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A Model-Theoretic Proof of Gödel's Theorem: Kripke's Notion of Fulfilment

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Mattias Granberg Olsson

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

A Model-Theoretic Proof of Gödel's Theorem: Kripke's Notion of Fulfilment

Mattias Granberg Olsson

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The notion of fulfilment of a formula by a sequence of numbers, an approximation of truth due to Kripke, is presented and subsequently formalised in the weak arithmetic theory $I\Sigma_1$, in some detail. After a number of technical results connecting the formalised notion to the meta-theoretical one a version of Gödel's Incompleteness Theorem, that no consistent, recursively axiomatisable, Σ_2-sound extension T of Peano arithmetic is complete, is shown by construction of a true Π_2-sentence and a model of T where it is false, yielding its independence from T. These results are then generalised to a more general notion of fulfilment, proving that $I\Sigma_1$ has no complete, consistent, recursively axiomatisable, Σ_2-sound extensions by a similar construction of an independent sentence. This generalisation comes at the cost of some naturality, however, and an explicit falsifying model will only be obtained under additional assumptions.

The aim of the thesis is to reproduce in some detail the notions and results developed by Kripke and Quinsey and presented by Quinsey and Putnam. In particular no novel results are obtained.

Contents

1 Introduction

In the well-known paper [2] (an English translation can be found in [3]), Kurt Gödel proved the (first version of the) theorem which is usually denoted by any nonempty combination of the labels “Gödel's” and “Incompleteness” juxtaposed by “Theorem”, usually summarised as “For any consistent recursively axiomatisable theory T containing a sufficient amount of arithmetic there is a sentence of its language which is neither derivable nor refutable in T”.[1] The usual proof of this fact is by arithmetisation of logic and T itself, that is by regarding (or coding) logical formulae as numbers and verifying that many of their properties (in particular that of membership in and derivability from T) can then be expressed in the language of arithmetic by rather simple (in a specific sense) formulae, which are in turn provable in T. This leads to what is known as the “Diagonal(isation) Lemma” or “Fixed Point Lemma”, that every property Φ of formulae has a fixed point, a sentence φ which is provably (in T) equivalent to the fact that Φ holds of φ, and setting Φ to be “Underivability in T” yields the theorem. The probably most cited instance of this theorem concerns Peano arithmetic (henceforth PA) and in particular states that PA is not the complete first-order theory of the natural numbers (i.e. there is a true statement of arithmetic which is underivable in PA), which can also be obtained by slightly less cumbersome arguments of the same spirit. E.g. [13] contains a readable survey of these results.

This could be argued not to be a satisfactory answer to the question whether PA proves all *mathematical* properties of the natural numbers, since the independent sentence above (and those of similar nature), while (in theory) explicitly defined, can not readily be seen to express some property of numbers of interest to, say, number theorists (and in fact “the” independent sentence might very well depend on the chosen encoding of formulae as numbers and the recursive axiomatisation of T, not giving credit to the definite article from a number theoretic point of view). The generally accepted first *purely mathematical* statements of arithmetic independent of PA are attributed to Jeff Paris and Leo Harrington in the late 1970's; in [9] they exhibited a combinatorial principle, an extension of the Finite Ramsey Theorem nowadays known as the Paris-Harrington principle (PH), which is true in $\mathbb{N}$ but unprovable in PA.

Various other independent statements and methods of their proof of independence have been produced over the years. This thesis will present one such method, called fulfilment, which is due to Saul Kripke. It has earlier been presented by Joseph Quinsey in [12] and Hilary Putnam in [10], and the aim of this thesis is to reproduce these results. The idea is, briefly, to define an approximation of truth in a recursively axiomatisable theory T (truth itself of course being undefinable by Tarski's Theorem) by a sequence of bounds for the variables of a sentence: the sentence is said to be fulfilled by the sequence if the universal quantifiers can be bounded by some numbers from the sequence and the existential quantifiers can then be bounded by the subsequent numbers in the sequence (we will give a precise definition in section 4). See also [10] for an illuminating account of this

[1] Thus we conform to the use of the term “Gödel's Incompleteness Theorem” to apply to a range of slightly different (at least technically) results, such as the Gödel-Rosser theorem. We will also use the terms “derivation” and “derivable” for the formal versions of “proof” and “proovable”.

idea in terms of winning strategies in an "evaluation-like" game played on the sequence and the sentence. Fulfilment turns out to be expressible in the language of arithmetic in T, and, under certain assumptions on T, one obtains a sentence independent of T by formalising "For every n, the first n axioms of T are fulfilled by a sequence of length n" (colloquially this states that "the axioms of T are approximately true").

One benefit of this approach is that we do not need to formalise the notion of proof in the theories we are considering, and the proof does not use the Diagonal Lemma. Another, and our main motivation, is that the proof of the independent sentence's non-derivability is by explicit construction of a model where it is false; for sound theories (i.e. theories true in $\mathbb{N}$) this model can be concretely defined. The dual, an example of a model where it is true, will also be given for sound theories. The drawback of this approach is that, instead of a provability predicate, we will in the most general case need a satisfaction predicate for Δ_0-formulae (formulae where every quantifier is bounded), which is not a considerably weaker requirement. We will also in all cases need a coding of sequences with some (provable) elementary properties, so we will require the theories we consider to include a weak induction schema, whence the proof will not be as generally applicable as the usual one. Moreover, the independent sentence we construct will be Π_2, as opposed to the Π_1-sentence given by the usual proof.

1.1 On the history of fulfilment

The notion of fulfilment is Saul Kripke's: in [8] he and Simon Kochen gives a proof of Gödel's Incompleteness Theorem for PA via the notion of bounded ultrapower. In a note therein it is remarked that he (Kripke) has an argument for generalising the method (and restricting the results) to theories other than PA, via "a concept of 'satisfying' formulas by finite sequences called fulfillability" to appear in a later paper. To the best of the author's knowledge, this paper was never (or has yet to be) published. This is supported by the few (again to the best of the author's knowledge) works presenting Kripke's proof, namely [12] and [10] (which also appears summarised in [11]). Indeed, in the latter Hilary Putnam comments that he publishes the paper in question "because Kripke's proof is *still* unpublished". In [12], Joseph Quinsey uses the method to derive a considerable number of results (old and new), which subsequently have been reproduced in other works, resulting in a few more mentions of the method. That being said, this method of proof appears to be largely unnoticed in the literature; [15] and [1] seems to be the prominent examples of other works relating it. A similar construction also appears in [14], though it is unknown (to the author) whether this is independent of the above.

1.2 Indicators

Another way to prove independence results in arithmetic (in particular for sound theories) is via the notion of indicators. We will describe this notion briefly here, since it has some relations to the subject of our work. We will base this exposition mainly on the one found in [7, ch. 14] where these relations are apparent, though we will use a generalised definition similar to the one in [4, ch. IV 3]. We refer to 1.4 and section 2 below for the terminology used in the following.

Let $T \supseteq \mathrm{I}\Sigma_1$ be an arithmetic theory and Q be a property of cuts of models of T. An *indicator* for Q in T is a T-provably Δ_1-function $[\mathsf{y}(\mathsf{x},\mathsf{y}){=}\mathsf{z}]$ such that

- $T \vdash \forall \mathsf{v} \forall \mathsf{y}{\leq}\mathsf{v} \forall \mathsf{x}{\leq}\mathsf{y} \forall \mathsf{u}{\leq}\mathsf{x}[\mathsf{y}(\mathsf{x},\mathsf{y}){\leq}\mathsf{y}(\mathsf{u},\mathsf{v})]$;
- for all nonstandard $\mathcal{M} \models T$ and $a, b \in \mathcal{M}$ with $a <_{\mathcal{M}} b$ is it the case that $\forall n \in \mathbb{N} : \mathcal{M} \models [\mathsf{y}(a,b){>}\overline{n}]$ if and only if there is a cut I in $\mathcal{M}$ with $a \in I \subseteq \mathcal{M}_{<b}$ which has the property Q.[2]

For our purposes, Q can be taken to be $I \models T'$ for some arithmetic theory T'.

For sound theories $T \supseteq \mathrm{I}\Sigma_1$ the existence of indicators immediately yield incompleteness results, via the following theorem ([4, Thm. 3.10, p. 248]): If $T \supseteq \mathrm{I}\Sigma_1$ is an arithmetic theory and $\mathtt{y}$ is an indicator for T in T then

1. $T \nvdash \forall \mathsf{x} \forall \mathsf{z} \exists \mathsf{y}[\mathsf{y}(\mathsf{x},\mathsf{y}){\geq}\mathsf{z}]$;
2. $\forall n \in \mathbb{N} : T \vdash \forall \mathsf{x} \exists \mathsf{y}[\mathsf{y}(\mathsf{x},\mathsf{y}){\geq}\overline{n}]$;
3. if T is sound then $\forall \mathsf{x} \forall \mathsf{z} \exists \mathsf{y}[\mathsf{y}(\mathsf{x},\mathsf{y}){\geq}\mathsf{z}]$ is independent of T;
4. $\{\mathsf{g}_n(\mathsf{x})\}_{n\in\mathbb{N}}$ defined by $T \vdash [\mathsf{y}(\mathsf{x},\mathsf{g}_n(\mathsf{x})){\geq}\overline{n}] \wedge \forall[\mathsf{y}{<}\mathsf{g}_n(\mathsf{x})][\mathsf{y}(\mathsf{x},\mathsf{y}){<}\overline{n}]$ (this defines a function by 2) for all $n \in \mathbb{N}$ are T-provably Δ_1-functions such that if $\mathtt{h}(\mathsf{x})$ is any other T-provably Δ_1-function then $T \vdash \forall \mathsf{x}[\mathtt{h}(\mathsf{x}){<}\mathsf{g}_n(\mathsf{x})]$ for some $n \in \mathbb{N}$.

Combinatorial statements like PH can be used to prove the existence of indicators for models of PA in PA by construction of "indiscernible sequences" $s : \mathbb{N} \longrightarrow \mathcal{M}$ (of a model $\mathcal{M}$ of PA) in the sense that

$$\mathcal{M} \models_e \exists \mathsf{v}_0{<}s(n_0) \forall \mathsf{v}_1{<}s(n_1) \cdots \vartheta$$

holds if and only if

$$\mathcal{M} \models_e \exists \mathsf{v}_0{<}s(m_0) \forall \mathsf{v}_1{<}s(m_1) \cdots \vartheta$$

for all increasing $\{n_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}_{>0}$ and $\{m_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}_{>0}$, Δ_0-formulae ϑ and evaluations e of $\exists \mathsf{v}_0 \forall \mathsf{v}_1 \cdots \vartheta$ in $\mathcal{M}_{<n_0-1} \cap \mathcal{M}_{<m_0-1}$ (see also [5, ch. 11.1] for a more general treatment of indiscernible sequence in model theory, including some connections to combinatorics). Since one may also require $s(i)^2 < s(i+1)$, these can in turn be used to construct cuts of the form

$$I = \{a \in \mathcal{M} \mid \exists i \in \mathbb{N} : \mathcal{M} \models a{<}s(i)\},$$

which will turn out to be initial substructures satisfying that

$$I \models_e \exists \mathsf{v}_0 \forall \mathsf{v}_1 \cdots \vartheta \Leftrightarrow \mathcal{M} \models_e \exists \mathsf{v}_0{<}s(i+1) \forall \mathsf{v}_1{<}s(i+2) \cdots \vartheta$$

for all Δ_0-formulae ϑ, all $i \in \mathbb{N}$ and evaluations e of $\exists \mathsf{v}_0 \forall \mathsf{v}_1 \cdots \vartheta$ in $\mathcal{M}_{<s(i)}$. In what follows we will construct models in a similar fashion, and the proof of non-derivability of our independent sentence will have certain aspects in common with the proof of non-derivability in 1 above (see for example Theorem 4.11).

[2] The standard notation for indicators seems to be a capital Y, but we will keep to our convention of denoting provable functions by lower case typewriter font letters.

1.3 Disposition

The disposition of this thesis will be as follows: In section 2 we introduce the arithmetic hierarchy of formulae and several weak arithmetic theories, most prominently Q, PA^- and $\mathrm{I}\Sigma_1$. We subsequently formalise set theory and logic in $\mathrm{I}\Sigma_1$, and show that many properties of formulae are provable in $\mathrm{I}\Sigma_1$. In section 3 we introduce the concept of sets and functions coded by objects in models of $\mathrm{I}\Sigma_1$ and derive some basic properties thereof. In section 4 we introduce the notion of initial-fulfilment briefly described above and use it to derive the incompleteness of consistent, recursively axiomatisable, Σ_2-sound extensions of PA. This is then generalised in section 5 to show that $\mathrm{I}\Sigma_1$ has no complete, consistent, recursively axiomatisable, Σ_2-sound extensions. Finally, section 6 briefly summarises these results and comments on similarities and differences to other proofs, as well as limitations and possible improvements.

1.4 Notation

We will use a convention similar to that in [4]: formulae will be particular numbers, in such a way that $\mathrm{I}\Sigma_1$ proves the basic properties of formulae (this might appear to be a circular definition, but bear in mind that we can define what numbers are formulae on the meta level (in $\mathbb{N}$) and then write down formulae formally defining these concepts in $\mathrm{I}\Sigma_1$ and prove all their relevant properties therein). Informally speaking, we consider first order logic with the logical symbols $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$, $=$ (for simplicity, $=$ is considered a logical symbol, and the equality rules are thus part of the rules of inference) and variables v_i for all $i \in \mathbb{N}$. The language of arithmetic, $\mathcal{L}_\mathrm{A}$, will in addition contain the non-logical symbols $\mathsf{0}$, S, $+$, $\cdot$ and $<$. The symbol $\leq$ will be used as an abbreviation: $\mathsf{x}\leq\mathsf{y}$ is $\mathsf{x}<\mathsf{y}\vee\mathsf{x}=\mathsf{y}$ (except as bound for a quantifier; see the comment after Lemma 2.21). In addition, parentheses (and) will be used in the presentation of formulae. They will not be considered as symbols of the language, but are used to clarify association in formulae and to distinguish between variables and constants on one hand and terms on the other. We will use different conventions regarding parentheses at the different levels of language (object language and the object language in the object language (which we will call the formal object language)); in the object language parentheses are treated mostly as reading aids to be omitted or included as is deemed fit, while in the formal object language they are treated more like operators, never to be left out. In the object language parentheses will also be used to denote substitutions in a formula (see below).

To distinguish the three different levels of language (the two above and the meta-language) we will use the following conventions (except were this is deemed to be confusing): In the meta-language all logical symbols are written in "blackboard bold" font (though we will seldom use logical symbols in the meta-language), while the variables and (defined) names will be subject to most usual conventions of math presentation, except as stated in what follows. We will use the lowercase Greek letters φ, ψ and ϑ to denote arbitrary formulae of the object language and τ, σ and ρ to denote arbitrary terms. The variables and (non-)logical symbols of the object language will be written in sans-serif font. While all variables are assumed to have a specific position within the enumeration v_i, we will in general suppress this, instead choosing as suggestive (lowercase) names for variables as possible; in case it matters these variable names will be considered meta variables. Thus, in a context where x, y et cetera occur without having been introduced, the statements should be read to hold for all variables, except that variables with different names are assumed to be distinct unless otherwise stated.

We will also have to introduce many abbreviations of formulae, which will be written in typewriter font. When introducing such an abbreviation of a formula, a list of variables are written out following the name of the formula, $\mathtt{F}(\mathsf{x}, \mathsf{y}, \mathsf{z})$; it is then tacitly understood that the free variables are among the listed variables, which in turn are v_0, v_1 etc. *in the order given following the name* $\mathtt{F}$, so that in the case under consideration x will denote v_0, y will denote v_1 and z will denote v_2. When introducing a (formula to be a) provable function we use an abbreviation of the form $[\mathtt{f}(\bar{\mathsf{x}})=\mathsf{y}]$, and the same conventions will be used except that y is v_0 and the x:s are v_1, v_2 etc. in order of appearance; we will subsequently often write simply "the formula $\mathtt{F}$" or "the provable function $\mathtt{f}$" when the number of variables are immaterial. Note that abbreviations of provable functions consists of lower-case letters throughout, while abbreviations of other formulae have names starting with a capital letter.

In line with these conventions, when substituting terms for variables in the formula $\mathtt{F}$ or provable function $\mathtt{f}$ we often write the term in question in place of the variable following the name $\mathtt{F}$ or $\mathtt{f}$ (like ordinary relation and function application). Substitution will otherwise be denoted as follows: If φ is a formula and $f : \mathbb{N} \rightharpoonup \mathrm{Term}$ then $\varphi(f)$ is the formula obtained by simultaneously substituting $f(i)$ for v_i in φ for all $i \in \mathrm{dom}(f)$ (note: we do not assume $\mathrm{dom}(f)$ to be finite). A tuple of terms will then be considered a function from an initial segment of $\mathbb{N}$ to Term. We use the shorthand $\varphi[v_{i_0}/\tau_0, \ldots, v_{i_n}/\tau_n] = \varphi(f)$ where $f(i_k) = \tau_k$ for all $k \leq n$. Moreover $\varphi(\tau_0, \ldots, \tau_n)$, where τ_i are terms, means $\varphi[v_0/\tau_0, \ldots, v_n/\tau_n]$. In the formal object language, finally, the formulae will be written in boldface formal font, and substitution of terms for variables will be denoted by defined (in the object language) operators.

The main exception to the above conventions is that we will use a somewhat longer standard equality sign $=$ in the meta-language instead of the "blackboard bold variant" $\mathsf{=}$, since introducing an unfamiliar sign for equality is deemed to be more confusing than clarifying. Cases where this practise itself is likely to lead to confusion will be avoided if possible. When equality is an abbreviation of a provable function, it is enclosed in brackets, like $[\mathtt{lh}(\mathsf{x})=\mathsf{y}]$. If $[\mathtt{f}(\bar{\mathsf{x}})=\mathsf{y}]$ and $[\mathsf{g}(\bar{\mathsf{z}})=\mathsf{y}]$ are such abbreviations, then $[\mathtt{f}(\bar{\mathsf{x}})=\mathsf{g}(\bar{\mathsf{z}})]$ is an abbreviation of $\forall\mathsf{y}([\mathtt{f}(\bar{\mathsf{x}})\leftrightarrow\mathsf{y}]\wedge[\mathsf{g}(\bar{\mathsf{z}})=\mathsf{y}])$, which is provably equivalent to $\exists\mathsf{y}([\mathtt{f}(\bar{\mathsf{x}})=\mathsf{y}]\wedge[\mathsf{g}(\bar{\mathsf{z}})=\mathsf{y}])$ given that the original expressions are provable functions (here $(\bar{\mathsf{x}})$ and $(\bar{\mathsf{z}})$ need not be disjoint). We extend these conventions to other predicates: if $[\mathsf{g}(\bar{\mathsf{x}})=\mathsf{y}]$ is a provable function and $\mathtt{F}(\bar{\mathsf{z}}, \mathsf{v})$ an abbreviation of a formula then $[\mathtt{F}(\bar{\mathsf{z}},\mathsf{g}(\bar{\mathsf{x}}))]$ will denote $\forall\mathsf{u}([\mathsf{g}(\bar{\mathsf{x}})=\mathsf{u}]\rightarrow\mathtt{F}(\bar{\mathsf{z}}, \mathsf{u}))$, where u is a variable not occurring in $[\mathsf{g}(\bar{\mathsf{x}})=\mathsf{y}]$ and $\mathtt{F}(\bar{\mathsf{z}}, \mathsf{v})$; to be canonical we can take $\mathsf{u} = \mathsf{v}_{[\mathsf{g}(\bar{\mathsf{x}})=\mathsf{y}]+\mathtt{F}(\bar{\mathsf{z}},\mathsf{v})}$. Note that this is provably equivalent to $\exists\mathsf{u}([\mathsf{g}(\bar{\mathsf{x}})=\mathsf{u}]\wedge\mathtt{F}(\bar{\mathsf{z}}, \mathsf{u}))$ (with same assumptions and conventions as above). Similarly, if φ is a formula then $[\varphi(\mathsf{g}(\bar{\mathsf{x}}))]$ is $\forall\mathsf{u}([\mathsf{g}(\bar{\mathsf{x}})=\mathsf{u}]\rightarrow\varphi(\mathsf{u}))$ for a fresh u. Numerals in the object language will be denoted by $\overline{n}$ (that is $\overline{0} = \mathsf{0}$, $\overline{k+1} = \mathsf{S}(\overline{k})$ for all k) while numerals in the formal object language will be denoted $\dot{\tau}$, where τ is a term.

Note that we will use juxtaposition to denote *both* multiplication of numbers and composition of formulae, except where this would lead to confusion, in which case juxtaposition will denote composition while the multiplication sign will be written out.

As an example of the above conventions, consider the following

$$\forall\mathcal{M} \models \mathrm{I}\Sigma_1 : \mathcal{M} \models [(\forall \boldsymbol{\mathsf{v}}_0 \exists \boldsymbol{\mathsf{v}}_{\bar{1}} \boldsymbol{\mathsf{S}}(\boldsymbol{\mathsf{v}}_0) {=} \boldsymbol{\mathsf{v}}_{\bar{1}}) {=} \overline{(\forall \mathsf{v}_0 \exists \mathsf{v}_1 \mathsf{S}(\mathsf{v}_0) {=} \mathsf{v}_1)}].$$

By the soundness and completeness theorems for first-order logic, the above merely states that $\mathrm{I}\Sigma_1$ proves the (formal) formula $\forall \boldsymbol{\mathsf{v}}_0 \exists \boldsymbol{\mathsf{v}}_{\bar{1}} \boldsymbol{\mathsf{S}}(\boldsymbol{\mathsf{v}}_0) {=} \boldsymbol{\mathsf{v}}_{\bar{1}}$ to be exactly the (numeral of) the formula $\forall \mathsf{v}_0 \exists \mathsf{v}_1 \mathsf{S}(\mathsf{v}_0) {=} \mathsf{v}_1$. This is a true fact, as we shall see.

A generic structure of $\mathcal{L}_\mathrm{A}$ will be denoted $\mathcal{M} = (M, 0_\mathcal{M}, \mathrm{S}_\mathcal{M}, +_\mathcal{M}, \cdot_\mathcal{M}, <_\mathcal{M})$, while the standard model of arithmetic is $\mathbb{N} = (\omega, 0, \mathrm{S}, +, \cdot, <)$; we will use similar notations for structures of other signatures. We will however drop the subscripts marking which particular structure we are working in whenever this is not inconvenient. Likewise we will often identify a structure and its underlying set, writing e.g. $a \in \mathcal{M}$ when a is an element of (the universe of) the structure $\mathcal{M}$. Consequently, we will generally not distinguish a structure from any of its reducts (except in cases were this convention is deemed likely to cause confusion). Nevertheless, we will let $\subseteq$ denote the substructure relation, instead of the subset relation, when used between structures of the same signature. In either case, $\subset$ is the corresponding strict relation.

Given an $\mathcal{L}_\mathrm{A}$-structure $\mathcal{M}$, an $\mathcal{L}_\mathrm{A}$ formula φ and a subset A of $\mathcal{M}$, an evaluation e of φ in A is a partial function from the set of variables to A defined for all free variables of φ. We write $\mathcal{M} \models_e \varphi$ when φ is true in $\mathcal{M}$ when the variables are interpreted according to e, defined by Tarski's conditions in the usual way; thus the truth of $\mathcal{M} \models_e \varphi$ only depends on the values of e for the free variables of φ. We write $\mathcal{M} \models \varphi$ when $\mathcal{M} \models_e \varphi$ for all evaluations e of φ. In case φ and e is explicitly given, say $[\mathtt{lh}(\mathsf{x}){=}\mathsf{y}]$ and $\{(\mathsf{x}, a), (\mathsf{y}, b)\}$, we will often write e.g. $\mathcal{M} \models [\mathtt{lh}(a){=}b]$ for $\mathcal{M} \models_e [\mathtt{lh}(\mathsf{x}){=}\mathsf{y}]$. Later on we will convolute these notations and write e.g. $\mathcal{M} \models_{e'} [\mathtt{lh}(\mathsf{x}){=}b]$ (where e' is defined for x). This will be taken to mean $\mathcal{M} \models_{e'^{b}_{\mathsf{z}}} [\mathtt{lh}(\mathsf{x}){=}\mathsf{z}]$ for some fresh variable z (like above, the choice can be made canonical), so that the assignment written "in the formula" is the one that takes precedence in case of an apparent conflict. Here f^b_a for a function f is defined by $f^b_a(c) = f(c)$ in case $c \neq a$ and $f^b_a(a) = b$. We will also use f^n to denote the nth iteration of the function f (where $f^0 = \mathrm{id}$). A finite function will mean a function with finite domain and $f : A \rightharpoonup B$ will mean that f is a partial function from A to B.

Since $\mathbb{N}$ can be canonically embedded as an initial substructure in any model of Robinson arithmetic Q (see Proposition 2.11), we will (for convenience of notation and without loss of generality) assume that $\mathbb{N}$ is an initial substructure of any such model. In a similar fashion, though much more straightforward, $\mathbb{N}$ will be considered an initial segment of any infinite discrete linear order (an order where every non-maximal element has a successor and every non-minimal a predecessor) with a least element.

We will thus need some notation for discrete linear orders, which we for technical reasons will assume to be nonempty. Suppose $\mathcal{L} = (L, <_\mathcal{L})$ is a discrete linear order. The partial functions $\mathcal{L} \rightharpoonup \mathcal{L}$ which maps an element to its successor (predecessor)will be denoted $\mathrm{S}_\mathcal{L}$ ($\mathrm{P}_\mathcal{L}$). We will also use the notations $\mathcal{L}_{<a} = \{b \in \mathcal{L} \,|\, b <_\mathcal{L} a\}$ and $\mathcal{L}_{>a} = \{b \in$

$\mathcal{L} \mid b >_{\mathcal{L}} a\}$. In case a has no successor (predecessor), i.e. is maximal (minimal), we write $\mathcal{L}_{>\mathrm{S}_{\mathcal{L}}(a)} = \varnothing$ ($\mathcal{L}_{<\mathrm{P}_{\mathcal{L}}(a)} = \varnothing$). Note that the above will in particular apply to models of weak theories of arithmetic and (consequently), as for the arithmetic structures above, we will often omit the subscript to $<$. If N is a subset (or a substructure) of a linear order $\mathcal{L}$ and $a \in \mathcal{L}$, then $N <_{\mathcal{L}} a$ will mean that $b <_{\mathcal{L}} a$ for all $b \in N$. Unless otherwise stated, an $\mathcal{L}_{\mathrm{A}}$-structure will also be considered as a structure ordered by $<_{\mathcal{M}}$ (if indeed this is an ordering), and the power-set of any set will be considered an order under $\subset$. The adjective "increasing" will always mean "monotone" relative to the (strict) orderings considered, so that for example an increasing sequence $s : \mathbb{N} \longrightarrow \mathcal{P}(M)$ means that $s(k) \subset s(k+1)$ for all $k \in \mathbb{N}$. A sequence in a structure $\mathcal{M}$ is a function from some discrete linear order $\mathcal{L}$ to $\mathcal{M}$. Thus a sequence in a model of PA^- is increasing if it is monotone from $<_{\mathcal{L}}$ to $<_{\mathcal{M}}$. A particular case of discrete linear orders are $(\mathbb{N}_{<n}, <)$ for $n \in \mathbb{N}$, which we will simply denote by n.

By a theory we will mean a set of sentences; thus a theory T_1 extends a theory T_2 simply if $T_2 \subseteq T_1$ (as sets). However, most theories we shall consider will in addition be deductively closed (anything derivable from the theory is an element of the theory). When speaking of an axiomatisation of a deductively closed theory T, we will mean a set of sentences the deductive closure of which is T. The reason we only consider closed axiomatisations is technical; it could probably be circumvented by using a primitive recursive function producing universal closures of formulae. Indeed, if we state that a theory T is axiomatised by some open set of formulae it should be understood to mean that T is axiomatised by their universal closures. We will call a theory T *recursively axiomatisable* if there is a primitive recursive enumeration of an axiomatisation of T, that is a primitive recursive $\mathrm{ax} : \mathbb{N} \longrightarrow \mathrm{Fmla}$ whose range is an axiomatisation of T (this will be justified in section 2, where we define Fmla as a Δ_1 (recursive) set of natural numbers). By Craig's trick and using basic facts from computability theory and decidability of the correctness of derivations (see for example [4, Thm. 2.29, p. 166], [7, p. 150] and [13, pp. 130–131]), this is equivalent to there being a (primitive) recursive (i.e. decidable) axiomatisation of T, as well as to T itself being recursively enumerable; this motivates our use of terminology. Since we strive to avoid notions of computability except as motivation, and will not formalise the notion of derivation, we will not go further into this. As we shall see, all we need is that "recursively axiomatisable" means that there is a T-provably $\Delta_1(T)$-function $\mathtt{ax}$ such that $\exists k \in \mathbb{N} : T \vdash [\mathtt{ax}(\overline{k}) = \overline{\varphi}]$ for all axioms φ of T and $T \vdash [\mathtt{ax}(\overline{k}) \neq \overline{m}]$ for all $k \in \mathbb{N}$ and $m \in \mathbb{N}$ which are not axioms of T.

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2 Preliminaries

The results given in this section are standard in the literature and will to a large extent be stated without proof. We will base our exposition mainly on those found in [4] and [7] with occasionally some additional material from [13], though we will deviate slightly and in particular choose other symbols as primitives.

2.1 The arithmetic hierarchy and some fragments of arithmetic

Two elements central to the discussions and results of this thesis are the notions of a *bounded formula* of $\mathcal{L}_{\mathrm{A}}$ and an *end-extension* of an $\mathcal{L}_{\mathrm{A}}$-structure. There are some interconnections between these two, as we shall see. We will first introduce the notion of initial segments of arbitrary structures of the symbol $<$.

Definition 2.1 (Initial segments)**.** Let $\mathcal{M}$ be a structure whose signature contains $<$. A subset I of $\mathcal{M}$ is an *initial segment* of $\mathcal{M}$ if it is closed under $<_{\mathcal{M}}$, that is if $a \in I$ then $b \in I$ for all $b \in \mathcal{M}$ such that $b <_{\mathcal{M}} a$.

Note that, in particular, an initial segment of a discrete linear order is again a discrete linear order.

Definition 2.2 (Cuts, initial substructures and end-extensions)**.** Let $\mathcal{M}$ be an $\mathcal{L}_{\mathrm{A}}$-structure. A subset I of $\mathcal{M}$ is a *cut* if it is an initial segment closed under $\mathrm{S}_{\mathcal{M}}$. If $\mathcal{N}$ is both a substructure and a cut of $\mathcal{M}$, then $\mathcal{N}$ is an *initial substructure* of $\mathcal{M}$ and $\mathcal{M}$ is an *end-extension* of $\mathcal{N}$, and we write $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M}$. That $\mathcal{N}$ is a proper initial substructure of $\mathcal{N}$ (that is, $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M}$ and $\mathcal{N} \neq \mathcal{M}$) will be denoted $\mathcal{N} \subset_{\mathrm{e}} \mathcal{M}$.

The following transitivity lemma is immediate.

Lemma 2.1. *If $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{K}$ are $\mathcal{L}_{\mathrm{A}}$-structures with $\mathcal{K} \subseteq_{\mathrm{e}} \mathcal{N}$ and $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M}$, then $\mathcal{K} \subseteq_{\mathrm{e}} \mathcal{M}$.*

Proof. Clearly $\mathcal{K} \subseteq \mathcal{M}$, so in particular $\mathcal{K}$ is closed under $\mathrm{S}_{\mathcal{M}}$. Let $b \in \mathcal{K}$ and $a \in \mathcal{M}$ satisfy $a <_{\mathcal{M}} b$. Then $b \in \mathcal{N}$ whence $a \in \mathcal{N}$ by $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M}$, so $a <_{\mathcal{N}} b$ since $\mathcal{N} \subseteq \mathcal{M}$. Thus $a \in \mathcal{K}$ by $\mathcal{K} \subseteq_{\mathrm{e}} \mathcal{N}$. □

Remark 1. This is a slight deviation from the terminology of our sources. In particular, the term "initial substructure" is nonstandard. In [4] the term "cut" has the same meaning as here (restricted to models of a simple theory of arithmetic), but the term "end-extension" does not require that the cut constitutes a substructure (that is, end-extensions are not extensions in the model theoretic sense). [7], on the other hand, uses "end-extension" and "cut" as we do, but denotes by "initial segment" that which we call "initial substructure". Both uses $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M}$ to mean that $\mathcal{M}$ is an end-extension of $\mathcal{N}$. There are also texts on model theory (see for example [5]) which uses "end-extension" similarly to the present one, but for structures of arbitrary signatures containing $<$.

The reason we do not use "initial segment" as in [7] is that we will reserve this term for the order-theoretic notion of Definition 2.1.

"Initial substructure" will turn out to be a rather natural restriction of the notion "substructure" for structures of arithmetic, akin to transitive models in set theory (see for example [6]). In particular more formulae will be absolute between structures in this relation. Similarly to set theory again, we may define a notion of bounded formulae, which will turn out to constitute a natural set of such formulae.

Definition 2.3 (Bounded formulae)**.** Given a formula $\varphi \in \mathcal{L}_{\mathrm{A}}$ and distinct variables x and y, we will use the following abbreviations:

$$\forall \mathsf{x}{<}\mathsf{y}\varphi = \forall \mathsf{x}(\mathsf{x}{<}\mathsf{y}{\to}\varphi)$$

and

$$\exists \mathsf{x}{<}\mathsf{y}\varphi = \exists \mathsf{x}(\mathsf{x}{<}\mathsf{y}{\wedge}\varphi).$$

A quantifier which occurs in one of the above contexts will be called *bounded*. A formula is bounded if all its quantifiers are bounded, that is, bounded formulae can be recursively defined as follows:

- All atomic formulae are bounded.
- If φ and ψ are bounded, then $\varphi{\wedge}\psi$, $\varphi{\vee}\psi$, $\neg\varphi$, $\forall \mathsf{x}{<}\mathsf{y}\varphi$ and $\exists \mathsf{x}{<}\mathsf{y}\varphi$ are bounded.

Bounded formulae will also be called Δ_0-formulae. This is extended to a classification of "all" formulae by complexity (that is, every formula will be equivalent to a Π_n or Σ_n formula for some $n \in \mathbb{N}$ in all extensions of a weak arithmetic theory $\mathrm{I}\Delta_0$), by the following definition.

Definition 2.4 (The arithmetic hierarchy)**.** The sets Σ_n and Π_n of arithmetic formulae are simultaneously defined by recursion:

- $\Sigma_0 = \Pi_0 = \Delta_0$.
- Having defined Π_k, we define Σ_{k+1} by structural recursion on formulae, as follows:
  - Every atomic formula is Σ_{k+1}.
  - $\varphi{\wedge}\psi$ is a Σ_{k+1}-formula if and only if $\varphi{\wedge}\psi$ is bounded.
  - $\varphi{\vee}\psi$ is a Σ_{k+1}-formula if and only if $\varphi{\vee}\psi$ is bounded.
  - $\neg\varphi$ is a Σ_{k+1}-formula if and only if $\neg\varphi$ is bounded.
  - $\exists \mathsf{x}\varphi \in \Sigma_{k+1}$ if and only if $\varphi \in \Sigma_{k+1}$.
  - $\forall \mathsf{x}\varphi \in \Sigma_{k+1}$ if and only if $\forall \mathsf{x}\varphi \in \Pi_k$.
- Having defined Σ_k, we define Π_{k+1} by structural recursion on formulae, as follows:
  - Every atomic formula is Π_{k+1}.
  - $\varphi{\wedge}\psi$ is a Π_{k+1}-formula if and only if $\varphi{\wedge}\psi$ is bounded.

– $\varphi \vee \psi$ is a Π_{k+1}-formula if and only if $\varphi \vee \psi$ is bounded.
– $\neg\varphi$ is a Π_{k+1}-formula if and only if $\neg\varphi$ is bounded.
– $\exists \mathsf{x}\varphi \in \Pi_{k+1}$ if and only if $\exists \mathsf{x}\varphi \in \Sigma_k$.
– $\forall \mathsf{x}\varphi \in \Pi_{k+1}$ if and only if $\varphi \in \Pi_{k+1}$.

This is the *arithmetic hierarchy* of formulae.

Unwinding this definition, we see that a formula is Σ_n (where $n > 0$) if and only if it is a Δ_0-formula preceded n blocks of quantifiers of the same kind beginning with a block of $\exists$'s, and dually for Π_n, where the blocks are allowed to be empty. The central facts of this hierarchy that we shall use are expressed in the next theorem, the exposition of which will benefit from an additional definition.

Definition 2.5 (Absoluteness)**.** Let $\mathcal{N} \subseteq \mathcal{M}$ be $\mathcal{L}_\mathrm{A}$-structures and φ an $\mathcal{L}_\mathrm{A}$-formula.

- If $\mathcal{N} \models_e \varphi \Rightarrow \mathcal{M} \models_e \varphi$ for all evaluations e of φ in $\mathcal{N}$ then φ is *upwards absolute* between $\mathcal{N}$ and $\mathcal{M}$.
- If $\mathcal{M} \models_e \varphi \Rightarrow \mathcal{N} \models_e \varphi$ for all evaluations e of φ in $\mathcal{N}$ then φ is *downwards absolute* between $\mathcal{N}$ and $\mathcal{M}$.
- If φ is both upwards and downwards absolute between $\mathcal{N}$ and $\mathcal{M}$, then φ is *absolute* between $\mathcal{N}$ and $\mathcal{M}$.

Remark 2. Note that the evaluation is always in the smaller structure (as the opposite would be potentially meaningless).

This usage of the terms upwards/downwards absolute is taken from [6] by analogy.

Theorem 2.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}_\mathrm{A}$-structures such that $\mathcal{N} \subseteq_\mathrm{e} \mathcal{M}$. Then:*

1. *if $\varphi \in \Delta_0$ then φ is absolute between $\mathcal{N}$ and $\mathcal{M}$;*
2. *if $\varphi \in \Sigma_1$ then φ is upwards absolute between $\mathcal{N}$ and $\mathcal{M}$;*
3. *if $\varphi \in \Pi_1$ then φ is downwards absolute between $\mathcal{N}$ and $\mathcal{M}$.*

A proof can be found in [7, pp. 24–25]. Indeed, the proof shows slightly more:

Lemma 2.3. *Let $\mathcal{N} \subseteq \mathcal{M}$ be $\mathcal{L}_\mathrm{A}$-structures.*

1. *If φ is upwards absolute between $\mathcal{N}$ and $\mathcal{M}$ then so is $\exists \mathsf{x}\varphi$.*
2. *If φ is downwards absolute between $\mathcal{N}$ and $\mathcal{M}$ then so is $\forall \mathsf{x}\varphi$.*

These notions will also be used to define theories which restricts the induction schema to certain levels in the hierarchy. There are several other families of theories defined by instead restricting some other general schema of PA to one of the above sets, of which we will consider one (see Definition 2.10).

Definition 2.6. Let $\mathcal{M}$ be a structure of $\mathcal{L}_{\mathrm{A}}$ and T be a theory of $\mathcal{L}_{\mathrm{A}}$. For each $n \in \mathbb{N}$, a formula φ is Σ_n in T if and only if $T \vdash \varphi \leftrightarrow \psi$ for some $\psi \in \Sigma_n$ with the same free variables. The set of such φ is denoted $\Sigma_n(T)$. Similarly for $\Pi_n(T)$, $\Sigma_n(\mathcal{M})$ and $\Pi_n(\mathcal{M})$ (mutatis mutandis). A formula φ is Δ_n in T (denoted $\varphi \in \Delta_n(T)$) if it is both Σ_n and Π_n in T. In the same way, $\Delta_n(\mathcal{M}) = \Sigma_n(\mathcal{M}) \cap \Pi_n(\mathcal{M})$.

That the arithmetic hierarchy gives a complexity to "all" formulae can now be stated as in the following proposition.

Proposition 2.4. *Let T be an $\mathcal{L}_{\mathrm{A}}$-theory and $\mathcal{M}$ an $\mathcal{L}_{\mathrm{A}}$-structure.*

- *For any $\varphi \in \mathcal{L}_{\mathrm{A}}$ there are $m, n \in \mathbb{N}$ such that $\varphi \in \Sigma_m(T)$ and $\varphi \in \Pi_n(T)$*
- *If $\varphi, \psi \in \Sigma_n(T)$ then $\varphi \wedge \psi, \varphi \vee \psi \in \Sigma_n(T)$, and similarly for $\Pi_n(T)$.*
- *If $\varphi \in \Sigma_n(T)$ then $\neg\varphi \in \Pi_n(T)$, and vice versa.*
- *$\Sigma_n(T) \subseteq \Delta_{n+1}(T) \subseteq \Pi_{n+1}(T)$ and $\Pi_n(T) \subseteq \Delta_{n+1}(T) \subseteq \Sigma_{n+1}(T)$.*
- *If $\varphi \in \Sigma_n(T)$ then $\forall \mathsf{x} \varphi \in \Pi_{n+1}(T)$, and if $\varphi \in \Pi_n(T)$ then $\exists \mathsf{x} \varphi \in \Sigma_{n+1}(T)$.*

The corresponding results hold for $\Sigma_n(\mathcal{M})$, $\Pi_m(\mathcal{M})$ and $\Delta_k(\mathcal{M})$.

The proofs are straightforward, see [7, pp. 79–80] for the first four.

Remark 3. Subsets of $\mathbb{N}$ defined by some formula in the arithmetic hierarchy will have computational properties linked to that formula's position in the hierarchy. For instance, $\Delta_1(\mathbb{N})$-formulae define recursive subsets of $\mathbb{N}$, while Σ_1-formulae define the recursively enumerable sets. So in a sense, the position in the hierarchy measures how far from being computable a certain notion is. This is one of the chief sources of interest of the hierarchy.

Remark 4. There are similar definitions in e.g. the language of set theory, with $\in$ instead of $<$ above, see [6].

The results on absoluteness in Theorem 2.2 carry over to arbitrary arithmetic theories.

Lemma 2.5. *Let $\mathcal{N} \subseteq \mathcal{M}$ both be models of the $\mathcal{L}_{\mathrm{A}}$-theory T such that Δ_0-formulae are absolute between $\mathcal{N}$ and $\mathcal{M}$. Then:*

1. *$\Sigma_1(T)$-formulae are upwards absolute between $\mathcal{N}$ and $\mathcal{M}$;*
2. *$\Pi_1(T)$-formulae are downwards absolute between $\mathcal{N}$ and $\mathcal{M}$;*
3. *$\Delta_1(T)$-formulae are absolute between $\mathcal{N}$ and $\mathcal{M}$;*

Proof. Let φ be a $\Sigma_1(T)$-formula and ψ a $\Pi_1(T)$-formula. Let $\xi \in \Sigma_1$ and $\vartheta \in \Pi_1$ have the same free variables as φ and ψ, respectively, and be such that $T \vdash \varphi \leftrightarrow \xi$ and $T \vdash \psi \leftrightarrow \vartheta$. Then

$$\mathcal{N} \models_{e_1} \varphi \Leftrightarrow \mathcal{N} \models_{e_1} \xi \Rightarrow \mathcal{M} \models_{e_1} \xi \Leftrightarrow \mathcal{M} \models_{e_1} \varphi$$

for all evaluations e_1 of φ in $\mathcal{N}$, and

$$\mathcal{M} \models_{e_2} \psi \Leftrightarrow \mathcal{M} \models_{e_2} \vartheta \Rightarrow \mathcal{N} \models_{e_2} \vartheta \Leftrightarrow \mathcal{N} \models_{e_2} \psi$$

for all evaluations e_2 of ψ in $\mathcal{N}$.

The claim on $\Delta_1(T)$-formulae follows by considering the case $\varphi = \psi$. □

Definition 2.7. Let $T \subseteq \mathcal{L}_{\mathrm{A}}$ be a theory. T is Σ_k*-sound* (Π_k-sound) if $T \vdash \varphi \Rightarrow \mathbb{N} \models \varphi$ for all Σ_k-sentences (Π_k-sentences). T is *sound* if it is Σ_k-sound for every $k \in \mathbb{N}$, equivalently if $\mathbb{N} \models T$.

Conversely, T is Σ_k*-complete* (Π_k-complete) if $\mathbb{N} \models \varphi \Rightarrow T \vdash \varphi$ for all Σ_k-sentences (Π_k-sentences) φ.[3]

We now turn to some weak theories of arithmetic, which are in fact enough to deduce many truths of $\mathbb{N}$.

Definition 2.8. The theory Q of *Robinson arithmetic* is the deductive closure of the following axioms

$$\forall \mathsf{x}(\mathsf{S}(\mathsf{x}) \neq 0) \tag{1}$$
$$\forall \mathsf{x}\forall \mathsf{y}(\mathsf{S}(\mathsf{x}) = \mathsf{S}(\mathsf{y}) \rightarrow \mathsf{x} = \mathsf{y}) \tag{2}$$
$$\forall \mathsf{x}(\mathsf{x} \neq 0 \rightarrow \exists \mathsf{y}(\mathsf{x} = \mathsf{S}(\mathsf{y}))) \tag{3}$$
$$\forall \mathsf{x}(\mathsf{x} + 0 = \mathsf{x}) \tag{4}$$
$$\forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} + \mathsf{S}(\mathsf{y}) = \mathsf{S}(\mathsf{x} + \mathsf{y})) \tag{5}$$
$$\forall \mathsf{x}(\mathsf{x} \cdot 0 = 0) \tag{6}$$
$$\forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} \cdot \mathsf{S}(\mathsf{y}) = (\mathsf{x} \cdot \mathsf{y}) + \mathsf{x}) \tag{7}$$
$$\forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} < \mathsf{y} \leftrightarrow \exists \mathsf{z}(\mathsf{x} + \mathsf{S}(\mathsf{z}) = \mathsf{y})) \tag{8}$$

where $\mathsf{x} = \mathsf{v}_0$, $\mathsf{y} = \mathsf{v}_1$ and $\mathsf{z} = \mathsf{v}_2$. The theory PA^- is axiomatised by the axioms of Q together with the following axioms

$$\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}((\mathsf{x} + \mathsf{y}) + \mathsf{z} = \mathsf{x} + (\mathsf{y} + \mathsf{z})) \tag{9}$$
$$\forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} + \mathsf{y} = \mathsf{y} + \mathsf{x}) \tag{10}$$
$$\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}((\mathsf{x} \cdot \mathsf{y}) \cdot \mathsf{z} = \mathsf{x} \cdot (\mathsf{y} \cdot \mathsf{z})) \tag{11}$$
$$\forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} \cdot \mathsf{y} = \mathsf{y} \cdot \mathsf{x}) \tag{12}$$
$$\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}(\mathsf{x} \cdot (\mathsf{y} + \mathsf{z}) = \mathsf{x} \cdot \mathsf{y} + \mathsf{x} \cdot \mathsf{z}) \tag{13}$$
$$\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}((\mathsf{x} < \mathsf{y} \wedge \mathsf{y} < \mathsf{z}) \rightarrow \mathsf{x} < \mathsf{z}) \tag{14}$$
$$\forall \mathsf{x}(\mathsf{x} \not< \mathsf{x}) \tag{15}$$
$$\forall \mathsf{x}\forall \mathsf{y}(\mathsf{x} < \mathsf{y} \vee \mathsf{x} = \mathsf{y} \vee \mathsf{y} < \mathsf{x}) \tag{16}$$

[3] N.B. Contrarily to soundness, *completeness* of a theory T means that $T \vdash \varphi$ or $T \vdash \neg\varphi$ for every sentence $\varphi \in \mathcal{L}_{\mathrm{A}}$, which is *not* the same as being Σ_k-complete for every k, since the only theory which is Σ_k-complete for every k is $\mathrm{Th}(\mathbb{N})$.

$$\forall x \forall y \forall z (x < y \leftrightarrow x + z < y + z) \tag{17}$$

$$\forall x \forall y \forall z ((0 < z \wedge x < y) \leftrightarrow x \cdot z < y \cdot z) \tag{18}$$

$$\forall x \forall y (y < x \leftrightarrow (S(y) < x \vee S(y) = x)) \tag{19}$$

where again $x = v_0$, $y = v_1$ and $z = v_2$.

Remark 5. Since our aim is not to investigate minimal axiomatisations of theories, we have not made an effort to keep the axiomatisations above free of redundancy.

As a simple example, which will be of use in itself later on, of what can be proved in PA^- we consider the following.

Lemma 2.6. $PA^- \vdash (\overline{2} \leq z \wedge x < z \wedge y < z) \rightarrow (x + y < z \cdot z \wedge x \cdot y < z \cdot z)$.

Proof. We reason in PA^-. Assume $\overline{2} \leq z$, $x<z$ and $y<z$. Then $0<z$ by (4), (10), (8) and (14). By (17) $x + y < x + z$. By (10) $x + z = z + x$. By (17) again $z + x < z + z$. By (7) and (6) $z + z = z \cdot \overline{2}$. By (12) $z \cdot \overline{2} = \overline{2} \cdot z$. Finally $\overline{2} \cdot z \leq z \cdot z$ by (18), whence $x+y<z \cdot z$ by (14).

Similarly, if $y=0$ then $x \cdot y=0$ by (6), $y \cdot z=0$ by (6) and (12) and $y \cdot z<z \cdot z$ by (18). If $0<y$ then $x \cdot y < z \cdot y$ by (18), $z \cdot y = y \cdot z$ by (12) and $y \cdot z < z \cdot z$ by (18) again, whence $x \cdot y < z \cdot z$ by (14). If $y<0$ then $y+S(u)=0$ for some u by (8) whence by (5) $S(y + u)=0$, which contradicts (1). These cases are exhaustive by (16). □

It will be of importance that any initial substructure of a model of PA^- is itself such a model. By Lemma 2.5 this will follow if we can show that all axioms of PA^- are $\Pi_1(T)$ for some simpler theory T which already has this property. Since most axioms of PA^- are Π_1 already as stated we have an obvious choice for T:

Lemma 2.7. *Let T be the theory axiomatised by all axioms of* PA^- *except* (3) *and* (8). *Then* (3) *and* (8) *are* $\Pi_1(T)$.

Proof. First consider (3). Clearly

$$\vdash (\forall x(x \neq 0 \rightarrow \exists y < x(x = S(y)))) \rightarrow (\forall x(x \neq 0 \rightarrow \exists y(x = S(y)))).$$

Conversely, reasoning in T now, assume (3). Take $x \neq 0$, whence there is y such that $x = S(y)$ by assumption. By (19) we get $y < x$, hence $\exists y < x(x = S(y))$. Thus

$$T \vdash (\forall x(x \neq 0 \rightarrow \exists y(x = S(y)))) \rightarrow (\forall x(x \neq 0 \rightarrow \exists y < x(x = S(y))))$$

as desired.

Next consider (8). Reasoning in T we prove this to be equivalent to

$$\forall x \forall y (x < y \leftrightarrow \exists z < y(x + S(z) = y)). \tag{20}$$

Assume (8) and take x and y. That $\exists z < y(x + S(z) = y) \rightarrow x<y$ is immediate by (8). Thus suppose $x < y$, whence $\exists z(x + S(z) = y)$ by (8). By (5) and (10)

$$z + S(x) = S(z + x) = S(x + z) = x + S(z) = y,$$

whence $\mathsf{z} < \mathsf{y}$ by (8). To sum up

$$T \vdash (\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} < \mathsf{y} \leftrightarrow \exists \mathsf{z}(\mathsf{x} + \mathsf{S}(\mathsf{z}) = \mathsf{y}))) \to (\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} < \mathsf{y} \leftrightarrow \exists \mathsf{z} < \mathsf{y}(\mathsf{x} + \mathsf{S}(\mathsf{z}) = \mathsf{y}))).$$

Conversely, assume (20) and take x and y. That $\mathsf{x} < \mathsf{y} \to \exists \mathsf{z}(\mathsf{x} + \mathsf{S}(\mathsf{z}) = \mathsf{y})$ is now immediate, so suppose $\exists \mathsf{z}(\mathsf{x} + \mathsf{S}(\mathsf{z}) = \mathsf{y})$. By (16), $\mathsf{y} \leq \mathsf{z}$ or $\mathsf{z} < \mathsf{y}$; in the latter case $\mathsf{x} < \mathsf{y}$ by (20) so we need only consider the former case. Again

$$\mathsf{z} + \mathsf{S}(\mathsf{x}) = \mathsf{S}(\mathsf{x} + \mathsf{z}) = \mathsf{y}$$

by (5) and (10), whence $\mathsf{y} \neq 0$ by (1) and $\mathsf{x} + \mathsf{z} < \mathsf{y}$ by (19). Hence

$$\mathsf{x} + \mathsf{z} < \mathsf{z} = 0 + \mathsf{z}$$

by (14), (4) and (10), whence $\mathsf{x} < 0$ by (17). Moreover, if $\mathsf{y} < 0$ then by (20) there is a $\mathsf{u} < 0$ such that $\mathsf{y} + \mathsf{S}(\mathsf{u}) = 0$, whence $\mathsf{S}(\mathsf{y} + \mathsf{u}) = 0$ by (5); this is absurd by (1). Hence $0 < \mathsf{y}$ by (16) again, whence $\mathsf{x} < \mathsf{y}$ by (14). This verifies that

$$T \vdash (\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} < \mathsf{y} \to \exists \mathsf{z} < \mathsf{y}(\mathsf{x} + \mathsf{S}(\mathsf{z}) = \mathsf{y}))) \leftrightarrow (\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x} < \mathsf{y} \leftrightarrow \exists \mathsf{z}(\mathsf{x} + \mathsf{S}(\mathsf{z}) = \mathsf{y})))$$

as claimed. □

As noted earlier, T in the above lemma is itself axiomatised by Π_1-formulae (T is even a $\forall$-theory). Thus we get the following corollary.

Corollary 2.8. *If $\mathcal{N}$ and $\mathcal{M}$ are $\mathcal{L}_{\mathrm{A}}$-structures and $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M} \models \mathrm{PA}^-$, then $\mathcal{N} \models \mathrm{PA}^-$.*

The axioms of Q and PA^- express well known properties of natural numbers. Of the two, Q seems to be the more standard and natural choice for a "minimal theory of arithmetic", for example Q is already Σ_1-complete, as will be shown below (Theorem 2.13). Unfortunately, it will not always be enough for our purposes; hence we consider the slightly stronger theory PA^-. There are two main reasons for this, that is, two properties of PA^- which will be important in the following which Q lacks. The first one is the above absoluteness property; the second is the following.

Proposition 2.9. *Models $\mathcal{M}$ of PA^- are discrete linear orderings with successor the interpretation of S in $\mathcal{M}$, so the notation $\mathrm{S}_{\mathcal{M}}$ for the latter is unambiguous.*

Proof. Let $\mathcal{M} \models \mathrm{PA}^-$. That $<_{\mathcal{M}}$ is a linear order is stated plainly in axioms (14), (15) and (16). That the order theoretic successor exists and equals the arithmetic successor is the content of axiom (19). $0_{\mathcal{M}}$ is the least element by (3) and (8) (and (4) and (10)), and the former also give that every other element of $\mathcal{M}$ is a successor, and thus has a predecessor. □

Many results, however, are formulated in terms of Q in the literature, and we have thus chosen to state results in terms of this theory whenever feasible. The following lemma and proposition serve as good examples.

Lemma 2.10. *Let $n \in \mathbb{N}$. Then* $\mathrm{Q} \vdash \forall \mathsf{x}(\mathsf{x} {<} \overline{n} \leftrightarrow \bigvee_{i<n} \mathsf{x} {=} \overline{i})$.

Proof of the above can be found in [4, p. 30].

Proposition 2.11. *For every model $\mathcal{M}$ of* Q *there is an $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M}$ such that $\mathbb{N} \cong \mathcal{N}$.*

Proof. Let $N = \{\overline{n}_{\mathcal{M}} \,|\, n \in \mathbb{N}\}$ and $f : \mathbb{N} \longrightarrow N$ be defined by $f(n) = \overline{n}_{\mathcal{M}}$ (that is, $f(n)$ is the interpretation of the closed term $\overline{n}$ in $\mathcal{M}$). By the axioms of Q, f respects $\mathrm{S}_{\mathcal{M}}$, $+_{\mathcal{M}}$ and $\cdot_{\mathcal{M}}$, whence N is the underlying set of a substructure $\mathcal{N}$ of $\mathcal{M}$. Moreover, $\mathrm{Q} \vdash \forall \mathsf{x}(\mathsf{x} {<} \overline{n} \leftrightarrow \bigvee_{i<n} \mathsf{x} {=} \overline{i})$ by above, whence $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M}$. Finally, this also implies that $\mathrm{Q} \vdash \overline{n} {<} \overline{m} \Leftrightarrow n < m$ for $n, m \in \mathbb{N}$, whence f is an isomorphism. □

Thus we can without loss of generality assume $\mathbb{N} \subseteq_{\mathrm{e}} \mathcal{M}$ for any model $\mathcal{M}$ of Q. As already noted in notations subsection (1.4), we will do so henceforth. We might occasionally comment on what differences omitting this assumption would require.

With the above proposition we can also give a (rather convoluted) proof that $\mathbb{N} \models \mathrm{PA}^-$ (under the assumption that PA^- is consistent): since $\mathcal{M} \models \mathrm{Q}$ for any $\mathcal{M} \models \mathrm{PA}^-$ we get $\mathbb{N} \subseteq_{\mathrm{e}} \mathcal{M}$ for such $\mathcal{M}$, whence $\mathbb{N} \models \mathrm{PA}^-$ by Corollary 2.8.

Proposition 2.12 (Soundness of PA^-)**.** $\mathbb{N} \models \mathrm{PA}^-$.

A more interesting (and less obvious) fact which follows from Proposition 2.11 is that Robinson arithmetic suffices to derive all Σ_1-truths of $\mathbb{N}$.

Theorem 2.13 (Σ_1-completeness of Q)**.** *Let $T \subseteq \mathcal{L}_{\mathrm{A}}$ be a sound extension of* Q *and $\varphi \in \mathcal{L}_{\mathrm{A}}$ be a $\Sigma_1(T)$ sentence true in $\mathbb{N}$. Then $T \vdash \varphi$.*

Proof. Let $\mathcal{M}$ be any model of T. By Proposition 2.11, $\mathbb{N} \subseteq_{\mathrm{e}} \mathcal{M}$, whence $\mathcal{M} \models \varphi$ by Lemma 2.5. Since $\mathcal{M} \models T$ was arbitrary, $T \vdash \varphi$ by the completeness theorem. □

For another proof (not using the soundness and completeness theorems) see [4, pp. 30–31].

Definition 2.9 ($\mathrm{I}\Sigma_n$)**.** $\mathrm{I}\Sigma_n$ is the theory axiomatised by the axioms of PA^- and the schema of induction restricted to Σ_n-formulae:

$$(\varphi(\mathsf{0}) \wedge \forall \mathsf{x}(\varphi(\mathsf{x}) \rightarrow \varphi(\mathsf{S}(\mathsf{x})))) \rightarrow \forall \mathsf{x} \varphi(\mathsf{x}) \tag{21}$$

where φ is a Σ_n-formula (with any number of free variables) and x is a variable which does not occur in φ. $\mathrm{I}\Sigma_0$ is often denoted $\mathrm{I}\Delta_0$.

Peano arithmetic, PA, is axiomatised by PA^- and the induction schema for all formulae (with the above restriction on the induction variable). Thus PA can be thought of as $\mathrm{I}\Sigma_\omega$.

Definition 2.10 (Collection)**.** $\mathrm{B}\Sigma_n$ is the theory axiomatised by the axioms of $\mathrm{I}\Delta_0$ and the schema of collection for Σ_n-formulae:

$$\forall \mathsf{u}((\forall \mathsf{x} {<} \mathsf{u} \exists \mathsf{y} \varphi) \rightarrow (\exists \mathsf{v} \forall \mathsf{x} {<} \mathsf{u} \exists \mathsf{y} {<} \mathsf{v} \varphi)) \tag{22}$$

where φ is a Σ_n-formula (with any number of free variables) and u and v do not occur free in φ.

Similarly, $S\Sigma_n$ is the theory axiomatised by the axioms of $I\Delta_0$ and strong collection for Σ_n-formulae:

$$\forall \mathsf{u} \exists \mathsf{v} \forall \mathsf{x} {<} \mathsf{u} ((\exists \mathsf{y} \varphi) \to (\exists \mathsf{y} {<} \mathsf{v} \varphi)) \tag{23}$$

where φ is a Σ_n-formula (with any number of free variables) and u and v do not occur free in φ.

The collection axioms are collected from set theory, where they express that the image of a set under a (class) relation is contained (i.e. collected) in(to) a set (to see this, replace $<$ by $\in$ in the formulae above), an alternative to the replacement axioms (see for example [6]).[4]

In what follows we shall need some instances of the lemmas below. Proofs of those which have no explicitly given reference may be found in e.g. [7, p. 82] or [4, pp. 63–70].

Lemma 2.14. *For every $n \in \mathbb{N}$, $I\Sigma_n$ proves all axioms of $B\Sigma_n$ and $B\Sigma_{n+1}$ proves all axioms of $I\Sigma_n$.*

For every $n > 0$, $I\Sigma_n$ and $S\Sigma_n$ are equivalent (the same theory), i.e. each proves all axioms of the other.

Definition 2.11 (Provable functions)**.** Let T be a theory and φ a formula with at least one free variable y. Then φ defines y as a *provable function* in T (of the remaining free variables of φ) if $T \vdash \exists \mathsf{y}(\varphi \wedge \forall \mathsf{v} \varphi[\mathsf{y}/\mathsf{v}] \to \mathsf{v}{=}\mathsf{y})$.

Since y (and T) is often clear from the context (see below and Subsection 1.4), we will usually say φ is a (T-)provable function.

We will sometimes write $[\mathtt{f}(\bar{\mathsf{x}}){=}\mathsf{y}]$ and similar for a general provable function. A particular case will be the following observation:

> If τ is a term and y is a variable which does not occur in τ, then $\tau{=}\mathsf{y}$ is a ($\emptyset$-)provable function.

We will use the term "(T-)provably Σ_k-function" (Π_k, $\Sigma_k(I\Sigma_k)$ etc.) to mean "Σ_k-formula which is (T-)provably a function", as in the following lemmas.

Lemma 2.15. *If $\mathtt{F}(\bar{\mathsf{z}}, \mathsf{v})$ is an abbreviation of a $\Sigma_k(T)$-formula and $[\mathsf{g}(\bar{\mathsf{x}}){=}\mathsf{y}]$ is a T-provably $\Sigma_k(T)$-function then $[\mathtt{F}(\bar{\mathsf{z}},\mathsf{g}(\bar{\mathsf{x}}))]$ is also $\Sigma_k(T)$.*

Proof. Let u be a new variable. Then

$$T \vdash [\mathtt{F}(\bar{\mathsf{z}},\mathsf{g}(\bar{\mathsf{x}}))] \leftrightarrow \exists \mathsf{u}([\mathtt{f}(\bar{\mathsf{x}}){=}\mathsf{u}] \wedge \mathtt{f}(\bar{\mathsf{z}}, \mathsf{u})).$$

□

[4] There appears to be some divergence of terminology between the subjects, however, since the above strong collection schema in [6] is called simply "collection" (which in standard set theory is equivalent to strong collection as above, by the separation schema); "collection" in our sense is used in [7] and [4], while the term "strong collection" is taken from [4].

Lemma 2.16. *In* $\mathrm{I}\Sigma_1$*, provably* Σ_1*-functions are* $\Delta_1(\mathrm{I}\Sigma_1)$ *and closed under definition by composition and primitive recursion.*

Proof. See [4, p. 48]. □

We will not state the above in greater detail here, but see also Lemma 2.31. Together with the fact that the zero, successor and projection functions are clearly provably Σ_1-functions, this yields the corollary below.

Corollary 2.17. *For every primitive recursive function* $f : \mathbb{N}^n \longrightarrow \mathbb{N}$ *there is a* $\mathrm{I}\Sigma_1$*-provably* $\Delta_1(\mathrm{I}\Sigma_1)$*-function* $[\mathtt{f}(\mathsf{x}_1, \ldots, \mathsf{x}_\mathsf{n}){=}\mathsf{y}]$ *(mimicking the construction of* f *as primitive recursive as in the previous lemma) such that* $\mathbb{N} \models [\mathtt{f}(\overline{k_1}, \ldots, \overline{k_n}){=}\overline{f(k_1, \ldots, k_n)}]$ *for all* $k_1, \ldots, k_n \in \mathbb{N}$*.*

This means that primitive recursive functions are what is called *provably recursive* in $\mathrm{I}\Sigma_1$. This is, however, somewhat misleading terminology, since if f is a primitive recursive function and $\mathtt{f}$ and $\mathtt{g}$ are $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-functions such that $\mathbb{N} \models [\mathtt{f}(\overline{k_1}, \ldots, \overline{k_n}){=}\overline{f(k_1, \ldots, k_n)}]$ and $\mathbb{N} \models [\mathtt{g}(\overline{k_1}, \ldots, \overline{k_n}){=}\overline{f(k_1, \ldots, k_n)}]$ for all $k_1, \ldots, k_n \in \mathbb{N}$, there is no guarantee that $\mathrm{I}\Sigma_1 \vdash \forall \mathsf{x}_1 \cdots \forall \mathsf{x}_n [\mathtt{f}(\mathsf{x}_1, \ldots, \mathsf{x}_n){=}\mathtt{g}(\mathsf{x}_1, \ldots, \mathsf{x}_n)]$, since the latter is a Π_1-sentence.[5] That said, many proofs of $\mathbb{N} \models [\mathtt{f}(\mathsf{x}_1, \ldots, \mathsf{x}_n){=}\mathtt{g}(\mathsf{x}_1, \ldots, \mathsf{x}_n)]$ uses only "Σ_1-induction", and so are translatable into $\mathrm{I}\Sigma_1$, especially if the corresponding definitions of f by primitive recursion as given by $\mathtt{f}$ and $\mathtt{g}$ are "natural". Thus many construction techniques of primitive recursive functions carry over to $\mathrm{I}\Sigma_1$-provably Δ_1-functions. Since we will only need that constructions by case distinction are valid we will state and prove this directly, without further reference to primitive recursive functions.

Lemma 2.18. *Let* $[\mathtt{f}(\bar{\mathsf{x}}){=}\mathsf{y}]$ *and* $[\mathtt{g}(\bar{\mathsf{x}}){=}\mathsf{y}]$ *be* $\mathrm{I}\Sigma_1$*-provably* $\Delta_1(\mathrm{I}\Sigma_1)$*-functions and* φ *be a* $\Delta_1(\mathrm{I}\Sigma_1)$*-formula whose free variables are among* $\bar{\mathsf{x}}$*. Then there is an* $\mathrm{I}\Sigma_1$*-provably* $\Delta_1(\mathrm{I}\Sigma_1)$*-function* $[\mathtt{h}(\bar{\mathsf{x}}){=}\mathsf{y}]$ *such that*

$$\mathrm{I}\Sigma_1 \vdash (\varphi{\to}[\mathtt{h}(\bar{\mathsf{x}}){=}\mathtt{f}(\bar{\mathsf{x}})]){\wedge}(\neg\varphi{\to}[\mathtt{h}(\bar{\mathsf{x}}){=}\mathtt{g}(\bar{\mathsf{x}})]). \tag{24}$$

Proof. Define $[\mathtt{h}(\bar{\mathsf{x}}){=}\mathsf{y}]$ to be the $\Delta_1(\mathrm{I}\Sigma_1)$-formula

$$(\varphi{\wedge}[\mathtt{f}(\bar{\mathsf{x}}){=}\mathsf{y}]){\vee}(\neg\varphi{\wedge}[\mathtt{g}(\bar{\mathsf{x}}){=}\mathsf{y}]).$$

We reason inside $\mathrm{I}\Sigma_1$ to show that $\mathtt{h}$ is a $\mathrm{I}\Sigma_1$-provable function satisfying (24).

Suppose φ. There is a y so that $[\mathtt{f}(\bar{\mathsf{x}}){=}\mathsf{y}]$, whence $[\mathtt{h}(\bar{\mathsf{x}}){=}\mathsf{y}]$ as well. If $[\mathtt{h}(\bar{\mathsf{x}}){=}\mathsf{w}]$ as well then we must have $\varphi(\bar{\mathsf{x}}){\wedge}[\mathtt{f}(\bar{\mathsf{x}}){=}\mathsf{w}]$ (since we cannot have $\neg\varphi$), so $\mathsf{w}{=}\mathsf{y}$ since $\mathtt{f}$ is a provable function.

If $\neg\varphi$ then similarly to above there is a unique y such that $[\mathtt{h}(\bar{\mathsf{x}}){=}\mathsf{y}]$. Thus $\mathtt{h}$ is a provable function.

Consequently (24) is meaningful, and the arguments above then verifies that it is true. □

[5] In fact, there are counterexamples to this, by Gödel's Incompleteness Theorems.

Since we will not need more general facts about provable recursion, we will not delve further into this theory here.

Lemma 2.19. *Let T be an $\mathcal{L}_{\mathrm{A}}$-theory and $k > 0$. If $[\mathtt{f}(\bar{\mathtt{x}})=\mathtt{y}]$ is a T-provably $\Delta_k(T)$-function then $[\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]$ and $[\mathtt{u}\leq\mathtt{f}(\bar{\mathtt{x}})]$ are $\Delta_k(T)$ (where $\mathtt{u}$ and $\mathtt{y}$ may be same variable).*

Proof. Let $\mathtt{v}$ be a fresh variable. Then

$$T \vdash [\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]\leftrightarrow\forall\mathtt{v}([\mathtt{f}(\bar{\mathtt{x}})=\mathtt{v}]\rightarrow\mathtt{u}<\mathtt{v}) \text{ and } T \vdash [\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]\leftrightarrow\exists\mathtt{v}([\mathtt{f}(\bar{\mathtt{x}})=\mathtt{v}]\wedge\mathtt{u}<\mathtt{v}).$$

Similarly

$$T \vdash [\mathtt{u}\leq\mathtt{f}(\bar{\mathtt{x}})]\leftrightarrow\forall\mathtt{v}([\mathtt{f}(\bar{\mathtt{x}})=\mathtt{v}]\rightarrow(\mathtt{u}<\mathtt{v}\vee\mathtt{u}=\mathtt{v})) \text{ and } T \vdash [\mathtt{u}\leq\mathtt{f}(\bar{\mathtt{x}})]\leftrightarrow\exists\mathtt{v}([\mathtt{f}(\bar{\mathtt{x}})=\mathtt{v}]\wedge(\mathtt{u}<\mathtt{v}\vee\mathtt{u}=\mathtt{v})).$$

□

Lemma 2.20. *Let T be an $\mathcal{L}_{\mathrm{A}}$-theory. If $k > 0$ and T proves Σ_k-collection then $\Sigma_k(T)$ and $\Pi_k(T)$ are closed under bounded quantifiers.*

Lemma 2.21. *Let $k > 0$, T be an $\mathcal{L}_{\mathrm{A}}$-theory and $[\mathtt{f}(\bar{\mathtt{x}})=\mathtt{y}]$ a $\Delta_k(T)$-function. If T proves Σ_k-collection then $\Sigma_k(T)$ and $\Pi_k(T)$ are closed under quantification bounded by $\mathtt{f}$, that is if $\varphi \in \Sigma_k(T)$ then $\forall[\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]\varphi \in \Sigma_k(T)$ and $\exists[\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]\varphi \in \Sigma_k(T)$, and similarly for $\Pi_k(T)$ (where $\mathtt{u}$ may be $\mathtt{y}$).*

Proof. Let $\mathtt{v}$ be a fresh variable and $\varphi \in \Sigma_k(T)$. Then

$$\begin{aligned}\forall[\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]\varphi &= \forall\mathtt{u}(\underbrace{[\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]}_{\Delta_k(T)}\rightarrow\varphi)\\ &\overset{T\vdash}{\leftrightarrow} \exists\mathtt{v}([\mathtt{f}(\bar{\mathtt{x}})=\mathtt{v}]\wedge\underbrace{\forall\mathtt{u}<\mathtt{v}\varphi}_{\Sigma_k(T)}) \in \Sigma_k(T)\end{aligned}$$

and

$$\exists[\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]\varphi = \exists\mathtt{u}(\underbrace{[\mathtt{u}<\mathtt{f}(\bar{\mathtt{x}})]}_{\Delta_k(T)}\wedge\varphi) \in \Sigma_k(T).$$

The proof for $\Pi_k(T)$ is completely symmetric (with $\rightarrow$ instead of $\wedge$). □

Note that the above in particular applies for provable functions defined by terms. Thus for many purposes we could equally well have allowed terms as bounds for quantifiers, which is indeed often the case. In particular we define $\forall\mathtt{x}\leq\mathtt{y}\varphi$ as $\forall[\mathtt{x}<\mathsf{S}(\mathtt{y})]\varphi$ and similarly for $\exists$.

The closedness results on $\Sigma_k(T)$ et cetera above will be used throughout the thesis. Thus we will seldom give explicit references to these results, but merely state "Let φ be the $\Sigma_k(T)$-formula ...", giving a definition from which the claim is immediate with the closedness results.

We close this subsection with two central facts of $\mathrm{I}\Sigma_k$.

Lemma 2.22 (Least number principle)**.** *Let φ be a Σ_k-formula or a Π_k-formula. Then*

$$\mathrm{I}\Sigma_k \vdash (\exists \mathsf{x}\varphi(\mathsf{x})) \rightarrow \exists \mathsf{x}(\varphi(\mathsf{x}) \wedge \forall \mathsf{y} < \mathsf{x} \neg \varphi(\mathsf{y})),$$

where x *and* y *do not occur in* φ*.*

Proof. See [4, Theorem 2.4, pp. 63–66]. □

Corollary 2.23. *The least number principle holds for $\Sigma_k(\mathcal{M})$ as well: Let $\mathcal{M} \models \Sigma_k$ and φ be a $\Sigma_k(\mathcal{M})$-formula. Then*

$$\mathcal{M} \models (\exists \mathsf{x}\varphi(\mathsf{x})) \rightarrow \exists \mathsf{x}(\varphi(\mathsf{x}) \wedge \forall \mathsf{y} < \mathsf{x} \neg \varphi(\mathsf{y})),$$

where x *and* y *do not occur in* φ*.*

Proof. Let ψ be a Σ_k-formula witnessing $\varphi \in \Sigma_k(\mathcal{M})$ such that x and y do not occur in ψ. Then

$$\mathcal{M} \models (\exists \mathsf{x}\psi(\mathsf{x})) \rightarrow \exists \mathsf{x}(\psi(\mathsf{x}) \wedge \forall \mathsf{y} < \mathsf{x} \neg \psi(\mathsf{y}))$$

by above, whence the same holds of φ since $\mathcal{M} \models \psi \leftrightarrow \varphi$. □

Lemma 2.24 (Overspill)**.** *Let $\mathcal{M} \models \mathrm{I}\Sigma_k$ and I be a proper cut of $\mathcal{M}$. Let $\varphi \in \Sigma_k$, x be a variable, e be an evaluation of the free variables of φ, except possibly x, in $\mathcal{M}$ and suppose $\mathcal{M} \models_{e^a_\mathsf{x}} \varphi$ for all $a \in I$. Then $\mathcal{M} \models_{e^b_\mathsf{x}} \varphi$ for some $b \in \mathcal{M} \setminus I$.*

Proof. See [7, Lemma 6.1, pp. 70–71]. While that lemma, as written, only applies to PA, the proof is the same (together with [4, Observation 1.15, p. 218]). □

Note that the above in particular applies to proper initial substructures of $\mathcal{M}$.

2.2 Arithmetisation of logic in $\mathrm{I}\Sigma_1$

We now turn to formalising logic within the theory $\mathrm{I}\Sigma_1$. We will mainly follow [4, ch. 1 sec. 1]. We will in general only state results, of interest and adapted to the current context; those proofs that are written out will be sketchy. However, to avoid unnecessary dependencies on specific choices, we will introduce some in reality superfluous notation (such as the predicate `Set` below; in [4] every number is (codes) a set). We also argue that this makes it plausible that other theories able to code this amount of set theory and arithmetic could have been used in place of $\mathrm{I}\Sigma_1$, see [12].

A preliminary result along these lines is the following.

Lemma 2.25 (Pairing)**.** *There are quantifier free $\mathrm{I}\Delta_0$-provable functions $[\langle \mathsf{x},\mathsf{y}\rangle = \mathsf{z}]$, $[\langle \mathsf{z}\rangle_\mathrm{l} = \mathsf{x}]$ and $[\langle \mathsf{z}\rangle_\mathrm{r} = \mathsf{y}]$ such that*

$$\mathrm{I}\Delta_0 \vdash [\langle \mathsf{x},\mathsf{y}\rangle = \mathsf{z}] \leftrightarrow ([\langle \mathsf{z}\rangle_\mathrm{l} = \mathsf{x}] \wedge [\langle \mathsf{z}\rangle_\mathrm{r} = \mathsf{y}])$$
$$\mathrm{I}\Delta_0 \vdash [\langle \mathsf{z}\rangle_\mathrm{l} \leq \mathsf{z}]$$
$$\mathrm{I}\Delta_0 \vdash [\langle \mathsf{z}\rangle_\mathrm{r} \leq \mathsf{z}]$$

Proof. The pairing formula $[\langle \mathsf{x},\mathsf{y}\rangle{=}\mathsf{z}]$ is $(\mathsf{x}+\mathsf{y})\cdot(\mathsf{x}+\mathsf{y}+1){+}\overline{2}{\cdot}\mathsf{y}{=}\overline{2}{\cdot}\mathsf{z}$ (with projections $[\langle \mathsf{z}\rangle_1{=}\mathsf{x}]$ and $[\langle \mathsf{z}\rangle_{\mathrm{r}}{=}\mathsf{y}]$ defined by $\exists \mathsf{y}[\langle \mathsf{x},\mathsf{y}\rangle{=}\mathsf{z}]$ and $\exists \mathsf{x}[\langle \mathsf{x},\mathsf{y}\rangle{=}\mathsf{z}]$ respectively). The proof that they have the required properties can be found in [4, 1.18, pp. 34–35]. $\square$

Definition 2.12. We define the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-functions $[\langle \mathsf{x}_0,\mathsf{x}_1,\ldots,\mathsf{x}_{k-1}\rangle^k{=}\mathsf{z}]$ and $[\langle \mathsf{z}\rangle^k_i{=}\mathsf{x}]$ for $i<k$, by recursion on k as follows:

- Define $[\langle\rangle^0{=}\mathsf{z}]$ to be $0{=}\mathsf{z}$, $[\langle \mathsf{x}_0\rangle^1{=}\mathsf{z}]$ to be $\mathsf{x}_0{=}\mathsf{z}$ and $[\langle \mathsf{z}\rangle^1_0{=}\mathsf{x}]$ to be $\mathsf{x}{=}\mathsf{z}$.
- For $l>0$ we define $[\langle \mathsf{x}_0,\mathsf{x}_1,\ldots,\mathsf{x}_l\rangle^{l+1}{=}\mathsf{z}]$ as $[\langle \mathsf{x}_0,\langle \mathsf{x}_1,\ldots,\mathsf{x}_l\rangle^l\rangle{=}\mathsf{z}]$. Moreover, we let $[\langle \mathsf{z}\rangle^{l+1}_0{=}\mathsf{x}]$ be $[\langle \mathsf{z}\rangle_1{=}\mathsf{x}]$ and $[\langle \mathsf{z}\rangle^{l+1}_{i+1}{=}\mathsf{x}]$ be $[\langle\langle \mathsf{z}\rangle_{\mathrm{r}}\rangle^l_i{=}\mathsf{x}]$.

That these formulae define k-tuples with corresponding projections now follows directly from the lemma above by induction on k.

2.2.1 Set theory

In $\mathrm{I}\Sigma_1$ we have the following $\Delta_1(\mathrm{I}\Sigma_1)$-formulae:

$$\mathtt{Set}(\mathsf{x}),\mathsf{x}\epsilon\mathsf{y},\mathsf{x}\sqsubset\mathsf{y},\mathsf{x}\sqsubseteq\mathsf{y},\mathtt{Fcn}(\mathsf{f}),\mathtt{Seq}(\mathsf{s}),$$

and the following provably $\Delta_1(\mathrm{I}\Sigma_1)$-functions:

$$[\varnothing{=}\mathsf{x}],[\{\mathsf{x}\}{=}\mathsf{y}],[(<\mathsf{x}){=}\mathsf{y}],[\mathsf{x}\sqcup\mathsf{y}{=}\mathsf{z}],[\mathsf{x}\backslash\mathsf{y}{=}\mathsf{z}],[\mathsf{x}{\times}\mathsf{y}{=}\mathsf{z}],$$
$$[\mathtt{dom}(\mathsf{f}){=}\mathsf{x}],[\mathtt{ran}(\mathsf{f}){=}\mathsf{x}],[\mathtt{lh}(\mathsf{s}){=}\mathsf{x}],[\mathtt{apl}(\mathsf{f},\mathsf{x}){=}\mathsf{y}],[(\mathsf{s})_{\mathsf{x}}{=}\mathsf{y}].$$

With the abbreviation $\forall \mathsf{x}\epsilon\mathsf{y}\varphi$ being $\forall \mathsf{x}{<}\mathsf{y}(\mathsf{x}\epsilon\mathsf{y}{\to}\varphi)$[6] and similarly for $\exists$ (extended by conventions similar to those for $<$, see subsection 1.4), which are meaningful courtesy of the first fact below, they satisfy the following properties:

Lemma 2.26.

$$\mathrm{I}\Sigma_1\vdash \mathsf{x}\epsilon\mathsf{y}{\to}\mathsf{x}{<}\mathsf{y}$$
$$\mathrm{I}\Sigma_1\vdash \mathsf{x}\sqsubseteq\mathsf{y}{\leftrightarrow}\forall \mathsf{u}\epsilon\mathsf{x}(\mathsf{u}\epsilon\mathsf{y})$$
$$\mathrm{I}\Sigma_1\vdash \mathsf{x}\sqsubseteq\mathsf{y}{\to}\mathsf{x}{\le}\mathsf{y}$$
$$\mathrm{I}\Sigma_1\vdash \mathsf{x}\sqsubset\mathsf{y}{\leftrightarrow}(\mathsf{x}\sqsubseteq\mathsf{y}\wedge\neg(\mathsf{x}{=}\mathsf{y}))$$
$$\mathrm{I}\Sigma_1\vdash [\mathtt{Set}(\{\mathsf{x}\})]\wedge\forall \mathsf{y}([\mathsf{y}\epsilon\{\mathsf{x}\}]{\leftrightarrow}\mathsf{y}{=}\mathsf{x})$$
$$\mathrm{I}\Sigma_1\vdash [\mathtt{Set}(\varnothing)]\wedge\forall \mathsf{x}\neg[\mathsf{x}\epsilon\varnothing]$$
$$\mathrm{I}\Sigma_1\vdash [\mathtt{Set}((<\mathsf{y}))]\wedge\forall \mathsf{x}([\mathsf{x}\epsilon(<\mathsf{y})]{\leftrightarrow}\mathsf{x}{<}\mathsf{y})$$
$$\mathrm{I}\Sigma_1\vdash (\mathtt{Set}(\mathsf{x})\wedge\mathtt{Set}(\mathsf{y})){\to}([\mathtt{Set}(\mathsf{x}\sqcup\mathsf{y})]\wedge\forall \mathsf{u}([\mathsf{u}\epsilon\mathsf{x}\sqcup\mathsf{y}]\leftrightarrow(\mathsf{u}\epsilon\mathsf{x}\vee\mathsf{u}\epsilon\mathsf{y})))$$
$$\mathrm{I}\Sigma_1\vdash (\mathtt{Set}(\mathsf{x})\wedge\mathtt{Set}(\mathsf{y})){\to}([\mathtt{Set}(\mathsf{x}\backslash\mathsf{y})]\wedge\forall \mathsf{u}([\mathsf{u}\epsilon\mathsf{x}\backslash\mathsf{y}]\leftrightarrow(\mathsf{u}\epsilon\mathsf{x}\wedge\neg(\mathsf{u}\epsilon\mathsf{y}))))$$
$$\mathrm{I}\Sigma_1\vdash (\mathtt{Set}(\mathsf{x})\wedge\mathtt{Set}(\mathsf{y})){\to}([\mathtt{Set}(\mathsf{x}{\times}\mathsf{y})]\wedge\forall \mathsf{u}([\mathsf{u}\epsilon\mathsf{x}{\times}\mathsf{y}]\leftrightarrow\exists \mathsf{v}\epsilon\mathsf{x}\exists \mathsf{w}\epsilon\mathsf{y}(\mathsf{u}{=}\langle \mathsf{v},\mathsf{w}\rangle)))$$
$$\mathrm{I}\Sigma_1\vdash \mathtt{Set}(\mathsf{x}){\to}([\mathtt{Set}(\mathtt{dom}(\mathsf{x}))]\wedge\forall \mathsf{u}([\mathsf{u}\epsilon\mathtt{dom}(\mathsf{x})]\leftrightarrow\exists \mathsf{z}\epsilon\mathsf{x}\exists \mathsf{v}{\le}\mathsf{z}(\mathsf{z}{=}\langle \mathsf{u},\mathsf{v}\rangle)))$$

[6] Note that if φ is $\Sigma_k(\mathrm{I}\Sigma_1)$ (or $\Pi_k(\mathrm{I}\Sigma_1)$), $\forall \mathsf{x}\epsilon\mathsf{y}\varphi$ is $\Sigma_k(\mathrm{I}\Sigma_1)$ ($\Pi_k(\mathrm{I}\Sigma_1)$) if $k>0$, otherwise $\Delta_1(\mathrm{I}\Sigma_1)$.

$$\mathrm{I}\Sigma_1 \vdash \mathtt{Set}(\mathsf{x}){\rightarrow}([\mathtt{Set}(\mathtt{ran}(\mathsf{x}))]{\wedge}\forall \mathsf{u}([\mathsf{u}\epsilon\mathtt{ran}(\mathsf{x})] \leftrightarrow \exists \mathsf{z}\epsilon\mathsf{x}\exists \mathsf{v}{\leq}\mathsf{z}(\mathsf{z}{=}\langle \mathsf{v},\mathsf{u}\rangle)))$$
$$\mathrm{I}\Sigma_1 \vdash \mathtt{Fcn}(\mathsf{f}){\leftrightarrow}(\mathtt{Set}(\mathsf{f}){\wedge}\forall \mathsf{p}\epsilon\mathsf{f}\exists \mathsf{x}\exists \mathsf{y}([\langle \mathsf{x},\mathsf{y}\rangle{=}\mathsf{p}]{\wedge}\forall \mathsf{z}([\langle \mathsf{x},\mathsf{z}\rangle\epsilon\mathsf{f}]{\rightarrow}\mathsf{z}{=}\mathsf{y})))$$
$$\mathrm{I}\Sigma_1 \vdash \mathtt{Fcn}(\mathsf{f}){\rightarrow}\forall[\mathsf{x}\epsilon\mathtt{dom}(\mathsf{f})]([\mathtt{apl}(\mathsf{f},\mathsf{x}){=}\mathsf{y}] \leftrightarrow [\langle \mathsf{x},\mathsf{y}\rangle\epsilon\mathsf{f}])$$
$$\mathrm{I}\Sigma_1 \vdash \mathtt{Seq}(\mathsf{s}){\leftrightarrow}(\mathtt{Fcn}(\mathsf{s}){\wedge}\exists \mathsf{x}{\leq}\mathsf{s}[\mathtt{dom}(\mathsf{s}){=}({<}\ \mathsf{x})])$$
$$\mathrm{I}\Sigma_1 \vdash \mathtt{Seq}(\mathsf{s}){\rightarrow}([\mathtt{lh}(\mathsf{s}){=}\mathsf{x}]{\leftrightarrow}[\mathtt{dom}(\mathsf{s}){=}({<}\ \mathsf{x})])$$
$$\mathrm{I}\Sigma_1 \vdash \mathtt{Seq}(\mathsf{s}){\rightarrow}\forall[\mathsf{x}{<}\mathtt{lh}(\mathsf{s})]([(\mathsf{s})_{\mathsf{x}}{=}\mathsf{y}] \leftrightarrow [\mathtt{apl}(\mathsf{s},\mathsf{x}){=}\mathsf{y}])$$
$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Seq}(\varnothing)]{\wedge}[\mathtt{lh}(\varnothing){=}0]$$
$$\mathrm{I}\Sigma_1 \vdash \mathtt{Seq}(\mathsf{s}){\rightarrow}([\mathtt{lh}(\mathsf{s}){\leq}\mathsf{s}]{\wedge}\forall[\mathsf{x}{<}\mathtt{lh}(\mathsf{s})][(\mathsf{s})_{\mathsf{x}}{<}\mathsf{s}])$$
$$\mathrm{I}\Sigma_1 \vdash \forall \mathsf{x}\exists \mathsf{y}(\mathsf{x}{<}\mathsf{y}{\wedge}\neg\mathtt{Seq}(\mathsf{y}))$$

We also have the following highly desirable result.

Lemma 2.27 (Extensionality)**.** $\mathrm{I}\Sigma_1$ *proves extensionality for sets and functions (and hence for sequences):*

$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Set}(\mathsf{x}){\wedge}\mathtt{Set}(\mathsf{y}){\wedge}\forall \mathsf{u}((\mathsf{u}\epsilon\mathsf{x}){\leftrightarrow}(\mathsf{u}\epsilon\mathsf{y}))){\rightarrow}\mathsf{x}{=}\mathsf{y},$$
$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Fcn}(\mathsf{f}){\wedge}\mathtt{Fcn}(\mathsf{g}){\wedge}[\mathtt{dom}(\mathsf{f}){=}\mathtt{dom}(\mathsf{g})]{\wedge}\forall[\mathsf{u}\epsilon\mathtt{dom}(\mathsf{f})][\mathtt{apl}(\mathsf{f},\mathsf{u}){=}\mathtt{apl}(\mathsf{g},\mathsf{u})]){\rightarrow}\mathsf{f}{=}\mathsf{g}.$$

Proof. The first claim is Corollary 1.38 of [4], the second follows from the first using the properties of sets stated above. □

Worth mentioning might be that the present construction does not use a pairing based on the language of set theory like e.g. the Kuratowski pairing, but the pairing defined above (Lemma 2.25) where every object (i.e. number) is also a pair of objects. Thus this is not a direct incorporation of a weak theory of sets in $\mathrm{I}\Sigma_1$.

Definition 2.13. The $\Delta_1(\mathrm{I}\Sigma_1)$-formula $\mathtt{Setq}(\mathsf{s})$ is

$$\mathtt{Seq}(\mathsf{s}){\wedge}\forall[\mathsf{i}{<}\mathtt{lh}(\mathsf{s})][\mathtt{Set}((\mathsf{s})_{\mathsf{i}})].$$

We also define the $\Delta_1(\mathrm{I}\Sigma_1)$-formulae $\mathtt{Incrseq}(\mathsf{s})$ and $\mathtt{Incrsetq}(\mathsf{s})$ to be

$$\mathtt{Seq}(\mathsf{s}){\wedge}\forall[\mathsf{i}{<}\mathtt{lh}(\mathsf{s})]([\mathsf{S}(\mathsf{i}){<}\mathtt{lh}(\mathsf{s})]{\rightarrow}[(\mathsf{s})_{\mathsf{i}}{<}(\mathsf{s})_{\mathsf{S}(\mathsf{i})}])$$

and

$$\mathtt{Setq}(\mathsf{s}){\wedge}\forall[\mathsf{i}{<}\mathtt{lh}(\mathsf{s})]([\mathsf{S}(\mathsf{i}){<}\mathtt{lh}(\mathsf{s})]{\rightarrow}[(\mathsf{s})_{\mathsf{i}}{\sqsubset}(\mathsf{s})_{\mathsf{S}(\mathsf{i})}])$$

respectively.

As the final piece of set theory that we shall require, we verify that with the above properties of sets we can always extend functions with a single pair of objects, and so in particular we can construct sets and functions of any (concrete) finite cardinality.

Definition 2.14. Let $[\mathtt{ext}(\mathsf{f},\mathsf{x},\mathsf{y}){=}\mathsf{g}]$ be the formula $[(\mathsf{f}\backslash\{\langle \mathsf{x},\mathtt{apl}(\mathsf{f},\mathsf{x})\rangle\}){\sqcup}\{\langle \mathsf{x},\mathsf{y}\rangle\}{=}\mathsf{g}]$.

This construction shares the principal properties of the corresponding operation on actual functions.

Lemma 2.28. $\mathtt{ext}$ *is a* $\mathrm{I}\Sigma_1$*-provably* $\Delta_1(\mathrm{I}\Sigma_1)$*-function satisfying*

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &\mathtt{Fcn}(\mathsf{f})\rightarrow([\mathtt{Fcn}(\mathtt{ext}(\mathsf{f},\mathsf{x},\mathsf{y}))]\wedge[\mathtt{dom}(\mathtt{ext}(\mathsf{f},\mathsf{x},\mathsf{y}))=\mathtt{dom}(\mathsf{f})\sqcup\{\mathsf{x}\}]\wedge\\ &\forall[\mathsf{z}\epsilon\mathtt{dom}(\mathtt{ext}(\mathsf{f},\mathsf{x},\mathsf{y}))]((\mathsf{z}=\mathsf{x}\rightarrow[\mathtt{apl}(\mathtt{ext}(\mathsf{f},\mathsf{x},\mathsf{y}),\mathsf{z})=\mathsf{y}])\wedge\\ &(\mathsf{z}\neq\mathsf{x}\rightarrow[\mathtt{apl}(\mathtt{ext}(\mathsf{f},\mathsf{x},\mathsf{y}),\mathsf{z})=\mathtt{apl}(\mathsf{f},\mathsf{z})]))),\\ \mathrm{I}\Sigma_1 \vdash &(\mathtt{Fcn}(\mathsf{f})\wedge[\mathsf{x}\epsilon\mathtt{dom}(\mathsf{f})])\rightarrow[\mathtt{ext}(\mathsf{f},\mathsf{x},\mathtt{apl}(\mathsf{f},\mathsf{x}))=\mathsf{f}].\end{aligned}$$

Proof. Since $\mathtt{ext}$ is a composition of $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-functions, it is itself an $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function. To verify the claimed properties we reason in $\mathrm{I}\Sigma_1$.

First let f, x, y and g be such that $\mathtt{Fcn}(\mathsf{f})$ and $[\mathtt{ext}(\mathsf{f},\mathsf{x},\mathsf{y})=\mathsf{g}]$. Then $\mathtt{Set}(\mathsf{g})$ by Lemma 2.26. Let $\mathsf{p}\epsilon\mathsf{g}$, then $[\mathsf{p}\epsilon(\mathsf{f}\backslash\{\langle\mathsf{x},\mathtt{apl}(\mathsf{f},\mathsf{x})\rangle\})]$ or $\mathsf{p}=\langle\mathsf{x},\mathsf{y}\rangle$. In the first case let z be such that $\mathsf{p}=\langle\mathsf{z},\mathtt{apl}(\mathsf{f},\mathsf{z})\rangle$; then $\mathsf{z}\neq\mathsf{x}$ and for all w such that $[\langle\mathsf{z},\mathsf{w}\rangle\epsilon\mathsf{g}]$ we must have $[\langle\mathsf{z},\mathsf{w}\rangle\epsilon\mathsf{f}]$, whence $[\mathtt{apl}(\mathsf{f},\mathsf{z})=\mathsf{w}]$. In the second case let z be such that $[\langle\mathsf{x},\mathsf{z}\rangle\epsilon\mathsf{g}]$; then either $[\langle\mathsf{x},\mathsf{z}\rangle\epsilon\mathsf{f}]$ but $[\langle\mathsf{x},\mathsf{z}\rangle\neq\langle\mathsf{x},\mathtt{apl}(\mathsf{f},\mathsf{x})\rangle]$, which is impossible since $\mathtt{Fcn}(\mathsf{f})$, or $[\langle\mathsf{x},\mathsf{z}\rangle=\langle\mathsf{x},\mathsf{y}\rangle]$, in which case $\mathsf{z}=\mathsf{y}$. Thus $\mathtt{Fcn}(\mathsf{g})$. Now let $[\mathsf{u}\epsilon\mathtt{dom}(\mathsf{g})]$. If $\mathsf{u}=\mathsf{x}$ then $[\langle\mathsf{u},\mathsf{y}\rangle\epsilon\mathsf{g}]$ by definition, whence $[\mathtt{apl}(\mathsf{g},\mathsf{u})=\mathsf{y}]$. If instead $\mathsf{u}\neq\mathsf{x}$ we must have $[\mathsf{u}\epsilon\mathtt{dom}(\mathsf{f})]$, whence $[\mathtt{apl}(\mathsf{g},\mathsf{u})=\mathtt{apl}(\mathsf{f},\mathsf{u})]$. This confirms the first claim.

Next let f, x and g be such that $[\mathsf{x}\epsilon\mathtt{dom}(\mathsf{f})]$ and $[\mathtt{ext}(\mathsf{f},\mathsf{x},\mathtt{apl}(\mathsf{f},\mathsf{x}))=\mathsf{g}]$. Then, by above, $[\mathtt{dom}(\mathsf{g})=\mathtt{dom}(\mathsf{f})]$. Suppose $[\mathsf{z}\epsilon\mathtt{dom}(\mathsf{f})]$. If $\mathsf{z}=\mathsf{x}$ then $[\mathtt{apl}(\mathsf{g},\mathsf{z})=\mathtt{apl}(\mathsf{f},\mathsf{z})]$, and if $\mathsf{z}\neq\mathsf{x}$ we have $[\mathtt{apl}(\mathsf{g},\mathsf{z})=\mathtt{apl}(\mathsf{f},\mathsf{z})]$, again by above. Hence $\mathsf{f}=\mathsf{g}$ by extensionality. ☐

Lemma 2.29. *Given* $k\in\mathbb{N}$ *there are* $\mathrm{I}\Sigma_1$*-provably* $\Delta_1(\mathrm{I}\Sigma_1)$*-functions* $[\{\mathsf{x}_0,\ldots,\mathsf{x}_{k-1}\}^k=\mathsf{y}]$ *and* $[\mathtt{fn}_k(\mathsf{x}_0,\ldots,\mathsf{x}_{k-1},\mathsf{y}_0,\ldots,\mathsf{y}_{k-1})=\mathsf{f}]$ *such that*

$$\mathrm{I}\Sigma_1 \vdash[\mathtt{Set}(\{\mathsf{x}_0,\ldots,\mathsf{x}_{k-1}\}^k)]\wedge\forall\mathsf{y}([\mathsf{y}\epsilon\{\mathsf{x}_0,\ldots,\mathsf{x}_{k-1}\}^k]\leftrightarrow\bigvee\nolimits_{0\leq i<k}\mathsf{y}=\mathsf{x}_i)$$

and

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash&([\mathtt{fn}_k(\mathsf{x}_0,\ldots,\mathsf{x}_{k-1},\mathsf{y}_0,\ldots,\mathsf{y}_{k-1})=\mathsf{f}]\wedge\bigwedge\nolimits_{0\leq i<j<k}\neg(\mathsf{x}_i=\mathsf{x}_j))\rightarrow\\ &(\mathtt{Fcn}(\mathsf{f})\wedge[\mathtt{dom}(\mathsf{f})=\{\mathsf{x}_0,\ldots,\mathsf{x}_{k-1}\}^k]\wedge\bigwedge\nolimits_{0\leq i<k}[\mathtt{apl}(\mathsf{f},\mathsf{x}_i)=\mathsf{y}_i])\end{aligned}$$

Proof. Define $[\{\}^0=\mathsf{y}]$ to be $[\varnothing=\mathsf{y}]$ and for all $k\in\mathbb{N}$ define $[\{\mathsf{x}_0,\ldots,\mathsf{x}_k\}^{k+1}=\mathsf{y}]$ to be $[(\cdots(\{\mathsf{x}_0\}\sqcup\cdots\sqcup\{\mathsf{x}_{k-1}\})\sqcup\{\mathsf{x}_k\})=\mathsf{y}]$. By the properties expressed in Lemma 2.26 and induction on k, the first statement above holds.

Now define $[\mathtt{fn}_k(\mathsf{x}_0,\ldots,\mathsf{x}_{k-1},\mathsf{y}_0,\ldots,\mathsf{y}_{k-1})=\mathsf{f}]$ by $[\{\langle\mathsf{x}_0,\mathsf{y}_0\rangle,\ldots,\langle\mathsf{x}_{k-1},\mathsf{y}_{k-1}\rangle\}^k=\mathsf{f}]$. Reasoning in $\mathrm{I}\Sigma_1$, let $\mathsf{x}_0,\ldots,\mathsf{x}_{k-1},\mathsf{y}_0,\ldots,\mathsf{y}_{k-1}$ satisfy $\bigwedge_{0\leq i<j<k}\neg(\mathsf{x}_i=\mathsf{x}_j)$ and f be such that $[\mathtt{fn}_k(\mathsf{x}_0,\ldots,\mathsf{x}_{k-1},\mathsf{y}_0,\ldots,\mathsf{y}_{k-1})=\mathsf{f}]$. Take p such that $\mathsf{p}\epsilon\mathsf{f}$; then $\bigvee_{0\leq i<k}[\langle\mathsf{x}_i,\mathsf{y}_i\rangle=\mathsf{p}]$ by above. Assuming $[\langle\mathsf{x}_n,\mathsf{y}_n\rangle=\mathsf{p}]$, let z be such that $[\langle\mathsf{x}_n,\mathsf{z}\rangle\epsilon\mathsf{f}]$. By above again $\bigvee_{0\leq i<k}[\langle\mathsf{x}_i,\mathsf{y}_i\rangle=\langle\mathsf{x}_n,\mathsf{z}\rangle]$, whence $[\langle\mathsf{x}_n,\mathsf{y}_n\rangle=\langle\mathsf{x}_n,\mathsf{z}\rangle]$ by $\bigwedge_{0\leq i<j<k}\neg(\mathsf{x}_i=\mathsf{x}_j)$ and Lemma 2.25, which subsequently gives $\mathsf{y}_n=\mathsf{z}$. Thus $\mathtt{Fcn}(\mathsf{f})$. Furthermore

$$[\mathsf{x}\epsilon\mathtt{dom}(\mathsf{f})]\leftrightarrow\exists\mathsf{v}[\langle\mathsf{x},\mathsf{v}\rangle\epsilon\mathsf{f}]\leftrightarrow\exists\mathsf{v}\bigvee\nolimits_{0\leq i<k}[\langle\mathsf{x}_i,\mathsf{y}_i\rangle=\langle\mathsf{x},\mathsf{v}\rangle]\leftrightarrow\bigvee\nolimits_{0\leq i<k}\mathsf{x}=\mathsf{x}_i,$$

so $[\mathtt{dom}(\mathsf{f})=\{\mathsf{x}_0,\ldots,\mathsf{x}_{k-1}\}^k]$. Since $\bigwedge_{0\leq i<k}[\langle \mathsf{x}_i,\mathsf{y}_i\rangle\epsilon\mathsf{f}]$ this implies $\bigwedge_{0\leq i<k}[\mathtt{apl}(\mathsf{f},\mathsf{x}_i)=\mathsf{y}_i]$ by Lemma 2.26. □

2.2.2 Terms and formulae

With set theory, we can define what are terms and formulae. Specifically, there are $\Delta_1(\mathrm{I}\Sigma_1)$-formulae $\mathtt{Var}(\mathsf{x})$, $\mathtt{Term}(\mathsf{t})$ and $\mathtt{Fmla}(\mathsf{f})$ and $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-functions $[(\mathsf{x})=\mathsf{t}]$, $[\mathbf{0}=\mathsf{x}]$, $[\mathbf{v}_\mathsf{x}=\mathsf{y}]$, $[(\mathbf{S}\mathsf{s})=\mathsf{t}]$, $[(\mathsf{r}+\mathsf{s})=\mathsf{t}]$, $[(\mathsf{r}\cdot\mathsf{s})=\mathsf{t}]$, $[(\mathsf{s}=\mathsf{t})=\mathsf{f}]$, $[(\mathsf{s}<\mathsf{t})=\mathsf{f}]$, $[(\mathsf{h}\wedge\mathsf{g})=\mathsf{f}]$, $[(\mathsf{h}\vee\mathsf{g})=\mathsf{f}]$, $[(\neg\mathsf{g})=\mathsf{f}]$, $[(\forall\mathbf{v}_\mathsf{x}\mathsf{g})=\mathsf{f}]$ and $[(\exists\mathbf{v}_\mathsf{x}\mathsf{g})=\mathsf{f}]$ such that

$$\begin{aligned}
&\mathrm{I}\Sigma_1 \vdash[\mathsf{x}<\mathbf{v}_\mathsf{x}],\\
&\mathrm{I}\Sigma_1 \vdash\mathtt{Var}(\mathsf{x})\leftrightarrow\exists\mathsf{i}<\mathsf{x}[\mathbf{v}_\mathsf{i}=\mathsf{x}],\\
&\mathrm{I}\Sigma_1 \vdash[\mathbf{v}_\mathsf{x}=\mathbf{v}_\mathsf{y}]\rightarrow\mathsf{x}=\mathsf{y},\\
&\mathrm{I}\Sigma_1 \vdash[\mathtt{Term}((\mathbf{0}))]\wedge[\mathbf{0}<(\mathbf{0})],\\
&\mathrm{I}\Sigma_1 \vdash\mathtt{Var}(\mathsf{x})\rightarrow([\mathtt{Term}((\mathsf{x}))]\wedge[\mathsf{x}<(\mathsf{x})]),\\
&\mathrm{I}\Sigma_1 \vdash\mathtt{Term}(\mathsf{t})\rightarrow([\mathtt{Term}((\mathbf{S}\mathsf{t}))]\wedge[\mathsf{t}<(\mathbf{S}\mathsf{t})]),\\
&\mathrm{I}\Sigma_1 \vdash(\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t}))\rightarrow([\mathtt{Term}((\mathsf{s}+\mathsf{t}))]\wedge[\mathsf{s}<(\mathsf{s}+\mathsf{t})]\wedge[\mathsf{t}<(\mathsf{s}+\mathsf{t})]),\\
&\mathrm{I}\Sigma_1 \vdash(\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t}))\rightarrow([\mathtt{Term}((\mathsf{s}\cdot\mathsf{t}))]\wedge[\mathsf{s}<(\mathsf{s}\cdot\mathsf{t})]\wedge[\mathsf{t}<(\mathsf{s}\cdot\mathsf{t})]),\\
&\mathrm{I}\Sigma_1 \vdash(\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t})) \rightarrow ([\mathtt{Fmla}((\mathsf{s}=\mathsf{t}))]\wedge[\mathsf{s}<(\mathsf{s}=\mathsf{t})]\wedge[\mathsf{t}<(\mathsf{s}=\mathsf{t})]),\\
&\mathrm{I}\Sigma_1 \vdash(\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t})) \rightarrow ([\mathtt{Fmla}((\mathsf{s}<\mathsf{t}))]\wedge[\mathsf{s}<(\mathsf{s}<\mathsf{t})]\wedge[\mathsf{t}<(\mathsf{s}<\mathsf{t})]),\\
&\mathrm{I}\Sigma_1 \vdash(\mathtt{Fmla}(\mathsf{f})\wedge\mathtt{Fmla}(\mathsf{g})) \rightarrow ([\mathtt{Fmla}((\mathsf{f}\wedge\mathsf{g}))]\wedge[\mathsf{f}<(\mathsf{f}\wedge\mathsf{g})]\wedge[\mathsf{g}<(\mathsf{f}\wedge\mathsf{g})]),\\
&\mathrm{I}\Sigma_1 \vdash(\mathtt{Fmla}(\mathsf{f})\wedge\mathtt{Fmla}(\mathsf{g})) \rightarrow ([\mathtt{Fmla}((\mathsf{f}\vee\mathsf{g}))]\wedge[\mathsf{f}<(\mathsf{f}\vee\mathsf{g})]\wedge[\mathsf{g}<(\mathsf{f}\vee\mathsf{g})]),\\
&\mathrm{I}\Sigma_1 \vdash\mathtt{Fmla}(\mathsf{f}) \rightarrow ([\mathtt{Fmla}((\neg\mathsf{f}))]\wedge[\mathsf{f}<(\neg\mathsf{f})]),\\
&\mathrm{I}\Sigma_1 \vdash\mathtt{Fmla}(\mathsf{f}) \rightarrow ([\mathtt{Fmla}((\forall\mathbf{v}_\mathsf{x}\mathsf{f}))]\wedge[\mathbf{v}_\mathsf{x}<(\forall\mathbf{v}_\mathsf{x}\mathsf{f})]\wedge[\mathsf{f}<(\forall\mathbf{v}_\mathsf{x}\mathsf{f})]),\\
&\mathrm{I}\Sigma_1 \vdash\mathtt{Fmla}(\mathsf{f}) \rightarrow ([\mathtt{Fmla}((\exists\mathbf{v}_\mathsf{x}\mathsf{f}))]\wedge[\mathbf{v}_\mathsf{x}<(\exists\mathbf{v}_\mathsf{x}\mathsf{f})]\wedge[\mathsf{f}<(\exists\mathbf{v}_\mathsf{x}\mathsf{f})]),\\
&\mathrm{I}\Sigma_1 \vdash\neg(\mathtt{Term}(\mathsf{x})\wedge\mathtt{Fmla}(\mathsf{x}))\\
&\mathrm{I}\Sigma_1 \vdash(\neg\mathtt{Term}(0))\wedge(\neg\mathtt{Fmla}(0)).
\end{aligned}$$

We also define $\mathtt{Atf}(\mathsf{f})$ to be $\exists\mathsf{s}<\mathsf{f}\exists\mathsf{t}<\mathsf{f}\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t}) \wedge ([(\mathsf{s}=\mathsf{t})=\mathsf{f}]\vee[(\mathsf{s}<\mathsf{t})=\mathsf{f}])$, so that

$$\mathrm{I}\Sigma_1 \vdash\mathtt{Atf}(\mathsf{f})\rightarrow\mathtt{Fmla}(\mathsf{f})$$

et cetera.

We can introduce the usual abbreviations for $\rightarrow$ and $\leftrightarrow$, but we will not since we will have little use for them, instead using them solely at the meta level.

Σ_1-induction now extends to (formal) terms and formulae.

Lemma 2.30 (Σ_1-$\mathtt{Term}$-induction and Σ_1-$\mathtt{Fmla}$-induction)**.** *For all Σ_1-formulae φ we have*

$$\begin{aligned}
\mathrm{I}\Sigma_1 \vdash&([\varphi((\mathbf{0}))] \wedge \forall\mathsf{x}[\varphi((\mathbf{v}_\mathsf{x}))]\wedge\\
&\quad\forall\mathsf{s}\forall\mathsf{t}((\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t})\wedge\varphi(\mathsf{s})\wedge\varphi(\mathsf{t}))\rightarrow([\varphi((\mathbf{S}\mathsf{t}))]\wedge[\varphi((\mathsf{s}+\mathsf{t}))]\wedge[\varphi((\mathsf{s}\cdot\mathsf{t}))])))\rightarrow\\
&\quad\forall\mathsf{t}(\mathtt{Term}(\mathsf{t})\rightarrow\varphi(\mathsf{t}))
\end{aligned}$$

where $\mathtt{t}$ *and* $\mathtt{s}$ *do not occur in* φ, *and*

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash & (\forall \mathsf{s}\forall \mathsf{t}((\mathtt{Term}(\mathsf{t})\wedge\mathtt{Term}(\mathsf{s}))\rightarrow[\varphi(\mathbf{(}\mathsf{s}\mathbf{=}\mathsf{t}\mathbf{)})]\wedge[\varphi(\mathbf{(}\mathsf{s}\mathbf{\leq}\mathsf{t}\mathbf{)})])\wedge\\ &\quad \forall \mathsf{f}\forall \mathsf{g}((\mathtt{Fmla}(\mathsf{f})\wedge\mathtt{Fmla}(\mathsf{g})\wedge\varphi(\mathsf{f})\wedge\varphi(\mathsf{g}))\rightarrow\\ &\qquad ([\varphi(\mathbf{(\neg}\mathsf{f}\mathbf{)})]\wedge[\varphi(\mathbf{(}\mathsf{f}\mathbf{\wedge}\mathsf{g}\mathbf{)})]\wedge[\varphi(\mathbf{(}\mathsf{f}\mathbf{\vee}\mathsf{g}\mathbf{)})]\wedge\forall \mathsf{x}([\varphi(\mathbf{(\forall v}_\mathsf{x}\mathsf{f}\mathbf{)})]\wedge[\varphi(\mathbf{(\exists v}_\mathsf{x}\mathsf{f}\mathbf{)})]))))\rightarrow\\ &\quad \forall \mathsf{f}(\mathtt{Fmla}(\mathsf{f})\rightarrow\varphi(\mathsf{f}))\end{aligned}$$

where f *and* g *do not occur in* φ.

In $\mathrm{I}\Sigma_1$ we can even define provably $\Delta_1(\mathrm{I}\Sigma_1)$-functions on terms and formulae by structural recursion with parameters.

Lemma 2.31 (Σ_1-$\mathtt{Term}$-recursion and Σ_1-$\mathtt{Fmla}$-recursion)**.** *If* $\mathtt{h}_{\mathbf{Var}}$, $\mathtt{h}_{\mathbf{0}}$, $\mathtt{h}_{\mathbf{S}}$, $\mathtt{h}_{+}$ *and* $\mathtt{h}_{\cdot}$ *as well as* $\mathtt{h}_{=}$, $\mathtt{h}_{<}$, $\mathtt{h}_{\wedge}$, $\mathtt{h}_{\vee}$, $\mathtt{h}_{\neg}$, $\mathtt{h}_{\forall}$ *and* $\mathtt{h}_{\exists}$ *are* $\mathrm{I}\Sigma_1$*-provably* $\Delta_1(\mathrm{I}\Sigma_1)$*-functions (with the correct number of parameters) then there are unique* $\mathrm{I}\Sigma_1$*-provably* $\Delta_1(\mathrm{I}\Sigma_1)$*-functions* $\mathtt{f}$ *and* $\mathtt{g}$ *such that*

$$\begin{aligned}&\mathrm{I}\Sigma_1 \vdash [\mathtt{f}(\bar{\mathsf{x}},\mathbf{(0)})=\mathtt{h}_{\mathbf{0}}(\bar{\mathsf{x}})]\\ &\mathrm{I}\Sigma_1 \vdash [\mathtt{f}(\bar{\mathsf{x}},\mathbf{(v}_\mathsf{y}\mathbf{)})=\mathtt{h}_{\mathbf{Var}}(\bar{\mathsf{x}},\mathbf{v}_\mathsf{y})]\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Term}(\mathsf{t})\rightarrow[\mathtt{f}(\bar{\mathsf{x}},\mathbf{(S}\mathsf{t}\mathbf{)})=\mathtt{h}_{\mathbf{S}}(\bar{\mathsf{x}},\mathtt{f}(\mathsf{t}))]\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t}))\rightarrow[\mathtt{f}(\bar{\mathsf{x}},\mathbf{(}\mathsf{s}\mathbf{+}\mathsf{t}\mathbf{)})=\mathtt{h}_{+}(\bar{\mathsf{x}},\mathtt{f}(\mathsf{s}),\mathtt{f}(\mathsf{t}))]\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t}))\rightarrow[\mathtt{f}(\bar{\mathsf{x}},\mathbf{(}\mathsf{s}\mathbf{\cdot}\mathsf{t}\mathbf{)})=\mathtt{h}_{\cdot}(\bar{\mathsf{x}},\mathtt{f}(\mathsf{s}),\mathtt{f}(\mathsf{t}))]\\ &\mathrm{I}\Sigma_1 \vdash \neg\mathtt{Term}(\mathsf{t})\rightarrow[\mathtt{f}(\bar{\mathsf{x}},\mathsf{t})=0]\end{aligned}$$

and

$$\begin{aligned}&\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t}))\rightarrow[\mathsf{g}(\bar{\mathsf{x}},\mathbf{(}\mathsf{s}\mathbf{=}\mathsf{t}\mathbf{)})=\mathtt{h}_{=}(\bar{\mathsf{x}},\mathsf{s},\mathsf{t})]\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s})\wedge\mathtt{Term}(\mathsf{t}))\rightarrow[\mathsf{g}(\bar{\mathsf{x}},\mathbf{(}\mathsf{s}\mathbf{<}\mathsf{t}\mathbf{)})=\mathtt{h}_{<}(\bar{\mathsf{x}},\mathsf{s},\mathsf{t})]\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f})\rightarrow[\mathsf{g}(\bar{\mathsf{x}},\mathbf{(\neg}\mathsf{f}\mathbf{)})=\mathtt{h}_{\neg}(\bar{\mathsf{x}},\mathsf{g}(\mathsf{f}))]\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f})\wedge\mathtt{Fmla}(\mathsf{g}))\rightarrow[\mathsf{g}(\bar{\mathsf{x}},\mathbf{(}\mathsf{f}\mathbf{\wedge}\mathsf{g}\mathbf{)})=\mathtt{h}_{\wedge}(\bar{\mathsf{x}},\mathsf{g}(\mathsf{f}),\mathsf{g}(\mathsf{g}))]\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f})\wedge\mathtt{Fmla}(\mathsf{g}))\rightarrow[\mathsf{g}(\bar{\mathsf{x}},\mathbf{(}\mathsf{f}\mathbf{\vee}\mathsf{g}\mathbf{)})=\mathtt{h}_{\vee}(\bar{\mathsf{x}},\mathsf{g}(\mathsf{f}),\mathsf{g}(\mathsf{g}))]\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f})\rightarrow[\mathsf{g}(\bar{\mathsf{x}},\mathbf{(\forall v}_\mathsf{y}\mathsf{f}\mathbf{)})=\mathtt{h}_{\forall}(\bar{\mathsf{x}},\mathbf{v}_\mathsf{y},\mathsf{g}(\mathsf{f}))]\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f})\rightarrow[\mathsf{g}(\bar{\mathsf{x}},\mathbf{(\exists v}_\mathsf{y}\mathsf{f}\mathbf{)})=\mathtt{h}_{\exists}(\bar{\mathsf{x}},\mathbf{v}_\mathsf{y},\mathsf{g}(\mathsf{f}))]\\ &\mathrm{I}\Sigma_1 \vdash \neg\mathtt{Fmla}(\mathsf{f})\rightarrow[\mathsf{g}(\bar{\mathsf{x}},\mathsf{f})=0].\end{aligned}$$

Among other things this, in a roundabout way, gives uniqueness of the "construction-trees" of (formal) formulae.

Lemma 2.32 (Uniqueness of formula-constructions)**.** $\mathrm{I}\Sigma_1$ *proves uniqueness of the con-*

struction of any formula:

$$\begin{aligned}
\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \leftrightarrow (&\mathtt{Atf}(\mathsf{f}) \vee \exists \mathsf{g}{<}\mathsf{f}(\mathtt{Fmla}(\mathsf{g}) \wedge [(\neg \mathsf{g}) = \mathsf{f}]) \vee \\
&\exists \mathsf{g}{<}\mathsf{f} \exists \mathsf{h}{<}\mathsf{f}(\mathtt{Fmla}(\mathsf{g}) \wedge \mathtt{Fmla}(\mathsf{h}) \wedge [(\mathsf{g} \wedge \mathsf{h}) = \mathsf{f}]) \vee \\
&\exists \mathsf{g}{<}\mathsf{f} \exists \mathsf{h}{<}\mathsf{f}(\mathtt{Fmla}(\mathsf{g}) \wedge \mathtt{Fmla}(\mathsf{h}) \wedge [(\mathsf{g} \vee \mathsf{h}) = \mathsf{f}]) \vee \\
&\exists \mathsf{g}{<}\mathsf{f} \exists \mathsf{x}{<}\mathsf{f}(\mathtt{Fmla}(\mathsf{g}) \wedge [(\forall \mathbf{v}_{\mathsf{x}} \mathsf{g}) = \mathsf{f}]) \vee \\
&\exists \mathsf{g}{<}\mathsf{f} \exists \mathsf{x}{<}\mathsf{f}(\mathtt{Fmla}(\mathsf{g}) \wedge [(\exists \mathbf{v}_{\mathsf{x}} \mathsf{g}) = \mathsf{f}])),
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t})) \to (&\forall \mathsf{r} \forall \mathsf{u}((\mathtt{Term}(\mathsf{r}) \wedge \mathtt{Term}(\mathsf{u}) \wedge [(\mathsf{r} = \mathsf{u}) = (\mathsf{s} = \mathsf{t})]) \to (\mathsf{r} = \mathsf{s} \wedge \mathsf{u} = \mathsf{t})) \wedge \\
&\neg \exists \mathsf{r} \exists \mathsf{u}(\mathtt{Term}(\mathsf{r}) \wedge \mathtt{Term}(\mathsf{u}) \wedge [(\mathsf{r} < \mathsf{u}) = (\mathsf{s} = \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f}(\mathtt{Fmla}(\mathsf{f}) \wedge [(\neg \mathsf{f}) = (\mathsf{s} = \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f} \exists \mathsf{g}(\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g}) \wedge [(\mathsf{f} \wedge \mathsf{g}) = (\mathsf{s} = \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f} \exists \mathsf{g}(\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g}) \wedge [(\mathsf{f} \vee \mathsf{g}) = (\mathsf{s} = \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f} \exists \mathsf{x}(\mathtt{Fmla}(\mathsf{f}) \wedge [(\forall \mathbf{v}_{\mathsf{x}} \mathsf{f}) = (\mathsf{s} = \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f} \exists \mathsf{x}(\mathtt{Fmla}(\mathsf{f}) \wedge [(\exists \mathbf{v}_{\mathsf{x}} \mathsf{f}) = (\mathsf{s} = \mathsf{t})])),
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t})) \to (&\forall \mathsf{r} \forall \mathsf{u}((\mathtt{Term}(\mathsf{r}) \wedge \mathtt{Term}(\mathsf{u}) \wedge [(\mathsf{r} < \mathsf{u}) = (\mathsf{s} < \mathsf{t})]) \to (\mathsf{r} = \mathsf{s} \wedge \mathsf{u} = \mathsf{t})) \wedge \\
&\neg \exists \mathsf{r} \exists \mathsf{u}(\mathtt{Term}(\mathsf{r}) \wedge \mathtt{Term}(\mathsf{u}) \wedge [(\mathsf{r} = \mathsf{u}) = (\mathsf{s} < \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f}(\mathtt{Fmla}(\mathsf{f}) \wedge [(\neg \mathsf{f}) = (\mathsf{s} < \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f} \exists \mathsf{g}(\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g}) \wedge [(\mathsf{f} \wedge \mathsf{g}) = (\mathsf{s} < \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f} \exists \mathsf{g}(\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g}) \wedge [(\mathsf{f} \vee \mathsf{g}) = (\mathsf{s} < \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f} \exists \mathsf{x}(\mathtt{Fmla}(\mathsf{f}) \wedge [(\forall \mathbf{v}_{\mathsf{x}} \mathsf{f}) = (\mathsf{s} < \mathsf{t})]) \wedge \\
&\neg \exists \mathsf{f} \exists \mathsf{x}(\mathtt{Fmla}(\mathsf{f}) \wedge [(\exists \mathbf{v}_{\mathsf{x}} \mathsf{f}) = (\mathsf{s} < \mathsf{t})])),
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \to (&\forall \mathsf{g}((\mathtt{Fmla}(\mathsf{g}) \wedge [(\neg \mathsf{g}) = (\neg \mathsf{f})]) \to \mathsf{g} = \mathsf{f}) \wedge \\
&\neg \exists \mathsf{s} \exists \mathsf{t}(\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t}) \wedge [(\mathsf{s} = \mathsf{t}) = (\neg \mathsf{f})]) \wedge \\
&\neg \exists \mathsf{s} \exists \mathsf{t}(\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t}) \wedge [(\mathsf{s} < \mathsf{t}) = (\neg \mathsf{f})]) \wedge \\
&\neg \exists \mathsf{g} \exists \mathsf{h}(\mathtt{Fmla}(\mathsf{g}) \wedge \mathtt{Fmla}(\mathsf{h}) \wedge [(\mathsf{g} \wedge \mathsf{h}) = (\neg \mathsf{f})]) \wedge \\
&\neg \exists \mathsf{g} \exists \mathsf{h}(\mathtt{Fmla}(\mathsf{g}) \wedge \mathtt{Fmla}(\mathsf{h}) \wedge [(\mathsf{g} \vee \mathsf{h}) = (\neg \mathsf{f})]) \wedge \\
&\neg \exists \mathsf{g} \exists \mathsf{x}(\mathtt{Fmla}(\mathsf{g}) \wedge [(\forall \mathbf{v}_{\mathsf{x}} \mathsf{g}) = (\neg \mathsf{f})]) \wedge \\
&\neg \exists \mathsf{g} \exists \mathsf{x}(\mathtt{Fmla}(\mathsf{g}) \wedge [(\exists \mathbf{v}_{\mathsf{x}} \mathsf{g}) = (\neg \mathsf{f})])),
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g})) \to (&\forall \mathsf{h} \forall \mathsf{k}((\mathtt{Fmla}(\mathsf{h}) \wedge \mathtt{Fmla}(\mathsf{k}) \wedge [(\mathsf{h} \wedge \mathsf{k}) = (\mathsf{f} \wedge \mathsf{g})]) \to (\mathsf{h} = \mathsf{f} \wedge \mathsf{k} = \mathsf{g})) \wedge \\
&\neg \exists \mathsf{s} \exists \mathsf{t}(\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t}) \wedge [(\mathsf{s} = \mathsf{t}) = (\mathsf{f} \wedge \mathsf{g})]) \wedge \\
&\neg \exists \mathsf{s} \exists \mathsf{t}(\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t}) \wedge [(\mathsf{s} < \mathsf{t}) = (\mathsf{f} \wedge \mathsf{g})]) \wedge \\
&\neg \exists \mathsf{h}(\mathtt{Fmla}(\mathsf{h}) \wedge [(\neg \mathsf{f}) = (\mathsf{f} \wedge \mathsf{g})]) \wedge \\
&\neg \exists \mathsf{h} \exists \mathsf{k}(\mathtt{Fmla}(\mathsf{h}) \wedge \mathtt{Fmla}(\mathsf{k}) \wedge [(\mathsf{h} \vee \mathsf{k}) = (\mathsf{f} \wedge \mathsf{g})]) \wedge \\
&\neg \exists \mathsf{h} \exists \mathsf{x}(\mathtt{Fmla}(\mathsf{h}) \wedge [(\forall \mathbf{v}_{\mathsf{x}} \mathsf{h}) = (\mathsf{f} \wedge \mathsf{g})]) \wedge \\
&\neg \exists \mathsf{h} \exists \mathsf{x}(\mathtt{Fmla}(\mathsf{h}) \wedge [(\exists \mathbf{v}_{\mathsf{x}} \mathsf{h}) = (\mathsf{f} \wedge \mathsf{g})])),
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g})) \to (&\forall \mathsf{h} \forall \mathsf{k}((\mathtt{Fmla}(\mathsf{h}) \wedge \mathtt{Fmla}(\mathsf{k}) \wedge [(\mathsf{h} \vee \mathsf{k}) = (\mathsf{f} \vee \mathsf{g})]) \to (\mathsf{h} = \mathsf{f} \wedge \mathsf{k} = \mathsf{g})) \wedge \\
&\neg \exists \mathsf{s} \exists \mathsf{t}(\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t}) \wedge [(\mathsf{s} = \mathsf{t}) = (\mathsf{f} \vee \mathsf{g})]) \wedge
\end{aligned}$$

$$
\begin{aligned}
&\neg\exists s\exists t(\mathtt{Term}(s)\wedge\mathtt{Term}(t)\wedge[(s<t)=(f\vee g)])\wedge\\
&\neg\exists h(\mathtt{Fmla}(h)\wedge[(\neg f)=(f\vee g)])\wedge\\
&\neg\exists h\exists k(\mathtt{Fmla}(h)\wedge\mathtt{Fmla}(k)\wedge[(h\wedge k)=(f\vee g)])\wedge\\
&\neg\exists h\exists x(\mathtt{Fmla}(h)\wedge[(\forall\mathbf{v}_x h)=(f\vee g)])\wedge\\
&\neg\exists h\exists x(\mathtt{Fmla}(h)\wedge[(\exists\mathbf{v}_x h)=(f\vee g)])),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{I}\Sigma_1\vdash\mathtt{Fmla}(f)\rightarrow(\forall g\forall y((&\mathtt{Fmla}(g)\wedge[(\forall\mathbf{v}_y g)=(\forall\mathbf{v}_x f)])\rightarrow(y=x\wedge g=f))\wedge\\
&\neg\exists s\exists t(\mathtt{Term}(s)\wedge\mathtt{Term}(t)\wedge[(s=t)=(\forall\mathbf{v}_x f)])\wedge\\
&\neg\exists s\exists t(\mathtt{Term}(s)\wedge\mathtt{Term}(t)\wedge[(s<t)=(\forall\mathbf{v}_x f)])\wedge\\
&\neg\exists g(\mathtt{Fmla}(g)\wedge[(\neg g)=(\forall\mathbf{v}_x f)])\wedge\\
&\neg\exists g\exists h(\mathtt{Fmla}(g)\wedge\mathtt{Fmla}(h)\wedge[(g\wedge h)=(\forall\mathbf{v}_x f)])\wedge\\
&\neg\exists g\exists h(\mathtt{Fmla}(g)\wedge\mathtt{Fmla}(h)\wedge[(g\vee h)=(\forall\mathbf{v}_x f)])\wedge\\
&\neg\exists g\exists y(\mathtt{Fmla}(g)\wedge[(\exists\mathbf{v}_y g)=(\forall\mathbf{v}_x f)]))
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{I}\Sigma_1\vdash\mathtt{Fmla}(f)\rightarrow(\forall g\forall y((&\mathtt{Fmla}(g)\wedge[(\exists\mathbf{v}_y g)=(\exists\mathbf{v}_x f)])\rightarrow(y=x\wedge g=f))\wedge\\
&\neg\exists s\exists t(\mathtt{Term}(s)\wedge\mathtt{Term}(t)\wedge[(s=t)=(\exists\mathbf{v}_x f)])\wedge\\
&\neg\exists s\exists t(\mathtt{Term}(s)\wedge\mathtt{Term}(t)\wedge[(s<t)=(\exists\mathbf{v}_x f)])\wedge\\
&\neg\exists g(\mathtt{Fmla}(g)\wedge[(\neg g)=(\exists\mathbf{v}_x f)])\wedge\\
&\neg\exists g\exists h(\mathtt{Fmla}(g)\wedge\mathtt{Fmla}(h)\wedge[(g\wedge h)=(\exists\mathbf{v}_x f)])\wedge\\
&\neg\exists g\exists h(\mathtt{Fmla}(g)\wedge\mathtt{Fmla}(h)\wedge[(g\vee h)=(\exists\mathbf{v}_x f)])\wedge\\
&\neg\exists g\exists y(\mathtt{Fmla}(g)\wedge[(\forall\mathbf{v}_y g)=(\exists\mathbf{v}_x f)])).
\end{aligned}
$$

Proof. The first claim is immediate by Σ_1-$\mathtt{Fmla}$-induction (see the proof of Lemma 2.34 for comparison). The others follow by Σ_1-recursion; we show a representative part of the statement for the disjunction as a template.

Let $[\mathtt{f}(f)=y]$ be the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function satisfying

$$
\begin{aligned}
&\mathrm{I}\Sigma_1\vdash(\mathtt{Term}(s)\wedge\mathtt{Term}(t))\rightarrow[\mathtt{f}((s=t))=0]\\
&\mathrm{I}\Sigma_1\vdash(\mathtt{Term}(s)\wedge\mathtt{Term}(t))\rightarrow[\mathtt{f}((s<t))=0]\\
&\mathrm{I}\Sigma_1\vdash\mathtt{Fmla}(f)\rightarrow[\mathtt{f}((\neg f))=0]\\
&\mathrm{I}\Sigma_1\vdash(\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g))\rightarrow[\mathtt{f}((f\wedge g))=0]\\
&\mathrm{I}\Sigma_1\vdash(\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g))\rightarrow[\mathtt{f}((f\vee g))=\langle f,g\rangle]\\
&\mathrm{I}\Sigma_1\vdash\mathtt{Fmla}(f)\rightarrow[\mathtt{f}((\forall\mathbf{v}_x f))=0]\\
&\mathrm{I}\Sigma_1\vdash\mathtt{Fmla}(f)\rightarrow[\mathtt{f}((\exists\mathbf{v}_x f))=0].
\end{aligned}
$$

Reasoning in $\mathrm{I}\Sigma_1$ now, if $\mathtt{Fmla}(f)$, $\mathtt{Fmla}(g)$, $\mathtt{Fmla}(h)$, $\mathtt{Fmla}(k)$ and $[(f\vee g)=(h\vee k)]$ then $[\langle f,g\rangle=\langle h,k\rangle]$ by definition of $\mathtt{f}$, whence $f=h$ and $g=k$ by Lemma 2.25. If e.g. there are s and t with $\mathtt{Term}(s)$ and $\mathtt{Term}(t)$ such that $[(f\vee g)=(s=t)]$ then $[\langle f,g\rangle=0]$, whereby $f=0$ and $g=0$, contradicting that $\neg\mathtt{Fmla}(0)$.

The remaining cases are similar. □

A similar statement holds for Term, but since we will not have use for this, we will omit it (and save some paper).

We can use this result to show that restrictions in the inductive clauses of the definition of Fmla can meaningfully be made to yield certain "subclasses" of Fmla. For instance we can formalise the notion of negation normal form, which will be of the utmost importance for the central constructions of this thesis.

Lemma 2.33. *There is a* $\Delta_1(\mathrm{I}\Sigma_1)$*-formula* $\mathtt{Nnf}(\mathsf{f})$ *which satisfies*

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash \mathtt{Nnf}(\mathsf{f}) \leftrightarrow (\mathtt{Fmla}(\mathsf{f}) \wedge (\mathtt{Atf}(\mathsf{f}) \vee \exists \mathsf{g}{<}\mathsf{f}(\mathtt{Atf}(\mathsf{g}) \wedge [(\neg\mathsf{g}){=}\mathsf{f}]) \vee &\\ \exists \mathsf{g}{<}\mathsf{f}\exists \mathsf{h}{<}\mathsf{f}(\mathtt{Nnf}(\mathsf{g}) \wedge \mathtt{Nnf}(\mathsf{h}) \wedge [(\mathsf{g}\wedge\mathsf{h}){=}\mathsf{f}]) \vee &\\ \exists \mathsf{g}{<}\mathsf{f}\exists \mathsf{h}{<}\mathsf{f}(\mathtt{Nnf}(\mathsf{g}) \wedge \mathtt{Nnf}(\mathsf{h}) \wedge [(\mathsf{g}\vee\mathsf{h}){=}\mathsf{f}]) \vee &\qquad (25)\\ \exists \mathsf{g}{<}\mathsf{f}\exists \mathsf{x}{<}\mathsf{f}(\mathtt{Nnf}(\mathsf{g}) \wedge [(\forall \mathbf{v}_{\mathsf{x}}\mathsf{g}){=}\mathsf{f}]) \vee &\\ \exists \mathsf{g}{<}\mathsf{f}\exists \mathsf{x}{<}\mathsf{f}(\mathtt{Nnf}(\mathsf{g}) \wedge [(\exists \mathbf{v}_{\mathsf{x}}\mathsf{g}){=}\mathsf{f}]))) &\end{aligned}$$

and an $\mathrm{I}\Sigma_1$*-provably* $\Delta_1(\mathrm{I}\Sigma_1)$*-function* $[\mathtt{nnf}(\mathsf{f}){=}\mathsf{g}]$ *satisfying*

$$\begin{aligned}&\mathrm{I}\Sigma_1 \vdash \mathtt{Atf}(\mathsf{f}) \rightarrow ([\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}] \wedge [\mathtt{nnf}((\neg\mathsf{f})){=}(\neg\mathtt{nnf}(\mathsf{f}))]),\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \rightarrow [\mathtt{nnf}((\neg(\neg\mathsf{f}))){=}\mathtt{nnf}(\mathsf{f})],\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g})) \rightarrow ([\mathtt{nnf}((\mathsf{f}\wedge\mathsf{g})){=}(\mathtt{nnf}(\mathsf{f})\wedge\mathtt{nnf}(\mathsf{g}))] \wedge\\ &\qquad [\mathtt{nnf}((\neg(\mathsf{f}\wedge\mathsf{g}))){=}(\mathtt{nnf}((\neg\mathsf{f}))\vee\mathtt{nnf}((\neg\mathsf{g})))]),\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g})) \rightarrow ([\mathtt{nnf}((\mathsf{f}\vee\mathsf{g})){=}(\mathtt{nnf}(\mathsf{f})\vee\mathtt{nnf}(\mathsf{g}))] \wedge \qquad (26)\\ &\qquad [\mathtt{nnf}((\neg(\mathsf{f}\vee\mathsf{g}))){=}(\mathtt{nnf}((\neg\mathsf{f}))\wedge\mathtt{nnf}((\neg\mathsf{g})))]),\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \rightarrow ([\mathtt{nnf}((\forall \mathbf{v}_{\mathsf{x}}\mathsf{f})){=}(\forall \mathbf{v}_{\mathsf{x}}\mathtt{nnf}(\mathsf{f}))] \wedge [\mathtt{nnf}((\neg(\forall \mathbf{v}_{\mathsf{x}}\mathsf{f}))){=}(\exists \mathbf{v}_{\mathsf{x}}\mathtt{nnf}((\neg\mathsf{f})))]),\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \rightarrow ([\mathtt{nnf}((\exists \mathbf{v}_{\mathsf{x}}\mathsf{f})){=}(\exists \mathbf{v}_{\mathsf{x}}\mathtt{nnf}(\mathsf{f}))] \wedge [\mathtt{nnf}((\neg(\exists \mathbf{v}_{\mathsf{x}}\mathsf{f}))){=}(\forall \mathbf{v}_{\mathsf{x}}\mathtt{nnf}((\neg\mathsf{f})))])\end{aligned}$$

and such that in addition $\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \rightarrow [\mathtt{Nnf}(\mathtt{nnf}(\mathsf{f}))]$ *and* $\mathrm{I}\Sigma_1 \vdash \mathtt{Nnf}(\mathsf{f}) \rightarrow [\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$.

Proof. Let $[\mathsf{c}_{\mathtt{Atf}}(\mathsf{x}){=}\mathsf{y}]$ be the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function $(\mathtt{Atf}(\mathsf{x}) \wedge \mathsf{y}{=}\overline{1}) \vee (\neg\mathtt{Atf}(\mathsf{x}) \wedge \mathsf{y}{=}0)$. Let $\mathsf{c}_{\mathtt{Nnf}}$ be the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function satisfying

$$\begin{aligned}&\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t})) \rightarrow [\mathsf{c}_{\mathtt{Nnf}}((\mathsf{s}{=}\mathsf{t})){=}\overline{1}],\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s}) \wedge \mathtt{Term}(\mathsf{t})) \rightarrow [\mathsf{c}_{\mathtt{Nnf}}((\mathsf{s}{<}\mathsf{t})){=}\overline{1}],\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g})) \rightarrow [\mathsf{c}_{\mathtt{Nnf}}((\mathsf{f}\wedge\mathsf{g})){=}\mathsf{c}_{\mathtt{Nnf}}(\mathsf{f}) \cdot \mathsf{c}_{\mathtt{Nnf}}(\mathsf{g})],\\ &\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g})) \rightarrow [\mathsf{c}_{\mathtt{Nnf}}((\mathsf{f}\vee\mathsf{g})){=}\mathsf{c}_{\mathtt{Nnf}}(\mathsf{f}) \cdot \mathsf{c}_{\mathtt{Nnf}}(\mathsf{g})],\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \rightarrow [\mathsf{c}_{\mathtt{Nnf}}((\neg\mathsf{f})){=}\mathsf{c}_{\mathtt{Atf}}(\mathsf{f})],\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \rightarrow [\mathsf{c}_{\mathtt{Nnf}}((\forall \mathbf{v}_{\mathsf{x}}\mathsf{f})){=}\mathsf{c}_{\mathtt{Nnf}}(\mathsf{f})],\\ &\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}) \rightarrow [\mathsf{c}_{\mathtt{Nnf}}((\exists \mathbf{v}_{\mathsf{x}}\mathsf{f})){=}\mathsf{c}_{\mathtt{Nnf}}(\mathsf{f})],\\ &\mathrm{I}\Sigma_1 \vdash \neg\mathtt{Fmla}(\mathsf{x}) \rightarrow [\mathsf{c}_{\mathtt{Nnf}}(\mathsf{x}){=}0].\end{aligned}$$

Define $\mathtt{Nnf}(\mathsf{f})$ to be the formula $[\mathsf{c}_{\mathtt{Nnf}}(\mathsf{f}){=}\overline{1}]$. It is now straightforward to verify the right to left direction of the first claim. To prove the other direction, note that $\mathrm{I}\Sigma_1 \vdash$

$\mathtt{Nnf}(\mathsf{f}){\rightarrow}\mathtt{Fmla}(\mathsf{f})$ is the contraposition of the last line above. Thus we use Σ_1-$\mathtt{Fmla}$-induction on f with the formula $\mathtt{F}(\mathsf{f})$ defined as

$$\begin{aligned}\mathtt{Nnf}(\mathsf{f}){\rightarrow}(&\mathtt{Atf}(\mathsf{f})\vee\exists \mathsf{g}{<}\mathsf{f}(\mathtt{Atf}(\mathsf{g})\wedge[(\neg\mathsf{g}){=}\mathsf{f}])\vee\\ &\exists\mathsf{g}{<}\mathsf{f}\exists\mathsf{h}{<}\mathsf{f}(\mathtt{Nnf}(\mathsf{g})\wedge\mathtt{Nnf}(\mathsf{h})\wedge[(\mathsf{g}\wedge\mathsf{h}){=}\mathsf{f}])\vee\\ &\exists\mathsf{g}{<}\mathsf{f}\exists\mathsf{h}{<}\mathsf{f}(\mathtt{Nnf}(\mathsf{g})\wedge\mathtt{Nnf}(\mathsf{h})\wedge[(\mathsf{g}\vee\mathsf{h}){=}\mathsf{f}])\vee\\ &\exists\mathsf{g}{<}\mathsf{f}\exists\mathsf{x}{<}\mathsf{f}(\mathtt{Nnf}(\mathsf{g})\wedge[(\forall\mathbf{v}_{\mathsf{x}}\mathsf{g}){=}\mathsf{f}])\vee\\ &\exists\mathsf{g}{<}\mathsf{f}\exists\mathsf{x}{<}\mathsf{f}(\mathtt{Nnf}(\mathsf{g})\wedge[(\exists\mathbf{v}_{\mathsf{x}}\mathsf{g}){=}\mathsf{f}])).\end{aligned}$$

We reason in $\mathrm{I}\Sigma_1$ to verify each conjunct of the relevant antecedent (see Lemma 2.30). We will need the fact that $\mathrm{I}\Sigma_1\vdash \mathsf{x}{\cdot}\mathsf{y}{=}\overline{1}{\rightarrow}(\mathsf{x}{=}\overline{1}\wedge\mathsf{y}{=}\overline{1})$.

- Assume that $\mathtt{Term}(\mathsf{s})$ and $\mathtt{Term}(\mathsf{t})$. Then $\mathtt{Atf}((\mathsf{s}{=}\mathsf{t}))$ and $\mathtt{Atf}((\mathsf{s}{<}\mathsf{t}))$ so that $[\mathtt{F}((\mathsf{s}{=}\mathsf{t}))]$ and $[\mathtt{F}((\mathsf{s}{<}\mathsf{t}))]$.
- Suppose that $\mathtt{Fmla}(\mathsf{f})$, $\mathtt{Fmla}(\mathsf{g})$, $\mathtt{F}(\mathsf{f})$ and $\mathtt{F}(\mathsf{g})$.

  Assume $[\mathtt{Nnf}((\neg\mathsf{f}))]$, that is $[\mathsf{c}_{\mathtt{Nnf}}((\neg\mathsf{f})){=}\overline{1}]$. Then $[\mathsf{c}_{\mathtt{Atf}}(\mathsf{f}){=}\overline{1}]$ by definition, whence

  $$\exists[\mathsf{g}{<}(\neg\mathsf{f})](\mathtt{Atf}(\mathsf{g})\wedge[(\neg\mathsf{g}){=}(\neg\mathsf{f})]).$$

  So $[\mathtt{F}((\neg\mathsf{f}))]$.

  Assume $[\mathtt{Nnf}((\mathsf{f}\wedge\mathsf{g}))]$. Then $[\mathsf{c}_{\mathtt{Nnf}}(\mathsf{f}){\cdot}\mathsf{c}_{\mathtt{Nnf}}(\mathsf{g}){=}\overline{1}]$, so that $[\mathsf{c}_{\mathtt{Nnf}}(\mathsf{f}){=}\overline{1}]$ and $[\mathsf{c}_{\mathtt{Nnf}}(\mathsf{g}){=}\overline{1}]$. Hence

  $$\exists[\mathsf{h}_1{<}(\mathsf{f}\wedge\mathsf{g})]\exists[\mathsf{h}_2{<}(\mathsf{f}\wedge\mathsf{g})](\mathtt{Nnf}(\mathsf{h}_1)\wedge\mathtt{Nnf}(\mathsf{h}_2)\wedge[(\mathsf{h}_1\wedge\mathsf{h}_2){=}(\mathsf{f}\wedge\mathsf{g})]),$$

  whereby $[\mathtt{F}((\mathsf{f}\wedge\mathsf{g}))]$.

  Similarly, assume $[\mathtt{Nnf}((\mathsf{f}\vee\mathsf{g}))]$, whence $[\mathtt{Nnf}(\mathsf{f})]$ and $[\mathtt{Nnf}(\mathsf{g})]$. Thus

  $$\exists[\mathsf{h}_1{<}(\mathsf{f}\vee\mathsf{g})]\exists[\mathsf{h}_2{<}(\mathsf{f}\vee\mathsf{g})](\mathtt{Nnf}(\mathsf{h}_1)\wedge\mathtt{Nnf}(\mathsf{h}_2)\wedge[(\mathsf{h}_1\vee\mathsf{h}_2){=}(\mathsf{f}\vee\mathsf{g})]),$$

  whence $[\mathtt{F}((\mathsf{f}\vee\mathsf{g}))]$.

  Assume $[\mathtt{Nnf}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}))]$. Then $[\mathsf{c}_{\mathtt{Nnf}}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})){=}\overline{1}]$, whence $[\mathsf{c}_{\mathtt{Nnf}}(\mathsf{f}){=}\overline{1}]$. So

  $$\exists[\mathsf{g}{<}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})]\exists[\mathsf{y}{<}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})](\mathtt{Nnf}(\mathsf{g})\wedge[(\forall\mathbf{v}_{\mathsf{y}}\mathsf{g}){=}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})]),$$

  from which $[\mathtt{F}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}))]$ follows.

  Assume $[\mathtt{Nnf}((\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}))]$, so that $[\mathtt{Nnf}(\mathsf{f})]$. Hence

  $$\exists[\mathsf{g}{<}(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})]\exists[\mathsf{y}{<}(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})](\mathtt{Nnf}(\mathsf{g})\wedge[(\exists\mathbf{v}_{\mathsf{y}}\mathsf{g}){=}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})]),$$

  and consequently $[\mathtt{F}((\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}))]$ follows.

Thus $\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}\mathtt{F}(\mathsf{f})$, which verifies (25).

Now let $[\mathtt{nnf}_2(\mathsf{f}){=}\mathsf{p}]$ be the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function satisfying

$$
\begin{aligned}
&\mathrm{I}\Sigma_1 \vdash \mathtt{Atf}(\mathsf{f}){\to}[\mathtt{nnf}_2(\mathsf{f}){=}\langle \mathsf{f},(\neg\mathsf{f})\rangle],\\
&\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}[\mathtt{nnf}_2((\neg\mathsf{f})){=}\langle\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{r}},\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{l}}\rangle],\\
&\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f}){\wedge}\mathtt{Fmla}(\mathsf{g})){\to}\\
&\qquad [\mathtt{nnf}_2((\mathsf{f}{\wedge}\mathsf{g})){=}\langle(\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{l}}{\wedge}\langle \mathtt{nnf}_2(\mathsf{g})\rangle_{\mathrm{l}}),(\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{r}}{\vee}\langle \mathtt{nnf}_2(\mathsf{g})\rangle_{\mathrm{r}})\rangle],\\
&\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(\mathsf{f}){\wedge}\mathtt{Fmla}(\mathsf{g})){\to}\\
&\qquad [\mathtt{nnf}_2((\mathsf{f}{\vee}\mathsf{g})){=}\langle(\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{l}}{\vee}\langle \mathtt{nnf}_2(\mathsf{g})\rangle_{\mathrm{l}}),(\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{r}}{\wedge}\langle \mathtt{nnf}_2(\mathsf{g})\rangle_{\mathrm{r}})\rangle],\\
&\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}[\mathtt{nnf}_2((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})){=}\langle(\forall\mathbf{v}_{\mathsf{x}}\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{l}}),(\exists\mathbf{v}_{\mathsf{x}}\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{r}})\rangle],\\
&\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}[\mathtt{nnf}_2((\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})){=}\langle(\exists\mathbf{v}_{\mathsf{x}}\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{l}}),(\forall\mathbf{v}_{\mathsf{x}}\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{r}})\rangle].
\end{aligned}
$$

Define $[\mathtt{nnf}(\mathsf{f}){=}\mathsf{g}]$ as $[\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{l}}{=}\mathsf{g}]$. Then $\mathrm{I}\Sigma_1 \vdash [\mathtt{nnf}((\neg\mathsf{f})){=}\langle \mathtt{nnf}_2(\mathsf{f})\rangle_{\mathrm{r}}]$, from which (26) follows directly.

Next we verify that

$$\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}([\mathtt{Nnf}(\mathtt{nnf}(\mathsf{f}))]{\wedge}[\mathtt{Nnf}(\mathtt{nnf}((\neg\mathsf{f})))])$$

by Σ_1-$\mathtt{Fmla}$-induction on f with the formula $[\mathtt{Nnf}(\mathtt{nnf}(\mathsf{f}))]{\wedge}[\mathtt{Nnf}(\mathtt{nnf}((\neg\mathsf{f})))]$. As above we reason in $\mathrm{I}\Sigma_1$ to prove each conjunct of the antecedent of the relevant instance of Lemma 2.30.

- Suppose $\mathtt{Term}(\mathsf{s})$ and $\mathtt{Term}(\mathsf{t})$. Then $\mathtt{Atf}((\mathsf{s}{=}\mathsf{t}))$ and $\mathtt{Atf}((\mathsf{s}{<}\mathsf{t}))$, whence $[\mathtt{nnf}((\mathsf{s}{=}\mathsf{t})){=}(\mathsf{s}{=}\mathsf{t})]$, $[\mathtt{nnf}((\mathsf{s}{<}\mathsf{t})){=}(\mathsf{s}{<}\mathsf{t})]$, $[\mathtt{nnf}((\neg(\mathsf{s}{=}\mathsf{t}))){=}(\neg(\mathsf{s}{=}\mathsf{t}))]$ and $[\mathtt{nnf}((\neg(\mathsf{s}{<}\mathsf{t}))){=}(\neg(\mathsf{s}{<}\mathsf{t}))]$ by (26), and $[\mathtt{Nnf}((\mathsf{s}{=}\mathsf{t}))]$, $[\mathtt{Nnf}((\mathsf{s}{<}\mathsf{t}))]$, $[\mathtt{Nnf}((\neg(\mathsf{s}{=}\mathsf{t})))]$ and $[\mathtt{Nnf}((\neg(\mathsf{s}{<}\mathsf{t})))]$ by (25).

- Suppose $\mathtt{Fmla}(\mathsf{f})$, $\mathtt{Fmla}(\mathsf{g})$, $[\mathtt{Nnf}(\mathtt{nnf}(\mathsf{f}))]$, $[\mathtt{Nnf}(\mathtt{nnf}((\neg\mathsf{f})))]$, $[\mathtt{Nnf}(\mathtt{nnf}(\mathsf{g}))]$ and $[\mathtt{Nnf}(\mathtt{nnf}((\neg\mathsf{g})))]$.

  Consider $(\neg\mathsf{f})$. Since $[\mathtt{nnf}((\neg(\neg\mathsf{f}))){=}\mathtt{nnf}(\mathsf{f})]$ by (26) we have $[\mathtt{Nnf}(\mathtt{nnf}((\neg\mathsf{f})))]$ and $[\mathtt{Nnf}(\mathtt{nnf}((\neg(\neg\mathsf{f}))))]$ by induction hypothesis.

  Consider $(\mathsf{f}{\wedge}\mathsf{g})$. We have $[\mathtt{nnf}((\mathsf{f}{\wedge}\mathsf{g})){=}(\mathtt{nnf}(\mathsf{f}){\wedge}\mathtt{nnf}(\mathsf{g}))]$ and $[\mathtt{nnf}((\neg(\mathsf{f}{\wedge}\mathsf{g}))){=}(\mathtt{nnf}((\neg\mathsf{f})){\vee}\mathtt{nnf}((\neg\mathsf{g})))]$ by (26), whence $[\mathtt{Nnf}(\mathtt{nnf}((\mathsf{f}{\wedge}\mathsf{g})))]$ and $[\mathtt{Nnf}(\mathtt{nnf}((\neg(\mathsf{f}{\wedge}\mathsf{g}))))]$ by (25).

  Now consider $(\mathsf{f}{\vee}\mathsf{g})$. We have $[\mathtt{nnf}((\mathsf{f}{\vee}\mathsf{g})){=}(\mathtt{nnf}(\mathsf{f}){\vee}\mathtt{nnf}(\mathsf{g}))]$ and $[\mathtt{nnf}((\neg(\mathsf{f}{\vee}\mathsf{g}))){=}(\mathtt{nnf}((\neg\mathsf{f})){\wedge}\mathtt{nnf}((\neg\mathsf{g})))]$ by (26), whence $[\mathtt{Nnf}(\mathtt{nnf}((\mathsf{f}{\vee}\mathsf{g})))]$ and $[\mathtt{Nnf}(\mathtt{nnf}((\neg(\mathsf{f}{\vee}\mathsf{g}))))]$ by (25).

  Next consider $(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})$. By (26), $[\mathtt{nnf}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})){=}(\forall\mathbf{v}_{\mathsf{x}}\mathtt{nnf}(\mathsf{f}))]$ and $[\mathtt{nnf}((\neg(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}))){=}(\exists\mathbf{v}_{\mathsf{x}}\mathtt{nnf}((\neg\mathsf{f})))]$. Thus $[\mathtt{Nnf}(\mathtt{nnf}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})))]$ and $[\mathtt{Nnf}(\mathtt{nnf}((\neg(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}))))]$, by (25).

  Finally consider $(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})$. We have $[\mathtt{nnf}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})){=}(\forall\mathbf{v}_{\mathsf{x}}\mathtt{nnf}(\mathsf{f}))]$ and $[\mathtt{nnf}((\neg(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}))){=}(\exists\mathbf{v}_{\mathsf{x}}\mathtt{nnf}((\neg\mathsf{f})))]$ by (26). Thus $[\mathtt{Nnf}(\mathtt{nnf}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})))]$ and $[\mathtt{Nnf}(\mathtt{nnf}((\neg(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}))))]$ by (25).

Hence $\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}([\mathtt{Nnf}(\mathtt{nnf}(\mathsf{f}))]{\wedge}[\mathtt{Nnf}(\mathtt{nnf}((\neg\mathsf{f})))])$, whence in particular $\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}[\mathtt{Nnf}(\mathtt{nnf}(\mathsf{f}))]$.

Finally we verify

$$\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}(\mathtt{Nnf}(\mathsf{f}){\to}[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}])$$

by Σ_1-$\mathtt{Fmla}$-induction on f with the formula $\mathtt{Nnf}(\mathsf{f}){\to}[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$. We reason in $\mathrm{I}\Sigma_1$ as before.

- Suppose $\mathtt{Term}(\mathsf{s})$ and $\mathtt{Term}(\mathsf{t})$. Then $\mathtt{Atf}((\mathsf{s}{=}\mathsf{t}))$ and $\mathtt{Atf}((\mathsf{s}{<}\mathsf{t}))$, whence $[\mathtt{nnf}((\mathsf{s}{=}\mathsf{t})){=}(\mathsf{s}{=}\mathsf{t})]$ and $[\mathtt{nnf}((\mathsf{s}{<}\mathsf{t})){=}(\mathsf{s}{<}\mathsf{t})]$ by (26).

- Suppose $\mathtt{Fmla}(\mathsf{f})$, $\mathtt{Fmla}(\mathsf{g})$, $\mathtt{Nnf}(\mathsf{f}){\to}[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$ and $\mathtt{Nnf}(\mathsf{g}){\to}[\mathtt{nnf}(\mathsf{g}){=}\mathsf{g}]$.

  Assume $[\mathtt{Nnf}((\neg\mathsf{f}))]$. Then (25) and Lemma 2.32 give $\mathtt{Atf}(\mathsf{f})$, whence $\mathtt{Nnf}(\mathsf{f})$. The induction hypothesis then gives $[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$, whereby $[\mathtt{nnf}((\neg\mathsf{f})){=}(\neg\mathsf{f})]$ by (26).

  Assume $[\mathtt{Nnf}((\mathsf{f}{\wedge}\mathsf{g}))]$. Then (25) and Lemma 2.32 give $\mathtt{Nnf}(\mathsf{f})$ and $\mathtt{Nnf}(\mathsf{g})$. By induction hypothesis we get $[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$ and $[\mathtt{nnf}(\mathsf{g}){=}\mathsf{g}]$, whence $[\mathtt{nnf}((\mathsf{f}{\wedge}\mathsf{g})){=}(\mathsf{f}{\wedge}\mathsf{g})]$ by (26).

  Assume $[\mathtt{Nnf}((\mathsf{f}{\vee}\mathsf{g}))]$. Like above, (25) and Lemma 2.32 give $\mathtt{Nnf}(\mathsf{f})$ and $\mathtt{Nnf}(\mathsf{g})$, whence $[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$ and $[\mathtt{nnf}(\mathsf{g}){=}\mathsf{g}]$. Thus $[\mathtt{nnf}((\mathsf{f}{\vee}\mathsf{g})){=}(\mathsf{f}{\vee}\mathsf{g})]$ by (26).

  Assume $[\mathtt{Nnf}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}))]$. Then (25) and Lemma 2.32 give $\mathtt{Nnf}(\mathsf{f})$, whence $[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$. Consequently $[\mathtt{nnf}((\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})){=}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})]$ by (26).

  Assume $[\mathtt{Nnf}((\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}))]$, whence $\mathtt{Nnf}(\mathsf{f})$ by (25) and Lemma 2.32. By induction hypothesis this implies $[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$, whence (26) gives $[\mathtt{nnf}((\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})){=}(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})]$.

Thus $\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\to}(\mathtt{Nnf}(\mathsf{f}){\to}[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}])$. Since $\mathrm{I}\Sigma_1 \vdash \mathtt{Nnf}(\mathsf{f}){\to}\mathtt{Fmla}(\mathsf{f})$, this simplifies to $\mathrm{I}\Sigma_1 \vdash \mathtt{Nnf}(\mathsf{f}){\to}[\mathtt{nnf}(\mathsf{f}){=}\mathsf{f}]$. □

Uniqueness of the construction trees of formulae (Lemma 2.32) also makes it meaningful to speak about "occurrences" of "symbols" in a formula. The above then confirms that $\mathtt{Nnf}$ coincides precisely with the meta-theoretic idea of negation normal form: formulae where negation occur in front of atomic formulae only.

With recursion we can also construct many familiar notions concerning terms and formulae, such as the set of variables free in a term or a formula and what is the value of a term given an evaluation for it.

Definition 2.15. Let $\mathtt{freevarT}$ be the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function satisfying

$$\begin{aligned}
&\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarT}((\mathbf{0})){=}\varnothing]\\
&\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarT}((\mathbf{v}_{\mathsf{x}})){=}\{\mathbf{v}_{\mathsf{x}}\}]\\
&\mathrm{I}\Sigma_1 \vdash \mathtt{Term}(\mathsf{t}){\to}[\mathtt{freevarT}((\mathbf{S}\mathsf{t})){=}\mathtt{freevarT}(\mathsf{t})]\\
&\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s}){\wedge}\mathtt{Term}(\mathsf{t})){\to}[\mathtt{freevarT}((\mathsf{s}{+}\mathsf{t})){=}\mathtt{freevarT}(\mathsf{s}){\sqcup}\mathtt{freevarT}(\mathsf{t})]\\
&\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(\mathsf{s}){\wedge}\mathtt{Term}(\mathsf{t})){\to}[\mathtt{freevarT}((\mathsf{s}{\cdot}\mathsf{t})){=}\mathtt{freevarT}(\mathsf{s}){\sqcup}\mathtt{freevarT}(\mathsf{t})]\\
&\mathrm{I}\Sigma_1 \vdash \neg\mathtt{Term}(\mathsf{x}){\to}[\mathtt{freevarT}(\mathsf{x}){=}0].
\end{aligned}$$

Similarly, let $\texttt{freevarF}$ be the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function satisfying

$$\mathrm{I}\Sigma_1 \vdash (\texttt{Term}(s)\wedge\texttt{Term}(t))\to[\texttt{freevarF}((\mathbf{s}{=}t))=\texttt{freevarT}(s)\sqcup\texttt{freevarT}(t)]$$
$$\mathrm{I}\Sigma_1 \vdash (\texttt{Term}(s)\wedge\texttt{Term}(t))\to[\texttt{freevarF}((\mathbf{s}{<}t))=\texttt{freevarT}(s)\sqcup\texttt{freevarT}(t)]$$
$$\mathrm{I}\Sigma_1 \vdash \texttt{Fmla}(f)\to[\texttt{freevarF}((\neg f))=\texttt{freevarF}(f)]$$
$$\mathrm{I}\Sigma_1 \vdash (\texttt{Fmla}(f)\wedge\texttt{Fmla}(g))\to[\texttt{freevarF}((f\wedge g))=\texttt{freevarF}(f)\sqcup\texttt{freevarF}(g)]$$
$$\mathrm{I}\Sigma_1 \vdash (\texttt{Fmla}(f)\wedge\texttt{Fmla}(g))\to[\texttt{freevarF}((f\vee g))=\texttt{freevarF}(f)\sqcup\texttt{freevarF}(g)]$$
$$\mathrm{I}\Sigma_1 \vdash \texttt{Fmla}(f)\to[\texttt{freevarF}((\forall\mathbf{v}_x f))=\texttt{freevarF}(f)\setminus\{\mathbf{v}_x\}]$$
$$\mathrm{I}\Sigma_1 \vdash \texttt{Fmla}(f)\to[\texttt{freevarF}((\exists\mathbf{v}_x f))=\texttt{freevarF}(f)\setminus\{\mathbf{v}_x\}]$$
$$\mathrm{I}\Sigma_1 \vdash \neg\texttt{Fmla}(x)\to[\texttt{freevarF}(x)=0].$$

Define $\texttt{Eval}(e,s)$ to be the formula

$$\texttt{Fcn}(e)\wedge(\forall[x\epsilon\texttt{dom}(f)]\texttt{Var}(x))\wedge$$
$$((\texttt{Term}(s)\wedge[\texttt{freevarT}(s)\sqsubseteq\texttt{dom}(e)])\vee(\texttt{Fmla}(s)\wedge[\texttt{freevarF}(s)\sqsubseteq\texttt{dom}(e)]))$$

and $[\texttt{val}(e,t)=x]$ the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function satisfying

$$\mathrm{I}\Sigma_1 \vdash [\texttt{val}(e,(\mathbf{0}))=0]$$
$$\mathrm{I}\Sigma_1 \vdash [\texttt{val}(e,(\mathbf{v}_x))=\texttt{apl}(e,\mathbf{v}_x)]$$
$$\mathrm{I}\Sigma_1 \vdash \texttt{Term}(t)\to[\texttt{val}(e,(\mathbf{S}t))=S(\texttt{val}(e,t))]$$
$$\mathrm{I}\Sigma_1 \vdash (\texttt{Term}(s)\wedge\texttt{Term}(t))\to[\texttt{val}(e,(s\mathbf{+}t))=\texttt{val}(e,s)+\texttt{val}(e,t)]$$
$$\mathrm{I}\Sigma_1 \vdash (\texttt{Term}(s)\wedge\texttt{Term}(t))\to[\texttt{val}(e,(s\mathbf{\cdot}t))=\texttt{val}(e,s)\cdot\texttt{val}(e,t)]$$
$$\mathrm{I}\Sigma_1 \vdash \neg\texttt{Term}(t)\to[\texttt{val}(e,t)=0].$$

Finally define $\texttt{Sent}(f)$ to be the formula $\texttt{Fmla}(f)\wedge[\texttt{freevarF}(f)=\varnothing]$.

Remark 6. Note that $\mathrm{I}\Sigma_1 \vdash (\texttt{Term}(s)\wedge\texttt{Term}(t)\wedge\texttt{Eval}(e,(s\mathbf{=}t)))\leftrightarrow(\texttt{Eval}(e,s)\wedge\texttt{Eval}(e,t))$ etc.

Given that we can evaluate a term, we would like to do the corresponding thing for formulae, that is define a truth predicate. This is of course impossible by Tarski's theorem, but we *can* define a truth (or as we shall call it, satisfaction) predicate for (formal) Δ_0-formulae, the class of which is itself defined by a $\Delta_1(\mathrm{I}\Sigma_1)$-formula.

Lemma 2.34. *In* $\mathrm{I}\Sigma_1$ *we have the* $\Delta_1(\mathrm{I}\Sigma_1)$*-formula* $\Delta_0(f)$ *which satisfies:*

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash \Delta_0(f)\leftrightarrow(\texttt{Fmla}(f)\wedge(&\texttt{Atf}(f)\vee\exists g{<}f(\Delta_0(g)\wedge[(\neg g)=f])\vee\\ &\exists g{<}f\exists h{<}f(\Delta_0(g)\wedge\Delta_0(h)\wedge[(g\wedge h)=f])\vee\\ &\exists g{<}f\exists h{<}f(\Delta_0(g)\wedge\Delta_0(h)\wedge[(g\vee h)=f])\vee\\ &\exists g{<}f\exists x{<}f\exists y{<}f(\Delta_0(g)\wedge x\neq y\wedge[(\forall\mathbf{v}_x((\neg(\mathbf{v}_x\mathbf{<}\mathbf{v}_y))\vee g))=f])\vee\\ &\exists g{<}f\exists x{<}f\exists y{<}f(\Delta_0(g)\wedge x\neq y\wedge[(\exists\mathbf{v}_x((\mathbf{v}_x\mathbf{<}\mathbf{v}_y)\wedge g))=f]))).\end{aligned}$$

Proof. This is (modulo a proof by induction) Lemma 1.68 of [4]. □

Proposition 2.35 (Satisfaction)**.** *There is a* $\Delta_1(\mathrm{I}\Sigma_1)$*-formula* $\mathtt{Sat}_0$ *such that*

$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(s)\wedge\mathtt{Term}(t)\wedge\mathtt{Eval}(e,\mathbf{(}s\mathbf{=}t\mathbf{)}))\to(\mathtt{Sat}_0(\mathbf{(}e,s\mathbf{=}t\mathbf{)})\leftrightarrow\mathtt{val}(e,s)=\mathtt{val}(e,t))$$
$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Term}(s)\wedge\mathtt{Term}(t)\wedge\mathtt{Eval}(e,\mathbf{(}s\mathbf{<}t\mathbf{)}))\to(\mathtt{Sat}_0(\mathbf{(}e,s\mathbf{<}t\mathbf{)})\leftrightarrow\mathtt{val}(e,s)<\mathtt{val}(e,t))$$
$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(f)\wedge[\mathtt{Eval}(e,\mathbf{(}\neg f\mathbf{)})])\to([\mathtt{Sat}_0(e,\mathbf{(}\neg f\mathbf{)})]\leftrightarrow\neg\mathtt{Sat}_0(e,f))$$
$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g)\wedge[\mathtt{Eval}(e,\mathbf{(}f\wedge g\mathbf{)})])\to([\mathtt{Sat}_0(e,\mathbf{(}f\wedge g\mathbf{)})]\leftrightarrow(\mathtt{Sat}_0(e,f)\wedge\mathtt{Sat}_0(e,g)))$$
$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g)\wedge[\mathtt{Eval}(e,\mathbf{(}f\vee g\mathbf{)})])\to([\mathtt{Sat}_0(e,\mathbf{(}f\vee g\mathbf{)})]\leftrightarrow(\mathtt{Sat}_0(e,f)\vee\mathtt{Sat}_0(e,g)))$$
$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(f)\wedge x\neq y\wedge[\mathtt{Eval}(e,\mathbf{(}\forall\mathbf{v}_x\mathbf{((}\neg\mathbf{(}\mathbf{v}_x\mathbf{<}\mathbf{v}_y\mathbf{))}\vee f\mathbf{))})])\to$$
$$([\mathtt{Sat}_0(e,\mathbf{(}\forall\mathbf{v}_x\mathbf{((}\neg\mathbf{(}\mathbf{v}_x\mathbf{<}\mathbf{v}_y\mathbf{))}\vee f\mathbf{))})]\leftrightarrow\forall[z<\mathtt{apl}(e,\mathbf{v}_y)][\mathtt{Sat}_0(\mathtt{ext}(e,\mathbf{v}_x,z),f)])$$
$$\mathrm{I}\Sigma_1 \vdash (\mathtt{Fmla}(f)\wedge x\neq y\wedge[\mathtt{Eval}(e,\mathbf{(}\exists\mathbf{v}_x\mathbf{((}\neg\mathbf{(}\mathbf{v}_x\mathbf{<}\mathbf{v}_y\mathbf{))}\vee f\mathbf{))})])\to$$
$$([\mathtt{Sat}_0(e,\mathbf{(}\exists\mathbf{v}_x\mathbf{((}\neg\mathbf{(}\mathbf{v}_x\mathbf{<}\mathbf{v}_y\mathbf{))}\vee f\mathbf{))})]\leftrightarrow\exists[z<\mathtt{apl}(e,\mathbf{v}_y)][\mathtt{Sat}_0(\mathtt{ext}(e,\mathbf{v}_x,z),f)])$$
$$\mathrm{I}\Sigma_1 \vdash \mathtt{Sat}_0(e,f)\to(\Delta_0(f)\wedge\mathtt{Eval}(e,f)).$$

Proof. See Theorem 1.70 of [4]. □

With the above definition of satisfaction, it is possible to prove that the truth value of a formula under an evaluation only depends on the values of the evaluation for the *free* variables of the formula, which we know to be true at the meta level. Like many propositions of this sort, the proof is just mimicking the meta proof, taking care to find appropriate bounds for quantifiers to make the induction go through.

Lemma 2.36.

$$\mathrm{I}\Sigma_1 \vdash \forall e\forall d\forall t((\mathtt{Term}(t)\wedge\mathtt{Eval}(e,t)\wedge\mathtt{Eval}(d,t)\wedge \forall[x\epsilon\mathtt{freevarT}(t)][\mathtt{apl}(e,x)=\mathtt{apl}(d,x)])\to [\mathtt{val}(e,t)=\mathtt{val}(d,t)])$$

Proof. By Σ_1-$\mathtt{Term}$-induction on $\mathtt{t}$ with the formula

$$(\mathtt{Eval}(e,t)\wedge\mathtt{Eval}(d,t)\wedge\forall[x\epsilon\mathtt{freevarT}(t)][\mathtt{apl}(e,x)=\mathtt{apl}(d,x)])\to[\mathtt{val}(e,t)=\mathtt{val}(d,t)].$$

We reason in $\mathrm{I}\Sigma_1$.

- Assume $\mathtt{Eval}(e,\mathbf{(0)})$, $\mathtt{Eval}(d,\mathbf{(0)})$ and $\forall[x\epsilon\mathtt{freevarT}(\mathbf{(0)})][\mathtt{apl}(e,x)=\mathtt{apl}(d,x)]$. By definition $[\mathtt{val}(e,\mathbf{(0)})=0]$ and $[\mathtt{val}(d,\mathbf{(0)})=0]$.

- Assume $\mathtt{Eval}(e,\mathbf{(}\mathbf{v}_y\mathbf{)})$, $\mathtt{Eval}(d,\mathbf{(}\mathbf{v}_y\mathbf{)})$ and $\forall[x\epsilon\mathtt{freevarT}(\mathbf{(}\mathbf{v}_y\mathbf{)})][\mathtt{apl}(e,x)=\mathtt{apl}(d,x)]$. Since $[\mathtt{freevarT}(\mathbf{(}\mathbf{v}_y\mathbf{)})=\{\mathbf{v}_y\}$[1]$]$ we have

$$[\mathtt{apl}(e,\mathbf{v}_y)=\mathtt{apl}(d,\mathbf{v}_y)],$$

  and since $[\mathtt{val}(e,\mathbf{(}\mathbf{v}_y\mathbf{)})=\mathtt{apl}(e,\mathbf{v}_y)]$ and $[\mathtt{val}(d,\mathbf{(}\mathbf{v}_y\mathbf{)})=\mathtt{apl}(d,\mathbf{v}_y)]$ we thus get

$$[\mathtt{val}(e,\mathbf{(}\mathbf{v}_y\mathbf{)})=\mathtt{val}(d,\mathbf{(}\mathbf{v}_y\mathbf{)})].$$

- Suppose $\texttt{Term}(\mathsf{t})$ and

  $$(\texttt{Eval}(\mathsf{e},\mathsf{t})\wedge\texttt{Eval}(\mathsf{d},\mathsf{t})\wedge\forall[\mathsf{x}\epsilon\texttt{freevarT}(\mathsf{t})][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})])\rightarrow[\texttt{val}(\mathsf{e},\mathsf{t})=\texttt{val}(\mathsf{d},\mathsf{t})].$$

  Assume $\texttt{Eval}(\mathsf{e},(\mathbf{S}\mathsf{t}))$, $\texttt{Eval}(\mathsf{d},(\mathbf{S}\mathsf{t}))$ and $\forall[\mathsf{x}\epsilon\texttt{freevarT}((\mathbf{S}\mathsf{t}))][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})]$. Since $[\texttt{freevarT}((\mathbf{S}\mathsf{t}))=\texttt{freevarT}(\mathsf{t})]$ we get $\texttt{Eval}(\mathsf{e},\mathsf{t})$, $\texttt{Eval}(\mathsf{d},\mathsf{t})$ and

  $$\forall[\mathsf{x}\epsilon\texttt{freevarT}(\mathsf{t})][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})],$$

  whence $[\texttt{val}(\mathsf{e},\mathsf{t})=\texttt{val}(\mathsf{d},\mathsf{t})]$ by induction hypothesis. Finally $[\texttt{val}(\mathsf{e},(\mathbf{S}\mathsf{t}))=\mathsf{S}(\texttt{val}(\mathsf{e},\mathsf{t}))]$ and $[\texttt{val}(\mathsf{d},(\mathbf{S}\mathsf{t}))=\mathsf{S}(\texttt{val}(\mathsf{d},\mathsf{t}))]$, whereby

  $$[\texttt{val}(\mathsf{e},(\mathbf{S}\mathsf{t}))=\texttt{val}(\mathsf{d},(\mathbf{S}\mathsf{t}))].$$

- Suppose $\texttt{Term}(\mathsf{s})$, $\texttt{Term}(\mathsf{t})$,

  $$(\texttt{Eval}(\mathsf{e},\mathsf{s})\wedge\texttt{Eval}(\mathsf{d},\mathsf{s})\wedge\forall[\mathsf{x}\epsilon\texttt{freevarT}(\mathsf{s})][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})])\rightarrow[\texttt{val}(\mathsf{e},\mathsf{s})=\texttt{val}(\mathsf{d},\mathsf{s})]$$

  and similarly for t. Assume $\texttt{Eval}(\mathsf{e},(\mathsf{s}+\mathsf{t}))$, $\texttt{Eval}(\mathsf{d},(\mathsf{s}+\mathsf{t}))$ and

  $$\forall[\mathsf{x}\epsilon\texttt{freevarT}((\mathsf{s}+\mathsf{t}))][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})].$$

  By definition $[\texttt{freevarT}((\mathsf{s}+\mathsf{t}))=\texttt{freevarT}(\mathsf{s})\sqcup\texttt{freevarT}(\mathsf{t})]$, whence $\texttt{Eval}(\mathsf{e},\mathsf{s})$, $\texttt{Eval}(\mathsf{d},\mathsf{s})$ and $\forall[\mathsf{x}\epsilon\texttt{freevarT}(\mathsf{s})][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})]$, and similarly for t. Thus $[\texttt{val}(\mathsf{e},\mathsf{s})=\texttt{val}(\mathsf{d},\mathsf{s})]$ and $[\texttt{val}(\mathsf{e},\mathsf{t})=\texttt{val}(\mathsf{d},\mathsf{t})]$ by induction hypothesis. Since $[\texttt{val}(\mathsf{e},(\mathsf{s}+\mathsf{t}))=\texttt{val}(\mathsf{e},\mathsf{s})+\texttt{val}(\mathsf{e},\mathsf{t})]$ and similarly for d, we get

  $$[\texttt{val}(\mathsf{e},(\mathsf{s}+\mathsf{t}))=\texttt{val}(\mathsf{d},(\mathsf{s}+\mathsf{t}))].$$

- The case $(\mathsf{s}\cdot\mathsf{t})$ is similar.

This concludes the induction and the proof. □

Lemma 2.37.

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash \forall\mathsf{e}\forall\mathsf{d}\forall\mathsf{f}((\Delta_0(\mathsf{f})\wedge\texttt{Eval}(\mathsf{e},\mathsf{f})\wedge\texttt{Eval}(\mathsf{d},\mathsf{f})\wedge\\ \forall[\mathsf{x}\epsilon\texttt{freevarF}(\mathsf{f})][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})])\rightarrow\\ (\texttt{Sat}_0(\mathsf{e},\mathsf{f})\leftrightarrow\texttt{Sat}_0(\mathsf{d},\mathsf{f}))).\end{aligned}$$

Proof. Using the lemma above we show

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash \forall\mathsf{n}\forall\mathsf{f}((\Delta_0(\mathsf{f})\wedge\mathsf{f}<\mathsf{n})\rightarrow\\ \forall[\mathsf{e}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]\forall[\mathsf{d}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]\\ ((\texttt{Eval}(\mathsf{e},\mathsf{f})\wedge\texttt{Eval}(\mathsf{d},\mathsf{f})\wedge\\ \forall[\mathsf{x}\epsilon\texttt{freevarF}(\mathsf{f})][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})])\rightarrow\\ (\texttt{Sat}_0(\mathsf{e},\mathsf{f})\leftrightarrow\texttt{Sat}_0(\mathsf{d},\mathsf{f}))))\end{aligned}$$

by Σ_1-$\texttt{Fmla}$-induction on f with the formula

$$\begin{aligned}&(\Delta_0(\mathsf{f})\wedge\mathsf{f}<\mathsf{n})\rightarrow\\&\forall[\mathsf{e}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]\forall[\mathsf{d}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]\\&\quad((\texttt{Eval}(\mathsf{e},\mathsf{f})\wedge\texttt{Eval}(\mathsf{d},\mathsf{f})\wedge\\&\qquad\forall[\mathsf{x}\epsilon\texttt{freevarF}(\mathsf{f})][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})])\rightarrow\\&\quad(\texttt{Sat}_0(\mathsf{e},\mathsf{f})\leftrightarrow\texttt{Sat}_0(\mathsf{d},\mathsf{f}))).\end{aligned}$$

We reason in $\mathrm{I}\Sigma_1$.

- Suppose $\texttt{Term}(\mathsf{s})$ and $\texttt{Term}(\mathsf{t})$. Let $[\mathsf{e}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]$ and $[\mathsf{d}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]$ be such that $\texttt{Eval}(\mathsf{e},(\mathsf{s}=\mathsf{t}))$, $\texttt{Eval}(\mathsf{d},(\mathsf{s}=\mathsf{t}))$ and $\forall[\mathsf{x}\epsilon\texttt{freevarF}((\mathsf{s}=\mathsf{t}))][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})]$. Since $[\texttt{freevarF}((\mathsf{s}=\mathsf{t}))=\texttt{freevarT}(\mathsf{s})\sqcup\texttt{freevarT}(\mathsf{t})]$ we get

  $$[\texttt{val}(\mathsf{e},\mathsf{s})=\texttt{val}(\mathsf{d},\mathsf{s})]$$

  and

  $$[\texttt{val}(\mathsf{e},\mathsf{t})=\texttt{val}(\mathsf{d},\mathsf{t})]$$

  by Lemma 2.36. Thus

  $$[\texttt{val}(\mathsf{e},\mathsf{s})=\texttt{val}(\mathsf{e},\mathsf{t})]\leftrightarrow[\texttt{val}(\mathsf{d},\mathsf{s})=\texttt{val}(\mathsf{d},\mathsf{t})],$$

  whence $\texttt{Sat}_0(\mathsf{e},(\mathsf{s}=\mathsf{t}))\leftrightarrow\texttt{Sat}_0(\mathsf{d},(\mathsf{s}=\mathsf{t}))$ by construction (Proposition 2.35).

- The case $(\mathsf{s}<\mathsf{t})$ is similar.

- Suppose $\texttt{Fmla}(\mathsf{f})$ and

  $$\begin{aligned}&(\Delta_0(\mathsf{f})\wedge\mathsf{f}<\mathsf{n})\rightarrow\\&\forall[\mathsf{e}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]\forall[\mathsf{d}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]\\&\quad((\texttt{Eval}(\mathsf{e},\mathsf{f})\wedge\texttt{Eval}(\mathsf{d},\mathsf{f})\wedge\\&\qquad\forall[\mathsf{x}\epsilon\texttt{freevarF}(\mathsf{f})][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})])\rightarrow\\&\quad(\texttt{Sat}_0(\mathsf{e},\mathsf{f})\leftrightarrow\texttt{Sat}_0(\mathsf{d},\mathsf{f}))).\end{aligned}$$

  Assume $[\Delta_0((\neg\mathsf{f}))]$ and $[(\neg\mathsf{f})<\mathsf{n}]$ and let $[\mathsf{e}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]$ and $[\mathsf{d}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]$ satisfy $\texttt{Eval}(\mathsf{e},(\neg\mathsf{f}))$, $\texttt{Eval}(\mathsf{d},(\neg\mathsf{f}))$ and $\forall[\mathsf{x}\epsilon\texttt{freevarF}((\neg\mathsf{f}))][\texttt{apl}(\mathsf{e},\mathsf{x})=\texttt{apl}(\mathsf{d},\mathsf{x})]$. By uniqueness of the construction of $(\neg\mathsf{f})$ and construction of Δ_0 we have $\Delta_0(\mathsf{f})$ and $\mathsf{f}<\mathsf{n}$. Furthermore, since $[\texttt{freevarF}((\neg\mathsf{f}))=\texttt{freevarF}(\mathsf{f})]$ we get $\texttt{Eval}(\mathsf{e},\mathsf{f})$, $\texttt{Eval}(\mathsf{d},\mathsf{f})$ and hence, by induction hypothesis,

  $$\texttt{Sat}_0(\mathsf{e},\mathsf{f})\leftrightarrow\texttt{Sat}_0(\mathsf{d},\mathsf{f}).$$

  By the construction of $\texttt{Sat}_0$, finally,

  $$\texttt{Sat}_0(\mathsf{e},(\neg\mathsf{f}))\leftrightarrow\texttt{Sat}_0(\mathsf{d},(\neg\mathsf{f})).$$

- Suppose $\texttt{Fmla}(\mathsf{f})$, $\texttt{Fmla}(\mathsf{g})$,

$$\begin{aligned}&(\Delta_0(\mathsf{f}){\wedge}\mathsf{f}{<}\mathsf{n}){\rightarrow}\\&\forall[\mathsf{e}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{n})]\forall[\mathsf{d}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{n})]\\&\quad((\texttt{Eval}(\mathsf{e},\mathsf{f}){\wedge}\texttt{Eval}(\mathsf{d},\mathsf{f}){\wedge}\\&\qquad\forall[\mathsf{x}\epsilon\texttt{freevarF}(\mathsf{f})][\texttt{apl}(\mathsf{e},\mathsf{x}){=}\texttt{apl}(\mathsf{d},\mathsf{x})]){\rightarrow}\\&\quad(\texttt{Sat}_0(\mathsf{e},\mathsf{f}){\leftrightarrow}\texttt{Sat}_0(\mathsf{d},\mathsf{f})))\end{aligned}$$

  and similarly for g. Assume $[\Delta_0((\mathsf{f}{\wedge}\mathsf{g}))]$ and $[(\mathsf{f}{\wedge}\mathsf{g}){<}\mathsf{n}]$, and let $[\mathsf{e}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{n})]$ and $[\mathsf{d}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{n})]$ be such that $\texttt{Eval}(\mathsf{e},(\mathsf{f}{\wedge}\mathsf{g}))$, $\texttt{Eval}(\mathsf{d},(\mathsf{f}{\wedge}\mathsf{g}))$ and $\forall[\mathsf{x}\epsilon\texttt{freevarF}((\mathsf{f}{\wedge}\mathsf{g}))][\texttt{apl}(\mathsf{e},\mathsf{x}){=}\texttt{apl}(\mathsf{d},\mathsf{x})]$. By construction of Δ_0 and the uniqueness of the construction of $(\mathsf{f}{\wedge}\mathsf{g})$ we have $\Delta_0(\mathsf{f})$, $\mathsf{f}{<}\mathsf{n}$, $\Delta_0(\mathsf{g})$ and $\mathsf{g}{<}\mathsf{n}$. That $[\texttt{freevarF}((\mathsf{f}{\wedge}\mathsf{g})){=}\texttt{freevarF}(\mathsf{f}){\sqcup}\texttt{freevarF}(\mathsf{g})]$ implies $\texttt{Eval}(\mathsf{e},\mathsf{f})$, $\texttt{Eval}(\mathsf{d},\mathsf{f})$ and $\texttt{Eval}(\mathsf{e},\mathsf{g})$, $\texttt{Eval}(\mathsf{d},\mathsf{g})$. By induction hypothesis these facts imply

$$\texttt{Sat}_0(\mathsf{e},\mathsf{f}){\leftrightarrow}\texttt{Sat}_0(\mathsf{d},\mathsf{f})$$

  and

$$\texttt{Sat}_0(\mathsf{e},\mathsf{g}){\leftrightarrow}\texttt{Sat}_0(\mathsf{d},\mathsf{g}),$$

  whence

$$\texttt{Sat}_0(\mathsf{e},(\mathsf{f}{\wedge}\mathsf{g})){\leftrightarrow}\texttt{Sat}_0(\mathsf{d},(\mathsf{f}{\wedge}\mathsf{g}))$$

  by the construction of $\texttt{Sat}_0$.

- The case $(\mathsf{f}{\vee}\mathsf{g})$ is similar.

- Suppose $\texttt{Fmla}(\mathsf{f})$ and

$$\begin{aligned}&(\Delta_0(\mathsf{f}){\wedge}\mathsf{f}{<}\mathsf{n}){\rightarrow}\\&\forall[\mathsf{e}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{n})]\forall[\mathsf{d}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{n})]\\&\quad((\texttt{Eval}(\mathsf{e},\mathsf{f}){\wedge}\texttt{Eval}(\mathsf{d},\mathsf{f}){\wedge}\\&\qquad\forall[\mathsf{x}\epsilon\texttt{freevarF}(\mathsf{f})][\texttt{apl}(\mathsf{e},\mathsf{x}){=}\texttt{apl}(\mathsf{d},\mathsf{x})]){\rightarrow}\\&\quad(\texttt{Sat}_0(\mathsf{e},\mathsf{f}){\leftrightarrow}\texttt{Sat}_0(\mathsf{d},\mathsf{f}))).\end{aligned}$$

  Assume $\Delta_0((\forall\mathbf{v}_\mathsf{y}\mathsf{f}))$ and $[(\forall\mathbf{v}_\mathsf{y}\mathsf{f}){<}\mathsf{n}]$. Take $[\mathsf{e}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{n})]$ and $[\mathsf{d}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{n})]$ with $\texttt{Eval}(\mathsf{e},(\forall\mathbf{v}_\mathsf{y}\mathsf{f}))$, $\texttt{Eval}(\mathsf{d},(\forall\mathbf{v}_\mathsf{y}\mathsf{f}))$ and $\forall[\mathsf{x}\epsilon\texttt{freevarF}((\forall\mathbf{v}_\mathsf{y}\mathsf{f}))][\texttt{apl}(\mathsf{e},\mathsf{x}){=}\texttt{apl}(\mathsf{d},\mathsf{x})]$. By the uniqueness of the construction of $(\forall\mathbf{v}_\mathsf{x}\mathsf{f})$ we have $\mathsf{f}{<}\mathsf{n}$ and, by the construction of Δ_0, there must be $\mathsf{g}{<}\mathsf{f}$ and $\mathsf{z}{<}\mathsf{f}$ with $\Delta_0(\mathsf{g})$, $\mathsf{z}{\neq}\mathsf{y}$ and

$$[((\neg(\mathbf{v}_\mathsf{y}{<}\mathbf{v}_\mathsf{z})){\vee}\mathsf{g}){=}\mathsf{f}].$$

Hence $\Delta_0(\mathsf{f})$. Moreover, $[\mathtt{freevarF}((\forall \mathbf{v}_y \mathsf{f}))=\mathtt{freevarF}(\mathsf{f})\backslash\{\mathbf{v}_y\}]$ and $[\mathbf{v}_z \epsilon \mathtt{freevarF}(\mathsf{f})]$, whence in particular

$$[\mathtt{apl}(\mathsf{e},\mathbf{v}_z)=\mathtt{apl}(\mathsf{d},\mathbf{v}_z)].$$

Now take $[\mathsf{a}<\mathtt{apl}(\mathsf{e},\mathbf{v}_z)]$, so that in particular $\mathsf{a}<\mathsf{n}$. Since

$$[\mathtt{dom}(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}))=\mathtt{dom}(\mathsf{e})\sqcup\{\mathbf{v}_y\}]$$

and

$$[\mathtt{dom}(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}))=\mathtt{dom}(\mathsf{d})\sqcup\{\mathbf{v}_y\}]$$

we have $[\mathtt{Eval}(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{f})]$ and $[\mathtt{Eval}(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{f})]$. If $[\mathsf{x}\epsilon\mathtt{freevarF}(\mathsf{f})]$ then either $[\mathsf{x}=\mathbf{v}_y]$ or $[\mathsf{x}\epsilon\mathtt{freevarF}((\forall \mathbf{v}_y \mathsf{f}))]$; in the first case $[\mathtt{apl}(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{x})=\mathsf{a}]$ and $[(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{x})=\mathsf{a}]$ by definition of $\mathtt{ext}$, in the latter $[\mathtt{apl}(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{x})=\mathtt{apl}(\mathsf{e},\mathsf{x})]$ and $[\mathtt{apl}(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{x})=\mathtt{apl}(\mathsf{d},\mathsf{x})]$. Thus

$$\forall[\mathsf{x}\epsilon\mathtt{freevarF}(\mathsf{f})][\mathtt{apl}(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{x})=\mathtt{apl}(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{x})].$$

Finally we have $[\mathbf{v}_y<(\forall \mathbf{v}_y \mathsf{f})]$, so $[\mathbf{v}_y<\mathsf{n}]$. Hence $[\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a})\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]$ and $[\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a})\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]$. Consequently the induction hypothesis applies, yielding

$$[\mathtt{Sat}_0(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{f})]\leftrightarrow[\mathtt{Sat}_0(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{f})].$$

Since $[\mathsf{f}=((\neg(\mathbf{v}_y<\mathbf{v}_z))\vee\mathsf{g})]$ this is equivalent to

$$((\neg[\mathsf{a}<\mathtt{apl}(\mathsf{e},\mathbf{v}_z)])\vee[\mathtt{Sat}_0(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{g})])\leftrightarrow((\neg[\mathsf{a}<\mathtt{apl}(\mathsf{d},\mathbf{v}_z)])\vee[\mathtt{Sat}_0(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{g})])$$

by construction of $\mathtt{Sat}_0$. As $[\mathsf{a}<\mathtt{apl}(\mathsf{e},\mathbf{v}_z)]$ and $[\mathsf{a}<\mathtt{apl}(\mathsf{d},\mathbf{v}_z)]$, the above statement is equivalent to

$$[\mathtt{Sat}_0(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{g})]\leftrightarrow[\mathtt{Sat}_0(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{g})]$$

in turn.

Since the choice of $[\mathsf{a}<\mathtt{apl}(\mathsf{e},\mathbf{v}_z)]$ was arbitrary, we conclude that

$$\forall[\mathsf{a}<\mathtt{apl}(\mathsf{e},\mathbf{v}_y)]([\mathtt{Sat}_0(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{g})]\leftrightarrow[\mathtt{Sat}_0(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{g})]).$$

As $[\mathtt{apl}(\mathsf{e},\mathbf{v}_z)=\mathtt{apl}(\mathsf{d},\mathbf{v}_z)]$, the above gives

$$\forall[\mathsf{a}<\mathtt{apl}(\mathsf{e},\mathbf{v}_y)][\mathtt{Sat}_0(\mathtt{ext}(\mathsf{e},\mathbf{v}_y,\mathsf{a}),\mathsf{g})]\leftrightarrow\forall[\mathsf{a}<\mathtt{apl}(\mathsf{d},\mathbf{v}_y)][\mathtt{Sat}_0(\mathtt{ext}(\mathsf{d},\mathbf{v}_y,\mathsf{a}),\mathsf{g})],$$

which by construction of $\mathtt{Sat}_0$ is equivalent to

$$[\mathtt{Sat}_0(\mathsf{e},(\forall \mathbf{v}_y \mathsf{f}))]\leftrightarrow[\mathtt{Sat}_0(\mathsf{d},(\forall \mathbf{v}_y \mathsf{f}))],$$

the desired conclusion.

- The case $(\exists \mathbf{v}_{\mathsf{y}}\mathsf{f})$ is similar.

This concludes the induction.

To verify the claim we reason in $I\Sigma_1$ again: Given f, e and d such that $\Delta_0(\mathsf{f})$, $\mathtt{Eval}(\mathsf{e},\mathsf{f})$, $\mathtt{Eval}(\mathsf{d},\mathsf{f})$ and $\forall[\mathsf{x}\epsilon\mathtt{freevarF}(\mathsf{f})][\mathtt{apl}(\mathsf{e},\mathsf{x})=\mathtt{apl}(\mathsf{d},\mathsf{x})]$, let $\mathsf{n}=\mathsf{S}(\mathsf{f}+\mathsf{e}+\mathsf{d})$. Then $\mathsf{f}<\mathsf{n}$, $\mathsf{e}<\mathsf{n}$ and $\mathsf{d}<\mathsf{n}$, whence in particular $\mathsf{x}<\mathsf{n}$ for all $[\mathsf{x}\epsilon\mathtt{dom}(\mathsf{e})]$ and $\mathsf{y}<\mathsf{n}$ for all $[\mathsf{y}\epsilon\mathtt{ran}(\mathsf{e})]$, and similarly for d. Thus $[\mathsf{e}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]$ and $[\mathsf{d}\sqsubseteq(<\mathsf{n})\times(<\mathsf{n})]$, whereby

$$\mathtt{Sat}_0(\mathsf{e},\mathsf{f})\leftrightarrow\mathtt{Sat}_0(\mathsf{d},\mathsf{f})$$

by above. □

Note that, until now, we have not presupposed anything about terms, formulae, et cetera that requires them to be any particular kind of objects, as long as they have the correct relationships to each other, and this is precisely what the above states is true of the sets $\mathtt{Term}(\mathbb{N})$ and $\mathtt{Fmla}(\mathbb{N})$ et cetera of numbers. Thus we will follow [4][7] to *define* formulae as those particular natural numbers that satisfy these properties. Indeed we can assume that this is the definition of "formula" we have been working with all along (since we could have defined these sets in $\mathbb{N}$ directly, without the detour via $I\Sigma_1$).

To make this precise we state that have defined terms and formulae via the following definition.

Definition 2.16 (Terms and formulae)**.** $\mathsf{0}$ is the number $x \in \mathbb{N}$ such that $\mathbb{N} \models [\mathbf{0}=\overline{x}]$.[8] Likewise v_i is the number $x \in \mathbb{N}$ such that $\mathbb{N} \models [\mathbf{v}_{\bar{i}}=\overline{x}]$. (This amounts to saying that $\mathbb{N} \models [\mathbf{0}=\overline{0}]$ and $\mathbb{N} \models [\mathbf{v}_{\bar{i}}=\overline{\mathsf{v}_i}]$ by definition.)

Terms are recursively defined as follows:

- The term $(\mathsf{0})$, often abbreviated $\mathsf{0}$, is defined by $\mathbb{N} \models [(\mathbf{0})=\overline{(\mathsf{0})}]$.
- For each $i \in \mathbb{N}$ the term (v_i), often abbreviated v_i, is defined by $\mathbb{N} \models [(\mathbf{v}_{\bar{i}})=\overline{(\mathsf{v})_i}]$.
- If τ is a term then $(\mathsf{S}\tau)$, abbreviated $\mathsf{S}(\tau)$, is a term, defined by $\mathbb{N} \models [(\mathbf{S}\overline{\tau})=\overline{(\mathsf{S}\tau)}]$.
- If σ and τ are terms then $(\sigma+\tau)$, abbreviated $\sigma+\tau$, is a term, defined by $\mathbb{N} \models [(\overline{\sigma}\mathbf{+}\overline{\tau})=\overline{(\sigma+\tau)}]$.
- If σ and τ are terms then $(\sigma\cdot\tau)$, abbreviated $\sigma\cdot\tau$, is a term, defined by $\mathbb{N} \models [(\overline{\sigma}\boldsymbol{\cdot}\overline{\tau})=\overline{(\sigma\cdot\tau)}]$.

The set of terms will be denoted Term.

Formulae are recursively defined as follows:

- If σ and τ are terms then $(\sigma=\tau)$ is a formula and defined by $\mathbb{N} \models [(\overline{\sigma}\mathbf{=}\overline{\tau})=\overline{(\sigma=\tau)}]$.
- If σ and τ are terms then $(\sigma<\tau)$ is a formula and defined by $\mathbb{N} \models [(\overline{\sigma}\mathbf{<}\overline{\tau})=\overline{(\sigma<\tau)}]$.

[7] who attributes the idea to Feferman

[8] Again, this last statement can be unwinded to yield some explicit (in terms of the basic arithmetic operations and relations on $\mathbb{N}$) statement about x, which is what we use as the definition of $\mathsf{0}$.

- If φ is a formula then $(\neg\varphi)$ is a formula, defined by $\mathbb{N} \models [(\neg\overline{\varphi}) = \overline{(\neg\varphi)}]$.
- If φ and ψ are formulae then $(\varphi\wedge\psi)$ is a formula, defined by $\mathbb{N} \models [(\overline{\varphi}\wedge\overline{\psi}) = \overline{(\varphi\wedge\psi)}]$.
- If φ and ψ are formulae then $(\varphi\vee\psi)$ is a formula, defined by $\mathbb{N} \models [(\overline{\varphi}\vee\overline{\psi}) = \overline{(\varphi\vee\psi)}]$.
- For each i, if φ is a formula then $(\forall \mathsf{v}_i\varphi)$ is a formula defined by $\mathbb{N} \models [(\forall \mathbf{v}_{\overline{i}}\overline{\varphi}) = \overline{(\forall \mathsf{v}_i\varphi)}]$.
- For each i, if φ is a formula then $(\exists \mathsf{v}_i\varphi)$ is a formula defined by $\mathbb{N} \models [(\exists \mathbf{v}_{\overline{i}}\overline{\varphi}) = \overline{(\exists \mathsf{v}_i\varphi)}]$.

The set of formulae will be denoted Fmla.

Since these are recursive definitions of subsets of $\mathbb{N}$ and each construct is greater (with respect to $<$) than its constituents (which is provable in $\mathrm{I}\Sigma_1$), we can prove properties of formulae by (unrestricted) induction. In particular we can prove that τ is a term if and only if $\mathbb{N} \models \mathtt{Term}(\overline{\tau})$ and φ is a formula if and only if $\mathbb{N} \models \mathtt{Fmla}(\overline{\varphi})$; in a similar fashion we see that φ is a Δ_0-formula if and only if $\mathbb{N} \models \mathtt{\Delta_0}(\overline{\varphi})$. Moreover, since we have uniqueness of formula constructions by Lemma 2.32, we can define functions and predicates by recursion on the construction of formulae the way we are used to (a similar result holds for terms, but we will not need it). Finally, this definition of terms and formulae justifies the definition of a recursively axiomatisable theory, since Term and Fmla are Δ_1-sets.

A result similar to the above for Δ_0-formulae would hold for negation normal formulae, if indeed we had given a definition of negation normal formulae at the meta level. Thus we can take this opportunity to simplify matters and define negation normal formulae to be exactly those formulae which satisfy $\mathtt{Nnf}$.

Definition 2.17. A formula φ is a *negation normal formula* (or *on negation normal form*) if $\mathbb{N} \models \mathtt{Nnf}(\overline{\varphi})$. The set of negation normal formulae will be denoted Nnf.

It is now straightforward, basically just applying de Morgan's laws, to verify that the function $\mathrm{nnf} : \mathrm{Fmla} \longrightarrow \mathrm{Nnf}$ defined so that $\mathbb{N} \models [\mathtt{nnf}(\overline{\varphi}) = \overline{\mathrm{nnf}(\varphi)}]$ to every formula gives a (logically) equivalent negation normal form.

Lemma 2.38. *For every formula φ,*

$$\vdash \varphi \leftrightarrow \mathrm{nnf}(\varphi).$$

Proof. We show that $\vdash \varphi \leftrightarrow \mathrm{nnf}(\varphi)$ and $\vdash \neg\varphi \leftrightarrow \mathrm{nnf}(\neg\varphi)$ for all φ by induction.

- Suppose ϑ is atomic; then $\mathbb{N} \models \mathtt{Atf}(\overline{\vartheta})$. Hence

  $$\mathbb{N} \models [\mathtt{nnf}(\overline{\vartheta}) = \overline{\vartheta}] \text{ and } \mathbb{N} \models [\mathtt{nnf}((\neg\overline{\vartheta})) = (\neg\overline{\vartheta})]$$

  by Lemma 2.33. Thereby $\mathrm{nnf}(\vartheta) = \vartheta$ and $\mathrm{nnf}(\neg\vartheta) = \neg\vartheta$ by Definition 2.16, whence

  $$\vdash \vartheta \leftrightarrow \mathrm{nnf}(\vartheta) \text{ and } \vdash \neg\vartheta \leftrightarrow \mathrm{nnf}(\neg\vartheta)$$

  trivially.

- Assume $\vdash \vartheta \leftrightarrow \text{nnf}(\vartheta)$ and $\vdash \neg\vartheta \leftrightarrow \text{nnf}(\neg\vartheta)$ and consider $\neg\vartheta$. By induction hypothesis

$$\vdash \neg\vartheta \leftrightarrow \text{nnf}(\neg\vartheta).$$

  Furthermore, Lemma 2.33 gives $\text{nnf}(\neg(\neg\vartheta)) = \text{nnf}(\vartheta)$, whence

$$\vdash \neg(\neg\vartheta) \leftrightarrow \text{nnf}(\neg(\neg\vartheta))$$

  since $\vdash \neg(\neg\vartheta) \leftrightarrow \vartheta$ and $\vdash \vartheta \leftrightarrow \text{nnf}(\vartheta)$ by induction hypothesis.

- Assume $\vdash \vartheta \leftrightarrow \text{nnf}(\vartheta)$, $\vdash \neg\vartheta \leftrightarrow \text{nnf}(\neg\vartheta)$, $\vdash \psi \leftrightarrow \text{nnf}(\psi)$ and $\vdash \neg\psi \leftrightarrow \text{nnf}(\neg\psi)$ and consider $\vartheta \wedge \psi$. By Lemma 2.33 we get $\text{nnf}(\vartheta \wedge \psi) = \text{nnf}(\vartheta) \wedge \text{nnf}(\psi)$ and $\text{nnf}(\neg(\vartheta \wedge \psi)) = \text{nnf}(\neg\vartheta) \vee \text{nnf}(\neg\psi)$. Now

$$\vdash \vartheta \wedge \psi \leftrightarrow \text{nnf}(\vartheta \wedge \psi) \text{ and } \vdash \neg(\vartheta \wedge \psi) \leftrightarrow \text{nnf}(\neg(\vartheta \wedge \psi))$$

  by induction hypothesis and de Morgan's laws.

- The disjunctive case is similar.

- Assume $\vdash \vartheta \leftrightarrow \text{nnf}(\vartheta)$ and $\vdash \neg\vartheta \leftrightarrow \text{nnf}(\neg\vartheta)$ and consider $\forall \mathsf{v}_i \vartheta$. By Lemma 2.33 we have $\text{nnf}(\forall \mathsf{v}_i \vartheta) = \forall \mathsf{v}_i \, \text{nnf}(\vartheta)$ and $\text{nnf}(\neg \forall \mathsf{v}_i \vartheta) = \exists \mathsf{v}_i \, \text{nnf}(\neg\vartheta)$, whereby

$$\vdash \forall \mathsf{v}_i \vartheta \leftrightarrow \text{nnf}(\forall \mathsf{v}_i \vartheta) \text{ and } \vdash \neg \forall \mathsf{v}_i \vartheta \leftrightarrow \text{nnf}(\neg \forall \mathsf{v}_i \vartheta)$$

  by induction hypothesis and de Morgan's laws.

- The existential case is similar.

This concludes the induction, and the proof. □

We have thus identified Term and Fmla as particular subsets of $\mathbb{N}$. The (in the author's experience) more usual practice is to let terms and formulae be whatever they are, and define a(n effective) Gödel numbering of them, that is a "computable" function $\ulcorner \cdot \urcorner : \text{Term} \cup \text{Fmla} \longrightarrow \mathbb{N}$, such that the Gödel number of a formula (term et cetera) is exactly the number given by the definitions above (or similar; there are many Gödel numberings). Thus, our approach amounts to an identification of formulae (terms) and their Gödel numbers. Both descriptions have their merits and their drawbacks, and our choice of definition is partly to avoid, or at least hide, the subtleties they might entail in what follows. The other reason is purely expository, since we will avoid the clutter of $\ulcorner \; \urcorner$ in our statements.

Remark 7. Since this might still (naturally) lead to some confusion, it can be worth to keep the following example in mind. While indeed

$$\mathbb{N} \models [\mathbf{0} = \overline{0}]$$

(that is, the numeral of the term $\mathsf{0}$ is interpreted (in $\mathbb{N}$, and hence in any other model of $\mathrm{I}\Sigma_1$) as the number which has the property of being the formal term $\mathbf{0}$), this is not to say that any of these are 0 (the number zero). In fact they are not, that is

$$\mathbb{N} \models \overline{\mathsf{0}} \neq \mathsf{0}$$

and

$$\mathbb{N} \models [\mathbf{0} \neq \mathsf{0}]$$

(however, $\mathbb{N} \models \overline{\mathsf{0}} = \mathsf{0}$ by definition).

Having verified the relationships between the construction of formulae of the object language and of the formal object language, we can likewise verify that these constructions and the satisfaction predicate have some sort of "inverse" behaviour. This is to be expected, to the degree that we at the meta level rarely bother proving similar results, instead taking it for granted that e.g. $\mathcal{M} \models \forall x \forall y < x \exists z < x (x = y + z)$ just in case there for all $a, b \in \mathcal{M}$ with $b <_{\mathcal{M}} a$ is a $c \in \mathcal{M}$ with $c <_{\mathcal{M}} a$ and $a = b +_{\mathcal{M}} c$. This is in fact not an issue, since in all concrete cases we only need to apply Tarski's truth definition an explicitly given finite number of times, which is trivial. The proof of the lemma below uses exactly this idea: via induction on the meta level (which should correspond roughly to our notion of an "explicitly given finite number" from the point of view of the object level) we verify that the formal object corresponding to a formula (that is, its numeral) is satisfied if and only if the formula is true.

Lemma 2.39. *Let $k \in \mathbb{N}$. For every $\mathcal{L}_{\mathrm{A}}$-term τ with variables among $\mathsf{v}_0, \ldots, \mathsf{v}_{k-1}$ we have that*

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\tau})] \wedge \\ ([\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\tau}) = \tau]).$$

Proof. Since this is a schema in τ (that is, a statement in the meta-language about real terms τ) we prove this by induction, this time not inside $\mathrm{I}\Sigma_1$ but at the meta level, on the construction of τ. Note that $\mathrm{I}\Sigma_1 \vdash \mathtt{Term}(\overline{\tau})$ for all terms τ by Σ_1-completeness.

- Consider the term $(\mathsf{0})$ (where we include the parentheses for clarity). Since $\mathbb{N} \models [(\mathbf{0}) = \overline{(\mathsf{0})}]$ by definition, $\mathrm{I}\Sigma_1 \models [(\mathbf{0}) = \overline{(\mathsf{0})}]$ by Σ_1-completeness. Hence

  $$\mathrm{I}\Sigma_1 \vdash [\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{(\mathsf{0})}) = (\mathsf{0})]$$

  by definition. Since $(\mathsf{0})$ has no free variables, and this is provable in $\mathrm{I}\Sigma_1$ (for $\overline{(\mathsf{0})}$ that is) by definition,

  $$\mathrm{I}\Sigma_1 \vdash [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{(\mathsf{0})})]$$

  vacuously.

- Let $i < k$ and consider the term (v_i) (where we again include the parentheses for clarity). Since $\mathrm{I}\Sigma_1 \vdash [\mathbf{v}_{\bar{i}} = \overline{\mathsf{v}_i}]$ we have

$$\mathrm{I}\Sigma_1 \vdash [\overline{\mathsf{v}_i} \epsilon \mathtt{dom}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}))],$$

whence

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{(\mathsf{v}_i)})]$$

by definition. Furthermore

$$\mathrm{I}\Sigma_1 \vdash ([\mathtt{apl}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\mathsf{v}_i}) = (\mathsf{v}_i)])$$

by definition of $\mathtt{fn}_k$, whence

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{(\mathsf{v}_i)}) = (\mathsf{v}_i)]$$

by definition of $\mathtt{val}$.

- Suppose σ is a term such that if no v_i for $i \geq k$ is free in σ,

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash & [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\sigma})] \wedge \\ & ([\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\sigma}) = \sigma]).\end{aligned}$$

Assume that no v_i for $i \geq k$ is free in $\mathsf{S}(\sigma)$. Like before, by Σ_1-completeness $\mathrm{I}\Sigma_1 \vdash [(\mathbf{S}\overline{\sigma}) = \overline{\mathsf{S}(\sigma)}]$. Moreover, by definition $\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarT}((\mathbf{S}\overline{\sigma})) = \mathtt{freevarT}(\overline{\sigma})]$, whence no v_i is free in σ for $i \geq k$. Taking this into account as well, we get

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\mathsf{S}(\sigma)})]$$

by induction hypothesis. Additionally

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash & [\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), (\mathbf{S}\overline{\sigma})) = \\ & \mathsf{S}(\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\sigma}))]\end{aligned}$$

by definition, and since $\mathrm{I}\Sigma_1 \vdash [\mathsf{S}(\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\sigma})) = \mathsf{S}(\sigma)]$ by induction hypothesis we get

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\mathsf{S}(\sigma)}) = \mathsf{S}(\sigma)].$$

- Suppose σ and ρ are terms such that if no v_i for $i \geq k$ is free in σ then

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash & [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\sigma})] \wedge \\ & ([\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1}), \overline{\sigma}) = \sigma])\end{aligned}$$

and similarly for ρ. Assume that no v_i with $i \geq k$ is free in $\sigma+\rho$. By Σ_1-completeness again, $\mathrm{I}\Sigma_1 \vdash [(\overline{\sigma}\boldsymbol{+}\overline{\rho})=\overline{\sigma+\rho}]$, and by definition

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarT}((\overline{\sigma}\boldsymbol{+}\overline{\rho}))=\mathtt{freevarT}(\overline{\sigma})\sqcup\mathtt{freevarT}(\overline{\rho})]$$

so that no v_i with $i \geq k$ is free in σ or ρ and

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\sigma+\rho})]$$

by induction hypothesis. Since furthermore

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash [&\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),(\overline{\sigma}\boldsymbol{+}\overline{\rho}))=\\ &\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\sigma})+\\ &\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\rho})],\end{aligned}$$

using the induction hypothesis again we get

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\sigma+\rho})=\sigma+\rho]$$

as desired.

- The case $\cdot$ is completely analogous to the above.

This concludes the induction, and the proof. □

Lemma 2.40. *Let $k \in \mathbb{N}$. For every Δ_0-formula φ such that the variables of φ are among $\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}$ we have*

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &[\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\varphi})]\wedge\\ &([\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\varphi})]\leftrightarrow\varphi).\end{aligned}$$

Proof. The subsequent induction on the construction of φ is very similar to the one in the lemma above (2.39), and hence we will omit most of it.

- Let σ and τ be terms and consider $\sigma=\tau$. Suppose its variables are among $\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}$. Since $\mathrm{I}\Sigma_1 \vdash [(\overline{\sigma}\boldsymbol{=}\overline{\tau})=\overline{\sigma=\tau}]$ and

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarF}((\overline{\sigma}\boldsymbol{=}\overline{\tau}))=\mathtt{freevarT}(\overline{\sigma})\sqcup\mathtt{freevarT}(\overline{\tau})]$$

we have that the variables of σ and τ are among $\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}$. Then

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &[\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\tau})]\wedge\\ &([\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\tau})=\tau])\end{aligned}$$

and similarly for σ, by Lemma 2.39. In particular

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarT}(\overline{\tau})\sqsubseteq\mathtt{dom}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}))]$$

and similarly for σ, whence

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\sigma{=}\tau})].$$

Furthermore, since

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash [&\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),(\overline{\sigma}\mathbf{=}\overline{\tau}))]\leftrightarrow\\ &\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\sigma})=\\ &\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\tau}),\end{aligned}$$

we get

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),(\overline{\sigma}\mathbf{=}\overline{\tau}))]\leftrightarrow\sigma{=}\tau.$$

- The case $\sigma{<}\tau$ is similar.
- The case $\neg\vartheta$ is similar to $\mathsf{S}(\sigma)$ of Lemma 2.39 and is omitted.
- The case $\vartheta\wedge\psi$ is similar to $\sigma{+}\rho$ of Lemma 2.39, and is omitted.
- The case $\vartheta\vee\psi$ is similar to $\sigma{+}\rho$ of Lemma 2.39, and is omitted.
- Suppose ϑ is a Δ_0-formula such that if no v_i for $i \geq k$ occurs in ϑ then

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &[\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})]\wedge\\ &([\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})]\leftrightarrow\vartheta).\end{aligned}$$

Consider $\forall\mathsf{v}_p{<}\mathsf{v}_q\vartheta$, and suppose its variables are among $\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}$. Thus the same holds of ϑ, and in addition $p, q < k$. By induction hypothesis, then,

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &[\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})]\wedge\\ &([\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})]\leftrightarrow\vartheta).\end{aligned}$$

In particular

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarF}(\overline{\vartheta})\sqsubseteq\mathtt{dom}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}))],$$

and

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarF}(\overline{\forall\mathsf{v}_p{<}\mathsf{v}_q\vartheta})\sqsubseteq\mathtt{dom}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}))]$$

since by definition

$$\mathrm{I}\Sigma_1 \vdash [(\forall\mathbf{v}_{\overline{p}}((\neg(\mathbf{v}_{\overline{p}}\mathbf{<}\mathbf{v}_{\overline{q}}))\vee\overline{\vartheta}))=\overline{\forall\mathsf{v}_p{<}\mathsf{v}_q\vartheta}]$$

and

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{freevarF}((\forall \mathbf{v}_{\overline{p}}((\neg(\mathbf{v}_{\overline{p}}<\mathbf{v}_{\overline{q}}))\vee\overline{\vartheta})))=(\{\mathbf{v}_{\overline{q}}\}\sqcup\mathtt{freevarF}(\vartheta))\setminus\{\mathbf{v}_{\overline{p}}\}].$$

This implies that

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\forall \mathsf{v}_p<\mathsf{v}_q\vartheta})].$$

Turning to the second conjunct to be verified, by definition

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &[\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\forall \mathsf{v}_p<\mathsf{v}_q\vartheta})]\leftrightarrow\\ &\forall[\mathsf{z}<\mathtt{apl}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\mathbf{v}_{\overline{q}})]\\ &\qquad[\mathtt{Sat}_0(\mathtt{ext}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\mathbf{v}_{\overline{p}},\mathsf{z}),\overline{\vartheta})],\end{aligned}$$

where z is a fresh variable. However, $\mathrm{I}\Sigma_1 \vdash [\mathtt{apl}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\mathbf{v}_{\overline{i}})=\mathsf{v}_i]$ for all $i<k$, whereby in particular

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &[\mathtt{ext}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\mathbf{v}_{\overline{p}},\mathsf{z})=\\ &\quad\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{p}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{z},\ldots,\mathsf{v}_{k-1})]\end{aligned}$$

by Lemmas 2.28 and 2.29, and consequently

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &[\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\forall \mathsf{v}_p<\mathsf{v}_q\vartheta})]\leftrightarrow\\ &\forall \mathsf{z}<\mathsf{v}_q[\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{p}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{z},\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})].\end{aligned}$$

Moreover

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &\forall \mathsf{z}<\mathsf{v}_q[\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{p}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{z},\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})]\leftrightarrow\\ &\forall \mathsf{v}_p<\mathsf{v}_q[\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})]\end{aligned}$$

since v_p is not free in $[\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{p}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{z},\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})]$. Thus

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash &[\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\forall \mathsf{v}_p<\mathsf{v}_q\vartheta})]\leftrightarrow\\ &\forall \mathsf{v}_p<\mathsf{v}_q[\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})].\end{aligned}$$

By induction hypothesis

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\vartheta})]\leftrightarrow\vartheta,$$

whence

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Sat}_0(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\forall \mathsf{v}_p<\mathsf{v}_q\vartheta})]\leftrightarrow\forall \mathsf{v}_p<\mathsf{v}_q\vartheta$$

as desired.

- The case $\exists \mathsf{x}{<}\mathsf{y}\vartheta$ is similar to the above.

This concludes the induction, and the proof. □

This is (essentially) Corollary 1.76 on p. 59 of [4]. In view of Lemma 2.37 this result is far from being the best possible, but it is sufficient for our purposes. In fact, it is probably less cumbersome to apply both 2.37 and the above whenever needed than finding a general formulation (and proof) which incorporates the first into the latter. A similar comment applies to Lemma 2.39 relative Lemma 2.36.

3 Coding of sets and functions in models of $\mathrm{I}\Sigma_1$

The material of this section could also be considered preliminaries, were it not for the fact that the author was unable to find an applicable reference to it. To remedy this, proofs will be given for all results, though they might be slightly less formalistic than in the previous section.

Definition 3.1. Let $\mathcal{M} \models \mathrm{I}\Sigma_1$.

1. For $m \in \mathcal{M}$ such that $\mathcal{M} \models \mathtt{Set}(m)$ we define the (set-)realisation of m as $\mathfrak{R}_\mathrm{s}(m)_\mathcal{M} = \{a \in \mathcal{M} \mid \mathcal{M} \models a \epsilon m\}$.
2. For $A \subseteq \mathcal{M}$ such that there is a $m \in \mathcal{M}$ with $\mathcal{M} \models \mathtt{Set}(m)$ and $A = \mathfrak{R}_\mathrm{s}(m)_\mathcal{M}$, we write $m = \mathfrak{F}_\mathrm{s}(A)_\mathcal{M}$. We say A is *coded* in $\mathcal{M}$.

The above definition is meaningful by extensionality, Lemma 2.27.

Remark 8. The code of a natural number n as a set (finite ordinal) in $\mathcal{M}$ will in general *not* be $\overline{n}_\mathcal{M}$.

Lemma 3.1. *If $\mathcal{M} \models \mathrm{I}\Sigma_1$, then all finite subsets of $\mathcal{M}$, as well as $\mathcal{M}_{<l}$ for $l \in \mathcal{M}$, are coded in $\mathcal{M}$.*

Proof. Let A be a finite set and $a_0, \ldots, a_{k-1}$ be an enumeration of A. Let $m \in \mathcal{M}$ be such that $\mathcal{M} \models [\{a_0, \ldots, a_{k-1}\}^k = m]$. By Lemma 2.29

$$\mathrm{I}\Sigma_1 \vdash [\mathtt{Set}(m)] \wedge \forall \mathsf{y}(\mathsf{y} \epsilon m \leftrightarrow \bigvee\nolimits_{0 \le i < k} \mathsf{y} = a_i),$$

whence $A = \mathfrak{R}_\mathrm{s}(m)_\mathcal{M}$, that is A is coded in $\mathcal{M}$.

Now let $l \in \mathcal{M}$. By Lemma 2.26, $\mathcal{M} \models [\mathtt{Set}((<l))] \wedge \forall \mathsf{x}([\mathsf{x} \epsilon (<l)] \leftrightarrow \mathsf{x} < l)$, so $\mathcal{M}_{<l}$ is coded in $\mathcal{M}$. □

As there will seldom be more than one structure under consideration, we will often drop the subscripts specifying the structure, as usual.

Lemma 3.2. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ and $A, B \subseteq \mathcal{M}$ be coded in $\mathcal{M}$. Then $A \subseteq B$ if and only if $\mathcal{M} \models \mathfrak{F}_\mathrm{s}(A) \sqsubseteq \mathfrak{F}_\mathrm{s}(B)$.*

Proof. Assume $A \subseteq B$ and $a \in \mathcal{M}$ satisfies $\mathcal{M} \models a \epsilon \mathfrak{F}_\mathrm{s}(A)$. Then $a \in A \subseteq B$, whence $\mathcal{M} \models a \epsilon \mathfrak{F}_\mathrm{s}(B)$. Hence $\mathcal{M} \models \mathfrak{F}_\mathrm{s}(A) \sqsubseteq \mathfrak{F}_\mathrm{s}(B)$.

Conversely, assume $\mathcal{M} \models \mathfrak{F}_\mathrm{s}(A) \sqsubseteq \mathfrak{F}_\mathrm{s}(B)$. Let $a \in A$. Then $\mathcal{M} \models a \epsilon \mathfrak{F}_\mathrm{s}(A)$, whence $\mathcal{M} \models a \epsilon \mathfrak{F}_\mathrm{s}(B)$ and consequently $a \in B$. Thus $A \subseteq B$. □

Definition 3.2. Let $\mathcal{M} \models \mathrm{I}\Sigma_1$.

1. For every $m, d \in \mathcal{M}$ such that $\mathcal{M} \models \mathtt{Fcn}(m)$ and $\mathcal{M} \models [\mathtt{dom}(m) = d]$ we define the (function-)realisation $\mathfrak{R}_\mathrm{f}(m)_\mathcal{M} : \mathfrak{R}_\mathrm{s}(d)_\mathcal{M} \longrightarrow \mathcal{M}$ by $\mathcal{M} \models [\mathtt{apl}(m,a) = \mathfrak{R}_\mathrm{f}(m)_\mathcal{M}(a)]$ for all $a \in \mathfrak{R}_\mathrm{s}(d)$.

2. For $f : \mathcal{M} \rightharpoonup \mathcal{M}$ with $f = \mathfrak{R}_{\mathrm{f}}(m)_{\mathcal{M}}$ for some $m \in \mathcal{M}$ with $\mathcal{M} \models \mathtt{Fcn}(m)$, we write $\mathfrak{F}_{\mathrm{f}}(f)_{\mathcal{M}} = m$ and say f is *coded* in $\mathcal{M}$.

Again, this is meaningful by extensionality.

Remark 9. Note that, if $\mathcal{M} \models \mathrm{I}\Sigma_1$ and $f : \mathcal{M} \rightharpoonup \mathcal{M}$ is coded in $\mathcal{M}$, then $\mathrm{dom}(f)$ is coded in $\mathcal{M}$ and $\mathcal{M} \models [\mathtt{dom}(\mathfrak{F}_{\mathrm{f}}(f)) = \mathfrak{F}_{\mathrm{s}}(\mathrm{dom}(f))]$.

Lemma 3.3. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$. Then every finite $f : \mathcal{M} \rightharpoonup \mathcal{M}$ is coded in $\mathcal{M}$.*

Proof. Let $a_0, \dots, a_{k-1}$ be an enumeration of the domain of f and $g \in \mathcal{M}$ be such that $\mathcal{M} \models [\mathtt{fn}_k(a_0, \dots, a_{k-1}, f(a_1), \dots, f(a_{k-1})) = g]$. By Lemma 2.29 we have $\mathcal{M} \models \mathtt{Fcn}(g)$, $\mathcal{M} \models [\mathtt{dom}(\mathtt{g}) = \{a_0, \dots, a_{k-1}\}^k]$ and $\mathcal{M} \models [\mathtt{apl}(g, a_i) = f(a_i)]$ for each $i < k$. With the aid of the previous lemma, $g = \mathfrak{F}_{\mathrm{f}}(f)$. ☐

Remark 10. Since we assume $\mathbb{N} \subseteq_{\mathrm{e}} \mathcal{M}$, the above in particular applies to evaluations. Moreover, for sequences $s : \mathcal{M}_{<m} \longrightarrow \mathcal{M}$ the definition of $\mathfrak{F}_{\mathrm{f}}(s)$ is rather elegant: $\mathfrak{F}_{\mathrm{f}}(s) \in \mathcal{M}$ the unique element so that $\mathcal{M} \models \mathtt{Seq}(\mathfrak{F}_{\mathrm{f}}(s)) \land [\mathtt{lh}(\mathfrak{F}_{\mathrm{f}}(s)) = m]$ and $\mathcal{M} \models [(\mathfrak{F}_{\mathrm{f}}(s))_i = s(i)]$ for all $i \in \mathcal{M}_{<m}$.

Lemma 3.4. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ and $f : \mathbb{N} \rightharpoonup \mathcal{M}$ be coded in $\mathcal{M}$. Then f is finite.*

Proof. Assume f is infinite. Then $\mathcal{M} \models \exists \mathsf{x}(\mathsf{x} > \overline{n} \land [\mathsf{x} \epsilon \mathfrak{F}_{\mathrm{s}}(\mathrm{dom}(f))])$. Since the latter is a Σ_1-formula, there is an $a \in \mathcal{M} \setminus \mathbb{N}$ such that $\mathcal{M} \models \exists \mathsf{x}(\mathsf{x} > a \land [\mathsf{x} \epsilon \mathfrak{F}_{\mathrm{s}}(\mathrm{dom}(f))])$, by Σ_1-overspill (Lemma 2.24). Thus there is a $b \in \mathcal{M} \setminus \mathbb{N}$ such that $b \in \mathrm{dom}(f)$, which is absurd. ☐

Again, the above in particular applies to evaluations.

Corollary 3.5. *For every $n \in \mathbb{N}$ and $f : n \longrightarrow \mathbb{N}$*

$$\mathrm{I}\Sigma_1 \vdash \exists \mathsf{s}(\mathtt{Seq}(\mathsf{s}) \land [\mathtt{lh}(\mathsf{s}) = \overline{n}] \land \bigwedge\nolimits_{i<n} ([(\mathsf{s})_{\overline{i}} = \overline{f(i)}]) \land \\ \forall \mathsf{t}((\mathtt{Seq}(\mathsf{t}) \land [\mathtt{lh}(\mathsf{t}) = \overline{n}] \land \bigwedge\nolimits_{i<n} ([(\mathsf{t})_{\overline{i}} = \overline{f(i)}])) \to \mathsf{t} = \mathsf{s})).$$

Definition 3.3. Let $\mathcal{M} \models \mathrm{I}\Sigma_1$.

1. For every $m, d \in \mathcal{M}$ such that $\mathcal{M} \models \mathtt{Fcn}(m)$, $\mathcal{M} \models \forall [\mathsf{x} \epsilon \mathtt{dom}(m)][\mathtt{Set}(\mathtt{apl}(m, \mathsf{x}))]$ and $\mathcal{M} \models [\mathtt{dom}(m) = d]$, we define the (set-function-)realisation of m, $\mathfrak{R}_{\mathrm{sf}}(m)_{\mathcal{M}} : \mathfrak{R}_{\mathrm{s}}(d)_{\mathcal{M}} \longrightarrow \mathcal{P}(\mathcal{M})$, by $\mathfrak{R}_{\mathrm{sf}}(m)_{\mathcal{M}}(a) = \mathfrak{R}_{\mathrm{s}}(\mathfrak{R}_{\mathrm{f}}(m)_{\mathcal{M}}(a))_{\mathcal{M}}$ for all $a \in \mathfrak{R}_{\mathrm{s}}(\mathtt{dom}(m))$.

2. For $f : \mathcal{M} \rightharpoonup \mathcal{P}(\mathcal{M})$ with $f = \mathfrak{R}_{\mathrm{sf}}(m)_{\mathcal{M}}$ for some $m \in \mathcal{M}$ with $\mathcal{M} \models \mathtt{Fcn}(m)$ and $\mathcal{M} \models \forall [\mathsf{x} \epsilon \mathtt{dom}(m)][\mathtt{Set}(\mathtt{apl}(m, \mathsf{x}))]$, we write $\mathfrak{F}_{\mathrm{f}}(f)_{\mathcal{M}} = m$ and say f is *coded* in $\mathcal{M}$.

Once again, extensionality guarantees that the above definition is meaningful.

Remark 11. If $f : \mathcal{M} \rightharpoonup \mathcal{P}(\mathcal{M})$ is such that each $f(a) \subseteq \mathcal{M}$ for $a \in \mathrm{dom}(f)$ is coded in $\mathcal{M}$ and $g : \mathcal{M} \longrightarrow \mathcal{M}$ defined by $g(a) = \mathfrak{F}_{\mathrm{s}}(f(a))$ is coded in $\mathcal{M}$, then f is coded in $\mathcal{M}$.

Having shown that we in many cases can find elements of a model $\mathcal{M}$ which this "believes" to be certain external objects, we conclude this subsection with a number of technical results concerning these notions.

Lemma 3.6. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ and e be an evaluation in $\mathcal{M}$ coded in $\mathcal{M}$. Then $\mathcal{M} \models_e [\mathfrak{F}_\mathrm{f}(e)(\mathbf{v}_{\bar{i}}){=}\mathsf{v}_i]$ for all $i \in \mathbb{N}$ such that e is defined for v_i.*

Proof. We know that

$$\mathcal{M} \models [\mathtt{apl}(\mathfrak{F}_\mathrm{f}(e),\overline{\mathsf{v}_i}){=}e(\mathsf{v}_i)] \tag{27}$$

and

$$\mathbb{N} \models [\mathbf{v}_{\bar{i}}{=}\overline{\mathsf{v}_i}]$$

by definitions. The latter gives

$$\mathcal{M} \models [\mathbf{v}_{\bar{i}}{=}\overline{\mathsf{v}_i}] \tag{28}$$

by Σ_1-completeness. Now, let v_k be outside the domain of e.[9] Then $\mathcal{M} \models_{e^{\mathfrak{F}_\mathrm{f}(e)}_{\mathsf{v}_k}} [\mathtt{apl}(\mathsf{v}_k,\overline{\mathsf{v}_i}){=}\mathsf{v}_i]$ by (27), whence $\mathcal{M} \models_{e^{\mathfrak{F}_\mathrm{f}(e)}_{\mathsf{v}_k}} [\mathtt{apl}(\mathsf{v}_k,\mathbf{v}_{\bar{i}}){=}\mathsf{v}_i]$ by (28), which is exactly $\mathcal{M} \models_e [\mathtt{apl}(\mathfrak{F}_\mathrm{f}(e),\mathbf{v}_{\bar{i}}){=}\mathsf{v}_i]$. □

Lemma 3.7. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$, ν be a term or a formula of $\mathcal{L}_\mathrm{A}$ and $e : \mathbb{N} \rightharpoonup \mathcal{M}$ be coded in $\mathcal{M}$.[10] Then e is an evaluation for ν if and only if $\mathcal{M} \models \mathtt{Eval}(\mathfrak{F}_\mathrm{f}(e),\overline{\nu})$.*

Proof. We prove the case when ν is a formula. The proof when ν is a term is almost completely identical.

First assume e is an evaluation for ν and consider $\mathfrak{F}_\mathrm{f}(e)$. Let $a \in \mathcal{M}$ be such that $\mathcal{M} \models [a\epsilon\mathtt{dom}(\mathfrak{F}_\mathrm{f}(e))]$, that is $a \in \mathrm{dom}(e)$, which implies that $a \in \mathrm{Var}$. Thus $\mathcal{M} \models \forall[\mathsf{x}\epsilon\mathtt{dom}(\mathfrak{F}_\mathrm{f}(e))]\mathtt{Var}(\mathsf{x})$. Moreover, if $b \in \mathcal{M}$ is such that $\mathcal{M} \models [b\epsilon\mathtt{freevarF}(\overline{\nu})]$, then $b <_\mathcal{M} \overline{\nu}$ and so $b \in \mathbb{N}$, whence $\mathbb{N} \models [b\epsilon\mathtt{freevarF}(\overline{\nu})]$ by $\Delta_1(\mathrm{I}\Sigma_1)$-absoluteness, that is b is a free variable of ν. Thus $b \in \mathrm{dom}(e)$, whence $\mathcal{M} \models [b\epsilon\mathtt{dom}(\mathfrak{F}_\mathrm{f}(e))]$. So $\mathcal{M} \models \forall[\mathsf{x}\epsilon\mathtt{freevarF}(\overline{\nu})][\mathsf{x}\epsilon\mathtt{dom}(\mathfrak{F}_\mathrm{f}(e))]$. Hence $\mathcal{M} \models \mathtt{Eval}(\mathfrak{F}_\mathrm{f}(e),\overline{\nu})$.

Now assume $\mathcal{M} \models \mathtt{Eval}(\mathfrak{F}_\mathrm{f}(e),\overline{\nu})$. Let $x \in \mathrm{dom}(e)$. Then $\mathcal{M} \models [x\epsilon\mathtt{dom}(\mathfrak{F}_\mathrm{f}(e))]$, whence $\mathcal{M} \models \mathtt{Var}(x)$. By absoluteness we get $\mathbb{N} \models \mathtt{Var}(x)$, that is x is a variable. Finally let x be a free variable of ν; then $\mathbb{N} \models [\overline{\mathsf{x}}\epsilon\mathtt{freevarF}(\overline{\nu})]$. By absoluteness again, $\mathcal{M} \models [\overline{\mathsf{x}}\epsilon\mathtt{freevarF}(\overline{\nu})]$, whence $\mathcal{M} \models [\overline{\mathsf{x}}\epsilon\mathtt{dom}(\mathfrak{F}_\mathrm{f}(e))]$. Thus $\mathsf{x} \in \mathrm{dom}(e)$. Therefore e is an evaluation of ν in $\mathcal{M}$. □

[9] This is slightly technical, but in (27) the (suppressed) variable will depend (so as to be different from) v_i. Since Lemma 3.4 assures e is finite, this is not a restriction.

[10] We cannot assume $e : \mathcal{M} \rightharpoonup \mathcal{M}$ because there is no (Σ_1) formulation of $\mathtt{Eval}$ which ensures that $\mathcal{M} \models \mathtt{Eval}(\mathfrak{F}_\mathrm{f}(e),\overline{\nu})$ is true only for e defined only for *real* variables and not also for elements v of $\mathcal{M} \setminus \mathbb{N}$ satisfying $\mathcal{M} \models \mathtt{Var}(v)$, by Σ_1-overspill.

The next lemmas could be considered model-theoretic analogues of Lemma 2.39 and Lemma 2.40.

Lemma 3.8. *Let* $\mathcal{M} \models \mathrm{I}\Sigma_1$ *and* τ *a term. Then*

$$\mathcal{M} \models_e [\tau{=}\mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau})]$$

for all coded evaluations e *of* τ *in* $\mathcal{M}$.

Proof. By the previous lemma $\mathcal{M} \models_e \mathtt{Eval}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau})$. Let $k \in \mathbb{N}$ be such that the free variables of τ are among $\mathsf{v}_0, \ldots, \mathsf{v}_{k-1}$. By Lemma 2.39 we then have

$$\begin{aligned}\mathcal{M} \models &[\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\tau})]\wedge\\ &([\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\tau}){=}\tau]),\end{aligned}$$

whence in particular

$$\begin{aligned}\mathcal{M} \models_e &[\mathtt{Eval}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\tau})]\wedge\\ &([\mathtt{val}(\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}),\overline{\tau}){=}\tau]).\end{aligned}$$

Let $f \in \mathcal{M}$ be such that $\mathcal{M} \models_e [\mathtt{fn}_k(\mathbf{v}_{\overline{0}},\ldots,\mathbf{v}_{\overline{k-1}},\mathsf{v}_0,\ldots,\mathsf{v}_{k-1}){=}f]$. By Lemma 2.36

$$\begin{aligned}\mathcal{M} \models &(\mathtt{Term}(\overline{\tau})\wedge\mathtt{Eval}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau})\wedge\mathtt{Eval}(f,\overline{\tau})\wedge\\ &\forall[\mathsf{x}\epsilon\mathtt{freevarT}(\overline{\tau})][\mathtt{apl}(\mathfrak{F}_{\mathrm{f}}(e),\mathsf{x}){=}\mathtt{apl}(f,\mathsf{x})])\to\\ &[\mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau}){=}\mathtt{val}(f,\overline{\tau})],\end{aligned}$$

whence

$$\mathcal{M} \models \forall[\mathsf{x}\epsilon\mathtt{freevarT}(\overline{\tau})][\mathtt{apl}(\mathfrak{F}_{\mathrm{f}}(e),\mathsf{x}){=}\mathtt{apl}(f,\mathsf{x})])\to[\mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau}){=}\mathtt{val}(f,\overline{\tau})].$$

Now let v_i be a free variable of τ; then $i < k$ by assumption. Thus $\mathcal{M} \models [\mathtt{apl}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\mathsf{v}_i}){=}e(\mathsf{v}_i)]$ and $\mathcal{M} \models_e [\mathtt{apl}(f,\overline{\mathsf{v}_i}){=}\mathsf{v}_i]$ by construction. Since the latter is equivalent to $\mathcal{M} \models_{e\restriction\{\mathsf{v}_i\}}$ $[\mathtt{apl}(f,\overline{\mathsf{v}_i}){=}\mathsf{v}_i]$, which is exactly $\mathcal{M} \models [\mathtt{apl}(f,\overline{\mathsf{v}_i}){=}e(\mathsf{v}_i)]$, we get $\mathcal{M} \models [\mathtt{apl}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\mathsf{v}_i}){=}\mathtt{apl}(f,\overline{\mathsf{v}_i})]$. Thus $\mathcal{M} \models \forall[\mathsf{x}\epsilon\mathtt{freevarF}(\overline{\tau})][\mathtt{apl}(\mathfrak{F}_{\mathrm{f}}(e),\mathsf{x}){=}\mathtt{apl}(f,\mathsf{x})]$, whence

$$\mathcal{M} \models \mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau}){=}\mathtt{val}(f,\overline{\tau})$$

by above. Since $\mathcal{M} \models_e [\tau{=}\mathtt{val}(f,\overline{\tau})]$ we get

$$\mathcal{M} \models_e [\tau{=}\mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau})]$$

as desired. □

Lemma 3.9. *Let* $\mathcal{M} \models \mathrm{I}\Sigma_1$ *and* φ *be a* Δ_0*-formula. Then*

$$\mathcal{M} \models_e \varphi \Leftrightarrow \mathcal{M} \models \mathtt{Sat}_0(\mathfrak{F}_{\mathrm{f}}(e),\overline{\varphi})$$

for all evaluations e *of* φ *in* $\mathcal{M}$ *coded in* $\mathcal{M}$.

Proof. As in the previous lemma, let $k \in \mathbb{N}$ be such that the free variables of φ are among $\mathsf{v}_0, \ldots \mathsf{v}_{k-1}$ and $f \in \mathcal{M}$ satisfy $\mathcal{M} \models_e [\mathsf{f}{=}\mathtt{fn}_k(\mathbf{v}_{\overline{0}}, \ldots, \mathbf{v}_{\overline{k-1}}, \mathsf{v}_0, \ldots, \mathsf{v}_{k-1})]$. Then $\mathcal{M} \models_e \mathtt{Eval}(f, \overline{\varphi}) \wedge \mathtt{Sat}_0(f, \overline{\varphi})$ by Lemma 2.40, $\mathcal{M} \models \mathtt{Eval}(\mathfrak{F}_{\mathrm{f}}(e), \overline{\varphi})$ by Lemma 3.7, and

$$\begin{aligned}\mathcal{M} \models &(\Delta_0(\overline{\varphi}) \wedge \mathtt{Eval}(\mathfrak{F}_{\mathrm{f}}(e), \overline{\varphi}) \wedge \mathtt{Eval}(f, \overline{\varphi}) \wedge \\ &\quad \forall [\mathsf{x} \epsilon \mathtt{freevarF}(\overline{\varphi})][\mathtt{apl}(\mathfrak{F}_{\mathrm{f}}(e), \mathsf{x}) {=} \mathtt{apl}(f, \mathsf{x})]) \rightarrow \\ &\quad (\mathtt{Sat}_0(\mathfrak{F}_{\mathrm{f}}(e), \overline{\varphi}) \leftrightarrow \mathtt{Sat}_0(f, \overline{\varphi})),\end{aligned}$$

by Lemma 2.37. Moreover $\mathcal{M} \models \forall [\mathsf{x} \epsilon \mathtt{freevarF}(\overline{\varphi})][\mathtt{apl}(\mathfrak{F}_{\mathrm{f}}(e), \mathsf{x}) {=} \mathtt{apl}(f, \mathsf{x})]$ like in the previous lemma. Thus

$$\mathcal{M} \models \mathtt{Sat}_0(\mathfrak{F}_{\mathrm{f}}(e), \overline{\varphi}) \leftrightarrow \mathtt{Sat}_0(f, \overline{\varphi}).$$

Since $\mathcal{M} \models_e \varphi \leftrightarrow \mathtt{Sat}_0(f, \overline{\varphi})$ we get $\mathcal{M} \models_e \varphi \leftrightarrow \mathtt{Sat}_0(\mathfrak{F}_{\mathrm{f}}(e), \overline{\varphi})$. This is equivalent to

$$\mathcal{M} \models_e \varphi \Leftrightarrow \mathcal{M} \models \mathtt{Sat}_0(\mathfrak{F}_{\mathrm{f}}(e), \overline{\varphi})$$

as desired. □

Lemma 3.10. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ and $f : \mathcal{M} \rightharpoonup \mathcal{M}$ be coded in $\mathcal{M}$. Then f^b_a is coded in $\mathcal{M}$, indeed $\mathcal{M} \models [\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(f), a, b) {=} \mathfrak{F}_{\mathrm{f}}(f^b_a)]$ for all $a, b \in \mathcal{M}$.*

Proof. Let $a, b \in \mathcal{M}$. By definition $\mathfrak{F}_{\mathrm{f}}(f) \in \mathcal{M}$ is such that

$$\mathcal{M} \models \mathtt{Fcn}(\mathfrak{F}_{\mathrm{f}}(f)) \wedge [\mathtt{dom}(\mathfrak{F}_{\mathrm{f}}(f)) {=} \mathfrak{F}_{\mathrm{s}}(\mathrm{dom}(f))]$$

and $\mathcal{M} \models [\mathtt{apl}(\mathfrak{F}_{\mathrm{f}}(f), x) {=} f(x)]$ for all $x \in \operatorname{dom} f$. Since

$$\mathcal{M} \models [\mathtt{dom}(\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(f), a, b)) {=} \mathtt{dom}(\mathfrak{F}_{\mathrm{f}}(f)) \sqcup \{a\}],$$

we thus have

$$\mathcal{M} \models [\mathtt{dom}(\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(f), a, b)) {=} \mathfrak{F}_{\mathrm{s}}(\mathrm{dom}(f^b_a))].$$

Moreover,

$$\mathcal{M} \models [\mathtt{apl}(\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(f), a, b), x) {=} f(x)]$$

for all $x \in \mathrm{dom}(f) \setminus \{a\}$ and

$$\mathcal{M} \models [\mathtt{apl}(\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(f), a, b), a) {=} b],$$

whence

$$\mathcal{M} \models [\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(f), a, b) {=} \mathfrak{F}_{\mathrm{f}}(f^b_a)]$$

as desired. □

Corollary 3.11. *Let* $\mathcal{M} \models \mathrm{I}\Sigma_1$*,* $k \in \mathbb{N}$ *and* e *be an evaluation in* $\mathcal{M}$*. Then* $\mathcal{M} \models [\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(e),\mathsf{v}_{\overline{k}},b)=\mathfrak{F}_{\mathrm{f}}(e^{b}_{\mathsf{v}_k})]$.

Proof. This follows from the above lemma with $a = \mathsf{v}_k = \overline{\mathsf{v}_k}_{\mathcal{M}}$. □

We recall that the above results have corresponding formulations in terms of realisations of elements of a model $\mathcal{M}$. They will be used interchangeably henceforth.

Remark 12. The corresponding statement for $\sqsubset$ follows by extensionality.

Lemma 3.12. *Let* $\mathcal{M} \models \mathrm{I}\Sigma_1$*,* $D, R \subseteq \mathcal{M}$ *and* $f : \mathcal{M} \rightharpoonup \mathcal{M}$ *be coded in* $\mathcal{M}$*. Then* $\mathrm{dom}(f) \subseteq D$ *and* $\mathrm{ran}(f) \subseteq R$ *if and only if* $\mathcal{M} \models [\mathfrak{F}_{\mathrm{f}}(f)\sqsubseteq\mathfrak{F}_{\mathrm{s}}(D)\times\mathfrak{F}_{\mathrm{s}}(R)]$*.*

Proof. By assumption, if $a \in \mathrm{dom}(f)$ then $a \in D$ whence $\mathcal{M} \models a\epsilon\mathfrak{F}_{\mathrm{s}}(D)$, so $\mathcal{M} \models [\mathtt{dom}(\mathfrak{F}_{\mathrm{f}}(f))\sqsubseteq\mathfrak{F}_{\mathrm{s}}(D)]$. In the same way $\mathrm{ran}(f) \subseteq R$ implies $\mathcal{M} \models [\mathtt{ran}(\mathfrak{F}_{\mathrm{f}}(f))\sqsubseteq\mathfrak{F}_{\mathrm{s}}(R)]$. Hence $\mathcal{M} \models [\mathfrak{F}_{\mathrm{f}}(f)\sqsubseteq\mathfrak{F}_{\mathrm{s}}(D)\times\mathfrak{F}_{\mathrm{s}}(R)]$. □

Lemma 3.13. *Let* $\mathcal{M} \models \mathrm{I}\Sigma_1$*,* $l \in \mathcal{M}$ *and* $f : \mathcal{M}_{<l} \longrightarrow \mathcal{P}(\mathcal{M})$ *be coded in* $\mathcal{M}$*. Then* $\mathcal{M} \models \mathtt{Setq}(\mathfrak{F}_{\mathrm{sf}}(f))$*. Furthermore,* $\mathcal{M} \models \mathtt{Incrsetq}(\mathfrak{F}_{\mathrm{sf}}(f))$ *if and only if* f *is increasing.*

Proof. We know that $\mathcal{M} \models [\mathtt{dom}(\mathfrak{F}_{\mathrm{sf}}(f))=(<l)]$ and $\mathcal{M} \models [(\mathfrak{F}_{\mathrm{sf}}(f))_{\overline{i}}=\mathfrak{F}_{\mathrm{s}}(f(i))]$ for all $i \in \mathcal{M}_{<l}$. Thus $\mathcal{M} \models \mathtt{Seq}(\mathfrak{F}_{\mathrm{sf}}(f))\wedge\forall \mathsf{i}<l[\mathtt{Set}((\mathfrak{F}_{\mathrm{sf}}(f))_{\mathsf{i}})]$. Moreover, if f is increasing then $f(i) \subset f(\mathrm{S}_{\mathcal{M}}(i))$ for all $i, \mathrm{S}_{\mathcal{M}}(i) \in \mathcal{M}_{<l}$, whence $\mathcal{M} \models \mathfrak{F}_{\mathrm{s}}(f(i))\sqsubset\mathfrak{F}_{\mathrm{s}}(f(\mathrm{S}_{\mathcal{M}}(i)))$ for such i. Conversely, if $\mathcal{M} \models \forall[\mathsf{i}<l]([\mathsf{S}(\mathsf{i})<l]\rightarrow\mathfrak{F}_{\mathrm{s}}(f(i))\sqsubset\mathfrak{F}_{\mathrm{s}}(f(\mathrm{S}_{\mathcal{M}}(i))))$ then $f(i) \subset f(\mathrm{S}_{\mathcal{M}}(i))$ for all $i, \mathrm{S}_{\mathcal{M}}(i) \in \mathcal{M}_{<l}$. □

Remark 13. The corresponding result for $f : \mathcal{M}_{<l} \longrightarrow \mathcal{M}$ and $\mathtt{Incrseq}$ is immediate.

4 Initial Fulfilment: Incompleteness of PA

We first present a slight generalisation of the argument in [10] probably due to Quinsey, see [12]. The idea is Kripke's, though he does not seem to have published more than a remark on it, see [8].The proof does not seem to be directly generalisable to work for theories not extending PA (for theories in the language of arithmetic extending Q), since an apparently unbounded number of instances of collection is required for the argument in subsection 4.2 to go through.

4.1 Definitions

Definition 4.1 (Initial-fulfilment)**.** Let $\mathcal{L}$ be a discrete linear order. Given $\mathcal{M} \models \mathrm{PA}^-$ and an $\mathcal{L}$-sequence $s : \mathcal{L} \longrightarrow \mathcal{M}$, we define for formulae φ to be *initial-fulfilled* by s with respect to an assignment e of φ in $\mathcal{M}$ at $i \in \mathcal{L}$ recursively as follows:

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \models_e \vartheta \qquad \text{if } \vartheta \text{ is atomic,}$$

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \neg\vartheta \Leftrightarrow \forall j \in \mathcal{L}_{\geq i} : \mathcal{M} \overset{s,j,e}{\not\models_{\mathrm{I}}} \vartheta,$$

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \curlywedge \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \psi,$$

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \vee \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \curlyvee \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \psi,$$

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \forall \mathsf{x} \vartheta \Leftrightarrow \forall j \in \mathcal{L}_{\geq i} \forall a \in \mathcal{M}_{<s(j)} : \mathcal{M} \overset{s,j,e^a_x}{\models_{\mathrm{I}}} \vartheta,$$

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \exists \mathsf{x} \vartheta \Leftrightarrow (\mathrm{S}_{\mathcal{L}}(i) \in \mathcal{L} \Rightarrow \exists a \in \mathcal{M}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}}(i),e^a_x}{\models_{\mathrm{I}}} \vartheta).$$

Note that the information $\mathcal{L}$ is implicit in s.

If φ is a sentence then φ is simply *initial-fulfilled* by s in $\mathcal{M}$ if $\mathcal{M} \overset{s,0,\emptyset}{\models_{\mathrm{I}}} \varphi$. If there is an *increasing* $\mathcal{L}$-sequence s which initial-fulfils the sentence φ in $\mathcal{M}$, then φ is *initial-$\mathcal{L}$-fulfillable* in $\mathcal{M}$.

Some differences to the definitions in [10] and [12] should be pointed out. Putnam (in [10]) uses a non-recursive formulation in terms of games, restricted to Π_{2n}-formulae on "normal form" in the arithmetical hierarchy (that is a Δ_0-formula preceded by a sequence of even length of alternating quantifiers, beginning with $\forall$) only, which thus has no need to consider different starting positions in the sequence; focus is more on effectiveness. Quinsey's definition (in [12]), on the other hand, uses terminal segments instead of positions in the sequence in the recursive clauses; thus a formula of the form $\forall \mathsf{x}\varphi$ (for instance) is fulfilled by a sequence if φ is fulfilled by all terminal segments of the sequence. The present formulation is chosen partly to emphasise the similarity between fulfilment and Kripke models (in particular the clause for negation). Moreover, while Quinsey's definition has some advantage of exposition at the meta level, the present one seems easier to formalise in $\mathrm{I}\Sigma_1$.We will not benefit from the similarity with Kripke models, however, since in the relevant cases all formulae will be on negation normal form, and for atomic ϑ we have $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \neg\vartheta \Leftrightarrow \mathcal{M} \overset{s,i,e}{\not\models_{\mathrm{I}}} \vartheta$. This will be elaborated upon in the next lemma.

Lemma 4.1. *Let $\mathcal{M} \models \mathrm{PA}^-$, $\mathcal{L}$ be a discrete linear order and $s : \mathcal{L} \longrightarrow \mathcal{M}$ be a sequence. If s is increasing and φ is on negation normal form then $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \varphi \Rightarrow \mathcal{M} \overset{s,j,e}{\models_{\mathrm{I}}} \varphi$ for all $i \in \mathcal{L}$, $j \in \mathcal{L}_{\geq i}$ and all evaluations e of φ in $\mathcal{M}$. A converse holds if instead φ is quantifier free; indeed $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \varphi \Leftrightarrow \mathcal{M} \models_e \varphi$ for $i \in \mathcal{L}$ and evaluations e for such φ.*

Proof. We prove the second claim first, by induction on φ.

- If ϑ is atomic then $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \models_e \vartheta$ for all $i \in \mathcal{L}$ and evaluations e of φ by definition.
- Suppose $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \models_e \vartheta$ for all $i \in \mathcal{L}$ and evaluations e of φ. Then

$$\begin{aligned}\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \neg\vartheta &\Leftrightarrow \forall j \in \mathcal{L}_{\geq i} \mathcal{M} \overset{s,j,e}{\not\models_{\mathrm{I}}} \vartheta \\ &\Leftrightarrow \forall j \in \mathcal{L}_{\geq i} \mathcal{M} \not\models_e \vartheta \\ &\Leftrightarrow \mathcal{M} \not\models_e \vartheta\end{aligned}$$

  for all $i \in \mathcal{L}$ and evaluations e of φ, where the last equivalence holds since $\mathcal{L} \neq \emptyset$.

- Assume $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \models_e \vartheta$ for all $i \in \mathcal{L}$ and evaluations e of ϑ and similarly for ψ. Then

$$\begin{aligned}\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi &\Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \mathbin{\&} \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \psi \\ &\Leftrightarrow \mathcal{M} \models_e \vartheta \mathbin{\&} \mathcal{M} \models_e \psi \\ &\Leftrightarrow \mathcal{M} \models_e \vartheta \wedge \psi\end{aligned}$$

  for all $i \in \mathcal{L}$ and evaluations e of $\vartheta \wedge \psi$.

- The disjunctive case is similar.

To verify the first claim we proceed by induction on φ.

- If ϑ is a literal then

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \models_e \vartheta \Leftrightarrow \mathcal{M} \overset{s,j,e}{\models_{\mathrm{I}}} \vartheta$$

  for all $i \in \mathcal{L}$, $j \in \mathcal{L}_{\geq i}$ and evaluations e of ϑ by above.

- Assume $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \overset{s,j,e}{\models_{\mathrm{I}}} \vartheta$ and $\mathcal{M} \overset{s,i,f}{\models_{\mathrm{I}}} \psi \Rightarrow \mathcal{M} \overset{s,j,f}{\models_{\mathrm{I}}} \psi$ for all $i \in \mathcal{L}$, $j \in \mathcal{L}_{\geq i}$ and evaluations e and f of ϑ and ψ, respectively. Then

$$\begin{aligned} \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi &\Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \psi \\ &\Rightarrow \mathcal{M} \overset{s,j,e}{\models_{\mathrm{I}}} \vartheta \wedge \mathcal{M} \overset{s,j,e}{\models_{\mathrm{I}}} \psi \\ &\Leftrightarrow \mathcal{M} \overset{s,j,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi \end{aligned}$$

for all $i \in \mathcal{L}$, $j \in \mathcal{L}_{\geq i}$ and evaluations e of $\vartheta \wedge \psi$.

- The disjunctive case is similar.

- Consider $\forall \mathsf{x} \vartheta$. Then

$$\begin{aligned} \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \forall \mathsf{x} \vartheta &\Leftrightarrow \forall k \in \mathcal{L}_{\geq i} \forall a \in \mathcal{M}_{<s(k)} : \mathcal{M} \overset{s,k,e^{\mathsf{x}}_a}{\models_{\mathrm{I}}} \vartheta \\ &\Rightarrow \forall k \in \mathcal{L}_{\geq j} \forall a \in \mathcal{M}_{<s(k)} : \mathcal{M} \overset{s,k,e^{\mathsf{x}}_a}{\models_{\mathrm{I}}} \vartheta \\ &\Leftrightarrow \mathcal{M} \overset{s,j,e}{\models_{\mathrm{I}}} \forall \mathsf{x} \vartheta \end{aligned}$$

for all $i \in \mathcal{L}$, $j \in \mathcal{L}_{\geq i}$ and evaluations e of ϑ.

- Suppose $\mathcal{M} \overset{s,l,f}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta$ for all $l \in \mathcal{L}$, $k \in \mathcal{L}_{\geq l}$ and evaluations f of ϑ. Let $i \in \mathcal{L}$, $j \in \mathcal{L}_{\geq i}$ and e be an evaluation of $\exists \mathsf{x} \vartheta$. Then

$$\begin{aligned} \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \exists \mathsf{x} \vartheta &\Leftrightarrow (\mathrm{S}_{\mathcal{L}}(i) \in \mathcal{L} \Rightarrow \exists a \in \mathcal{M}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}}(i),e^{\mathsf{x}}_a}{\models_{\mathrm{I}}} \vartheta) \\ &\Rightarrow (\mathrm{S}_{\mathcal{L}}(j) \in \mathcal{L} \Rightarrow \exists a \in \mathcal{M}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}}(i),e^{\mathsf{x}}_a}{\models_{\mathrm{I}}} \vartheta) \\ &\Rightarrow (\mathrm{S}_{\mathcal{L}}(j) \in \mathcal{L} \Rightarrow \exists a \in \mathcal{M}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}}(j),e^{\mathsf{x}}_a}{\models_{\mathrm{I}}} \vartheta) \\ &\Rightarrow (\mathrm{S}_{\mathcal{L}}(j) \in \mathcal{L} \Rightarrow \exists a \in \mathcal{M}_{<s(\mathrm{S}_{\mathcal{L}}(j))} : \mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}}(j),e^{\mathsf{x}}_a}{\models_{\mathrm{I}}} \vartheta) \\ &\Leftrightarrow \mathcal{M} \overset{s,j,e}{\models_{\mathrm{I}}} \exists \mathsf{x} \vartheta. \end{aligned}$$

where the last implication (second last row) holds since s is increasing.

□

As it has already been remarked, the above lemma shows that for formulae on negation normal form, we could have used the definition

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \neg \vartheta \Leftrightarrow \mathcal{M} \overset{s,i,e}{\not\models_{\mathrm{I}}} \vartheta$$

of a negated formula being fulfilled by s at i.

We continue by showing some degree of independence of fulfilment on the length on the sequence and the particular model, just like we have just shown independence of the actual place in the sequence.

Lemma 4.2. *Let $\mathcal{M} \models \mathrm{PA}^-$, $\mathcal{L}$ be a discrete linear order and s an $\mathcal{L}$-sequence in $\mathcal{M}$. Let $\mathcal{L}' \subseteq \mathcal{L}$ be closed under the successor function in $\mathcal{L}$ and $s' = s \restriction \mathcal{L}'$. Then $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \varphi \Rightarrow \mathcal{M} \overset{s',i,e}{\models_{\mathrm{I}}} \varphi$ for every negation normal formula φ, all evaluations e of φ and all $i \in \mathcal{L}'$.*

Proof. The proof is by induction on φ.

- Let ϑ be a literal. Then

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \models_e \vartheta \Leftrightarrow \mathcal{M} \overset{s',i,e}{\models_{\mathrm{I}}} \vartheta$$

  for all evaluations e of ϑ and $i \in \mathcal{L}'$, by the previous lemma.

- Suppose ϑ and ψ are such that $\mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \overset{s',k,f}{\models_{\mathrm{I}}} \vartheta$ for all evaluations f of ϑ and $k \in \mathcal{L}'$, and similarly for ψ. Then

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \curlywedge \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \psi \Rightarrow \mathcal{M} \overset{s',i,e}{\models_{\mathrm{I}}} \varphi \curlywedge \mathcal{M} \overset{s',i,e}{\models_{\mathrm{I}}} \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi$$

  for all evaluations e of $\vartheta \wedge \psi$ and $i \in \mathcal{L}'$.

- The disjunctive case is similar.

- Assume ϑ is such that $\mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \overset{s',k,f}{\models_{\mathrm{I}}} \vartheta$ for all evaluations f of ϑ and $k \in \mathcal{L}'$. Then

$$\begin{aligned}
\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \forall \mathsf{x} \vartheta &\Leftrightarrow \forall j \in \mathcal{L}_{>i} \forall a \in \mathcal{M}_{<s(j)} : \mathcal{M} \overset{s,j,e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta \\
&\Rightarrow \forall j \in \mathcal{L}'_{>i} \forall a \in \mathcal{M}_{<s'(j)} : \mathcal{M} \overset{s,j,e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta \\
&\Rightarrow \forall j \in \mathcal{L}'_{>i} \forall a \in \mathcal{M}_{<s'(j)} : \mathcal{M} \overset{s',j,e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta \\
&\Leftrightarrow \mathcal{M} \overset{s',i,e}{\models_{\mathrm{I}}} \forall \mathsf{x} \vartheta
\end{aligned}$$

  for all evaluations e of $\forall \mathsf{x} \vartheta$ and $i \in \mathcal{L}'$.

- Let ϑ satisfy $\mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \overset{s',k,f}{\models_{\mathrm{I}}} \vartheta$ for all evaluations f of ϑ and $k \in \mathcal{L}'$. Let e be an evaluation of $\exists \mathsf{x} \vartheta$ and $i \in \mathcal{L}'$. Suppose $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \exists \mathsf{x} \vartheta$ and that i has a successor in $\mathcal{L}'$. Hence it is not maximal in $\mathcal{L}$, whence it has a successor there as well, and by assumption on the closedness of $\mathcal{L}'$, $\mathrm{S}_{\mathcal{L}'}(i) = \mathrm{S}_{\mathcal{L}}(i)$. Consequently there is an $a \in \mathcal{M}_{<s'(\mathrm{S}_{\mathcal{L}'}(i))}$ such that

$$\mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}'}(i),e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta.$$

By induction hypothesis

$$\mathcal{M} \overset{s',\mathrm{S}_{\mathcal{L}'}(i),e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta,$$

whence

$$\mathcal{M} \overset{s',i,e}{\models_{\mathrm{I}}} \exists \mathsf{x}\vartheta.$$

This concludes the induction, and the proof. □

Lemma 4.3. *Let $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M} \models \mathrm{PA}^-$, $\mathcal{L}$ be a discrete linear order and $s : \mathcal{L} \longrightarrow \mathcal{N}$ be a sequence. Then $\mathcal{N} \overset{s,i,e}{\models_{\mathrm{I}}} \varphi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \varphi$ for every formula φ, all evaluations e of φ in $\mathcal{N}$ and all $i \in \mathcal{L}$.*

Proof. The proof is by induction on φ.

- Let ϑ be an atomic formula. Then

$$\mathcal{N} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{N} \models_e \vartheta \Leftrightarrow \mathcal{M} \models_e \vartheta \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta$$

  for all evaluations e of ϑ in $\mathcal{N}$ and all $i \in \mathcal{L}$, by definition.

- Suppose ϑ satisfies $\mathcal{N} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta$ for all evaluations f of ϑ in $\mathcal{N}$ and all $k \in \mathcal{L}$. Then

$$\mathcal{N} \overset{s,i,e}{\models_{\mathrm{I}}} \neg\vartheta \Leftrightarrow \forall j \in \mathcal{L}_{\geq i} : \mathcal{N} \overset{s,j,e}{\not\models_{\mathrm{I}}} \vartheta \Leftrightarrow \forall j \in \mathcal{L}_{\geq i} : \mathcal{M} \overset{s,j,e}{\not\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \neg\vartheta$$

  for all evaluations e of $\neg\vartheta$ in $\mathcal{N}$ and $i \in \mathcal{L}$.

- Let ϑ and ψ be such that $\mathcal{N} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta$ for all evaluations f of ϑ in $\mathcal{N}$ and all $k \in \mathcal{L}$, and similarly for ψ. We have

$$\mathcal{N} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi \Leftrightarrow \mathcal{N} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \mathbin{\&} \mathcal{N} \overset{s,i,e}{\models_{\mathrm{I}}} \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \mathbin{\&} \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi$$

  for all evaluations e of $\vartheta \wedge \psi$ in $\mathcal{N}$ and $i \in \mathcal{L}$.

- The disjunctive case is similar.

- Assume ϑ is such that $\mathcal{N} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta$ for all evaluations f of ϑ in $\mathcal{N}$ and all $k \in \mathcal{L}$. Then

$$\begin{aligned}\mathcal{N} \overset{s,i,e}{\models_{\mathrm{I}}} \forall \mathsf{x}\vartheta &\Leftrightarrow \mathbb{V} j \in \mathcal{L}_{\geq i} \mathbb{V} a \in \mathcal{N}_{<s(j)} : \mathcal{N} \overset{s,j,e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta \\ &\Leftrightarrow \mathbb{V} j \in \mathcal{L}_{\geq i} \mathbb{V} a \in \mathcal{N}_{<s(j)} : \mathcal{M} \overset{s,j,e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta \\ &\Leftrightarrow \mathbb{V} j \in \mathcal{L}_{\geq i} \mathbb{V} a \in \mathcal{M}_{<s(j)} : \mathcal{M} \overset{s,j,e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta \\ &\Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \forall \mathsf{x}\vartheta\end{aligned}$$

for all evaluations e of $\forall \mathsf{x}\vartheta$ in $\mathcal{N}$ and all $i \in \mathcal{L}$, where the left to right direction of the third equivalence follows since $\mathcal{N}_{<b} = \mathcal{M}_{<b}$ for $b \in \mathcal{N}$ by $\mathcal{N} \subseteq_{\mathrm{e}} \mathcal{M}$.

- Let ϑ satisfy $\mathcal{N} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Leftrightarrow \mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta$ for all evaluations f of ϑ in $\mathcal{N}$ and all $k \in \mathcal{L}$. Then

$$\begin{aligned}\mathcal{N} \overset{s,i,e}{\models_{\mathrm{I}}} \exists \mathsf{x}\vartheta &\Leftrightarrow (\mathrm{S}_{\mathcal{L}}(i) \in \mathcal{L} \Rightarrow \mathbb{E} a \in \mathcal{N}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{N} \overset{s,\mathrm{S}_{\mathcal{L}}(i),e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta) \\ &\Leftrightarrow (\mathrm{S}_{\mathcal{L}}(i) \in \mathcal{L} \Rightarrow \mathbb{E} a \in \mathcal{N}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}}(i),e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta) \\ &\Leftrightarrow (\mathrm{S}_{\mathcal{L}}(i) \in \mathcal{L} \Rightarrow \mathbb{E} a \in \mathcal{M}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}}(i),e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta) \\ &\Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \exists \mathsf{x}\vartheta\end{aligned}$$

for all evaluations e of $\exists \mathsf{x}\vartheta$ in $\mathcal{N}$ and all $i \in \mathcal{L}$.

This concludes the induction, and the proof. □

Having now derived the basic properties of fulfilment we will turn to the corresponding notions inside $\mathrm{I}\Sigma_1$; with some care we can make essentially the same definitions work, and show them to yield the same result for coded sequences. With this formalised notion of fulfilment we will be able to construct a sentence independent of a given theory T. Similarly to how the formalisation of provability is used to construct an independent sentence in the proof of Gödel's Second Incompleteness Theorem, this sentence will express that T has a certain correctness property (in Gödel's theorem it is consistency, here it is fulfillability of the axioms by arbitrarily long sequences).

Proposition 4.4. *There is a* $\Delta_1(I\Sigma_1)$*-formula* $\mathtt{IFulf}$ *with the following properties:*

$$\begin{aligned}
I\Sigma_1 \vdash\ & (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\sqsubseteq(<n)\times(<s)]\wedge\mathtt{Term}(r)\wedge\mathtt{Term}(t)\wedge[\mathtt{Eval}(e,(r{=}t))]\wedge[(r{=}t)\leq n])\to\\
& ([\mathtt{IFulf}(s,i,e,n,(r{=}t))]\leftrightarrow([\mathtt{Eval}(e,(r{=}t))]\wedge[\mathtt{val}(e,r)=\mathtt{val}(e,t)]))\\
I\Sigma_1 \vdash\ & (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\sqsubseteq(<n)\times(<s)]\wedge\mathtt{Term}(r)\wedge\mathtt{Term}(t)\wedge[\mathtt{Eval}(e,(r{=}t))]\wedge[(r{<}t)\leq n])\to\\
& ([\mathtt{IFulf}(s,i,e,n,(r{<}t))]\leftrightarrow([\mathtt{Eval}(e,(r{<}t))]\wedge[\mathtt{val}(e,r)<\mathtt{val}(e,t)]))\\
I\Sigma_1 \vdash\ & (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\sqsubseteq(<n)\times(<s)]\wedge\mathtt{Fmla}(f)\wedge[\mathtt{Eval}(e,(\neg f))]\wedge[(\neg f)\leq n])\to\\
& ([\mathtt{IFulf}(s,i,e,n,(\neg f))]\leftrightarrow\forall[j<\mathtt{lh}(s)](i\leq j\to\neg\mathtt{IFulf}(s,j,e,n,f)))\\
I\Sigma_1 \vdash\ & (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\sqsubseteq(<n)\times(<s)]\wedge\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g)\wedge[\mathtt{Eval}(e,(f\wedge g))]\wedge[(f\wedge g)\leq n])\to\\
& ([\mathtt{IFulf}(s,i,e,n,(f\wedge g))]\leftrightarrow(\mathtt{IFulf}(s,i,e,n,f)\wedge\mathtt{IFulf}(s,i,e,n,g)))\\
I\Sigma_1 \vdash\ & (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\sqsubseteq(<n)\times(<s)]\wedge\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g)\wedge[\mathtt{Eval}(e,(f\vee g))]\wedge[(f\vee g)\leq n])\to\\
& ([\mathtt{IFulf}(s,i,e,n,(f\vee g))]\leftrightarrow(\mathtt{IFulf}(s,i,e,n,f)\vee\mathtt{IFulf}(s,i,e,n,g)))\\
I\Sigma_1 \vdash\ & (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\sqsubseteq(<n)\times(<s)]\wedge\mathtt{Fmla}(f)\wedge[\mathtt{Eval}(e,(\forall\mathbf{v}_x f))]\wedge[(\forall\mathbf{v}_x f)\leq n])\to\\
& ([\mathtt{IFulf}(s,i,e,n,(\forall\mathbf{v}_x f))]\leftrightarrow\forall[j<\mathtt{lh}(s)](i\leq j\to\forall[a<(s)_j][\mathtt{IFulf}(s,j,\mathtt{ext}(e,\mathbf{v}_x,a),n,f)]))\\
I\Sigma_1 \vdash\ & (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\sqsubseteq(<n)\times(<s)]\wedge\mathtt{Fmla}(f)\wedge[\mathtt{Eval}(e,(\exists\mathbf{v}_x f))]\wedge[(\exists\mathbf{v}_x f)\leq n])\to\\
& ([\mathtt{IFulf}(s,i,e,n,(\exists\mathbf{v}_x f))]\leftrightarrow([S(i)<\mathtt{lh}(s)]\to\exists[a<(s)_{S(i)}][\mathtt{IFulf}(s,S(i),\mathtt{ext}(e,\mathbf{v}_x,a),n,f)])).
\end{aligned}$$

Proof. The proof is rather lengthy and divided into several parts. The point is that we for each formula, sequence et cetera construct a sequence containing the information whether the formula is fulfilled for (at least) all admissible values of the evaluation and index of the sequence. We do this by introducing a number of auxiliary $I\Sigma_1$-provably $\Delta_1(I\Sigma_1)$-functions in three steps, at each step constructing new functions by recursion from the ones at the previous step. First, $c_=$, $c_<$, $c_\neg$, $c_\wedge$, $c_\vee$, $c_\forall$ and $c_\exists$ are the $I\Sigma_1$-provably $\Delta_1(I\Sigma_1)$-functions which satisfy:

$$\begin{aligned}
I\Sigma_1 \vdash\ & ([\mathtt{val}(e,r)=\mathtt{val}(e,t)]\to[c_=(e,r,t)=\overline{1}])\wedge(\neg[\mathtt{val}(e,r)=\mathtt{val}(e,t)]\to[c_=(e,r,t)=0]),\\
I\Sigma_1 \vdash\ & ([\mathtt{val}(e,r)<\mathtt{val}(e,t)]\to[c_<(e,r,t)=\overline{1}])\wedge(\neg[\mathtt{val}(e,r)<\mathtt{val}(e,t)]\to[c_<(e,r,t)=0]),\\
I\Sigma_1 \vdash\ & (\forall[j<\mathtt{lh}(s)](i\leq j\to[(q)_{\langle j,e\rangle}=0])\to[c_\neg(s,i,e,q)=\overline{1}])\wedge\\
& (\neg\forall[j<\mathtt{lh}(s)](i\leq j\to[(q)_{\langle j,e\rangle}=0])\to[c_\neg(s,i,e,q)=0]),\\
I\Sigma_1 \vdash\ & ((a=\overline{1}\wedge b=\overline{1})\to[c_\wedge(a,b)=\overline{1}])\wedge(\neg(a=\overline{1}\wedge b=\overline{1})\to[c_\wedge(a,b)=0]),\\
I\Sigma_1 \vdash\ & ((a=\overline{1}\vee b=\overline{1})\to[c_\vee(a,b)=\overline{1}])\wedge(\neg(a=\overline{1}\vee b=\overline{1})\to[c_\vee(a,b)=0]),\\
I\Sigma_1 \vdash\ & ((\forall[j<\mathtt{lh}(s)](i\leq j\to\forall[a<(s)_j][(q)_{\langle j,\mathtt{ext}(e,\mathbf{v}_x,a)\rangle}=\overline{1}]))\to[c_\forall(s,i,e,x,q)=\overline{1}])\wedge\\
& (\neg\forall[j<\mathtt{lh}(s)](i\leq j\to\forall[a<(s)_j][(q)_{\langle j,\mathtt{ext}(e,\mathbf{v}_x,a)\rangle}=\overline{1}])\to[c_\forall(s,i,e,x,q)=0]),\\
I\Sigma_1 \vdash\ & (([S(i)<\mathtt{lh}(s)]\to\exists[a<(s)_{S(i)}][(q)_{\langle S(i),\mathtt{ext}(e,\mathbf{v}_x,a)\rangle}=\overline{1}])\to[c_\exists(s,i,e,x,q)=\overline{1}])\wedge\\
& (\neg([S(i)<\mathtt{lh}(s)]\to\exists[a<(s)_{S(i)}][(q)_{\langle S(i),\mathtt{ext}(e,\mathbf{v}_x,a)\rangle}=\overline{1}])\to[c_\exists(s,i,e,x,q)=0]).
\end{aligned}$$

Next we introduce $\mathtt{tseq}_=$, $\mathtt{tseq}_<$, $\mathtt{tseq}_\neg$, $\mathtt{tseq}_\wedge$, $\mathtt{tseq}_\vee$, $\mathtt{tseq}_\forall$ and $\mathtt{tseq}_\exists$ by stipulating that:

$$\begin{aligned}
&I\Sigma_1 \vdash [\mathtt{tseq}_{=}(0,r,t)=\varnothing]\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{=}(S(l),r,t)=\mathtt{ext}(\mathtt{tseq}_{=}(l,r,t),l,c_{=}(\langle l\rangle_r,r,t))],\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{<}(0,r,t)=\varnothing]\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{<}(S(l),r,t)=\mathtt{ext}(\mathtt{tseq}_{=}(l,r,t),l,c_{<}(\langle l\rangle_r,r,t))],\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\neg}(s,0,q)=\varnothing]\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\neg}(s,S(l),q)=\mathtt{ext}(\mathtt{tseq}_{\neg}(s,l,q),l,c_{\neg}(s,\langle l\rangle_l,\langle l\rangle_r,q))],\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\wedge}(0,q,w)=\varnothing]\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\wedge}(S(l),q,w)=\mathtt{ext}(\mathtt{tseq}_{\wedge}(l,q,w),l,c_{\wedge}((q)_l,(w)_l))],\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\wedge}(0,q,w)=\varnothing]\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\vee}(S(l),q,w)=\mathtt{ext}(\mathtt{tseq}_{\vee}(l,q,w),l,c_{\vee}((q)_l,(w)_l))],\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\forall}(s,0,x,q)=\varnothing]\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\forall}(s,S(l),x,q)=\mathtt{ext}(\mathtt{tseq}_{\forall}(s,l,x,q),l,c_{\forall}(s,\langle l\rangle_l,\langle l\rangle_r,x,q))],\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\exists}(s,0,x,q)=\varnothing]\\
&I\Sigma_1 \vdash [\mathtt{tseq}_{\exists}(s,S(l),x,q)=\mathtt{ext}(\mathtt{tseq}_{\exists}(s,l,x,q),l,c_{\exists}(s,\langle l\rangle_l,\langle l\rangle_r,x,q))].
\end{aligned}$$

Now we can define `iful` as the $I\Sigma_1$-provably Δ_1-function which satisfies:

$$\begin{aligned}
&I\Sigma_1 \vdash (\mathtt{Term}(r)\wedge\mathtt{Term}(t))\rightarrow[\mathtt{iful}(s,n,(r{=}t))=\mathtt{tseq}_{=}(\langle\mathtt{lh}(s),S((<n)\times(<s))\rangle,r,t)]\\
&I\Sigma_1 \vdash (\mathtt{Term}(r)\wedge\mathtt{Term}(t))\rightarrow[\mathtt{iful}(s,n,(r{<}t))=\mathtt{tseq}_{<}(\langle\mathtt{lh}(s),S((<n)\times(<s))\rangle,r,t)]\\
&I\Sigma_1 \vdash \mathtt{Fmla}(f)\rightarrow[\mathtt{iful}(s,n,(\neg f))=\mathtt{tseq}_{\neg}(s,\langle\mathtt{lh}(s),S((<n)\times(<s))\rangle,\mathtt{iful}(s,n,f))]\\
&I\Sigma_1 \vdash (\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g))\rightarrow\\
&\qquad[\mathtt{iful}(s,n,(f\wedge g))=\mathtt{tseq}_{\wedge}(\langle\mathtt{lh}(s),S((<n)\times(<s))\rangle,\mathtt{iful}(s,n,f),\mathtt{iful}(s,n,g))]\\
&I\Sigma_1 \vdash (\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g))\rightarrow\\
&\qquad[\mathtt{iful}(s,n,(f\vee g))=\mathtt{tseq}_{\vee}(\langle\mathtt{lh}(s),S((<n)\times(<s))\rangle,\mathtt{iful}(s,n,f),\mathtt{iful}(s,n,g))]\\
&I\Sigma_1 \vdash \mathtt{Fmla}(f)\rightarrow[\mathtt{iful}(s,n,(\forall \mathbf{v}_x f))=\mathtt{tseq}_{\forall}(s,\langle\mathtt{lh}(s),S((<n)\times(<s))\rangle,x,\mathtt{iful}(s,n,f))]\\
&I\Sigma_1 \vdash \mathtt{Fmla}(f)\rightarrow[\mathtt{iful}(s,n,(\exists \mathbf{v}_x f))=\mathtt{tseq}_{\exists}(s,\langle\mathtt{lh}(s),S((<n)\times(<s))\rangle,x,\mathtt{iful}(s,n,f))].
\end{aligned}$$

Since it is defined by composition and primitive recursion from $I\Sigma_1$-provably Δ_1-functions, `iful` is an $I\Sigma_1$-provably Δ_1-function. We will show that `iful` has the following properties:

$$\begin{aligned}
&I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge\mathtt{Fmla}(f))\rightarrow\\
&\qquad([\mathtt{Seq}(\mathtt{iful}(s,n,f))]\wedge[\mathtt{lh}(\mathtt{iful}(s,n,f))=\langle\mathtt{lh}(s),S((<n)\times(<s))\rangle]\wedge \qquad (29)\\
&\qquad\qquad\forall[i<\mathtt{lh}(s)]\forall[e\le(<n)\times(<s)][(\mathtt{iful}(s,n,f))_{\langle i,e\rangle}<\overline{2}])\\
&I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\le(<n)\times(<s)]\wedge\mathtt{Term}(r)\wedge\mathtt{Term}(t))\rightarrow \qquad (30)\\
&\qquad([(\mathtt{iful}(s,n,t{=}r))_{\langle i,e\rangle}=\overline{1}]\leftrightarrow[\mathtt{val}(e,r)=\mathtt{val}(e,t)])\\
&I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\le(<n)\times(<s)]\wedge\mathtt{Term}(r)\wedge\mathtt{Term}(t))\rightarrow \qquad (31)
\end{aligned}$$

$$([(\mathtt{iful}(s,n,t{<}r))_{\langle i,e\rangle}{=}\overline{1}]\leftrightarrow[\mathtt{val}(e,r)<\mathtt{val}(e,t)])$$

$$I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\leq(<n)\times(<s)]\wedge\mathtt{Fmla}(f))\rightarrow \quad (32)$$
$$([(\mathtt{iful}(s,n,(\neg f)))_{\langle i,e\rangle}{=}\overline{1}]\leftrightarrow\forall[j<\mathtt{lh}(s)](i\leq j\rightarrow[(\mathtt{iful}(s,n,f))_{\langle j,e\rangle}{=}0]))$$

$$I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\leq(<n)\times(<s)]\wedge\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g))\rightarrow \quad (33)$$
$$([(\mathtt{iful}(s,n,(f\wedge g)))_{\langle i,e\rangle}{=}\overline{1}]\leftrightarrow([(\mathtt{iful}(s,n,f))_{\langle i,e\rangle}{=}\overline{1}]\wedge[(\mathtt{iful}(s,n,g))_{\langle i,e\rangle}{=}\overline{1}]))$$

$$I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\leq(<n)\times(<s)]\wedge\mathtt{Fmla}(f)\wedge\mathtt{Fmla}(g))\rightarrow \quad (34)$$
$$([(\mathtt{iful}(s,n,(f\vee g)))_{\langle i,e\rangle}{=}\overline{1}]\leftrightarrow([(\mathtt{iful}(s,n,f))_{\langle i,e\rangle}{=}\overline{1}]\vee[(\mathtt{iful}(s,n,g))_{\langle i,e\rangle}{=}\overline{1}]))$$

$$I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\leq(<n)\times(<s)]\wedge\mathtt{Fmla}(f))\rightarrow$$
$$([(\mathtt{iful}(s,n,(\forall \mathbf{v}_x f)))_{\langle i,e\rangle}{=}\overline{1}]\leftrightarrow \quad (35)$$
$$\forall[j<\mathtt{lh}(s)](i\leq j\rightarrow\forall[a<(s)_j][(\mathtt{iful}(s,n,f))_{\langle j,\mathtt{ext}(e,\mathbf{v}_x,a)\rangle}{=}\overline{1}]))$$

$$I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge[i<\mathtt{lh}(s)]\wedge[e\leq(<n)\times(<s)]\wedge\mathtt{Fmla}(f))\rightarrow$$
$$([(\mathtt{iful}(s,n,(\exists \mathbf{v}_x f)))_{\langle i,e\rangle}{=}\overline{1}]\leftrightarrow \quad (36)$$
$$([S(i)<\mathtt{lh}(s)]\rightarrow\exists[a<(s)_{S(i)}][(\mathtt{iful}(s,n,f))_{\langle S(i),\mathtt{ext}(e,\mathbf{v}_x,a)\rangle}{=}\overline{1}])).$$

To this end, we begin by verifying that each of the functions tseq satisfies the corresponding properties. First we argue in $I\Sigma_1$ to show that

$$I\Sigma_1 \vdash (\mathtt{Seq}(s)\wedge[\mathtt{lh}(s){=}l]\wedge x<\overline{2}\wedge\forall i<l[(s)_i<\overline{2}])\rightarrow \quad (37)$$
$$([\mathtt{Seq}(\mathtt{ext}(s,l,x))]\wedge[\mathtt{lh}(\mathtt{ext}(s,l,x)){=}S(l)]\wedge\forall i\leq l[(\mathtt{ext}(s,l,x))_i<\overline{2}]).$$

Thus suppose $\mathtt{Seq}(s)\wedge[\mathtt{lh}(s){=}l]\wedge x<\overline{2}\wedge\forall i<l[(s)_i<\overline{2}]$ and consider $\mathtt{ext}(s,l,x)$. We have $[\mathtt{Fcn}(\mathtt{ext}(s,l,x))]$, $[\mathtt{dom}(\mathtt{ext}(s,l,x)){=}\mathtt{dom}(s)\sqcup\{l\}]$, $\forall i<l[\mathtt{apl}(\mathtt{ext}(s,l,x),i){=}\mathtt{apl}(s,i)]$ and $[\mathtt{apl}(\mathtt{ext}(s,l,x),l){=}x]$ by Lemma 2.28. Consequently

$$\forall i\leq l[(\mathtt{ext}(s,l,x))_i<\overline{2}]$$

and, since $[(<l)\sqcup\{l\}{=}(<S(l))]$, also

$$[\mathtt{Seq}(\mathtt{ext}(s,l,x))]\wedge[\mathtt{lh}(\mathtt{ext}(s,l,x)){=}S(l)]$$

by Lemma 2.26.

Next we use (37) inductively for each of the functions tseq.

- We reason in $I\Sigma_1$ to show that

$$I\Sigma_1 \vdash [\mathtt{Seq}(\mathtt{tseq}_{=}(l,r,t))]\wedge[\mathtt{lh}(\mathtt{tseq}_{=}(l,r,t)){=}l]\wedge$$
$$\forall k<l([(\mathtt{tseq}_{=}(l,r,t))_k<\overline{2}]\wedge \quad (38)$$
$$([(\mathtt{tseq}_{=}(l,r,t))_k{=}\overline{1}]\leftrightarrow[\mathtt{val}(\langle k\rangle_r,r){=}\mathtt{val}(\langle k\rangle_r,t)]))$$

  by Σ_1-induction on l.

– Since $[\mathtt{tseq}_{=}(0,r,t)=\varnothing]$ we have

$$[\mathtt{Seq}(\mathtt{tseq}_{=}(0,r,t))],\ [\mathtt{lh}(\mathtt{tseq}_{=}(0,r,t))=0]$$

and

$$\begin{aligned}&\forall k{<}0([(\mathtt{tseq}_{=}(0,r,t))_i{<}\overline{2}]\wedge\\&\quad([(\mathtt{tseq}_{=}(0,r,t))_k{=}\overline{1}]\leftrightarrow[\mathtt{val}(\langle k\rangle_r,r){=}\mathtt{val}(\langle k\rangle_r,t)]))\end{aligned}$$

vacuously, by Lemma 2.26.

– Suppose

$$\begin{aligned}&[\mathtt{Seq}(\mathtt{tseq}_{=}(l,r,t))]\wedge[\mathtt{lh}(\mathtt{tseq}_{=}(l,r,t)){=}l]\wedge\\&\forall k{<}l([(\mathtt{tseq}_{=}(l,r,t))_k{<}\overline{2}]\wedge\\&\quad([(\mathtt{tseq}_{=}(l,r,t))_k{=}\overline{1}]\leftrightarrow[\mathtt{val}(\langle k\rangle_r,r){=}\mathtt{val}(\langle k\rangle_r,t)])).\end{aligned}$$

Then since $[\mathtt{tseq}_{=}(S(l),r,t){=}\mathtt{ext}(\mathtt{tseq}_{=}(l,r,t),l,c_{=}(\langle l\rangle_r,r,t))]$ we have

$$\begin{aligned}&[\mathtt{Seq}(\mathtt{tseq}_{=}(S(l),r,t))]\wedge[\mathtt{lh}(\mathtt{tseq}_{=}(S(l),r,t)){=}S(l)]\wedge\\&\forall k{<}S(l)[(\mathtt{tseq}_{=}(S(l),f))_k{<}\overline{2}]\end{aligned}$$

by definition of $c_{=}$ and (37). Now let $k < S(l)$; then $k < l$ or $k = l$. In the first case

$$[(\mathtt{tseq}_{=}(S(l),r,t))_k{=}\overline{1}]\leftrightarrow[\mathtt{val}(\langle k\rangle_r,r){=}\mathtt{val}(\langle k\rangle_r,t)]$$

by Lemma 2.28 and induction hypothesis. In the latter case we have that $[(\mathtt{tseq}_{=}(S(l),r,t))_k{=}c_{=}(\langle l\rangle_r,r,t)]$, whereby

$$[(\mathtt{tseq}_{=}(S(l),r,t))_k{=}\overline{1}]\leftrightarrow[\mathtt{val}(\langle k\rangle_r,r){=}\mathtt{val}(\langle r\rangle_1,t)]$$

by definition of $c_{=}$.

This concludes the induction and verifies (38).

- By Σ_1-induction on l we also have

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash&[\mathtt{Seq}(\mathtt{tseq}_{<}(l,r,t))]\wedge[\mathtt{lh}(\mathtt{tseq}_{<}(l,r,t)){=}l]\wedge\\&\forall k{<}l([(\mathtt{tseq}_{<}(l,r,t))_k{<}\overline{2}]\wedge\\&\quad([(\mathtt{tseq}_{<}(l,r,t))_k{=}\overline{1}]\leftrightarrow[\mathtt{val}(\langle k\rangle_r,r){<}\mathtt{val}(\langle k\rangle_r,t)]))\end{aligned}\tag{39}$$

in essentially the same way as for (38), whence we omit the proof.

- We reason in $\mathrm{I}\Sigma_1$ to show that

$$\begin{aligned}\mathrm{I}\Sigma_1 \vdash&[\mathtt{Seq}(\mathtt{tseq}_{\neg}(s,l,q))]\wedge[\mathtt{lh}(\mathtt{tseq}_{\neg}(s,l,q)){=}l]\wedge\\&\forall k{<}l([(\mathtt{tseq}_{\neg}(s,l,q))_k{<}\overline{2}]\wedge\\&\quad([(\mathtt{tseq}_{\neg}(s,l,q))_k{=}\overline{1}]\leftrightarrow\forall[j{<}\mathtt{lh}(s)]([\langle k\rangle_1{\le}j]\to[(q)_{\langle j,\langle k\rangle_r\rangle}{=}0])))\end{aligned}\tag{40}$$

by Σ_1-induction on l.

– Since $[\mathtt{tseq}_\neg(\mathsf{s},0,\mathsf{q})=\varnothing]$,

$$[\mathsf{Seq}(\mathtt{tseq}_\neg(\mathsf{s},0,\mathsf{q}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\neg(\mathsf{s},0,\mathsf{q}))=0]\wedge$$
$$\forall\mathsf{k}<0([(\mathtt{tseq}_\neg(\mathsf{s},0,\mathsf{q}))_\mathsf{k}<\overline{2}]\wedge$$
$$([(\mathtt{tseq}_\neg(\mathsf{s},0,\mathsf{q}))_\mathsf{k}=\overline{1}]\leftrightarrow\forall[\mathsf{j}<\mathtt{lh}(\mathsf{s})]([\langle\mathsf{k}\rangle_1\leq\mathsf{j}]\rightarrow[(\mathsf{q})_{\langle\mathsf{j},\langle\mathsf{k}\rangle_\mathrm{r}\rangle}=0])))$$

holds as for $\mathtt{tseq}_=$.

– Suppose

$$[\mathsf{Seq}(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{l},\mathsf{q}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{l},\mathsf{q}))=\mathsf{l}]\wedge$$
$$\forall\mathsf{k}<\mathsf{l}([(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{l},\mathsf{q}))_\mathsf{k}<\overline{2}]\wedge$$
$$([(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{l},\mathsf{q}))_\mathsf{k}=\overline{1}]\leftrightarrow\forall[\mathsf{j}<\mathtt{lh}(\mathsf{s})]([\langle\mathsf{k}\rangle_1\leq\mathsf{j}]\rightarrow[(\mathsf{q})_{\langle\mathsf{j},\langle\mathsf{k}\rangle_\mathrm{r}\rangle}=0]))).$$

By definition $[\mathtt{tseq}_\neg(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{q})=\mathtt{ext}(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{l},\mathsf{q}),\mathsf{l},\mathsf{c}_\neg(\mathsf{s},\langle\mathsf{l}\rangle_1,\langle\mathsf{l}\rangle_\mathrm{r},\mathsf{q}))]$ which together with $[\mathsf{c}_\neg(\mathsf{s},\langle\mathsf{l}\rangle_1,\langle\mathsf{l}\rangle_\mathrm{r},\mathsf{q})<\overline{2}]$ implies

$$[\mathsf{Seq}(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{q}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{q}))=\mathsf{S}(\mathsf{l})]\wedge$$
$$\forall\mathsf{k}<\mathsf{S}(\mathsf{l})[(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{q}))_\mathsf{k}<\overline{2}]$$

by (37). Now take $\mathsf{k}<\mathsf{S}(\mathsf{l})$. If $\mathsf{k}<\mathsf{l}$ then

$$[(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{q}))_\mathsf{k}=\overline{1}]\leftrightarrow\forall[\mathsf{j}<\mathtt{lh}(\mathsf{s})]([\langle\mathsf{k}\rangle_1\leq\mathsf{j}]\rightarrow[(\mathsf{q})_{\langle\mathsf{j},\langle\mathsf{k}\rangle_\mathrm{r}\rangle}=0])$$

by induction hypothesis. If $\mathsf{k}=\mathsf{l}$ then $[(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{q}))_\mathsf{k}=\mathsf{c}_\neg(\mathsf{s},\langle\mathsf{k}\rangle_1,\langle\mathsf{k}\rangle_\mathrm{r},\mathsf{q})]$ on the other hand, whence

$$[(\mathtt{tseq}_\neg(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{q}))_\mathsf{k}=\overline{1}]\leftrightarrow\forall[\mathsf{j}<\mathtt{lh}(\mathsf{s})]([\langle\mathsf{k}\rangle_1\leq\mathsf{j}]\rightarrow[(\mathsf{q})_{\langle\mathsf{j},\langle\mathsf{k}\rangle_\mathrm{r}\rangle}=0])$$

by definition of $\mathsf{c}_\neg$.

- We reason in $\mathrm{I}\Sigma_1$ to show that

$$\mathrm{I}\Sigma_1\vdash[\mathsf{Seq}(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{w}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{w}))=\mathsf{l}]\wedge$$
$$\forall\mathsf{k}<\mathsf{l}([(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{w}))_\mathsf{k}<\overline{2}]\wedge \tag{41}$$
$$([(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{w}))_\mathsf{k}=\overline{1}]\leftrightarrow([(\mathsf{q})_\mathsf{k}=\overline{1}]\wedge[(\mathsf{w})_\mathsf{k}=\overline{1}])))$$

by Σ_1-induction on l.

– Since $[\mathtt{tseq}_\wedge(0,\mathsf{q},\mathsf{r})=\varnothing]$,

$$[\mathsf{Seq}(\mathtt{tseq}_\wedge(0,\mathsf{q},\mathsf{r}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\wedge(0,\mathsf{q},\mathsf{r}))=0]\wedge$$
$$\forall\mathsf{k}<0([(\mathtt{tseq}_\wedge(0,\mathsf{q},\mathsf{r}))_\mathsf{k}<\overline{2}]\wedge$$
$$([(\mathtt{tseq}_\wedge(0,\mathsf{q},\mathsf{w}))_\mathsf{k}=\overline{1}]\leftrightarrow([(\mathsf{q})_\mathsf{k}=\overline{1}]\wedge[(\mathsf{w})_\mathsf{k}=\overline{1}])))$$

holds as before.

– Suppose

$$\begin{aligned}&[\mathtt{Seq}(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{r}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{r}))=\mathsf{l}]\wedge\\&\forall\mathsf{k}<\mathsf{l}([(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{r}))_\mathsf{k}<\overline{2}]\wedge\\&\qquad([(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{w}))_\mathsf{k}=\overline{1}]\leftrightarrow([(\mathsf{q})_\mathsf{k}=\overline{1}]\wedge[(\mathsf{w})_\mathsf{k}=\overline{1}]))).\end{aligned}$$

Since $[\mathtt{tseq}_\wedge(\mathsf{S}(\mathsf{l}),\mathsf{q},\mathsf{r})=\mathtt{ext}(\mathtt{tseq}_\wedge(\mathsf{l},\mathsf{q},\mathsf{r}),\mathsf{l},\mathsf{c}_\wedge((\mathsf{q})_\mathsf{l},(\mathsf{r})_\mathsf{l}))]$ and $[\mathsf{c}_\wedge((\mathsf{q})_\mathsf{l},(\mathsf{r})_\mathsf{l})<\overline{2}]$ this implies

$$\begin{aligned}&[\mathtt{Seq}(\mathtt{tseq}_\wedge(\mathsf{S}(\mathsf{l}),\mathsf{q},\mathsf{r}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\wedge(\mathsf{S}(\mathsf{l}),\mathsf{q},\mathsf{r}))=\mathsf{S}(\mathsf{l})]\wedge\\&\forall\mathsf{k}<\mathsf{S}(\mathsf{l})[(\mathtt{tseq}_\wedge(\mathsf{S}(\mathsf{l}),\mathsf{q},\mathsf{r}))_\mathsf{k}<\overline{2}]\end{aligned}$$

by (37). Moreover,

$$[(\mathtt{tseq}_\wedge(\mathsf{S}(\mathsf{l}),\mathsf{q},\mathsf{w}))_\mathsf{k}=\overline{1}]\leftrightarrow([(\mathsf{q})_\mathsf{k}=\overline{1}]\wedge[(\mathsf{w})_\mathsf{k}=\overline{1}])$$

for all $\mathsf{k}<\mathsf{l}$ by induction hypothesis, and

$$[(\mathtt{tseq}_\wedge(\mathsf{S}(\mathsf{l}),\mathsf{q},\mathsf{w}))_\mathsf{l}=\overline{1}]\leftrightarrow([(\mathsf{q})_\mathsf{k}=\overline{1}]\wedge[(\mathsf{w})_\mathsf{k}=\overline{1}])$$

by definition of $\mathsf{c}_\wedge$.

- The proof that

$$\begin{aligned}\mathrm{I}\Sigma_1\vdash&[\mathtt{Seq}(\mathtt{tseq}_\vee(\mathsf{l},\mathsf{q},\mathsf{r}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\vee(\mathsf{l},\mathsf{q},\mathsf{r}))=\mathsf{l}]\wedge\\&\forall\mathsf{k}<\mathsf{l}([(\mathtt{tseq}_\vee(\mathsf{l},\mathsf{q},\mathsf{r}))_\mathsf{k}<\overline{2}]\wedge\\&\qquad([(\mathtt{tseq}_\vee(\mathsf{l},\mathsf{q},\mathsf{w}))_\mathsf{k}=\overline{1}]\leftrightarrow([(\mathsf{q})_\mathsf{k}=\overline{1}]\vee[(\mathsf{w})_\mathsf{k}=\overline{1}])))\end{aligned}\tag{42}$$

is almost identical to the Σ_1-induction for $\mathtt{tseq}_\wedge$ above.

- We now show that

$$\begin{aligned}\mathrm{I}\Sigma_1\vdash&[\mathtt{Seq}(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))=\mathsf{l}]\wedge\\&\forall\mathsf{k}<\mathsf{l}([(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))_\mathsf{k}<\overline{2}]\wedge\\&\qquad([(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))_\mathsf{k}=\overline{1}]\leftrightarrow\\&\qquad\forall[\mathsf{j}<\mathtt{lh}(\mathsf{s})]([\langle\mathsf{k}\rangle_1\leq\mathsf{j}]\to\forall[\mathsf{a}<(\mathsf{s})_\mathsf{j}][(\mathsf{q})_{\langle\mathsf{j},\mathtt{ext}(\langle\mathsf{k}\rangle_\mathrm{r},\mathbf{v}_\mathsf{x},\mathsf{a})\rangle}=\overline{1}])))\end{aligned}\tag{43}$$

by Σ_1-induction on l. We reason in $\mathrm{I}\Sigma_1$.

– Since $[\mathtt{tseq}_\forall(\mathsf{s},0,\mathsf{q},\mathsf{x})=\varnothing]$, the base case holds true as in the earlier proofs.

– Suppose

$$\begin{aligned}&[\mathtt{Seq}(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{q},\mathsf{x}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{q},\mathsf{x}))=\mathsf{l}]\wedge\\&\forall\mathsf{k}<\mathsf{l}([(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{q},\mathsf{x}))_\mathsf{k}<\overline{2}]\wedge\\&\qquad([(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))_\mathsf{k}=\overline{1}]\leftrightarrow\forall[\mathsf{j}<\mathtt{lh}(\mathsf{s})]([\langle\mathsf{k}\rangle_1\leq\mathsf{j}]\to\forall[\mathsf{a}<(\mathsf{s})_\mathsf{j}][(\mathsf{q})_{\langle\mathsf{j},\mathtt{ext}(\langle\mathsf{k}\rangle_\mathrm{r},\mathbf{v}_\mathsf{x},\mathsf{a})\rangle}=\overline{1}]))).\end{aligned}$$

Since $[\mathtt{tseq}_\forall(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}){=}\mathtt{ext}(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}),\mathsf{l},\mathsf{c}_\forall(\mathsf{s},\langle\mathsf{l}\rangle_1,\langle\mathsf{l}\rangle_\mathrm{r},\mathsf{x},\mathsf{q}))]$ and we have $[\mathsf{c}_\forall(\mathsf{s},\langle\mathsf{l}\rangle_1,\langle\mathsf{l}\rangle_\mathrm{r},\mathsf{x},\mathsf{q}){<}\overline{2}]$, we get

$$[\mathtt{Seq}(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q})){=}\mathsf{S}(\mathsf{l})]\wedge\\ \forall\mathsf{k}{<}\mathsf{S}(\mathsf{l})[(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}))_\mathsf{k}{<}\overline{2}]$$

by (37). We also get

$$[(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}))_\mathsf{k}{=}\overline{1}]{\leftrightarrow}\forall[\mathsf{j}{<}\mathtt{lh}(\mathsf{s})](([\langle\mathsf{k}\rangle_1{\le}\mathsf{j}]{\rightarrow}\forall[\mathsf{a}{<}(\mathsf{s})_\mathsf{j}][(\mathsf{q})_{\langle\mathsf{j},\mathtt{ext}(\langle\mathsf{k}\rangle_\mathrm{r},\mathbf{v}_\mathsf{x},\mathsf{a})\rangle}{=}\overline{1}])$$

for all $\mathsf{k}{<}\mathsf{l}$ by induction hypothesis and

$$[(\mathtt{tseq}_\forall(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}))_\mathsf{l}{=}\overline{1}]{\leftrightarrow}\forall[\mathsf{j}{<}\mathtt{lh}(\mathsf{s})](([\langle\mathsf{l}\rangle_1{\le}\mathsf{j}]{\rightarrow}\forall[\mathsf{a}{<}(\mathsf{s})_\mathsf{j}][(\mathsf{q})_{\langle\mathsf{j},\mathtt{ext}(\langle\mathsf{l}\rangle_\mathrm{r},\mathbf{v}_\mathsf{x},\mathsf{a})\rangle}{=}\overline{1}])$$

by definition of $\mathsf{c}_\forall$.

- We finally show that

$$\begin{aligned}\mathrm{I}\Sigma_1\vdash&[\mathtt{Seq}(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q})){=}\mathsf{l}]\wedge\\ &\forall\mathsf{k}{<}\mathsf{l}([(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))_\mathsf{k}{<}\overline{2}]\wedge\\ &\quad([(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))_\mathsf{k}{=}\overline{1}]{\leftrightarrow}\\ &\quad([\mathsf{S}(\langle\mathsf{k}\rangle_1){<}\mathtt{lh}(\mathsf{s})]{\rightarrow}\exists[\mathsf{a}{<}(\mathsf{s})_{\mathsf{S}(\langle\mathsf{k}\rangle_1)}][(\mathsf{q})_{\langle\mathsf{S}(\langle\mathsf{k}\rangle_1),\mathtt{ext}(\langle\mathsf{k}\rangle_1,\mathbf{v}_\mathsf{x},\mathsf{a})\rangle}{=}\overline{1}])))\end{aligned}\tag{44}$$

by Σ_1-induction on l, reasoning in $\mathrm{I}\Sigma_1$.

  - Since $[\mathtt{tseq}_\exists(\mathsf{s},0,\mathsf{q},\mathsf{x}){=}\varnothing]$, the base case is true.
  - Suppose that

$$\begin{aligned}&[\mathtt{Seq}(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q})){=}\mathsf{l}]\wedge\\ &\forall\mathsf{k}{<}\mathsf{l}([(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))_\mathsf{k}{<}\overline{2}]\wedge\\ &\quad([(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}))_\mathsf{k}{=}\overline{1}]{\leftrightarrow}\\ &\quad([\mathsf{S}(\langle\mathsf{k}\rangle_1){<}\mathtt{lh}(\mathsf{s})]{\rightarrow}\exists[\mathsf{a}{<}(\mathsf{s})_{\mathsf{S}(\langle\mathsf{k}\rangle_1)}][(\mathsf{q})_{\langle\mathsf{S}(\langle\mathsf{k}\rangle_1),\mathtt{ext}(\langle\mathsf{k}\rangle_1,\mathbf{v}_\mathsf{x},\mathsf{a})\rangle}{=}\overline{1}]))).\end{aligned}$$

Since $[\mathtt{tseq}_\exists(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}){=}\mathtt{ext}(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{l},\mathsf{x},\mathsf{q}),\mathsf{l},\mathsf{c}_\exists(\mathsf{s},\langle\mathsf{l}\rangle_1,\langle\mathsf{l}\rangle_\mathrm{r},\mathsf{x},\mathsf{q}))]$ and like before $[\mathsf{c}_\exists(\mathsf{s},\langle\mathsf{l}\rangle_1,\langle\mathsf{l}\rangle_\mathrm{r},\mathsf{x},\mathsf{q}){<}\overline{2}]$, we get

$$[\mathtt{Seq}(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}))]\wedge[\mathtt{lh}(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q})){=}\mathsf{S}(\mathsf{l})]\wedge\\ \forall\mathsf{k}{<}\mathsf{S}(\mathsf{l})[(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}))_\mathsf{k}{<}\overline{2}]$$

by (37). Furthermore,

$$[(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}))_\mathsf{k}{=}\overline{1}]{\leftrightarrow}\\ ([\mathsf{S}(\langle\mathsf{k}\rangle_1){<}\mathtt{lh}(\mathsf{s})]{\rightarrow}\exists[\mathsf{a}{<}(\mathsf{s})_{\mathsf{S}(\langle\mathsf{k}\rangle_1)}][(\mathsf{q})_{\langle\mathsf{S}(\langle\mathsf{k}\rangle_1),\mathtt{ext}(\langle\mathsf{k}\rangle_\mathrm{r},\mathbf{v}_\mathsf{x},\mathsf{a})\rangle}{=}\overline{1}])$$

for all $\mathsf{k}{<}\mathsf{l}$ by induction hypothesis and

$$[(\mathtt{tseq}_\exists(\mathsf{s},\mathsf{S}(\mathsf{l}),\mathsf{x},\mathsf{q}))_\mathsf{l}{=}\overline{1}]{\leftrightarrow}\\ ([\mathsf{S}(\langle\mathsf{l}\rangle_1){<}\mathtt{lh}(\mathsf{s})]{\rightarrow}\exists[\mathsf{a}{<}(\mathsf{s})_{\mathsf{S}(\langle\mathsf{l}\rangle_1)}][(\mathsf{q})_{\langle\mathsf{S}(\langle\mathsf{l}\rangle_1),\mathtt{ext}(\langle\mathsf{l}\rangle_\mathrm{r},\mathbf{v}_\mathsf{x},\mathsf{a})\rangle}{=}\overline{1}])$$

by definition of $\mathsf{c}_\exists$.

We now verify the properties of $\mathtt{iful}$ expressed in (29)–(36), reasoning in $I\Sigma_1$ again. Thus suppose $\mathtt{Seq}(\mathsf{s})$, $[\mathsf{i}{<}\mathtt{lh}(\mathsf{s})]$ and $[\mathsf{e}{\leq}({<}\ \mathsf{n})\times({<}\ \mathsf{s})]$, so $[\langle \mathsf{i},\mathsf{e}\rangle{<}\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle]$ by Lemma 2.25.

- Let r and t be such that $\mathtt{Term}(\mathsf{r})$ and $\mathtt{Term}(\mathsf{t})$. Then

  $$[\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{=}\mathsf{t})){=}\mathtt{tseq}_{=}(\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle,(\mathsf{r}{=}\mathsf{t}))]$$

  and

  $$[\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{<}\mathsf{t})){=}\mathtt{tseq}_{<}(\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle,(\mathsf{r}{<}\mathsf{t}))].$$

  Thus we have

  $$[\mathtt{Seq}(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{=}\mathsf{t})))]\wedge[\mathtt{lh}(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{=}\mathsf{t}))){=}\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle]\wedge \\ [(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{=}\mathsf{t})))_{\langle\mathsf{i},\mathsf{e}\rangle}{<}\overline{2}]\wedge([(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{=}\mathsf{t})))_{\langle\mathsf{i},\mathsf{e}\rangle}{=}\overline{1}]\leftrightarrow[\mathtt{val}(\mathsf{e},\mathsf{r}){=}\mathtt{val}(\mathsf{e},\mathsf{t})])$$

  by (38), since $[\langle\mathsf{i},\mathsf{e}\rangle{<}\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle]$ and $[\langle\langle\mathsf{i},\mathsf{e}\rangle\rangle_{\mathrm{r}}{=}\mathsf{e}]$. Similarly

  $$[\mathtt{Seq}(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{<}\mathsf{t})))]\wedge[\mathtt{lh}(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{<}\mathsf{t}))){=}\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle]\wedge \\ [(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{<}\mathsf{t})))_{\langle\mathsf{i},\mathsf{e}\rangle}{<}\overline{2}]\wedge([(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{r}{<}\mathsf{t})))_{\langle\mathsf{i},\mathsf{e}\rangle}{=}\overline{1}]\leftrightarrow[\mathtt{val}(\mathsf{e},\mathsf{r}){<}\mathtt{val}(\mathsf{e},\mathsf{t})])$$

  by (39).

- Assume $\mathtt{Fmla}(\mathsf{f})$ and consider $(\neg\mathsf{f})$. By definition we have

  $$[\mathtt{iful}(\mathsf{s},\mathsf{n},(\neg\mathsf{f})){=}\mathtt{tseq}_{\neg}(\mathsf{s},\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle,\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))],$$

  whereby

  $$[\mathtt{Seq}(\mathtt{iful}(\mathsf{s},\mathsf{n},(\neg\mathsf{f})))]\wedge[\mathtt{lh}(\mathtt{iful}(\mathsf{s},\mathsf{n},(\neg\mathsf{f}))){=}\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle]\wedge \\ [(\mathtt{iful}(\mathsf{s},\mathsf{n},(\neg\mathsf{f})))_{\langle\mathsf{i},\mathsf{e}\rangle}{<}\overline{2}]\wedge \\ ([(\mathtt{iful}(\mathsf{s},\mathsf{n},(\neg\mathsf{f})))_{\langle\mathsf{i},\mathsf{e}\rangle}{=}\overline{1}]\leftrightarrow\forall[\mathsf{j}{<}\mathtt{lh}(\mathsf{s})](\mathsf{i}{\leq}\mathsf{j}\rightarrow[(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))_{\langle\mathsf{j},\mathsf{e}\rangle}{=}0]))$$

  by (40), since $[\langle\langle\mathsf{i},\mathsf{e}\rangle\rangle_{\mathrm{l}}{=}\mathsf{i}]$ and $[\langle\langle\mathsf{i},\mathsf{e}\rangle\rangle_{\mathrm{r}}{=}\mathsf{e}]$.

- Assume $\mathtt{Fmla}(\mathsf{f})$ and $\mathtt{Fmla}(\mathsf{g})$ and consider $(\mathsf{f}\wedge\mathsf{g})$. Then by definition

  $$[\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{f}\wedge\mathsf{g})){=}\mathtt{tseq}_{\wedge}(\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle,\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}),\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{g}))].$$

  Hence

  $$[\mathtt{Seq}(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{f}\wedge\mathsf{g})))]\wedge[\mathtt{lh}(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{f}\wedge\mathsf{g}))){=}\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}(({<}\ \mathsf{n})\times({<}\ \mathsf{s}))\rangle]\wedge \\ [(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{f}\wedge\mathsf{g})))_{\langle\mathsf{i},\mathsf{e}\rangle}{<}\overline{2}]\wedge \\ ([(\mathtt{iful}(\mathsf{s},\mathsf{n},(\mathsf{f}\wedge\mathsf{g})))_{\langle\mathsf{i},\mathsf{e}\rangle}{=}\overline{1}]\leftrightarrow([(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))_{\langle\mathsf{i},\mathsf{e}\rangle}{=}\overline{1}]\wedge[(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{g}))_{\langle\mathsf{i},\mathsf{e}\rangle}{=}\overline{1}]))$$

  by (41), like before.

- Similarly, if $\texttt{Fmla}(\mathsf{f})$ and $\texttt{Fmla}(\mathsf{g})$ then

$$[\texttt{iful}(\mathsf{s,n,}(\mathsf{f}\vee\mathsf{g}))=\texttt{tseq}_{\vee}(\langle\texttt{lh}(\mathsf{s}),\mathsf{S}((<\mathsf{n})\times(<\mathsf{s}))\rangle,\texttt{iful}(\mathsf{s,n,f}),\texttt{iful}(\mathsf{s,n,g}))]$$

by definition, whence

$$[\texttt{Seq}(\texttt{iful}(\mathsf{s,n,}(\mathsf{f}\vee\mathsf{g})))]\wedge[\texttt{lh}(\texttt{iful}(\mathsf{s,n,}(\mathsf{f}\vee\mathsf{g})))=\langle\texttt{lh}(\mathsf{s}),\mathsf{S}((<\mathsf{n})\times(<\mathsf{s}))\rangle]\wedge$$
$$[(\texttt{iful}(\mathsf{s,n,}(\mathsf{f}\vee\mathsf{g})))_{\langle\mathsf{i,e}\rangle}<\overline{2}]\wedge$$
$$([(\texttt{iful}(\mathsf{s,n,}(\mathsf{f}\vee\mathsf{g})))_{\langle\mathsf{i,e}\rangle}=\overline{1}]\leftrightarrow([(\texttt{iful}(\mathsf{s,n,f}))_{\langle\mathsf{i,e}\rangle}=\overline{1}]\vee[(\texttt{iful}(\mathsf{s,n,g}))_{\langle\mathsf{i,e}\rangle}=\overline{1}]))$$

by (42).

- Assume $\texttt{Fmla}(\mathsf{f})$. Since

$$[\texttt{iful}(\mathsf{s,n,}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}))=\texttt{tseq}_{\forall}(\mathsf{s},\langle\texttt{lh}(\mathsf{s}),\mathsf{S}((<\mathsf{n})\times(<\mathsf{s}))\rangle,\mathsf{x},\texttt{iful}(\mathsf{s,n,f}))]$$

by definition,

$$[\texttt{Seq}(\texttt{iful}(\mathsf{s,n,}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})))]\wedge[\texttt{lh}(\texttt{iful}(\mathsf{s,n,}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})))=\langle\texttt{lh}(\mathsf{s}),\mathsf{S}((<\mathsf{n})\times(<\mathsf{s}))\rangle]\wedge$$
$$[(\texttt{iful}(\mathsf{s,n,}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})))_{\langle\mathsf{i,e}\rangle}<\overline{2}]\wedge$$
$$([(\texttt{iful}(\mathsf{s,n,}(\forall\mathbf{v}_{\mathsf{x}}\mathsf{f})))_{\langle\mathsf{i,e}\rangle}=\overline{1}]\leftrightarrow$$
$$\forall[\mathsf{j}<\texttt{lh}(\mathsf{s})](\mathsf{i}\leq\mathsf{j}\rightarrow\forall[\mathsf{a}<(\mathsf{s})_{\mathsf{j}}][(\texttt{iful}(\mathsf{s,n,f}))_{\langle\mathsf{j},\texttt{ext}(\mathsf{e},\mathbf{v}_{\mathsf{x}},\mathsf{a})\rangle}=\overline{1}]))$$

by (43) (compare to $(\neg\mathsf{f})$ above).

- Similarly, if $\texttt{Fmla}(\mathsf{f})$ again, we have

$$[\texttt{iful}(\mathsf{s,n,}(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}))=\texttt{tseq}_{\exists}(\mathsf{s},\langle\texttt{lh}(\mathsf{s}),\mathsf{S}((<\mathsf{n})\times(<\mathsf{s}))\rangle,\mathsf{x},\texttt{iful}(\mathsf{s,n,f}))]$$

by definition, whence

$$[\texttt{Seq}(\texttt{iful}(\mathsf{s,n,}(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})))]\wedge[\texttt{lh}(\texttt{iful}(\mathsf{s,n,}(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})))=\langle\texttt{lh}(\mathsf{s}),\mathsf{S}((<\mathsf{n})\times(<\mathsf{s}))\rangle]\wedge$$
$$[(\texttt{iful}(\mathsf{s,n,}(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})))_{\langle\mathsf{i,e}\rangle}<\overline{2}]\wedge$$
$$([(\texttt{iful}(\mathsf{s,n,}(\exists\mathbf{v}_{\mathsf{x}}\mathsf{f})))_{\langle\mathsf{i,e}\rangle}=\overline{1}]\leftrightarrow$$
$$([\mathsf{S}(\mathsf{i})<\texttt{lh}(\mathsf{s})]\rightarrow\exists[\mathsf{a}<(\mathsf{s})_{\mathsf{S}(\mathsf{i})}][(\texttt{iful}(\mathsf{s,n,f}))_{\langle\mathsf{S}(\mathsf{i}),\texttt{ext}(\mathsf{e},\mathbf{v}_{\mathsf{x}},\mathsf{a})\rangle}=\overline{1}]))$$

by (44).

To verify (29) ((30)–(36) being immediate from the above), we note that by Lemma 2.32

$$\begin{aligned}\mathrm{I}\Sigma_1\vdash\texttt{Fmla}(\mathsf{f})\leftrightarrow(&\texttt{Atf}(\mathsf{f})\vee(\exists\mathsf{g}<\mathsf{f}\texttt{Fmla}(\mathsf{g})\wedge[(\neg\mathsf{g})=\mathsf{f}])\vee\\&(\exists\mathsf{g}<\mathsf{f}\exists\mathsf{h}<\mathsf{f}\texttt{Fmla}(\mathsf{g})\wedge\texttt{Fmla}(\mathsf{h})\wedge[(\mathsf{g}\wedge\mathsf{h})=\mathsf{f}])\vee\\&(\exists\mathsf{g}<\mathsf{f}\exists\mathsf{h}<\mathsf{f}\texttt{Fmla}(\mathsf{g})\wedge\texttt{Fmla}(\mathsf{h})\wedge[(\mathsf{g}\vee\mathsf{h})=\mathsf{f}])\vee\\&(\exists\mathsf{g}<\mathsf{f}\exists\mathsf{x}<\mathsf{f}\texttt{Fmla}(\mathsf{g})\wedge[(\forall\mathbf{v}_{\mathsf{x}}\mathsf{g})=\mathsf{f}])\vee\\&(\exists\mathsf{g}<\mathsf{f}\exists\mathsf{x}<\mathsf{f}\texttt{Fmla}(\mathsf{g})\wedge[(\exists\mathbf{v}_{\mathsf{x}}\mathsf{g})=\mathsf{f}])),\end{aligned}$$

whence

$$\mathrm{I}\Sigma_1 \vdash \mathtt{Fmla}(\mathsf{f}){\rightarrow}([\mathtt{Seq}(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))]\wedge [\mathtt{lh}(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f})){=}\langle\mathtt{lh}(\mathsf{s}),\mathsf{S}((<\mathsf{n})\times(<\mathsf{s}))\rangle]\wedge \forall[\mathsf{i}{<}\mathtt{lh}(\mathsf{s})]\forall[\mathsf{e}{\leq}(<\mathsf{n})\times(<\mathsf{s})][(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))_{\langle \mathsf{i},\mathsf{e}\rangle}{<}\overline{2}])$$

by the foregoing, as desired. This verifies (29).

We can now define $\mathtt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{f})$ to be the formula

$$[(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))_{\langle \mathsf{i},\mathsf{e}\rangle}{=}\overline{1}]$$

and verify each of the statements of the proposition, all of whose proofs will use the fact that $\mathrm{I}\Sigma_1 \vdash [\mathsf{e}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{s})]{\rightarrow}[\mathsf{e}{\leq}(<\mathsf{n})\times(<\mathsf{s})]$ by Lemma 2.26. We reason in $\mathrm{I}\Sigma_1$.

- Suppose $\mathtt{Seq}(\mathsf{s})$, $[\mathsf{i}{<}\mathtt{lh}(\mathsf{s})]$, $[\mathsf{e}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{s})]$, $\mathtt{Term}(\mathsf{r})$, $\mathtt{Term}(\mathsf{t})$, $[\mathtt{Eval}(\mathsf{e},\boldsymbol{(}\mathsf{r}\boldsymbol{=}\mathsf{t}\boldsymbol{)})]$, and $[\boldsymbol{(}\mathsf{r}\boldsymbol{=}\mathsf{t}\boldsymbol{)}{\leq}\mathsf{n}]$. Then

  $$[\mathtt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\boldsymbol{(}\mathsf{r}\boldsymbol{=}\mathsf{t}\boldsymbol{)})]{\leftrightarrow}[\mathtt{val}(\mathsf{e},\mathsf{r}){=}\mathtt{val}(\mathsf{e},\mathsf{t})]$$

  by (30).

- Suppose $\mathtt{Seq}(\mathsf{s})$, $[\mathsf{i}{<}\mathtt{lh}(\mathsf{s})]$, $[\mathsf{e}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{s})]$, $\mathtt{Term}(\mathsf{r})$, $\mathtt{Term}(\mathsf{t})$, $[\mathtt{Eval}(\mathsf{e},\boldsymbol{(}\mathsf{r}\boldsymbol{=}\mathsf{t}\boldsymbol{)})]$, and $[\boldsymbol{(}\mathsf{r}\boldsymbol{=}\mathsf{t}\boldsymbol{)}{\leq}\mathsf{n}]$. Then

  $$[\mathtt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\boldsymbol{(}\mathsf{r}\boldsymbol{<}\mathsf{t}\boldsymbol{)})]{\leftrightarrow}[\mathtt{val}(\mathsf{e},\mathsf{r}){<}\mathtt{val}(\mathsf{e},\mathsf{t})]$$

  by (31).

- Now suppose $\mathtt{Seq}(\mathsf{s})$, $[\mathsf{i}{<}\mathtt{lh}(\mathsf{s})]$, $[\mathsf{e}{\sqsubseteq}(<\mathsf{n})\times(<\mathsf{s})]$, $\mathtt{Fmla}(\mathsf{f})$, $[\mathtt{Eval}(\mathsf{e},\boldsymbol{(\neg}\mathsf{f}\boldsymbol{)})]$ and $[\boldsymbol{(\neg}\mathsf{f}\boldsymbol{)}{\leq}\mathsf{n}]$. Assume

  $$[\mathtt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\boldsymbol{(\neg}\mathsf{f}\boldsymbol{)})],$$

  that is $[(\mathtt{iful}(\mathsf{s},\mathsf{n},\boldsymbol{(\neg}\mathsf{f}\boldsymbol{)}))_{\langle \mathsf{i},\mathsf{e}\rangle}{=}\overline{1}]$. Then

  $$\forall[\mathsf{j}{<}\mathtt{lh}(\mathsf{s})](\mathsf{i}{\leq}\mathsf{j}{\rightarrow}[(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))_{\langle \mathsf{j},\mathsf{e}\rangle}{=}0])$$

  by (32), whence in particular

  $$\forall[\mathsf{j}{<}\mathtt{lh}(\mathsf{s})](\mathsf{i}{\leq}\mathsf{j}{\rightarrow}\neg\mathtt{IFulf}(\mathsf{s},\mathsf{j},\mathsf{e},\mathsf{n},\mathsf{f})).$$

  Conversely, assuming the above we have

  $$\forall[\mathsf{j}{<}\mathtt{lh}(\mathsf{s})](\mathsf{i}{\leq}\mathsf{j}{\rightarrow}[(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))_{\langle \mathsf{j},\mathsf{e}\rangle}{\neq}\overline{1}])$$

  whence

  $$\forall[\mathsf{j}{<}\mathtt{lh}(\mathsf{s})](\mathsf{i}{\leq}\mathsf{j}{\rightarrow}[(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))_{\langle \mathsf{j},\mathsf{e}\rangle}{=}0])$$

  since $\forall[\mathsf{j}{<}\mathtt{lh}(\mathsf{s})][(\mathtt{iful}(\mathsf{s},\mathsf{n},\mathsf{f}))_{\langle \mathsf{j},\mathsf{e}\rangle}{<}\overline{2}]$ by (29). The latter is equivalent to

  $$[\mathtt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\boldsymbol{(\neg}\mathsf{f}\boldsymbol{)})]$$

  by (32).

- Suppose $\texttt{Seq}(\mathsf{s})$, $[\mathsf{i}{<}\texttt{lh}(\mathsf{s})]$, $[\mathsf{e}{\sqsubseteq}(<\mathsf{n}){\times}(<\mathsf{s})]$, $\texttt{Fmla}(\mathsf{f})$, $\texttt{Fmla}(\mathsf{g})$, $[\texttt{Eval}(\mathsf{e},\boldsymbol{(}\mathsf{f}{\wedge}\mathsf{g}\boldsymbol{)})]$ and $[\boldsymbol{(}\mathsf{f}{\wedge}\mathsf{g}\boldsymbol{)}{\leq}\mathsf{n}]$. Then

  $$[\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\boldsymbol{(}\mathsf{f}{\wedge}\mathsf{g}\boldsymbol{)})]{\leftrightarrow}(\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{f}){\wedge}\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{g}))$$

  by unwinding the definition and applying (33).

- In the same way, if $\texttt{Seq}(\mathsf{s})$, $[\mathsf{i}{<}\texttt{lh}(\mathsf{s})]$, $[\mathsf{e}{\sqsubseteq}(<\mathsf{n}){\times}(<\mathsf{s})]$, $\texttt{Fmla}(\mathsf{f})$, $\texttt{Fmla}(\mathsf{g})$, $[\texttt{Eval}(\mathsf{e},\boldsymbol{(}\mathsf{f}{\vee}\mathsf{g}\boldsymbol{)})]$ and $[\boldsymbol{(}\mathsf{f}{\vee}\mathsf{g}\boldsymbol{)}{\leq}\mathsf{n}]$, then

  $$[\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\boldsymbol{(}\mathsf{f}{\vee}\mathsf{g}\boldsymbol{)})]{\leftrightarrow}(\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{f}){\vee}\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{g})).$$

  by (34).

- Now suppose $\texttt{Seq}(\mathsf{s})$, $[\mathsf{i}{<}\texttt{lh}(\mathsf{s})]$, $[\mathsf{e}{\sqsubseteq}(<\mathsf{n}){\times}(<\mathsf{s})]$, $\texttt{Fmla}(\mathsf{f})$, $[\texttt{Eval}(\mathsf{e},\boldsymbol{(}\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}\boldsymbol{)})]$ and $[\boldsymbol{(}\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}\boldsymbol{)}{\leq}\mathsf{n}]$. Then

  $$[\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\boldsymbol{(}\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}\boldsymbol{)})]{\leftrightarrow}\forall[\mathsf{j}{<}\texttt{lh}(\mathsf{s})](\mathsf{i}{\leq}\mathsf{j}{\to}\forall[\mathsf{a}{<}(\mathsf{s})_{\mathsf{j}}][\texttt{IFulf}(\mathsf{s},\mathsf{j},\texttt{ext}(\mathsf{e},\mathbf{v}_{\mathsf{x}},\mathsf{a}),\mathsf{n},\mathsf{f})])$$

  by definition and (35).

- Finally suppose $\texttt{Seq}(\mathsf{s})$, $[\mathsf{i}{<}\texttt{lh}(\mathsf{s})]$, $[\mathsf{e}{\sqsubseteq}(<\mathsf{n}){\times}(<\mathsf{s})]$, $\texttt{Fmla}(\mathsf{f})$, $[\texttt{Eval}(\mathsf{e},\boldsymbol{(}\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}\boldsymbol{)})]$ and $[\boldsymbol{(}\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}\boldsymbol{)}{\leq}\mathsf{n}]$. Again

  $$[\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\boldsymbol{(}\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}\boldsymbol{)})]{\leftrightarrow}(\mathsf{S}(\mathsf{i}){<}\texttt{lh}(\mathsf{s}){\to}\exists[\mathsf{a}{<}(\mathsf{s})_{\mathsf{S}(\mathsf{i})}][\texttt{IFulf}(\mathsf{s},\mathsf{S}(\mathsf{i}),\texttt{ext}(\mathsf{e},\mathbf{v}_{\mathsf{x}},\mathsf{a}),\mathsf{n},\mathsf{f})])$$

  by definition and (36).

□

Remark 14. As can be seen from the proof, nothing in the definition of $\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{f})$ requires $\texttt{Eval}(\mathsf{e},\mathsf{f})$ or $\mathsf{f}{\leq}\mathsf{n}$, or even $[\mathsf{e}{\sqsubseteq}(<\mathsf{n}){\times}(<\mathsf{s})]$, for the inductive relations expressed in the theorem. We make these requirements partly to ensure the notion is stable; if $[\mathsf{e}{\sqsubseteq}(<\mathsf{n}){\times}(<\mathsf{s})]$, $\mathsf{f}{\leq}\mathsf{n}$ and $\mathsf{n}{\leq}\mathsf{m}$ then $\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{f}){\leftrightarrow}\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{m},\mathsf{f})$ since $[\mathsf{e}{\leq}(<\mathsf{n}){\times}(<\mathsf{s})]$ at all stages of the induction. Moreover, these requirements restrict to the cases we will use where $\texttt{IFulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{f})$ has the intended meaning. We will not prove these facts, but see Proposition 4.5.

We can now define

$$\texttt{IFbl}(\mathsf{s},\mathsf{f}) = \texttt{Incrseq}(\mathsf{s}){\wedge}\texttt{Sent}(\mathsf{f}){\wedge}\texttt{IFulf}(\mathsf{s},0,\varnothing,\texttt{nnf}(\mathsf{f}),\texttt{nnf}(\mathsf{f})).$$

We thus have "two" notions of fulfilment: one in the meta-language about the object language and one in the object language about the formal object language. As the similarities suggest, these notions coincide in the intended cases where both are applicable.

Proposition 4.5. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ and $s, l \in \mathcal{M}_{>0}$ be such that $\mathcal{M} \models \texttt{Seq}(s)$ and $\mathcal{M} \models [\texttt{lh}(s){=}l]$. Let $m \in \mathcal{M}$. Given a formula $\varphi \leq_{\mathcal{M}} m$ (which is of course trivially the case if $m \notin \mathbb{N}$) and a finite evaluation e of φ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$, we have $\mathcal{M} \models \texttt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{\varphi})$ iff $\mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \varphi$, for all $i \in \mathcal{M}_{<l}$.*

Proof. First, for φ and e as in the statement, $\mathrm{dom}(e) \subseteq \mathcal{M}_{<m}$ and $\mathrm{ran}(e) \subseteq \mathcal{M}_{<s}$, whence $\mathcal{M} \models [\mathfrak{F}_{\mathrm{f}}(e) \sqsubseteq (<m) \times (<s)]$ by Lemma 3.12. Moreover, such an e will satisfy $\mathcal{M} \models \mathtt{Eval}(\mathfrak{F}_{\mathrm{f}}(e), \overline{\varphi})$ by Lemma 3.7.

The rest of the proof is by induction on φ.

- Suppose σ and τ are terms such that $(\sigma{=}\tau) \leq_{\mathcal{M}} m$. Then

$$\begin{aligned}
\mathcal{M} \models \mathtt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{(\sigma{=}\tau)}) &\Leftrightarrow \mathcal{M} \models_e [\mathtt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,(\overline{\sigma}{=}\overline{\tau}))] \\
&\Leftrightarrow \mathcal{M} \models_e [\mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\sigma}){=}\mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau})] && \text{by 4.4} \\
&\Leftrightarrow \mathcal{M} \models_e \sigma{=}\tau && \text{by 3.8} \\
&\Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \sigma{=}\tau && \text{by definition}
\end{aligned}$$

  for all finite evaluations e of $(\sigma{=}\tau)$ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$ and all $i \in \mathcal{M}_{<l}$.

- Suppose σ and τ are terms such that $(\sigma{<}\tau) \leq_{\mathcal{M}} m$. Then

$$\begin{aligned}
\mathcal{M} \models \mathtt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{(\sigma{<}\tau)}) &\Leftrightarrow \mathcal{M} \models_e [\mathtt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,(\overline{\sigma}{<}\overline{\tau}))] \\
&\Leftrightarrow \mathcal{M} \models_e [\mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\sigma}){<}\mathtt{val}(\mathfrak{F}_{\mathrm{f}}(e),\overline{\tau})] && \text{by 4.4} \\
&\Leftrightarrow \mathcal{M} \models_e \sigma{<}\tau && \text{by 3.8} \\
&\Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \sigma{<}\tau && \text{by definition}
\end{aligned}$$

  for all finite evaluations e of $(\sigma{<}\tau)$ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$ and all $i \in \mathcal{M}_{<l}$.

- Let $\vartheta \leq_{\mathcal{M}} m$ be a formula such that $\mathcal{M} \models \mathtt{IFulf}(s,k,\mathfrak{F}_{\mathrm{f}}(f),m,\overline{\vartheta}) \Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),k,f}{\models_{\mathrm{I}}} \vartheta$ for all finite evaluations f of ϑ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$ and all $k \in \mathcal{M}_{<l}$. Suppose $(\neg\vartheta) \leq_{\mathcal{M}} m$. Then

$$\begin{aligned}
\mathcal{M} \models \mathtt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{(\neg\vartheta)}) &\Leftrightarrow \mathcal{M} \models [\mathtt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,(\neg\overline{\vartheta}))] \\
&\Leftrightarrow \mathcal{M} \models \forall \mathsf{j}{<}l(i{\leq}\mathsf{j}{\to}\neg\mathtt{IFulf}(s,\mathsf{j},\mathfrak{F}_{\mathrm{f}}(e),m,\overline{\vartheta})) \\
&\Leftrightarrow \forall j \in \mathcal{M}_{<l} \cap \mathcal{M}_{\geq i} : \mathcal{M} \not\models \mathtt{IFulf}(s,j,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{\vartheta}) \\
&\Leftrightarrow \forall j \in \mathcal{M}_{<l} \cap \mathcal{M}_{\geq i} : \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),j,e}{\not\models_{\mathrm{I}}} \varphi \\
&\Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \neg\vartheta
\end{aligned}$$

  for all finite evaluations e of $\neg\vartheta$ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$ and all $i \in \mathcal{M}_{<l}$.

- Let $\vartheta, \psi \leq_{\mathcal{M}} m$ be formulae such that $\mathcal{M} \models \mathtt{IFulf}(s,k,\mathfrak{F}_{\mathrm{f}}(f),m,\overline{\vartheta}) \Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),k,f}{\models_{\mathrm{I}}} \vartheta$ for all finite evaluations f of ϑ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$ and all

$k \in \mathcal{M}_{<l}$, and similarly for ψ. Suppose $(\vartheta\wedge\psi) \leq_{\mathcal{M}} m$. Then

$$\begin{aligned}\mathcal{M} \models \texttt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{(\vartheta\wedge\psi)}) &\Leftrightarrow \mathcal{M} \models [\texttt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\textbf{(}\overline{\vartheta}\textbf{∧}\overline{\psi}\textbf{)})]\\ &\Leftrightarrow \mathcal{M} \models \texttt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{\vartheta})\wedge\texttt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{\psi})\\ &\Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \vartheta \mathbin{\mathrm{\Lambda}} \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \psi\\ &\Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \vartheta\wedge\psi\end{aligned}$$

for all finite evaluations e of $\vartheta\wedge\psi$ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$ and all $i \in \mathcal{M}_{<l}$.

- The disjunctive case is again similar.

- Let $\vartheta \leq_{\mathcal{M}} m$ be a formula such that $\mathcal{M} \models \texttt{IFulf}(s,k,\mathfrak{F}_{\mathrm{f}}(f),n,\overline{\vartheta}) \Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),k,f}{\models_{\mathrm{I}}} \vartheta$ for all finite evaluations f of ϑ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$ and all $k \in \mathcal{M}_{<l}$. Suppose $(\forall\mathsf{v}_n\vartheta) \leq_{\mathcal{M}} m$ (so that in particular $\mathsf{v}_n <_{\mathcal{M}} m$) and let e be a finite evaluation of ($\forall\mathsf{v}_n\vartheta$ in $\mathcal{M}_{<s}$) undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$. Then given $k \in \mathcal{M}_{<l}$ and $b \in \mathcal{M}_{<\mathfrak{R}_{\mathrm{f}}(s)(k)} \subset \mathcal{M}_{<s}$, $\mathfrak{F}_{\mathrm{f}}(e^b_{\mathsf{v}_n})$ is an evaluation of ϑ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$, and moreover $\mathcal{M} \models [\texttt{ext}(\mathfrak{F}_{\mathrm{f}}(e),\mathbf{v}_{\overline{n}},b)\text{=}\mathfrak{F}_{\mathrm{f}}(e^b_{\mathsf{v}_n})]$ by Corollary 3.11. Thus

$$\begin{aligned}&\mathcal{M} \models \texttt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{(\forall\mathsf{v}_n\vartheta)})\\ \Leftrightarrow&\mathcal{M} \models [\texttt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\textbf{(}\forall\mathbf{v}_{\overline{n}}\overline{\vartheta}\textbf{)})]\\ \Leftrightarrow&\mathcal{M} \models \forall\mathsf{j}<l(i\leq\mathsf{j}\rightarrow\forall\mathsf{a}<(s)_{\mathsf{j}}[\texttt{IFulf}(s,i,\texttt{ext}(\mathfrak{F}_{\mathrm{f}}(e),\mathbf{v}_{\overline{n}},\mathsf{a}),m,\overline{\vartheta})])\\ \Leftrightarrow&\forall j \in \mathcal{M}_{<l} \cap \mathcal{M}_{\geq i}\forall a \in \mathcal{M}_{<\mathfrak{R}_{\mathrm{f}}(s)(j)} : \mathcal{M} \models [\texttt{IFulf}(s,i,\texttt{ext}(\mathfrak{F}_{\mathrm{f}}(e),\mathbf{v}_{\overline{n}},a),m,\overline{\vartheta})]\\ \Leftrightarrow&\forall j \in \mathcal{M}_{<l} \cap \mathcal{M}_{\geq i}\forall a \in \mathcal{M}_{<\mathfrak{R}_{\mathrm{f}}(s)(j)} : \mathcal{M} \models [\texttt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e^a_{\mathsf{v}_n}),m,\overline{\vartheta})]\\ \Leftrightarrow&\forall j \in \mathcal{M}_{<l} \cap \mathcal{M}_{\geq i}\forall a \in \mathcal{M}_{<\mathfrak{R}_{\mathrm{f}}(s)(j)} : \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),j,e^a_{\mathsf{v}_n}}{\models_{\mathrm{I}}} \vartheta\\ \Leftrightarrow&\mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \forall\mathsf{v}_n\vartheta\end{aligned}$$

for all $i \in \mathcal{M}_{<l}$.

- Let $\vartheta \leq_{\mathcal{M}} m$ be a formula such that $\mathcal{M} \models \texttt{IFulf}(s,k,\mathfrak{F}_{\mathrm{f}}(f),m,\overline{\vartheta}) \Leftrightarrow \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),k,f}{\models_{\mathrm{I}}} \vartheta$ for all finite evaluations f of ϑ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$. Suppose $(\exists\mathsf{v}_n\vartheta) \leq_{\mathcal{M}} m$ (so $\mathsf{v}_n <_{\mathcal{M}} m$). Furthermore, let e be a finite evaluation of $(\exists\mathsf{v}_n\vartheta)$ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$. Then for $b \in \mathcal{M}_{<s}$, $\mathfrak{F}_{\mathrm{f}}(e^b_{\mathsf{v}_n})$ is also an evaluation of ϑ in $\mathcal{M}_{<s}$ undefined for variables $\mathsf{x} \geq_{\mathcal{M}} m$ and $\mathcal{M} \models$

$[\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(e),\mathbf{v}_{\overline{n}},b)=\mathfrak{F}_{\mathrm{f}}(e^{b}_{\mathsf{v}_n})]$. Consequently

$$
\begin{aligned}
&\mathcal{M} \models \mathtt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{(\exists \mathsf{v}_n\vartheta)})\\
\Leftrightarrow&\mathcal{M} \models [\mathtt{IFulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\mathbf{(}\exists \mathbf{v}_{\overline{n}}\overline{\vartheta}\mathbf{)})]\\
\Leftrightarrow&\mathcal{M} \models \mathsf{S}(i)<l\rightarrow\exists \mathsf{a}<(s)_{\mathsf{S}(i)}[\mathtt{IFulf}(s,\mathsf{S}(i),\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(e),\mathbf{v}_{\overline{n}},\mathsf{a}),m,\overline{\vartheta})]\\
\Leftrightarrow&\mathrm{S}_{\mathcal{M}}(i) \in \mathcal{M}_{<l} \Rightarrow \exists a \in \mathcal{M}_{<\mathfrak{R}_{\mathrm{f}}(s)(\mathrm{S}_{\mathcal{M}}(i))} : \mathcal{M} \models [\mathtt{IFulf}(s,\mathsf{S}(i),\mathtt{ext}(\mathfrak{F}_{\mathrm{f}}(e),\mathbf{v}_{\overline{n}},a),m,\overline{\vartheta})]\\
\Leftrightarrow&\mathrm{S}_{\mathcal{M}}(i) \in \mathcal{M}_{<l} \Rightarrow \exists a \in \mathcal{M}_{<\mathfrak{R}_{\mathrm{f}}(s)(\mathrm{S}_{\mathcal{M}}(i))} : \mathcal{M} \models \mathtt{IFulf}(s,\mathsf{S}(i),\mathfrak{F}_{\mathrm{f}}(e^{a}_{\mathsf{v}_n}),m,\overline{\vartheta})\\
\Leftrightarrow&\mathrm{S}_{\mathcal{M}}(i) \in \mathcal{M}_{<l} \Rightarrow \exists a \in \mathcal{M}_{<\mathfrak{R}_{\mathrm{f}}(s)(\mathrm{S}_{\mathcal{M}}(i))} : \mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),\mathrm{S}_{\mathcal{M}}(i),e^{a}_{\mathsf{v}_n}}{\models_{\mathrm{I}}} \vartheta\\
\Leftrightarrow&\mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),i,e}{\models_{\mathrm{I}}} \exists \mathsf{v}_n\vartheta
\end{aligned}
$$

for all $i \in \mathcal{M}_{<l}$.

This concludes the induction, and the proof. □

We will use this notion of fulfilment to construct new models of $\mathrm{I}\Sigma_1$ with particular properties, by restricting to subsets of a given model bounded by elements of the sequence fulfilling a statement. To ensure that the result of this procedure is indeed a model of $\mathrm{I}\Sigma_1$, or at least an $\mathcal{L}_{\mathrm{A}}$-structure, we require that the sequences increase quick enough that sums and products of elements of the model less than some element of a sequence will be less than the subsequent element of the sequence. For reasons that will become apparent, we also require that the length of the sequence is smaller than its first element.

Definition 4.2 (Bounded initial partial models)**.** Let $\mathcal{M} \models \mathrm{PA}^{-}$ and $l \in \mathcal{M}_{>0}$. A sequence $s : \mathcal{M}_{<l} \longrightarrow \mathcal{M}$ is an *bounded initial partial model* (reminiscent of Putnam's term "finite partial model" from [11]) if

$$l <_{\mathcal{M}} s(0) \text{ and } s(\mathrm{S}_{\mathcal{M}}(i)) >_{\mathcal{M}} s(i)^2_{\mathcal{M}}$$

for all non-maximal $i \in \mathcal{M}_{<l}$. If $l \in \mathbb{N}$ and $s(i) \in \mathbb{N}$ for all $i \in \mathcal{M}_{<l}$ then s is a *finite* initial partial model.

The formal counterpart of this notion is given in $\mathrm{I}\Sigma_1$ by the $\Delta_1(\mathrm{I}\Sigma_1)$-formula $\mathtt{IBpm}(\mathsf{s})$ defined as

$$\mathtt{Seq}(\mathsf{s})\wedge[0<\mathtt{lh}(\mathsf{s})]\wedge[\mathtt{lh}(\mathsf{s})<(\mathsf{s})_0]\wedge\forall[\mathsf{i}<\mathtt{lh}(\mathsf{s})]([\mathsf{S}(\mathsf{i})<\mathtt{lh}(\mathsf{s})]\rightarrow(\mathsf{s})^2_{\mathsf{i}}<(\mathsf{s})_{\mathsf{S}(\mathsf{i})}).$$

Note that bounded initial partial models are increasing, and if the $\mathrm{I}\Sigma_1$-model $\mathcal{M} \models \mathtt{IBpm}(s)$, then $\mathfrak{R}_{\mathrm{f}}(s)_{\mathcal{M}}$ is a bounded initial partial model, and vice versa.

4.1.1 The initial-Gödel-Kripke sentence

Throughout the rest of this section, let T be a consistent and recursively axiomatisable arithmetic theory extending $\mathrm{I}\Sigma_1$, and let ax be some primitive recursive enumeration of an axiomatisation of T. Since ax is then provably recursive in $\mathrm{I}\Sigma_1$, let $\mathtt{ax}(\mathsf{x})$ be some

$\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function which coincides with ax in $\mathbb{N}$ (see Corollary 2.17 and the succeeding comment). It is worth pointing out that the independent sentence we will construct will depend upon this choice of the formula $[\mathtt{ax}(\mathsf{x})=\mathsf{y}]$.

Definition 4.3 (The initial-Gödel-Kripke sentence)**.** $\mathtt{IK}(\mathsf{x})$, where $\mathtt{K}$ is for "Kripke" and $\mathtt{I}$ is for "initial", is the formula $\exists \mathsf{s}(\mathtt{IBpm}(\mathsf{s})\wedge[\mathtt{lh}(\mathsf{s})=\mathsf{S}(\mathsf{x})]\wedge\forall \mathsf{y}<\mathsf{x}\mathtt{IFbl}(\mathsf{s},\mathtt{ax}(\mathsf{y})))$ stating that "the first x axioms have (are fulfilled by) a bounded initial partial model of length $\mathsf{S}(\mathsf{x})$".

The following formula is the *initial-Gödel-Kripke sentence*, which we shall prove to be independent of T, under certain circumstances:

$$\forall \mathsf{x}\mathtt{IK}(\mathsf{x}).$$

This proof of independence is naturally divided into two parts: showing that $\forall \mathsf{x}\mathtt{IK}(\mathsf{x})$ cannot be refuted in T and showing that it cannot be proved in T.

4.2 The initial-Gödel-Kripke sentence is not refutable

The aim of this subsection is to show that the initial-Gödel-Kripke sentence is not refutable in T if T is Σ_2-sound and "resembles" PA closely enough (to be made precise). Our approach is to show that $\mathbb{N}$, which is then a model of the Σ_2-part of T, is also a model of $\forall \mathsf{x}\mathtt{IK}(\mathsf{x})$. This will be the conclusion of Theorem 4.8. First we need two technical results concerning $\mathrm{S}\Sigma_n$, the first rather general, the second more specific to the current context. By Lemma 2.14 these could have just as well have been formulated for $\mathrm{I}\Sigma_n$, if $n>0$.

Lemma 4.6. *Let $k,n\in\mathbb{N}$, φ be a Σ_n-formula, $\mathsf{x}_0,\dots,\mathsf{x}_{k-1}$, y, u and v be distinct variables such that u and v do not occur in φ. Then*

$$\mathrm{S}\Sigma_n\vdash\forall\mathsf{u}\exists\mathsf{v}\forall\mathsf{x}_0<\mathsf{u}\cdots\forall\mathsf{x}_{k-1}<\mathsf{u}((\exists\mathsf{y}\varphi)\rightarrow\exists\mathsf{y}<\mathsf{v}\varphi).$$

Proof. Let z and w be distinct new variables and ψ be the formula

$$\forall\mathsf{x}_0\leq\mathsf{z}\cdots\forall\mathsf{x}_{k-1}\leq\mathsf{z}((\bigwedge\nolimits_{i<k}[\langle\mathsf{z}\rangle^k_i=\mathsf{x}_i])\rightarrow\varphi).$$

This is a Σ_n-formula, whence $\mathrm{S}\Sigma_n\vdash\forall\mathsf{w}\exists\mathsf{v}\forall\mathsf{z}<\mathsf{w}((\exists\mathsf{y}\psi)\rightarrow\exists\mathsf{y}<\mathsf{v}\psi)$.

We now reason in $\mathrm{S}\Sigma_n$: Given u let $\mathsf{w}=\mathsf{S}(\langle\mathsf{u},\mathsf{u},\dots,\mathsf{u}\rangle^k)$. Then there is a v such that $\forall\mathsf{z}<\mathsf{w}((\exists\mathsf{y}\psi)\rightarrow\exists\mathsf{y}<\mathsf{v}\psi)$. Now take $\mathsf{x}_0<\mathsf{u},\dots,\mathsf{x}_{k-1}<\mathsf{u}$ and assume that $\exists\mathsf{y}\varphi$. Let $\mathsf{z}=\langle\mathsf{x}_0,\dots,\mathsf{x}_{k-1}\rangle^k$. Then $\mathsf{z}<\mathsf{w}$ and $\exists\mathsf{y}\psi$, whence there is a $\mathsf{y}<\mathsf{v}$ so that ψ holds. Since $\mathsf{x}_0\leq\mathsf{z}\wedge\cdots\wedge\mathsf{x}_{k-1}\leq\mathsf{z}$ and $(\bigwedge_{i<k}[\langle\mathsf{z}\rangle^k_i=\mathsf{x}_i])$ we have φ.

Thus $\mathrm{S}\Sigma_n\vdash\forall\mathsf{u}\exists\mathsf{v}\forall\mathsf{x}_0<\mathsf{u}\cdots\forall\mathsf{x}_{k-1}<\mathsf{u}((\exists\mathsf{y}\varphi)\rightarrow\exists\mathsf{y}<\mathsf{v}\varphi)$. □

Lemma 4.7. *Let $k\in\mathbb{N}$ and Φ be a finite set of negation normal $\Sigma_k(\mathrm{S}\Sigma_k)$-formulae. Let $\mathcal{M}\models\mathrm{S}\Sigma_k$. (In truth, we only need strong collection for the existential sub-formulae of Φ.) Then for every $n\in\mathbb{N}_{>0}$ there is a bounded initial partial model $s:n\longrightarrow\mathcal{M}$ such that $\mathcal{M}\overset{s,i,e}{\models_{\mathrm{I}}}\varphi$ for all $\varphi\in\Phi$, $i<n$ and evaluations e of φ in $\mathcal{M}_{<s(i)}$ such that $\mathcal{M}\models_e\varphi$.*

Proof. Let E be the (finite) set of (literal) existential sub-formulae of formulae from Φ. Let, for $\psi \in E$, $v(\psi)$ be the index of the largest free variable of ψ, $w(\psi)$ be the index of the "outermost" bound variable in ψ (which by definition is bound by an existential quantifier) and $\widetilde{\psi}$ be ψ without this quantifier, so that $\psi = \exists \mathsf{v}_{w(\psi)}\widetilde{\psi}$. Moreover, for every such ψ, let $\widehat{\psi}$ be the formula $\forall \mathsf{v}_0{<}\mathsf{v}_\psi \cdots \forall \mathsf{v}_{v(\psi)}{<}\mathsf{v}_\psi(\psi{\to}\exists \mathsf{v}_{w(\psi)}{<}\mathsf{v}_{\psi+1}\widetilde{\psi})$. Note that v_ψ and $\mathsf{v}_{\psi+1}$ do not occur in ψ, and thus are the only free variables of $\widehat{\psi}$.

By Lemma 4.6 we have $\mathcal{M} \models \forall \mathsf{v}_\psi \exists \mathsf{v}_{\psi+1}\widehat{\psi}$. Let $G_\psi : \mathcal{M} \longrightarrow \mathcal{M}$ be a Skolem function for $\exists \mathsf{v}_{\psi+1}\widehat{\psi}$, so that

$$\mathcal{M} \models_e \exists \mathsf{v}_{\psi+1}\widehat{\psi} \Rightarrow \mathcal{M} \models_{e^{G_\psi(e(\mathsf{v}_\psi))}_{\mathsf{v}_{\psi+1}}} \widehat{\psi} \tag{45}$$

for all evaluations e of $\exists \mathsf{v}_{\psi+1}\widehat{\psi}$ in $\mathcal{M}$ (such a construction is possible by the axiom of choice).

Now fix $n \in \mathbb{N}_{>0}$. Define $s : n \longrightarrow \mathcal{M}$ by

$$\begin{aligned} s(0) &= n+1 \\ s(i+1) &= \mathrm{S}_{\mathcal{M}}\left(\max_{\mathcal{M}}(\{s(i)^2\} \cup \bigcup_{\psi \in E}\{G_\psi(s(i))\})\right) \end{aligned}$$

for all $i < n-1$. Note that, since E is finite, in each step the maximum is taken over a finite set, so s is well defined. Thus s is a bounded initial partial model.

We now show that for any φ which is a sub-formula of some formula of Φ, if $i < n$ and e is an evaluation of φ in $\mathcal{M}_{<s(i)}$ such that $\mathcal{M} \models_e \varphi$, then $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \varphi$, by induction on φ.

- Suppose ϑ is a literal and e is an evaluation of ϑ such that $\mathcal{M} \models_e \vartheta$. Then $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta$ for all $i < n$ by Lemma 4.1.

- Let ϑ and ψ be such that $\mathcal{M} \models_f \vartheta \Rightarrow \mathcal{M} \overset{s,j,f}{\models_{\mathrm{I}}} \vartheta$ for all $j < n$ and evaluations f of ϑ in $\mathcal{M}_{<s(j)}$, and the corresponding for ψ. Then

$$\mathcal{M} \models_e \vartheta{\wedge}\psi \Leftrightarrow \mathcal{M} \models_e \vartheta \mathbin{\&} \mathcal{M} \models_e \psi \Rightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \mathbin{\&} \mathcal{M} \overset{s,i,e}{\models} \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta{\wedge}\psi$$

  for all $i < n$ and evaluations e of $\vartheta{\wedge}\psi$ in $\mathcal{M}_{<s(i)}$.

- The disjunctive case is once again similar.

- Assume ϑ is such that $\mathcal{M} \models_f \vartheta \Rightarrow \mathcal{M} \overset{s,j,f}{\models_{\mathrm{I}}} \vartheta$ for all $j < n$ and evaluations f of ϑ in $\mathcal{M}_{<s(j)}$. Let $i < n$, e be an evaluation of $\forall \mathsf{x}\vartheta$ in $\mathcal{M}_{<s(i)}$ and suppose $\mathcal{M} \models_e \forall \mathsf{x}\vartheta$. Take $l < n$, $l \geq i$ and $a \in \mathcal{M}_{<s(l)}$. Then e^a_{x} is an evaluation of ϑ in $\mathcal{M}_{<s(l)}$ and $\mathcal{M} \models_{e^a_{\mathsf{x}}} \vartheta$, whence

$$\mathcal{M} \overset{s,l,e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta$$

by induction hypothesis. Since l and $a \in \mathcal{M}_{<s(l)}$ were arbitrary with $l < n$ and $l > i$, $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \forall \mathsf{x}\vartheta$.

- Let ϑ be such that $\mathcal{M} \models_f \vartheta \Rightarrow \mathcal{M} \overset{s,j,f}{\models_{\mathrm{I}}} \vartheta$ for all $j < n$ and evaluations f of ϑ in $\mathcal{M}_{<s(j)}$, and consider $\exists \mathsf{x}\vartheta$. Note $w(\exists \mathsf{x}\vartheta) = \mathsf{x}$. By assumption this is a sub-formula of some $\psi \in \Phi$, whence $\exists \mathsf{x}\vartheta \in E$. Thus we have $G_{\exists \mathsf{x}\vartheta}$ as defined above. Let $i < n$ and e be an evaluation of $\exists \mathsf{x}\vartheta$ in $\mathcal{M}_{<s(i)}$ such that $\mathcal{M} \models_e \exists \mathsf{x}\vartheta$. If $i + 1 = n$ then $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \exists \mathsf{x}\vartheta$ vacuously, so assume $i + 1 < n$. Let $a = G_{\exists \mathsf{x}\vartheta}(s(i))$ which by construction is in $\mathcal{M}_{<s(i+1)}$. Recall that

  $$\mathcal{M} \models \exists \mathsf{v}_{\psi+1} \forall \mathsf{v}_0 {<} s(i) \cdots \forall \mathsf{v}_{v(\psi)} {<} s(i) (\exists \mathsf{x}\vartheta \rightarrow \exists \mathsf{x} {<} \mathsf{v}_{\psi+1} \vartheta),$$

  whence

  $$\mathcal{M} \models \forall \mathsf{v}_0 {<} s(i) \cdots \forall \mathsf{v}_{v(\psi)} {<} s(i) (\exists \mathsf{x}\vartheta \rightarrow \exists \mathsf{x} {<} a\, \vartheta)$$

  by (45). Since e is an evaluation of $\exists \mathsf{x}\vartheta$ in $\mathcal{M}_{<s(i)}$ we get

  $$\mathcal{M} \models_e \exists \mathsf{x}\vartheta \rightarrow \exists \mathsf{x} {<} a\, \vartheta,$$

  whence

  $$\mathcal{M} \models_e \exists \mathsf{x} {<} a\, \vartheta.$$

  Thus there is a $b <_{\mathcal{M}} a$ such that

  $$\mathcal{M} \models_{e^b_{\mathsf{x}}} \vartheta,$$

  whence, since e^b_{x} is an evaluation of ϑ in $\mathcal{M}_{<s(i+1)}$,

  $$\mathcal{M} \overset{s,i+1,e^b_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta$$

  by induction hypothesis. Thus $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \exists \mathsf{x}\vartheta$.

This concludes the induction.

In particular, for any $\varphi \in \Phi$, $i < n$ and evaluation e of φ in $\mathcal{M}_{<s(i)}$ we have that $\mathcal{M} \models_e \varphi$ implies $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \varphi$. □

Theorem 4.8. *Suppose T is Σ_2-sound. Then $T \nvdash \neg \forall \mathsf{x} \mathtt{IK}(\mathsf{x})$ in case any of the following holds:*

1. PA $\subseteq T$,

2. $\mathbb{N} \models T$

Proof. First, let $\mathcal{M}$ be a model of $T + \mathrm{PA}$, which exists in either case. Let $n \in \mathbb{N}$. Since PA proves strong collection, the above lemma gives a bounded initial partial model $s : n+1 \longrightarrow \mathcal{M}$ with $\mathcal{M} \overset{s,0,\emptyset}{\models_{\mathrm{I}}} \mathrm{nnf}(\mathrm{ax}(i))$ for all $i < n$. This is a finite sequence, whence $\mathfrak{F}_{\mathrm{f}}(s)$ exists and satisfies $\mathcal{M} \models \mathtt{IBpm}(\mathfrak{F}_{\mathrm{f}}(s)) \wedge [\mathtt{lh}(\mathfrak{F}_{\mathrm{f}}(s)) = \overline{n+1}]$. By Proposition 4.5 this means that

$$\mathcal{M} \models \mathtt{IFulf}(\mathfrak{F}_{\mathrm{f}}(s),0,\mathfrak{F}_{\mathrm{f}}(\emptyset),\overline{\mathrm{nnf}(\mathrm{ax}(i))},\overline{\mathrm{nnf}(\mathrm{ax}(i))}),$$

that is

$$\mathcal{M} \models [\mathtt{IFulf}(\mathfrak{F}_{\mathrm{f}}(s),0,\varnothing,\mathtt{nnf}(\mathtt{ax}(\bar{i})),\mathtt{nnf}(\mathtt{ax}(\bar{i})))]$$

since $\mathcal{M} \models [\varnothing = \mathfrak{F}_{\mathrm{f}}(\emptyset)]$ and $\mathcal{M} \models [\mathtt{nnf}(\mathtt{ax}(\bar{i})) = \overline{\mathrm{nnf}(\mathrm{ax}(i))}]$ (by Σ_1-completeness), for all $i < n$. Furthermore, Σ_1-completeness also gives $\mathcal{M} \models \mathtt{Sent}(\mathtt{ax}(\bar{i}))$ for all $i < n$. Thus $\mathcal{M} \models \mathtt{IFbl}(\mathfrak{F}_{\mathrm{f}}(s),\mathtt{ax}(\bar{i}))$ for all $i < n$. To sum up

$$\mathcal{M} \models \exists \mathtt{s} \mathtt{IBpm}(\mathtt{s}) \wedge [\mathtt{lh}(\mathtt{s}) = \overline{n+1}] \wedge \forall \mathtt{y} < \overline{n} \mathtt{IFbl}(\mathtt{s},\mathtt{ax}(\mathtt{y})).$$

Hence $\mathcal{M} \models \mathtt{IK}(\overline{n})$ for all $n \in \mathbb{N}$.

We next show that, in either case, $\mathbb{N} \models \forall \mathtt{x} \mathtt{IK}(\mathtt{x})$.

1. Let $\mathcal{M}$ be a model of T; by above $\mathcal{M} \models \mathtt{IK}(\overline{n})$ for all $n \in \mathbb{N}$. Hence $T \vdash \mathtt{IK}(\overline{n})$ for all $n \in \mathbb{N}$. Since T is Σ_1-sound, $\mathbb{N} \models \mathtt{IK}(\overline{n})$ for all $n \in \mathbb{N}$

2. Since $\mathbb{N} \models T + \mathrm{PA}$, the above gives $\mathbb{N} \models \mathtt{IK}(\overline{n})$ for all $n \in \mathbb{N}$.

Now assume $T \vdash \neg \forall \mathtt{x} \mathtt{IK}(\mathtt{x})$, whence $\mathbb{N} \models \neg \forall \mathtt{x} \mathtt{IK}(\mathtt{x})$ as T is Σ_2-sound. This contradicts the above. □

Remark 15. From the perspective of incompleteness the first alternative in the above theorem is the stronger, since if T is sound and complete then certainly $\mathrm{PA} \subseteq T$.

Note also that the proof shows that $\forall \mathtt{x} \mathtt{IK}(\mathtt{x})$ is true (in $\mathbb{N}$).

4.3 The initial-Gödel-Kripke sentence is not derivable

We finally turn to verifying the non-provability of the initial-Gödel-Kripke sentence, thereby establishing its independence. The idea is, given a nonstandard model of T where the sentence is true (or at least not too false), to use the sequence given by the sentence to construct a structure where the sentence is false. To ensure that this structure is itself a model of T, we will need the following result.

Lemma 4.9. *Let $\mathcal{M} \models \mathrm{PA}^-$, $\mathcal{L}$ be a discrete linear order without maximal element and $s : \mathcal{L} \longrightarrow \mathcal{M}$ be an increasing sequence unbounded (or cofinal) in $\mathcal{M}$ (that is $\forall a \in \mathcal{M} \exists i \in \mathcal{L} : a <_{\mathcal{M}} s(i)$). Then $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \varphi \Rightarrow \mathcal{M} \models_e \varphi$ for every negation normal arithmetic formula φ, all evaluations e of φ and all $i \in \mathcal{L}$.*

Proof. The proof is by induction on φ.

- Let ϑ be a literal. Then $\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \models_e \vartheta$ for all evaluations e of ϑ and $i \in \mathcal{L}$, by Lemma 4.1.

- Suppose ϑ and ψ are such that $\mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \models_f \vartheta$ for all evaluations f of ϑ and $k \in \mathcal{L}$, and the corresponding for ψ. Then

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \wedge \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \vartheta \mathbin{\bar\wedge} \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \psi \Rightarrow \mathcal{M} \models_e \vartheta \mathbin{\bar\wedge} \mathcal{M} \models_e \psi \Leftrightarrow \mathcal{M} \models_e \vartheta \wedge \psi$$

  for all evaluations e of $\varphi \wedge \psi$ and $i \in \mathcal{L}$.

- The disjunctive case is the same, mutatis mutandis.

- Let ϑ be such that $\mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \models_f \vartheta$ for all evaluations f of ϑ and $k \in \mathcal{L}$, and consider $\forall \mathsf{x} \vartheta$. Let e be an evaluation of $\forall \mathsf{x} \vartheta$ and $i \in \mathcal{L}$ and suppose

$$\mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \forall \mathsf{x} \vartheta. \tag{46}$$

  Also let $a \in \mathcal{M}$. Since s unbounded in $\mathcal{M}$, there is a $j \in \mathcal{L}$ such that $a <_{\mathcal{M}} s(j)$; since s is increasing, we can choose $j \geq_{\mathcal{L}} i$. By (46) we thus have

$$\mathcal{M} \overset{s,j,e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta$$

  which by induction hypothesis implies that

$$\mathcal{M} \models_{e^a_{\mathsf{x}}} \vartheta.$$

  Since $a \in \mathcal{M}$ was arbitrary, $\mathcal{M} \models_e \forall \mathsf{x} \vartheta$.

- Suppose ϑ satisfies $\mathcal{M} \overset{s,k,f}{\models_{\mathrm{I}}} \vartheta \Rightarrow \mathcal{M} \models_f \vartheta$ for all evaluations f of ϑ and $k \in \mathcal{L}$. Then

$$\begin{aligned} \mathcal{M} \overset{s,i,e}{\models_{\mathrm{I}}} \exists \mathsf{x} \vartheta &\Leftrightarrow \bar\exists a \in \mathcal{M}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{M} \overset{s,\mathrm{S}_{\mathcal{L}}(i),e^a_{\mathsf{x}}}{\models_{\mathrm{I}}} \vartheta \\ &\Rightarrow \bar\exists a \in \mathcal{M}_{<s(\mathrm{S}_{\mathcal{L}}(i))} : \mathcal{M} \models_{e^a_{\mathsf{x}}} \vartheta \\ &\Rightarrow \mathcal{M} \models_e \exists \mathsf{x} \vartheta \end{aligned}$$

  for all evaluations e of $\exists \mathsf{x} \vartheta$ and $i \in \mathcal{L}$, since $\mathcal{L}$ has no maximum.

This concludes the induction, and the proof. □

Corollary 4.10. *Let $\mathcal{M} \models PA^-$, $\mathcal{L}$ be a discrete linear order without maximum and $s : \mathcal{L} \longrightarrow \mathcal{M}$ be an unbounded increasing sequence such that s initial-fulfils the negation normal sentence φ in $\mathcal{M}$. Then $\mathcal{M} \models \varphi$.*

Proof. Immediate by the lemma. □

Theorem 4.11. *Suppose T has a nonstandard model $\mathcal{M}$ with $\mathcal{M} \models \mathtt{IK}(m)$ for some nonstandard $m \in \mathcal{M}$. Then there is a nonstandard model $\mathcal{N}$ of T such that $\mathcal{N} \not\models \forall \mathtt{x} \mathtt{IK}(\mathtt{x})$. In particular $T \nvdash \forall \mathtt{x} \mathtt{IK}(\mathtt{x})$.*

Proof. Since $\mathtt{IK}(\mathtt{x}) = \exists \mathtt{s}(\mathtt{IBpm}(\mathtt{s}) \wedge [\mathtt{lh}(\mathtt{s}){=}\mathtt{S}(\mathtt{x})] \wedge \forall \mathtt{y}{<}\mathtt{x}\mathtt{IFbl}(\mathtt{s},\mathtt{ax}(\mathtt{y})))$ is a $\Sigma_1(\mathrm{I}\Sigma_1)$-formula and $\mathcal{M} \models \mathrm{I}\Sigma_1$, let $s \in \mathcal{M}$ be least (by the least number principle for $\Sigma_1(\mathcal{M})$-formulae, 2.23) so that

$$\mathcal{M} \models \mathtt{IBpm}(s) \wedge [\mathtt{lh}(s){=}\mathtt{S}(m)] \wedge \forall \mathtt{y}{<}m\mathtt{IFbl}(s,\mathtt{ax}(\mathtt{y})).$$

Consider $f = \mathfrak{R}_{\mathrm{f}}(s) \restriction \mathbb{N}$; then $f(k)^2_{\mathcal{M}} <_{\mathcal{M}} f(\mathrm{S}_{\mathcal{M}}(k))$ for all $k \in \mathbb{N}$. Define $N = \bigcup_{k \in \mathbb{N}} \mathcal{M}_{<f(k)}$. Since $a, b <_{\mathcal{M}} f(k)$ implies $\mathrm{S}_{\mathcal{M}}(a), a +_{\mathcal{M}} b, a \cdot_{\mathcal{M}} b <_{\mathcal{M}} f(k)^2 <_{\mathcal{M}} f(k+1)$ (since $f(0) >_{\mathcal{M}} m >_{\mathcal{M}} 2$) by Lemma 2.6, N is closed under $\mathrm{S}_{\mathcal{M}}$, $+_{\mathcal{M}}$ and $\cdot_{\mathcal{M}}$. Furthermore, if $a \in N$ and $b \in \mathcal{M}$ is such that $b <_{\mathcal{M}} a$, then $b <_{\mathcal{M}} f(k)$ for some $k \in \mathbb{N}$, whence $b \in N$. Thus $N \subseteq \mathcal{M}$ is (the underlying set of) an initial substructure $\mathcal{N}$, which is a model of PA^- by Corollary 2.8.

Now let φ be the negation normal form of some axiom of T, that is $\varphi = \mathrm{nnf}(\mathrm{ax}(n))$ for some $n \in \mathbb{N}$. Since m is nonstandard, $n <_{\mathcal{M}} m$. By choice of s we thus have

$$\mathcal{M} \models [\mathtt{IFulf}(s,0,\varnothing,\overline{\varphi},\overline{\varphi})].$$

Proposition 4.5 then gives $\mathcal{M} \overset{\mathfrak{R}_{\mathrm{f}}(s),0,\emptyset}{\models_{\mathrm{I}}} \varphi$, whence $\mathcal{M} \overset{f,0,\emptyset}{\models_{\mathrm{I}}} \varphi$ by Lemma 4.2 and thus $\mathcal{N} \overset{f,0,\emptyset}{\models_{\mathrm{I}}} \varphi$ by Lemma 4.3. Now Lemma 4.9 guarantees $\mathcal{N} \models \varphi$. Since this holds for all axioms of T, we have $\mathcal{N} \models T$.

Note however that $s \notin \mathcal{N}$, since for no $k \in \mathbb{N}$ do we have $\mathcal{M} \models [s < \mathtt{apl}(s,\overline{k})]$; in fact $a <_{\mathcal{M}} s$ for all $a \in \mathcal{N}$. On the other hand $m \in \mathcal{N}$ since $m <_{\mathcal{M}} s(0)$[11]. Hence if $\mathcal{N} \models \forall \mathtt{x}\mathtt{IK}(\mathtt{x})$ there is a $t \in \mathcal{N}$ such that $\mathcal{N} \models \mathtt{IBpm}(t) \wedge [\mathtt{lh}(t){=}\mathtt{S}(m)] \wedge \forall \mathtt{y}{<}m\mathtt{IFbl}(t,\mathtt{ax}(\mathtt{y}))$. But this is a $\Delta_1(\mathrm{I}\Sigma_1)$-formula, whence $\mathcal{M} \models \mathtt{IBpm}(t) \wedge [\mathtt{lh}(t){=}\mathtt{S}(m)] \wedge \forall \mathtt{y}{<}m\mathtt{IFbl}(t,\mathtt{ax}(\mathtt{y}))$ by Lemma 2.5. Since $t <_{\mathcal{M}} s$, this contradicts the choice of s. Consequently $\mathcal{N} \not\models \forall \mathtt{x}\mathtt{IK}(\mathtt{x})$. □

Corollary 4.12. *If T is Σ_2-sound and extends* PA, *then T is incomplete.*

Proof. By Theorem 4.8, $T + \forall \mathtt{x}\mathtt{IK}(\mathtt{x})$ has a model, which without loss of generality can be chosen nonstandard by the Löwenheim-Skolem theorems. Then Theorem 4.11 gives a nonstandard model of $T + \neg\forall \mathtt{x}\mathtt{IK}(\mathtt{x})$. □

Remark 16. As Putnam remarks in [10], these results have the following sharpening in case $T + \mathrm{PA}$ is consistent: If $\mathcal{M}$ is any nonstandard model of $T + \mathrm{PA}$ then $\mathcal{M} \models \mathtt{IK}(\overline{n})$ for all $n \in \mathbb{N}$ as in the proof of Theorem 4.8, whence there is a nonstandard $m \in \mathcal{M}$ so that $\mathcal{M} \models \mathtt{IK}(m)$ by Σ_1-overspill (Lemma 2.24). The proof of the above theorem then

[11] This is the principal reason for this requirement

yields a (nonstandard) model of $T + \mathrm{PA} + \neg\forall \mathtt{x}\mathtt{IK}(\mathtt{x})$. In case T is sound we thus have an explicit model of $T + \forall \mathtt{x}\mathtt{IK}(\mathtt{x})$, namely $\mathbb{N}$, and by above can construct an explicit model of $T + \neg\forall \mathtt{x}\mathtt{IK}(\mathtt{x})$ by choosing $\mathcal{M}$ as an appropriate ultrapower of $\mathbb{N}$.

The above theorems, while establishing incompleteness for PA itself, are still somewhat limited; for if the proof of non-refutability of $\forall \mathtt{x}\mathtt{IK}(\mathtt{x})$ in T is to work, then T must be either sound or prove the collection schema (see Theorem 4.8). As far as incompleteness is concerned, this means that T is an extension of PA (sound theories are consistent with PA, so if T is to be complete then $\mathrm{PA} \subseteq T$). The reason why collection is required can be seen in the prof of Lemma 4.7, where a bounded initial partial model is constructed by recursion using a finite set of Skolem functions. The issue here is that, if an element m of the sequence is ever nonstandard, then the next element of the sequence should be an upper bound to the values of the Skolem functions in the (infinite) initial segment defined by m, and without (the appropriate instances of) collection there is no reason why such an element would exist. Furthermore, we have to apply this lemma to any finite collection of the axioms, whence the full collection schema seems the only plausible alternative.

One remedy to this is to generalise the notion of initial fulfilment so that, instead of requiring that quantifiers be instantiated in initial segments given by the elements of some sequence, we can instantiate them in some element of a (more general) sequence of sets. This will be the approach of the next section.

5 Fulfilment: Incompleteness of $\mathrm{I}\Sigma_1$

We now turn to the full notion of fulfilment where we replace the initial segments of initial-fulfilment by arbitrary sets. While able to prove incompleteness of arbitrary recursively axiomatisable extensions of $\mathrm{I}\Sigma_1$, this generalisation carries a cost, as the counter model we will construct in Theorem 5.11 (similarly to in Theorem 4.11) will not be an initial substructure of the original model, and so we can not appeal to Δ_0-absoluteness for initial substructures to make the proof go through. We will instead have the independent-to-be sentence *express* Δ_0-absoluteness between any of its models and the bounded partial model it claims to exist. Nevertheless, most arguments will be essentially identical to those of section 4 and will be omitted.

The material in this section was presented by Quinsey in [12], where it is attributed mainly to Kripke. Like in the previous section, our approach is a combination of the ones in [12] and [10], with the addition of a number of technical proofs.

5.1 Definitions

Definition 5.1. Let $\mathcal{L}$ be a discrete linear order. Given a $\mathcal{L}_\mathrm{A}$-structure $\mathcal{M}$ and an $\mathcal{L}$-sequence $s : \mathcal{L} \longrightarrow \mathcal{P}(\mathcal{M})$, we define for formulae φ to be fulfilled by s with respect to an assignment e in $\mathcal{M}$ at $i \in \mathcal{L}$ recursively as follows:

$$\begin{aligned}
\mathcal{M} &\overset{s,i,e}{\models} \vartheta \Leftrightarrow \mathcal{M} \models_e \vartheta && \text{if } \vartheta \text{ is atomic,}\\
\mathcal{M} &\overset{s,i,e}{\models} \neg\vartheta \Leftrightarrow \forall j \in \mathcal{L}_{\geq i} : \mathcal{M} \overset{s,j,e}{\not\models} \vartheta, \\
\mathcal{M} &\overset{s,i,e}{\models} \vartheta \wedge \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models} \vartheta \wedge \mathcal{M} \overset{s,i,e}{\models} \psi, \\
\mathcal{M} &\overset{s,i,e}{\models} \vartheta \vee \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models} \vartheta \vee \mathcal{M} \overset{s,i,e}{\models} \psi, \\
\mathcal{M} &\overset{s,i,e}{\models} \forall \mathsf{x}\vartheta \Leftrightarrow \forall j \in \mathcal{L}_{\geq i} \forall a \in s(j) : \mathcal{M} \overset{s,j,e^a_x}{\models} \vartheta, \\
\mathcal{M} &\overset{s,i,e}{\models} \exists \mathsf{x}\vartheta \Leftrightarrow (\mathrm{S}_\mathcal{L}(i) \in \mathcal{L} \Rightarrow \exists a \in s(\mathrm{S}_\mathcal{L}(i)) : \mathcal{M} \overset{s,\mathrm{S}_\mathcal{L}(i),e^a_x}{\models} \vartheta).
\end{aligned}$$

If φ is a sentence then φ is simply *fulfilled* by s in $\mathcal{M}$ if $\mathcal{M} \overset{s,0,\emptyset}{\models} \varphi$. If there is an increasing $\mathcal{L}$-sequence s which fulfils the sentence φ in $\mathcal{M}$, then φ is $\mathcal{L}$-*fulfillable* in $\mathcal{M}$.

As can be seen from the definitions, initial-fulfilment by a sequence s of elements of a model $\mathcal{M}$ of PA^- is the same as fulfilment by the sequence of initial segments of $\mathcal{M}$ defined by the elements of s, hence the term initial-fulfilment. The reason we require $\mathcal{M} \models \mathrm{PA}^-$ in the definition of initial-fulfilment is that we want $\mathcal{M}$ to be (linearly) ordered by $<_\mathcal{M}$ for the definition of initial-fulfilment to make sense; this is not an issue for (general) fulfilment.

The differences between the two notions will result in a slightly more convoluted formalisation of this concept in the theory $\mathrm{I}\Sigma_1$, and some additional issues in finding a model witnessing $T \not\vdash \forall \mathsf{x}\mathsf{K}(\mathsf{x})$ (to be defined, see Definition 5.3), since a model defined as the union of all elements of such a sequence will not necessarily be an initial substructure of $\mathcal{M}$.

Remark 17. As in Definition 4.1, we remark on the similarity between this definition and that of truth in a Kripke model (of e.g. constructive mathematics, see for example [16]).

Lemma 5.1. *Let $\mathcal{M}$ be an $\mathcal{L}_{\mathrm{A}}$-structure, $\mathcal{L}$ a discrete linear order and $s : \mathcal{L} \longrightarrow \mathcal{P}(\mathcal{M})$. If s is increasing and φ is on negation normal form then $\mathcal{M} \overset{s,i,e}{\models} \varphi \Rightarrow \mathcal{M} \overset{s,j,e}{\models} \varphi$ for all $i \in \mathcal{L}$, $j \in \mathcal{L}_{\geq i}$ and all evaluations e of φ in $\mathcal{M}$. If instead φ is quantifier free we have $\mathcal{M} \overset{s,i,e}{\models} \varphi \Leftrightarrow \mathcal{M} \models_e \varphi$ for all such i and e.*

Proof. The proof is the same as that of Lemma 4.1, mutatis mutandis. □

As in the case of initial fulfilment, this shows that as far as negation normal formulae are concerned, we could have used the definition

$$\mathcal{M} \overset{s,i,e}{\models} \neg\vartheta \Leftrightarrow \mathcal{M} \overset{s,i,e}{\not\models} \vartheta$$

of a negated formula being fulfilled by s at i.

Lemma 5.2. *Let $\mathcal{M}$ be an $\mathcal{L}_{\mathrm{A}}$-structure, $\mathcal{L}$ be a discrete linear order and s an increasing $\mathcal{L}$-sequence in $\mathcal{P}(\mathcal{M})$. Let $\mathcal{L}' \subseteq \mathcal{L}$ be closed under the successor function in $\mathcal{L}$ and $s' = s \restriction \mathcal{L}'$. Then $\mathcal{M} \overset{s,i,e}{\models} \varphi \Rightarrow \mathcal{M} \overset{s',i,e}{\models} \varphi$ for every negation normal formula φ, all evaluations e of φ in $\mathcal{M}$ and all $i \in \mathcal{L}'$.*

Proof. The same proof as Lemma 4.2, mutatis mutandis. □

Lemma 5.3. *Let $\mathcal{N} \subset \mathcal{M}$ be $\mathcal{L}_{\mathrm{A}}$-structures , $\mathcal{L}$ be a discrete linear order and $s : \mathcal{L} \longrightarrow \mathcal{P}(\mathcal{N})$. Then $\mathcal{N} \overset{s,i,e}{\models} \varphi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models} \varphi$ for every formula φ, all evaluations e of φ in $\mathcal{N}$ and all $i \in \mathcal{L}$.*

Proof. The proof is similar to (in fact slightly more straightforward than) that of Lemma 4.3, and is omitted. □

In the same way as initial-fulfilment, these notions are formalisable within $\mathrm{I}\Sigma_1$.

Proposition 5.4. *Thereis a* $\Delta_1(\mathrm{I}\Sigma_1)$*-formula* $\mathtt{Fulf}$ *with the following properties:*

$\mathrm{I}\Sigma_1 \vdash (\mathtt{Setq}(\mathsf{s}) \wedge [\mathsf{i}<\mathtt{lh}(\mathsf{s})] \wedge [\mathsf{e} \sqsubseteq (<\mathsf{n}) \times (<\mathsf{s})] \wedge \mathtt{Term}(\mathsf{r}) \wedge \mathtt{Term}(\mathsf{t}) \wedge [\mathtt{Eval}(\mathsf{e},\mathbf{(}\mathsf{r}=\mathsf{t}\mathbf{)})] \wedge [\mathbf{(}\mathsf{r}=\mathsf{t}\mathbf{)} \leq \mathsf{n}]) \rightarrow$
$([\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathbf{(}\mathsf{r}=\mathsf{t}\mathbf{)})] \leftrightarrow ([\mathtt{Eval}(\mathsf{e},\mathbf{(}\mathsf{r}=\mathsf{t}\mathbf{)})] \wedge [\mathtt{val}(\mathsf{e},\mathsf{r})=\mathtt{val}(\mathsf{e},\mathsf{t})]))$

$\mathrm{I}\Sigma_1 \vdash (\mathtt{Setq}(\mathsf{s}) \wedge [\mathsf{i}<\mathtt{lh}(\mathsf{s})] \wedge [\mathsf{e} \sqsubseteq (<\mathsf{n}) \times (<\mathsf{s})] \wedge \mathtt{Term}(\mathsf{r}) \wedge \mathtt{Term}(\mathsf{t}) \wedge [\mathtt{Eval}(\mathsf{e},\mathbf{(}\mathsf{r}<\mathsf{t}\mathbf{)})] \wedge [\mathbf{(}\mathsf{r}<\mathsf{t}\mathbf{)} \leq \mathsf{n}]) \rightarrow$
$([\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathbf{(}\mathsf{r}<\mathsf{t}\mathbf{)})] \leftrightarrow ([\mathtt{Eval}(\mathsf{e},\mathbf{(}\mathsf{r}<\mathsf{t}\mathbf{)})] \wedge [\mathtt{val}(\mathsf{e},\mathsf{r})<\mathtt{val}(\mathsf{e},\mathsf{t})]))$

$\mathrm{I}\Sigma_1 \vdash (\mathtt{Setq}(\mathsf{s}) \wedge [\mathsf{i}<\mathtt{lh}(\mathsf{s})] \wedge [\mathsf{e} \sqsubseteq (<\mathsf{n}) \times (<\mathsf{s})] \wedge \mathtt{Fmla}(\mathsf{f}) \wedge [\mathtt{Eval}(\mathsf{e},\mathbf{(}\neg\mathsf{f}\mathbf{)})] \wedge [\mathbf{(}\neg\mathsf{f}\mathbf{)} \leq \mathsf{n}]) \rightarrow$
$([\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathbf{(}\neg\mathsf{f}\mathbf{)})] \leftrightarrow \forall[\mathsf{j}<\mathtt{lh}(\mathsf{s})](\mathsf{i}\leq\mathsf{j} \rightarrow \neg\mathtt{Fulf}(\mathsf{s},\mathsf{j},\mathsf{e},\mathsf{n},\mathsf{f})))$

$\mathrm{I}\Sigma_1 \vdash (\mathtt{Setq}(\mathsf{s}) \wedge [\mathsf{i}<\mathtt{lh}(\mathsf{s})] \wedge [\mathsf{e} \sqsubseteq (<\mathsf{n}) \times (<\mathsf{s})] \wedge \mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g}) \wedge [\mathtt{Eval}(\mathsf{e},\mathbf{(}\mathsf{f}\wedge\mathsf{g}\mathbf{)})] \wedge [\mathbf{(}\mathsf{f}\wedge\mathsf{g}\mathbf{)} \leq \mathsf{n}]) \rightarrow$
$([\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathbf{(}\mathsf{f}\wedge\mathsf{g}\mathbf{)})] \leftrightarrow (\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{f}) \wedge \mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{g})))$

$\mathrm{I}\Sigma_1 \vdash (\mathtt{Setq}(\mathsf{s}) \wedge [\mathsf{i}<\mathtt{lh}(\mathsf{s})] \wedge [\mathsf{e} \sqsubseteq (<\mathsf{n}) \times (<\mathsf{s})] \wedge \mathtt{Fmla}(\mathsf{f}) \wedge \mathtt{Fmla}(\mathsf{g}) \wedge [\mathtt{Eval}(\mathsf{e},\mathbf{(}\mathsf{f}\vee\mathsf{g}\mathbf{)})] \wedge [\mathbf{(}\mathsf{f}\vee\mathsf{g}\mathbf{)} \leq \mathsf{n}]) \rightarrow$
$([\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathbf{(}\mathsf{f}\vee\mathsf{g}\mathbf{)})] \leftrightarrow (\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{f}) \vee \mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathsf{g})))$

$\mathrm{I}\Sigma_1 \vdash (\mathtt{Setq}(\mathsf{s}) \wedge [\mathsf{i}<\mathtt{lh}(\mathsf{s})] \wedge [\mathsf{e} \sqsubseteq (<\mathsf{n}) \times (<\mathsf{s})] \wedge \mathtt{Fmla}(\mathsf{f}) \wedge [\mathtt{Eval}(\mathsf{e},\mathbf{(}\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}\mathbf{)})] \wedge [\mathbf{(}\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}\mathbf{)} \leq \mathsf{n}]) \rightarrow$
$([\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathbf{(}\forall\mathbf{v}_{\mathsf{x}}\mathsf{f}\mathbf{)})] \leftrightarrow \forall[\mathsf{j}<\mathtt{lh}(\mathsf{s})](\mathsf{i}\leq\mathsf{j} \rightarrow \forall[\mathsf{a}\epsilon(\mathsf{s})_{\mathsf{j}}][\mathtt{Fulf}(\mathsf{s},\mathsf{j},\mathtt{ext}(\mathsf{e},\mathbf{v}_{\mathsf{x}},\mathsf{a}),\mathsf{n},\mathsf{f})]))$

$\mathrm{I}\Sigma_1 \vdash (\mathtt{Setq}(\mathsf{s}) \wedge [\mathsf{i}<\mathtt{lh}(\mathsf{s})] \wedge [\mathsf{e} \sqsubseteq (<\mathsf{n}) \times (<\mathsf{s})] \wedge \mathtt{Fmla}(\mathsf{f}) \wedge [\mathtt{Eval}(\mathsf{e},\mathbf{(}\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}\mathbf{)})] \wedge [\mathbf{(}\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}\mathbf{)} \leq \mathsf{n}]) \rightarrow$
$([\mathtt{Fulf}(\mathsf{s},\mathsf{i},\mathsf{e},\mathsf{n},\mathbf{(}\exists\mathbf{v}_{\mathsf{x}}\mathsf{f}\mathbf{)})] \leftrightarrow ([\mathsf{S}(\mathsf{i})<\mathtt{lh}(\mathsf{s})] \rightarrow \exists[\mathsf{a}\epsilon(\mathsf{s})_{\mathsf{S}(\mathsf{i})}][\mathtt{Fulf}(\mathsf{s},\mathsf{S}(\mathsf{i}),\mathtt{ext}(\mathsf{e},\mathbf{v}_{\mathsf{x}},\mathsf{a}),\mathsf{n},\mathsf{f})])).$

Proof. The proof is the same as that of Proposition 4.4, with a number of occurrences of $\mathtt{Seq}$ and $<$ replaced by $\mathtt{Setq}$ and ϵ, respectively (since these are all $\Delta_1(\mathrm{I}\Sigma_1)$ concepts, this replacement does not alter the construction). □

We can now define

$$\mathtt{Fbl}(\mathsf{s},\mathsf{f}) = \mathtt{Incrsetq}(\mathsf{s}) \wedge \mathtt{Sent}(\mathsf{f}) \wedge [\mathtt{Fulf}(\mathsf{s},0,\varnothing,\mathtt{nnf}(\mathsf{f}),\mathtt{nnf}(\mathsf{f}))]$$

With this, we are almost ready to construct the Gödel-Kripke sentence which we will show to be independent of $\mathrm{I}\Sigma_1$.

First however, as in the case of initial fulfilment we should confirm that this formal definition is adequate: that it captures exactly what is meant by a sequence fulfilling a formula. The similarities of the definitions, and of the corresponding ones for initial fulfilment, of course make this rather plausible.

Proposition 5.5. *Let* $\mathcal{M} \models \mathrm{I}\Sigma_1$ *and* $s, l \in \mathcal{M}_{>0}$ *be such that* $\mathcal{M} \models \mathtt{Setq}(s)$ *and* $\mathcal{M} \models [\mathtt{lh}(s)=l]$. *Let* $m \in \mathcal{M}$. *Given a formula* $\varphi \leq_{\mathcal{M}} m$ *and a finite evaluation* e *of* φ *in* $\mathcal{M}_{<s}$ *undefined for variables* $\mathsf{x} \geq_{\mathcal{M}} m$, *we have* $\mathcal{M} \models \mathtt{Fulf}(s,i,\mathfrak{F}_{\mathrm{f}}(e),m,\overline{\varphi})$ *iff* $\mathcal{M} \overset{\mathfrak{R}_{\mathrm{sf}}(s),i,e}{\models} \varphi$, *for all* $i \in \mathcal{M}_{<l}$.

Proof. The proof is essentially the same as that of Proposition 4.5. The important differences are that $\mathfrak{F}_{\mathrm{f}}(s)$ is replaced by $\mathfrak{F}_{\mathrm{sf}}(s)$ and that $\mathcal{M}_{<\mathfrak{R}_{\mathrm{f}}(s)(k)}$ is replaced by its subset $\mathfrak{R}_{\mathrm{sf}}(s)(k)$ (since $\mathrm{I}\Sigma_1 \vdash \mathsf{x}\epsilon\mathsf{y} \rightarrow \mathsf{x}<\mathsf{y}$ and $\mathrm{I}\Sigma_1 \vdash \mathtt{Setq}(\mathsf{s}) \rightarrow \mathtt{Seq}(\mathsf{s})$). □

Definition 5.2 (Bounded partial models)**.** Let $\mathcal{M} \models \mathrm{PA}^-$ and $l \in \mathcal{M}_{>0}$. A sequence $s : \mathcal{M}_{<l} \longrightarrow \mathcal{P}(\mathcal{M})$ is a *bounded partial model* if $0, l \in s(0)$ and

$$\mathrm{S}_{\mathcal{M}}(a), a +_{\mathcal{M}} b, a \cdot_{\mathcal{M}} b \in s(\mathrm{S}_{\mathcal{M}}(i))$$

for all $a, b \in s(i)$, for all non-maximal $i \in \mathcal{M}_{<l}$. If in addition $l \in \mathbb{N}$ and $s(i)$ is finite for every $i \in \mathcal{M}_{<l}$ we call s a *finite* partial model (the term is introduced in [11]). We furthermore define the corresponding notion in $\mathrm{I}\Sigma_1$ by the $\Delta_1(\mathrm{I}\Sigma_1)$-formula $\mathtt{Bpm}(\mathsf{s})$:

$$\begin{aligned}&\mathtt{Setq}(\mathsf{s}) \wedge [0 < \mathtt{lh}(\mathsf{s})] \wedge [0 \epsilon (\mathsf{s})_0] \wedge [\mathtt{lh}(\mathsf{s}) \epsilon (\mathsf{s})_0] \wedge \\ &\forall [\mathsf{i} < \mathtt{lh}(\mathsf{s})]([\mathsf{S}(\mathsf{i}) < \mathtt{lh}(\mathsf{s})] \rightarrow \\ &\qquad (\forall [\mathsf{x} \epsilon (\mathsf{s})_{\mathsf{i}}] \forall [\mathsf{y} \epsilon (\mathsf{s})_{\mathsf{i}}][\mathsf{S}(\mathsf{x}) \epsilon (\mathsf{s})_{\mathsf{S(i)}}] \wedge [\mathsf{x} + \mathsf{y} \epsilon (\mathsf{s})_{\mathsf{S(i)}}] \wedge [\mathsf{x} \cdot \mathsf{y} \epsilon (\mathsf{s})_{\mathsf{S(i)}}]))).\end{aligned}$$

As for bounded initial partial models, we note that a bounded partial model is in particular increasing.

5.1.1 The Gödel-Kripke sentence

Now fix a consistent and recursively axiomatised arithmetic theory $T \supseteq \mathrm{I}\Sigma_1$, a primitive recursive enumeration ax of an axiomatisation of T and a $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function $\mathtt{ax}(\mathsf{x})$ coinciding with ax in $\mathbb{N}$. As for initial-fulfilment in 4.1.1, the independent sentence will depend upon this choice of the formula $[\mathtt{ax}(\mathsf{x}) = \mathsf{y}]$.

Definition 5.3 (The Gödel-Kripke sentence)**.** The $\Delta_1(\mathrm{I}\Sigma_1)$-formula $\mathtt{Tr}_0(\mathsf{s})$ is

$$\begin{aligned}\forall [\mathsf{f} < \mathtt{lh}(\mathsf{s})] \forall [\mathsf{i} < \mathtt{lh}(\mathsf{s})] \forall [\mathsf{e} \sqsubseteq (< \mathsf{f}) \times (\mathsf{s})_{\mathsf{i}}]((\mathtt{Eval}(\mathsf{e}, \mathsf{f}) \wedge \Delta_0(\mathsf{f}) \wedge \mathtt{Sat}_0(\mathsf{e}, \mathsf{f})) \rightarrow \\ [\mathtt{Fulf}(\mathsf{s}, \mathsf{i}, \mathsf{e}, \mathtt{nnf}(\mathsf{f}), \mathtt{nnf}(\mathsf{f}))]).\end{aligned}$$

Now, the formula $\mathtt{K}(\mathsf{x})$ is

$$\exists \mathsf{s}(\mathtt{Bpm}(\mathsf{s}) \wedge [\mathtt{lh}(\mathsf{s}) = \mathsf{S}(\mathsf{x})] \wedge \mathtt{Tr}_0(\mathsf{s}) \wedge \forall \mathsf{y} < \mathsf{x} \mathtt{Fbl}(\mathsf{s}, \mathtt{ax}(\mathsf{y}))).$$

The *Gödel-Kripke sentence* is the sentence

$$\forall \mathsf{x} \mathtt{K}(\mathsf{x}),$$

which we shall prove to be independent of T.

The formula $\mathtt{Tr}_0$ is Quinsey's construction from [12], somewhat adapted to the current context, and codes "Δ_0-absoluteness" for the bounded partial model $\mathfrak{F}_{\mathrm{sf}}(s)$.

We now turn to the proof of the independence of $\mathtt{K}$ from T as above. The proof of its non-provability and non-refutability follows the same strategies as in the case of initial-fulfilment, though the individual theorems, and their proofs, are slightly different.

5.2 The Gödel-Kripke sentence is not refutable

Lemma 5.6. *Let $\mathcal{M} \models \mathrm{Q}$ and Φ be a finite set of negation normal formulae of $\mathcal{L}_{\mathrm{A}}$. Then for every $n \in \mathbb{N}_{>0}$ there is a finite partial model $s : n \longrightarrow \mathcal{P}(\mathcal{M})$ such that $\mathcal{M} \overset{s,i,e}{\models} \varphi$ for all $\varphi \in \Phi$, $i < n$ and evaluations e of φ in $s(i)$ with $\mathcal{M} \models_e \varphi$.*

Proof. Fix $n \in \mathbb{N}_{>0}$. Let E be the (finite) set of (literal) existential sub-formulae of formulae from Φ. Let, for $\psi \in E$, $v(\psi)$ be the index of the largest free variable of ψ. Moreover, for every such ψ, let $F_\psi : \mathcal{M}^{v(\psi)} \longrightarrow \mathcal{M}$ be a Skolem function associated to ψ, so that if $\psi = \exists \mathsf{x}\vartheta$ then

$$\mathcal{M} \models_e \psi \Rightarrow \mathcal{M} \models_{e_{\mathsf{x}}^{F_\psi(e(\mathsf{v}_0),e(\mathsf{v}_1),\ldots,e(\mathsf{v}_{v(\psi)}))}} \vartheta$$

for all evaluations e of ψ in $\mathcal{M}$ (the intended construction of course makes F_ψ invariant under changes of the evaluations of variables which are not free in ψ, but this is not an issue). Define $s : n \longrightarrow \mathcal{P}(\mathcal{M})$ by

$$\begin{aligned} s(0) &= \mathcal{M}_{\leq n} \\ s(k+1) &= \{a +_{\mathcal{M}} b \,|\, a, b \in s(k)\} \cup \{a \cdot_{\mathcal{M}} b \,|\, a, b \in s(k)\} \cup \bigcup_{\psi \in E} F_\psi[s(k)^{v(\psi)}] \end{aligned}$$

for all $k < n-1$. Note that, since $\mathcal{M} \models \mathrm{Q}$, $s(0)$ is a finite set, and if $s(k)$ is finite then so is $s(k+1)$ (since E is). Moreover, $0, 1, n \in s(0)$, and if $1, a \in s(k)$ then $1 = 1 +_{\mathcal{M}} 0 \in s(k+1)$ and $\mathrm{S}_{\mathcal{M}}(a) = a +_{\mathcal{M}} 1 \in s(k+1)$ whence $\mathrm{S}_m[s(k)] \subseteq s(k+1)$. Thus s is a finite partial model.

We show by induction that for any φ which is a sub-formula of some formula of Φ, if $i < n$ and e is an evaluation of φ in $s(i)$ with $\mathcal{M} \models_e \varphi$, then $\mathcal{M} \overset{s,i,e}{\models} \varphi$.

- Suppose ϑ is a literal and e is an evaluation of ϑ such that $\mathcal{M} \models_e \vartheta$. Then $\mathcal{M} \overset{s,i,e}{\models} \vartheta$ for all $i < n$ by Lemma 5.1.

- Let ϑ and ψ be such that $\mathcal{M} \models_f \vartheta \Rightarrow \mathcal{M} \overset{s,k,f}{\models} \vartheta$ for all $k < n$ and evaluations f of ϑ in $s(k)$, and the corresponding for ψ. Then

$$\mathcal{M} \models_e \vartheta \wedge \psi \Leftrightarrow \mathcal{M} \models_e \vartheta \wedge \mathcal{M} \models_e \psi \Rightarrow \mathcal{M} \overset{s,i,e}{\models} \vartheta \wedge \mathcal{M} \overset{s,i,e}{\models} \psi \Leftrightarrow \mathcal{M} \overset{s,i,e}{\models} \vartheta \wedge \psi$$

  for all $i < n$ and evaluations e of $\vartheta \wedge \psi$ in $s(i)$.

- The disjunctive case is once again similar.

- Assume ϑ is such that $\mathcal{M} \models_f \vartheta \Rightarrow \mathcal{M} \overset{s,k,f}{\models} \vartheta$ for all $k < n$ and evaluations f of ϑ in $s(k)$. Let $i < n$ and e be an evaluation of $\forall \mathsf{x}\vartheta$ in $s(i)$, and suppose $\mathcal{M} \models_e \forall \mathsf{x}\vartheta$.

Also let $j \in n_{\geq i}$ and $a \in s(j)$; then $\mathcal{M} \models_{e^a_{\mathsf{x}}} \vartheta$. As e^a_{x} is an evaluation of ϑ in $s(j)$ this implies

$$\mathcal{M} \overset{s,j,e^a_{\mathsf{x}}}{\models} \vartheta$$

by induction hypothesis. Since $j \in n_{\geq i}$ and $a \in s(j)$ was arbitrary, $\mathcal{M} \overset{s,i,e}{\models} \forall\mathsf{x}\vartheta$.

- Let ϑ be such that $\mathcal{M} \models_f \vartheta \Rightarrow \mathcal{M} \overset{s,k,f}{\models} \vartheta$ for all $k < n$ and evaluations f of ϑ in $s(k)$, and consider $\exists\mathsf{x}\vartheta$. By assumption this is a sub-formula of some $\psi \in \Phi$, whence $\exists\mathsf{x}\vartheta \in E$. Thus $F_{\exists\mathsf{x}\vartheta}$ as above is a Skolem function for $\exists\mathsf{x}\vartheta$. Let $i < n$, e be an evaluation of $\exists\mathsf{x}\vartheta$ in $s(i)$ and $a = F_{\exists\mathsf{x}\vartheta}(e(\mathsf{v}_0), e(\mathsf{v}_1), \ldots, e(\mathsf{v}_{v(\exists\mathsf{x}\vartheta)}))$, and suppose $\mathcal{M} \models_e \exists\mathsf{x}\vartheta$. If $i + 1 \geq n$ then $\mathcal{M} \overset{s,i,e}{\models} \exists\mathsf{x}\vartheta$ vacuously, otherwise $a \in s(i+1)$ by the definition of s. In this case e^a_{x} is an evaluation of ϑ in $s(i+1)$, and since $\mathcal{M} \models_{e^a_{\mathsf{x}}} \vartheta$ by the definition of $F_{\exists\mathsf{x}\vartheta}$, $\mathcal{M} \overset{s,i+1,e^a_{\mathsf{x}}}{\models} \vartheta$ by induction hypothesis. Thus $\mathcal{M} \overset{s,i,e}{\models} \vartheta$.

This concludes the induction. In particular $\mathcal{M} \overset{s,i,e}{\models} \varphi$ for all $i < n$ and all evaluations e of φ in $s(i)$, for every $\varphi \in \Phi$. □

Proposition 5.7. *Let $\mathcal{M} \models T$. Then for every $n \in \mathbb{N}$, $\mathcal{M} \models \mathtt{K}(\overline{n})$.*

Proof. Take $n \in \mathbb{N}$. Let A be the set of negation normal forms of axioms 0 to $n-1$ of T. Additionally, let B be the set of negation normal forms of Δ_0-formulae less than n and $\Phi = A \cup B$. Apply Lemma 5.6 to $\mathcal{M}$, Φ and $n+1$; thus there is a finite partial model $s : n+1 \longrightarrow \mathcal{P}(\mathcal{M})$ such that $\mathcal{M} \overset{s,i,e}{\models} \varphi$ for all $i \leq n$ and evaluations e of φ in $s(i)$ with $\mathcal{M} \models_e \varphi$, for all $\varphi \in \Phi$. In particular $\mathcal{M} \overset{s,0,\emptyset}{\models} \varphi$ for all $\varphi \in A$.

Thus if $i < n$ then $\mathrm{nnf}(\mathrm{ax}(i)) \in A$ and $\mathcal{M} \models \mathtt{Sent}(\mathtt{ax}(\bar{i}))$, the latter by Σ_1-completeness. Furthermore, since s is finite and consists of finite sets, $\mathfrak{F}_{\mathrm{sf}}(s)$ exists and hence satisfies

$$\mathcal{M} \models [\mathtt{Fulf}(\mathfrak{F}_{\mathrm{sf}}(s),0,\varnothing,\mathtt{nnf}(\mathtt{ax}(\bar{i})),\mathtt{nnf}(\mathtt{ax}(\bar{i})))]$$

by Proposition 5.5. Thus $\mathcal{M} \models \mathtt{Fbl}(\mathfrak{F}_{\mathrm{sf}}(s),\mathtt{ax}(\bar{i}))$. Since $i < n$ was arbitrary, $\mathcal{M} \models \forall\mathsf{i} < \overline{n}\mathtt{Fbl}(\mathfrak{F}_{\mathrm{sf}}(s),\mathtt{ax}(\mathsf{i}))$.

Now let $\varphi, i \leq n$ and $e \in \mathcal{M}$ be such that $\mathcal{M} \models e{<}({<}\overline{\varphi}){\times}(\mathfrak{F}_{\mathrm{sf}}(s))_{\bar{i}}$; then $\mathfrak{R}_{\mathrm{f}}(e) \subseteq \varphi \times s(i)$ by Lemma 3.12. Suppose $\mathcal{M} \models \mathtt{Eval}(e,\overline{\varphi}){\wedge}\Delta_0(\overline{\varphi}){\wedge}\mathtt{Sat}_0(e,\overline{\varphi})$. Then (by Σ_1-completeness) φ is a Δ_0-formula, whence $\mathrm{nnf}(\varphi) \in B$. By Lemma 3.7 $\mathfrak{R}_{\mathrm{f}}(e)$ is an evaluation of φ in $s(i)$ and thus $\mathcal{M} \models_{\mathfrak{R}_{\mathrm{f}}(e)} \varphi$ by Lemma 3.9. By definition of nnf, $\mathfrak{R}_{\mathrm{f}}(e)$ is also an evaluation of $\mathrm{nnf}(\varphi)$ and $\mathcal{M} \models_{\mathfrak{R}_{\mathrm{f}}(e)} \mathrm{nnf}(\varphi)$. Since $\mathrm{nnf}(\varphi) \in B$ this implies

$$\mathcal{M} \overset{s,\mathfrak{R}_{\mathrm{f}}(e),i}{\models} \mathrm{nnf}(\varphi),$$

whence $\mathcal{M} \models \mathtt{Fulf}(\mathfrak{F}_{\mathrm{sf}}(s),\bar{i},e,\overline{\mathrm{nnf}(\varphi)},\overline{\mathrm{nnf}(\varphi)})$ by Proposition 5.5. With i, φ and e being arbitrary with the mentioned properties, we have $\mathcal{M} \models \mathtt{Tr}_0(\mathfrak{F}_{\mathrm{sf}}(s))$.

Thus $\mathcal{M} \models \exists\mathsf{s}(\mathtt{Bpm}(\mathsf{s}){\wedge}[\mathtt{lh}(\mathsf{s}){=}\overline{n}]{\wedge}\mathtt{Tr}_0(\mathsf{s}){\wedge}\forall\mathsf{y}{<}\mathsf{x}\mathtt{Fbl}(\mathsf{s},\mathtt{ax}(\mathsf{y})))$, i.e. $\mathcal{M} \models \mathtt{K}(\overline{n})$. □

Theorem 5.8. *If T is Σ_2-sound then*

$$T \nvdash \neg\forall \mathtt{x}\mathtt{K}(\mathtt{x}).$$

Proof. By Proposition 5.7 (and the completeness theorem), $T \vdash \mathtt{K}(\overline{n})$ for all $n \in \mathbb{N}$. Since $\mathtt{K}(\overline{n})$ is a $\Sigma_1(\mathrm{I}\Sigma_1)$-formula, $\mathbb{N} \models \mathtt{K}(\overline{n})$ for all $n \in \mathbb{N}$, whence $\mathbb{N} \models \forall\mathtt{x}\mathtt{K}(\mathtt{x})$. But if $T \vdash \neg\forall\mathtt{x}\mathtt{K}(\mathtt{x})$ then $\mathbb{N} \models \neg\forall\mathtt{x}\mathtt{K}(\mathtt{x})$, since this is a $\Sigma_2(\mathrm{I}\Sigma_1)$-formula and T is Σ_2-sound; hence $T \nvdash \neg\forall\mathtt{x}\mathtt{K}(\mathtt{x})$ as claimed. □

As in section 5, the proof shows that $\forall\mathtt{x}\mathtt{K}(\mathtt{x})$ is true (in $\mathbb{N}$).

5.3 The Gödel-Kripke sentence is not derivable

Lemma 5.9. *Let $\mathcal{M}$ be an $\mathcal{L}_\mathrm{A}$-structure and $\mathcal{L}$ be a discrete linear order without maximal element and $s : \mathcal{L} \longrightarrow \mathcal{P}(\mathcal{M})$ be an increasing sequence unbounded in $\mathcal{M}$ (that is $\mathcal{M} = = \bigcup_{i\in\mathcal{L}} s(i)$). Then $\mathcal{M} \overset{s,i,e}{\models} \varphi \Rightarrow \mathcal{M} \models_e \varphi$ for every negation normal arithmetic formula φ, all evaluations e of φ and all $i \in \mathcal{L}$.*

Proof. The proof is essentially the same as that of Lemma 4.9. □

Corollary 5.10. *Let $\mathcal{M}$ be an $\mathcal{L}_\mathrm{A}$-structure and $\mathcal{L}$ be a discrete linear order without maximal element. Suppose $s : \mathcal{L} \longrightarrow \mathcal{P}(\mathcal{M})$ is an increasing sequence unbounded in $\mathcal{M}$ such that s fulfils the negation normal sentence φ in $\mathcal{M}$. Then $\mathcal{M} \models \varphi$.*

Proof. Immediate by the lemma. □

Theorem 5.11. *There is a nonstandard model $\mathcal{N}$ of T such that $\mathcal{N} \not\models \forall\mathtt{x}\mathtt{K}(\mathtt{x})$.*

Proof. Let $\mathcal{M} \models T$ be nonstandard. By Proposition 5.7 $\mathcal{M} \models \mathtt{K}(\overline{n})$ for all $n \in \mathbb{N}$. Then $\mathcal{M} \models \mathtt{K}(m)$ for some $m \in \mathcal{M} \setminus \mathbb{N}$ by Σ_1-overspill (Lemma 2.24). By the least number principle for Σ_1-formulae (Lemma 2.22) in $\mathcal{M}$, there is a least $s \in \mathcal{M}$ such that $\mathcal{M} \models \mathtt{Bpm}(s)\wedge[\mathtt{lh}(s){=}m]\wedge\mathtt{Tr}_0(s)\wedge\forall\mathtt{y}{<}m\mathtt{Fbl}(s,\mathtt{ax}(\mathtt{y}))$. Let $f = \mathfrak{R}_\mathrm{sf}(s) \restriction \mathbb{N}$ and $\mathcal{N} = \bigcup_{i\in\mathbb{N}} f(i)$.

Now let $\vartheta \in \mathcal{L}_\mathrm{A}$ be on negation normal form. Then we have the following chain of implications, for any finite evaluation e of φ in $\mathcal{N}$ and $i \in \mathbb{N}$:

$$\begin{aligned}
\mathcal{M} \models \mathtt{Fulf}(s,\bar{i},\mathfrak{F}_\mathrm{f}(e),\overline{\vartheta},\overline{\vartheta}) &\Rightarrow \mathcal{M} \overset{\mathfrak{R}_\mathrm{sf}(s),i,e}{\models} \vartheta && \text{by Prp. 5.5}\\
&\Rightarrow \mathcal{M} \overset{f,i,e}{\models} \vartheta && \text{by Lma. 5.2}\\
&\Rightarrow \mathcal{N} \overset{f,i,e}{\models} \vartheta && \text{by Lma. 5.3}\\
&\Rightarrow \mathcal{N} \models_e \vartheta && \text{by Lma. 5.9.}
\end{aligned}$$

Consider an axiom φ (in the chosen axiomatisation) of T. Then $\varphi = \mathrm{ax}(k)$ for some $k \in \mathbb{N}$, whence $\mathcal{M} \models \mathtt{Fbl}(s,\overline{\varphi})$. Thus $\mathcal{M} \models [\mathtt{Fulf}(s,0,\varnothing,\overline{\mathrm{nnf}(\varphi)},\overline{\mathrm{nnf}(\varphi)})]$. By above we get $\mathcal{N} \models \mathrm{nnf}(\varphi)$, that is $\mathcal{N} \models \varphi$. Since φ was an arbitrary axiom of T, $\mathcal{N} \models T$.

Now let ϑ be a Δ_0-formula and e' an evaluation of ϑ in $\mathcal{N}$. Suppose $\mathcal{M} \models_{e'} \vartheta$. Let e be the restriction of e' to the variables occurring in ϑ; then e is a finite evaluation of ϑ in $\mathcal{N}$ and in particular is coded in $\mathcal{M}$. Moreover, $\mathrm{ran}(e) \subseteq f(i) = \mathfrak{R}_{\mathrm{sf}}(s)_{\mathcal{M}}(i) = \mathfrak{R}_{\mathrm{s}}(\mathfrak{R}_{\mathrm{f}}(s)_{\mathcal{M}}(i))_{\mathcal{M}}$ for some $i \in \mathbb{N}$, and since $\mathcal{M} \models [(s)_{\overline{i}}{=}\mathfrak{R}_{\mathrm{f}}(s)_{\mathcal{M}}(i)]$ and $\mathrm{dom}(e) \subseteq \mathcal{M}_{<\vartheta}$ we get $\mathcal{M} \models \mathfrak{F}_{\mathrm{f}}(e)_{\mathcal{M}} \sqsubseteq (<\overline{\vartheta}) \times (s)_{\overline{i}}$ by Lemma 3.12. Furthermore, $i, \vartheta < m$ since m is nonstandard, $\mathcal{M} \models \mathtt{\Delta}_0(\overline{\vartheta})$ by Σ_1-completeness, $\mathcal{M} \models \mathtt{Eval}(\mathfrak{F}_{\mathrm{f}}(e)_{\mathcal{M}}, \overline{\vartheta})$ by Lemma 3.7 and $\mathcal{M} \models \mathtt{Sat}_0(\mathfrak{F}_{\mathrm{f}}(e)_{\mathcal{M}}, \overline{\vartheta})$ by 3.9. Since $\mathcal{M} \models \mathtt{Tr}_0(s)$ we thus get

$$\mathcal{M} \models \mathtt{Fulf}(s, \overline{i}, \mathfrak{F}_{\mathrm{f}}(e), \overline{\vartheta}, \overline{\vartheta}),$$

whence $\mathcal{N} \models_{e'} \vartheta$ by above. Thus Δ_0-formulae are downwards absolute between $\mathcal{N}$ and $\mathcal{M}$, and since negated Δ_0-formulae are (tautologically) equivalent to Δ_0-formulae, Δ_0-formulae are absolute between $\mathcal{N}$ and $\mathcal{M}$. Consequently $\Sigma_1(T)$-formulae are upwards absolute between $\mathcal{N}$ and $\mathcal{M}$ by Lemma 2.5.

Finally, since $m \in f(0)$, $m \in \mathcal{N}$. Assume towards a contradiction that $\mathcal{N} \models \mathtt{K}(m)$ and let t be a witness thereof, that is $\mathcal{N} \models \mathtt{Bpm}(t) \wedge [\mathtt{lh}(t){=}m] \wedge \mathtt{Tr}_0(t) \wedge \forall \mathtt{y}{<}m \mathtt{Fbl}(t, \mathtt{ax}(\mathtt{y}))$. Since this is a $\Delta_1(\mathrm{I}\Sigma_1)$-formula, $\mathcal{M} \models \mathtt{Bpm}(t) \wedge [\mathtt{lh}(t){=}m] \wedge \mathtt{Tr}_0(t) \wedge \forall \mathtt{y}{<}m \mathtt{Fbl}(t, \mathtt{ax}(\mathtt{y}))$ as well. Moreover, $t \in \mathcal{N}$ implies $t \in f(k) = \mathfrak{R}_{\mathrm{sf}}(s)(k)$ for some $k \in \mathbb{N}$, whence $\mathcal{M} \models [t \epsilon (s)_{\overline{k}}]$ like before; in particular $\mathcal{M} \models t{<}s$. But this contradicts that s was the least witness of $\mathcal{M} \models \mathtt{K}(m)$ in $\mathcal{M}$. Hence $\mathcal{N} \not\models \mathtt{K}(m)$. □

Corollary 5.12. *If T is Σ_2-sound then T is incomplete.*

Proof. By Theorems 5.8 and 5.11. □

Remark 18. As in section 4, for sound theories T the above results can be sharpened to yield explicit models of $T + \forall \mathtt{x} \mathtt{K}(\mathtt{x})$ and $T + \neg \forall \mathtt{x} \mathtt{K}(\mathtt{x})$.

6 Summary and concluding remarks

The results of sections 4 and 5, as summarised in Corollaries 4.12 and 5.12, verifies that Σ_2-sound recursively axiomatisable extensions of PA and $I\Sigma_1$, respectively, are incomplete. These are rather weak incompleteness results as compared to those obtained by the standard Gödel-Rosser proof via diagonalisation (see for example [13, pp. 135-140]). However, this method of proof still deserves some attention, since it is a rather explicit model-theoretic construction which is, at the same time, applicable to a wide range of theories. In the case of a sound theory such as PA the proofs can even be modified to yield truly explicit models witnessing its incompleteness. Also, these proofs appears to be slightly easier than the standard one (though the independent sentence is more complex, Π_2 instead of Π_1), since we do not need to formalise the notion of formal derivation. This is in particular true for the proof via initial fulfilment which does not even use the satisfaction predicate $\mathtt{Sat}_0$. It might be possible, however, to remove the dependence on $\mathtt{Sat}_0$ even in the proof of section 5. Namely, by the MRDP-theorem for $I\Sigma_1$ (see [4, Theorem 3.25 p. 97]), every $\Sigma_1(I\Sigma_1)$-formula is equivalent in $I\Sigma_1$ to a formula of the form $\exists x \varphi$ with the same free variables, such that φ is quantifier free. This indicates that $\mathtt{Tr}_0$, the only place where $\mathtt{Sat}_0$ is used, is in fact not needed in the proof of Theorem 5.11, since quantifier free formulae are upwards absolute between a structure and any of its extensions. Whether this would constitute a simplification of the proof finally depends on your view of which of the proof of the MRDP-theorem and the construction of $\mathtt{Sat}_0$ is the simpler one. A final note along this line: MRDP is provable even in $I\Delta_0(\exp)$, so if all other constructions used are possible in $I\Delta_0(\exp)$ as well (with the exception of $\mathtt{Fulf}$, this is done in [13] (using other coding techniques than the ones we have used here) and it is plausible that $\mathtt{Fulf}$ can be given a Δ_1-definition in $I\Delta_0(\exp)$ as well, since the construction mainly involves a number of 0-1-sequences of predetermined length) then the proof of section 5 should generalise to $I\Delta_0(\exp)$ (the languages differ slightly, but this should not be a problem). This of course also applies to initial fulfilment which, as far as arithmetic is concerned, seems the more elegant construction.

As indicated above, these results should also hold for languages other than $\mathcal{L}_A$ as long as "recursively axiomatisable theory" is meaningful (so the language cannot be uncountable, for instance), there are only finitely many function symbols and we have a hierarchy of formulae such that $\mathtt{Sat}_0$ is Δ_1-definable. This is, approximately, actually one of the approaches of [12], which also uses a more set-theoretic framework in general, which seems more natural for the proofs of section 5. That is, while the ordinary proof of the Incompleteness Theorem uses logic coded in arithmetic (possibly partly via set theory as we have done here), the more natural approach to this proof might be to code logic directly in (weak) set theory. Thus, this take on the proof emphasises that the incompleteness theorem is applicable to theories which "contains a sufficient amount of set theory" rather than "a sufficient amount of arithmetic".

The differences between the approaches of sections 4 and 5, minor though they may seem, might also merit some comments. First, note that the proof by initial fulfilment does not seem to involve any kind of self-reference or diagonalisation; the independent sentence expresses properties of the theory it is independent from, not of the sentence itself or any of its constituents. On the contrary, the proof by (general) fulfilment could be said to involve at least a weak form of self-reference (or at least an undesirable degree of impredicativity, though not diagonalisation), as the formula $\mathtt{Tr}_0(\mathsf{s})$ expresses that "every true Δ_0-formula (with parameters) is fulfilled by s", among which are, of course, the Δ_0-formulae witnessing that $\mathtt{Tr}_0(\mathsf{s})$ is $\Delta_1(\mathrm{I}\Sigma_1)$; this is an essential part of the proof. Thus there seems to be some at least anecdotal evidence of the parole "the nicer the class of independent sentences, the smaller the class of theories it is applicable to". This is also supported by the Paris-Harrington theorem, which has the nicest (concrete and natural) independent sentence and applies to the smallest class of theories of the proofs here considered. Likewise, the results of [8] seems to indicate that for PA we could have used an even simpler definition of initial fulfilment, where each quantifier only increases the position in the sequence by one. The same thing is suggested by the proof via indicators constructed from combinatorial principles (see [7, ch. 14])

Concerning indicators, the proof of 4.11 also seems to indicate that the $\mathrm{I}\Sigma_1$-provably $\Delta_1(\mathrm{I}\Sigma_1)$-function $[\mathsf{y}(\mathsf{a},\mathsf{b}){=}\mathsf{s}]$

$$\begin{aligned}&(\mathtt{IBpm}(\mathsf{s})\wedge\mathsf{s}<\mathsf{b}\wedge[\mathtt{lh}(\mathsf{s})=\mathsf{a}]\wedge\forall\mathsf{x}<\mathsf{a}\mathtt{IFbl}(\mathsf{s},\mathtt{ax}(\mathsf{x}))\wedge\\ &\qquad\qquad\forall\mathsf{t}<\mathsf{s}(([\mathtt{lh}(\mathsf{t})=\mathsf{a}]\wedge\mathtt{IBpm}(\mathsf{t}))\rightarrow\neg\forall\mathsf{x}<\mathsf{a}\mathtt{IFbl}(\mathsf{t},\mathtt{ax}(\mathsf{x}))))\\ &\vee(\forall\mathsf{t}<\mathsf{b}\neg([\mathtt{lh}(\mathsf{t})=\mathsf{a}]\wedge\mathtt{IBpm}(\mathsf{t})\wedge(\forall\mathsf{x}<\mathsf{a}\mathtt{IFbl}(\mathsf{t},\mathtt{ax}(\mathsf{x}))))\wedge\mathsf{s}=0)\end{aligned}$$

(that is, $\mathsf{y}(\mathsf{a},\mathsf{b})$ is the smallest $\mathtt{IBpm}$ less than b which has length a and fulfils all of the first $\mathsf{a}-\overline{1}$ axioms, if such a thing exists, otherwise 0) is an indicator of T in $\mathrm{I}\Sigma_1$, where T and $\mathtt{ax}$ are as in section 4.

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