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Postprint

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Citation for the original published paper (version of record):

Buijsman, S. (2018)

Are The Natural Numbers Fundamentally Ordinals?

Philosophy and phenomenological research

Access to the published version may require subscription.

N.B. When citing this work, cite the original published paper.

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<http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-155302>

ARE THE NATURAL NUMBERS FUNDAMENTALLY ORDINALS?

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Abstract

There are two ways of thinking about the natural numbers: as ordinal numbers or as cardinal numbers. It is, moreover, well-known that the cardinal numbers can be defined in terms of the ordinal numbers. Some philosophies of mathematics have taken this as a reason to hold the ordinal numbers as (metaphysically) fundamental. By discussing structuralism and neo-logicism we argue that one can empirically distinguish between accounts that endorse this fundamentality claim and those that do not. In particular, we argue that if the ordinal numbers are metaphysically fundamental then it follows that one cannot acquire cardinal number concepts without appeal to ordinal notions. On the other hand, without this fundamentality thesis that would be possible. This allows for an empirical test to see which account best describes our actual mathematical practices. We then, finally, discuss some empirical data that suggests that we can acquire cardinal number concepts without using ordinal notions. However, there are some important gaps left open by this data that we point to as areas for

future empirical research.

With just a little reflection it is clear that there are two roles which the natural numbers play. First of all, they can function as cardinal numbers. The natural numbers can be used to specify the number of objects falling under a concept. This is the role in which they appear when a number is given in answer to the question ‘how many Fs are there?’. Second, the natural numbers can function as ordinal numbers. In this case they specify the position of an item in an ordering, for example, to say that a particular person finished third. If one wants to be even more explicit that the same number is used in such a case, consider that the same thing can be said with ‘this person ended up being number three’. Rather than giving the size of a set, ordinal numbers give the position, and so answer questions of the form ‘which F is this?’.

The dual role of the natural numbers has long been known, just as it has long been known that it is possible to define cardinal numbers in terms of ordinal numbers. Cantor (1883) introduced the necessary machinery for defining a cardinal number based on an ordinal number in a well-ordering. According to Cantor (1895) the ‘general concept’ of a cardinal number, “arises from [a set] M when we make abstraction of the nature of its various elements m and of the order in which they are given [...] Since every single element m , if we abstract from its nature, becomes a ‘unit’, the cardinal number $\overline{\overline{M}}$ is a definite [set] composed of units.” (Cantor, 1895, §1) He gives a parallel account for the general concept of an ordinal, which “results from $\overline{\overline{M}}$ if we only abstract from the nature of the elements m and retain the order of precedence among them.” (Cantor, 1895, §7) Thus, on Cantor’s abstractionist story, we first abstract from any discerning properties of the elements

of a set, other than their order, to obtain its ordinal, and then abstract from their order to obtain the corresponding cardinal. Thus construed, a cardinal number will arise from an ordinal number.¹

In the more modern context of the standard set theory ZFC, natural numbers are also defined in terms of their ordinal role. This is typically done through the von Neumann ordinals. These build up the natural numbers through the powerset operation: 0 is $\emptyset$, 1 is $\{\emptyset\}$, 2 is $\{\emptyset, \{\emptyset\}\}$, etc. The cardinal number of a set can then be defined as the least ordinal number whose members are in one-to-one correspondence with the members of the set. For example, a set A with two members is assigned the ordinal number $\{\emptyset, \{\emptyset\}\}$, as that is the least (and, as is true for the finite case, only) ordinal for which a one-to-one correspondence can be constructed. One member of A is put in correspondence with the empty set and the other member is put in correspondence with the set containing the empty set. Thus, the cardinal numbers are defined in terms of the ordinal numbers.

It is now tempting to say that the result of this account is that the ordinal numbers are, in some sense, more fundamental than the cardinal numbers. This was, for example, the conclusion reached by Dummett:

[I]f Frege had paid more attention to Cantor's work, he would have understood what it revealed, that the notion of an ordinal number is more fundamental than that of a cardinal number. This is true even in the finite case; after all, when we count the strokes of a clock, we are assigning an ordinal number rather than a cardinal. If Frege had understood this, he would therefore have characterised the natural

¹For more on this widely accepted interpretation of Cantor's abstractionism, see Hallett (1984, p.134, 139-140).

numbers as finite ordinals rather than as finite cardinals (Dummett, 1991, p.293)

As is to be expected, there is more than one way in which one may understand this idea. It is possible to understand it along semantic lines, as Linnebo (2009) does. He discusses how we semantically individuate the natural numbers to make singular reference to them. In particular, whether the criterion of identity we use in doing so is based on the cardinal or the ordinal conception of the numbers. Whichever criterion is used, that criterion is then more fundamental when it comes to how we (semantically) individuate the natural numbers. Two other ways of understanding fundamentality are presented by Heck:

Two sorts of responses to this question [why Hume's Principle (HP)² is more significant than PA] come immediately to mind. The first sort of reply we might call metaphysical. It would hold that among the truths themselves, as it were, some are more fundamental than others, and that HP is, as it happens, more fundamental than the axioms of PA.

...

The other sort of reply is an epistemological one: On this view, HP somehow grounds our *knowledge* of arithmetic. (Heck, 2000, p.190)

The question we address in this paper is whether Heck's metaphysical idea of fundamentality, incorporated into some but not all philosophies of mathematics, is correct. Of course, the cardinal numbers, however they are construed, admit a natural ordering. But that leaves open the question whether the notion of cardinal

²Hume's principle is typically stated as $Nx.Fx = Nx.Gx \leftrightarrow F \approx G$, where $F \approx G$ means that there is a one-to-one correspondence between F and G, and Nx is a non-logical operator to be read as 'the number of'.

number *by itself* is autonomous with respect to a notion of ordinal number. In order to answer this question, we first discuss logicism, neo-logicism, and two versions of structuralism, to clarify the different ways in which philosophies of mathematics may incorporate/avoid the idea that ordinal numbers are more fundamental than cardinal numbers. The idea is that in some cases, a philosophy of mathematics has room for (mathematically useful) autonomous cardinal numbers under different notions of autonomy that we borrow from Linnebo and Pettigrew (2011). The most important of these notions of autonomy for the rest of the paper is that of conceptual autonomy. For, in order to test whether it is correct to make the ordinal numbers more fundamental, we discuss some of the empirical differences between accounts where the cardinal numbers are conceptually autonomous and accounts where they are not. Finally, we offer some tentative empirical evidence in favour of the claim that the cardinal numbers are conceptually autonomous.

1 Notions of autonomy

The different ways of understanding the notion of fundamentality that we briefly mentioned are intended to clarify the kind of view we are interested in here. We focus on the view that the notion of ordinal number is metaphysically more fundamental than the notion of cardinal number, as this is a view that one finds embedded in some philosophies of mathematics. Such a view, as one would expect, leads to restrictions on the degree of autonomy of the notion of cardinal number. Roughly speaking, if the notion of cardinal number is reduced to that of ordinal number, then that notion of cardinal number is not independent. To make this more precise we use the different notions of autonomy introduced by Linnebo

and Pettigrew:

Suppose T_1 and T_2 are theories: not the formal theories of mathematical logic, but rather accounts of a particular part of reality.

- T_1 has *logical autonomy* with respect to T_2 if it is possible to formulate T_1 without appealing to notions that belong to T_2 .
- T_1 has *conceptual autonomy* with respect to T_2 if it is possible to understand T_1 without first understanding notions that belong to T_2 .
- T_1 has *justificatory autonomy* with respect to T_2 if it is possible to motivate and justify the claims of T_1 without appealing to T_2 , or to justifications that belong to T_2 . (Linnebo and Pettigrew, 2011, p.241-242, original italics)

This distinction is easily applied to the current question. A notion of cardinal number is logically autonomous with respect a notion of ordinal number if it can be formulated without using that ordinal notion. The notion of cardinal number in ZFC discussed above is, for example, not logically autonomous from the notion of a von Neumann ordinal. Next, a notion of cardinal number is conceptually autonomous with respect to a notion of ordinal number if it can be understood without first understanding that notion of ordinal number. Finally, a notion of cardinal number has justificatory autonomy with respect to a notion of ordinal number if claims about it can be justified without appeal to that of ordinal notion. With these distinctions in place, we can discuss several views on the natural numbers and see what kind(s) of autonomy they allow for. We also discuss whether

these notions are mathematically useful, since there being an (in some sense) autonomous notion of cardinal number can only make an empirical difference if it is somehow useful for pure and/or applied mathematics.

1.1 (Neo-)logicism

One of the more explicit claims regarding fundamentality is found in the paper by Heck quoted above. He argues for his metaphysical notion of fundamentality, and so claims that Hume's Principle is more fundamental than the Peano Axioms.

What I have argued here is that recognition of the truth of something very much like HP is required if one is even to have a concept of cardinality. If so, then what Frege's Theorem shows is that the fact that the finite cardinals form an initial segment of an ω -sequence is implicit in our very concept of cardinal number – 'implicit' in the sense that their forming such a sequence is logically required by the character of our concept of cardinal number. (Heck, 2000, p.204)

Heck argues on broadly empirical grounds that HP is needed to have a concept of cardinality, arguments which we will not discuss here. What is relevant is his argument that, as a result, the concept of cardinal number already involves the concept of ordinal number. As he says, from HP it follows that the second-order Peano Axioms hold, by Frege's Theorem. In other words, from HP it follows that there is an ordering on the cardinal numbers such that the cardinal numbers form the initial segment of an ω -sequence. The cardinal numbers automatically function as ordinal numbers and there is no way in which we can have a concept of the cardinal numbers where they do not function as such.

The question, now, is whether this means that the notion of cardinal number is not conceptually autonomous with respect to that of an ordinal number/initial segment of an ω -sequence. This will depend on whether one thinks that this notion of cardinal number has been understood even when one hasn't grasped that they also form an initial segment of an ω -sequence. There are, of course, plenty of logical consequences of HP and it would thus be strange to claim that one has only understood the corresponding notion of cardinal number if all of these logical consequences are understood. The mere fact that the Peano axioms can be shown to follow from HP is not enough for the claim that the corresponding ordinal notion has to be antecedently understood.

An argument in favour of thinking that this particular consequence would need to be understood is the significance of the fact that the cardinal numbers form an initial segment of an ω -sequence. It seems that if this is automatically true about any finite cardinal, then this ordering on the cardinal numbers is an important part of what they are. Consequently, something would be missing from one's conception of a cardinal number if they are not also understood as initial segments of an ω -sequence. In other words, something would be missing from one's understanding of a cardinal number if they are not also (implicitly) understood as ordinal numbers. The notion of a cardinal number would not be conceptually autonomous from that of an ordinal number.

Admittedly this argument is fairly weak. For that reason it is important that there is a more promising route to the idea that the notion of a cardinal number based on HP is not conceptually autonomous from the notion of an ordinal number. This has to do with the fact that HP cannot function as a definition of cardinal numbers in its standard, unrestricted form (i.e. as using second-order variables

ranging over all concepts). The problem with this unrestricted form, as it has been raised by Boolos (1997) and Rumfitt (2001), is that in this case the concept of self-identity would also have a cardinal number. If this concept receives a cardinal number, then the resulting theory is incompatible with ZFC. According to ZFC only sets can have cardinal numbers, and so if a concept F has a cardinal number, there has to be a set of all the F s. This means that for the concept of self-identity to have a cardinal number there has to be a set of everything that is self-identical. That, in turn, means that there has to be a set of all sets and, via the axiom of specification, a set of all sets that do not contain themselves. This, of course, is ruled out by the argument leading to Russell's Paradox.

Even so, the unrestricted version of HP is the one that underlies the concept of cardinal number of Frege's classical logicism.³ On the view of Frege (1884) (and, without HP, that of Russell (1903, 1919)) cardinal numbers are equivalence classes under the relation of equinumerosity. As long as that notion of cardinal number is used, it cannot be used together with ZFC. This goes back to our earlier remark that the notion of cardinal number in question has to be mathematically useful. The classical logicist notion is not mathematically useful. It cannot be combined with ZFC⁴, and even in Russell's alternative type-theoretic framework there are problems. Within Russell's type-theory the cardinal numbers are type-dependent, meaning that there are different cardinal numbers for each type (so long as there are enough objects to support them). In neither case are the cardinal numbers mathematically useful, even though the notion of a cardinal number as an equiv-

³Our thanks to a reviewer for suggesting a discussion of classical logicism and pointing out that this notion of cardinal number is not mathematically useful.

⁴For Frege due to the above clash, for Russell because cardinal numbers end up being proper classes and thus can't support functions from cardinal numbers to other sets.

alence class of the relation of equinumerosity is autonomous (under all the above notions of autonomy) from the notion of an ordinal number. To state, understand, and justifiably believe claims about equivalence classes over the relation of equinumerosity doesn't require use of a notion of ordinal number. Yet this notion will not do, because it is useless for both pure and applied mathematics.

The neo-logicist, in contrast, attempts to restrict HP, so as to avoid this clash with ZFC. The issue, and also our argument against the autonomy of the notion of cardinal number, is how to affect this restriction of HP. The current state of work on this question suggests that HP can only be restricted through an appeal to ordinal notions. For, at the moment there are two main proposals on the market due to Wright and Shapiro, and Rumfitt. Wright's first attempt was to restrict the initial second-order variables of HP by appealing to the notion of indefinite extensibility (Dummett, 1991, p.316-319). His idea is that there will be a determinate answer to the question How many Fs are there? if and only if the concept F is definite, where a concept F is definite if it is not indefinitely extensible. But to say that a concept F is definite means that F is equinumerous with a member of the iterative hierarchy. This suggests that the notion of indefinite extensibility can be rendered precise only in set-theoretical terms, and this is no good news if HP is to be epistemologically innovative, for its understanding would be parasitic on a prior understanding of set theory.

Shapiro and Wright offer a way past this problem. In essence, their proposal is as follows:

Let P be a concept of items of a certain type τ . Typically, τ will be the (or a) type of individual objects. Let Π be a concept of concepts of

type τ items. Let us say that P is *indefinitely extensible with respect to* Π if and only if there is a function F from items of the same type as P to items of type τ such that if X is any sub-concept of P such that ΠX then

1. FX falls under the concept P ,
2. it is not the case that FX falls under the concept X , and
3. $\Pi X'$, where X' is the concept instantiated just by FX and every item which instantiates X (i.e., $\forall x(X'x \equiv (Xx \vee x = FX))$); in set-theoretic terms, X' is $(X \cup \{FX\})$. (Shapiro and Wright, 2006, p.266)

For example, Px iff x is a finite ordinal (or cardinal) number; ΠX iff there are only finitely many X s; FX is the successor of the largest X . So being a finite ordinal (or cardinal) is indefinitely extensible with respect to ‘finite’.

But this is a relativized form of indefinite extensibility. How can we have a non-relative account of indefinite extensibility? Here is what Wright and Shapiro propose: where λ ranges over ordinals, let us say that “ P is *up-to- λ -extensible with respect to* Π just in case P and Π meet the conditions for the relativized notion as originally defined but λ places a limit on the series of Π -preserving iterations of F to any sub-concept X of P such that ΠX ” (Shapiro and Wright, 2006, p.269). Now, define *proper indefinite extensibility with respect to* Π as follows: “ P is *properly indefinitely extensible with respect to* Π just in case P meets the conditions for the relativized notion as originally defined and there is no λ such that P is up-to- λ -extensible with respect to Π .” (Shapiro and Wright, 2006, p.269). Finally: “ P is

indefinitely extensible (simpliciter) just in case there is a Π such that P is properly indefinitely extensible with respect to Π ” (Shapiro and Wright, 2006, p.269).

For our purposes, the main point here is that the above characterization of indefinite extensibility makes use of ordinal numbers, for indefinite extensibility simpliciter is defined in terms of indefinite extensibility proper, which is, in turn, defined in terms of ordinally bounded extensibility, i.e. up-to- λ -extensibility with respect to a concept Π .⁵ This would render the notion of cardinality to be introduced by HP dependent on that of ordinality. Thus, our theory of cardinality in terms of HP would be neither conceptually nor logically autonomous with respect to ordinality.

Rumfitt’s proposal (Rumfitt, 2001, 2003) for restricting HP is as follows. He tries to show how one could build into HP the thesis that it is in the nature of the cardinal numbers to be arranged in a definite order. In doing so, Rumfitt makes use of the ordinary notion of a tally; i.e. a concept whose instances are used to assign cardinalities to concepts. He then goes on to argue that there will be a determinate answer to the question ‘How many Fs are there?’ if and only if the concept F is equinumerous with an initial segment of a tally, where the instances of the tally should be well-ordered, and, in addition, each successive instance of the tally under the ordering should not exhibit “cardinal redundancy”, in the sense that each successive instance under the ordering should represent a new cardinality. As Rumfitt points out, we cannot “assign natural numbers to concepts using the sequence of numerals ‘1, 2, a , 3, ...’ where ‘ a ’ indicated the cardinality of twosomes, for this would destroy the natural relation between a concept’s cardinality and the place in the tally of the numeral used to indicate that cardinality”

⁵Our thanks to Bob Hale for discussions about this point.

(Rumfitt, 2003, p.216). In sum, Rumfitt's proposal is a concept has a cardinal number when it is equinumerous with a bounded initial segment of a concept under a well-ordering which does not exhibit cardinal redundancy. This suggests that the cardinal numbers belong to just those concepts that are equinumerous with the predecessors of an initial ordinal – a point that is well accommodated by ZFC. This, again, shows that if the only way to appropriately restrict HP is by using ordinal notions, then the notion of cardinal number is not even logically autonomous from the notion of ordinal number. It is not possible to define the notion of cardinal number (as based on HP) without using ordinal notions in the definition. Consequently, the notion of cardinal number (as defined in terms of HP) cannot be understood without an antecedent understanding of the notion of ordinal number. This notion of cardinal number is not conceptually autonomous with respect to the notion of ordinal number. It is, however, mathematically useful, as evidenced by Frege's Theorem: the result that a reformulated version of the Dedekind-Peano axioms can be proved in Frege Arithmetic (Boolos, 1995). Neologicism, by abandoning the further claim that cardinal numbers are equivalence classes, avoids the problems with the logicist notion of cardinal number but also fails to arrive at a conceptually autonomous notion of cardinal number.

1.2 Structuralism

Structuralism is another good candidate for an account where the notion of cardinal number is not autonomous with respect to that of an ordinal number, although this depends on the details of one's structuralist metaphysics. Furthermore, to make the discussion more concrete, we will focus on Shapiro's *ante rem* structuralism

as defended in Shapiro (1997) and elaborated in Shapiro (2006, 2008, 2012). The general structuralist position is that mathematical objects are characterized by their structural properties. That claim is common to all realist versions of structuralism, so what sets *ante rem* structuralism apart is the further claim that the structures to which mathematical objects belong (in which they are places) exist independently of any mathematical or non-mathematical systems that exemplify these structures. What this means is that the natural numbers are positions in these *ante rem* structures:

The essence of a natural number is its *relations* to other natural numbers ... The number 2, for example, is no more and no less than the second position in the natural number structure; 6 is the sixth position.
(Shapiro, 1997, p.72, original italics)

If this claim is correct, then the structuralist considers ordinal numbers to be fundamental. When it comes down to it, the natural numbers essentially are ordinal numbers. Cardinal numbers can, of course, be defined in the usual way on the basis of these ordinal numbers. They are just not metaphysically fundamental. Consequently, the notion of a cardinal number also has very little autonomy on this view.

To begin with, there is no way of formulating or understanding what a natural number is without appeal to the notion of an ordinal number on this interpretation of *ante rem* structuralism. Since the claim is that the notion of ordinal number captures the defining characteristics of a natural number, those two are not autonomous with respect to each other. If the natural number structure and the ordinal number structure coincide, then we are in the same situation as the

ZFC-theorist above. Cardinal numbers are defined in terms of ordinal numbers, and so are not logically autonomous with respect to the notion of ordinal number. Furthermore, understanding what a cardinal number is seems to have to go through such a definition. There is no independent structure of just cardinal numbers, and so there is nothing independent one might grasp in order to understand the concept of a cardinal number, or might figure in the justification of claims about cardinal numbers. The notion of a cardinal number, as defined based on the structuralist ontology suggested by this passage from Shapiro, leaves no room for autonomy with respect to the notion of an ordinal number.

There is, however, a complication. While Shapiro did make this claim about the nature of the natural numbers, he also suggested (Shapiro, 1997, ch.4) that there are finite cardinal structures containing no ordering between their positions. These structures are now better known under the heading of structures with *utter indiscernibles*, as an important feature of these finite cardinal structures is that the positions cannot be distinguished from each other (except for the fact that they are non-identical). Finite cardinal structures, since they are structures with the same number of positions as the relevant finite cardinal but no relations between those positions, are structures with utter indiscernibles. Another clear example of such a structure, presented by Leitgeb and Ladyman (2008), is that of a graph with two nodes and no edges. The two nodes in the graph, while distinct, cannot be distinguished from each other on the basis of any properties.

Utter indiscernibles are, however, far from uncontroversial. They violate the principle of identity of indiscernibles, in addition to the fact that it is hardly trivial to explain how we could refer to positions in such structures. There has, therefore, been a lively debate about the admissibility of such structures. Opponents of the

existence of utter indiscernibles include Button (2006), Hellman (2005), Keränen (2001), Macbride (2006), and Saunders (2003, 2006). On the other hand, Shapiro (2006, 2008, 2012) has defended the idea that there are structures with utter indiscernibles, as have Leitgeb and Ladyman (2008).

For this reason, we consider two versions of structuralism: a version with and a version without utter indiscernibles. The version without utter indiscernibles is the one discussed above, on which the natural numbers coincide with the ordinal numbers, and where there are no independent structures for cardinal numbers. More precisely, on this version, for any given structure with utter indiscernibles, we add a well-order on its positions. That is, we embed it into another structure that has a well-order. From the perspective of this new structure, the positions of the original structure will be discernible on the basis of their place in the well-order. (It should be noted that in many structures, such embeddings crucially depend on the axiom of global choice)

When there are structures with utter indiscernibles, the situation is different. In this case, the finite cardinal structures do offer metaphysically independent structures of cardinal numbers. The notion of a (finite) cardinal number can be formulated, independently from the of an ordinal number, in terms of these finite cardinal structures; i.e. each cardinal number is a position in cardinal structures with utter indiscernibles. Furthermore, the concept of a finite cardinal can therefore be understood independently from the concept of an ordinal number. To take a simple case, we can understand the concept of the cardinal number 4 on the basis of the exemplifications of the finite cardinal structure with four distinct positions. What these exemplifications have in common is their cardinal number, and so through abstraction (to the finite cardinal structure) we can get to the notion

of their cardinal number. The story will be more complicated for larger finite cardinals, such as three billion, since we clearly do not understand those on the basis of exemplifications of finite cardinal structures. However, that complication holds just as much for ordinal structures, and so whatever story one gives for finite ordinal structures can in turn be used for finite cardinal structures. Presumably, we can even justify claims about finite cardinals by appeal to finite cardinal structures alone, thus avoiding ordinal numbers in the justificatory case as well.

Finally, the notion of cardinal number on either version of structuralism is mathematically useful. This may not be immediately clear for structuralism with utter indiscernibles. One may think that finite cardinal structures cannot be applied, since then the application (the mapping of the positions in the structure to objects in the world) seems to offer a way to distinguish the positions in the structure. Such identification of the utter indiscernibles isn't possible, since without a mathematical/structural way to differentiate between them there is also no way to know which position was mapped to which object. All we know is that one position was mapped to one object, another position to another object, etc. Finite cardinal structures can be applied, even though the application leaves it unclear exactly which position is mapped to which object. For the application all we need is numerical distinctness, and positions in structures with utter indiscernibles are still numerically distinct. At least for applied mathematics, both types of structuralism are mathematically useful.

So, to sum up, we argued that (among mathematically useful notions of cardinal number) the notion of cardinal number is not conceptually autonomous from the notion of an ordinal number if it is understood along the lines of neo-logicism or structuralism without utter indiscernibles. In both these cases, it is necessary

to understand the concept of an ordinal number before one can understand the concept of a cardinal number. This absence of conceptual autonomy is not common to all accounts of the natural numbers. Structuralism with utter indiscernibles does have a notion of cardinal number that is conceptually autonomous from the notion of ordinal number. The question for the rest of this paper is whether this difference is helpful in deciding between these theories.

2 Distinguishing between cardinal and ordinal numbers

We argued that different philosophies of mathematics give us different notions of cardinal number. Some of these notions are autonomous with respect to the notion of an ordinal number and some of them are not. As mentioned earlier, the particular kind of autonomy we focus on in this second half is that of conceptual autonomy, as we think that it is easiest to derive empirical consequences from that notion. So, in what way would we see a difference between notions of cardinal number that are and those that are not conceptually autonomous with respect to the notion of ordinal number? Our basic idea is that the difference will show up in *concept acquisition*. If there is conceptual autonomy, then it is possible to understand the concept of cardinal number without understanding the concept of ordinal number. So, in the case of conceptual autonomy, it is possible to acquire the concept of cardinal number without having acquired the concept of ordinal number. More generally, it will be possible to acquire the concept of cardinal number without having recourse to the concept of ordinal number during concept

acquisition. Similarly, if there is no conceptual autonomy, then it is not possible to acquire the concept of cardinal number without use of the concept of ordinal number during the acquisition process.

There is one worry that needs to be dispelled before proceeding to the details of this suggestion. As we have tried to make clear, the notion of cardinal number used by the different philosophies of mathematics is not held constant. Neo-logicists use a different notion of cardinal number than structuralists. Within structuralism, too, the notion of cardinal number used differs depending on one's position regarding utter indiscernibles. It is, for that reason, not possible to show that any one notion of cardinal number is wrong by pointing to empirical data on concept acquisition. If we find out that the concept of cardinal number cannot be acquired independently from the concept of ordinal number, then this does not show that there is something philosophically wrong with the notion employed by structuralism with utter indiscernibles. Rather, it shows that this notion does not fit the concept of cardinal number that is acquired by the relevant community. It may very well be that a different community does acquire a concept of cardinal number that fits this notion, e.g. a community taught by such structuralists.

Yet it does make sense, in our opinion, to look at which notion of cardinal number best fits the notion found among laymen and professional mathematicians (note that these might well be different). As pointed out earlier, this is not relevant in all cases. Hale and Wright (2000), for example, are interested in offering an account of how it is possible to acquire *a priori* knowledge of arithmetic, and not necessarily of how this happens in actual practice. We, along with Linnebo (2009) and Heck (2000), have a different goal. Our interest is specifically on understanding how to understand the actual practice of arithmetic, and in this

paper, we focus on the practice of arithmetic by laymen – i.e. those who are not professional mathematicians. When looking for a philosophy of mathematics that best describes these practices, a relevant question is which notion of cardinal number best fits the concept of cardinal number prevalent in this community. In order to determine that, we can look at how this concept of cardinal number is acquired. If it can only be acquired via the concept of ordinal number, then it best fits a notion that is not conceptually autonomous, and vice versa. The difference in conceptual autonomy can therefore be exploited to determine what kind of account is the best when it comes to describing our actual mathematical practices.

With this goal in mind, it makes sense to look at concept acquisition, as there is a fair amount of empirical work related to the acquisition of number concepts (for a discussion of this work see Buijsman (forthcoming)). Some of that literature will be discussed in the next section. For now, though, we want to briefly consider other options to decide between positions based on whether or not they claim that the ordinal numbers are metaphysically more fundamental than the cardinal numbers. While we looked at conceptual autonomy, there are of course two other kinds of autonomy that might be of help. We suspect, however, that logical autonomy will be unhelpful. Contrary to concept acquisition it is difficult to directly study whether laymen have a notion of cardinal number that can be formulated without appeal to ordinal notions. This possibility should not be related to how laymen are able to formulate this notion themselves. They may not have reflected sufficiently on their notions of cardinal and ordinal number. Furthermore, it might be the case that philosophical accounts use concepts not grasped by the laymen even though the account gives a faithful description of the practices. When giving a truth-conditional semantics, for example, it is hardly a reasonable criticism to say

that truth-conditions do not accurately describe the semantic values of expressions because the speakers do not know what truth-conditions are. Focussing on logical autonomy is therefore unlikely to produce good empirical reasons either way.

Justificatory autonomy is more promising, but here too there are difficulties in finding appropriate empirical data. The basic difference between accounts here is whether it is possible to justify claims about cardinal numbers without using justifications for claims about ordinal numbers. That is, whether or not every justification for a claim about cardinal numbers builds on a justification for a claim about ordinal numbers. This means that there will be a difference in the kind of epistemology that can fit into the philosophy of mathematics. That difference is nicely illustrated by the two versions of structuralism we discussed. When there are utter indiscernibles, there are also finite cardinal structures. On the version without utter indiscernibles, no such structures are admitted into the ontology. As a result, it is only possible to justify claims about cardinal numbers on the basis of exemplifications of finite cardinal structures when there are utter indiscernibles. Otherwise, one has to reason via exemplifications of finite ordinal structures in order to justify claims about finite cardinals.

It seems to us that it is in part an empirical question whether we acquire justified beliefs about cardinal numbers without using justifications about claims regarding ordinal numbers. One part of that question is whether people are subjectively justified in their beliefs about cardinal numbers in this way. In other words, whether people are justified in these beliefs in this way and also (consciously) base these beliefs on those considerations. What people base their beliefs on is an empirical matter and it may well turn out that there are cases where people base their beliefs about cardinal numbers only on factors independent of ordinal

notions. Should this happen in a large number of cases then this would generate some pressure to say that even in these cases, people are subjectively justified. We do, after all, have *prima facie* reasons to think that people are subjectively justified in the majority of their arithmetical beliefs. Finding out what people base their beliefs on is, however, a difficult thing to do. so While we think that this strategy could lead to good results, it is for that reason not pursued here. Instead we focus on concept acquisition, which we turn to next.

3 An empirical reason for conceptual autonomy

The particular instance of concept acquisition that we think supports the idea that laymen use a notion of cardinal number conceptually autonomous with respect to the notion of ordinal number is the acquisition of the number *one*. Note that a single case where the concept of a cardinal number is acquired independently from any concepts of ordinal numbers is sufficient as an argument that we should have a philosophical account that allows for such a notion of cardinal number. We do not rule out that the other number concepts are acquired via ordinal concepts, but if there is at least one instance where this is not the case, then one's metaphysics should allow for that case.

We focus on the number *one* since the acquisition of this concept is widely thought to happen on the basis of a wider ability to distinguish between sets based on their size.⁶ Before this wider ability is acquired, children are, for example, able to distinguish situations where one object is presented to them from situations where three objects are presented. They can choose at above chance level to crawl

⁶This case is also discussed in Buijsman (forthcoming)

towards three cookies instead of one cookie (Feigenson and Carey, 2003). However, they are unable to distinguish between situations with one object and situations with four objects. When presented with one cookie on the left and four on the right, children are at chance when choosing to crawl in either direction (Feigenson and Carey, 2003). The ability to distinguish between one and three is based on an innate object tracking system that is able to follow up to three items at a time (Feigenson et al, 2004). Since the object tracking system doesn't have room for more than three items at any one time, some sort of cognitive dissonance in how the stimuli are processed causes infants to make no distinction between situations with one and situations with four objects.

That changes quite suddenly, and for currently unknown reasons, around 22 months. Children at this age start to succeed at comparing 1 to 4 and other larger numbers, but still fail at comparisons involving, for example, 2 objects and 4 objects. For that reason it has typically been classified as a general ability to distinguish between collections with one item ('singular sets' in the lingo of the psychologists) and collections with more than one item (Sarnecka and Lee, 2009). It is at this stage not linked to any number concepts, or at least not to the word *one*. As the current evidence stands, the idea is that this ability underlies our acquisition of the cardinal concept ONE. There are two reasons for this.

First, the pace at which the cardinal concept ONE is acquired is influenced by whether or not the language which the child speaks explicitly marks a singular/plural distinction. So, it matters for the speed of acquisition whether the distinction between collections with one item and collections with more than one item is grammatically marked. English does this, but Japanese does not explicitly mark the distinction in the grammar. English speaking children consequently ac-

quire the cardinal concept ONE faster than Japanese speaking children (Sarnecka et al, 2007). The concept in question is the cardinal number concept, as all the tests used to see if the number concept has been acquired involve children answering ‘how many?’ questions or giving an experimenter a set number of items (this holds for the vast majority of experiments cited here).

Related to this, children start marking the singular/plural distinction in language around the same time at which they start succeeding at tasks involving the wider ability to distinguish between collections with one and those with more than one item (Barner et al, 2007; Barner and Bachrach, 2010). This means that the relation between the grammatical singular/plural distinction and concept acquisition also links to this wider ability. It is the wider ability that, presumably, allows children to grammatically mark the singular/plural distinction and that in turn helps with acquiring the cardinal number concept ONE. Hence, the first reason for thinking that this acquisition process is based on the wider ability to distinguish between the two situations.

Second, the acquisition of the concept ONE is faster in children with a larger general vocabulary. Children acquire the concept faster if they know more words, even if those words are in no way related to the numbers (Negen and Sarnecka, 2012). This evidence supports the above argument. If there is a wider ability that supports the acquisition of the cardinal concept ONE, then greater opportunity to exercise this wider ability should lead to an easier time in acquiring the cardinal number concept. Basically, if something like attending to the singular/plural distinction leads to the acquisition of the concept ONE, then a larger vocabulary with more opportunities of marking this as a distinction independent from particular words will make it easier to acquire the number concept. We therefore reiterate

the hypothesis presented in Buijsman (forthcoming) that the acquisition of the cardinal number concept ONE is based on an ability to evaluate claims of the form $\exists!x Fx$ (i.e. that there is an F, and nothing else is an F). The correct evaluation of such claims allows one to grasp the singular/plural distinction and, with the help of syntactic clues (Syrett et al, 2012), assign the right interpretation to adult usage of the word *one*. In this way the cardinal concept ONE is acquired.

Now, why should this evidence be a reason to think that children are here acquiring a concept of a cardinal number without using a concept of an ordinal number in the process? The interpretation that the acquisition process is based on the ability to evaluate the quantifier $\exists!x$ is what we want to point to in response. Note that being able to evaluate this quantifier is sufficient to acquire the cardinal concept ONE. There is no need for (further) ordinal notions to make the step from the quantifier to the cardinal number concept. Besides, at the time at which children acquire the cardinal concept ONE they still fail at distinguishing collections with two items from collections with four items. There are no other numbers to which they could relate the cardinal number concept, as children also do not grasp the concept ZERO at this time.

There are, however, a few points where one can criticize the argument that there is a conceptually autonomous cardinal notion because the cardinal concept ONE is acquired on the basis of being able to evaluate $\exists!x$. The first issue is whether or not we are able to evaluate this quantifier on the basis of ordinal notions. If we should interpret this ability in a way which brings in ordinal notions, then we clearly do not have an acquisition process which does not involve ordinal notions. Of course, this too is an empirical question. What underlies this ability to distinguish between collections with one item and collections with more than

one item? If there is something to do with ordinal notions in the answer to this question – which is possible, as there are some indications that infants and animals are able to process ordinality (Cassia et al, 2012; Rugani and de Hevia, 2017) – then we do not have conceptually autonomous cardinal notions after all. The fact that the wider ability has been demonstrated in relation to cardinal notions, such as the grammatical singular/plural distinction, is unfortunately of little help in answering this question. As we mentioned earlier, we actually have very little idea what underpins this sudden ability to evaluate $\exists!x$. Admittedly this leaves a gap in our argument, one that can only be filled by more experimental research.

Another point that should be raised is that our argument has to rely on the claim that being able to evaluate $\exists!x$ is the only thing that matters in the process of acquiring the cardinal concept ONE. The reasons we gave, however, only support the claim that this ability is an important part in this process. There is no clear evidence to support the idea that this is the only thing that matters. It is tempting to think that this claim holds nevertheless, since being able to evaluate the quantifier is sufficient. Even so, more evidence is needed to support the idea that this is the only factor. One particularly important kind of evidence is related to the acquisition of ordinal number concepts. So far the cognitive psychologists have focussed exclusively on cardinal number concepts. Presumably, this is because they continued their research from the object tracking system and the Approximate Number System (ANS), where the latter is sensitive exclusively to cardinality but only up to a certain ratio (Feigenson et al, 2004). The ANS allows us to distinguish between collections with four and eight objects, but not, say, between collections with four and collections with five objects. Due to this focus very little is known about ordinal number concepts. No tests have been done to

see when children are able to ‘give the n th item’. If children do not possess any ordinal number concepts at the moment when they acquire the cardinal number concept ONE then that would strengthen our argument in favour of a conceptually autonomous notion of cardinal number. If, however, they already have an ordinal number concept ONE (or perhaps even several ordinal number concepts), then the situation is decidedly more complex.

This goes back to our earlier remark that at the time when children acquire the cardinal concept ONE there are no other number concepts to relate it to. This is correct as long as one thinks of cardinal number concepts. However, that was not in question. All philosophical accounts grant that the cardinal number concepts, once grasped, are not characterized by their relation to other cardinal concepts. The absence of other cardinal number concepts is therefore little reason to think that the cardinal number concept ONE is acquired without using ordinal notions. To reiterate, that can only be decided once we know more about children’s ordinal number concepts.

One more thing should be remarked in this context, namely that the fact that children are already familiar with more than one number word at the time at which they acquire the cardinal concept ONE does not show that ordinal notions played a role. Yes, they typically can count up to ten and are even able to enumerate objects using these number words. That is, they can point to objects in succession and go through the list of number words at the same time. This doesn’t mean that ordinal notions are used in the acquisition process, as there is very little evidence that children interpret these number words as anything more than a sequence just like the sequence of days, letters, and so on. There is no numerical interpretation given to the words (at least no cardinal interpretation). They can, therefore, make

very little use of these number words during concept acquisition. For the rest, we will have to await further evidence.

4 Conclusion

We started this paper with the question whether there is any way in which one can empirically distinguish between positions in the philosophy of mathematics that hold that ordinal numbers are metaphysically more fundamental than cardinal numbers and positions that do not. As we have tried to show, these two positions can be pulled apart in virtue of the fact that they lead to different claims about the autonomy of the notion of a cardinal number (because they lead to different notions of cardinal number). We then argued that two different claims are clearly empirically distinguishable. First, the claim that cardinal numbers are conceptually autonomous with respect to the ordinal numbers can be tested by looking at the way(s) in which we can acquire cardinal number concepts. Second, the claim that cardinal numbers are justificatorily autonomous with respect to the ordinal numbers might be tested by looking at the way in which people base their arithmetical beliefs. We want to stress again that neither of these tests show that there is something philosophically wrong with whatever position comes out as in conflict with the empirical data. The only thing these tests show is which philosophical position best describes the cardinal notions of the people studied.

The second half of the paper then presented some evidence for thinking that laymen might have a notion of cardinal number that is conceptually autonomous from the notion of ordinal number. If the evidence we presented is right, then children acquire the cardinal number concept ONE on the basis of being able to

evaluate the quantifier $\exists!x$. This suggests that, if the quantifier is understood in purely cardinal terms, we acquire a cardinal number concept without using ordinal notions. In other words, that we have a cardinal notion which is conceptually autonomous from ordinal notions and so that a view such as *ante rem* structuralism with utter indiscernibles better describes our cardinal number concepts than views such as neo-logicism and *ante rem* structuralism without utter indiscernibles. However, we did identify two places where this argument is in need of further empirical evidence. First, it is unclear that the quantifier is understood in purely cardinal terms. That is, it is unclear what allows us to evaluate this quantifier, which means that ordinal notions might still play a role here after all. Second, very little is known about our acquisition of ordinal number concepts. Therefore, it might be that as we learn more about these concepts we find out that ordinal number concepts figure in the acquisition of the cardinal concept ONE after all. While we have to await further empirical evidence to find out, for now we hope to have established a clear role for that evidence in the philosophy of mathematics.

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