Decomposition of perverse sheaves

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Abstract

This PhD thesis consists in three papers in which we describe irreducibility conditions and the number of factors in a composition series of certain perverse sheaves. We study some particular cases, providing examples and showing how to explicitly use perverse sheaves to obtain precise results. The aim is to add to the class of concrete applications of perverse sheaves and exploit their role in the cohomology of hyperplane arrangements. In the three papers the perverse sheaves considered are given by the derived direct image of locally constant sheaves defined in the complement U of a hyperplane arrangement. In Paper I, we start with a locally constant rank 1 sheaf on U and use a category equivalence, developed by MacPherson and Vilonen, to obtain a criterion for the irreducibility in terms of a multi-index that determines the locally constant sheaf. We then determine the number of decomposition factors when the irreducibility conditions are not satisfied. In Paper II we consider the constant sheaf on U, show that the number of decomposition factors of the direct image is given by the Poincaré polynomial of the hyperplane arrangement, and furthermore describe them as certain local cohomology sheaves and give their multiplicity. In Paper III, we use the Riemann-Hilbert correspondence and D-module calculations to determine a condition describing when the direct image of a locally constant sheaf contains a decomposition factor as a perverse sheaf that has support on a certain flat of the hyperplane arrangement.

Keywords: *Sheaf cohomology, hyperplane arrangements, D-modules.*

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Abstract

This PhD thesis consists in three papers in which we describe irreducibility conditions and the number of factors in a composition series of certain perverse sheaves. We study some particular cases, providing examples and showing how to explicitly use perverse sheaves to obtain precise results. The aim is to add to the class of concrete applications of perverse sheaves and exploit their role in the cohomology of hyperplane arrangements. In the three papers the perverse sheaves considered are given by the derived direct image of locally constant sheaves defined in the complement $U := \mathbb{C}^n \setminus \mathcal{A}$ of a hyperplane arrangement $\mathcal{A}$. In Paper I, we start with a locally constant rank 1 sheaf $\mathcal{L}_a$ in $\mathbb{C}^2 \setminus \mathcal{A}$ and use a category equivalence, developed by MacPherson and Vilonen, to obtain a criterion for the irreducibility of $Rj_*\mathcal{L}_a$ in terms of a multi-index $a \in \mathbb{C}^n$ associated to the sheaf $\mathcal{L}_a$. We then determine the number of decomposition factors when the irreducibility conditions are not satisfied. In Paper II we define a constant sheaf $\mathbb{C}$ in the complement U of $\mathcal{A} \subset \mathbb{C}^n$, show that the number of decomposition factors of $Rj_*\mathbb{C}_U$ is given by the Poincaré polynomial of the hyperplane arrangement $\mathcal{A}$ and describe them as certain local cohomology sheaves and give their multiplicity. In Paper III, we use the Riemann-Hilbert correspondence and D-module calculations to determine a condition for which $Rj_*\mathcal{L}_a$ contains a decomposition factor as perverse sheaf that has support on a certain flat of the hyperplane arrangement.

Sammanfattning

Denna avhandling består av tre uppsatser i vilka vi beskriver villkor för irreducibilitet och antal dekompositionsfaktorer i en dekompositionsserie för vissa perversa kärvar. Vi studerar några speciella situationer som exemplifierar explicita beräkningar med perversa kärvar. Avsikten är att öka antalet konkreta tillämpningar av perversa kärvar, i synnerhet tillämpa dem för kohomologi av hyperplanarrangemang.

I alla de tre uppsatserna är de perversa kärvar som studeras den härledda direkta bilden av en lokalkonstant kärve på komplementet $U := \mathbb{C}^n \setminus \mathcal{A}$ av ett hyperplanarrangemang $\mathcal{A}$. I uppsats I, är utgångspunkten en lokalkonstant rang 1 kärve $\mathcal{L}_a$ i $\mathbb{C}^2 \setminus \mathcal{A}$, och med hjälp av en kategoriekvivalens, utvecklad av MacPherson and Vilonen, erhåller vi ett kriterium för irreducibiliteten av $Rj_*\mathcal{L}_a$ i termer av multiindexet $a \in \mathbb{C}^n$ associerat till kärven $\mathcal{L}_a$. Dessutom bestäms antalet dekompositionsfaktorer och deras stöd, i den situation när irreducibilitetsvillkoret inte är uppfyllt.

I uppsats II bestämmer vi för den konstanta kärven $\mathbb{C}$ på komplementet U av $\mathcal{A} \subset \mathbb{C}^n$ explicit antalet dekompositionsfaktorer till $Rj_*\mathbb{C}_U$ som Poincarépolynomet av hyperplansarrangemanget. Dessutom identifieras dekompositionsfaktorerna; de är kärvar av lokala kohomologigrupper på flator, och faktorernas multiplicitet beräknas.

I uppsats III, använder vi Riemann-Hilbert korrespondensen tillsammans med D-modulberäkningar för att ge ett (partiellt) villkor för att dekompositionsserien av $Rj_*\mathcal{L}_a$ innehåller en faktor som har support på en given flata(snitt av hyperplan) i hyperplanarrangemanget.

Sumário

Esta Tese de Doutoramento é composta por uma introdução alargada e três artigos. Na Introdução são apresentadas algumas definicões e resultados conhecidos, sob uma forma que se adequa ao trabalho que desenvolvemos. Nos artigos descrevemos as condições para a irredutibilidade de certos feixes perversos e, no caso dessas condições não serem cumpridas, o número de factores de decomposição das correspondentes séries de composição.

Foram estudados alguns casos particulares, em que se forneceram exemplos e se mostrou como efectivamente fazer uso dos feixes perversos de modo a obter resultados precisos. O objectivo foi contribuir para a classe das aplicações concretas dos feixes perversos e no sue papel na co-homologiade arranjos de hiperplanos.

No Artigo I, começamos por considerar um arranjo central de linhas plano $\bigcup_{i=1}^{n} L_i \subset \mathbb{C}^2$. No seu complementar $X = \mathbb{C}^2 - \bigcup_{i=1}^{n} L_i$ definimos um feixe localmente constante de espaços vectoriais $\mathcal{L}_a$ associado a um multi-índice $a \in \mathbb{C}^n$. Usando a descrição de MacPherson e Vilonen da categoria dos feixes perversos ([17] and [18]) obtemos um critério para a irredutibilidade e o número de factores de decomposção da imagem directa $Rj_*\mathcal{L}_a$ como um feixe perverso, onde $j : X \to \mathbb{C}^2$ é a inclusão canónica.

No Artigo II, o nosso objecto inicial é um arranjo de hiperplanos $\mathcal{A}$ em $\mathbb{C}^n$. Seja $Rj_*\mathbb{C}_U$ a imagem directa no complemento U do arranjo $\mathcal{A}$. Tomando $Rj_*\mathbb{C}_U$ como um feixe perverso, mostramos que o seu número de factores de decomposição é dado pelo Polinómio de Poincaré do arranjo. Apresentamos também uma descrição dos factores de decomposição de $Rj_*\mathbb{C}_U$ como certos feixes de co-homologia local e indicamos a sua multiplicidade.

No Artigo III voltamos a tomar um arranjo de hiperplanos $\mathcal{A}$ em $\mathbb{C}^n$ e usamos o espaço de módulos de fibrados de linhas de característica 1 em U e D-módulos para determinar uma condição para que que a imagem directa $Rj_*\mathcal{L}_a$ contenha um feixe perverso como factor de decomposição com suporte na variedade linear F do arranjo de hiperplanos.

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this PhD thesis.

PAPER I: **Decomposition of Perverse Sheaves on Plane Line Arrangements** [1]
R. Bøgvad and I. Gonçalves, Communications in Algebra, Volume 46, Issue 6

PAPER II: **Length and Decomposition of the Cohomology of the Complement to a Hyperplane Arrangement**
R. Bøgvad and I. Gonçalves, *Preprint, arXiv:1703.07662*

PAPER III: **Support of Decomposition Factors of Direct Images of Line Bundles on the Open Complement of a Hyperplane Configuration**
R. Bøgvad and I. Gonçalves

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[1] This paper is mostly contained in a more detailed version as part of the Licentiate thesis ([10]).

Contents

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1. Introduction

To study a complicated object, one tries to see how it is built from simpler objects. In an abelian category these simpler objects may consist in irreducible objects. This PhD thesis is concerned with decomposition of direct images of perverse sheaves. We start by presenting a general overview of the main definitions and properties, in particular we will describe perverse sheaves. Our general reference in this is the work by Beĭlinson, Bernstein and Deligne [3].

1.1 Sheaves

1.1.1 Abelian categories

Recall that an abelian category is a category in which all hom-sets are abelian groups and the composition of morphisms is bilinear, all finite limits and colimits (in particular kernels and cokernels) exist and have certain good properties, the main being the following:

Let $\mathcal{C}$ be an abelian category, $A, B \in \mathcal{C}$ and f the morphism $A \to B$. The fundamental defining property of an abelian category is:

$$cok(ker(f) \to A) \xrightarrow{\cong} ker(B \to cok(f)), \qquad (1.1)$$

which for categories of modules over a ring reduces to one of the isomorphism theorems: $A/ker(f) \cong im(f)$.

Hence in an abelian category we may talk about injection, subobjects and simple objects. The category of perverse sheaves is abelian and it has the property that there is a finite decomposition series for each object (see below subsection 1.2.3).

1.1.2 Sheaves and Operations

A general reference for this section is [12], in particular pp.223-225. In our work we only consider sheaves with sections that are finite-dimensional vector spaces over $\mathbb{C}$. There is a direct equivalence between complex representations of the fundamental group and locally constant sheaves, given by the monodromy representation on stalks.

First recall that a locally constant sheaf on a simply connected and path-connected space is constant. We will always assume that the spaces considered satisfy these conditions. Let then $\mathcal{L}$ be a locally constant sheaf on a topological space V. For a fix point $x_0 \in V$, consider an element $\gamma \in \pi_1(V,x_0)$, represented by a path $\gamma(t), t \in [0,1]$. The gluing property of sheaves allow us to identify $\mathcal{L}_{x_0}$ with $\mathcal{L}_{\gamma(t)}$, independently of the the homotopy class of $\gamma(t)$.

Proposition 1.1.1. *Let V be a path-connected topological space and $\pi_1(V,x_0)$ be its fundamental group with base point x_0. The monodromy representation of $\pi_1(V,x_0)$ on the stalk $\mathcal{L}_{x_0}$, defines an equivalence between the category of local systems $\mathcal{L}$ on a space V and the category of finite dimensional complex representations of the fundamental group of V.*

Example 1.1.1. Consider a union of lines L_i, $i = 1,...,n$, through the origin in $\mathbb{C}^2$. Let Γ_i correspond to a loop around L_i. These loops generate the fundamental group of the space $\mathbb{C}^2 \setminus \cup_{i=1}^n L_i$, which has the following presentation:

$$\pi_1(\mathbb{C}^2 \setminus \cup_{i=1}^n L_i) = \langle \Gamma_1, \ldots, \Gamma_n \rangle / R,$$

where R is the group generated by the (cyclic) relations

$$\Gamma_1\Gamma_2 \ldots \Gamma_n = \Gamma_2 \ldots \Gamma_n\Gamma_1 = \Gamma_n\Gamma_1 \ldots \Gamma_{n-1}.$$

Then, locally constant sheaves $\mathcal{L}$ of rank 1 on $\mathbb{C}^2 \setminus \cup_{i=1}^n L_i$ are classified up to isomorphism by the element $a = (a_1,...,a_n) \in \mathbb{C}^n$, such that for the monodromy representation

$$\Gamma_i \mathbf{e} = a_i \mathbf{e} \tag{1.2}$$

where $\mathcal{L}_{x_0} = \mathbb{C}\mathbf{e}$ is the stalk at x_0.

The example above, showing the direct relation between the local systems on a space V and the finite dimensional representations of the fundamental group of the same space V, provides the starting point of Paper I (see section 2.1.1).

Operations

Let X be a topological space, U an open subset of X and F the closed complement. We will denote by j the inclusion of U in X and by i the inclusion of F in X. Let $\mathcal{S}h(X)$ (respectively $\mathcal{S}h(U)$ and $\mathcal{S}h(F)$) denote the category of sheaves on X (respectively on U and F).

Then there are the following basic functors that relate the categories above:

- $j_! : \mathcal{S}h(U) \to \mathcal{S}h(X)$: extension by 0 (exact);

- $j^* : \mathcal{S}h(X) \to \mathcal{S}h(U)$: restriction (exact, also denoted by $j^!$);
- $j_* : \mathcal{S}h(U) \to \mathcal{S}h(X)$: direct image (left exact);
- $i^* : \mathcal{S}h(X) \to \mathcal{S}h(F)$: restriction (exact);
- $i_* : \mathcal{S}h(F) \to \mathcal{S}h(X)$: direct image (exact, also denoted by $i_!$);
- $i^! : \mathcal{S}h(X) \to \mathcal{S}h(F)$: sections with support in F (left exact).

1.2 Perverse sheaves

The main object of this PhD thesis might be described in the following way: consider a topological space X, a closed subset S and a direct image functor $j : X - S \to X$. Let $\mathcal{L}_a$ be a a locally constant sheaf in the open subspace $X - S$. We will be interested in certain characteristics of the perverse sheaf $Rj_*\mathcal{L}_a$ in different settings.

1.2.1 The category of complexes and the derived category

Let $\mathcal{A}$ denote the category of sheaves on a topological space X and $K(\mathcal{A})$ the homotopy category of complexes over $\mathcal{A}$. Let $\mathcal{Q}$ be the class of $K(\mathcal{A})$ consisting of all quasi-isomorphisms. The category obtained by formally inverting the class $\mathcal{Q}$ of quasi-isomorphisms is the (bounded below) derived category $D(\mathcal{A})$ of $\mathcal{A}$ (see [12], pp. 430-435).

Any bounded below complex $A^\bullet$ admits a quasi-isomorphism $f : A^\bullet \to I^\bullet$ into a bounded below complex of injective objects, $I^\bullet$ (an injective resolution of $A^\bullet$, that is a right resolution whose all elements are injective)(see [12], pp.40). By working with injective resolutions the derived category becomes more manageable. The definition of derived functors is an example.

Definition 1.2.1. ([21], pp.14) Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$. Let $A^\bullet$ be a complex of $\mathcal{A}$ and $A^\bullet \cong I(A^\bullet)$ an injective resolution. Then define $RF(A^\bullet) := F(I(A^\bullet))$. This establishes a functor

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

called a right derived functor of F. The i-th right derived functor

$$R^iF : D^+(\mathcal{A}) \to \mathcal{B}$$

is defined by $R^iF = H^i \circ RF$.

If $A^\bullet \in K(\mathcal{A})$, the shifted complex $A^\bullet[m]$ is defined to be the complex that in degree n is A^{n+m}, and has differentials that are those of $A^\bullet$ multiplied by $-m$. If $f : A^\bullet \to B^\bullet$ is a chain map then the mapping cone is the complex $M^\bullet := A^\bullet[1] \oplus B$. It sits in a sequence

$$A^\bullet \xrightarrow{f} B^\bullet \to M^\bullet, \tag{1.3}$$

which can be continued by a map $M^\bullet \to A^\bullet[1] \to B^\bullet[1] \to ...$, and therefore often written as

$$A^\bullet \xrightarrow{f} B^\bullet \to M^\bullet \xrightarrow{+1} ...$$

We will denote the mapping cone as above by $Cone(f)$. Any sequence in the derived category that is homotopy equivalent to a sequence of type (1.3) in $K(\mathcal{A})$ is called a distinguished triangle. This descends to the derived category $D(\mathcal{A})$, where again a distinguished triangle is a sequence that is isomorphic to one of the type (1.3).

These two features, the translation functor $A^\bullet \mapsto A^\bullet[1]$ and the set of distinguished triangles, make $D(\mathcal{A})$ into a *triangulated category* (see the axioms that have to be satisfied in [3]).

1.2.2 Definition of perverse sheaves

The definition of perverse sheaves can be presented in different ways according to the properties we want to explore, but every definition demands a certain level of abstraction and a rather complex previous technical work. We are going to present the definition from [3].

Let X be topological space and $D = D_X$ the derived category of sheaves on it. The definition of perverse sheaves is based on the concept of a t-structure, which comprehends a triangulated category D, two full subcategories ${}^pD^{\leq 0}$ and ${}^pD^{\geq 0}$ and perverse truncation functors, $\tau_p^{\leq 0} : D \to {}^pD^{\leq 0}$ and $\tau_p^{\geq 0} : D \to {}^pD^{\geq 0}$. More generally, we may define the truncation functors as $\tau_p^{\leq n} X^\bullet := (\tau_p^{\leq 0}(X^\bullet[-n]))[n]$ and $\tau_p^{\geq n} X^\bullet := (\tau_p^{\geq 0}(X^\bullet[-n]))[n]$.

The category of perverse sheaves $\mathcal{M}(X)$, in the case, $D = D_X$ is a full subcategory of D, corresponding to its heart (or coeur): $\mathcal{M}(X) := {}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X)$.

$\mathcal{M}(X)$ turns out to be an abelian category and therefore we can find kernels and cokernels. Let $f : Q^\bullet \to R^\bullet$ be a map of perverse sheaves. The kernel and cokernel are defined through the perverse truncation functors as:

$$ker(f) = (\tau_p^{\leq -1}(Cone(f))^\bullet)[-1]$$

and

$$coker(f) = \tau_p^{\geq 0}(Cone(f))^\bullet.$$

In $\mathcal{M}(X)$ is also defined a cohomological functor ${}^pH^0 := \tau_p^{\leq 0}\tau_p^{\geq 0}$. Similarly, in a more general setting, we define ${}^pH^m := \tau_p^{\leq m}\tau_p^{\geq m}$.

We will now describe this more precisely. Note that the most important of the above concepts is the concept of truncation – using it the others may be defined.

t-category

Definition 1.2.2. ([9], pp.125) A t-category is a triangulated category D, with two strictly full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ of the category D, such that, by setting $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$, one has the following properties:

- $Hom(X^\bullet;Y^\bullet) = 0$ if $X^\bullet \in D^{\leq 0}$ and $Y^\bullet \in D^{\geq 1}$;
- $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$;
- for any object $X^\bullet \in D$, there is a distinguished triangle

$$A^\bullet \to X^\bullet \to B^\bullet \xrightarrow{+1} A^\bullet[+1]$$

  with the object $A^\bullet$ in $D^{\leq 0}$ and the object $B^\bullet$ in $D^{\geq 1}$.

We say that $(D^{\leq 0}, D^{\geq 0})$ is a t-structure over D. Its heart is the full subcategory $\mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$.

In the above definition $A^\bullet := \tau^{\leq 0}X^\bullet$ and $B^\bullet := \tau^{\geq 1}X^\bullet$. As referred before, truncation functors may be defined as $\tau^{\leq n}X^\bullet := (\tau^{\leq 0}(X^\bullet[-n]))[n]$ and $\tau^{\geq n}X^\bullet := (\tau^{\geq 0}(X^\bullet[-n]))[n]$. For these functors we have the following proposition.

Proposition 1.2.3. *([3], pp.29) The inclusion of $D^{\leq n}$ in D admits a right adjoint $\tau^{\leq n}$, and the inclusion of $D^{\geq n}$ admits a left adjoint $\tau^{\geq n}$. For every $X^\bullet$ in D, there exists a unique morphism $d \in Hom^1(\tau^{\geq 1}X^\bullet, \tau^{\leq 0}X^\bullet)$ such that*

$$\tau^{\leq 0}X^\bullet \to X^\bullet \to \tau^{\geq 1}X^\bullet \xrightarrow{d}$$

is a distinguished triangle. Apart from isomorphism, this is the unique distinguished triangle $(A^\bullet, X^\bullet, B^\bullet)$ with $A^\bullet$ in $D^{\leq 0}$ and $B^\bullet$ in $D^{\geq 1}$.

The simplest example of a t-structure on D_X is the one induced by $\tau^{\geq 0}$ and $\tau^{\leq 0}$ being the ordinary truncation operators on complexes. In this case the heart is just the category of sheaves on X. By using shifted truncation operators one gets a t-structure with a heart that is the the category of sheaves on X, but now shifted to a fix degree. We call this the *shifted trivial t-structure*. The idea of using perversities is now to shift differently according to the different strata in a stratification of X.

An at first view easy, but fundamental concept, is the one of perversity. We will only use a certain perversity, but it makes the construction clearer if we introduce it more generally.

Definition 1.2.4. Let X be a topological space and Σ a stratification of X. A perversity is a map $\underline{p} : \Sigma \to \mathbb{Z}$ (to each stratum an integer is associated).

The stratification that we are going to use will be by complex subvarieties S_i, $i = 1, \dots, d$ and the perversity that we will use will be the middle perversity $p(S_i) = -dim_{\mathbb{C}} S_i$.

Gluing t-categories using perversities

(For the following construction see [3], pp. 48-58.) The truncation in the category D_X has to be understood inductively from the truncation τ_U, associated to a t-structure in the category D_U, and τ_F in D_F, where $U \subset X$ is an open subset and $F = X \setminus U$ the closed complement. Consider $(D_U^{\leq 0}, D_U^{\geq 0})$ a t-structure on D_U and $(D_F^{\leq 0}, D_F^{\geq 0})$ a t-structure on D_F. Then define:

$$D^{\leq 0} := \{K \in D \mid j^* K \in D_U^{\leq 0} \text{ and } i^* K \in D_F^{\leq 0}\}$$

$$D^{\geq 0} := \{K \in D \mid j^* K \in D_U^{\geq 0} \text{ and } i^! K \in D_F^{\geq 0}\}$$

Proposition 1.2.5. *$(D^{\leq 0}, D^{\geq 0})$ is a t-structure over D.*

We say that we glue the t-structures over U and F.

Now, given a perversity, the idea is to build the t-structure inductively, starting with $U_1 = S_0$ and $F_1 = S_1$ ($S_0, S_1 \in \Sigma$), and using the t-structures on D_{U_1} and D_{F_1}, that are trivial t-structures that are shifted according to the perversities p_{U_1} and p_{F_1}, to build a t-structure on $X \setminus X_2 = S_0 \cup S_1$. In the next step one considers $X \setminus X_3 = X \setminus X_2 \cup (X_2 \setminus X_3)$, where one, by the previous inductive step, has a t-structure on D_{U_2}, with $U_2 = X \setminus X_2$ and uses the trivial t-structure on D_{F_2}, with $F_2 = X_2 \setminus X_3 = S_2$ shifted according to p_{S_2}.

We now describe this equivalently, just using the strata.

Lemma 1.2.6. *The subcategory ${}^pD^{\leq 0}(X)$ (resp. ${}^pD^{\geq 0}(X)$) of $D(X)$ is the subcategory given by the complexes $K^\bullet$ (resp. $K^\bullet$ in $D^+(X)$) such that for each stratum S, being i_S the inclusion of S in X, one has $H^n(i_S^* K^\bullet) = 0$ for $n > p(S)$ (resp. $H^n(i_S^! K^\bullet) = 0$ for $n < p(S)$).*

Definition 1.2.7. The category $\mathcal{M}(p, X)$ of p-perverse sheaves over X is the category ${}^pD^{\leq 0} \cap {}^pD^{\geq 0}$ (the heart of the t-structure $({}^pD^{\leq 0}, {}^pD^{\geq 0})$).

Remark 1.2.8. To a locally constant sheaf $\mathcal{L}$ defined on a topological space of even dimension, X, we can "attach" a perverse sheaf structure (for more details see [3], pp.63,64). If all the strata S_i, $i = 1,\ldots,d$ have even dimension in a complex space X, there exists a perversity defined in terms of their complex dimension, such that, for every stratum S, $p(S_i) = -dim_{\mathbb{C}} S_i$. If X is a complex variety of dimension d, the locally constant sheaf $\mathcal{L}$ placed in degree $-d$ corresponds to a perverse sheaf, since the cohomology groups satisfy the conditions in Lemma 1.2.6.

Truncation

We just describe the truncation for the case of two strata, an open U and a closed F. Let $j : U \to X$ and $i : F \to X$. To these strata we assume that we have perversities, where p_U is the perversity of U and p_F the perversity of F. For $\mathcal{K}^\bullet$ on U, $\tau_U^{\geq i}(\mathcal{K}^\bullet)$ in D_U is given by the shifted trivial truncation $\tau_U^{\geq i} := \tau^{\geq i+p_U}(\mathcal{K}^\bullet)$. For $\mathcal{L}^\bullet$ on F, $\tau_F^{\geq i}(\mathcal{L}^\bullet)$ is given by $\tau_F^{\geq i} := \tau^{\geq i+p_F}(\mathcal{L}^\bullet)$ in D_F. Similarily with the other truncations.

To get the perverse truncation which is associated to the glued t-structure:

- start with an object $X^\bullet$ in D, and choose $Y^\bullet$ in a way so that we have the distinguished triangle

$$(Y^\bullet, X^\bullet, j_*\tau_U^{\geq 1} j^* X^\bullet);$$

- then we define $A^\bullet$ such that the following is also a distinguished triangle

$$(A^\bullet, Y^\bullet, i_*\tau_F^{\geq 1} i^* Y^\bullet);$$

- finally we define $B^\bullet$ so that we have the third distinguished triangle (using the composition $A^\bullet \to Y^\bullet \to X^\bullet$ as the first map)

$$(A^\bullet, X^\bullet, B^\bullet).$$

Clearly all of this constructions may be done using mapping cones. It is not difficult to prove from Propositions 1.2.2 and 1.2.3 that $A^\bullet$ is in $D^{\leq 0}$ and $B^\bullet$ in $D^{\geq 1}$, and so the perverse truncation is determined as $A^\bullet = \tau_p^{\leq 0} X^\bullet$ and $B^\bullet = \tau_p^{\geq 1} X^\bullet$.

The relation between the perverse truncation and the kernel and cokernel

In the case of the trivial truncation it is easy to relate it with the kernel and cokernel of a morphism in the heart, i.e. a map between sheaves. There is an equivalent result for the perverse truncation and thus this gives a way of computing the kernel and cokernel for a morphism of complexes in the derived category.

Let $Z^\bullet$ be the mapping cone of the morphism $f : Q^\bullet \to R^\bullet$, between two perverse sheaves. Then we have maps

$$Q^\bullet \to R^\bullet \to Z^\bullet$$

and

$$Z^\bullet[-1] \to Q^\bullet \to R^\bullet.$$

Following the results of [3], pp.27-31, we have that:

$$ker(f) = (\tau_p^{\leq -1} Z^\bullet)[-1] \qquad , \qquad coker(f) = \tau_p^{\geq 0} Z^\bullet,$$

or more precisely the kernel of f is the composition

$$(\tau_p^{\leq -1} Z^\bullet)[-1] \to Z^\bullet[-1] \to Q^\bullet,$$

and the cokernel is the composition

$$R^\bullet \to Z^\bullet \to \tau_p^{\geq 0} Z^\bullet.$$

For our purposes we rewrite this as

$$ker(f) = (\tau_p^{\leq -1} Z^\bullet)[-1] = (\tau_p^{\leq 0} Z^\bullet[-1][1])[-1] = \tau_p^{\leq 0}(Z^\bullet[-1]) \tag{1.4}$$

$$coker(f) = \tau_p^{\geq 0} Z^\bullet = \left(\tau_p^{\geq 1}(Z^\bullet[-1])\right)[1] \tag{1.5}$$

In particular, one should note that the kernel and cokernel are possible to calculate in terms of the truncation operators.

Lemma 1.2.9. *Suppose $A \xrightarrow{\theta} B \to C$ is a distinguished triangle of perverse sheaves and that $\theta : A \to B$ is an injection in $\mathcal{M}(X)$. Then $C = coker\theta$.*

Remark 1.2.10. Let $Q^\bullet$ and $R^\bullet$ be complexes that are different from zero only in degree 0. Therefore the mapping cone is the complex $Z^\bullet : Q^\bullet \to R^\bullet$. Let us apply the definitions of kernel and cokernel given above to this $Z^\bullet$, but using the normal truncation, which corresponds to the perversity $p \equiv 0$. $Z^\bullet[-1]$ will be a complex different from zero in degrees 0, 1 and the result of $\tau^{\leq 0}(Z^\bullet[-1])$ will be precisely $ker(f)$ (in degree 0). The result of $\tau^{\geq 1}(Z^\bullet[-1])$ is, according to the definition of (trivial) truncation functors, $coker(f)$, in degree 1. After the last shifting, $\left(\tau^{\geq 1}(Z^\bullet[-1])\right)[1]$, the result will be $coker(f)$ in degree 0. Hence the truncation functor gives the kernel and cokernel for the trivial t-structure.

1.2.3 Irreducibility and composition series

Our main concern in this work is to define the conditions for the irreducibility of given perverse sheaves in different contexts and, in the case those conditions are not satisfied, the number of factors in its composition series.

Definition 1.2.11 ([14], Def.1.2.18)**.** Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$.

- An isomorphism class of a monomorphism with target X is called a subobject of X.
- An isomorphism class of an epimorphism with source X is called a quotient of X.

Definition 1.2.12 ([14], Ex.8.20)**.** Let $\mathcal{C}$ be an abelian category. An object X in $\mathcal{C}$ is irreducible (or simple) if it is not isomorphic to 0 and any subobject of X is either X or 0. A sequence

$$X = X_0 \supset X_1 \supset \ldots \supset X_{n-1} \supset X_n = 0$$

is a composition series if the quotient X_i/X_{i+1} is irreducible for all i with $0 \leq i < n$.

We call the integer n the length of the object X. We will denote the number of factors in a composition series of an object X by $c(X)$ and so $c(X) = n$.

Every perverse sheaf has a finite composition series whose successive quotients are irreducible perverse sheaves ([3]). As stated, the fact that $\mathcal{M}(X)$ is an abelian category implies that every morphism has a kernel and a cokernel. The explicit construction of kernels and cokernels and Lemma 1.2.9 above allow us to associate a decomposition series to the perverse sheaves we will study.

1.2.4 Grothendieck Group

The Grothendieck group is an abstract construction that can be defined for an abelian category. As mentioned before, the category of perverse sheaves is abelian and so has short exact sequences. In that case, exact sequences of perverse sheaves correspond to distinguished triangles:

$$A \to B \to C \xrightarrow{+1} A[1] \to$$

Definition 1.2.13. ([11], pp.95) Let $\mathcal{C}$ be an abelian category. Let F be the free abelian group generated by representatives of the isomorphism classes of objects in $Ob(\mathcal{C})$. We denote by the symbol $[N]$ the representative of $N \in Ob(\mathcal{C})$. Let F_0 be the subgroup generated by $[M] + [L] = [N]$ for all exact sequences $0 \to M \to N \to L \to 0$ in $\mathcal{C}$. The Grothendieck group $G(\mathcal{C})$ is by definition the factor group F/F_0.

From the definition above we get the following properties:

- $[0] = 0$ (take $N = L$);
- if $M \cong N$, then $0 \to M \xrightarrow{\cong} N \to 0 \to 0$, and so $[M] = [N]$;
- $[M \oplus L] = [M] + [L]$ (take $N = M \oplus L$).

One of the fundamental features of the Grothendieck construction is that it allow us to keep track of the building blocks of the objects we are studying, while disregarding their relations.

1.3 The Poincaré Polynomial

In its more general form the Poincaré polynomial is a diversified tool based in the notion of dimension of graded algebraic structures defined in a given space.

Definition 1.3.1 ([20], Def.4.117)**.** Let $\mathbb{K}$ be a field. Suppose $M = \oplus_{q\in\mathbb{Z}} M_q$ is a finitely generated graded vector space and each M_q is finite dimensional over $\mathbb{K}$. Then, the Poincaré series is

$$Poin(M,t) = \sum_{q\in\mathbb{Z}} (dim_{\mathbb{K}} M_q) t^q$$

We will work with the Poincaré polynomial of an hyperplane arrangement. Let $M(\mathcal{A})$ denote the complement of the arrangement $\mathcal{A}$.

Definition 1.3.2 ([20]. Def.5.92)**.** The Poincaré polynomial of the complement of the arrangement $\mathcal{A}$ is

$$Poin(M(\mathcal{A}),t) = \sum_{p\geq 0} dim H^p(M(\mathcal{A})) t^p$$

Inspired by the fact that the Poincaré polynomial is a combinatorial hyperplane invariant ([20]) and the results of a work from T. Oaku ([19]) relating the length of the first local cohomology group of a polynomial ring with the Poincaré polynomial, we reached a result that describes the number of decomposition factors of the direct image of a locally constant sheaf defined in the complement of a hyperplane arrangement in terms of its Poincaré polynomial. One of the main features of Poincaré polynomial appears in the work of Orlik and Terao, who extended results from V.I. Arnold and Brieskorn, proving that the dimension of the cohomology groups of the complement depend only on the lattice of intersections of the hyperplanes.

Let V be a topological space and $\mathcal{A} = \{H_0, \ldots H_m\}$ be an hyperplane arrangement in V.

Definition 1.3.3. Let $L(\mathcal{A})$ denote the set of all intersections of the arrangement $\mathcal{A}$. We consider that $V \in L(\mathcal{A})$ as the empty intersection.

We define in $L(\mathcal{A})$ a reverse inclusion partial order:

$$F \leq G \Leftrightarrow G \subseteq F$$

Definition 1.3.4. Let $F, G, K \in L(\mathcal{A})$. The Möbius function $\mu_{\mathcal{A}} : L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z}$ is defined recursively as:

$$\mu(F,G) = \begin{cases} 1 \quad \text{if } F = G \\ -\sum_{F \leq K < G} \mu(F,K) \quad \text{if } F < G \\ 0 \quad \text{otherwise} \end{cases}$$

For $F \in L(\mathcal{A})$, we denote $\mu(F) = \mu(V,F)$.

The following Lemma gives us a concrete, explicit way of computing the Poincaré polynomial of a hyperplane arrangement.

Lemma 1.3.5 ([20], Def.2.48))**.** *The Poincaré polynomial of $\mathcal{A}$ is defined by*

$$Poin(\mathcal{A},t) = \sum_{F \in L} \mu(F)(-t)^{r(F)}$$

where t is an indeterminate and $r(F)$ is the codimension of F in V.

Example 1.3.1. Consider the braid arrangement $\mathcal{B}_3$ in $\mathbb{C}^3$ consisting in 3 hyperplanes $B_{i,j} = \{x \in \mathbb{C}^3 \mid x_i = x_j, 1 \leq i < j \leq 3\}$.

For this arrangement the intersection lattice is

$$L(\mathcal{B}_3) = \{\mathbb{C}^3, B_{1,2}, B_{1,3}, B_{2,3}, B_{1,2} \cap B_{1,3} \cap B_{2,3}\}$$

Each one of the elements of $L(\mathcal{B}_3)$ has the following Möbius function:

$$\mu(\mathbb{C}^3) = 1, \quad \mu(B_{1,2}) = \mu(B_{1,3}) = \mu(B_{2,3}) = -1, \quad \mu(B_{1,2} \cap B_{1,3} \cap B_{2,3}) = 2$$

Finally,

$$Poin(\mathcal{B}_3,t) = 2t^2 + 3t + 1$$

Theorem 1.3.6 ([20], Thm.5.93)**.** *Let $\mathcal{A}$ be a complex arrangement in V and $M(\mathcal{A})$ its complement. Then,*

$$Poin(M(\mathcal{A}),t) = Poin(\mathcal{A},t).$$

The Poincaré polynomial can also be defined for an intersection lattice L that is not directly associated to an hyperplane arrangement. For a lattice of subvarieties in a space V, we define a partial order as above and the Poincaré is obtained in the same way.

One of the main properties of the Poincaré polynomial is that it satisfies the Deletion-Restriction Theorem.

Definition 1.3.7. Let $\mathcal{A} = \{H_0, \dots H_m\}$ be an hyperplane arrangement in V. We define the deleted arrangement as $\mathcal{A}' = \{H_1, \dots H_m\} = \mathcal{A} \setminus H_0$ and the restricted arrangement as $\mathcal{A}'' = \{H_0 \cap H_i \mid H_i \in \mathcal{A}' \text{ and } H_0 \cap H_i \neq H_0\}$. The arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ define a triple of arrangements.

Theorem 1.3.8 (Deletion-Restriction Theorem)**.** *([20], Thm.2.56) If $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ define a triple of arrangements, then*

$$Poin(\mathcal{A}, t) = Poin(\mathcal{A}', t) + tPoin(\mathcal{A}'', t)$$

It is possible to prove a statement similar to the Deletion-Restriction Theorem for the Möbius function. We will use results from K. Jewell ([13]) to outline a brief explanation of the result.

We consider a triple of hyperplane arrangements as before and let (L, L', L'') be the intersection posets. In [13] there is associated to this situation, and to each $F \in L$, certain simplicial complexes $(K[F], K'[F], K''[F])$ that are subcomplexes of the nerves of the arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. Each of these complexes contains a subcomplex $K(F), K'(F)$ and $K''(F)$ respectively, and there is a long exact sequence in relative homology:

$$\dots \to H_m(K'[F], K'(F)) \to H_m(K[F], K(F)) \to H_{m-1}(K''[F], K''(F)) \to \dots \tag{1.6}$$

Denote by $r_L(F) = codimF$. Then, as hyperplane arrangements are instances of what is called a geometric poset arrangement in [13, section 3], the following result applies:

Theorem 1.3.9. *([13], Thm.3.3)*

$$H_p(K[F], K(F)) = \begin{cases} \mathbb{Z}^{|\mu_L(F)|} \; \textit{if } p = r_L(F) - 1 \\ 0 \; \textit{if } p \neq r_L(F) - 1 \end{cases}$$

Since $\mathcal{A}'$ and $\mathcal{A}''$ are again hyperplane arrangements, this applies as well to L' and L''. So only three homology groups are non-zero in the sequence (1.6) and their ranks are the absolute value of the appropriate Möbius function.

Recalling standard properties of short exact sequences, it then follows from (1.6) that we have:

$$rk(H_m(K[F], K(F))) = rk(H_m(K'[F], K'(F))) + rk(H_{m-1}(K'[F], K'(F))).$$

And so, using the statement in Theorem 1.3.9, we conclude that

$$| \mu_L(F) | = | \mu_L'(F) | + | \mu_L''(F) | .$$

This property is used in paper 2. There is probably a more immediate proof of this property, but the paper of Jewell was were we somewhat indirectly found the statement.

1.4 Verdier Duality

The Verdier Duality consists in a generalization of Poincaré Duality. Its description is based in the dualizing complex and, among its main features, is the compatibility with direct and inverse images. We present here the Local Form of Verdier Duality since is the one that is applied to complex of sheaves.

Let X and Y be topological manifolds, $A^\bullet$ and $B^\bullet$ complex of sheaves in X and Y, respectively, and $f : X \to Y$ a continuous map. Verdier Duality is induced by the following adjunction property.

Theorem 1.4.1. *([5], Thm.7.17) In $D^b(Y)$ we have the canonical isomorphism*

$$RHom^\bullet(Rf_!A^\bullet, B^\bullet) \cong Rf_*RHom^\bullet(A^\bullet, f^!B^\bullet).$$

To define Verdier Duality it is convenient to introduce the dualizing complex.

Definition 1.4.2. The dualizing complex $w_X \in D^b(X)$ is defined by

$$w_X := a^!(A_{pt})$$

where $a = a_X : X \to \{pt\}$ is the projection to a point.

Let $D_X A^\bullet \in D^b(X)$ denote the dualizing functor (for more details see [14], Def. 3.1.6). Define $D_X A^\bullet = RHom(A^\bullet, w_X)$. This is the Verdier dual of $A^\bullet$.

There are some basic and important properties of the Verdier dual. Let $A^\bullet$ be a complex in $D^b(X)$, then:

1. $D_X : D^b(X) \to D^b(X)$ is a contravariant functor;
2. $D_X^2 A^\bullet = A^\bullet$;
3. $D_X(A^\bullet[n]) = D_X(A^\bullet)[-n]$, for any $n \in \mathbb{Z}$;
4. let $f : X \to Y$ be a continuous map, then:
   - $D_X f^* A^\bullet \cong f^! D_Y A^\bullet$;
   - $Rf_* D_X A^\bullet \cong D_Y Rf_! A^\bullet$;
   - $Rf_! D_X A^\bullet \cong D_Y Rf_* A^\bullet$, for $A^\bullet$ and $Rf_! D_X A^\bullet$ constructible.

This means that results on the decomposition of Rf_* can be related to results on $Rf_!$ by duality, a fact that is used in Paper 3.

1.5 The Weyl Algebra

In this section our references are [8] and [15].

Let k denote a field of characteristic zero and $k[\mathbf{x}]$ the ring of polynomials in n variables $\mathbf{x} = (x_1, \ldots, x_n)$. We will use the multi-index representation for the monomials in this ring, $x^\alpha = x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$ for $\alpha \in \mathbb{N}^n$.

Definition 1.5.1. The n-th Weyl algebra $D = k\langle \mathbf{x}, \partial \rangle$ is defined as the associative k-algebra on $2n$ symbols $\mathbf{x} = (x_1, \ldots, x_n)$ and $\partial = (\partial_1, \ldots, \partial_n)$ with relations

$$
\begin{aligned}
x_i x_j &= x_j x_i, && \text{for all} 1 \leq i, j \leq n; \\
\partial_i \partial_j &= \partial_j \partial_i, && \text{for all} \quad 1 \leq i, j \leq n; \\
\partial_i x_j &= x_j \partial_i, && \text{for all} \quad 1 \leq i, j \leq n,\ i \neq j; \\
\partial_i x_i &= x_i \partial_i + 1, && \text{for all} \quad 1 \leq i \leq n.
\end{aligned}
$$

The dimension of a finitely generated D-module is the degree of a certain Hilbert polynomial associated to it (for more details see [8], Chapter 9). Bernstein's inequality says that $dim(M) \geq n$ if M is non-zero.

Definition 1.5.2. A finitely generated D-module M over the n-th Weyl algebra D is holonomic if either $M = 0$ or $dim\, M = n$.

Holonomic D-modules present interesting properties, in particular, they form an abelian subcategory of the category of D-modules. They are artinian and noetherian and therefore their decomposition series have a finite length.

We describe now the holonomic module M_β^α that corresponds to $Rj_*\mathcal{L}_a$ ($a = exp(2\pi i\alpha)$) and which has been studied in connection with Bernstein-Sato polynomials (see Walther [22]).

The initial setting is a hyperplane arrangement

$$A = \{H_1 = V(\beta_1), \ldots, H_m = V(\beta_m)\}$$

in $\mathbb{C}^n$, where each $\beta_i : \mathbb{C}^n \to \mathbb{C}$, $i = 1, \ldots, m$ is a non-zero polynomial of degree 1. Define $U = \mathbb{C}^n \setminus V(\beta)$, where $\beta = \prod_{i=1}^m \beta_i$.

The coordinate ring $\mathcal{O}_U$ of U is the localization $\mathbb{C}[x]_\beta := \mathbb{C}[x_1, \ldots, x_n]_\beta$. This corresponds to a holonomic A_n-module, where A_n is the n-th Weyl algebra. Consider now the formal D-module corresponding to the multivalued function $\beta^\alpha = \beta_1^{\alpha_1} \ldots \beta_m^{\alpha_m}$.

Definition 1.5.3. ([2], Def.1.3) The module M_β^α is a the free rank 1 $\mathbb{C}[x]_\beta$-module on the generator β^α, where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$. It is furthermore an A_n-module, defining

$$\partial_j(\beta^\alpha) = \sum_{i=1}^{m} \alpha_i \frac{\partial_j(\beta_i)}{\beta_i} \beta^\alpha$$

for $j = 1, 2, \ldots, n$ and extending to an action of A_n on M^α_β.

One of the main results about holonomic D-modules is the Riemann-Hilbert correspondence. Denote by $\widetilde{D}_X$ the sheaf of rings of finite order holomorphic linear differential operators. Let $D^b_h(\widetilde{D}_X)$ denote the bounded derived category of holonomic complexes and $D^b_{rh}(\widetilde{D}_X)$ its subcategory of regular holonomic complexes (see [6], Chapter VIII).

Theorem 1.5.4. *([9], Thm.5.3.3) [The Riemann-Hilbert Correspondence] Consider the triangulated category $D^b_{rh}(\widetilde{D}_X)$ endowed with the natural t-structure and the triangulated category of constructible sheaves $D^b_c(X)$ endowed with the middle perversity t-structure. Then the de Rham functor*

$$D^b_{rh}(\widetilde{D}_X) \xrightarrow{DR} D^b_c(X)$$

is t-exact and establishes an equivalence of categories which commutes with direct images, inverse images and duality. In particular:

1. *DR induces an equivalence of categories between the abelian category $RH(D_X)$ of regular holonomic $\widetilde{D}_X$-modules on X and the abelian category of middle perversity perverse sheaves $Perv(X)$;*

2. *For any complex $M^\bullet \in D^b_{rh}(\widetilde{D}_X)$, one has an isomorphism*

$$DR(H^m(M^\bullet)) = {}^pH^m(DR(M^\bullet)).$$

Here ${}^pH^m$ is a cohomological functor ${}^pH^m : D^b_c(X) \to Perv(X)$, that was defined in Section 1.2.2.

As the category of perverse sheaves, the category of holonomic D-modules admits Grothendieck's six operations. Under the de Rahm functor it is possible to describe the correspondent operations in the two categories. Since any module over the Weyl algebra can be extended to a module over the differential operators with holomorphic coefficients $\widetilde{D}_X$, and this extension is an exact functor, we have, in particular, the following Proposition.

Proposition 1.5.5. *$DRM^\alpha_\beta = Rj_*\mathcal{L}_a$ (where j_* is the direct image $j_* : \mathcal{S}h(U) \to \mathcal{S}h(\mathbb{C}^n)$ and $\mathcal{L}_a$ is the locally constant rank 1 sheaf on U, which is defined by the fact that the monodromy around each H_i is given by multiplication by $a_i = exp(2\pi i \alpha_i)$).*

2. Summary of Papers

2.1 Summary of Paper I

This paper is published in ”Communications in Algebra”, Volume 46, Issue 6, DOI:10.1080/00927872.2017.1399410 ([4]). These results are partially contained in a more detailed version as part of my Licentiate thesis ([10]).

2.1.1 Problem and Results

In Paper I we present conditions for the irreducibility of the perverse direct image of a locally constant sheaf and the number of factors of its composition series, in the case the referred conditions are not satisfied.

The main definitions and properties of perverse sheaves we use are based in the work by Beilinson, Bernstein and Deligne, [3].

The initial setting is the space $\mathbb{C}^2 - \cup_{i=1}^{m} L_i$, where $\cup_{i=1}^{m} L_i$ represents a central line arrangement. Consider a rank 1 locally constant sheaf $\mathcal{L}_a$, determined, up to isomorphism by $a \in \mathbb{C}^m$ (see Example 1.1.1) and the inclusion $j : \mathbb{C}^2 - \cup_{i=1}^{m} L_i \to \mathbb{C}^2$. Our main problem was, since we know that the perverse sheaf $Rj_*\mathcal{L}_a$ has a finite decomposition series, to describe it in terms of $a \in \mathbb{C}^m$.

Theorem 2.1.1. *Assume that $m > 2$. The perverse sheaf $Rj_*\mathcal{L}_a$, where $j : \mathbb{C}^2 - \cup_{i=1}^{n} S_i \to \mathbb{C}^2$, is irreducible if, and only if, both of the following conditions are satisfied:*

- *$a_i \neq 1$, for all $i = 1, \ldots, m$;*
- *$\Pi_{i=1}^{m} a_i \neq 1$.*

In the second part we assume that $a_1, \ldots, a_k = 1$ and $a_{k+1}, \ldots, a_n \neq 1$. In this case we know that $Rj_*\mathcal{L}_a$ is not irreducible and we investigate the number of factors of its composition series, $c(Rj_*\mathcal{L}_a)$. Imposing different conditions on the product $\Pi_{i=1}^{m} a_i$ and on the value of k, we obtain models of the decomposition factors of $Rj_*\mathcal{L}_a$ and its number.

Theorem 2.1.2. *Assume that $a_1, \ldots, a_k = 1$ and $a_{k+1}, \ldots, a_m \neq 1$.*

- *If $k = m$, then $c(Rj_*\mathcal{L}_a) = 2m$*

- *If* $k < m$ *and* $\Pi_{i=1}^{m} a_i = 1$, *then* $c(Rj_*\mathcal{L}_a) = m + k - 1$.
- *If* $k < m$ *and* $\Pi_{i=1}^{m} a_i \neq 1$, *then* $c(Rj_*\mathcal{L}_a) = k + 1$.

2.1.2 Methods

In order to obtain the results above, we use a quiver description of irreducible perverse sheaves, first introduced by Deligne and later developed by MacPherson and Vilonen in the papers [16], [17] and [18]. In this quiver presentation we consider an abelian category $(F,G;T)$ whose objects are pairs $(A,B) \in Obj\mathcal{A} \times Obj\mathcal{B}$ together with a commutative triangle

$$\begin{array}{ccc} FA & \xrightarrow{\quad T_A \quad} & GA \\ & {\scriptstyle m}\searrow \qquad \nearrow{\scriptstyle n} & \\ & B & \end{array}$$

and whose morphisms are pairs $(a,b) \in Mor\mathcal{A} \times Mor\mathcal{B}$ such that

$$\begin{array}{ccccc} FA & & \xrightarrow{\quad T_A \quad} & & GA \\ & {\scriptstyle m}\searrow & & \nearrow{\scriptstyle n} & \\ {\scriptstyle Fa}\downarrow & & B & & \downarrow{\scriptstyle Ga} \\ & & \downarrow{\scriptstyle b} & & \\ FA' & & \xrightarrow{\quad T_{A'} \quad} & & GA' \\ & {\scriptstyle m'}\searrow & & \nearrow{\scriptstyle n'} & \\ & & B' & & \end{array}$$

We translate, to this new language, the definitions and results previously described in the category of perverse sheaves, in particular the concept of irreducibility and the functors ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$. As an intermediate step, we follow the work of MacPherson and Vilonen in [18], associating the functors F,G,T above, to the functors ψ (nearby cycles), ψ_c (nearby cycles with compact support) and Φ (vanishing cycles), respectively. Then we follow a two-step process, considering the inclusions, j^1 and j^2:

$$\mathbb{C}^2 - S \xrightarrow{j^1} \mathbb{C}^2 - \{0\} \xrightarrow{j^2} \mathbb{C}^2$$

The irreducibility conditions of ${}^p j^1_* \mathcal{L}_a$ are given by a direct application of a category equivalence established by MacPherson and Vilonen (Prop. 2.6). To determine the irreducibility conditions of ${}^p j^2_* \mathcal{L}_a$, we combine Lemma 2.7 and

Lemma 2.8, in a longer and more detailed process. It is interesting to point out that we start with a rather abstract problem and we manage to simplify it in a way that our final result comes from a computation of the determinant of a certain matrix. We conclude that the irreducibility conditions depend only on the values that $a \in C^m$ might assume. This method also allow us to give a quiver description of the decomposition factors of $Rj_*\mathcal{L}_a$, when the conditions of Theorem 2.1.1 are not satisfied.

2.2 Summary of Paper II

This paper can be accessed in arXiv:1703.07662.

2.2.1 Problem and Results

In Paper II our initial main goal is to describe the number of decomposition factors of the perverse direct image of a constant sheaf in terms of an invariant of the hyperplane, the Poincaré polynomial. Note that this contrasts with the problem in Paper I, in which a locally constant sheaf $\mathcal{L}_a$ is considered. In this paper we were primarily inspired by the work of Oaku, [19]. The references for the initial concepts were Iversen, [12] and Orlik and Terao, [20].

Our initial space was the complement of a hyperplane arrangement $H_0,\dots,H_m$ in $\mathbb{C}^n$. We considered the inclusion $j:\mathbb{C}^n\setminus\cup_{i=0}^m H_i\to\mathbb{C}^n$ and a constant sheaf on $\mathbb{C}^n\setminus\cup_{i=0}^m H_i$.

Theorem 2.2.1. *Let $\mathcal{A}$ be a hyperplane arrangement with hyperplanes H_i, $i=0,\dots,m$. Let $j:\tilde{U}:=\mathbb{C}^n\setminus\cup_{i=0}^m H_i\to\mathbb{C}^n$ be the inclusion of the complement to the arrangement, and $\mathbb{C}_{\tilde{U}}$ the constant sheaf on $\tilde{U}$. Then*

$$c(Rj_*\mathbb{C}_{\tilde{U}}[n])=\Pi(\mathcal{A},1)=\sum_{F\in L(\mathcal{A})}|\mu(F)|.$$

In the second part we take a more detailed view over the composition series of the perverse sheaf $Rj_*\mathbb{C}_{\tilde{U}}[n]$ and we describe it as an element in the Grothendieck group $G(\mathcal{A})$ of $Perv(\mathbb{C}^n)$. This is the free abelian group on a symbol $[K]$ for each perverse sheaf, modulo the relations $[M]+[K]=[N]$ for any short exact sequence $M\hookrightarrow N\twoheadrightarrow K$. $G(\mathcal{A})$ is a free abelian group with a basis corresponding to the set of irreducible objects.

We denote by F a flat of the arrangement $\mathcal{A}$ and define $N_F=i_*\mathbb{C}_F[\dim_{\mathbb{C}}F]\in Perv(X)$.

Proposition 2.2.2.

$$[Rj_*\mathbb{C}_{\tilde{U}}[n]]=\sum_{F\in L(\mathcal{A})}|\mu(F)|[N_F], \tag{2.1}$$

where $\mu(F)$ is the Möbius function on the intersection lattice $L(\mathcal{A})$.

2.2.2 Methods

We start by defining a restricted and deleted arrangement, $\mathcal{A}' := \mathcal{A} - H_0$ in $\mathbb{C}^n$ and $\mathcal{A}'' := \{H_0 \cap H \mid H \in \mathcal{A}' \text{ and } H_0 \cap H \neq 0\}$, respectively. Let $U = \mathbb{C}^n \setminus H_0$ and $V = \mathbb{C}^n \setminus \cup_{i=1}^m H_i$.

We then consider a Mayer-Vietoris sequence of the right derived direct images of the constant sheaf associated to the space $U \cap V = \mathbb{C}^n \setminus \cup_{i=1}^m H_i$. In the category of perverse sheaves this translates to a short exact sequence.

Since the number of decomposition factors of a perverse sheaf is the sum of the number of decomposition factors of perverse sheaves in a short exact sequence, we may show that:

Lemma 2.2.3. *The length c of the decomposition series of the sheaves in the short exact sequence satisfies*

$$c(i_{U\cap V*}i^!_{U\cap V}\mathbb{C}[n+1]) = 1 + c(i_{V*}i^!_V\mathbb{C}[n+1]) + c(i_{U\cup V*}i^!_{U\cup V}\mathbb{C}[n+2]),$$

where $i_Y : \mathbb{C}^n \setminus Y \to \mathbb{C}^n$ represents the closed inclusion.

The proof of the theorem proceeds by induction on the number of hyperplanes, using that there is a similar recursive formula for the Poincaré polynomial. To get the more precise result on multiplicities we use the fact, due to K. Jewell, [13], that, the a priori long exact Mayer-Vietoris sequence, splits into short exact sequences, implying an inductive formula for the Möbius numbers.

The proof of Proposition 2.2.2 is based on the previous results and on the fact that the Möbius function satisfies a version of the Deletion-Restriction Theorem:

$$|\mu_{\mathcal{A}}(F)| = |\mu_{\mathcal{A}'}(F)| + |\mu_{\mathcal{A}''}(F)|.$$

We can again use induction to prove the desired result.

2.3 Summary of Paper III

2.3.1 Problem and Results

In Paper III conditions for the existence of decomposition factors of the perverse sheaf $Rj_*L^\gamma[n]$ that have support on a flat are described (note the change of notation from paper I, L^γ here is the same as L_a there). In paper II we saw that when L^γ was trivial (i.e. a constant sheaf), there are decomposition factors with support on any flat. The situation is more differentiated when L^γ is arbitrary.

We use recent work from N. Budur [7] concerning the description of the locus M_s, the set of γ such that Rj_*L^γ is irreducible as a perverse sheaf, in terms of torii .

We consider a free hyperplane arrangement in $\mathbb{C}^n$, $\mathcal{A} = \{H_1, \dots, H_m\}$, and the inclusion $j : U := \mathbb{C}^n \setminus \cup_{i=1}^m H_i \to \mathbb{C}^n$. Let $M(\mathcal{A})$ be the moduli space of rank 1 locally constant sheaves on U. We have that $M(\mathcal{A})$ is isomorphic to the torus $\mathbb{C}^{*m}$. We denote by L^γ the sheaf corresponding to $\gamma = (\gamma_1, \dots \gamma_m) \in \mathbb{C}^{*m}$.

Let F be a flat in $\mathcal{A}$. We define:

- $J(F) = \{H \in \mathcal{A} : H \supset F\}$;
- $s_F(\gamma) = \{\prod_{H_i \in J(F)} \gamma_i\}$;
- $T_F = \{\gamma \in M(\mathcal{A}) : s_F(\gamma) = 1\}$, which is a subtorus of $\mathbb{C}^{*m}$.

We obtain the following result:

Theorem 2.3.1. *i) If K is a decomposition factor of $Rj_*L^\gamma[n]$, such that $F := suppK$ satisfies condition (A) relative to $Rj_*L^\gamma[n]$, then $\gamma \in T_F$.*

ii) Conversely, if $\gamma \in T_F$, where F is a dense flat, but $\gamma \notin T_{F'}$, for all flats F' such that $X \neq F' \neq F$, then $Rj_*L^\gamma[n]$ has a decomposition factor with support on F.

The condition (A) is a slight variation of saying that K is the top (i. e. is the maximal element in a decomposition series) modulo perverse sheaves with support of smaller dimension on F. In particular the top of $Rj_*L^\gamma[n]$ in a decomposition series always satisfies condition (A).

2.3.2 Methods

We consider a holonomic module M^α over the Weyl algebra, that, under the Riemann-Hilbert correspondence, is associated to the perverse sheaf $Rj_*L^\gamma[n]$. Hence its decomposition series as a Weyl algebra module corresponds to the decomposition series of $Rj_*L^\gamma[n]$ as a perverse sheaf. The relation between α and γ is that, $\gamma_i = exp(2\pi i\alpha_i)$. Let β_i be a defining polynomial of the affine hyperplane H_i.

Definition 2.3.2. ([1], Def.1.3) The module M^α is the free rank 1 $\mathbb{C}[x]_\beta$-module on the generator β^α, where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m$. It is an A_n-module, defining

$$\partial_j(\beta^\alpha) = \sum_{i=1}^m \alpha_i \frac{\partial_j(\beta_i)}{\beta_i} \beta^\alpha$$

for $j = 1, 2, \dots, m$ and extending to an action of A_n on M^α.

We then consider the closed inclusions of suitable generic lines $i_K : K \to \mathbb{C}^n$, and the effect of $i_K^!$ on M^α. This functor on the category of D-modules is related to local cohomology, and is easy to work with for closed inclusions. We then deduce the theorem using information on M^α from the behaviour of similar modules defined on lines.

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