader variety of mathematical abilities (Hiebert et al., 2003; Juter & Sriraman, 2011; Krutetskii, 1976; Roberts & Tayeh, 2007), the selected problems belonged to two mathematical areas. Furthermore, both problems could be solved with algebraic methods or by particular approaches.

However, when selecting the problems, I faced a considerable dilemma. Being aware of that a mathematical task can be considered a non-routine problem based on the novelty and the challenge it represents to the individual who is solving it (e.g. Carlson & Bloom, 2005; Garofalo, 1993; Lester, 1994; Pólya, 1966; Schoenfeld, 1985), I could not be certain that the selected tasks would be perceived as non-routine by each participant. Thus, the selection of problems was guided by some sensitive assumptions that were grounded in the following understandings. Firstly, the participants were studying from textbooks (Szabo, Larson, Viklund, & Marklund, 2009a) that were authored by my colleagues and me; that is, I was familiar with the tasks that they have been facing during their mathematics education. Secondly, I have participated in their mathematics classes and discussed with the participants during a period of four months, and thirdly, I consulted their mathematics teacher about the proposed problems.

Thus, based on the above described assumptions, Problem 1 (P1) and Problem 2 (P2), were selected for the second phase of data collection.

Problem 1: In a semicircle, as shown in the figure, are drawn two additional semicircles. Is the perimeter of the large semicircle longer, shorter or equal to the sum of the perimeters of the two smaller semicircles? Justify your answer.

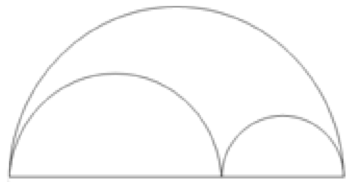


Figure 3. The figure associated with Problem 1.

Problem 2: Mary and Peter want to buy a CD. At the store they realise that Mary has 24 SEK less and Peter has 2 SEK less than the price of the CD. Even when they put their money together they would not be able to afford the CD. What is the cost of the CD and how much money has each person?

The aim of the empirical study associated with the third phase of data collection, was to observe the role of mathematical memory and students' individual problem-solving abilities at occasions occurring over a relatively large period of time. As mentioned, gifted and high-achieving students typically do not recall the contextual information of a problem after the problem is solved, but – as an impact of their mathematical memory – they remember several months later the methods applied to solve the problem (Krutetskii, 1976). Thus, in an attempt to investigate expressions of mathematical memory and the structure of mathematical abilities in high-achieving students over an extended period of time, the study reported in Paper IV was carried out approximately a year after the first observation. In addition, in order to avoid that the participants will recall the problems proposed at the first occasion (P1 and P2) mainly because of recalling the circumstances for the first observation as an unusual element in their activities – that is, not because of the retention of structure or the methods used at P1 – I continued to participate as a mathematical friend in their classroom during the year between the two occasions.

Accordingly, in order to highlight the nature of mathematical memory, a main criterion was to select a task, which was non-routine, but could be solved by methods similar to those applied at P1. Thus, based on assumptions similar to those associated with the selection of problems for the second phase of data collection – that is, after continuing to interact with the participants and observing their learning in the subject, analysing their textbooks (Szabo, Larson, Viklund, & Marklund, 2009b) authored by my colleagues and me, and consulting their mathematics teacher – Problem 3 (P3) was selected for the third phase of data collection.

Problem 3: In a square we draw two arbitrary contiguous squares, according to the figure. Is the perimeter of the large square longer, shorter or equal to the sum of the perimeters of the two smaller squares? Answer the question without measuring the figure. Justify your answer.

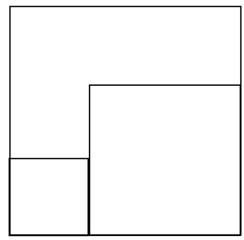


Figure 4. The figure associated with Problem 3.

As seen above, P3 is also a geometrical problem (similar to P1) which can be solved with both algebraic methods and particular approaches.

Additionally, another task, Problem 4 (P4), was proposed during the third phase of data collection. The aim of P4 was to uncover the structure of the participants' mathematical abilities during problem solving over an extended period of time. Accordingly, when selecting it, the main criterion was to propose a non-routine problem, which required methods that were not similar to those used to solve the previous problems (P1 and P2). Besides, also P4 was possible to approach with both particular and algebraic methods. Thus, at the second observation, that occurred approximately a year after the first one, the following problem was proposed.

Problem 4: A dinner was attended by the same number of women and men. A bowl of rice was served for every two people, every bowl of soup was served to every three people and every bowl of meat to every four people. The guests used 52 bowls in total. How many men and how many women, respectively, were at the dinner?

However, in the context it is important to mention, that participants' activities associated with Problem 4 did not constitute a part of the studies included in this thesis – therefore, the data collection associated with P4 will not be discussed to a further extent in this thesis.

The test of selected tasks

Prior to the different phases of data collection, the selected tasks underwent an a-priori testing with a corresponding group of high-achieving students at a different Swedish upper secondary school.

The testing prior to the second phase of data collection confirmed that both tasks could be categorized as well-suited for the mathematical knowledge of the participants and, importantly, as non-routine problems for

16-17-year-old high-achievers in Swedish upper secondary school. Further, it was confirmed that students in the corresponding group applied both particular and general approaches when solving P1 and P2. Moreover, the solutions to the circles problem (P1) indicated that particular approaches could be developed into general solutions during problem solving.

A year later, the testing prior to the third phase of data collection, beside demonstrating that P3 was matching the mathematical knowledge of participants, confirmed that the students solved P3 with methods similar to those applied at P1. That is, regardless of applying particular or algebraic methods, the students solved both problems by using formulae for perimeters of circles and squares. In that way, the a-priori testing of the three proposed tasks (P1, P2 and P3) demonstrated their suitability for the studies included in this thesis.

Ethical considerations

The display of appropriate and well-grounded ethical aspects is considered to be a foundation stone in scientific research. An accurate investigation of a focused topic and the presentation of new and relevant results may be considered pivotal from a researcher point of view, but those aspects should be carefully balanced against the requirement to protect those individuals who participate in the research (Larsson, 2005).

The ethical considerations associated with the second and third phases of data collection, described below, are in concordance with the guidelines from the Swedish Research Council [Vetenskapsrådets forskningsetiska principer] (2009).

Prior to the involvement in the study, every participant was informed orally and received printed information about the upcoming study. In addition, during the oral information, the students were able to pose questions according to their concerns associated with the study. After receiving letters of consent signed by their legal guardians, the students were informed that they will be able participate in the study.

Neither the school that the students attend nor the region in which the school is located are mentioned in any part of this thesis. Moreover, in my oral communication with colleagues and friends, I avoided to mention either participants, or their school, by their real names. In transcripts, which are written in Swedish, the participants are referred as X1, X2, X3, X4, X5 and X6, respectively; however, in the English language papers, the participants are mentioned by English pseudonyms. Moreover, the participants' mathematics teacher, who has been partially involved in this thesis, is not named in the transcriptions.

Another ethical issue concerns the transcriptions of participants' verbal utterances. During problem solving and reflective interviews, the participants expressed themselves in ways which may be described as careless, that is, the recordings show that they often said "ba" instead of "bara" or "asså" instead of the correct form “alltså” – which in English may be interpreted as the use of “combo” in informal language instead of “combination”. However, because the use of the mentioned words had no connections to participants’ mathematical skills or abilities, in my ambition to make the transcribed utterances more understandable, I decided to transcribe such incorrectly pronounced words in their correct grammatical form. Yet, apart from these remarks, in order to not miss any utterances during problem solving, all recorded material was transcribed verbatim. The analyses of the studies were carried out in private places or in environments where individuals present could identify neither the participants nor the circumstances of the empirical studies. The digital pens and booklets, just as collected data and all transcriptions, were stored in safes both at my workplace and at my home.

Finally, it should also be mentioned, that participants were informed repeatedly that they had the right to abandon the studies at any time and without further motivation. However, in order to express my appreciation for their in contribution the studies – which occurred at participants’ spare time – each participant received a cinema ticket to an optional film after each completed observation; that is, every participant received two tickets, one year apart. The tickets were purchased from my own money, and the participants were not informed prior to the observations that they will receive any form of compensation.

Analysis

As shown in the previous section, the methodology used in this thesis, according to the nature of the investigated research questions, took two forms. Thus, the methodology applied in the empirical studies on problem-solving abilities of high-achieving students – that is, individual observations, reflective interviews and digital recordings – had a different focus compared to the methodology applied in the systematic review of the research literature, which was supported by methods for retrieving research papers.

Besides, in the context it should also be noted that the methods of data collection may influence the results of a given study. Therefore, it is important to maintain an internal harmony between research questions, data collection and methods of analysis – moreover, in order to establish an optimal harmony within a research project, the research questions should determine the methods for data collection and thereby also the methods of analysis (Howe & Eisenhart, 1990; Larsson, 2005).

Accordingly, the different natures of the research questions and corresponding methodologies lead to two different methods of analysis. Thus, I will first present the analysis connected to the first phase of data collection and next, I will describe the analysis related to the second and third phases.

Analysis associated with the first phase of data collection

Principally drawing on the results of the search algorithm of EDS, in combination with the examination of mentioned anthologies (Johnsen & Kendrick, 2005; Sriraman & Lee, 2011), the data collection was carried out according to the methodological considerations described in the previous section. After this, in order to identify those educational practices recommended to promote gifted pupils' development in the mathematics classroom and to determine which practices are supported by empirical evidence, the following phases of analysis were performed.

The search for articles on mathematics education for gifted pupils in the six journals under consideration, that is, in Gifted Child Quarterly, Gifted Education International, High Ability Studies, Journal for the Education of the Gifted, Roeper Review and The Journal of Secondary Gifted Education – by using the terms "mathematics", "mathematics education", "math education" and "mathematical problem solving" – resulted in 3 037 articles distributed as seen in Table 1. The search terms are noted as it follows, mathematics = S1, mathematics education = S2, math education = S3, mathematical problem solving = S4.

Journal	S1	S2	S3	S4
Gifted Child Quarterly	760	62	4	27
Gifted Education International	468	23	6	24
High Ability Studies	193	4	1	6
Journal for the Education of the Gifted	372	21	2	16
Roeper Review	706	41	7	18
The Journal of Secondary Gifted Education	241	23	3	9

Table 1. Retrieved articles from respective peer reviewed journals.

The search algorithm of EDS, for peer reviewed articles published between 1966 and 2014, returned 776 results for the terms "mathematically gifted" and 352 results for "mathematically talented". Additionally, the examination of the anthologies on the education on mathematically gifted pupils, edited by Johnsen and Kendrick (2005), by Lerman (2014) and by Sriraman and Lee (2011) resulted in a further 28 papers. The process of analysis contained three phases, as described below.

During the first phase, as mentioned, up to the first 80 returned papers of every search result were selected for further reading. For example, I selected the first 80 retrieved articles from Gifted Child Quarterly among 760 in the category S1, but all 62 articles in category S2 (Table 1). This way of selecting papers resulted in 937 papers, which together with articles selected from other sources (mentioned anthologies) gave a total sum of 965 papers. During this phase, I read all 965 abstracts in order to find papers focusing educational practices intended to develop gifted pupils' in school mathematics. After reading the abstracts, editorial texts, book reviews and duplicate papers were excluded from the analysis. Further, articles associated with the following categories were also excluded: the importance of parents and the impact of parenting styles, gifted pupils with learning disabilities, home schooling and extracurricular programs, historical perspectives on gifted pupils, gifted pupils belonging to subgroups, editorial board papers, interviews with researchers, curricular considerations for gifted pupils, gifted children in preschool, gifted adults, students at tertiary level, programs directed towards teachers or pre-service teachers and papers about pupils who are gifted in other subjects than mathematics. After excluding the mentioned articles, the number of articles for analysis was reduced to 177.

During the second phase of the analysis, inspired by Graneheim and Lundman (2004) and van Leeuwen (2005), I performed a *qualitative content analysis* of the selected papers. This type of analysis may be performed with different foci, according to the intentions of the given study; that is, the analysis depends on whether the ambitions of the researcher are data-driven or theory-driven (Kvale & Brinkmann, 2009). The systematic review was not located in a specific framework and its aim was not to validate or extend a theoretical framework (Hsieh & Shannon, 2005). Indeed, the aim of the first phase of data collection was to identify pedagogical and organizational methods in the selected literature. Accordingly, the content analysis was data-driven by my ambition to identify, code and categorize the basic patterns in the selected articles. Thus, during this phase of the analysis, the articles were read thoroughly in order to scrutinize their content for those broad categories in which the articles could be divided into, with respect to gifted pupils' development in mathematics, as well as to pedagogical and organizational practices linked to their mathematical development. After an average of three attentive readings of each selected article, the content analysis indicated the following *broad and relatively distinct contextual categories*:

- gifted pupils' *performances* in school mathematics,
- gifted pupils' *social situation* in school and *gender differences* between those pupils,
- *teachers' perceptions* on mathematically gifted pupils,
- *definition* and *identification* of mathematical giftedness, as well as national programs for mathematically gifted pupils, and
- *motivational* and *cognitive characteristics* of gifted pupils.

Prior to the next phase of the qualitative content analysis, by noticing that the identification and definition of mathematical giftedness are not in the focus for the research questions, I decided to exclude the articles within these categories. This step reduced the number of analysed papers to 135.

During the third phase of the analysis, by delimiting common pedagogical and organizational patterns in the content of the papers, it became obvious that the above broad categories were not mutually exclusive with respect to pedagogical and organizational topics discussed in corresponding papers. That is, a substantial number of articles associated with gifted pupils' performances in mathematics, with their social situation in school and gender differences between gifted pupils, with teachers' perceptions on mathematically gifted pupils and with motivational and cognitive characteristics of gifted pupils, could be associated with some more well-defined categories in the context of school mathematics. For example, many of the articles focusing *mathematical performances* of gifted pupils could be associated with categories referred to as pedagogical and organizational practices in the *heterogeneously grouped* mathematics classroom (e.g. Dimitriadis, 2012; Leikin, 2010, 2014; Reed, 2004; Rotiger & Fello, 2005; Tucker, 1982), *acceleration programs* for gifted pupils (e.g. Rogers, 2007; Sowell, 1993; Threlfall & Hargreaves, 2008), *ability grouping* of gifted pupils (e.g. Dimitriadis, 2012; George, 1976; Preckel, Götz, & Frenzel, 2010; Rogers, 2007; Sowell, 1993), and gifted pupils' *attitudes* towards *different learning environments* (e.g. Hunt, 1996; Li & Adamson, 1992; Preckel et al., 2010; Robinson, 1990; Sowell, 1993). Moreover, in similar ways, also the articles belonging to the remaining broad and relatively distinct categories could be associated with more accurate pedagogical and organizational practices. For example, some of the papers discussing *gender differences* could be associated with pedagogical and organizational methods focusing *heterogeneously grouped* classes (e.g. Hong & Aqui, 2004; Li & Adamson, 1992; Schober, Reimann, & Wagner, 2004), while other papers focusing gender differences, could be associated with the topic of gifted pupils' *attitudes* towards *different learning environments* (e.g. Freeman, 2004; Stutler, 2005).

Accordingly, during the last phase of the analysis, the analysed papers were structured according to the following pedagogical and organizational practices:

- gifted pupils' in the *heterogeneously grouped* mathematics classroom,
- *acceleration programs* for gifted pupils,
- *ability grouping* of gifted pupils,
- gifted pupils' *attitudes* towards *different learning environments*

Finally, and importantly, it should also be mentioned that according to editorial reasons, such as the limitation of articles published in the current journal, not all analysed papers were included in Paper I.

The reliability of the analysis

After performing the above described analysis, its reliability was tested. Reliability is a concept rooted in and used for evaluating quantitative research that is also frequently used in the qualitative paradigm (e.g. Golafshani, 2003).

With respect to reliability in qualitative studies, some researchers suggest that "the most important test of any qualitative study is its quality" (Golafshani, 2003, p. 601) or that "reliability has no relevance in qualitative research" (Stenbacka, 2001, p. 552), because if "a qualitative study is discussed with reliability as a criterion, the consequence is rather that the study is no good" (Stenbacka, 2001, p. 552). In a different vein, other researchers indicate that reliability in qualitative research should be ensured by examining it for its trustworthiness (Seale, 1999) or that concepts from quantitative research should be adapted to the conditions of qualitative research (Strauss & Corbin, 1990). Moreover, it is also emphasized that the reliability of a qualitative study should be tested by *triangulation* – that is, by combining multiple methods of data collection and data analysis – and that the applied methods should depend on the criteria of the respective study (e.g. Golafshani, 2003).

Thus, in order to test the reliability of the analysis, after performing the earlier data analysis on my own I cooperated with a senior professor from the same institution with whom I was associated at the time of the first phase of data collection. During this process, the senior professor searched for articles in the electronic database according to the same principles that I have applied. The verification demonstrated that the senior professor retrieved the same number of articles in every search category. After verifying the distributed quantities of the retrieved articles, she performed an additional test by analysing the content of ten randomly selected articles from arbitrary chosen search categories. After this additional test, we concluded that the senior professor and I came to the same conclusions with respect to identified patterns and delimited categories in the randomly selected articles. In conclusion, it seems that the data analysis associated with the first phase of data collection was performed at an acceptable level.

Analysis associated with the second and third phases of data collection

As mentioned, the analyses performed after the second and third phases of data collection had a different focus compared to the analysis connected to the first phase of data collection. In the empirical studies associated with the second and third phases of data collection, when analysing the participants' activities, a main concern was to develop a clear and structured approach that permitted a correct identification of the components of Krutetskii's (1976) model of mathematical ability and to map the interaction between the identified abilities. Consequently, a particular analytical framework was developed for the second and third phases of data collection.

The analytical framework

As mentioned, both psychological and cognitive studies (e.g. Davis, 1978; Davis et al., 2000; Gagné, 1970; Krutetskii, 1976; Nyberg & Bäckman, 2009; Skemp, 1976; Sternberg & Sternberg, 2012) indicate that understanding depends on the character of the knowledge; that is, there is a qualitative difference between the ability to comprehend a given mathematical area in order to develop generalized knowledge and problem-solving methods, characterized as *mathematical memory*, and the ability to apply algorithms or to remember numbers. Also, Krutetskii (1976) points out that:

> … mathematical abilities are abilities to use mathematical material to form generalized, curtailed, flexible, and reversible associations and systems of them. (Krutetskii, 1976, p. 352)

Thus, when framing the analytical framework, particular attention was directed towards a special component of the *ability to process* mathematical material, that is, the ability to generalize "mathematical objects, relationships, and operations" (Krutetskii, 1976, p. 341), frequently associated with mathematical memory:

> It is impossible to imagine any system of instruction where, say, the ability to generalize or mathematical memory would not be included in the structure of mathematical abilities. (Krutetskii, 1976, p. 351)

Correspondingly, the presence of *the ability to generalize* mathematical objects, relations and operations in the problem-solving process was included as a specific component in the analytical framework.

In another respect, the a-priori testing of the proposed problems on corresponding groups of students confirmed Krutetskii's observation that the ability described as *general synthetic component*, i.e. the "mathematical cast of mind", was not observable during problem solving. Moreover, Krutetskii also indicated that the general synthetic component of mathematical ability is typical for mathematically gifted students (Krutetskii, 1976, pp. 350–351). Thus, being aware of that the participants were high-achievers in mathematics, but not tested for mathematical giftedness, and in addition, noticing that the general synthetic component was not observable during the a-priori testing of problems, I decided not to examine the occurrence of the general synthetic component. Consequently, this particular ability was not included in the analytical framework.

Finally, the a-priori testing and analysis of the proposed problems indicated that four components of Krutetskii's complex mathematical ability could be identified during problem solving. These were:

(O) The ability to *orientate* oneself with respect to the problem

- For example, the student demonstrates an understanding of the structure of the problem, represents the problem in appropriate symbolic forms that highlight the relationship between the given entities and underlying variables. This definition resonates closely with Krutetskii's ability to *obtain and formalize* mathematical information.

(P) The ability to *process* mathematical information

- For example, the student uses an appropriate problem-solving strategy, the student performs well-known methods for problem-solving by using logical, systematic and sequential thinking. This reflects the regulative element of Krutetskii's ability to process mathematical information.

(G) The ability to *generalize* mathematical objects, relations and operations

- For example, the student understands or observes that the obtained particular solution may represent or prompt a general solution of the problem. This reflects the generalization element of Krutetskii's ability to process mathematical information.

(M) The ability to *retain and recall* mathematical information

- For example, the student recalls a method for calculating the area of a geometric shape or that a given task can be written as an equation or an inequality. This definition resonates closely with Krutetskii's *mathematical memory*.

Analysing the problem-solving activities of the participants

The digital recording of problem-solving activities resulted in exact linear reproductions of the participants written solutions, drawings and verbal utterances. This was especially useful when performing *qualitative content analysis* inspired by Graneheim and Lundman (2004) and van Leeuwen (2005). As mentioned, qualitative content analysis may be directed with different objectives – that is, it can be data-driven or theory-driven (Kvale & Brinkmann, 2009) – depending of the design and ambition of the actual study. Given that the analysis associated with the second and the third phases of data collection is largely built on a framework which was developed by Krutetskii (1976), the analysis was theory-driven in its implementation of literature-derived codes (Xenofontos & Andrews, 2012). A main purpose of theory-driven content analysis is typically to "validate or extend conceptually a theoretical framework or theory" (Hsieh & Shannon, 2005, p. 1281).

Additionally, with respect to its pivotal elements – that is, *identifying*, *coding* and *categorizing* basic patterns in the data – qualitative content analysis can be performed at different levels of abstraction. Thus, during the analysis of problem-solving activities, the data was partitioned into *episodes* of different length, according to the natural flow of the observed processes. In the context of this thesis, an episode should be interpreted as a part of the problem-solving activity that contains at least one displayed mathematical ability. At first, the method highlighted those abilities that were directly expressed during problems-solving. However, this part of the analysis demonstrated that some episodes of problem-solving activities could not be associated with abilities focused in the analytical framework in a mutually exclusive manner. Consequently, the data from the digital recordings of problem-solving activities was combined with data from reflective interviews. The method of analysis will be exemplified with data associated with Linda's solution of Problem 1 (P1).

When Linda looked at the presented task, she sat quietly, without writing or drawing anything for 30 seconds. Then, she said:

> Linda: Thus, eh... Oh, and here we are after all just using what radius they have and such. One would...

After this episode, at 45 seconds from start, she drew three semicircles and started to solve the problem by not saying much. Later on, she presented a correct solution of P1. Thus, according to the analysis of her problem-solving activity, the observed mathematical abilities were summarized in a working document as shown in Table 2. When categorising the patterns in the empirical material, I used the labels from the analytical framework, i.e. the episodes were coded with O, P, G and M, respectively.

Time	Observational notes	Oral utterances	Abilities
0.00	*Sits quietly, draws or writes nothing*		
0.30		Linda: Thus, eh... Oh, and here we are after all just using what radius they have and such. One would...	O or M?
0.45	*Draws three semicircles*		O
0.55		Linda: Basically, what we try to get here, is…. If I have… If we have a diameter and then… Is basically…	O
1.00	*Recalls a method or a formula?* $\pi\, d$		O or M?
1.08	$\pi\,($		O or M?
1.11 1.40	$d - x) - \pi\, x$	Linda: If pi times the diameter is the same as pi times, like, the diameter minus that length minus x. And then plus pi times x.	P

Table 2. Working document displaying Linda's abilities when solving P1.

As seen above, when performing a qualitative content analysis of her actions and verbal utterances, it was not unproblematic to understand if she was only obtaining and formalizing mathematical material (O) at 0.30, at 1.00 and at 1.08 or if she was also using her mathematical memory to recall a method for solving the problem. Conversely, for example, it was more certain that at 0.45 and 0.55 she tried to formalize the context of the problem (O) by drawing semicircles and explaining her actions, or that, between 1.11 and 1.40 she was processing (P) the formalized material.

It might be also interesting to notice that, between 1.11 and 1.40 she ended her sentence by saying "then plus pi times x" even though she writes "$- \pi\, x$"; which might have influenced the next step in her problem solving.

Another interesting detail is that at 0.55 Linda switches from "I have" to "we have", where the latter form is usual in a mathematical context, for example, in textbooks. This might have occurred because she became aware of the instruction that she should explain her actions loudly and therefore started to use an expression which is a part of the mathematical discourse.

However, even though the analysis indicated some unexpected aspects of Linda's actions, as mentioned, after analysing the above exemplified process it was not certain which abilities were present in the respective episodes starting at 0.30, at 1.00 and at 1.08.

Consequently, in the next stage of the analysis, I analysed the content of the reflective interview that was performed directly after solving P1. During, the interview, after declaring that she has not seen the proposed task, when explaining her actions from the beginning of her problem solving, Linda pointed on the three above mentioned semicircles and the formalized material by stating:

> Linda: What I saw first and foremost, when I look at the task, the first I saw it is that I have to look at the diameters.
>
> …
>
> Linda: And then you express it simply as that, well, expressing their different diameters as something of each other.
>
> Interviewer: Yes.
>
> Linda: It is similar to another task that I like very much...
>
> ...
>
> Linda: Like there, when solving that, the first thing to do… it is making formulas... How different triangles and squares... how the inside of it looks.
>
> Interviewer: Yes... hm.
>
> Linda: Just like there, if you express different sides through... and take one side minus the other, just like in that problem...

By stating that "It is similar to another task that I like very much" and then continuing with "Like there, when solving that, the first thing to do… it is making formulas... How different triangles and squares... how the inside of it looks" indicates that Linda was consciously recalling and explaining her methods. Therefore, it seems reasonable to assume that elements from explicit memory, that is, mathematical memory, were present in the above exemplified episode of Linda's problem-solving activity (e.g. Davis et al., 2000; Krutetskii, 1976; Nyberg & Bäckman, 2009).

In addition, when asked which problem she relates the proposed task to, she drew a square inside a right triangle and said that P1 (with the context of semicircles) was similar to the problem with the square in the triangle. Later, when explaining the similarities between P1 and the mentioned task, she stated "Like there, when solving that, the first thing to do… it is making formulas... How different triangles and squares... how the inside of it looks.", thus, it seems that she applied the same generalized method at both problems (Krutetskii, 1976, p. 296). In this way, by combining the data from problem solving and reflective interviews, I was able to understand some of episodes that could not be explained based only on the analysis of problem-solving activities.

Thus, it was reasonable to assume that during the first 30 seconds of her problem solving, Linda both obtained and formalized mathematical information (O) *and* drew on her mathematical memory (M), by recalling the task (with a square in a triangle) and its method associated with Problem 1. In this way, it became more plausible that in similar situations – for example, between 1.00 and 1.11, when referring to the applied problem-solving method – she again used her mathematical memory (M).

Consequently, the combined analysis resulted in tables where every episode, that lasted at least one second, and the time period for its occurrence, was related to the mathematical abilities focused in this thesis. I will exemplify this with the combined data from Linda, as seen in Table 3.

Time	Observational notes	Oral utterances	Reflective interview	Abilities
0.00	*Sits quietly, draws or writes nothing.*		Linda: What I saw first and foremost, when I look at the task, the first I saw it is that I have to look at the diameters. Linda: It is similar to another task that I like very much...	O and M
0.30		Linda: Thus, eh... Oh, and here we are after all just using what radius they have and such. One would...	Linda: … I look at it, and then I see, so, I see a lot of arcs of circles that stays over, on, the same segment, this segment is what they have in common.	O
0.45	*Draws three semicircles*			O
0.55		Linda: Basically, what we try to get here, is…. If I have… If we have a diameter and then… Is basically…		O
1.00 1.08	πd π (		Linda: And then you express it simply as that, well, expressing their different diameters as something of each other. Linda: Like there, when solving that, the first thing to do… it is making formulas... How different triangles and squares... how the inside of it looks.	M

1.11 1.40	$d-x)-\pi x$	Linda: If pi times the diameter is the same as pi times, like, the diameter minus that length minus *x*. And then plus pi times *x*.	Linda: Just like there, if you express different sides through... and take one side minus the other, just like in that problem...	P

Table 3. Combined data from Linda, displaying her mathematical abilities when solving P1.

As exemplified in Table 3, all participants' actions were analysed by identifying and categorising basic patterns in the empirical content and, importantly, by combining data from problem solving with data from reflective interviews. Thus, the analysis resulted in tables similar to Table 3, where the individual problem-solving activities of the participants were displayed in terms of associated mathematical abilities from the analytical framework.

The reliability of the analysis

As mentioned in the section concerning the reliability of the data analysis performed after the first phase of data collection, reliability is a concept with origins in quantitative research, that has become frequently applied in qualitative studies (e.g. Golafshani, 2003).

As discussed in the above mentioned section, based on different opinions among researchers regarding how reliability should be interpreted in the qualitative paradigm (e.g. Golafshani, 2003; Seale, 1999; Stenbacka, 2001; Strauss & Corbin, 1990), I performed a reliability test of the data analysis associated with the second and third phases of data collection. Thus, after completing the analysis of the empirical material on my own, by adapting the testing methods to the second and third phases of data collection, I tested the reliability of the data analysis in veins similar to the verification process associated with the first phase of data collection.

The test was performed in cooperation with a professor and a lecturer working at the same institution with which I was associated during the second and third phases of data collection. The professor and the lecturer verified the analysed material by analysing the digital recordings (from observations and reflective interviews) connected to the problem-solving process of two randomly selected participants. When performing the analyses, they used the analytical framework that was constructed for the purpose of the second and third data collection phases. The professor and the lecturer carried out the analyses independently of each other. This test indicated that the lecturer, the professor and I drew the same conclusions in 93% of the episodes, that is, the mathematical abilities associated with respective episodes in the participants' problem-solving processes were identical in 93% of the total number of verified episodes. Thus, based on the relatively high proportion of coinciding conclusions, it seems that the reliability of the data analysis associated with the second and third phases of data collection holds an acceptable level.

Summary of the included papers

Paper I: Mathematics education for gifted pupils – a survey of research

Given the fact that Paper I is written in Swedish, in order to offer a better understanding for English readers, in this section I will present a relatively extended and comprehensive summary of the paper.

The first paper is a review article that reports on the data collection phase associated with the systematic review of peer reviewed papers about mathematics education for gifted pupils. Drawing on the indication that mathematic teachers are experiencing a considerable level of uncertainty regarding practices intended to develop gifted pupils in mathematics (e.g. Lester & Schroeder, 1983; Reed, 2004; Rogers, 2007; Rotiger & Fello, 2005), one aim of Paper I was to investigate those pedagogical and organizational practices associated with gifted pupils' education in mathematics that are recommended by the research field. Another aim of Paper I was to expose the ways recommended practices are supported by empirical evidence.

As described in the methodology section, Paper I focuses on the intersection of the research fields on mathematics education and on education of gifted pupils. Following the objectives of a systematic review (e.g. Wu et al., 2012; Ziegler & Raul, 2000), the search for relevant papers was performed according to the structure described in the methodology section.

The retrieved papers were analysed through three main phases. During the first phase of the analysis, after selecting the most relevant 965 papers, I read through all abstracts and excluded – among duplicates, editorial texts and book reviews – those papers which were not focusing the mathematics education of gifted pupils in a school context. After this phase, 177 papers remained in the analysis process.

During the second phase, by applying qualitative content analysis (e.g. Graneheim & Lundman, 2004; van Leeuwen, 2005), the remaining articles were scrutinized for basic patterns which could be associated with the *education of gifted* pupils in the *mathematics classroom*. The content analysis – consisting of stages of identifying, coding and categorizing the basic patterns of a relatively unfamiliar literature – was data-driven (Kvale & Brinkmann, 2009). Accordingly, the analysis demonstrated some broad categories concerning gifted pupils' development in mathematics. Consequently, the content of the papers could be associated with gifted pupils' *performances* in school mathematics, to their *social situation* in school, to *gender differences* between these pupils, to *definition* and *identification* of mathematical giftedness, to *national programs* for mathematically gifted pupils, to *teachers' perceptions* of mathematically gifted pupils, and to *motivational* and *cognitive characteristics* of gifted pupils.

Prior to the third phase, the papers focusing *definition* and *identification* of mathematical giftedness – because these topics were not converging towards the objectives of the review – were excluded from the analysis. This measure left 135 articles in the process.

Finally, during the third phase, a deeper content analysis uncovered that the contents of the papers in the above mentioned broad categories were not mutually exclusive, that is, a substantial number of selected papers could be associated with more than one well-defined pedagogical and organizational practices for gifted pupils in the mathematics classroom. After the deeper analysis of the content of the papers in the broad categories, the following four pedagogical and organizational categories were identified: gifted pupils' in the *heterogeneously grouped mathematics classroom*, *acceleration programs* for gifted pupils, *ability grouping* of gifted pupils, gifted pupils' *attitudes* towards *different learning environments*. Accordingly, the results are presented in the above mentioned four categories.

Concerning the *heterogeneously grouped* mathematics classroom, the article indicates that regardless of school system, the teaching of gifted pupils in heterogeneous classes involves considerable challenges for the teachers (e.g. Reed, 2004; Rotiger & Fello, 2005). Therefore, it is suggested that teaching should be differentiated in these classrooms, in order to meet the needs of the gifted (e.g. Dimitriadis, 2012; Leikin, 2010, 2014; Reed, 2004; Rogers, 2007; Rotiger & Fello, 2005; Tucker, 1982). With respect to teachers' attitude towards gifted students, it is recommended that teachers expect the pupils to work hard, use general ideas in concrete situations, love mathematics and encourage pupils to use their imagination, teach pupils to think systematically, critically and without prejudices, and that they appreciate the success of pupils (e.g. Leikin, 2014).

With respect to differentiated teaching in the heterogeneous classroom, the article suggests that the following practices may be taken into consideration: differentiated instructions, which include a faster progression through the curricula; flexible cluster groupings, where gifted pupils solve problems adapted to their understanding and expectations; working with a modified content of the studied mathematical area; enrichment within certain mathematical topics that allow further, adapted investigations for gifted pupils, or contact with mentors and readily accessible computers with software programs which enable gifted students to advance at their own rate (e.g. Leikin, 2010; Rotiger & Fello, 2005). And with respect to differentiated instructions, it is suggested that problems presented to the whole class are adjusted to the gifted in terms of extension and application within the studied topic, that gifted pupils are confronted with open-ended tasks, and, that gifted pupils should select mathematical problems by themselves (e.g. Leikin, 2010; Reed, 2004).

In addition, Paper I shows that most of the articles discussing instruction in the heterogeneous classroom are based on reviews of the literature (e.g. Leikin 2010, 2014; Rotiger & Fello, 2005). That is, they do not typically report empirical studies. However, Tucker (1982) draws his recommendations on the experience of several decades spent working with mathematics teachers, while Reed (2004) investigates the effects of differentiated instructions on one pupil, who, after performing the offered tasks remains uninterested in mathematics. Subsequently, Reed concludes that the pupil, in the ninth grade, might have experienced too little differentiated learning too late in his schooling (Reed, 2004). In this context, it should also be mentioned that Dimitriadis (2012), reporting on empirical results concerning a relatively small number of mathematically gifted pupils, indicates that pull-out grouping and mentoring have more benefits for the development of these pupils than differentiated instructions or ability grouping within heterogeneous classes.

With respect to gender differences, the article indicates that both gifted girls and boys prefer to work individually in heterogeneous groups, while gifted boys – unlike gifted girls – prefer competitive approaches (e.g. Li & Adamson, 1992).

Finally, studies investigating students in secondary school (e.g. Händel, Vialle, & Ziegler, 2013), demonstrate that mathematically gifted students are perceived as more intelligent and meticulous, but less sociable than students gifted in sports or literature. The mentioned findings are confirmed also by Coleman and Cross (2014) by stressing that many students experience giftedness as a social handicap and that some of them – in their intention to be socially accepted by their classmates – are minimizing the impact of their talent in the interaction with their peers.

Unlike the somehow divergent results concerning heterogeneously grouped classes, the results regarding *acceleration* in mathematics are more convergent in their nature. Thus, by drawing on a review of programs for mathematically gifted pupils, performed by Sowell (1993), Paper I displays that a considerable number of gifted pupils improved considerably their results in mathematics after participating in acceleration programs. Additionally, both Rogers (2007) – referring to several empirical studies about acceleration – and Leikin (2010) suggest that mathematically gifted pupils should encounter acceleration programs at some occasions during their schoolyears.

Furthermore, when analysing successful acceleration programs, it seems that their success is conditioned by some important prerequisites. For example, the participation in these programs should be voluntary for pupils, the participants should be carefully tested for their mathematical performances and, the teaching in these programs should be adapted to the knowledge and potential of the participants (George, 1976; Sowell, 1993).

Moreover, the article indicates that acceleration in mathematics has predominantly beneficial effects on gifted pupils who are characterized as precocious (Sowell, 1993) and that it is not unreasonable to assume that young mathematically gifted pupils are essentially precocious (Threlfall & Hargreaves, 2008).

In addition, it is also proposed that acceleration combined with enrichment programs adapted to the pupils' cognitive characteristics have synergy effects on the development of mathematically gifted pupils (Lester & Schroeder, 1983; Sowell, 1993). And moreover, some studies suggest – without drawing on empirical data – that enrichment programs have more benefits than acceleration (e.g. Tucker, 1982).

Concerning the effects of *ability grouping*, the article indicates that mathematically gifted pupils in primary school, when studying mathematics in homogenous groups for one year, perform better than mathematically gifted pupils in heterogeneous classes (Sowell, 1993). These findings are explained by that ability groupings are offering opportunities for pupils to socialize with and feel mathematically challenged by peers with similar abilities and cognitive characteristics. Moreover, Paper I indicates that the topics proposed for gifted students in ability groupings in secondary school, should focus more on *why* one should do something instead of *how* to do it, have *more depth* and *less breadth* of coverage, and, offer more *challenge* (Tretter, 2005). And it is also recommended that teachers who work with mathematically gifted pupils in ability groupings are well prepared in order to meet the needs of these pupils (Dimitriadis, 2012).

The paper also highlights that even though the academic self-confidence of gifted secondary school students is not particularly affected during longer periods of ability grouping (Preckel et al., 2010), long-term ability groupings at elementary school (settings) are less beneficial for pupils placed in the highest-achieving group – concerning particularly girls' attitudes towards mathematics – where also mathematically gifted pupils are supposed to be placed (e.g. Boaler, Wiliam, & Brown, 2000).

Another perspective focused in Paper I is gifted pupils' *attitudes* towards *different learning environments.* Accordingly, the paper presents arguments concerning cooperative learning and individual work for gifted pupils.

In that regard, by drawing on empirical studies, it is indicated that gifted pupils, prefer individual settings instead of *cooperative learning* in the heterogeneous mathematical classroom (e.g. Hunt, 1996; Li & Adamson, 1992; Robinson, 1990). These findings are motivated by the observation that gifted pupils cannot receive differentiated instructions during cooperative learning in heterogeneous groups (e.g. Li & Adamson, 1992; Robinson, 1990) and thereby cannot progress according to their potential (e.g. Hunt, 1996; Robinson, 1990). In addition, gifted pupils experience imbalance when cooperating with their non-gifted peers, which often cause frustration at the individual level (e.g. Hunt, 1996), and, importantly, they do not perform at the same excellent level as in homogenous groupings (e.g. Hunt, 1996; Li & Adamson, 1992). Further, it is indicated that gifted pupils in middle school not only perform better in homogenous groupings, they also appreciate the opportunity to cooperate with and get mathematically challenged by peers who perform at the same level, and, that in these groups they can progress according to their potential (Hunt, 1996).

Concerning *motivational* aspects, the paper indicates that mathematically gifted 9–10-year-old pupils are more intrinsically orientated towards mathematics and display lower levels of anxiety within the subject, but also, that they are attributing less importance to external factors and own efforts in the context of their good mathematical performances and less importance to external factors and abilities in the context of their shortcomings in mathematics (Vlahovic-Stetic, Vizek Vidovic, & Arambasic, 1999).

Additionally, in the perspective of gender differences, it is suggested that mathematically gifted girls in middle school benefit substantially from homogenous groupings based on gender (Stutler, 2005). That is, mathematically gifted girls not only performed significantly better in the single-sex group, they also enjoyed competing with each other and experienced less anxiety (Stutler, 2005). In the context, it is also shown that mathematically gifted boys in the single-sex group (parallel to the mentioned group of girls) did not experience more motivation or improved their results in mathematics (Stutler, 2005). Further, it is indicated that gifted girls in secondary school, in both heterogeneous and homogenous groupings, experience more anxiety towards the subject then boys in corresponding groupings, but also that the mathematical performances of girls in homogenous groups are slightly better than the results of girls in heterogeneous groups (e.g. Schober et al., 2004).

Finally, Paper I proposes that gender related achievements in and attitudes towards mathematics are culturally and socially biased. Thereby, drawing on Freeman's (2004) argumentation, it is suggested that performances of both girls and boys are in concordance with the cultural and social expectations of respective societies. Moreover, the paper indicates that gifted adolescent girls – despite displaying more effort within the subject – experience more anxiety towards and feeling helpless more often when working with mathematics than the corresponding group of boys (e.g. Schober et al., 2004; Hong & Aqui, 2004).

The findings of the systematic review are discussed in terms of empirical and methodological considerations. Thus, by noting that the recommendations of selected papers do not converge with regard to pedagogical and organizational practices, the article discusses these findings in terms of empirical data in the selected papers. In that vein, it is indicated that the differences among recommended practices might depend on the data the papers are based on; that is, more than the half of selected papers are based on empirical studies while other papers are based on theoretical considerations or on reviews of the research field.

Further, Paper I mentions that almost 80 percent of the empirical studies have applied intelligence tests in order to identify gifted learners, and, importantly, that a substantial part of the displayed findings concerns generally gifted learners performing well in mathematics, that is, not specifically mathematically gifted pupils. In the context, it is also emphasized that identification methods for giftedness are culturally dependent and geographically differentiated. Also, the special prerequisites for successful acceleration programs are discussed, and it is emphasized that participants in these programs should be tested for mathematical giftedness, that the participation should be voluntary and that the subject content and the teaching should be adapted to the respective group of gifted pupils.

Finally, the suggestion that gifted pupils prefer to work alone in heterogeneous classrooms is associated to empirical findings which suggest that these pupils both perform better and enjoy more to work in homogenous settings. In addition, the paper indicates that there are no practices which are applicable on all gifted pupils – because these pupils constitute a divergent group – and that pedagogical and organizational practices should be carefully chosen with regard to the particular abilities of each gifted pupil.

Paper II: Examining the interaction of mathematical abilities and mathematical memory: A study of problem-solving activity of high-achieving Swedish upper secondary students

The second paper, which draws on data collected during phase two of the study, is associated with students' mathematical abilities (Krutetskii, 1976) during problem solving from the following two perspectives:

- The evidence and the interaction of mathematical abilities when high-achieving students solve non-routine mathematical problems.
- The role of the mathematical memory in the process of solving non-routine mathematical problems.

As argued in the methodology section, the investigation of mathematical abilities is conditioned by some important criteria. Thus, drawing on the aims of the study – that is, the exposure of interaction between mathematical abilities – high-achieving students from upper secondary school were observed and interviewed in the context of problem solving. In order to facilitate the manifestation of mathematical abilities and, specifically, to display the role of the mathematical memory, the participants solved two non-routine problems (the circles problem and the compact disc problem), which, in order to offer the participants opportunities to express a broader variety of mathematical abilities, belonged to two mathematical areas. The participants solved the proposed problems individually, in a private room at their school. They were also asked to write down or to communicate orally every step in their problem solving. In order to avoid that important parts of their cognitive process will not be exhibited, a contextual (reflective) interview was conducted after every problem-solving activity.

When solving the problems, each participant confirmed that he or she had not seen the problems before and that they were of a non-routine character. And even though they have had no time limits, none of the participants needed more than 14 minutes to solve either problem. As mentioned, the problem-solving activity of every participant was collected in an exact linear reproduction of written solutions, drawings and verbal utterances.

The analysis of the empirical material resulted in matrices where every episode – that is, a part of the problem-solving activity which contains at least one displayed ability – as well as the time period for its occurrence was associated with the mathematical abilities presented in the analytical framework. However, during some episodes, particularly in the beginning of the process, when participants did not write or say anything, the applied methodology was not sufficient to separate closely interrelated abilities – for example, the ability to obtain and formalize mathematical material (O) and mathematical memory (M).

Paper II indicates that every problem-solving activity contains three main phases, which occur in a chronologically analogous manner. Accordingly, every activity begins with an *orientation phase* which comprehends both the ability to obtain and formalize mathematical information and mathematical memory. Directly after the orientation phase follows a phase when the participants *process the formalized mathematical information* by logical thinking and reversibility in mental processes while striving for clarity and simplicity during operations – drawing on the displayed abilities, this is a *processing phase*. In addition, every problem-solving activity ends with a *checking phase* of processing mathematical information (P) – during this phase, participants check the correctness of their results.

Another aspect of the interaction between mathematical abilities was observable at the solution of P2, where three participants faced unexpected difficulties when the initially formalized equations lead to inequalities. At this moment – even though all participants were familiar with inequalities – they became stressed and made relatively simple mistakes before returning to the *orientation phase*, which was once again followed by the *processing phase*. Some participants went through this shifting of phases three times before solving the relatively simple inequalities. During the interviews, the participants admitted that they experienced stress because the equations lead to inequalities:

> Earl: That´s I was a little surprised when it was… on the inequality you solved it.
>
> Erin: It is always difficult to start thinking outside the box... It feels like your mind goes blank.
>
> Linda: Because I get so… When I start with equations ... then I really want to solve it with equations.
>
> Sebastian: This kind of tasks usually requires an equation.

With respect to the general occurrence of the focused abilities, the most frequently exposed ability was the *processing* of information (P), followed by the ability to *obtain and formalize* information (O) and *mathematical* *memory* (M). Further, the paper indicates that the ability to generalize mathematical relationships and solutions (G) – typically attributable to gifted students – which, according to the a-priori testing of the problems was observable at the solution of the circles problem, was not observed during the study. That is, despite the fact that every student who offered particular approaches to the circles problem, that is, Erin, Larry and Sebastian, was asked to develop general solutions, none of them succeeded in so doing.

Finally, the paper indicates that participants used their mathematical memory (M) basically during the orientation phase and mostly in combination with the ability to obtain and formalize mathematical information (O). Thus, the main function of mathematical memory (M) in the participants' activities was the selection and recalling of appropriate methods of problem-solving. However, Paper II indicates that mathematical memory (M), despite its restricted occurrence, has a significant role in the process due to the following two reasons. Firstly, because the participants selected problem-solving methods in the orientation phase of the process, and secondly, because the participants displayed considerable difficulties when trying to modify or substitute selected methods. For example, even though four of six students using the same method at P2 faced stress and uncertainty – three of those committing the same and relatively simple error – before returning to the orientation phase for a repeated formalization of the information (O), none of them abandoned their initially selected method.

The findings of the study are discussed in both educational and cognitive perspectives. Thereby, it is mentioned that the three identified common phases of problem-solving activities – the orientation phase, followed by a phase of processing the information, and the completing phase of checking the results – display similarities to the respective problem-solving frameworks proposed by Pólya (1966) and Mason et al. (1982).

Moreover, with respect to the role of mathematical memory, by drawing on the observation that – despite experiencing stress, committing simple mistakes and re-actualizing the context of the problem repeatedly – participants did not abandon initially chosen problem-solving methods, the paper discusses some important differences between typically high-achievers and mathematically gifted students.

Additionally, the participants' attachment to initially selected methods and their inflexible ways of approaching the proposed problems are discussed in a cognitive context. Hence, these findings are interpreted as automatized processes retrieved to working memory – but also discussed in the light of a basic function of the cerebral cortex, that is, the subordination of all new information to previous experiences. Further, the difficulties experienced at the orientation phase, when trying to differentiate the apparently simultaneous abilities O and M, are debated in the context of cognitive neuroscience and related to the extremely high speed in which information from long-term memory is retrieved to working memory.

Finally, the lack of the ability to generalize mathematical relationships and solutions (G) is elevated in the perspectives of convergent thinking and structural similarities (Tan & Sriraman, 2017).

Paper III: Uncovering the relationship between mathematical ability and problem solving performance of Swedish upper secondary school students

The third paper is framed by the same data associated with Paper II. That is, Paper III reports on an investigation of the dynamic between mathematical abilities (Krutetskii, 1976) and problem-solving performances from the following perspectives:

- The approaches of non-routine problems by high-achieving upper-secondary students.
- The relationship between mathematical abilities and problem-solving performances in the context of non-routine problems.

As mentioned, the participants were observed during their individual problem-solving activities and interviewed directly after those activities. Two non-routine problems (the circles problem and the compact disc problem) were presented. In order to offer participants a broader range of opportunities to express their mathematical abilities, the problems belonged to two different mathematical areas. The participants, who were given as much time as they needed, solved the problems individually in a private room at their school. During observation, clinical interviews were conducted, that is, the participants were encouraged to solve the problems in a think out loud manner. Moreover, in order to minimize the risk that significant parts of cognitive processes associated with the solution of the problems would not be communicated, a reflective interview was conducted after every problem-solving activity.

When asked, every participant confirmed that the problems were challenging and non-routine, thereby fulfilling one of the objectives of the study, that is, to observe high-achieving students when solving challenging, non-routine problems.

When applying the analytical framework on the empirical material, participants' written solutions and verbal utterances were partitioned into episodes of varying length determined by the natural flow of their actions. Also, the transcriptions of the reflective interviews were examined for patterns which could be related to the mentioned episodes. The combined data from observations and reflective interviews was summarized in working documents, where every episode was then scrutinized for evidence of the four abilities from the analytical framework. This process, assuming that at least one ability could be associated with every episode, could have resulted in 15 possible forms of episode; four coded singly, six coded doubly, four coded with three abilities and one coded with all four. Despite this potential, only certain episodes – mostly in the beginning of the problem-solving process and basically involving mathematical memory (M) and the ability to obtain and formalize mathematical material (O) – were double-coded, all other episodes were single coded. However, no episode was associated with the ability to generalize mathematical objects, relations and operations (G). The abilities occurring during respective episodes and the time period for their exposure resulted in an exact linear description of every individual problem-solving activity.

The analysis resulted in matrixes where the total time associated with each form of observed episode was calculated as a percentage of the total time that a student spent on the respective problem. In the next phase, in order to understand how different abilities interacted during the problem-solving process, an individual schema was constructed for each student's attempt. In these schemes, each participant's activities were shown at the individual level and those consecutive episodes which were coded in the same way were integrated into a single, more extensive, episode.

Paper III discloses that all participants used the abilities to obtain and formalize mathematical information (O) and to process the formalized information (P) to a much greater extent than the ability to recall mathematical relationships and problem-solving methods (M).

When confronted with the circles problem, the ability to obtain and formalize information (O) had an average display of 29% of time on this problem. This ranged from 16% (in the cases of Erin and Larry) to a relatively high proportion of Earl's process at 43%. The processing of the information (P) was exhibited at an average of 49% of all time spent on this problem; the processing (P) oscillated between Earl's 28% and Linda's 62% during the participant's problem-solving activities. Compared with the previously mentioned abilities, the participants used their mathematical memory (M) in much smaller proportions; in addition, the recalling of problem solving methods and relationships (M) was observed typically closely related to the ability to obtain and formalize information (O). Thus, the simultaneous display of these two abilities (O, M) was observed at an average of 13%, while the solitary exposure of M occurred at only at 7% of the total time.

Additionally, when solving the circles problem, the sum of columns O and P were approximately equal. That is, most of participants spent individually constant proportions of their problem-solving time on these two abilities, where low proportions of one ability were compensated by high proportions of the other. Finally, and interestingly, it seems that those participants who spent less time on orientating (O) themselves – Erin and Sebastian at 16% – and those who orientated themselves while processing the information (O, P) – Sebastian at 3% and Larry at 20% – failed to solve the problem.

When solving the compact disc problem, similar to the solution of the circles problem, participants kept most of their focus on obtaining and formalizing information (O) and on processing formalized information (P). Moreover, the sum of their respective display of O and P were again at constant proportions; that is, for example, Heather dedicated small proportions of time to O and high proportions to P, while Earl and Erin, respectively, focused more on O compared to P.

However, unlike their activities associated with the circles problem, when solving the compact disc problem, participants spent more time on obtaining and formalizing the information (O), that is, O was observed at an average proportion of 38% concerning all activities. The proportion of O ranged from Heather's 19% to Erin's 67%, and, when combined with other activities, the proportion of *orientation phases* rose to 46%. Somehow surprisingly, the proportion of the activities corresponding to the processing information (P) was approximately equal to the proportion displayed at the circles problem. That is, when solving the compact disc problem, the participants spent an average of 48% of their time on processing the information (P), which, with the inclusion of episodes when P was observed simultaneously with M or O, rose to 52%.

Similar to the activities associated with the circles problem, the time spent on recalling methods for problem-solving and mathematical relationships (M) was considerably less than the time dedicated to obtaining (O) and processing (P) the information. Thus, M was observed uniquely at an average of 3%, and in combination with O at 8% of the total time. These low proportions of M were somehow surprising, considering that four participants used repeatedly their ability to obtain and formalize the mathematical information (O), an ability closely associated with the recalling of methods and relationships (M).

Finally, when comparing the individual schemas of the two different problems, three important similarities are highlighted. Specifically, that all participant began their process with (O, M), that every process contained at least one cycle incorporating O and M or their simultaneous display of (O, M) and, that every problem-solving activity ended with P, because the participants re-actualized the context of the given problem in order to check obtained results.

Interestingly, despite the individually constant sums of the proportions of orientating (O) and processing (P) episodes – where the proportions of mentioned abilities typically oscillated at the individual level during the two different activities – the most successful problem-solver, Linda, showed consistent proportions of these abilities. That is, she spent approximately one fourth of her time on obtaining and formalizing the information (O) and three fifths of total time on processing the data (P). Conversely, participants who experienced difficulties during problem solving, for example Erin, shifted from low O and high P at the circles problem to the inverse percentage at the compact disc problem.

However, when looking at the outcome of the problem-solving activities, it is not unproblematic to find a direct connection between the time spent on respective abilities and correct solutions. For example, Paper III indicates that more time at the orientation phase lead to successful solutions of both problems in the case of Earl and the compact disc problem in the case of Erin, while more time spent at the orientation phase of the circles problem had the opposite effect in the case of Larry.

When discussing the results, it is mentioned that three participants – Earl, Linda and Heather – were consistent in their algebraic approaches to the proposed problems and solved both problems successfully. The other three participants – Erin, Larry and Sebastian – tried to solve the circles problem with particular, mainly numerical, methods, without presenting complete solutions. However, when confronted with the compact disc problem – which could be solved with both particular and algebraic approaches – Erin's numerical method led to an acceptable solution, Larry switched to an algebraic method and solved the problem in a satisfactory manner, while Sebastian, who approached the problem with a particular method, while reasoning out loud, could not solve it accurately, despite the fact that he tried to present an algebraic solution. Thus, Paper III suggests that algebraic approaches were more successful than particular ones, not least because they contained general methods and mathematical relationships (M) which facilitated the processing of the formalized information (P).

Further, by drawing on the observation that every participant dedicated a relatively large proportion of total time to orientating activities, Paper III emphasizes the importance of the orientation phase of the problem-solving process. The substantial proportion of processing activities (P) is discussed in the perspective of particular and algebraic approaches, while the small proportion spent on recalling mathematical relationships and problem-solving methods (M) is highlighted from a student perspective.

Finally, by comparing the presented findings to results of studies on Swedish students' performances concerning word problems (e.g. Palm, 2008; Lithner, 2011) and by highlighting the importance of the teaching in mathematics, Paper III problematizes the absence of the ability to generalize mathematical objects, relations and operations (G). Moreover, given that the participants were high-achievers in upper secondary school, the unexpected difficulties associated with the inequalities at the compact disc problem are discussed in the context of the TIMSS 2011 (Foy, Arora, & Stanco, 2013a, 2013b) – thereby the article indicates that Swedish students generally find inequalities difficult.

Paper IV: Mathematical memory revisited: mathematical problem solving by high achieving students

The fourth paper reports on the third phase of data collection and investigated the role of mathematical memory and the structure of participants' mathematical abilities during the solving of non-routine problems at two different occasions, which occurred approximately one year apart. Thus, based on Krutetskii's (1976) definitions of mathematical ability, the aims of the study were:

- To identify the structure of mathematical abilities when high-achieving pupils solve structurally similar non-routine problems.
- To examine the role of mathematical memory during the mentioned problem-solving activities.

Beyond the methodological considerations mentioned at the second phase of data collection – for example, the selection process of participants, the framing of observation methods and the construction of non-routine mathematical problems – with respect to an accurate investigation of the mathematical abilities, the design of the third phase of data collection required some additional factors. Thus, drawing on the characteristics of mathematical memory (Krutetskii, 1976) – that gifted and high-achieving students typically forget the context of a problem but also several months later are able to recall the methods applied when solving it – one criterion was to present a problem which was non-routine, but could be solved with similar methods as the problem confronted a year ago. Consequently, after the mentioned a-priori testing of the constructed problems, P1 (the circles problem) and P3 (the squares problem) was selected for this study.

Analogous to the previously presented papers, Paper IV reports on the same group of high-achieving students from upper secondary school during problem solving. The students solved the problems individually, in a private room at their school, and were offered as much time as they needed to solve the problems. During problem solving, in order to avoid the risk that important cognitive actions will not be communicated, the participants were encouraged to solve the problems in a think-aloud manner. Additionally, if they did not say, draw or write anything for a while I posed supplementary questions, as described in the methodology section. The observations were carried out during single days, approximately one year apart; that is, first P1 (the circles problem) was solved and a year later P3 (the squares problem) was solved. In addition, in order to prevent participants' memories when solving P3 being stimulated by recollections of P1, I continued to interact with them occasionally at their mathematics classes during the time period between the two observations.

During the interviews, by confirming one of the criteria for the study, every participant stated that the proposed problems were non-routine. During the analysis, similar to the empirical studies presented in Papers II and III, as a first step, participants' written solutions and verbal utterances were partitioned into episodes of varying length determined by the natural flow of their actions. In the following phase, all transcriptions were scrutinized by identifying, coding and categorizing the basic patterns in the mentioned episodes. Finally, the data from the observations was combined with data from interviews with focus on the four abilities from the, for these studies developed, analytical framework.

The combined analysis emerged in matrixes where every episode and the time period for its manifestation was associated with the focused mathematical abilities. However, as mentioned, at some episodes, particularly in the orientation phase of the problem-solving process, the abilities to obtain and formalize mathematical information (O) and the mathematical memory (M) were exhibited in an apparently simultaneous manner, thus, with the methods applied in these studies, it was extremely difficult to differentiate those abilities.

Paper IV indicates that the participants spent most of their time on obtaining and formalizing mathematical information (O) and on processing the formalized information (P) while mathematical memory (M) was present at small proportions of summarized activities and mostly in during the orientation phase. However, the paper also indicates that the proportion of time spent on activities displaying mathematical memory decreased during the second observation. That is, even though mathematical memory (M) was still closely related to the ability to obtain and formalize mathematical information (O) at 10% of the total time, in his singular form, M was observed at only 0.5%.

Nevertheless, differently from the previously performed studies, Paper IV offers a remarkable result concerning the ability to generalize mathematical relations and operations (G). That is, all participants who presented numerical solutions, namely, Erin Sebastian and Larry, were encouraged to develop their numerical results into general solutions. Yet, when solving the circles problem, none of the participants were able to generalize the numerical results (G). However, a year later, during the reflective interview associated with the squares problem, Erin developed her particular approaches into an acceptable general, algebraic solution (G). Afterwards, seemingly pleased over her performance, she stated:

> Erin: I've never made a general solution like this ... But it was fun ... Especially when it concluded in something.

Discussing the role of mathematical memory (M), Krutetskii (1976) indicates that gifted students are able to remember problem-solving methods, which they applied relatively long time ago. However, when asked if they have solved similar problems before, only Earl and Larry among the six participants recalled the circles problem, which was presented a year before.

> Earl: We got a very similar task last year, when we had the circle and that semicircle.

Larry: We did a pretty similar task last time, when it was something like this, something with the radius or diameter on them.

Subsequently, and somehow expected, after recalling the previously solved problem, both Earl and Larry applied the *same method as* they applied a year before: that is Earl applied *identical algebraic methods* at both problems and Larry approached both tasks with the *same particular method.*

Moreover, Paper IV indicates that even though none of the other participants associated the squares problem with the circles problem, every participant approached both tasks in *individually identical ways*. For example, Linda related both problems to the same geometric problem – concerning a square drawn in a triangle and apparently very different from the proposed problems – which, in her case, represents a frequently applied generalized method at non-routine geometric problems.

Linda: I will bring up the same task as last time, with triangles and squares. It is a bit the same thing ... I connect very often geometrical tasks to that. I have written that solution many times and I can see every step in the process in front of me.

As seen above, Linda, influenced by her mathematical memory (Krutetskii, 1976, p. 296), states that "It is a bit the same thing" before applying the generalized method on the squares problem.

As mentioned, all participants approached both problems in identical ways. That is, Heather, after obtaining and formalizing the information, used the *same variables* in her *identical algebraic approaches*. Erin approached both tasks by *reasoning*, testing *numerical values* and applying *particular solutions*, while Sebastian *reasoned* carefully and *tested* an algebraic formalization before requesting the use of *numerical values* at both occasions. Finally, in veins similar to the findings of Paper II, this paper displays that, despite its very limited presence, mathematical memory seems to have a pivotal role in the problem-solving process, essentially because the participants selected problem-solving approaches in the beginning of the process and did not change them later.

When discussing the findings of Paper IV, the essential role of mathematical memory (M) is presented in different perspectives. For example, after highlighting that only two of six participants recalling the previously solved problem, mathematical memory (M) is discussed in the perspective of Krutetskii's (1976) descriptions of the topic.

Moreover, drawing on the remarkable conclusion that every pupil approached both problems in individually similar ways, the paper suggests that the participants, when confronted with non-routine problems, rely on apparently inflexible problem-solving methods, which are applied regardless of their outcome. Moreover, in the context it is also discussed that the stability of applied methods could have had the consequence that the participants dedicated a larger proportion of their problem-solving activities for processing the mathematical information (P) during the second observation.

Finally, the paper discusses – from the perspective of the concept of convergent thinking (Tan & Sriraman, 2017) – that, when offered additional opportunities, Erin, for the first time in her mathematics education, could perform a successful generalization of obtained particular results.

Discussion

This thesis had two different objectives. Firstly, to review the literature on pedagogical and organizational practices intended for gifted pupils' education in mathematics and to determine which of recommended practices are supported by empirical evidence. Secondly, to examine the interaction of mathematical ability and mathematical memory during problem solving.

Consequently, Paper I, which is a review article reporting on the first phase of data collection, offers a survey of the research concerning mathematics education for gifted pupils. The remaining three papers, based on empirical studies related to the second and third phases of data collection, display perspectives on the interaction of mathematical abilities and the role of mathematical memory in the context of non-routine mathematical problems.

Accordingly, the discussion related to the papers included in this thesis, will be structured according to the above mentioned main objectives. Thus, in this section, I will discuss the findings of the respective phases of data collection and I will present some general reflections about the data collection. Further, I will propose some implications of presented papers for mathematics education and I will offer some suggestions for future research on the discussed topics.

Discussion of the findings of included papers

Mathematics education for gifted pupils – a survey of research

The results presented in Paper I emerged after analysing the content of selected research papers in the context of pedagogical and organizational practices recommended to promote gifted pupils' development in mathematics. Subsequently, the recommendations were structured according to four main topics, that is, gifted pupils in the *heterogeneously grouped* mathematics classroom, *acceleration programs* for gifted pupils, *ability grouping* of gifted pupils, gifted pupils' *attitudes* towards different learning environments.

The analysis shows that recommendations structured in the above categories are predominantly convergent, which may facilitate the discernment of useful practices – however, it is also indicated that every category includes a few papers whose recommendations are different from the majority of recommendations.

In this section, some particular aspects of the mentioned recommendations will be discussed in terms of empirical data.

As mentioned, the analysis showed that it is reasonable to assume that specific pedagogical and organizational practices have significant potential to develop gifted pupils in mathematics, that gifted girls experience some aspects of mathematics education differently compared to the corresponding group of boys and, that some, but not all, mathematically gifted adolescents face difficulties in their social interaction.

For example, it has been indicated that teachers face considerable challenges when working with gifted learners in heterogeneous classrooms (e.g. Reed, 2004; Rotiger & Fello, 2005) and that certain practices, for example, differentiated teaching, are recommended by several selected papers. However, despite evidence that differentiated teaching – for example, differentiated instructions, enrichment, flexible cluster groupings, working with a modified content or working with software that allows pupils to advance in their own pace – is recommended for heterogeneous classrooms in a substantial number of papers (e.g. Dimitriadis, 2012; Leikin, 2010, 2014; Reed, 2004; Rogers, 2007; Rotiger & Fello, 2005; Tucker, 1982), the analysis indicated that that a clear majority of these papers are based on reviews of the research field or on authors' own experiences of gifted education, that is, only a few papers report on empirical studies performed by the authors (e.g. Dimitriadis, 2012; Reed, 2004). In a further differentiation, empirical studies indicated that both pull-out grouping and mentoring are more beneficial for the performances of gifted learners than differentiated instructions in heterogeneous classes (Dimitriadis, 2012).

However, empirical studies are unambiguous concerning the reluctance that gifted pupils expose towards cooperative learning in heterogeneous classes, thereby indicating that these pupils prefer to work in homogenous groupings, that, besides offering adequate developmental opportunities within the subject, also facilitate social-networking and cooperation with peers who are experiencing mathematics in similar ways (e.g. Hunt, 1996; Li & Adamson, 1992; Robinson, 1990).

From a different perspective, Paper I identified several articles reporting on the benefits of well-balanced acceleration programs (e.g. George, 1976; Leikin, 2010; Rogers, 2007; Sowell, 1993) in contrast to single papers stating that acceleration is inappropriate for gifted learners and it is applied mostly in order to release the challenge that teachers face in mixed ability classes (e.g. Tucker, 1982). Accordingly, with respect to this controversy, it should be noted that, for example, Sowell (1993) analyses the effects of 14 completed acceleration programs for gifted pupils, while Tucker draws his suggestion on his own experience from elementary education when concluding that "even though bright children can go faster, this author feels that acceleration accomplishes nothing really useful" (Tucker, 1982, p. 12). Hence, as exemplified above, it seems that the differences between these controversial recommendations can be explained by the fact that respective papers draw their findings on relatively dissimilar grounds.

With respect to another objective of Paper I, that is, the display of the ways recommended practices are supported by empirical evidence, it should be emphasized that around 60% of the 135 analysed papers are based on empirical studies. Consequently, 40% of papers are based on literature reviews, theoretical reflections, reports from the research field and personal experiences. In addition, when concerning empirical studies, it should be noted that almost 80% of those that identified gifted learners, report on approaches which can be considered psychometric. Finally, importantly, it should also be highlighted that those empirical studies that identified gifted learners have principally identified generally gifted pupils – in almost 75 % of the cases – rather than mathematically gifted pupils. Consequently, it is not unreasonable to assume that most of the recommendations presented in Paper I concern gifted pupils in general, that is, not pupils who are particularly gifted in mathematics.

Another explanation to divergent recommendations within the focused categories, might depend on the ways respective papers define gifted learners. For example, with respect to gender differences – drawing on results indicating that innate gender differences are minimal in gifted individuals – the empirical papers emphasize that gifted adolescent girls, despite their substantial effort associated with the learning of the subject and high levels of intrinsic motivation, experienced more anxiety and feel more helpless in the context of school mathematics compared to corresponding groups of boys (e.g. Hong & Aqui, 2004; Schober et al., 2004). However, with respect to gender differences and pupils' performances in mathematics, some papers (e.g. Schober et al., 2004) indicate that gifted adolescent girls and boys perform similarly in heterogeneous groupings, while other papers suggest that gifted girls generally outperform gifted boys in mathematics (e.g. Freeman, 2004). In this case, while the former article (Schober et al., 2004) describes an empirical study, where gifted learners were identified by syllogism tests – which are useful as psychometric measures of intelligence (Sternberg, 1982, p. 269) – the latter reports on the results of British students who achieved A-, B- and C-marks at the nationwide examination for General Certificate of School Education (GCSE), that is, students who are not identified as gifted or high-achieving. Therefore, based on the observation that there is a lack of sufficiently strong correlation between giftedness and mathematical performances (Juter & Sriraman, 2011) it might be understandable that the mentioned studies report relatively different findings.

Further, even though it seems that some practices are recommended by the majority of the analysed papers, there is another aspect of findings presented in Paper I that should be emphasized. Namely, that frequently recommended practices should meet some important criteria, when applied in an educational context.

For example, based on the relative convergence within the respective topics, it is reasonable to assume that both acceleration and ability groupings are beneficial for the performances and the social development of gifted pupils. However, the analysis of successful acceleration programs shows that participants need to be carefully chosen to these programs, that participants should be identified as mathematically gifted, that participation should be voluntary, that teaching should be adapted to the participants' knowledge and abilities, and that, acceleration should not be organized during periods longer than a year. In this context, it should be highlighted that mathematically gifted children (aged 9–10) seem to be fundamentally precocious (Threlfall & Hargreaves, 2008) and thereby particularly well-matched for sensible acceleration programs (Sowell, 1993). Finally, it should also be noted that general acceleration in mathematics in which all learners are offered favourable conditions to study courses on levels higher than their ordinary mathematics courses are particularly disadvantageous for pupils' attitudes towards the subject (Sheffield, 2015).

Similar to acceleration programs, ability grouping should be conditioned by the selection of challenging tasks, that offer more depth and less breadth in a certain topic, and, by the suggestion that teachers who work with these students should be well prepared for the needs and characteristics of the gifted (e.g. Dimitriadis, 2012; Tretter, 2005). Yet, regardless of the ability groupings that gifted learners are placed in, it should also be emphasized that some of them experience difficulties in their social interaction with non-gifted peers (Coleman & Cross, 2014).

In conclusion, based on the analysis presented in Paper I, it is reasonable to assume that the situation of gifted learners in the mathematics classroom is not unproblematic and that there are no easily adaptable practices in order to develop their mathematical abilities or to improve their performances.

As indicated, gifted learners generally prefer to work alone in heterogeneous groupings, feel comfortable to be engaged in challenging problems, sometimes face difficulties in their interaction with non-gifted peers and seem to enjoy working with pupils who experience mathematics in similar ways. However, these findings are often differentiated with respect to particular categories of gifted learners – therefore, it is suggested that the identification of gifted learners is pivotal in the context of those practices that are intended to meet their needs. For example, some findings concern mathematically gifted young pupils, other results focus on generally gifted adolescent girls while others are intended to meet the needs of secondary students in homogenous groupings. Thus, based on the assumption that gifted learners do not constitute a homogenous group (e.g. Hoeflinger, 1998; Pitta-Pantazi, 2017; Rogers, 2007), it is reasonable to assume that there is no universal pedagogical or organizational practice which is applicable on all gifted pupils. Consequently, every pedagogical or organizational practice should be adapted to the requirements and characteristics of the concerned pupils.

In addition, drawn on the observation that not every gifted pupil is high-achiever in mathematics, it is also reasonable to assume that, because of the diversified nature of identification methods, there are, and there will still exist, many gifted pupils which are not identified, and thereby will not benefit of recommended practices (e.g. Persson, 2010; Rotiger & Fello, 2005; Ziegler & Raul, 2000). Consequently, there is an emerging need of more empirical research on mathematically gifted pupils.

Finally, with respect to the displayed complexity of the focused topic and to the bluntness of our research instruments, it seems rational that every intended pedagogical or organizational practice is based on a common agreement between the school and the respective group of pupils and it should lead to that each pupil experience personal satisfaction in mathematics.

The interaction of mathematical abilities, the role of mathematical memory during problem solving and the relationship between mathematical abilities and problem-solving performances

One aim of the empirical papers associated with the second and third phases of data collection was to investigate the structure of mathematical abilities and to display the role of mathematical memory during problem solving. In addition, Paper III intended to uncover the relationship between mathematical abilities and problem-solving performances. The second and third phases of data collection, based on observations of six high-achieving students from Swedish upper secondary school in the context of non-routine problems, were drawn essentially on mathematical abilities delimited by Krutetskii (1976), widely agreed in the research field.

In an attempt to differentiate the discussion, I will highlight the results presented in Papers II, III and IV from the following three perspectives: the *interaction of* observed *mathematical abilities*, the *relationship* between *mathematical abilities and* problem-solving *performances*, and the *role of mathematical memory* during problem solving.

Accordingly, with respect to the *interaction of* observed *mathematical abilities*, the results indicate that – regardless of applied methods and levels of mathematical accuracy during problem solving – the participants demonstrated important similarities in the way they approached the proposed problems. Thus, three similar phases were identified in participants' problem-solving activities. That is, every participant started to solve the problems by being engaged in an *orientation phase*, where the ability to *obtain and formalize* mathematical information (O) was closely associated with *mathematical memory* (M). The orientation phase was followed immediately by a *phase of processing the formalized information*, where the participants, by displaying logical and reversible thinking, applied mathematical rules and principles in order to solve the problem, and every problem-solving activity was completed by a *checking phase*, in which the results and contextual information were tested for their correctness.

Consequently, by emphasising the chronological order of the mentioned common phases, the analyses showed that participants solved the problems with clear parallels to established problem-solving frameworks, for example, the three stages of *entry*, *attack* and *review* in Mason et al.'s (1982) model or the respective phases of *understanding* the problem, *developing* a plan, *carrying out* the plan and *looking back* at the completed solution in Pólya´s

(1966) framework. For example, with respect to Pólya's (1966) model, the initial stages of understanding the problem and developing a plan could be interpreted as the *orientation* phase, the stage of carrying out the plan could be associated with the phase of *processing* formalized information and the stage of looking back at the solution could be matched with the phase of *checking* the obtained results.

Further, the analyses indicate that, during the orientation phase, the abilities O and M were closely interrelated and extremely hard to differentiate with the methods used in this thesis. Thereby, despite the more advanced technology for recording problem-solving activities, the analyses confirmed the difficulties faced by Krutetskii's (1976) team in the given situation. Nevertheless, these findings may deserve an explanation from a different perspective. With respect to cognitive theories, problem solving can be viewed in terms of parallel processes in the working memory, where the involved information-units are kept for approximately 30 seconds. Subsequently, in order to counteract the relatively short period of time – in veins similar to all cognitive activities – the information is re-actualized, that is, essential information-units are repeatedly retrieved to working memory (e.g. Buckner & Wheeler, 2001; Nyberg et al., 2003; Olson et al., 2009). Accordingly, it might be reasonable to assume that participants' activities involved parallel cognitive actions during orientation phase, associated with the ability to obtain and formalize information (O) and to mathematical memory (M). Further, drawn on the fact that information-units are retrieved to working memory at extremely high speeds, it might be understandable that the methods applied in this thesis were not sufficient to discern respective cognitive actions.

Another finding associated with the interaction of mathematical abilities was observed during the solution of P2, when initially formalized equations lead to unexpected inequalities. In these situations, the participants experienced stress, made relatively simple mistakes and became insecure. Furthermore, the participants discontinued the processing of information (P) (they stopped to solve the problem) and returned to another phase of orientation, in order to once again formalize the information (O) before processing (P) it. However, despite that some of participants went through this shifting of phases more than one time, they seemed to find it difficult to abandon the initially chosen problem-solving methods, for example, methods for equation-solving. By this, some participants exposed strategies that may be described as inflexible and conformist, and thereby typical for high-achievers (Brandl, 2011), rather than flexible, out-of-the-box approaches attributed to the mathematically gifted (Leikin, 2014).

Nevertheless, in a perspective of cognitive neuroscience, the mentioned inflexible behaviour might be explained by the function of cerebral cortex, regarding automated knowledge (Ingvar, 2009). In that respect, automated processes – due to their inflexible nature – access working memory in a highly efficient effective manner because they tend not to interact with other working memory processes (Ingvar, 2009; Shipp, 2007). Thus, it is not unreasonable to assume that the inflexibility of the participants – high-achievers with a substantial subject knowledge – occurred because they were unable to adjust their automated equation-solving procedures.

Another finding which might need further attention is the absence of the ability to generalize mathematical relations and operations (G) at the second phase of data collection. As mentioned, there were six opportunities, three of them occurring at the circles problem and three at squares problem (at the third phase of data collection) when the participants, after presenting numerical results, were asked if they could generalize their results. Accordingly, drawn on the condition that all of them were high-achieving students who, according to Krutetskii (1976), should be able to generalize numerical approaches, it was somehow unexpected that none of Erin, Larry and Sebastian could generalize their particular solutions of the circles problem. However, a year later, when solving the squares problem and offered additional occasions to reflect, Erin performed a successful generalization of her particular approaches.

Thus, even though a closer examination of the ability to generalize (G) was not in the focus of this thesis, it should be mentioned that Krutetskii (1976) identified four levels of the ability to generalize mathematical relations and operations. According to Krutetskii (1976), mathematically able students displaying the highest level of the focused ability, are capable to generalize "mathematical material correctly and immediately, "on the spot"" (Krutetskii, 1976, p. 255), while less able students, at the lowest level, are not able to "generalize mathematical material according to essential features even with help from experimenter" (Krutetskii, 1976, p. 254). In sum, drawn on Krutetskii's differentiation, it might be reasonable to assume that some participants, even though they were high-achievers, performed at levels below the highest level of generalization at the time of the second phase of data collection. However, based on the dynamic nature of the mathematical abilities (Krutetskii, 1976), it is not unreasonable to assume, that they used other well-developed mathematical abilities in order to compensate for the lower levels of the ability to generalize (G).

However, Erin's successfully performed generalization during the third phase of data collection, indicates that the offered additional opportunities to reflect over general solutions might have influenced her actions. Therefore, it should be noted that with respect to gifted students, the ability to generalize mathematical relations and operations (G) develops over time (Sriraman, 2003, 2004a, 2004b, 2004c). For example, when engaged in appropriately designed mathematical activities, focusing the same generality over a period of several months – by displaying appropriate forms of *convergent thinking* (Guilford, 1967) – gifted students were able to discern "invariant principles or properties, as well as to formulate generalizations from seemingly different situations by focusing on structural properties during abstraction" (Tan & Sriraman, 2017, p. 118). Accordingly, it is not unreasonable to assume that

Erin's success at the squares problem emerged because the additional opportunities to reflect over her problem solving engaged her in some form of convergent thinking.

In conclusion, drawing on the conditions of the second phase of data collection – solving only two problems, which were not containing patterns of the same generality, and with relatively little time given for reflection – it might be reasonable to assume that the absence of the ability to generalize mathematical relations (G) depends more on that participants were not given appropriate opportunities to develop their ability to generalize (G) than on their mathematical abilities. Additionally, in similar veins, drawing on the indicated correlation between the teaching of the subject and the development of mathematical abilities (Krutetskii, 1976; Usiskin, 2000) it might also be rational to assume that the participants who were not familiar with algebraic methods, have not been exposed to teaching containing enough elements of general problem-solving strategies.

With respect to the *relationship between mathematical abilities and problem-solving performances*, Paper III shows that activities associated with the orientation phase were important for participants' problem-solving performances. This finding can be explained partly by that participants selected their methods during orientation phase and found it extremely hard to modify them, and, partly by that participants, based on the progress of their problem solving, shifted to orientation phase several times during their activities. Consequently, the relatively high proportion of time dedicated to orientating activities (47% of the total time) suggests that participants found it important to understand and formalize (O) the problem before starting to process (P) the formalized information.

Further, the displayed shifting between the phases of orientating and processing activities indicates that the participants were able to monitor and regulate their cognitive actions by considerable levels of understanding associated with their decisions (Andrews & Xenofontos, 2015). Thereby, the participants demonstrated significant patterns of "flexibility in mental processes" (Krutetskii, 1976, p. 350), which is an ability typical for gifted students.

Another finding of Paper III concerns the ability to process mathematical information (P). Characteristically, this ability occurred at fewer occasions than other abilities; yet, episodes dominated by processing activities were longer than other episodes and exhibited at more than the half (53%) of the total time.

In addition, by confirming findings of previous studies (e.g. Cai & Lester, 2005) the analyses indicate that algebraic approaches were more successful than particular approaches leading to numerical results. Thus, it is not unreasonable to assume that the proportion of time spent on processing information (P) and the shifting between phases of orientation and processing were dependent on the ways students formalized information (O) during the orientation phase. Accordingly, students who represented their task at both occasions – that is Earl, Linda and Heather – symbolically were not only more efficient, but also more confident during problem solving. For example, Earl – despite the stress and insecurity experienced when solving the compact disc problem – displayed considerable levels of perseverance in applying an algebraic method, which eventually lead to a correct solution of the proposed problem. Besides, it should be stressed, that all participants, even in cases when not presenting appropriate solutions, showed persistence in their activities – thereby exposing confidence that the proposed problems were worth solving, a pivotal attribute of the problem-solving competence (Andrews & Xenofontos, 2015).

Further, Paper III indicates that participants approached non-routine problems in ways convergent to those described in widely agreed problem-solving frameworks (Garofalo & Lester, 1985; Kapa, 2001; Mason et al., 1982; Nunokawa, 2005; Pólya, 1966; Singer & Voica, 2013). And it should also be mentioned, that the participants' perseverance associated with their relatively successful attempts at solving inequalities are to some extent contradictory to results of international assessments (Foy et al., 2013a, 2013b) showing that Swedish students experience difficulties when confronted with inequalities.

With respect to *mathematical memory* (M), Papers II and IV indicate that, despite its relatively small proportion of the problem-solving process, mathematical memory has a pivotal role in the participants' activities. Specifically, Paper IV shows that M was displayed at 5% and 0.5% in its solitary form during the observations occurring a year apart – and at 12%, respectively 10%, when closely related to other abilities. In addition, M was closely related to the ability to obtain and formalize mathematical information (O) and mainly associated with activities that occurred during the orientation phase.

Thus, drawn on the observation that participants found it difficult to abandon or to even modify initially selected methods – and on the indication that the quality of their problem-solving process was influenced by selected methods – it is not unreasonable to assume that the recalling of methods (M) was critical for the outcome of the participants' problem-solving activities. In addition, the third phase of data collection indicates that participants, even though not remembering the previously solved problem, approached both problems in identical ways at the individual level.

Accordingly, the relatively unexpected findings, concerning seemingly inflexible methods and approaches selected in the early phases of the participants' activities, might merit a further discussion. Thus, the early selection of methods and the great similarities demonstrated at participants' individual problem-solving processes at the circles problem and the squares problem, might be explained by the means of how the human brain is responding to all encountered information. Specifically, from a perspective of cognitive neuroscience, all new information is assimilated in relation to previous experiences and knowledge – thereby, the number of possible interpretations of the information is dependent on the individual's background (e.g. Buckner &

Wheeler, 2001; Ingvar, 2009; Nyberg et al., 2003; Shipp, 2007). Thus, it might be rational to assume that – based on their relatively good subject knowledge and influenced by their experience of mathematical problem solving – the participants, when confronted with non-routine problems, inclined to select methods which they felt familiar with. For example, Linda, who did not recall the circles problem when solving the squares problem, associated both problems to another task – containing a square and a triangle in an apparently very different context – and moreover, in veins similar to mathematically gifted pupils (Krutetskii, 1976, p. 296), she felt that both problems were very similar.

Thus, it seems that the main function of mathematical memory (M) during the solving of non-routine problems was to recall problem-solving methods and, importantly, when recalling those, the participants recalled methods that they were familiar with. And it might be also reasonable to assume that the displayed stability of individual approaches to the squares problem made recalling problem-solving methods (M) during the orientation phase more efficient, thereby increasing the presence of ability to process information (P) during the process. Therefore, with respect to the relative inflexibility of selected approaches and to the small proportion of time M was displayed during problem solving, it seems that mathematical memory (M) has a key role in the participants' problem-solving process.

In conclusion, the empirical studies report some findings on mathematical abilities in the context of problem solving that might deserve a short summary. Firstly, it should be noted, that the participants' problem-solving activities contained the three common phases of *orientation*, *processing* and *checking*. Secondly, that participants – even though they solved non-routine problems by exploiting mathematical abilities and problem-solving behaviours that are characteristic for mathematically gifted students – offered relatively inflexible individual approaches to the proposed problems. Thirdly, that mathematical memory is used mostly in the beginning of the problem-solving process and seems to be crucial for the selection of problem-solving methods. Fourthly, that participants approached non-routine problems in ways they felt familiar with and, importantly, which were similar at the individual level at observations occurring one year apart. Fifthly, that it seems that some participants might need more time and additional opportunities in order to develop their ability to generalize mathematical relations and operations. And finally, that participants who used algebraic methods were more efficient than those who applied particular approaches.

Finally, it should be accentuated, that the second and third phases of data collection are based on problem-solving activities of six Swedish students and thereby difficult to interpret in a larger perspective or in a different educational system (e.g. Lave & Wenger, 1991). Therefore, it is not unreasonable to assume that there might exist a substantial number of perspectives on mathematical abilities, not displayed in this thesis, that high-achievers from upper secondary school would exhibit when solving non-routine problems.

General reflections about the included papers

Mathematics education for gifted pupils – a survey of research

As indicated in the previous section, the results associated with the first phase of data collection are essentially dependent on those papers that have been selected and analysed with the intention to identify and categorize practices that are recommended by researchers in the field in order to develop gifted pupils' mathematical performances.

Thus, the presented results depend basically on the methodological considerations associated with the first phase of data collection (e.g. Grant & Booth, 2009; Wu et al., 2012). With respect to that, it might be possible that a larger diversity of search terms, that is, terms supplementary to those that were applied in the systematic searches, would have led to the opportunity to retrieve additional relevant articles. However, after analysing 965 selected abstracts in order to find relevant papers prior to the qualitative content analysis, I noticed a substantial overlapping of papers retrieved within the different search categories; that is, several papers were retrieved in more than one search category. Consequently, I am not convinced that a larger variety of search terms in the English language would offer access to a significantly increased number of articles that are relevant to the studied topic.

However, after discussing the findings of Paper I with colleagues, it is reasonable to assume that the use of search terms in additional languages would have resulted in an additional number of relevant articles. Thus, the absence of search terms in various languages should be considered a shortcoming of the first phase of data collection. Nevertheless, in the content it should also be mentioned that, when framing the data collection, even if I speak fluently several languages, I did not feel confident enough in my language skills in order to perform a qualitative content analysis of peer reviewed articles in some additional lingua franca – for example, in German or in French – of the research community.

It has been also indicated (e.g. Grant & Booth, 2009; Wu et al., 2012) that the quality of review articles is influenced by the competency of those who perform the analysis of the collected data. In that respect, it is reasonable to assume that if the analysis of selected papers would have been performed by a different researcher, the results might have been somehow divergent from the results presented by this author. Nevertheless, as mentioned in the analysis section of this thesis, after performing the analyses, I verified the reliability of the performed analyses by asking a senior professor to analyse the content of ten randomly selected papers. Afterwards, the senior professor came to the same conclusions concerning identified patterns and categories in the compared papers as I did. Thus, even though it seems that the reliability of the analyses reaches an acceptable level, my lack of experience within the actual research field – and moreover, this being the first review article that I worked on – might have influenced the decisions I took and the conclusions I came to during the analysis process. Hence, my relatively modest experience of the research field and especially my lack of experience of systematic reviews should also be considered as potential limitations of the first phase of data collection.

Another perspective that might deserve attention is that, due to editorial reasons, 42 initially selected articles, discussing identification and definitions of mathematical giftedness, were excluded prior to the third phase of the analysis process. Thus, it might be reasonable to assume that if those articles were included in the further analysis, then some supplementary aspects of gifted pupils learning in mathematics would have been displayed in Paper I.

Another aspect, that should be mentioned, concerns dialogues with colleagues about the first phase of data collection and the feedback I got from peer reviewers before publishing the article. During these interactions, I was asked why I chose to write this article in Swedish, considering that a publication in English would probably reach a larger scale of readers and thereby could offer a greater exposure of my work. The answer to that question has a basically rational and a somehow emotional background. Thus, being aware of that I received substantial support from the Swedish educational system both in my profession and within this research project – combined with my awareness of the difficulties that Swedish teachers face when confronted with literature written in English – I decided that the minimum I could do in order to reciprocate the support that I received, was to write this paper in Swedish, in that way making it more accessible to the Swedish community.

Finally, it should be accentuated that the results of Paper I are based on a data collection which was restricted to peer reviewed papers written in English and, importantly, that the analysis of the selected literature was performed by a doctoral student with a modest experience of the research field and with no previous experience of writing review articles. Therefore, the presented results should be interpreted in the perspective of these circumstances. However, even though I feel relatively confident in presenting my findings, given the large variety of the results and the relative divergence among them, it is reasonable to assume that there are, and, importantly, there will still be published articles whose recommendations may differ from the pedagogical and organizational practices included in the review article.

The interaction of mathematical abilities, the role of mathematical memory during problem solving and the relationship between mathematical abilities and problem-solving performances

As mentioned, the findings displayed in Papers II, III and IV are based on observations and interviews performed during the second and third phases of data collection. Thus, when considering possible interpretations of these findings, it might be beneficial to discuss the methodological considerations that lead to the data collection and to the analyses that emerged in presented results.

In this respect, it should be emphasized that the applied methods for data collection worked out satisfactory and in line with the expectations prior to the performed studies. That is, the participants were motivated during problem solving and made considerable efforts in order to solve the proposed problems in a think aloud manner. Also, the communication with the participants functioned appropriately during both problem solving and subsequent reflective interviews. In a similar vein, the applied technology, that is, the digital pens, functioned well and it seems that the participants were not affected by the technical details during problem solving. Moreover, the digital recordings of problem-solving activities and interviews offered an accurate linear display of every problem-solving activity, in that way, the analyses of the collected data were facilitated considerably.

However, when examining the first two recorded observations and reflective interviews during the second phase of the data collection, I noticed that I was not accurate enough in my communication with the participants, that is, some of my questions could be interpreted as leading questions and certain statements could be understood in more than one way. Accordingly, being aware of these deficiencies, I tried to improve my interaction during the upcoming observations and interviews. And, fortunately, the digital recordings indicate that my interaction with the participants was more accurate during the remaining phases of data collection.

Another issue that might deserve attention is that – according to the guidelines of observations and clinical interviews (Ginsburg, 1981) – during the observations I treated the participants respectfully and, importantly, I did not let them feel that I assessed the quality of the presented solutions. For example, even though an appropriate solution of P1 (the circles problem) requires general methods, when the students offered particular solutions, I did not insist that they should solve the problem in a more general manner or told them that their solution was not accurate enough. Instead, I asked them to consider whether the presented particular solution was the only solution, or whether there could exist other, more general solutions to the problem. And when the participants were not able to present a general solution – being aware of the possible negative impact of mathematics related affect, especially emotions, on pupils' cognitive actions (e.g. Antognazza, Di Martino, Pellandini, & Sbaragli, 2015; Hannula, Opt' Eynde, Schlöglmann, & Wedege, 2007) – I tried to encourage them by saying that presented particular solutions could also be viewed as solutions to the given problem. Moreover, not even in situations when I understood that the applied method would not lead to a correct solution, I interrupted the participants in their activities – instead, I waited until they declared that their problem-solving activity was completed.

And even though participants were informed that their performances during the empirical studies will not have any consequence for their marks in mathematics – according to my discernment during observations and to the analyses of digital recordings – every participant was reliably engaged in the problem-solving process. That is, all of them exhibited the ambition to solve the proposed problems in an accurate manner.

However, even though the digital recordings offered precise connections between participants' actions and verbal utterances with seconds' accuracy, as mentioned in the analysis section, the absence of writings, drawings or utterances was not possible to interpret without combining the data from observations and reflective interviews. Thus, as expected, the analysis of those episodes when the participants did not write or say anything, was a significantly difficult task. Nevertheless, the combined analysis of data from problem solving and interviews indicated that during certain episodes – which occurred mostly during the orientation phase – the ability to obtain and formalize mathematical information (O) and mathematical memory (M) were closely related to each other. Yet, even though the applied digital technology delivered accurate recordings, it was not possible to differentiate these abilities, or to display their interaction, during the analysis process.

Accordingly, it is not unreasonable to assume that during the orientation phase, participants evaluated the problems in the light of their previous problem-solving experiences; that is, it seems that mathematical memory (M) influenced the ability to obtain mathematical information (O) (e.g. Buckner & Wheeler, 2001; Ingvar, 2009; Nyberg et al., 2003; Shipp, 2007). Moreover, as previously mentioned, it seems rational that the diverse information in the participants working memory was maintained by parallel processes and retrieved at extremely high speeds from long-term memory (e.g. Buckner & Wheeler, 2001; Nyberg et al., 2003; Olson et al., 2009). Thus, the methods applied during these data collection phases were not sufficiently accurate to differentiate the participants' cognitive actions associated with the mentioned abilities. Accordingly, the applied digital technology might be considered as a shortcoming of the second and third phases of data collection.

A different perspective, which might deserve further attention is associated with a notable difference between the second and third phase of data collection. That is, even though none of the participants was able to generalize (G) numerical results at the circles problem, when offered additional opportunities, Erin could present an appropriate generalization of her results (G) a year later, at the squares problem. Therefore, it is not unreasonable to assume that during the second phase of data collection not all participants were able to performs generalizations "on the spot" (Krutetskii, 1976, p. 255) according to Krutetskii's (1976) highest level of the ability to generalize mathematical material. Moreover, it seems that Erin's successful generalization of the squares problem occurred because she has been offered additional opportunities to reflect on her particular solutions, that might have also engaged her in convergent thinking (Guilford, 1967). Thus, even though the ability to generalize was not in focus of the design of the second and third phases of data collection, it is rational to assume that different methodological considerations might have displayed the participants' ability to generalize mathematical information (G) in a more accurate manner.

Additional methodological considerations that might be discussed are the relatively low number of participants and the difficulty level of the presented problems. Thus, it seems to be obvious that a larger number of participants or participants from several schools could offer a more comprehensive understanding of the ways mathematical abilities interact during problem solving or how mathematical memory influences the problem-solving activities of high-achieving students (Battista et al., 2009). However, given that this research project was supposed to completed during a restricted period of time, in combination with the condition that I was supposed to collect all data by myself, I am not sure that I would have been able to perform the data collection in a more efficient way – especially with respect to that one aim of the studies was to observe students in the context of non-routine problems. Hence, if the number of participants would have been larger, I would most probably not be able to perform all observations during single days. And if the studies would have been conducted during several days at respective occasions one year apart, the risk would have increased that the participants would communicate about the proposed problems with each other and thereby the problems would not have been perceived as novel tasks. Accordingly, with respect to the precondition that I was going to conduct every observation by myself, it was pivotal for the design of the study that the data collection was performed during single days.

As mentioned above, the difficulty level of the presented problems might also merit some additional reasoning. In that respect, one may argue that some of the problems were not challenging enough for high-achievers from upper secondary school, and that more difficult problems could have uncovered different aspects of mathematical abilities. However, both the a-priori testing of the proposed problems, as well as the second and third phases of data collection showed that the participants perceived the problems as non-routine, and thereby challenging, and that some of them faced difficulties when solving them. Additionally, if the respective opening problems (the circles problem and the squares problem) would have been too difficult – thereby increasing the risk that participants would not be able to solve them – then eventual failures would have influenced the participants negatively when solving upcoming problems, that is, the compact disc problem and P4, the dinner problem, (e.g. Antognazza et al., 2015; Hannula et al., 2007). In conclusion, I am not convinced that an increased number of participants, or a selection of more difficult problems would have facilitated the implementation of the second and third phases of data collection.

Finally, it should be emphasized that the findings presented in Paper II, Paper III and Paper IV are restricted by the technology applied for data collection and the above debated methodological cornerstones. Therefore, it should also be underlined, that other or more participants as well as a different selection of problems would have most probably been able to display interactions between mathematical abilities or characteristics of the mathematical memory that were not displayed in this thesis. Further, the findings associated with these phases of data collection should not be interpreted in a more general perspective or transposed to a context different from the Swedish school system (e.g. Lave & Wenger, 1991).

Implications for mathematics education

Mathematics education for gifted pupils – a survey of research

Considering the relative diversity of pedagogical and organizational methods that are recommended for the mathematics education of gifted pupils displayed in Paper I, I will try to summarize the implications of this systematic review for mathematics education according to the following four topics.

Firstly, and importantly, considering that gifted or mathematically gifted pupils constitute a relatively divergent group (e.g. Borland, 2005; Hoeflinger, 1998; Pitta-Pantazi, 2017; Rogers, 2007) it is important to stress that there does not exist a pedagogical or organizational method that will be beneficial for all gifted pupils. Thus, the school system that aims to meet the needs of gifted learners in mathematics, should identify the needs and characteristics of those individuals for whom the developmental methods are intended for. And given the fact that some mathematically gifted pupils are witnessing about a problematic social interaction with their non-gifted peers (Coleman & Cross, 2014) and that teachers who work with gifted pupils should be familiar with the attributes of the gifted (Dimitriadis, 2012; Karp & Bengmark, 2011; Leikin, 2011; Tretter, 2005; Usiskin, 2000), I would recommend that every developmental measure is based on mutual respect between the gifted pupil and the respective school system.

The second implication arises from the indications that mathematically gifted pupils are experiencing boredom when working with repetitive tasks (e.g. Krutetskii, 1976; Rogers, 2007) and are reluctant towards cooperative learning in heterogeneous groupings (e.g. Hunt, 1996; Li & Adamson, 1992; Robinson, 1990). Thus, these indications, combined with the observation that a considerable number of empirical studies display substantial benefits associated with well-balanced acceleration programs (e.g. George, 1976; Rogers, 2007; Sowell, 1993) and ability groupings (e.g. Dimitriadis, 2012; Rogers, 2007; Tretter, 2005) lead to the recommendation that gifted pupils should be able to participate in acceleration programs or to work in ability groupings during their schoolyears. However, in order to be considered successful, these practices should fulfil some important criteria. That is, the participation in acceleration programs or in ability groupings should be voluntary for the pupils, the participants should be carefully chosen, the teaching should be adapted to the participants' level and characteristics and the teachers who are working with these students should be well prepared for the characteristics of excelling learners. Additionally, empirical studies indicate that acceleration programs should not be longer than a school year and that, when working within ability groupings, the pupils should be engaged in problems that are challenging and offer more depth and less breadth in a specific mathematical area (e.g. Dimitriadis, 2012; George, 1976; Sowell, 1993; Tretter, 2005).

Thirdly, partially based on the observation that cooperative learning in heterogeneous groupings is a problematic issue for gifted students (e.g. Hunt, 1996; Li & Adamson, 1992; Robinson, 1990) it seems that there exist successful practices aimed to meet the needs of gifted students in the heterogenous classroom. With that respect, differentiated teaching may be recommended in heterogenous classrooms. That is, the teacher should adapt the teaching for the needs of gifted pupils by applying some of the following methods: differentiated instructions; faster progression through the curricula; enrichment activities within carefully chosen topics; flexible cluster groupings, where the pupils encounter a modified content of the studied topic; or working with software that allows them to advance at their own pace (e.g. Dimitriadis, 2012; Leikin, 2010, 2014; Reed, 2004; Rogers, 2007; Rotiger & Fello, 2005; Tucker, 1982). Moreover, as indicated by Leikin (2010) and Rotiger and Fello (2005), if these practices are combined, their benefits may increase due to synergy effects.

The fourth potential implication for education is based on the circumstance that mathematically gifted girls in middle school, when studying in a homogenous grouping based on gender, performed significantly better than the corresponding group of boys (Stutler, 2005). Besides, while the boys did not experience more motivation or improved their results in mathematics, the girls enjoyed working more competitively and experienced less anxiety (Stutler, 2005). Thus, if the purpose of a school system is to improve the performances of mathematically gifted girls in middle school, it might be a suitable organizational method to organize single-sex groups. Nevertheless, in that case it should also be taken in consideration that the corresponding group of boys may not benefit from this practice.

Finally, it should be highlighted that, according to the analysis performed in the first phase of data collection, there is more empirical evidence for the success of well-prepared acceleration programs and ability groupings compared to differentiated instructions, cluster groupings and enrichment programs applied in heterogeneous classrooms (e.g. Borland 2005; Dimitriadis, 2012; George, 1976; Rogers, 2007; Sowell, 1993; Tretter, 2005). Accordingly, as a last suggestion, I would recommend that the respective school system and ultimately its teachers analyse the background and conditions of intended practices, in order to select those practices which have the highest potential to improve the situation of those pupils who they are intended for.

The interaction of mathematical abilities, the role of mathematical memory during problem solving and the relationship between mathematical abilities and problem-solving performances

The implications of the empirical studies associated with the second and third phases of data collection for mathematics education are somehow convergent and can be summarized as it follows.

One suggestion is based on the indication that mathematical memory has a crucial role in the problem-solving process. That is, influenced by their mathematical memory, a) the participants selected their problem-solving methods during the initial phase of the process, b) the participants selected methods that they felt familiar with and were identical at the individual level when solving problems one year apart, and c) the participants found extremely difficult to modify the initially selected methods.

Thus, the observation that participants selected individually identical methods that they felt familiar with, in combination with the fact that participants who applied algebraic methods were more successful than those who applied particular methods, leads to the suggestion that students should be more familiar with algebraic methods in the context of non-routine problems. That is, based on the indication that mathematical abilities are developed during mathematical activities and dependent of the teaching of the subject (e.g. Krutetskii, 1976; Usiskin, 2000), I would recommend that pupils are deliberately taught algebraic methods when working with word problems. Thus, in concordance with previous findings (e.g. Cai & Lester, 2005) I would like to recommend that pupils encounter mathematical problems that might be solved in qualitatively different levels, that is, problems that, by displaying the advantages of general approaches, motivates them to use algebraic methods instead of depending on particular and numerical methods. And when concerning classroom practices associated with problem-solving activities, results from previous studies (e.g. Ambrus & Barczi-Veres, 2016; Kilpatrick, 2016) indicate that pupils learn mathematics more efficient when they solve problems in cooperation with their peers.

Another suggestion arises from the indication that even though participants experienced several difficulties during problem solving, they found it very difficult to abandon or modify their initially selected methods. Thus, it seems that the applied methods were non-flexible with respect to included processes. Hence, even if the non-flexible behaviour of the participants might be explained by studies focusing memory functions from the perspective of cognitive neuroscience (e.g. Buckner & Wheeler, 2001; Ingvar, 2009; Nyberg et al., 2003; Olson et al., 2009; Shipp, 2007), another suggestion would be that pupils, during their mathematics education, should develop problem-solving methods that are flexible with respect to included operations and symbols. Accordingly, it seems reasonable that pupils, in order to develop their inductive reasoning and their flexible approaches to problems, should work with mathematical problems without given strategies (e.g. Ambrus & Barczi-Veres, 2016; Cai & Lester, 2005; Lithner, 2008). Further, classroom discussions about the problematic nature of analogical reasoning – when earlier acquired mathematical relations are applied at atypical word problems (e.g. Degrande et al., 2016) – should also lead to the use of more flexible problem-solving approaches by the pupils.

The next suggestion is grounded in the findings regarding the generalization of numerical results associated with the second and third phases of data collection. As mentioned, when solving P1, none of the three participants who presented numerical solutions could develop those results into general solutions. However, during the reflective interview after working with the squares problem, when additional opportunities to reflect were offered, one participant (Erin) could develop her numerical into a general solution. Somehow surprisingly, according to Erin, it was the first time in her life that she developed an algebraic solution from numerical values. Thus, in concordance with studies that demonstrate that – if working under appropriately designed conditions – talented students are able to develop their ability to generalize (Sriraman, 2003, 2004a, 2004b, 2004c), I would like to recommend that students, at least at some occasions during their education, will be engaged in solving series of tasks that focus on the same generality. As indicated by Tan and Sriraman (2017), working with these series of tasks should initiate pupils in convergent thinking (Guilford, 1967), and thereby lead to that they learn to assemble structural similarities in ways that eliminates superficial similarities – and in that way to the expected generalisations.

My final recommendation is based both on the results of included empirical studies and on the very nature of mathematics. Consequently, I would like to suggest that, regardless of the nature of the teaching, pupils should be taught that mathematics is about to identify and, if possible, to generalize the discerned patterns in a given material, but also, that mathematics is about being able to specify our results every time it is needed. In that way, it is not unreasonable to assume that mathematics will make more sense for pupils, both in their everyday life and in other school subjects.

Future research

Based on the studies included in this thesis, in this section I will present my suggestions for future research within the respective fields.

The findings presented in Paper I indicate that around 60% of analysed papers were grounded in empirical studies, while the remaining part of papers were based on, for example, literature reviews, theoretical propositions that were supported by analysis of the research field or the authors' personal experiences of gifted learners.

Accordingly, a plausible general reflexion would be that there is a need for more empirical studies that will investigate the characteristics of mathematics education of gifted learners.

Furthermore, when discussing the educational practices that are investigated through empirical studies, the analysis shows that the majority of papers reports on acceleration programs or on different kinds of pull-out groupings for gifted learners (e.g. Dimitriadis, 2012; George, 1976; Sowell, 1993; Tretter, 2005). That is, there are only a few empirical studies that focus on practices that are recommended by the research field and intended to meet the needs of gifted learners in heterogenous classrooms, for example, differentiated instructions, enrichment programs and cluster groupings (e.g. Dimitriadis, 2012; Reed, 2004). Thus, based on the indication that teachers generally find it difficult to develop the mathematical abilities of gifted pupils in heterogeneous classes (e.g. Leikin & Stanger, 2011; Shayshon et al., 2014), it seems that additional empirical studies are necessary in order to investigate the effects of recommended practices in heterogeneous classrooms.

With respect to that, it seems reasonable to suggest that case studies, that may involve multiple cases of generally gifted or mathematically gifted pupils – in order to contrast a given theoretical proposition (Yin, 2015) – should be carried out. In addition, importantly – in order to, as far as possible, avoid the problematic nature of correlation between multiple influences – these case studies should investigate the effects of a single recommended practice at the time, for example, the effects of differentiated instructions in the heterogeneous classroom. In that way, it might be possible to bring much-needed empirical evidence for the effects of practices that are recommended mostly based on theoretical foundations.

Further, concerning the quality of the systematic review presented in Paper I, based on the observation that giftedness and mathematics education are culturally, socially and politically construed (e.g. Borland, 2005; Callahan & Miller, 2005; Gyarmathy, 2013; Freeman, 2004; Jeltova & Grigorenko, 2005; Karp, 2017; Karp & Bengmark, 2011; Leikin, 2014; Mönks & Katzko, 2005; Ziegler & Raul, 2000) it is reasonable to assume that there is a need of systematic reviews that include papers on several languages. In that way, it might be possible to uncover aspects of mathematical education for gifted learners that were not exhibited in this thesis, which focused research published in English.

The findings presented in the papers discussing the interaction of mathematical abilities, the role of mathematical memory during problem solving and the relationship between abilities and problem-solving performances indicate some directions for future research that are somehow different from those suggested above.

Thus, according to the indication that participants – influenced by their mathematical memory – approached non-routine problems in ways that were identical at the individual level, for a further investigation of the nature of mathematical memory, studies observing a larger number of gifted or high-achieving pupils solving an increased number of non-routine problems at different qualitative levels should be carried out. In that way, we might bring light over attributes of the mathematical memory that were not possible to discern in this thesis.

Another suggestion for future research is based on the difficulties experienced when trying to differentiate the ability to obtain and formalize mathematical material (O) from mathematical memory (M) during the initial phases of participants' problem-solving activities. It seems that these difficulties were experienced partly due to cognitive and neuroscientific characteristics of human memory (e.g. Buckner & Wheeler, 2001; Ingvar, 2009; Nyberg et al., 2003; Olson et al., 2009; Shipp, 2007) and partly due to the limitations of the technology applied during the second and third phases of data collection.

Nevertheless, because the present thesis was not able to display the interaction of the mentioned mathematical abilities during the initial phase of problem solving, I would like to suggest that future studies with the intention to investigate the nature of mathematical ability will combine methodological traditions from several research fields, for example, methods associated with qualitative research – such as, collective case studies (Bassey, 1999) – with methods from the field of cognitive neuroscience (e.g. M. Leikin et al., 2013; R. Leikin et al., 2017). Based on some encouraging results from studies performed after the data collection of this thesis was completed (e.g. M. Leikin et al., 2013, 2014; R. Leikin et al., 2017), it is reasonable to assume that methods from above mentioned research fields have the potential to uncover aspects of mathematical abilities and mathematical memory that were not exposed in this thesis.

Sammanfattning

Denna avhandling är baserad på fyra artiklar som presenteras i det inledande avsnittet och diskuteras mer detaljerat genom avhandlingen. En av dessa artiklar behandlar begåvade elevers matematikundervisning och övriga tre artiklar fokuserar högpresterande elevers matematiska förmågor vid lösning av matematiska problem som är av icke-rutinkaraktär. Artiklarna rapporterar om tre faser av datainsamling.

Huvudsyftet med den första fasen av datainsamlingen var att undersöka de pedagogiska och organisatoriska metoder kopplade till begåvade elevers matematikundervisning som är rekommenderade av forskningsfältet. Ett annat syfte var att ta reda på rekommenderade metoders koppling till empiriska studier.

Den andra fasen av datainsamlingen syftade dels till att undersöka hur högpresterande elevers matematiska förmågor interagerar under matematisk problemlösning dels till att klarlägga det matematiska minnet roll under den nämnda processen. Datainsamlingens tredje fas hade särskilt fokus på matematiska minnets roll i den interaktion av matematiska förmågor som sker när högpresterande elever löser matematiska problem med ungefär ett års mellanrum. Av analytiska skäl var även problemen presenterade under den tredje fasen av datainsamlingen av icke-rutinkaraktär men kunde lösas med liknande metoder.

Eftersom syftet med och metoderna kopplade till den första fasen av datainsamlingen skiljer sig betydligt från den andra och tredje fasen, så kommer jag att presentera artiklarna kopplade till respektive faser av datainsamling i två skilda resonemang.

Baserad på observationen att begåvade elevers matematikundervisning innebär betydande utmaningar för varje enskilt skolsystem och på indikationen att den stora mängden rekommenderade pedagogiska metoder, som är ämnade att stimulera dessa elevers kunskapsutveckling, leder till att lärarna är osäkra på vilka metoder som är bäst lämpade för den egna praktiken, den första datainsamlingsfasen – som fokuserar pedagogiska och organisatoriska metoder kopplade till begåvade elevers matematikundervisning – resulterade i en systematisk forskningsöversikt. Denna översikt bygger på en inledande grövre analys av innehållet av 965 forskingsatiklar som har lett till en utförlig kvalitativ innehållsanalys av de 135 artiklar (bland de 965) som var bäst lämpade för studiens syfte. Resultatet av denna analys presenteras i artikeln Matematikundervisning för begåvade elever – en forskningsöversikt, som i denna avhandling betecknas som Paper I.

Den systematiska översikten visar dels att begåvade elever i matematikklassrummet utgör en divergent grupp, dels att det inte finns pedagogiska och organisatoriska metoder som passar för alla dessa elever. Vidare, med den nämnda inledande reservationen, indikerar översikten att det finns några pedagogiska och organisatoriska metoder som bör ha positiv inverkan på kunskapsutvecklingen hos vissa grupper begåvade elever. Enligt analysen kunde innehållet i analyserade artiklar och tillhörande rekommenderade metoder struktureras enligt följande kategorier: begåvade elever i det *heterogena matematikklassrummet*, *acceleration* av begåvade elever, *nivågrupperingar* av begåvade elever, samt begåvade elevers *attityder till olika arbetssätt*.

Översikten visar att begåvade elever bör undervisas i det heterogena klassrummet framför allt genom att tillämpa differentierade instruktioner, till exempel bör läraren: anpassa uppgifter som presenteras för hela klassen till de begåvade elevernas nivå; använda så kallade ”öppna matematiska problem” som eleverna kan lösa på sin egen prestationsnivå; och, utmana begåvade elever genom fördjupning i ett matematiskt område som de är intresserade av. Vidare, när det gäller undervisningens utformning, så rekommenderas det att begåvade elever tillåts arbeta i snabbare takt och att de har tillgång till mentorer som de kan diskutera djupare matematiska problemställningar med, att det organiseras flexibla grupperingar i klassrummet beroende på uppgifternas svårighetsgrad och enligt elevernas prestationsnivå, samt, att begåvade elever har tillgång till teknologi som erbjuder möjligheter att utforska komplexa matematiska frågeställningar.

När det gäller accelerationsprogram – dvs. program som är på en, för elevens ålder, avancerad nivå, i vilka eleven deltar vid exempelvis några tillfällen i veckan – indikerar analysen att dessa program kan ha en positiv inverkan på begåvade elevers kunskapsutveckling. Analysen är däremot tydlig i resonemanget att framgångsrika accelerationsprogram bör uppfylla en rad viktiga kriterier. Följaktligen bör dessa program rikta sig till elever som har identifierats som matematiskt begåvade och bygga på innehåll och undervisning som är anpassade till elevernas förkunskaper, dessutom bör programmen vara frivilliga och tidsbegränsade, dvs. program som inte sträcker sig över perioder som är längre än ett skolår åt gången. Vidare indikerar artikeln att acceleration verkar fungera utmärkt framför allt för ”precocious”, dvs. tidigt utvecklade elever – dessutom presenterar artikeln argument för att matematiskt begåvade elever i 9–10 års ålder kan betraktas som just tidigt utvecklade elever.

I ett vidare perspektiv visar analysen att även välbalanserade nivågrupperingar kan leda till att begåvade elever upplever matematiken som mer meningsfull och höjer sina resultat i ämnet. I det avseendet indikeras det att begåvade grundskoleelever som studerar matematik i prestationshomogena grupper under ett års tid – och arbetar med utmanande matematiska problem, som har mer fokus på varför man gör något istället för hur man gör det – presterar betydligt bättre än begåvade elever som studerar i heterogena klassrum. Dessutom visar analysen att begåvade elever som studerar i prestationshomogena grupper upplever att de får bättre möjligheter till att knyta sociala band till deras kamrater samt blir stimulerade matematiskt.

När det gäller könsskillnader mellan begåvade elever som studerar i heterogena klassrum, visar den systematiska översikten att det finns ytterst få skillnader relaterade till elevernas kön när det gäller deras resultat i matematik, men också, att de skillnader som finns mellan flickors och pojkars prestationer kan härledas till de sociala och kulturella grunder som kännetecknar de olika skolsystem som eleverna studerar i. I sammanhanget bör det också nämnas att begåvade flickor som studerar på gymnasiet anstränger sig mer, samt är mer ängsliga inför matematiken, jämfört med motsvarande grupp pojkar.

Emellertid visar analysen att matematiskt begåvade mellanstadieelever som har undervisats i könshomogena grupper uppvisar skillnader både i prestationer i ämnet och i deras uppfattning av det sociala sammanhanget i respektive grupper. Det vill säga, matematiskt begåvade flickor som har studerat i flickgruppen trivdes bättre, upplevde mer motivation och höjde sina resultat i ämnet jämfört med flickor som har studerat i den könsblandade gruppen. Däremot, när det gäller motsvarande grupper pojkar, fanns det inga skillnader mellan respektive gruppers motivation, attityd och matematiska prestationer.

När det gäller begåvade elevers attityder till olika arbetssätt under matematiklektionerna respektive matematiskt begåvade elevers sociala interaktion i det heterogena klassrummet, indikerar artikeln att dessa elevers situation inte är oproblematiskt. I det avseendet antyder analysen att det förekommer att matematiskt begåvade elever upplever utanförskap i heterogena klassrum samt att de ibland befinner sig i en utsatt position i klassens sociala sammanhang – allt detta leder till att de försöker minimalisera effekterna av deras begåvning i interaktionen med deras kamrater. Vidare är de analyserade studierna i stort sett överens om att begåvade elever uppvisar ovilja mot grupparbete i heterogena grupper och jobbar helst på egen hand i det heterogena klassrummet, men också i att dessa elever uppskattar att ingå i prestationshomogena grupper där de kan arbeta i egen takt, tillsammans med kamrater som betraktar matematiken på samma sätt som de själva gör.

Med avseende på artiklarnas empiriska grunder, visar analysen att runt 60 procent av artiklarna rapporterar om empiriska studier medan övriga artiklar är huvudsakligen baserade på forskningsöversikt, författarnas analys av forskningsfältet eller deras egna erfarenheter av matematikundervisning för begåvade elever. Det mesta av empirin är associerat till acceleration och nivågrupperingar samt till begåvade elevers attityd till olika arbetssätt. I motsatt förhållande, minst antal empiriska studier är förknippade med metoder som avser undervisning i det heterogena klassrummet.

Vidare, i sammanhanget är det viktigt att nämna att, när det gäller identifiering av begåvade elever i empiriska studier, så observerades i tre fjärdedelar av studierna allmänt begåvade elever, dvs. inte elever som är typiskt matematiskt begåvade. Följaktligen – även om det finns indikationer på att nämnda elevgrupper representerar mängder med en omfattande andel gemensamma element – är det inte orimligt att anta att många av resultaten som rapporteras i Paper I avser elever som är allmänt begåvade och inte specifikt matematiskt begåvade elever.

Den systematiska forskningsöversikten indikerar att det behövs fler empiriska studier som undersöker de rekommenderade metodernas effekt på begåvade elevers kunskapsutveckling. Artikeln avslutas med antagandet att det är osannolikt att framtida studier kommer att presentera metoder som passar för alla begåvade elever i matematikklassrummet. Följaktligen är artikelns rekommendation att varje pedagogisk och organisatorisk åtgärd baseras på ömsesidig respekt mellan lärare och elever, samt att dessa åtgärder leder till att varje elev känner personlig tillfredställelse i samband med sina matematiska aktiviteter.

Övriga tre artiklar i denna avhandling redogör för de empiriska studier som grundar sig i den andra och tredje fasen av datainsamlingen. Artiklarna som är betecknade som Paper II, Paper III och Paper IV redogör för högpresterande gymnasieelevers matematiska problemlösning. Den presenterade teoretiska bakgrunden gällande allmän begåvning, matematiska förmågor, matematiskt minne och matematisk problemlösning, har lett till att den andra och tredje fasen av datainsamlingen har bestått av observationer och kontextuella intervjuer i samband med att nämnda elever löste matematiska problem.

För att möjliggöra exponering av flera matematiska förmågor, och framför allt för att deltagarnas matematiska minne ska bli observerbart, under den andra fasen av datainsamlingen löste eleverna matematiska problem som var relativt nya för dem, dvs. problem av icke-rutinkaraktär. Dessa problem hörde till två olika matematiska områden och kunde lösas med både partikulära och allmänna metoder. Under den tredje fasen av datainsamlingen, med det uttalade syftet att apostrofera det matematiska minnets karaktär, löste deltagarna, med ungefär ett års mellanrum, två icke-rutinuppgifter som kunde lösas med liknande metoder – även dessa problem kunde lösas både med partikulära och allmänna metoder.

Varje deltagare löste problemen individuellt och utan tidsbegränsning. Efter varje avslutad problemlösningsaktivitet har det genomförts individuella kontextuella intervjuer där deltagarna erbjöds möjligheten att förklara sina handlingar och tankegångar i samband med den nyss avslutade aktiviteten.

Resultaten i artiklarna som rapporterar om den andra och tredje fasen av datainsamlingen kan relateras till framför allt fem huvudsakliga områden.

För det första, visar de empiriska studierna att deltagarnas problemlösningsprocesser innehåller tre gemensamma faser och att dessa faser uppträder i en bestämd kronologisk ordning. Följaktligen börjar varje deltagares process med en orienteringsfas, där förmågorna att insamla och formalisera matematisk information (O) samt att minnas matematisk information (M) är närvarande. Orienteringsfasen följs direkt av en fas där förmågan att bearbeta den nyligen formaliserade informationen (P) är mest framträdande; under denna fas uppvisar deltagarna kognitiva attribut som till exempel logiskt, systematiskt och sekventiellt tänkande, flexibilitet i tänkandet samt en strävan efter att förenkla och förkorta resonemanget i problemlösningen. Vidare visar analysen att varje problemlösningsprocess avslutas med en ny fas där deltagarna bearbetar matematisk information (P) med syftet att kontrollera de erhållna resultatens riktighet. Utöver dessa tre gemensamma faser interagerade deltagarnas matematiska förmågor enligt mer oregelbundna mönster. Genom att framhäva dessa tre faser av analyserade problemlösningsprocesser, indikeras det i Paper III att deltagarna löser nya matematiska problem enligt modeller för problemlösning som har sina grunder i matematikdidaktisk forskning och som rekommenderas av forskningsfältet.

Det andra området fokuserar det matematiska minnets roll i problemlösningsprocessen. De nämnda empiriska studierna indikerar dels att det matematiska minnet (M) är intimt sammankopplat med förmågan att insamla och formalisera matematisk information (O), dels att M är huvudsakligen närvarande i den inledande fasen av problemlösningsprocessen, när deltagarna väljer problemlösningsmetoder. Analysen visar att det matematiska minnet observerades under minst andel tid jämfört med övriga matematiska förmågor under problemlösningen. Vidare indikerar studierna som redovisas i Paper II och Paper IV, att det matematiska minnet verkar ha en avgörande roll i deltagarnas aktiviteter. Den sistnämnda slutsatsen baseras på iakttagelsen att deltagarna valde problemslösningsmetoder i den inledande fasen av processen samt att deltagarna uppvisade betydande svårigheter när de, på grund av metodernas tillkortakommanden, behövde ändra de initialt valda metoderna. Detta var mest påtagligt under lösningen av P2, när de initialt valda ekvationerna ledde till – enligt deltagarnas utsagor – oväntade olikheter. I dessa situationer upplevde deltagarna betydande nivåer av stress och osäkerhet, vilket ledde till att de återvände till orienteringsfasen för att på nytt formalisera den matematiska informationen (O). Orienteringsfasen följdes återigen av en fas där bearbetningen av den formaliserade matematiska informationen (P) var mest framträdande. Men trots att vissa av deltagarna har gått igenom denna skiftning av faser ett flertal gånger, så uppvisade de betydande svårigheter när de behövde ändra sina initialt valda metoder. I den bemärkelsen indikerar både Paper II och Paper IV att elevernas problemslösningsmetoder var relativt rigida med avseende på ingående procedurer.

Det tredje området relaterar till effektiviteten av deltagarnas problemlöningsaktiviteter. I det avseendet visar artiklarna som rapporterar om den andra fasen av datainsamlingen, att elever som använde allmänna, dvs. algebraiska metoder var mer framgångsrika än elever som valde partikulära lösningsmetoder och som ledde till numeriska lösningar. Vidare visar analysen som presenteras i Paper III, att deltagarna, oavsett vilka metoder de använde under sina respektive problemlösningsaktiviteter, uppvisade beteenden som är karaktäristiska för begåvade elever. Till exempel, under de tidigare nämnda skiftningarna mellan orienteringsfasen – där O och M interagerar – och fasen av bearbetning av den formaliserade informationen (P), demonstrerade deltagarna att de kunde kontrollera och reglera sina kognitiva handlingar med betydande nivåer av förståelse i samband med dessa beslut. Och det har även noterats att deltagarnas ihärdighet och självförtroende, som de uppvisade när de, trots den upplevda stressen och osäkerheten under vissa av de observerade processerna, fortsatte att lösa problemen, är attribut som är typiska för begåvade elever.

Det fjärde området fokuserar förmågan att generalisera matematiska samband (G). Eftersom två av problemen (P1 och P3) skulle lösas i så kallade allmänna fall, deltagarna som presenterade numeriska resultat blev tillfrågade om de kunde utveckla sina numeriska resultat till allmänna lösningar. Under observationerna som ägde rum i samband med P1, kunde ingen av de tre deltagarna (Erin, Larry och Sebastian) som har erhållit numeriska resultat utveckla dessa till allmänna lösningar, följaktligen blev förmågan att generalisera matematiska samband (G) inte observerbar under den andra fasen av datainsamlingen. Ett år senare, under den tredje fasen av datainsamlingen, vid lösningen av P3, presenterade nämnda elever återigen partikulära lösningar och blev på nytt erbjudna möjligheten att utveckla dessa resultat till allmänna lösningar. Två av deltagarna (Larry och Sebastian) kunde inte heller vid detta tillfälle presentera allmänna lösningar, medan Erin, efter att under den kontextuella intervjun har fått ytterligare tillfällen att resonera, utvecklade sina partikulära resultat till en allmän lösning. Alltså blev förmågan att generalisera matematiska samband (G) observerbar under den tredje fasen av datainsamlingen. Dessa iakttagelser kan möjligtvis förklaras med att förmågan att generalisera matematiska samband (G) hos nämnda elever inte var tillräckligt välutvecklad under den andra fasen av datainsamlingen. Och Erins väl genomförda generalisering av resultaten vid P3 – enligt egen utsago var det för första gången som hon har genomfört något liknande, vilket hon var mycket nöjd med – kan tyda på att hon i sin matematikundervisning inte har mött tillräckligt många tillfällen som har motiverat henne till att generalisera partikulära resultat eller till att använda algebraiska metoder för att lösa problem av icke-rutinkaraktär.

De femte området som resultaten kan härledas till fokuserar deltagarnas problemlösning i ett längre tidsperspektiv, dvs. det som var observerbart under den tredje fasen av datainsamlingen. Som det redan nämnts, under den aktuella datainsamlingen, löste deltagarna två problem (P1 och P3) med ungefär ett års mellanrum och problemen, trots att de var av icke-rutinkaraktär för deltagarna, kunde lösas med liknande metoder. Analysen av deltagarnas aktiviteter visar att det var endast två bland de sex deltagarna som relaterade det senare problemet (P3) till förra årets problem (P1). På det sättet motsäger studien som rapporteras i Paper IV delvis indikationen (Krutetskii, 1976) att matematiskt högpresterande elever från gymnasiet minns den generella strukturen av ett givet problem även relativt lång tid efter att de har arbetat med det. Vidare visar analysen – bortsett från att det var endast två deltagare som associerade P3 till P1 – att varje deltagare har löst respektive problem med hjälp av metoder som var likadana på den individuella nivån. Dessutom visade de kontextuella intervjuerna att de applicerade individuella metoderna var sådana som deltagarna kände sig trygga med oavsett om de ledde till problemens korrekta lösning. På det sättet, beroende på tidigare nämnda observationer, dvs. att deltagarna valde problemlösningsmetoder influerade av sina matematiska minnen samt att de hade mycket svårt för att ändra sina initialt valda metoder, bekräftar datainsamlingens tredje fas att det matematiska minnets inverkan på problemlösningsprocessens effektivitet är betydelsefull.

Baserat på studierna som genomfördes under den andra och tredje fasen av datainsamlingen, presenterar denna avhandling även några förslag som avser matematikundervisningen.

En av dessa rekommendationer är att matematiklärare bör uppmuntra elever till att lösa problem med hjälp av allmänna, algebraiska metoder, eftersom dessa metoder är mer effektiva än partikulära metoder. Och när det gäller undervisningens utformning, så är rekommendationen – i likhet med flera andra studiers rekommendation – att elever uppmuntras till att lösa matematiska problem genom att arbeta i mindre grupper i klassrummet, eftersom interaktionen som är förknippad med den arbetsformen har god inverkan på elevers problemlösningsförmåga och utvecklar deras matematiska kunskaper.

En annan inrådan grundas i observationen att deltagarna hade mycket svårt att modifiera eller ersätta sina initialt valda problemlösningsmetoder. Följaktligen, med syftet att uppmuntra elever till att utveckla mer flexibla problemlösningsmetoder, föreslås det att eleverna under sina matematiklektioner tidvis arbetar med problem som inte kan lösas med hjälp av rutinmässiga strategier eller med analoga resonemang.

Ett annat förslag är baserat på att några av deltagarna upplevde betydande svårigheter när de skulle utveckla sina partikulära resultat till allmänna lösningar. Följaktligen – även i det här fallet i överensstämmelse med andra studier – föreslås det att högpresterande elever, åtminstone vid något tillfälle under sin matematikutbildning, arbetar med väl genomtänkta serier av matematiska problem som fokuserar ett allmänt samband eller en typisk egenskap inom ett matematiskt område. På det sättet är det rimligt att anta att dessa elever kommer att identifiera de nödvändiga mönstren i dessa problemserier och därmed utveckla sin förmåga att generalisera matematiska samband.

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