EXACT COMPLETION AND CONSTRUCTIVE THEORIES OF SETS

JACOPO EMMENEGGER AND ERIK PALMGREN

Abstract. In the present paper we use the theory of exact completions to study categorical properties of small setoids in Martin-Löf type theory and, more generally, of models of the Constructive Elementary Theory of the Category of Sets, in terms of properties of their subcategories of choice objects (i.e. objects satisfying the axiom of choice). Because of these intended applications, we deal with categories that lack equalisers and just have weak ones, but whose objects can be regarded as collections of global elements. In this context, we study the internal logic of the categories involved, and employ this analysis to give a sufficient condition for the local cartesian closure of an exact completion. Finally, we apply this result to show when an exact completion produces a model of CETCS.

1. Introduction

Following a tradition initiated by Bishop [7], the constructive notion of set is taken to be a collection of elements together with an equivalence relation on it, seen as the equality of the set. In Martin-Löf type theory this is realised with the notion of setoid, which consists of a type together with a type-theoretic equivalence relation on it [22]. An ancestor of this construction can be found in Gandy's interpretation of the extensional theory of simple types into the intensional one [13]. A category-theoretic counterpart is provided by the exact completion construction $\mathbb{C}_{\mathrm{ex}}$, which freely adds quotients of equivalence relations to a category $\mathbb{C}$ with (weak) finite limits [9, 11]. As shown by Robinson and Rosolini, and further clarified by Carboni, the effective topos can be obtained using this construction [23, 8]. The authors of [6] then advocated the use of exact completions as an abstract framework where to study properties of categories of partial equivalence relations, which are widely used in the semantics of programming languages. For these reasons, this construction has been extensively studied and has a robust theory [14, 10, 3, 4], at least when $\mathbb{C}$ has finite limits, whereas its behaviour is less understood when $\mathbb{C}$ is only assumed to have weak finite limits.

The relevance of the latter case comes from the fact that setoids in Martin-Löf type theory arise as the exact completion of the category of closed types, which does have finite products but only weak equalisers (what we will call a quasi-cartesian category), meaning

Department of mathematics, Stockholm University. SE-106 91 Stockholm, Sweden.

E-mail addresses: emmenegger@math.su.se, palmgren@math.su.se.

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that a universal arrow exists but not necessarily uniquely. However, this category of types has some other features: it validates the axiom of choice and it has a proof-relevant internal logic with a strong existential quantifier. These features have been investigated by the second author in [19], where this internal logic is called categorical BHK-interpretation.

More generally, the same situation arises for any model of the Constructive Elementary Theory of the Category of Sets (CETCS), a first order theory introduced by the second author in [20] in order to formalise properties of the category of sets in the informal set theory used by Bishop. In fact, this theory provides a finite axiomatisation of the theory of well-pointed locally cartesian closed pretoposes with enough projectives and a natural numbers object. Therefore, any model $\mathbb{E}$ of CETCS is the exact completion of its projective objects, which form a quasi-cartesian category $\mathbb{P}$. As for closed types in Martin-Löf type theory, these are objects satisfying a categorical version of the axiom of choice, and the internal logic of $\mathbb{E}$ on the projectives is (isomorphic to) the categorical BHK-interpretation of intuitionistic first order logic in $\mathbb{P}$.

The aim of the present paper is to isolate certain properties of a quasi-cartesian category $\mathbb{C}$ that will ensure that its exact completion is a model of CETCS while, at the same time, making sure that these properties are satisfied by the category of closed types in Martin-Löf type theory. In fact, for some of the properties defining a model $\mathbb{E}$ of CETCS, an equivalent formulation in terms of projectives of $\mathbb{E}$ is already known, as in the case for pretoposes [14], or follows easily from known results, as for natural numbers objects [8, 6]. However, in the general case of weak finite limits (or just quasi-cartesian categories), a complete characterisation of local cartesian closure in terms of a property of the projectives is still missing.

The first contribution of this paper consists of a condition on a category which is sufficient for the local cartesian closure of its exact completion. This condition is a categorical formulation of Aczel's Fullness Axiom from Constructive Zermelo-Fraenkel set theory (CZF) [1, 2], and it is satisfied by the category of closed types. A complete characterisation of local cartesian closure for an exact completion $\mathbb{C}_{\mathrm{ex}}$ is given by Carboni and Rosolini in [10], but it has been recently discovered that the argument used requires finite limits in $\mathbb{C}$ [12]. Another sufficient condition, which applies to those exact completions arising from certain homotopy categories, has been recently given by van den Berg and Moerdijk [5]. We formulate our notion of Fullness, and the proof of local cartesian closure, in the context of well-pointed quasi-cartesian categories in order to match some aspects of set theory, like extensionality. However, a suitably generalised version of our formulation of Fullness in fact reduces to Carboni and Rosolini's characterisation in the presence of finite limits, and is tightly related to van den Berg and Moerdijk's condition as well [12].

In CZF minus Subset Collection, the Fullness Axiom is equivalent to Subset Collection. Hence it is instrumental in the construction of Dedekind real numbers in CZF and it implies Exponentiation [2]. It states the existence of a full set F of total relations (i.e. multi-valued functions) from a set A to a set B, where a set F is full if every total relation from A to B has a subrelation in F, i.e. if $F \subseteq \mathrm{TR}(A, B)$ and

$$\forall R \in \mathrm{TR}(A, B)\ \exists S \in F\ S \subseteq R,$$

where $\mathrm{TR}(A, B) := \{ R \subseteq A \times B \mid \forall a \in A\, \exists b \in B\ (a, b) \in R \}$ is the class of total relations from A to B. Since functional relations are minimal among total relations, a full set must contain all graphs of functions, however it is not a (weak) exponential as it may also contain non-functional relations. We will use a characterisation of local cartesian closure in terms of closure under families of partial functional relations as in [20] and, similarly, we will formulate a version of the Fullness Axiom in terms of families of partial pseudo-relations (i.e. non-monic relations). The key aspect of the proof is the very general universal property of a full set (or of a full family of partial pseudo-relations), which endows the internal (proof-relevant) logic with implication and universal quantification.

The second contribution of the paper is a complete characterisation of well-pointed exact completions in terms of their projectives. We relate well-pointedness, which amounts to extensionality with respect to global elements, with certain choice principles, namely versions of the axiom of unique choice in $\mathbb{C}_{\mathrm{ex}}$ and the axiom of choice in $\mathbb{C}$. We also exploit this correspondence to simplify the internal logic of the categories under consideration, and the exact completion construction itself. In the related context of quotient completions of elementary doctrines, an analogous result relating choice principles is obtained by Maietti and Rosolini in [16].

The paper is understood as being formulated in an essentially algebraic theory for category theory over intuitionistic first order logic, as the one presented in [20]. However, we believe that all the results herein can be formalised in intensional Martin-Löf type theory using E-categories [22], and this is indeed the case for those regarding the category of setoids. A step towards this goal is made in [21], where CETCS is formulated in a dependently typed first-order logic, which can be straightforwardly interpreted in Martin-Löf type theory.

The paper is organised as follows. Section 2 is an overview of already known facts. Here we recall the category-theoretic concepts needed to illustrate the exact completion construction, define the categories of small setoids and small types in Martin-Löf type theory, which will be the main intended examples throughout the paper, and relate the setoid construction to the exact completion of small types.

In Section 3 we consider the concept of elemental category, which is needed to formulate the constructive version of well-pointedness satisfied by models of CETCS, and which allows to regard objects as collections of (global) elements. Indeed, in abstract categorical terminology, it amounts to say that the global section functor is conservative, however we avoid this formulation since it refers to the category of sets, and prefer an elementary definition instead. The main result of this section is a characterisation of elemental exact completions as those arising from categories satisfying a version of the axiom of choice.

Section 4 contains the main result of the paper, namely our categorical formulation of Aczel's Fullness Axiom and the proof that it implies the local cartesian closure of the exact completion. In this section we fully exploit the simplifications of the internal logic and the exact completion construction given by elementality, as well as the proof relevance of the internal logic given by the BHK-interpretation.

Finally, in Section 5 we recall the axioms of CETCS from [20] and discuss how its models are exact completions of their choice objects (i.e. projective objects). We then use the results from the previous sections, and already known ones, to show when an exact completion produces a model of CETCS.

2. Exact and quasi-cartesian categories

An *equivalence relation* in a category $\mathbb{C}$ with finite limits is a subobject $r\colon R \hookrightarrow X \times X$ such that there are (necessarily unique) arrows witnessing reflexivity, symmetry and transitivity as in the following diagrams

$$
(1)\qquad
\begin{array}{ccc}
 & & R \\
 & \overset{\rho}{\nearrow} & \downarrow r \\
X & \xrightarrow[\Delta_X]{} & X \times X
\end{array}
\qquad
\begin{array}{ccc}
 & & R \\
 & \overset{\sigma}{\nearrow} & \downarrow r \\
R & \xrightarrow[\langle r_2, r_1\rangle]{} & X \times X
\end{array}
\qquad
\begin{array}{ccc}
 & & R \\
 & \overset{\tau}{\nearrow} & \downarrow r \\
R \times_X R & \xrightarrow[\langle r_1 p_1, r_2 p_2\rangle]{} & X \times X
\end{array}
$$

where $R \xleftarrow{p_1} R \times_X R \xrightarrow{p_2} R$ is a pullback of $R \xrightarrow{r_2} X \xleftarrow{r_1} R$. Subobjects obtained by pulling back an arrow along itself are always equivalence relations, these are called *kernel pairs*. A diagram of the form $R \rightrightarrows X \to Y$ is *exact* if it is a coequaliser diagram and $R \rightrightarrows X$ is the kernel pair of $X \to Y$. In such a situation, the arrow $X \to Y$ is called *quotient* of the equivalence relation $R \hookrightarrow X \times X$.

Definition 2.1. A category is *exact* if it has finite limits, and pullback-stable quotients of equivalence relations. An exact category is a *pretopos* if it has disjoint and pullback-stable finite sums, and the initial object is strict.

Example 2.2. Let **ML** be Martin-Löf type theory with rules for $\sum$-types, $\prod$-types, identity types $=_X$, sum types $+$, natural numbers $\mathbb{N}$, finite sets $\mathbb{N}_k$ and a universe $(\mathsf{U}, T(\cdot))$ closed under the previous type formers. For simplicity, we will leave the decoding type constructor $T(\cdot)$ implicit. Proposition 7.1 in [18] proves that the E-category of setoids Std in Martin-Löf type theory is a pretopos in **ML**. Since this is the motivating example for this paper, we recall here its construction.

An E-category is a formulation of category in Martin-Löf type theory that avoids equality on objects: its objects are given by a type, while the arrows between two objects form a setoid, i.e. a type equipped with an equivalence relation which is understood as the equality between arrows. For more details on E-categories and categories in Martin-Löf type theory we refer to [22].

Objects of Std are *small setoids*, that is, pairs $X \coloneqq (X_0, X_1)$ such that

$$X_0 : \mathsf{U} \quad \text{and} \quad X_1 : X_0 \to X_0 \to \mathsf{U},$$

and X_1 is an equivalence relation (i.e. it has proof-terms for reflexivity, symmetry and transitivity). We will write $X_1(x, x')$ as $x \sim_X x'$ and omit its proof-terms.

The type of arrows $X \to Y$ consists of *extensional functions*, that is, function terms $f : X_0 \to Y_0$ together with a closed term of type

$$\prod_{x, x' : X_0} (x \sim_X x' \to f(x) \sim_Y f(x')),$$

and two such arrows $f, g \colon X \to Y$ are equal if there is a closed term

$$h : \prod_{x : X_0} f(x) \sim_Y g(x).$$

Identity arrows and composition are defined in the obvious way using application and λ-abstraction.

Remark 2.3. It can be shown that in Std quotients are just surjective functions (cf. Section 3), i.e. extensional functions $f \colon X \to Y$ such that

$$\prod_{y : Y_0} \sum_{x : X_0} f(x) \sim_Y y$$

is inhabited in the empty context. The type-theoretic axiom of choice then yields $s : Y_0 \to X_0$ such that $fs(y) \sim_Y y$ for every $y : Y_0$ but, contrary to what happens in a category of sets in (a model of) ZFC, this is not a section of f in Std, since it is not necessarily extensional. For a discussion regarding the relation between the type-theoretic axiom of choice and setoids we refer to [17].

However, since the identity type $=_{Y_0}$ is the minimal reflexive relation on Y_0, function terms with domain a setoid of the form $(Y_0, =_{Y_0})$ are automatically extensional. Hence, in the above case, the function term s gives rise to a section of f in Std as soon as the equivalence relation on Y is given by the identity type.

Moreover, arrows of the form $(Y_0, =_{Y_0}) \to (Y_0, \sim_Y)$ whose underlying function term is the identity are trivially surjective, hence every setoid is the surjective image of a setoid for which the axiom of choice holds. This principle is know as the Presentation Axiom [1, 2].

The situation described in the previous remark is captured, in abstract category theory, by the notion of having enough projectives. Recall that an arrow f is a *cover* if, whenever it factors as $f = gh$ with g monic, then g is in fact an iso, and that an object P is *projective* if, for every cover $X \to Y$ and every arrow $P \to Y$, there is $P \to X$ such that the obvious triangle commutes. In an exact category, covers coincide with quotients.

Definition 2.4. A *projective cover* of an object $X \in \mathbb{C}$ is given by a projective object P and a cover $P \to X$. A *projective cover* of $\mathbb{C}$ is a full subcategory $\mathbb{P}$ of projective objects such that every $X \in \mathbb{C}$ has a projective cover $P \in \mathbb{P}$. $\mathbb{C}$ has *enough projectives* if it has a projective cover.

Projective covers are not necessarily closed under limits that may exist in $\mathbb{C}$. However they do have a weak limit of every diagram that has a limit in $\mathbb{C}$ [11], where a weak limit is defined in the same way as a limit but dropping uniqueness of the universal arrow. Indeed, if $L \in \mathbb{C}$ is a limit in $\mathbb{C}$ of a diagram $\mathcal{D}$ in $\mathbb{P}$, then any projective cover $P \in \mathbb{P}$ of L is a weak limit of $\mathcal{D}$ in $\mathbb{P}$: given any cone over $\mathcal{D}$ with vertex $Q \in \mathbb{P}$, the weak universal $Q \to P$ is obtained lifting the universal $Q \to L$ along the cover $P \to L$ using projectivity of Q.

Nevertheless, in the rest of the paper we shall be interested in subcategories of projectives which are closed under finite products, so we introduce the following definition.

Definition 2.5. A category is *quasi-cartesian* if it has finite products and weak equalisers.

Remark 2.6. As for the case of limits, a category with finite products has all weak finite limits if and only if it has weak equalisers if and only if it has weak pullbacks.

Remark 2.7. Quasi-cartesian categories come naturally equipped with a proof-relevant internal logic. This interpretation has been investigated by the second author in [19], where it is called *categorical BHK-interpretation*, due to its similarities to the propositions-as-types correspondence. Since this internal logic will be one of the main tool in the proof of our main result, we briefly review it here.

Recall that, given two arrows $f\colon Y \to X$ and $g\colon Z \to X$, $f \leq g$ means that there is $h\colon Y \to Z$ such that $gh = f$. This defines a preorder on $\mathbb{C}/X$, we denote by $\mathrm{Psub}_{\mathbb{C}}(X)$ its order reflection and call its elements *presubobjects* (these are called variations or weak subobjects in [15]). Presubobjects are used for the interpretation of predicates. Since weak limits are unique up to presubobject equivalence, weak pullbacks can be used to interpret weakening and substitution. For the same reason, we can interpret equality with weak equalisers and conjunction with weak pullbacks, while postcomposition provides an interpretation for the existential quantifier. Hence regular logic has a sound interpretation into any quasi-cartesian category.

Example 2.8. Remark 2.3 shows that the full subcategory of setoids of the form $(X_0, =_{X_0})$ is a projective cover of Std. In fact, it can be seen as the embedding in Std of another E-category in **ML**, namely the E-category of small types Type. Its type of objects is the universe U, the type of arrows from X_0 to Y_0 is the function type $X_0 \to Y_0$, and two arrows $f, g\colon X_0 \to Y_0$ are equal if there is a closed term

$$h : \prod_{x:X_0} f(x) =_{Y_0} g(x).$$

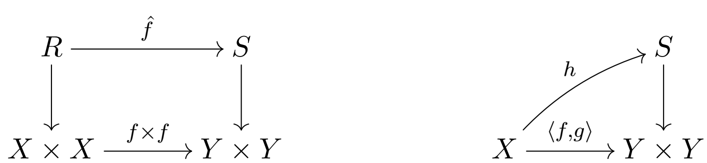
A product of two objects $X_0, Y_0 : \mathsf{U}$ is given by the dependent sum type $\sum_{X_0} Y_0 : \mathsf{U}$, which is written $X_0 \times Y_0$ when Y_0 does not depend on X_0, and a weak equaliser of two arrows $f, g\colon X_0 \to Y_0$ is given by the type $\sum_{x:X_0}(f(x) =_{Y_0} g(x)) : \mathsf{U}$ together with the first projection into X_0.

Hence Type is quasi-cartesian, and the embedding $X_0 \mapsto (X_0, =_{X_0})$ preserves all finite products, since the identity type $(x, y) =_{X_0 \times Y_0} (x', y')$ is (type-theoretically) equivalent to the type $(x =_{X_0} x') \times (y =_{Y_0} y')$.

Remark 2.9. Type has not arbitrary finite limits, since their existence would imply the derivability of Uniqueness of Identity Proofs (UIP) in **ML** for all small types. Indeed, given a small type $X_0 : \mathsf{U}$ in **ML**, the existence of an equaliser for every pair $x, x'\colon \mathbf{1} \to X_0$ would yield a mere equivalence relation $E : X_0 \to X_0 \to \mathsf{U}$ (i.e. an equivalence relation such that $u =_{E(x,x')} v$ for every $u, v : E(x, x')$ and $x, x' : X_0$) together with a function term $f : \Pi_{x,x':X} E(x, x') \to x =_X x'$. Theorem 7.2.2 in [24] would then imply $\mathrm{UIP}(X_0)$.

The embedding of Type into Std corresponds in categorical terms to a construction called *exact completion*, that embeds every category with weak finite limits into an exact category. This construction is due to Carboni and Vitale [11] and we will describe it in the case of a quasi-cartesian category $\mathbb{C}$.

Objects of the *exact completion* $\mathbb{C}_{\mathrm{ex}}$ are pseudo-equivalence relations in $\mathbb{C}$, that is arrows $r\colon R \to X \times X$ such that there are (not necessarily unique) arrows for reflexivity, symmetry and transitivity as in (1), where now the domain of τ is just a weak pullback of r_1 and r_2. Arrows $(R \to X \times X) \to (S \to Y \times Y)$ in $\mathbb{C}_{\mathrm{ex}}$ are equivalence classes $[f]$ of arrows $f\colon X \to Y$ in $\mathbb{C}$ such that there is an arrow $\hat{f}\colon R \to S$ making the left-hand diagram below commute, and where $f, g\colon X \to Y$ are equivalent if there is $h\colon X \to S$ making the right-hand diagram below commute.



The functor $\Gamma\colon \mathbb{C} \to \mathbb{C}_{\mathrm{ex}}$ mapping an object $X \in \mathbb{C}$ to the diagonal on X is full and faithful and preserves all the finite limits which exist in $\mathbb{C}$. The image of this embedding is a projective cover of $\mathbb{C}_{\mathrm{ex}}$, and every exact category with enough projectives is in fact an exact completion. In the case of quasi-cartesian categories, this characterisation assumes the following form.

Theorem 2.10 ([11]). *Every exact category with enough projectives which are closed under finite products is the exact completion of a quasi-cartesian category, namely its subcategory of projectives. Conversely, every quasi-cartesian category appears as a projective cover, closed under finite products, of its exact completion.*

Example 2.11. It follows from the fact that Std is a pretopos and the observation in Remark 2.3 that Type is a projective cover of it, that Std can be seen as the exact completion of Type as a quasi-cartesian category.

Remark 2.12. The isomoprhism of posets

$$\mathrm{Sub}_{\mathbb{C}_{\mathrm{ex}}}(\Gamma X) \cong \mathrm{Psub}_{\mathbb{C}}(X),$$

which follows from the theory of exact completions [11], commutes with the regular-logic structure on both posets. Hence it guarantees that the internal logic of $\mathbb{E}$ on a projective is still the BHK-interpretation in $\mathbb{C}$ of intuitionistic logic.

3. Elemental categories

An object G in a category $\mathbb{C}$ is called a *strong generator* if an arrow $f\colon X \to Y$ is an iso whenever

$$(\forall y\colon G \to Y)(\exists! x\colon G \to X)\, fx = y.$$

An object G is *separating* if for any pair of arrows $f, g\colon X \to Y$, $f = g$ whenever

$$(\forall x\colon G \to X)\, fx = gx.$$

Definition 3.1. A category $\mathbb{C}$ with a terminal object is *elemental* if the terminal object is a strong generator and is separating

The terminology comes from the fact that objects in elemental categories can be regarded, to a certain extent, as collections of global elements. In particular, this simplifies the internal logic of an elemental category, as shown in Propositions 3.7 and 3.8 and Corollary 3.9.

We will denote global elements $x\colon \mathbf{1} \to X$ as $x \in X$ and simply call them elements. Moreover, if $f\colon X \to Y$ and $y \in Y$, we will write $y \mathrel{\epsilon} f$ if there is $x \in X$ such that $fx = y$. An arrow $f\colon X \to Y$ will be called *injective* if $(\forall x, x' \in X)(fx = fx' \implies x = x')$, while it will be called *surjective* if $(\forall y \in Y)\, y \mathrel{\epsilon} f$. Notice that the terminal object is a strong generator if and only if an arrow is iso exactly when it is injective and surjective. Finally, we will say that an object Y in $\mathbb{C}$ is a *choice object* if every surjection $f\colon X \to Y$ has a section, i.e. an arrow $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$.

Example 3.2. Sets in (a model of) ZFC form a category in which every one-element set is both separating and a strong generator. Moreover, the Axiom of Choice implies that every object is a choice object (i.e. every surjective function has a section).

Example 3.3. Because of the type-theoretic axiom of choice, all objects in Type are choice objects, (as well as all setoids $(X_0, =_{X_0})$ in Std). Lemma 3.5 and Theorem 3.12 then imply that, respectively, Type and Std are elemental.

In the following lemma we collect some immediate results.

Lemma 3.4. *Let $\mathbb{C}$ be a category with a terminal object.*

(i) *If the terminal object is separating, then every surjection is epic and every injection is monic.*

(ii) *The terminal object is projective if and only if every cover is surjective.*
(iii) *If the terminal object is a strong generator, then every surjection is a cover. The converse holds if every injection is monic.*

In the presence of weak equalisers, we can derive elementality from a categorical choice principle.

Lemma 3.5. *Let $\mathbb{C}$ be a quasi-cartesian category. If every object is a choice object, then $\mathbb{C}$ is elemental.*

Proof. Since every surjection has a section, it follows that every surjection is a cover. Therefore it is enough to show that the terminal object is separating, since elementality will follow from 3.4*(iii)*. Let $f, g\colon X \to Y$ be such that $fx = gx$ for every $x \in X$, and let $e\colon E \to X$ be a weak equaliser for f and g. Since f and g coincide on elements, e is surjective, hence it has a section $s\colon X \to E$. Therefore $f = fes = ges = g$ as required. □

Since an equaliser is the same as a monic weak equaliser, with a similar argument we can also prove the following.

Lemma 3.6. *Let $\mathbb{C}$ be a category with finite limits. Then $\mathbb{C}$ is elemental if and only if every surjection is a cover.*

The following result proves that extensionality of presubobjects is equivalent to a categorical choice principle, in every quasi-cartesian category.

Proposition 3.7. *Let $\mathbb{C}$ be a quasi-cartesian category and consider the following.*
(i) *Every surjection has a section.*
(ii) *For every object X and arrows a, b with codomain X,*

$$a \leq b \qquad \textit{if and only if} \qquad (\forall x \in X)(x \,\epsilon\, a \Longrightarrow x \,\epsilon\, b).$$

(iii) *For every pseudo-relation $r\colon R \to X \times Y$*

$$(\forall x \in X)(\exists y \in Y)\,\langle x, y\rangle \,\epsilon\, r \implies (\exists f\colon X \to Y)(\forall x \in X)\,\langle x, fx\rangle \,\epsilon\, r.$$

Statements (i) and (ii) are equivalent and imply statement (iii). If the terminal object is separating, then they are equivalent.

Proof. *(i) ⇒ (ii)* The direction from left to right always holds, so let us assume that $(\forall x \in X)(x \,\epsilon\, a \Longrightarrow x \,\epsilon\, b)$ and observe that it amounts to the surjectivity of any weak pullback of b along a. Hence there is a section of it which, in turn, yields an arrow witnessing $a \leq b$.

(ii) ⇒ (i) This follows from the fact that an arrow $f\colon X \to Y$ is surjective precisely when $(\forall y \in Y)(y \,\epsilon\, id_Y \Longrightarrow y \,\epsilon\, f)$ and that any arrow witnessing $id_Y \leq f$ is a section of f.

(i) ⇒ (iii) Immediate from the fact that $(\forall x \in X)(\exists y \in Y)\,\langle x, y\rangle \,\epsilon\, r$ amounts to surjectivity of $r_1\colon R \to X$.

(iii) ⇒ (ii) As before, we only need to show the direction from right to left. Let $r\colon R \to A \times B$ be given as a weak pullback of a and b and observe that $(\forall x \in X)(x \,\epsilon\, a \Longrightarrow x \,\epsilon\, b)$ implies $(\forall u \in A)(\exists v \in B)\,\langle u, v\rangle \,\epsilon\, r$. Therefore we obtain an arrow $f\colon A \to B$ such that $bfu = au$ for every $u \in A$. If the terminal object is separating, this implies $a \leq b$ as required. □

In the presence of finite limits, an analogous equivalence holds between elementality (which can be identified with 3.8*(i)*, thanks to Lemma 3.6), an extensionality principle for subobjects (3.8*(ii)*), and a form of unique choice (3.8*(iii)*). This equivalence generalises Propositions 4.3 and 4.4 in [20].

Proposition 3.8. *Let $\mathbb{C}$ be a category with finite limits, and consider the following.*

(i) *Every surjective mono has a section.*
(ii) *For every object X and monos a, b with codomain X,*

$$a \leq b \qquad \text{if and only if} \qquad (\forall x \in X)(x \,\epsilon\, a \Longrightarrow x \,\epsilon\, b).$$

(iii) *For every relation $r\colon R \hookrightarrow X \times Y$*

$$(\forall x \in X)(\exists! y \in Y)\, \langle x, y\rangle \,\epsilon\, r \implies (\exists f\colon X \to Y)(\forall x \in X)\, \langle x, fx\rangle \,\epsilon\, r.$$

Statements (i) and (ii) are equivalent, and imply statement (iii). If the terminal object is separating, then they are equivalent.

An immediate consequence of Proposition 3.7 is that the internal logic of an elemental quasi-cartesian category is determined, up to presubobject equivalence, by global elements. Here we distinguish internal connectives and quantifiers by adding a dot on top of them and, to increase readability, we commit the common abuse of dealing with representatives instead of actual presubobjects. A similar result for the usual categorical interpretation of logic is in Theorem 5.6 in [20], which can be seen as a consequence of Proposition 3.8.

Corollary 3.9. *Let be a quasi-cartesian category where every object is a choice object, and let $a, b \in \mathrm{Psub}_{\mathbb{C}}(X)$, $f, g\colon Y \to X$ and $r \in \mathrm{Psub}_{\mathbb{C}}(X \times Y)$, then:*

(i) $y \,\epsilon\, f^{-1}a$ *if and only if* $fy \,\epsilon\, a$,
(ii) $y \,\epsilon\, (f \doteq g)$ *if and only if* $fy = gy$,
(iii) $x \,\epsilon\, (a \mathbin{\dot\wedge} b)$ *if and only if* $x \,\epsilon\, a \wedge x \,\epsilon\, b$,
(iv) $x \,\epsilon\, \dot\exists_Y r$ *if and only if* $(\exists y \in Y)\, \langle x, y\rangle \,\epsilon\, r$,

and the presubobjects obtained by $f^{-1}, \doteq, \dot\wedge$ and $\dot\exists$ are uniquely determined by the universal closure of the previous relations.

Proof. We prove the statement for $\dot\wedge$, the other proofs are similar. Of course if $c\colon C \to X$ is a representative of $a \mathbin{\dot\wedge} b$, then the equivalence in point 3 must hold for every $x \in X$. Conversely, suppose that $(\forall x \in X)(x \,\epsilon\, c \Longleftrightarrow x \,\epsilon\, a \wedge x \,\epsilon\, b)$ and let $p\colon P \to X$ be a representative of $a \mathbin{\dot\wedge} b$ (e.g. a weak pullback of a and b). Then $x \,\epsilon\, c \Longleftrightarrow x \,\epsilon\, p$ for every $x \in X$, so 3.7.*(ii)* implies that c is also a representative of $a \mathbin{\dot\wedge} b$. □

Proposition 3.7 also allows for a simpler construction of the exact completion of a quasi-cartesian category, when every object is a choice object. Denote with $\sim_r$ the relation induced on the elements of X by a pseudo-relation $r\colon R \to X \times X$, i.e.

$$x \sim_r x' \iff \langle x, x'\rangle \,\epsilon\, r.$$

Corollary 3.10. *Let $\mathbb{C}$ be a quasi-cartesian category where every object is a choice object. Then for every pseudo-relation $r\colon R \to X \times X$:*

(i) *r is reflexive if and only if $\sim_r$ is reflexive,*
(ii) *r is symmetric if and only if $\sim_r$ is symmetric,*
(iii) *r is transitive if and only if $\sim_r$ is transitive.*

Proof. We prove point 3, the others being easier. One direction is straightforward, so let us assume transitivity of $\sim_r$, i.e.

$$(\forall x, x', x'' \in X)(x \sim_r x' \wedge x' \sim_r x'' \Longrightarrow x \sim_r x''). \tag{2}$$

Consider the following weak pullback

$$\begin{array}{ccc} P & \xrightarrow{p_2} & R \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow{r_2} & X \end{array}$$

and define $p \coloneqq \langle r_1 p_1, r_2 p_2 \rangle \colon P \to X \times X$. Transitivity of r amounts to show that $p \leq r$, and thanks to 3.7.*(ii)* it is enough to show that $\langle x, x'' \rangle \,\epsilon\, p \Longrightarrow \langle x, x'' \rangle \,\epsilon\, r$. But $\langle x, x'' \rangle \,\epsilon\, p$ implies that there is $x' \in X$ such that $x \sim_r x'$ and $x' \sim_r x''$, hence $\langle x, x'' \rangle \,\epsilon\, r$ from (2). ☐

Corollary 3.11. *Let $\mathbb{C}$ be a quasi-cartesian category where every object is a choice object and let $r \colon R \to X \times X$ and $s \colon S \to Y \times Y$ be two pseudo-equivalence relations.*

(i) An arrow $f \colon X \to Y$ is also an arrow $r \to s$ in $\mathbb{C}_{ex}$ if and only if,

$$(\forall x, x' \in X)(x \sim_r x' \implies fx \sim_s fx').$$

(ii) Two arrows $f, g \colon r \to s$ in $\mathbb{C}_{ex}$ are equal in $\mathbb{C}_{ex}$ if and only if,

$$(\forall x \in X)\, fx \sim_s gx.$$

Finally, we can prove that elemental exact completions are precisely those exact categories with a projective cover consisting of choice objects. In light of the equivalences in Propositions 3.7 and 3.8, this result should be compared with the equivalence, proved in [16], between elementary doctrines satisfying the Axiom of Choice and elementary quotient completions satisfying the Axiom of Unique Choice.

Theorem 3.12. *Let $\mathbb{C}$ be a quasi-cartesian category. Then $\mathbb{C}_{ex}$ is elemental if and only if every object in $\mathbb{C}$ is a choice object.*

Proof. Let us first prove that, if $\mathbb{C}_{\text{ex}}$ is elemental, then every surjection in $\mathbb{C}$ splits. Observe that, if $f \colon X \to Y$ is surjective in $\mathbb{C}$, then $f \colon \Delta_X \to \Delta_Y$ is surjective in $\mathbb{C}_{\text{ex}}$, hence a cover because of elementality. But Δ_Y is projective in $\mathbb{C}_{\text{ex}}$, therefore we get a section $g \colon \Delta_Y \to \Delta_X$ which is a section of f in $\mathbb{C}$ as well.

In order to prove the other implication, thanks to Lemma 3.4 it is enough to show that in $\mathbb{C}_{\text{ex}}$ every monic surjection has a section. To this aim, let $r \colon R \to X \times X$ and $s \colon S \to Y \times Y$ be two pseudo-equivalence relations in $\mathbb{C}$ and let $f \colon X \to Y$ be such that $x \sim_r x' \Longrightarrow fx \sim_s fx'$ for every $x, x' \in X$. Assume that f is monic and surjective in $\mathbb{C}_{\text{ex}}$. In particular

$$(\forall y \in Y)(\exists x \in X)\, \langle fx, y \rangle \,\epsilon\, s,$$

hence 3.7.*(iii)* yields $g \colon Y \to X$ such that $\langle fgy, y \rangle \,\epsilon\, s$ for all $y \in Y$. If $y \sim_s y'$, then $fgy \sim_s fgy'$ and injectivity of f implies $gy \sim_r gy'$, so g is an arrow $s \to r$ from 3.11.*(i)*, and a section of f in $\mathbb{C}_{\text{ex}}$ from 3.11.*(ii)*. ☐

4. Fullness and exponentiation

This section contains the main contribution of the paper, which provides a sufficient condition on the choice objects of an elemental exact completion that ensures the local cartesian closure of the latter.

We begin recalling a characterisation, for elemental categories, of local cartesian closure in terms of closure under families of partial functional relations [20].

Definition 4.1. Let $Y \xrightarrow{g} X \xrightarrow{f} I$ be two arrows in a category $\mathbb{C}$ with finite products. A pair $h\colon J \to I$, $r\colon R \to J \times X \times Y$

(a) is a *family of partial sections of* g if for every $j \in J$, $x \in X$ and $y \in Y$,

$$\langle j, x, y\rangle \,\epsilon\, r \implies gy = x,$$

(b) has *domains indexed by* f if for every $j \in J$ and $x \in X$

$$fx = hj \iff (\exists y \in Y)\, \langle j, x, y\rangle \,\epsilon\, r,$$

(c) is *functional* if for every $j \in J$, $x, x' \in X$ and $y, y' \in Y$

$$\langle j, x, y\rangle \,\epsilon\, r \wedge \langle j, x', y'\rangle \,\epsilon\, r \wedge x = x' \implies y = y'.$$

Definition 4.2. Let $Y \xrightarrow{g} X \xrightarrow{f} I$ be two arrows in a category $\mathbb{C}$ with finite limits. A pair $h\colon J \to I$ and $r\colon R \hookrightarrow J \times X \times Y$ is a *family of functional relations over* f, g if it satisfies properties (a)–(c) from Definition 4.1. If J is terminal, then $h \in I$ will be called the *domain index* of r and $\langle r_2, r_3\rangle\colon R \hookrightarrow X \times Y$ will be called a *functional relation*.

A *universal dependent product for* f, g is a family of functional relations $\phi\colon F \to I$ and $\alpha\colon P \hookrightarrow F \times X \times Y$ over f, g such that, for every functional relation $r\colon R \hookrightarrow X \times Y$ over f, g with domain index $i \in I$, there is a unique $c \in F$ such that $\phi c = i$ and for all $x \in X$ and $y \in Y$

$$\langle c, x, y\rangle \,\epsilon\, \alpha \iff \langle x, y\rangle \,\epsilon\, r. \tag{3}$$

Remark 4.3. Intuitively, the property defining a universal depend product amounts to say that the arrow ϕ contains a code c for ever functional relation over f, g. In an elemental category with finite limits, functional relations coincide with arrows (see Proposition 3.8) and Theorem 6.8 in [20] proves that, in such a category, having all universal dependent products is equivalent to local cartesian closure.

Considering pseudo-relations instead of relations, and dropping functionality we obtain the following version of Aczel's notion of full set.

Definition 4.4. Let $Y \xrightarrow{g} X \xrightarrow{f} I$ be arrows in a quasi-cartesian category. A pair $h\colon J \to I$, $r\colon R \to J \times X \times Y$ is a *family of pseudo-relations over* f, g if it satisfies properties (a) and (b) from Definition 4.1. If J is terminal, then $h \in J$ will be called the *domain index* of r, and $\langle r_2, r_3\rangle\colon R \to X \times Y$ will just be called a pseudo-relation over f, g.

A *full family of pseudo-relation over* f, g is a family of pseudo-relations $\phi\colon F \to I$ and $\alpha\colon P \to F \times X \times Y$ over f, g such that, for every pseudo-relation $r\colon R \to X \times Y$ over f, g with domain index $i \in I$, there is $c \in F$ such that $\phi c = i$ and for all $x \in X$ and $y \in Y$

$$\langle c, x, y\rangle \,\epsilon\, \alpha \implies \langle x, y\rangle \,\epsilon\, r. \tag{4}$$

A quasi-cartesian category is *closed for pseudo-relations* if it has a full family of pseudo-relations for any pair of composable arrows.

Notice that in (4) only one direction of the implication is required, as opposed to the bi-implication in (3). This is to mimic the behaviour of a full set in CZF as defined by Aczel (see [1] pg. 58 or [2]): a (total) relation is not necessarily an element of a full set F, but it contains as subrelation an element of F.

Example 4.5. Proposition 4.10 below proves that the E-category of types is closed for pseudo-relations.

The next two results show that closure for pseudo-relations endows the internal logic of a quasi-cartesian category with implication and universal quantification.

Lemma 4.6. *Let $\mathbb{C}$ be a quasi-cartesian category where every object is a choice object. If $\mathbb{C}$ is closed for pseudo-relations, then for every $f\colon X \to I$ there is a right adjoint to $f^{-1}\colon \mathrm{Psub}_{\mathbb{C}}(I) \to \mathrm{Psub}_{\mathbb{C}}(X)$.*

Proof. As for weak pullbacks, it is easy to see that full families are unique up to presubobject equivalence. This defines an order-preserving function $\forall_f\colon \mathrm{Psub}_{\mathbb{C}}(X) \to \mathrm{Psub}_{\mathbb{C}}(I)$. Given an arrow $g\colon Y \to X$, let $\phi\colon F \to I$ and $\alpha\colon P \to F \times X \times Y$ be a full family of pseudo-relations over f, g. We need to show that $h \leq \phi \iff f^{-1}h \leq g$ for every $h\colon Z \to I$, and we will make use of the statement in 3.7.*(ii)* in doing so.

Assume $h \leq \phi$. We have that $x \,\epsilon\, f^{-1}h$ implies $fx \,\epsilon\, h \leq \phi$, so there is $c \in F$ such that $\phi c = fx$ and, from 4.2.(b), we obtain $y \in Y$ such that $\langle c, x, y\rangle \,\epsilon\, \alpha$. In particular, $gy = x$, i.e. $x \,\epsilon\, g$.

Suppose now that $f^{-1}h \leq g$. For every $i \,\epsilon\, h$ we have $f^{-1}i \leq f^{-1}h \leq g$, so there is $e\colon Z' \to Y$ such that $ge = f^{-1}i$, where Z' is the domain of $f^{-1}h$. It is easy to see that $\langle f^{-1}i, e\rangle\colon Z' \to X \times Y$ is a pseudo-relation over f, g with domain index $i \in I$. In particular, there is $c \in F$ such that $\phi c = i$, i.e. $i \,\epsilon\, \phi$. ☐

Recall from Remark 2.7 that regular logic is valid under the BHK-interpretation in any quasi-cartesian category. From the above lemma and results in [19] we obtain the following.

Corollary 4.7. *Let $\mathbb{C}$ be a quasi-cartesian category which is closed for pseudo-relations and where every object is a choice object. Then the $(\top, \wedge, \Rightarrow, \exists, \forall)$-fragment of intuitionistic first order logic is valid under the BHK-interpretation in $\mathbb{C}$.*

Remark 4.8. With the same hypothesis as the previous corollary, we can extend Corollary 3.9. Subobjects obtained by $\dot{\Rightarrow}$ and $\dot{\forall}$ are determined, up to presubobject equivalence, by the universal closure of the following relations:

5. $x \,\epsilon\, (a \dot{\Rightarrow} b)$ if and only if $x \,\epsilon\, a$ implies $x \,\epsilon\, b$,
6. $x \,\epsilon\, \dot{\forall}_Y r$ if and only if $(\forall y \in Y)\, \langle x, y\rangle \,\epsilon\, r$.

The following theorem proves that closure for pseudo-relations provides a sufficient condition for the local cartesian closure of an elemental exact completion.

Theorem 4.9. *Let $\mathbb{C}$ be a quasi-cartesian category where every object is a choice object. If $\mathbb{C}$ is closed for pseudo-relations, then $\mathbb{C}_{ex}$ is locally cartesian closed.*

The proof exploits the proof-relevance of the BHK-interpretation, as well as the characterisations of the internal logic and the exact completion construction provided by elementality (Corollaries 3.9 to 3.11 and Remark 4.8) in order to isolate the functional relations from a suitable full family of pseudo-relations, and to define an equivalence relation to identify point-wise equal functional relations.

Proof. We will show that $\mathbb{C}_{\mathrm{ex}}$ has all universal dependent products. Thanks to Corollaries 3.10 and 3.11 we can regard objects X in $\mathbb{C}_{\mathrm{ex}}$ as pairs $(X_0, \sim_X)$ where $\sim_X$ is a pseudo equivalence relation on the elements X_0, and arrows $f\colon X \to Y$ in $\mathbb{C}_{\mathrm{ex}}$ as arrows $f\colon X_0 \to Y_0$ in $\mathbb{C}$ such that $fx \sim_Y fx'$ whenever $x \sim_X x'$.

Let $Y \xrightarrow{g} X \xrightarrow{f} I$ be a pair of composable arrows in $\mathbb{C}_{\mathrm{ex}}$, define two pseudo-relations $\tau\colon T_0 \to X_0 \times I_0$ and $\sigma\colon S_0 \to Y_0 \times T_0$ in $\mathbb{C}$ by the formulas

$$fx \sim_I i \qquad \text{and} \qquad gy \sim_X \tau_1 t, \tag{5}$$

respectively, for $i \in I_0, x \in X_0, y \in Y_0$ and $t \in T_0$, and let $\phi\colon F_0 \to I_0$, $\alpha\colon P_0 \to F_0 \times T_0 \times S_0$ be a full family of pseudo-relations over $S_0 \xrightarrow{\sigma_2} T_0 \xrightarrow{\tau_2} I_0$. This means that, for every $c \in F_0, t \in T_0$ and $s \in S_0$,

$$\langle c, t, s\rangle \,\epsilon\, \alpha \Longrightarrow \sigma_2 s = t, \tag{6}$$

$$\tau_2 t = \phi c \Longleftrightarrow (\exists s \in S_0)\, \langle c, t, s\rangle \,\epsilon\, \alpha. \tag{7}$$

Let $\gamma\colon G_0 \to F_0$ be a presubobject of F_0 defined by the formula

$$(\forall t, t' \in T_0)(\forall s, s' \in S_0)(\langle c, t, s\rangle \,\epsilon\, \alpha \wedge \langle c, t', s'\rangle \,\epsilon\, \alpha \wedge \tau_1 t \sim_X \tau_1 t' \Rightarrow \sigma_1 s \sim_Y \sigma_1 s'), \tag{8}$$

for $c \in F_0$, and let $\beta\colon Q_0 \to G_0 \times X_0 \times Y_0$ be the pseudo-relation defined by the formula

$$(\exists t \in T_0)(\exists s \in S_0)(\tau_1 t = x \wedge \sigma_1 s = y \wedge \langle \gamma u, t, s\rangle \,\epsilon\, \alpha), \tag{9}$$

for $u \in G_0, x \in X_0$ and $y \in Y_0$.

Define now an equivalence relation $u \sim_G u'$ on G_0 as the conjunction of $\phi\gamma u \sim_I \phi\gamma u'$ and

$$(\forall t, t' \in T_0)(\forall s, s' \in S_0)(\langle \gamma u, t, s\rangle \,\epsilon\, \alpha \wedge \langle \gamma u', t', s'\rangle \,\epsilon\, \alpha \wedge \tau_1 t \sim_X \tau_1 t' \Rightarrow \sigma_1 s \sim_Y \sigma_1 s'). \tag{10}$$

Reflexivity follows from (8) and reflexivity of $\sim_I$, and symmetry is trivial. To verify transitivity, assume $u \sim_G u' \sim_G u''$. Then $\phi\gamma u \sim_I \phi\gamma u''$ follows immediately, so let $t, t'' \in T_0$ and $s, s'' \in S_0$ be such that $\langle \gamma u, t, s\rangle \,\epsilon\, \alpha$, $\langle \gamma u'', t'', s''\rangle \,\epsilon\, \alpha$ and $\tau_1 t \sim_X \tau_1 t''$. We need to show that $\sigma_1 s \sim_Y \sigma_1 s''$. From (7) and the definition of τ in (5) we have $f\tau_1 t \sim_I \tau_2 t = \phi\gamma u \sim_I \phi\gamma u'$, so there is $t' \in T_0$ such that $\tau t' = \langle \tau_1 t, \phi\gamma u'\rangle$ and (7) yields $s' \in S_0$ such that $\langle \gamma u', t', s'\rangle \,\epsilon\, \alpha$. But we also have $\tau_1 t = \tau_1 t'$ and $\tau_1 t' \sim_X \tau_1 t''$, hence $\sigma_1 s \sim_Y \sigma_1 s'$ and $\sigma_1 s' \sim_Y \sigma_1 s''$ from (10) and the assumption $u \sim_G u' \sim_G u''$. We have thus established that $G \coloneqq (G_0, \sim_G)$ is an object in $\mathbb{C}_{\mathrm{ex}}$ and $\phi\gamma$ is an arrow $G \to I$ in $\mathbb{C}_{\mathrm{ex}}$.

Define an equivalence relation $q \sim_Q q'$ on Q_0 as

$$\beta_1 q \sim_G \beta_1 q' \wedge \beta_2 q \sim_X \beta_2 q' \tag{11}$$

which makes $Q \coloneqq (Q_0, \sim_Q)$ an object of $\mathbb{C}_{\mathrm{ex}}$ and β_1 and β_2 arrows $Q \to G$ and $Q \to X$, respectively. We need to check that it also makes β_3 an arrow $Q \to Y$ in $\mathbb{C}_{\mathrm{ex}}$. For $q, q' \in Q_0$ we have, from (9) that there are $t, t' \in T_0$ and $s, s' \in S_0$ such that

$$\langle \beta_2, \beta_3\rangle q = \langle \tau_1 t, \sigma_1 s\rangle \quad \langle \beta_2, \beta_3\rangle q' = \langle \tau_1 t', \sigma_1 s'\rangle \quad \langle \gamma\beta_1 q, t, s\rangle \,\epsilon\, \alpha \quad \text{and} \quad \langle \gamma\beta_1 q', t', s'\rangle \,\epsilon\, \alpha.$$

If $q \sim_Q q'$, then $\beta_1 q \sim_G \beta_1 q'$ and $\tau_1 t = \beta_2 q \sim_X \beta_2 q' = \tau_1 t'$, which in turn imply $\beta_3 q = \sigma_1 s \sim_Y \sigma_1 s' = \beta_3 q'$ as required.

This gives us a pair of arrows $\phi\gamma\colon G \to I$ and $\beta\colon Q \hookrightarrow G \times X \times Y$, where the latter is monic because of (11). We now need to show that this pair is a universal dependent product for f, g. Let us first remark that in $\mathbb{C}_{\mathrm{ex}}$ the membership relation ϵ is different from the one in $\mathbb{C}$: we denote the former with $\tilde{\epsilon}$ and continue denoting the latter as ϵ. In particular, we have:

$$b \,\tilde{\epsilon}\, f \quad \Longleftrightarrow \quad (\exists b' \in B_0)(b \sim_B b' \wedge b' \,\epsilon\, f). \tag{12}$$

We start showing that the pair $\phi\gamma, \beta$ is a family of functional relations over f, g by checking the three properties in Definition 4.1.

(a) $\phi\gamma, \beta$ is a family of sections of g: If $\langle u, x, y\rangle \,\tilde{\epsilon}\, \beta$, then there are $u' \sim_G u$, $x' \sim_X x$ and $y' \sim_Y y$ such that $\langle u', x', y'\rangle \,\epsilon\, \beta$. From (9) we obtain $t \in T_0$ and $s \in S_0$ such that $\tau_1 t = x'$, $\sigma_1 s = y'$ and $\langle \gamma u', t, s\rangle \,\epsilon\, \alpha$ and, from (6), $gy \sim_X g\sigma_1 s \sim_X \tau_1\sigma_2 s \sim_X x$.

(b) $\phi\gamma, \beta$ has domains indexed by f: Reasoning as above, and using (7), it is easy to see that $\langle u, x, y\rangle \,\tilde{\epsilon}\, \beta$ implies $fx \sim_I \phi\gamma u$. Conversely, let $x \in X_0$ be such that $fx \sim_I \phi\gamma u$. From (5) we obtain $t \in T_0$ such that $\tau t = \langle x, \phi\gamma u\rangle$ and, from (7), we get $s \in S_0$ such that $\langle \gamma u, t, s\rangle \,\epsilon\, \alpha$, that is, $\langle u, x, \sigma_1 s\rangle \,\tilde{\epsilon}\, \beta$.

(c) $\phi\gamma, \beta$ is functional: Let $\langle u, x, y\rangle, \langle u, x', y'\rangle \,\tilde{\epsilon}\, \beta$ be such that $x \sim_X x'$. As in point (a), we obtain $t, t' \in T_0$ and $s, s' \in S_0$ such that $\tau_1 t \sim_X x$, $\tau_1 t' \sim_X x'$, $\sigma_1 s \sim_Y y$, $\sigma_1 s' \sim_Y y'$, $\langle \gamma u, t, s\rangle \,\epsilon\, \alpha$ and $\langle \gamma u, t', s'\rangle \,\epsilon\, \alpha$. Hence $y \sim_Y y'$ from (8).

It remains to show that the pair $\phi\gamma\colon G \to I$, $\beta\colon Q \hookrightarrow G \times X \times Y$ has the required universal property. Let $r\colon R \hookrightarrow X \times Y$ be a functional relation over f, g with domain index $i_0 \in I_0$, i.e. such that, for every $x, x' \in X_0$ and $y, y' \in Y_0$,

$$\langle x, y\rangle \,\tilde{\epsilon}\, r \Longrightarrow gy \sim_X x, \tag{13}$$

$$fx \sim_I i_0 \Longleftrightarrow (\exists y \in Y)\, \langle x, y\rangle \,\tilde{\epsilon}\, r, \tag{14}$$

$$\langle x, y\rangle \,\tilde{\epsilon}\, r \,\wedge\, \langle x', y'\rangle \,\tilde{\epsilon}\, r \,\wedge\, x \sim_X x' \Longrightarrow y \sim_Y y'. \tag{15}$$

Properties (13) and (14) above ensure that the pseudo-relation $r'\colon R'_0 \to T_0 \times S_0$ defined by the formula

$$\sigma_2 s = t \wedge \tau_2 t = i_0 \wedge \langle \tau_1 t, \sigma_1 s\rangle \,\tilde{\epsilon}\, r. \tag{16}$$

is a pseudo-relation over τ_2, σ_2 with domain index $i_0 \in I_0$, hence from fullness of ϕ and α we get $c \in F_0$ such that $\phi c = i_0$ and

$$\langle c, t, s\rangle \,\epsilon\, \alpha \implies \langle t, s\rangle \,\epsilon\, r' \tag{17}$$

for every $t \in T, s \in S_0$.

Using (15), (16) and (17) it is easy to see that $c \in F_0$ satisfies (8), hence there is $u \in G_0$ such that $\gamma u = c$. We now need to show that

$$\langle u, x, y\rangle \,\tilde{\epsilon}\, \beta \iff \langle x, y\rangle \,\tilde{\epsilon}\, r. \tag{18}$$

Suppose $\langle u, x, y\rangle \,\tilde{\epsilon}\, \beta$, hence there are $u' \sim_G u$, $x' \sim_X x$ and $y' \sim_Y y$ such that $\langle u', x', y'\rangle \,\epsilon\, \beta$. From (9) we obtain $t' \in T_0$ and $s' \in S_0$ such that $\tau_1 t' = x'$, $\sigma_1 s' = y'$ and $\langle \gamma u', t', s'\rangle \,\epsilon\, \alpha$. On the other hand, the family $\phi\gamma, \beta$ has domains indexed by f and, in particular, $fx \sim_I \phi\gamma u$. So there are $t \in T_0$ such that $\tau t = \langle x, \phi\gamma u\rangle$ and, from (7), $s \in S_0$ such that $\langle \gamma u, t, s\rangle \,\epsilon\, \alpha$. It follows from (17) and (16) that $\langle x, \sigma_1 s\rangle \,\tilde{\epsilon}\, r$. Since $u \sim_G u'$ and $\tau_1 t = x \sim_X x' = \tau_1 t'$, (10) implies $\sigma_1 s \sim_Y \sigma_1 s' = y' \sim_Y y$. Hence $\langle x, y\rangle \,\tilde{\epsilon}\, r$.

For the converse, suppose $\langle x, y\rangle \,\tilde{\epsilon}\, r$. Then $fx \sim_I i_0 = \phi\gamma u$ and, since the family $\phi\gamma, \beta$ has domains indexed by f, there is $y' \in Y_0$ such that $\langle u, x, y'\rangle \,\tilde{\epsilon}\, \beta$. But then $\langle x, y'\rangle \,\tilde{\epsilon}\, r$, and (15) implies $y \sim_Y y'$. Hence $\langle u, x, y\rangle \,\tilde{\epsilon}\, \beta$.

It only remains to show uniqueness of $u \in G$. Suppose that $u' \in G_0$ is such that $\phi\gamma u' \sim_I i_0$ and satisfies (18) for all $x \in X_0$ and $y \in Y_0$. Clearly $\phi\gamma u \sim_I \phi\gamma u'$. Let $t, t' \in T_0$ and $s, s' \in S_0$ be such that $\langle \gamma u, t, s\rangle, \langle \gamma u', t', s'\rangle \,\epsilon\, \alpha$ and $\tau_1 t \sim_X \tau_1 t'$. Hence $\langle u, \tau_1 t, \sigma_1 s\rangle, \langle u', \tau_1 t', \sigma_1 s'\rangle \,\tilde{\epsilon}\, \beta$ from (9) and, since both u and u' satisfy (18), we obtain $\langle \tau_1 t, \sigma_1 s\rangle, \langle \tau_1 t', \sigma_1 s'\rangle \,\tilde{\epsilon}\, r$. Functionality of r (15) implies $\sigma_1 s \sim_Y \sigma_1 s'$, hence $u \sim_G u'$ as required. □

Proposition 4.10. *In* **ML** *the E-category* Type *is closed for pseudo-relations.*

Proof. Let $Y \xrightarrow{g} X \xrightarrow{f} I$ be arrows in Type. For $i : I$ and $x : X$ define

$$f^-(i) :\equiv \sum_{x:X} f(x) =_I i, \qquad \text{and} \qquad g^-(x) :\equiv \sum_{y:Y} g(y) =_X x,$$

and form the closed types

$$F :\equiv \sum_{i:I} \prod_{u:f^-(i)} g^-(\mathrm{pr}_1(u)) \qquad \text{and} \qquad P :\equiv \sum_{v:F} \sum_{x:X} f(x) =_I \phi(v)$$

where $\phi :\equiv \mathrm{pr}_1 \colon F \to I$. Finally, define $\varepsilon \colon P \to Y$ and $\alpha \colon P \to F \times X \times Y$ as

$$\varepsilon(v, x, s) :\equiv \mathrm{pr}_1((\mathrm{pr}_2 v)(x, s)) \qquad \text{and} \qquad \alpha(v, x, s) :\equiv (v, x, \varepsilon(v, x, s)).$$

If $(v, x, y) \,\epsilon\, \alpha$, then there is $s : f(x) =_I \phi(v)$ such that $\varepsilon(v, x, s) =_Y y$ and

$$\mathrm{pr}_2((\mathrm{pr}_2 v)(x, s)) : g(\varepsilon(v, x, s)) =_X x,$$

so 4.2(a) is satisfied, while 4.2(b) follows immediately from the definition of equality of arrows in Type. Hence the pair ϕ, α is a family of pseudo-relations over f, g.

Let now $r \colon R \to X \times Y$ be a pseudo-relation over f, g with domain index $i : I$. Property 4.2(b) implies that

$$\prod_{u:f^-(i)} \sum_{t:R} r_1(t) =_X \mathrm{pr}_1(u)$$

is inhabited, therefore the type-theoretic axiom of choice yields a function term $k : f^-(i) \to R$ such that

$$\prod_{u:f^-(i)} r_1(k(u)) =_X \mathrm{pr}_1(u).$$

Property 4.2(a) implies that there is a closed term

$$m : \prod_{u:f^-(i)} g(r_2(k(u))) =_X \mathrm{pr}_1(u),$$

Hence we can define a function term $h : \prod_{f^-(i)} g^-(\mathrm{pr}_1(u))$ as $h(u) :\equiv (r_2(k(u)), m(u))$, thus obtaining a term $c :\equiv (i, h) : F$. Clearly $\phi(c) =_I i$, we need to show that for all $x : X$ and $y : Y$

$$(c, x, y) \,\epsilon\, \alpha \implies (x, y) \,\epsilon\, r.$$

Suppose that there is $s : f(x) =_I \phi(s)$ such that $(c, x, s) : P$ and $\varepsilon(c, x, s) =_Y y$, hence

$$y =_Y \varepsilon(c, x, s) =_Y \mathrm{pr}_1(h(x, s)) =_Y r_2(k(x, s)).$$

Since moreover $r_1(k(x, s)) =_X \mathrm{pr}_1(x, s) =_X x$, we can conclude $(x, y) \,\epsilon\, r$ as required. □

5. Models of CETCS as exact completions

The Constructive Elementary Theory of the Category of Sets (CETCS) is expressed in a three-sorted language for category theory and is based on a suitable essentially algebraic formalisation of category theory over intuitionistic first-order logic. We refer to [20] for more details. We now recall the axioms of CETCS.

(C1) Finite limits and finite colimits exist.
(C2) Any pair of composable arrows has a universal dependent product.
(C3) There is a natural numbers object.
(C4) Elementality.
(C5) For any object X there are a choice object P and a surjection $P \to X$.
(C6) The initial object $\mathbf{0}$ has no elements.
(C7) The terminal object is indecomposable: in any sum diagram $i\colon X \to S \leftarrow Y : j$, $z \in S$ implies $z \,\epsilon\, i$ or $z \,\epsilon\, j$.
(C8) In any sum diagram $i\colon \mathbf{1} \to S \leftarrow \mathbf{1} : j$, the arrows i and j are different.
(C9) Any arrow can be factored as a surjection followed by a mono.
(C10) Every equivalence relation is a kernel pair.

Remark 5.1. Theorem 6.10 in [20] characterises models of CETCS in terms of standard categorical properties, proving that CETCS provides a finite axiomatisation of the theory of well-pointed locally cartesian closed pretoposes with a natural numbers object and enough projectives. Recall that a pretopos is *well-pointed* if the terminal object is projective, indecomposable, non-degenerate (i.e. $\mathbf{0} \not\cong \mathbf{1}$) and a strong generator.

In particular, since the terminal object is projective and a strong generator, we have from Lemma 3.4 that covers and surjections coincide, hence the projectives mentioned above are precisely the choice objects given by axiom (C5). Using cartesian closure, it is also easy to see that these are closed under finite products. Hence we obtain the following result as a consequence of Theorem 2.10.

Corollary 5.2. *Choice objects in a model of CETCS form a quasi-cartesian category, and every model of CETCS is the exact completion of its choice objects.*

Remark 5.3. Choice objects in models of CETCS are in general not closed under all finite limits. This follows from Remark 2.9 and Corollary 5.10 below.

Using the results in the previous sections, we can isolate those properties that a quasi-cartesian category has to satisfy in order to arise as a subcategory of choice objects in a model of CETCS.

We begin recalling from [14] that an exact completion $\mathbb{C}_{\mathrm{ex}}$ is a pretopos if and only if $\mathbb{C}$ has finite sums and is weakly lextensive, meaning that finite sums interacts well with weak limits. More precisely, a quasi-cartesian category $\mathbb{C}$ with finite sums is *weakly lextensive* if

(a) sums are disjoint and the initial object is strict,
(b) it is distributive, i.e. $(X \times Y) + (X \times Z) \cong X \times (Y + Z)$,
(c) if $E_X \to X \rightrightarrows Z$ and $E_Y \to Y \rightrightarrows Z$ are weak equalisers, then so is $E_X + E_Y \to X + Y \rightrightarrows Z$.

In fact, as observed by Gran and Vitale, the exact completion of a weakly lextensive category coincides with the pretopos completion.

Example 5.4. Type is weakly lextensive. Strictness of the initial object is immediate from the elimination rule of the empty type, while disjointness of sums follows from the type-theoretic equivalences

$$\begin{aligned} \mathrm{inl}(x) =_{X+Y} \mathrm{inl}(x') \ &\simeq\ x =_X x', \\ \mathrm{inr}(y) =_{X+Y} \mathrm{inr}(y') \ &\simeq\ y =_Y y', \\ \mathrm{inl}(x) =_{X+Y} \mathrm{inr}(y) \ &\simeq\ \mathbf{0}. \end{aligned}$$

See for example [24]. For distributivity, it is enough to show that the function $(X \times Y) + (X \times Z) \to X \times (Y + Z)$ defined by $+$-elimination is injective and surjective, which is straightforward using the elimination rules of the types involved. The last property also follows from $+$-elimination, and the fact that a weak equaliser of $f, g \colon X \to Y$ is logically equivalent in Type/X to $\mathrm{pr}_1 \colon \sum_{x:X} f(x) =_Y g(x) \to X$.

Remark 5.5. Recall that a *natural numbers object* is an object N together with $0 \in N$ and $s \colon N \to N$ such that, for any other triple $X, x \in X, f \colon X \to X$, there is a unique $g \colon N \to X$ such that $g0 = x$ and $gs = fs$. If we drop uniqueness of g, then we obtain a *weak natural numbers object.*

If $N = (N_0, {\sim_N})$ is a natural numbers object in an elemental $\mathbb{C}_{\mathrm{ex}}$, then N_0 is a weak natural numbers object in $\mathbb{C}$. Conversely, a weak natural numbers object in $\mathbb{C}$ is a weak natural numbers object in $\mathbb{C}_{\mathrm{ex}}$ as well. Proposition 5.1 in [6] proves that a cartesian closed category with equalisers and a weak natural numbers object also has a natural numbers object.

Example 5.6. The type of natural numbers $\mathbb{N} : \mathsf{U}$ in **ML** provides Type with a natural numbers object. The existence of a universal arrow is an immediate consequence of the elimination rule of $\mathbb{N}$ (i.e. recursion on natural numbers) and, since the equality of arrows in Type is point-wise propositional equality, such an arrow is in fact unique.

Proposition 5.7. *Let $\mathbb{C}$ be a quasi-cartesian category with finite sums. Then $\mathbb{C}_{ex}$ is well-pointed if and only if the terminal object in $\mathbb{C}$ is non-degenerate and indecomposable and every object in $\mathbb{C}$ is a choice object.*

Proof. We already know that the terminal object in $\mathbb{C}_{\mathrm{ex}}$ is projective and, from Theorem 3.12, that it is a strong generator if and only if every object in $\mathbb{C}$ is a choice object. For non-degeneracy the equivalence follows from the fact that the embedding $\Gamma \colon \mathbb{C} \to \mathbb{C}_{\mathrm{ex}}$ is conservative and preserves terminal and initial objects.

If the terminal object is indecomposable in $\mathbb{C}_{\mathrm{ex}}$, then clearly it is so in $\mathbb{C}$ as well. To show the other implication, let us assume elementality of $\mathbb{C}_{\mathrm{ex}}$ (although it can be easily proved also without it). Every element $z \in X + Y$ in $\mathbb{C}_{\mathrm{ex}}$ is also an element of $X_0 + Y_0$ in $\mathbb{C}$. Indecomposability of $\mathbf{1}$ in $\mathbb{C}$ implies $z \,\epsilon\, i_0$ or $z \,\epsilon\, j_0$, hence we have $z \,\epsilon\, i$ in the first case, and $z \,\epsilon\, j$ in the second case, where i_0, j_0 (resp. i, j) are the coproduct injections of $X_0 + Y_0$ (resp. of $X + Y$). $\square$

Example 5.8. The unit type $\mathbf{1}$ in **ML** is clearly a non-degenerate and indecomposable terminal object in Type. Indeed, the types

$$(\mathbf{1} \to \mathbf{0}) \to \mathbf{0} \quad \text{and} \quad \prod_{u:X+Y} \big(\sum_{x:X} \mathrm{inl}(x) = u + \sum_{y:Y} \mathrm{inr}(y) = u\big)$$

are both inhabited by a closed term.

We can now collect all the properties seen so far which, all together, provide a sufficient condition ensuring that an exact completion construction will give rise to a model of CETCS.

Theorem 5.9. *Let $\mathbb{C}$ be a quasi-cartesian category with finite sums. Then $\mathbb{C}_{ex}$ is a model of CETCS if*

(i) every object in $\mathbb{C}$ is a choice object,
(ii) the terminal object in $\mathbb{C}$ is non-degenerate and indecomposable,
(iii) $\mathbb{C}$ is weakly lextensive and closed for pseudo-relations,
(iv) $\mathbb{C}$ has a weak natural numbers object.

Proof. Well-pointedness of $\mathbb{C}_{\text{ex}}$ follows from Proposition 5.7, while Proposition 2.1 in [14] and Theorem 4.9 imply that $\mathbb{C}_{\text{ex}}$ is a locally cartesian closed pretopos. The existence of enough projectives is automatic from the completion process, and the existence of a natural numbers object is ensured by Proposition 5.1 in [6] and Remark 5.5. By Theorem 6.10 in [20], well-pointed locally cartesian closed pretoposes with enough projectives and a natural numbers object are precisely the models of CETCS. □

Corollary 5.10. *In* **ML***, the category of small types* Type *is a quasi-cartesian category with finite sums that satisfies properties (i)–(iv) from the previous theorem. Hence the category of small setoids* Std *is a model of CETCS.*

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