

ON THE LOCAL CARTESIAN CLOSURE OF EXACT COMPLETIONS

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ABSTRACT. A characterisation of cartesian closure of exact completions as a property of the projective objects was given by Carboni and Rosolini. We show that the argument used to prove that characterisation is equivalent to the projectives being closed under binary products (equivalently, being internally projective). The property in question is the existence of weak simple products (a slight strengthening of weak exponentials) and the argument used relies on two claims: that weak simple products endow the internal logic with universal quantification, and that an exponential is the quotient of a weak exponential. We show that either these claims hold if and only if the projectives are internally projectives, which entails that Carboni and Rosolini's characterisation only applies to *ex/lex* completions. We then argue that this limitation depends on the universal property of weak simple products, and derive from this observation an alternative notion, which we call generalised weak simple product. We conclude by showing that existence of generalised weak simple products in the subcategory of projectives is equivalent to the cartesian closure of the exact category, thus obtaining a complete characterisation of (local) cartesian closure for exact completions of categories with weak finite limits.

1. INTRODUCTION

The aim of the present paper is twofold. We firstly show that the characterisation of (locally) cartesian closed exact completions in terms of a property of the projectives given in [7] holds if and only if the category of projectives is closed under binary products (respectively, pullbacks). We then discuss how to avoid the problem in the original argument by A. Carboni and G. Rosolini, and present a complete characterisation that is valid in the most general case.

Let \mathbb{E} be an exact category with enough projectives and let $\overline{\mathbb{P}}$ be its full subcategory of projectives. The authors of [7] identify the existence of weak simple products in $\overline{\mathbb{P}}$ (a slight strengthening of the notion of weak exponential) as being equivalent to the cartesian closure of \mathbb{E} . In particular, they argue that (1) weak simple products provide right adjoints to inverse images along product projections, and that (2) an exponential of $A, B \in \mathbb{E}$ is covered by the exponential of covers of A and B . However in Proposition 3.4, Lemma 4.6, and Remark 4.7, we show that either claims are equivalent to the closure of $\overline{\mathbb{P}}$ under binary products.

The results in [7] then provide an elementary characterisation of locally cartesian closed *ex/lex* completions [6, 5, 4], that is, when $\overline{\mathbb{P}}$ is closed under finite limits. This characterisation then still improves on the non-elementary one presented in [16] which requires $\overline{\mathbb{P}}$ to be infinitary *lex*tensive, and it does fulfil the main motivation of [7], that is, providing a common general reason for the local cartesian closure of the effective topos, the category of equilogical spaces and the exact completion of topological spaces. However, it is no longer valid whenever $\overline{\mathbb{P}}$ (and hence every other projective cover) fails to have some strong limit or, equivalently, whenever a projective (object or arrow) fails to be internally projective.

In Example 2.2 we describe a cartesian closed exact category with enough projectives (a presheaf topos, actually), whose projectives are not closed under product. This example was originally conceived by T. Trimble as a simple case where not every projective object is internally projective, and is

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introduced in the nLab web page on the Presentation Axiom (PA, [1, 2]) as a proof that not every topos validating PA also validates the internal version of it [14, 13]. Indeed, the existence of enough projectives can be regarded as the category-theoretic formulation of PA.

More natural examples are homotopy categories and categories arising from intensional type theories, both of which do have finite products but only have weak equalisers (and weak pullbacks). Moreover, these are the only examples of genuine weakly finitely complete categories which the author is aware of, where by genuine we mean not originally arising as a projective cover of a finitely complete category. This fact motivates the need for a characterisation of (local) cartesian closure in exact completions of categories with weak finite limits.

We accomplish this goal observing that what makes weak simple products unsuitable for the weak limits case is their universal property with respect to arrows determined by projections, and that these do not comprise all arrows out of a weak product. Recall that a weak product in \mathbb{P} of two projectives X and Y is a cover $V \rightarrow X \times Y$ with $V \in \mathbb{P}$, and an arrow $V \rightarrow Z$ out of a weak product is determined by projections if it induces an arrow $X \times Y \rightarrow Z$ in \mathbb{E} . However, we cannot expect in general a cover $V \rightarrow Z$ in \mathbb{P} of an arrow $X \times Y \rightarrow B$ in \mathbb{E} to be determined by projections: in fact a cover in \mathbb{P} of $\text{id}_{X \times Y}$ is determined by projections if and only if $X \times Y$ is projective. It seems then that the only tool left to us to characterise covers of arrows $X \times Y \rightarrow B$ is the characterisation of objects and arrows in \mathbb{E} in terms of projectives, which ultimately provides the equivalence $\mathbb{P}_{\text{ex}} \simeq \mathbb{E}$ [8]: objects in \mathbb{E} are quotients of pseudo equivalence relations $X_1 \rightrightarrows X_0$ in \mathbb{P} , and arrows $A \rightarrow B$ are arrows $X_0 \rightarrow Y_0$ which preserve the equivalence relation, i.e. with a tracking $X_1 \rightarrow Y_1$.

However, expressing cartesian closure in these terms would yield a notion which, we believe, would be very cumbersome and difficult to work with (especially in the case of local cartesian closure). The key observation to tackle this issue is the fact that it is enough to restrict to arrows with a projective domain, i.e. such that the pseudo relation on X_0 is given by the identity arrow (cf. Lemma 4.5). In this way we obtain the notion of generalised weak simple product in Definition 4.9, and prove in Theorem 4.13 that the cartesian closure of \mathbb{E} is equivalent to the existence of generalised weak simple products in \mathbb{P} . The cartesian closure of slices then follows by a standard descent argument. Moreover, in the presence of binary products, the notion of generalised weak simple product reduces to Carboni and Rosolini's weak simple product and, in turn, so does our characterisation in the case of ex/lex completions.

We found the problem in Carboni and Rosolini's argument while trying to apply it to a category of types arising from Martin-Löf type theory, but the solution presented in this paper did not follow immediately. Instead we formulated, together with E. Palmgren, a condition inspired by P. Aczel's Fullness Axiom from Constructive Set Theory [1, 2] and used it to show when an exact completion produces a model of the Constructive Elementary Theory of the Category of Sets [10] (a constructive version of Lawvere's ETCS [11, 12, 15]). However, this fullness condition only provides a sufficient condition for local cartesian closure, hence the results in the present paper improve on the situation in [10], allowing for a complete characterisation of models of CETCS in terms of properties of their choice objects. On the other hand, this fullness condition is, we believe, much simpler to state (and verify) than the existence of generalised weak simple products in every slice. Moreover, it naturally arises, under mild assumptions, in certain homotopy categories (including the homotopy categories of spaces and CW-complexes) [9].

The paper is organised as follows. In Section 2 we recall some background notions and results, besides fixing notations. Section 3 discusses the limitation in claim (1) and reformulates the main results from that paper. Section 4 begins by showing how to modify the notion of weak simple product in order to obtain right adjoints to product projections. We then argue that this is still not enough, by proving that also claim (2) requires closure under binary products. We thus reach the notion of generalised weak simple product, and prove the general characterisation.

2. BACKGROUND

2.1. Projective covers. A projective cover of an exact category \mathbb{E} is a full subcategory \mathbb{P} such that

- (a) every object X of \mathbb{P} is (regular) projective, i.e. $\mathbb{E}(X, _): \mathbb{E} \rightarrow \mathbf{Set}$ preserves regular epis, and
- (b) every object A in \mathbb{E} is covered by a projective in \mathbb{P} , i.e. there is regular epi $X \rightarrow A$ with X in \mathbb{P} .

\mathbb{E} has *enough projectives* if it has a projective cover. If we know an arrow to be a regular epi, we will write it as \rightarrow . We will say that an object X *covers* another object A if there is a regular epi $X \rightarrow A$ and, similarly, we will refer to regular epis also as covers. If moreover $X \in \mathbb{P}$, then we will refer to $X \rightarrow A$ as a \mathbb{P} -cover of A . The last six letters of the alphabet and J will denote objects in \mathbb{P} .

The characterisation of exact completions from [8] in terms of projective objects allows us to regard any category with weak finite limits as a projective cover of an exact category, namely its exact completion. Therefore, throughout the paper, \mathbb{P} will denote a category with weak finite limits and \mathbb{E} its exact completion or, equivalently, \mathbb{E} will denote an exact category with enough projectives and \mathbb{P} a projective cover of it.

Recall that \mathbb{P} does not necessarily consists of all projectives of \mathbb{E} , indeed the following are equivalent [8]:

- (1) \mathbb{P} is the full subcategory of projectives in \mathbb{E} ;
- (2) \mathbb{P} is closed under retracts in \mathbb{E} ;
- (3) \mathbb{P} is Cauchy complete, i.e. every idempotent splits in \mathbb{P} .

We denote by $\overline{\mathbb{P}}$ the Cauchy completion of \mathbb{P} (a.k.a. the splitting of idempotents of \mathbb{P} [8]), i.e. the full subcategory of \mathbb{E} on its projectives.

Another result which we will use without explicit mention, but which is instrumental in characterising properties of \mathbb{E} in terms of properties of \mathbb{P} and vice versa, is that subobjects in \mathbb{E} of a projective $X \in \mathbb{P}$ are isomorphic to the order reflection of \mathbb{P}/X [8]. More generally, for any $n > 0$ and any tuple X_1, \dots, X_n of projectives in \mathbb{P} , $\text{Sub}_{\mathbb{E}}(X_1 \times \dots \times X_n)$ is isomorphic to the order reflection of the category of cones in \mathbb{P} over that tuple, i.e. the comma category $\Delta \downarrow (X_1, \dots, X_n)$, where $\Delta: \mathbb{P} \rightarrow \mathbb{P}^n$ is the diagonal and $(X_1, \dots, X_n): \mathbb{1} \rightarrow \mathbb{P}^n$. For brevity, we denote this comma category as $\mathbb{P}/(X_1, \dots, X_n)$ and its order reflection as $(\mathbb{P}/(X_1, \dots, X_n))_{\text{po}}$.

2.2. Weak limits. \mathbb{P} always has weak finite limits. Intuitively, a weak limit is like a limit but without uniqueness of the universal arrow. More precisely, an object is weakly terminal if every object has an arrow into it, and a *weak limit* of a diagram D is a cone over D which is weakly terminal among cones over D . For simplicity, we will still speak of the universal property of a weak limit, although it should be called weakly universal property.

A weak limit W of a diagram $D: \mathbb{D} \rightarrow \mathbb{P}$ is computed by taking a \mathbb{P} -cover of the limit L in \mathbb{E} . For this reason, an arrow $f: W \rightarrow A$ induces a (necessarily unique) arrow $L \rightarrow A$ if and only if f coequalises every pair of arrows (in \mathbb{P}) coequalised by all the projections of the weak limit. If this is the case, f is said to be *determined by projections*. For example, let $X \xleftarrow{p} W \xrightarrow{q} Y$ be a weak product of X and Y in \mathbb{P} , then $f: W \rightarrow A$ induces an arrow $X \times Y \rightarrow A$ in \mathbb{E} if and only if, for any pair of arrows $h, k: V \rightarrow W$ in \mathbb{P} , $ph = pk$ and $qh = qk$ imply $fh = fk$.

Weak limits are certainly not unique up to isomorphism, however they are uniquely determined in the order reflection of the corresponding category of cones. In particular, for any $f: X \rightarrow Y$ in \mathbb{P} , weak pullbacks along f induce a functor $f^*: (\mathbb{P}/Y)_{\text{po}} \rightarrow (\mathbb{P}/X)_{\text{po}}$ which is in fact isomorphic to inverse images along f in \mathbb{E} via $(\mathbb{P}/X)_{\text{po}} \cong \text{Sub}(X)$. When f is a projection of a weak product of X and Y , we denote f^* as $\times_X^w: (\mathbb{P}/Y)_{\text{po}} \rightarrow (\mathbb{P}/(X, Y))_{\text{po}}$.

It is easy to see that a limit in \mathbb{E} is projective (hence in $\overline{\mathbb{P}}$) if and only if it is a retract of a weak limit W in \mathbb{P} . Note that, however, W has no (weakly) universal property in \mathbb{E} . Whenever we need to refer to a weak limit W as an object of \mathbb{E} we should call it a quasi limit, since a *quasi limit* is, by definition, any cover of a limit. However we will often abuse terminology, referring to it as a weak limit in both contexts.

A \mathbb{P} -covering square is a square of the form

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & Y \\ \downarrow p & & \downarrow q \\ A & \xrightarrow{f} & B \end{array}$$

where q and the canonical arrow $X \rightarrow A \times_B Y$ are both \mathbb{P} -covers. In this case we will say that \bar{f} \mathbb{P} -covers f .

Finally, a diagram $K \rightrightarrows A \rightarrow B$ is exact if it is a coequaliser and $K \rightrightarrows A$ is the kernel pair of $A \rightarrow B$. We will then say that it is *quasi exact* if it is a coequaliser and the obvious square is a covering square.

2.3. Cartesian closed exact completions. Similarly to what happens with limits, if \mathbb{E} is cartesian closed, a \mathbb{P} -cover $W \rightarrow Y^X$ of the exponential of a pair of projective objects $X, Y \in \mathbb{P}$ gives rise to a *weak exponential of X and Y* in \mathbb{P} , that is, a diagram

$$\begin{array}{ccccc} W & \longleftarrow & V & \longrightarrow & X \\ & & \downarrow & & \\ & & Y & & \end{array}$$

such that the top row is a weak product, the arrow $V \rightarrow Y$ is determined by projections and which is weakly terminal among diagrams of this form, that is, for any other weak product $W' \leftarrow V' \rightarrow X$ and arrow $V' \rightarrow Y$ determined by projections, there are $W' \rightarrow W$ and $V' \rightarrow V$ making everything commute.

An exact category which is also cartesian closed not only has exponentials but all simple products, that is, right adjoints to pullback along product projections:

$$\mathbb{E}/I \xleftarrow[\Pi_A]{(_) \times A} \mathbb{E}/(I \times A),$$

which are definable from the adjunctions $(_) \times A \dashv (_)^A$ pulling back along the unit.

The simple product functor Π_A restricts to subobjects, endowing the internal logic of \mathbb{E} with universal quantification $\forall_A: \text{Sub}(I \times A) \rightarrow \text{Sub}(I)$. In particular, for any two projectives $J, X \in \mathbb{P}$, the weak product functor \times_X^w has a right adjoint $\forall_X^w: (\mathbb{P}/(J, X))_{\text{po}} \rightarrow (\mathbb{P}/J)_{\text{po}}$. In fact, the proof of Lemma 2.6 in [7] shows that the converse is true as well: inverse images along product projections are left adjoints if and only if \times_X^w is left adjoint for every $X \in \mathbb{P}$.

2.4. Internal projectives. An object X in a category with binary products is *internally projective* if, for every object T , cover $A \rightarrow B$ and arrow $T \times X \rightarrow B$, there are an object U and arrows $U \rightarrow T$ and $U \times X \rightarrow A$ such that

$$\begin{array}{ccc} U \times X & \longrightarrow & A \\ \downarrow & & \downarrow \\ T \times X & \longrightarrow & B \end{array}$$

commutes.

In the case of an exact category \mathbb{E} with enough projectives, we may assume that $U \rightarrow T$ is a \mathbb{P} -cover. We also have that the following are equivalent for a projective X :

- (1) X is internally projective,
- (2) the functor $(_) \times X: \mathbb{E} \rightarrow \mathbb{E}$ restricts to a functor $\mathbb{P} \rightarrow \mathbb{P}$.
- (3) the functor $(_) \times X: \mathbb{E} \rightarrow \mathbb{E}$ preserves projectives.

If moreover \mathbb{E} is cartesian closed, then the following are equivalent:

- (1) X is internally projective,

- (2) the exponential functor $(_)^X$ preserves covers,
- (3) the simple product functor Π_X preserves covers.

In (2) and (3) we may also replace “covers” with “ \mathbb{P} -covers”.
Projectives and internal projectives need not coincide.

Lemma 2.1. *(1) Internal projectives are projective if and only if $\mathbf{1}$ is projective.
(2) Projectives are internally projective if and only if $\overline{\mathbb{P}}$ is closed under binary products.
In particular, they coincide if and only if $\overline{\mathbb{P}}$ is closed under finite products.*

We now consider a simple example, due to T. Trimble [14, 13], of a category where this fails.

Example 2.2. Let \mathbb{C} be the poset with underlying set given by the natural numbers \mathbb{N} plus two more elements a and b , and the order given by the reflective and transitive closure of the relations $n < n + 1$, $n < a$ and $n < b$, for every $n \in \mathbb{N}$. We may thus regard \mathbb{C} as the natural numbers with two distinct infinity points.

Consider the topos of presheaves on \mathbb{C} , and let us denote $\coprod_{x \in S} \mathbf{y}x$ as $\mathbf{y}S$, where \mathbf{y} is the Yoneda embedding and S is any subset of objects in \mathbb{C} . The unique arrow $\mathbf{y}\mathbb{C} \rightarrow \mathbf{1}$ is a cover and the image factorisation of the arrow $\mathbf{y}\mathbb{N} \rightarrow \mathbf{1}$ is given by $\mathbf{y}\mathbb{N} \rightarrow \mathbf{y}a \times \mathbf{y}b \hookrightarrow \mathbf{1}$. Indeed, for every $x \in \mathbb{C}$ we have

$$\mathbf{y}a \times \mathbf{y}b(x) \cong \begin{cases} \emptyset, & \text{if } x \in \{a, b\}, \\ \{*\}, & \text{if } x \in \mathbb{N}, \end{cases} \quad \mathbf{y}\mathbb{N}(x) \cong \begin{cases} \emptyset, & \text{if } x \in \{a, b\}, \\ \{n \in \mathbb{N} \mid x \leq n\}, & \text{if } x \in \mathbb{N}, \end{cases}$$

and $\mathbf{y}\mathbb{C}(x) \cong \{y \in \mathbb{C} \mid x \leq y\}$

and the three functors map arrows of \mathbb{C} to the obvious inclusions.

It is then clear that $\mathbf{1}$ and $\mathbf{y}a \times \mathbf{y}b$ cannot be projective, since the existence of a global element in $\mathbf{y}\mathbb{C}$ would imply $a = b$ and the existence of a natural transformation $\mathbf{y}a \times \mathbf{y}b \rightarrow \mathbf{y}\mathbb{N}$ would imply the existence of an element in \mathbb{N} larger than any other natural number.

3. CARBONI AND ROSOLINI'S CHARACTERISATION

3.1. Weak simple products. Taking a \mathbb{P} -cover of a simple product of $Y \rightarrow J \times X$, with $J, X, Y \in \mathbb{P}$, gives rise to what Carboni and Rosolini call a weak simple product [7, Proposition 2.4].

Definition 3.1 ([7], 2.1). Let $J \xleftarrow{f} Y \xrightarrow{g} X$ be a span in \mathbb{P} . A *weak simple product of f and g* is a commutative diagram

$$\begin{array}{ccccc} W & \longleftarrow & V & & \\ \downarrow & & \downarrow & \searrow & \\ J & \longleftarrow & Y & \longrightarrow & X \end{array}$$

where $W \leftarrow V \rightarrow X$ is a weak product and $V \rightarrow Y$ is determined by projections, which is weakly terminal among diagrams with these properties: for any other commutative diagram

$$\begin{array}{ccccc} W' & \longleftarrow & V' & & \\ \downarrow & & \downarrow & \searrow & \\ J & \longleftarrow & Y & \longrightarrow & X \end{array}$$

where $W' \leftarrow V' \rightarrow X$ is a weak product and $V' \rightarrow Y$ is determined by projections, there are arrows $W' \rightarrow W$ and $V' \rightarrow V$ making everything commute.

Remark 3.2. \mathbb{P} has weak simple products if and only if $\overline{\mathbb{P}}$ has weak simple products. This is a straightforward verification once we observe that, given two projective covers \mathbb{P} and \mathbb{P}' , any span

$J' \leftarrow Y' \rightarrow X'$ in \mathbb{P}' fits into a commutative diagram

$$\begin{array}{ccccc}
 & & J' & \longleftarrow & Y' & \longrightarrow & X' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 J & \longleftarrow & Y & \longrightarrow & X & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 J' & \longleftarrow & Y' & \longrightarrow & X' & &
 \end{array}$$

where the two front squares are \mathbb{P} -covering.

This same remark will apply to all other weakly universal diagrams in \mathbb{P} which we will encounter.

If \mathbb{P} has weak simple products, then it is possible to define functors

$$\mathbf{w}_X : \text{Sub}_{\mathbb{E}}(J \times X) \rightarrow \text{Sub}_{\mathbb{E}}(J)$$

for any two projectives $J, X \in \mathbb{P}$ [7]: for a subobject $a: A \hookrightarrow J \times X$, $\mathbf{w}_X(a) \in \text{Sub}_{\mathbb{E}}(J)$ is defined by taking the image factorisation of the arrow $W \rightarrow J$ obtained as a weak simple product of the span $J \leftarrow Y \rightarrow X$, where $Y \rightarrow A$ is a \mathbb{P} -cover. Notice that the definition of $\mathbf{w}_X(a)$ does not depend (up to isomorphism) on the \mathbb{P} -cover of A .

Lemma 2.6 from [7] claims that \mathbf{w}_X is right adjoint to $(_) \times X$. However this is true if and only if X is internally projective. Indeed, suppose for the moment that \mathbb{E} is cartesian closed. Then for any subobject $a: A \hookrightarrow J \times X$ and \mathbb{P} -cover $Y \rightarrow A$, we have the following commuting diagram

$$(1) \quad \begin{array}{ccc}
 \Pi_X Y & \longrightarrow & \forall_X A \\
 \downarrow & & \downarrow \\
 \mathbf{w}_X A & \hookrightarrow & J
 \end{array}$$

which shows that \mathbf{w}_X coincides with \forall_X if and only if the top arrow is a cover. Since this argument does not depend on the particular \mathbb{P} -cover of A , we obtain that $\mathbf{w}_X \cong \forall_X$ if and only if X is internally projective. In the particular case of presheaves on the natural numbers with two infinity points from Example 2.2, it is not difficult to check that the functor \mathbf{w}_{y_a} is indeed not right adjoint to $(_) \times y_a: \text{Sub}(y_b) \rightarrow \text{Sub}(y_a \times y_b)$, by computing $\mathbf{w}_{y_a}(y_a \times y_b) \cong y_a \times y_b$.

More generally, the same result can be proved only assuming the existence of weak simple products in \mathbb{P} , without using the cartesian closure of \mathbb{E} .

Lemma 3.3. *Let J and X be objects in \mathbb{P} . If \mathbf{w}_X is right adjoint to $(_) \times X: \text{Sub}(X) \rightarrow \text{Sub}(J \times X)$, then $J \times X$ is projective.*

Proof. Assume $(_) \times X \dashv \mathbf{w}_X$ and let $p: U \rightarrow J \times X$ be a \mathbb{P} -cover. We will show that p has a section, thus concluding that $J \times X$ is projective. Consider the following weak simple product diagram

$$\begin{array}{ccccc}
 W & \xleftarrow{v_1} & V & & \\
 \downarrow & & \downarrow & \searrow & \\
 B & & U & & X \\
 \downarrow & & \downarrow & & \\
 J & \xleftarrow{p_1} & U & \xrightarrow{p_2} & X
 \end{array}$$

The adjoint relation implies $\text{id}_J \leq \mathbf{w}_X(\text{id}_{J \times X})$ and, since J is projective, we obtain an arrow $h: J \rightarrow W$ over B and, in turn, an arrow $k: U \rightarrow V$ such that $v_1 k = h p_1$ and $v_2 k = p_2$. The arrow $ek: U \rightarrow U$ is determined by projections, as e is so by definition, and therefore it induces an arrow $s: J \times X \rightarrow U$. We have that $p_1 s p = p_1 e k = p_1$, which implies $p_1 s = \text{pr}_1$, and similarly we get $p_2 s = \text{pr}_2$. Hence $ps = \text{id}_{J \times X}$ as required. \square

The converse is true as well.

Proposition 3.4. *Let X be an object in \mathbb{P} . The following are equivalent.*

- (1) *The functor \mathbf{w}_X is right adjoint to $(_) \times X: \text{Sub}(J) \rightarrow \text{Sub}(J \times X)$ for every $J \in \mathbb{P}$.*
- (2) *The functor $(_) \times X: \mathbb{E} \rightarrow \mathbb{E}$ preserves projective objects.*
- (3) *The object X is internally projective.*

Proof. It only remains to prove that Item 3 implies Item 1. Observe first that $\mathbf{w}_X(a) \times X \leq a$ always holds for any $a \in \text{Sub}_{\mathbb{E}}(J \times X)$, therefore we only need to show that $b \leq \mathbf{w}_X(b \times X)$ for every $b \in \text{Sub}(J)$. To this aim, let $p: U \rightarrow B \times X$ be a \mathbb{P} -cover and let $W \rightarrow J$ be a weak simple product of $J \leftarrow U \rightarrow X$, so that $\mathbf{w}_X(b \times X)$ is the image factorisation of $W \rightarrow J$. Internal projectivity of X yields a \mathbb{P} -cover $y: Y \rightarrow B$ and an arrow $u: Y \times X \rightarrow U$ such that $pu = y \times X$. The composition of any \mathbb{P} -cover $V \rightarrow Y \times X$ with u is an arrow $V \rightarrow U$ determined by projections. Hence the weak universal property of weak simple products yields an arrow $Y \rightarrow W$ over J , which induces an arrow $b \rightarrow \mathbf{w}_X(b \times X)$ as required. \square

Corollary 3.5. *The following conditions are equivalent.*

- (1) *For every $J, X \in \mathbb{P}$, there is an adjunction $(_) \times X \dashv \mathbf{w}_X$.*
- (2) *$\overline{\mathbb{P}}$ is closed under binary products.*
- (3) *Every projective object is internally projective.*

3.2. Reformulation of the characterisation. This Corollary entails that the proof of Carboni and Rosolini's characterisation of cartesian closed exact completions requires the projectives to be also internally projectives. Therefore it does characterise (local) cartesian closure only for exact completions of categories with binary products (respectively, pullbacks). As observed in [7], in this case the notion of weak simple product can be reformulated as the existence, for every $y: Y \rightarrow J \times X$, of an arrow $w: W \rightarrow J$ and a natural surjection

$$\overline{\mathbb{P}}/J(_, w) \longrightarrow \overline{\mathbb{P}}/(J \times X)(_ \times X, y).$$

Theorem 2.5 from [7] has to be reformulated as follows.

Theorem 3.6 ([7], 2.5). *Assume that $\overline{\mathbb{P}}$ has binary products. Then \mathbb{E} is cartesian closed if and only if $\overline{\mathbb{P}}$ has weak simple products.*

For the local cartesian closure, weak simple products in every slice of $\overline{\mathbb{P}}$ are required. This condition is equivalent to requiring the existence of the more familiar weak dependent products: if $\overline{\mathbb{P}}$ has pullbacks, a *weak dependent product* of two arrows $f: X \rightarrow J$ and $g: Y \rightarrow X$ is given by an arrow $w: W \rightarrow J$ and a natural surjection

$$\overline{\mathbb{P}}/J(_, w) \longrightarrow \overline{\mathbb{P}}/X(f^*_, g).$$

Theorem 3.3 from [7] then has to be reformulated as follows.

Theorem 3.7 ([7], 3.3). *Assume that $\overline{\mathbb{P}}$ has pullbacks. Then \mathbb{E} is locally cartesian closed if and only if $\overline{\mathbb{P}}$ has weak dependent products.*

In particular, the exact completion of a finitely complete category \mathbb{C} is locally cartesian closed if and only if \mathbb{C} has weak dependent products.

We now discuss how to recover an elementary characterisation of the existence of right adjoints to weak products, but we will see that this is still not sufficient to provide a complete characterisation of cartesian closure for ex/wlex completions. We will then show how to generalise the notion of an arrow determined by projections, in order to obtain a new (weakly) universal property of certain diagrams of projectives which will provide the required elementary characterisation.

4. CHARACTERISING LOCAL CARTESIAN CLOSURE

4.1. Pseudo simple products. Diagram (1) suggests a possible way to isolate a property of the projectives which would allow us to recover the right adjoint \forall_X , namely taking a \mathbb{P} -cover of \forall_X instead of Π_X . As we will see in Proposition 4.2, this amounts to dropping the requirement that the weak evaluation in a weak simple product is determined by projections. Since, in a regular category, pseudo relations arise as covers of relations, we name these covers of subobjects of the form $\forall_X a$ pseudo simple products.

Definition 4.1. Let $J \xleftarrow{f} Y \xrightarrow{g} X$ be a span in \mathbb{P} . A *pseudo simple product of f and g* is a commutative diagram

$$\begin{array}{ccccc} W & \longleftarrow & V & & \\ \downarrow & & \downarrow & \searrow & \\ J & \longleftarrow & Y & \longrightarrow & X \end{array}$$

where $W \leftarrow V \rightarrow X$ is a weak product, which is weakly terminal among diagrams with these properties: for any other commutative diagram

$$\begin{array}{ccccc} W' & \longleftarrow & V' & & \\ \downarrow & & \downarrow & \searrow & \\ J & \longleftarrow & Y & \longrightarrow & X \end{array}$$

where $W' \leftarrow V' \rightarrow X$ is a weak product, there are arrows $W' \rightarrow W$ and $V' \rightarrow V$ making everything commute.

Proposition 4.2. *If \mathbb{E} is cartesian closed, then \mathbb{P} has pseudo simple products.*

Proof. Let $J \leftarrow Y \rightarrow X$ be a span in \mathbb{P} and consider its image factorisation $Y \rightarrow I \hookrightarrow J \times X$. Let $W \rightarrow \forall_X I$ be a \mathbb{P} -cover ad let $V \rightarrow Y$ \mathbb{P} -cover $W \times X \rightarrow \forall_X I \times X \rightarrow I$, i.e. such that

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W \times X & \longrightarrow & I \end{array}$$

is a \mathbb{P} -covering square. This defines a pseudo simple product. Indeed, consider a diagram as the second one in Definition 4.1. Since $I \hookrightarrow J \times X$ is monic, $V' \rightarrow Y \rightarrow I$ is determined by projections so it induces $W' \times X \rightarrow I$. Thus we get an arrow $W' \rightarrow \forall_X I$ over J which lifts, by projectivity of W' , to $W' \rightarrow W$. The arrow $V' \rightarrow V$ is obtained similarly, lifting the arrow into the pullback $(W \times X) \times_I Y$ defined by $V' \rightarrow Y$ and the composite $V' \rightarrow W' \times X \rightarrow W \times X$. \square

The universal property of a pseudo simple product is exactly what is needed to provide a right adjoint to the functor $\times_X^w : (\mathbb{P}/J)_{\text{po}} \rightarrow (\mathbb{P}/(J, X))_{\text{po}}$. The last observation in Section 2.3 then yields the following.

Lemma 4.3. *If \mathbb{P} has pseudo simple products, then \mathbb{E} has right adjoints to inverse images along products projections.*

We now have two necessary conditions, namely existence of weak exponentials and of pseudo simple products. The right adjoints to (weak) product functors, provided by the latter, allow us to define an equivalence relation on a weak exponential W of two projectives X and Y . The quotient of this equivalence relation will indeed be an exponential of X and Y in \mathbb{E} , as shown in the first part of the proof of Theorem 2.5 from [7]. Hence we do have the following.

Proposition 4.4. *If \mathbb{P} has weak exponentials and pseudo simple products, then \mathbb{E} has exponentials of projectives.*

At this point, the probably most natural strategy to define an exponential B^A is to do as in the second half of the proof of Theorem 2.5 in [7] and exploit the existence of exponentials of projectives in order to internalise the very definition of arrows $A \rightarrow B$ in the exact completion. This amounts to take two quasi-exact sequences $X_1 \rightrightarrows X_0 \rightarrow A$ and $Y_1 \rightrightarrows Y_0 \rightarrow B$, with $X_i, Y_i \in \mathbb{P}$, $i = 0, 1$, extract from $Y_0^{X_0}$ all those arrows that have a tracking $X_1 \rightarrow Y_1$, and then identify two such arrows if they are Y_1 -related.

This strategy can be seen as a two-step construction, where firstly the case of a projective domain is treated, and then the general case is derived. This means that we first define an exponential of $X \in \mathbb{P}$ and $B \in \mathbb{E}$ as the coequaliser below

$$(2) \quad Y_1^X \rightrightarrows Y_0^X \twoheadrightarrow E,$$

where $Y_1 \rightrightarrows Y_0 \rightarrow B$ is quasi-exact with $Y_0, Y_1 \in \mathbb{P}$ —this would imply in particular that every projective is exponentiable in \mathbb{E} —and then we use this fact to construct a general exponential.

As the following lemma shows, the second step represents no problem.

Lemma 4.5. *\mathbb{E} is cartesian closed if and only if every object in \mathbb{P} is exponentiable in \mathbb{E} .*

Proof. One direction is trivial, let us assume then that every projective is exponentiable. Let A and B be two objects in \mathbb{E} , let $X_1 \rightrightarrows X_0 \rightarrow A$ be quasi-exact with $X_0, X_1 \in \mathbb{P}$ and consider the following equaliser

$$E \hookrightarrow B^{X_0} \rightrightarrows B^{X_1}.$$

We will prove that E is an exponential of A and B .

The evaluation arrow $e: E \times A \rightarrow B$ is obtained from the universal property of the coequaliser and the commutativity of the diagram below.

$$\begin{array}{ccccc} & E \times X_1 & \rightrightarrows & E \times X_0 & \twoheadrightarrow & E \times A \\ & \swarrow & & \downarrow & & \downarrow \\ B^{X_1} \times X_1 & \longleftarrow & B^{X_0} \times X_1 & \rightrightarrows & B^{X_0} \times X_0 & \longrightarrow & B \\ & \searrow & & & & \swarrow & \end{array}$$

Given $C \in \mathbb{E}$ and $f: C \times A \rightarrow B$, there is a unique arrow $g': C \rightarrow B^{X_0}$ making the obvious triangle commute. This unique arrow factors through $E \hookrightarrow B^{X_0}$ since the left-hand diagram below commutes and B^{X_1} is an exponential. The resulting arrow $g: C \rightarrow E$ satisfies $e(g \times A) = f$ because of the commutativity of the right-hand diagram below.

$$\begin{array}{ccc} C \times X_1 \rightarrow B^{X_0} \times X_1 \rightrightarrows B^{X_1} \times X_1 & & C \times X_0 \twoheadrightarrow C \times A \\ \downarrow \downarrow & \downarrow \downarrow & \downarrow \\ C \times X_0 \rightarrow B^{X_0} \times X_0 & & C \times A \twoheadrightarrow B^{X_0} \times X_0 \leftarrow E \times X_0 \\ \downarrow & \searrow & \downarrow \\ C \times A \twoheadrightarrow B & & B \twoheadrightarrow E \times A \end{array}$$

Uniqueness of g follows from uniqueness of g' and monicity of $E \hookrightarrow B^{X_0}$. \square

Unfortunately, it turns out that the success of the first step is again equivalent to the projectives being internally projective.

Lemma 4.6. *Let X be an object in \mathbb{P} and suppose that X is exponentiable in \mathbb{E} . Then the following are equivalent.*

- (1) X is internally projective.
- (2) For every B in \mathbb{E} and every \mathbb{P} -cover $Y \rightarrow B$, $Y^X \rightarrow B^X$ is also regular epic.
- (3) For every B in \mathbb{E} there is a \mathbb{P} -cover $Y \rightarrow B$ such that $Y^X \rightarrow B^X$ is also regular epic.

Proof. We only need to prove that 3 implies 1, the other two implications being obvious. Let $A \rightarrow B$ be any regular epi in \mathbb{E} . Consider a \mathbb{P} -cover $Z \rightarrow A$, then the composite $Z \rightarrow A \rightarrow B$ is also a \mathbb{P} -cover, so that there is an arrow $Y \rightarrow Z$ over B . We then have the following commuting diagram

$$\begin{array}{ccc} Y^X & \longrightarrow & Z^X \\ \downarrow & \swarrow & \downarrow \\ B^X & \longleftarrow & A^X \end{array}$$

which clearly shows that $A^X \rightarrow B^X$ is regular epic. \square

Hence we conclude that the construction described in (2) produces an exponential for every $B \in \mathbb{E}$ if and only if X is internally projective. In the case of Example 2.2, indeed, there are no covers $(\mathbf{y}\mathbb{N})^{\mathbf{y}a} \rightarrow (\mathbf{y}a \times \mathbf{y}b)^{\mathbf{y}a}$, because $(\mathbf{y}\mathbb{N})^{\mathbf{y}a} \cong \mathbf{y}\mathbb{N}$ and $(\mathbf{y}a \times \mathbf{y}b)^{\mathbf{y}a} \cong \mathbf{y}b$.

Remark 4.7. As for the equivalence described in Section 3.1, it is possible to rephrase Lemma 4.6 only assuming weak exponentials in \mathbb{P} .

Let $X \in \mathbb{P}$ and $B \in \mathbb{E}$ and say that an object W in \mathbb{P} is a *quasi exponential of X and B* if there is an arrow $e: W \times X \rightarrow B$ and, for every object $Z \in \mathbb{P}$ and arrow $f: Z \times X \rightarrow B$, there is an arrow $g: Z \rightarrow W$ such that $e(g \times X) = f$.

Suppose that \mathbb{P} has weak exponentials and let X be an object in \mathbb{P} . Then the following are equivalent.

- (1) X is internally projective.
- (2) For every B in \mathbb{E} and every \mathbb{P} -cover $Y \rightarrow B$, any weak exponential of X and Y in \mathbb{P} is a quasi exponential of X and B .
- (3) For every B in \mathbb{E} there are a \mathbb{P} -cover $Y \rightarrow B$ and a weak exponential W of X and Y in \mathbb{P} such that W is a quasi exponential of X and B .

4.2. Generalised weak simple products. Contrary to what happens in the case of an ex/lex completion, Lemma 4.6 and Remark 4.7 clearly show that we cannot in general expect the quotient in (2) to produce an exponential. In fact, the \mathbb{P} -cover of an exponential is rather a quasi exponential as defined in Remark 4.7. In order to capture the essential properties of such a \mathbb{P} -cover only in terms of projectives, we find convenient to introduce some additional terminology.

An *equality* for a weak product $X \leftarrow V_0 \rightarrow Y$ is a pseudo equivalence relation $V_1 \rightrightarrows V_0$ whose quotient in \mathbb{E} is $X \times Y$, i.e. a weak limit of the diagram

$$\begin{array}{ccc} V_0 & & V_0 \\ \downarrow & \swarrow & \downarrow \\ X & & Y \end{array}$$

Given a pseudo equivalence relation $z_1, z_2: Z_1 \rightrightarrows Z_0$ and a weak product $X \leftarrow V_0 \rightarrow Y$, an arrow $V_0 \rightarrow Z_0$ *preserves projections with respect to z_1, z_2* if there is $V_1 \rightarrow Z_1$ making

$$\begin{array}{ccccc} & V_1 & \dashrightarrow & Z_1 & \\ & \searrow & & \searrow & \\ & & V_0 & \longrightarrow & Z_0 \\ & \swarrow & & \swarrow & \\ V_0 & \longrightarrow & & & Z_0 \end{array}$$

commute, where $V_1 \rightrightarrows V_0$ is an equality for the weak product. Note that this definition does not depend on the particular equality for V_0 . When the pseudo equivalence relation on Z_0 is clear from the context, we will just say that $V_0 \rightarrow Z_0$ preserves projections.

Remark 4.8. An arrow $V_0 \rightarrow Z_0$ out of a weak product of X and Y preserves projections with respect to z_1, z_2 if and only if it induces a (unique) arrow $X \times Y \rightarrow C$ in \mathbb{E} , where C is the quotient of z_1, z_2 . In particular, the following are equivalent.

- (1) $V_0 \rightarrow Z_0$ is determined by projections.
- (2) $V_0 \rightarrow Z_0$ preserves projections with respect to $\text{id}_{Z_0}, \text{id}_{Z_0}$.
- (3) $V_0 \rightarrow Z_0$ preserves projections with respect to any pseudo equivalence relation z_1, z_2 .

Moreover, if all projectives are internally projective, then every arrow which preserves projections with respect to z_1, z_2 is Z_1 -related to an arrow determined by projections.

Definition 4.9. Let $y_1, y_2: Y_1 \rightrightarrows Y_0$ be a pseudo equivalence relation and let $J \xleftarrow{f} Y_0 \xrightarrow{g} X$ be a span such that both f and g coequalise y_1, y_2 . A *generalised weak simple product of f and g with respect to y_1, y_2* consists of a commutative diagram

$$\begin{array}{ccc} W & \longleftarrow & V \\ \downarrow & & \downarrow \searrow \\ J & \longleftarrow & Y_0 \longrightarrow X \end{array}$$

where $W \leftarrow V \rightarrow X$ is a weak product and $V \rightarrow Y_0$ preserves projections with respect to y_1, y_2 , such that, for any other commutative diagram

$$\begin{array}{ccc} W' & \longleftarrow & V' \\ \downarrow & & \downarrow \searrow \\ J & \longleftarrow & Y_0 \longrightarrow X \end{array}$$

where $W' \leftarrow V' \rightarrow X$ is a weak product and $V' \rightarrow Y_0$ preserves projections, there are arrows $W' \rightarrow W$ and $V' \rightarrow V$ making everything commute.

- Remark 4.10.**
- (1) A generalised weak simple product of f and g with respect to $\text{id}_{Y_0}, \text{id}_{Y_0}$ is just a weak simple product of f and g .
 - (2) Let $Y_1 \rightrightarrows Y_0$ be a weak limit of

$$\begin{array}{ccc} Y_0 & & Y_0 \\ \downarrow & \searrow & \downarrow \\ J & & X \end{array}$$

then a generalised weak simple product of f and g with respect to $Y_1 \rightrightarrows Y_0$ is a pseudo simple product of f and g .

- (3) Suppose that every projective is internally projective. Then the last observation in Remark 4.8 and item (4) below imply that, in this case, a weak simple product is also a generalised weak simple product, and Theorems 4.13 and 4.16 reduce to Theorems 3.6 and 3.7, respectively.
- (4) There is an apparently weaker notion of generalised weak simple product, which in fact turns out to be equivalent to the one we considered in Definition 4.9. It consists of requiring, instead of two arrows $W' \rightarrow W$ and $V' \rightarrow V$ making everything commute, the existence of three arrows $W' \rightarrow W$, $V' \rightarrow V$ and $V' \rightarrow Y_1$ making the two diagrams below commute.

$$\begin{array}{ccc} W' & \longleftarrow & V' \\ \vdots \downarrow & & \downarrow \searrow \\ W & \longleftarrow & V \longrightarrow X \end{array} \quad \begin{array}{ccc} V & \longleftarrow \cdots & V' \\ \downarrow & & \downarrow \searrow \\ Y_0 & \longleftarrow & Y_1 \longrightarrow Y_0, \end{array}$$

that is, the arrow $V' \rightarrow V$ makes the two weak evaluations into Y_0 not equal, but only Y_1 -related. Indeed, given W and $V \rightarrow Y_0$ satisfying such a weaker condition, we may replace the weak product $W \leftarrow V \rightarrow X$ with another weak product $W \leftarrow U \rightarrow X$ which satisfies the condition in Definition 4.9. The object $U \in \mathbb{P}$ may be obtained as the top-left corner in a \mathbb{P} -covering

square

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ Y_0 & \longrightarrow & B, \end{array}$$

where the lower horizontal arrow is a coequaliser of $Y_1 \rightrightarrows Y_0$.

We may also define the notion of *generalised weak exponential* of a projective X and a pseudo equivalence relation $y_1, y_2: Y_1 \rightrightarrows Y_0$ as an object W , a weak product $W \leftarrow V \rightarrow X$ and an arrow $V \rightarrow Y_0$ which preserves projections such that, for any other object W' , weak product $W \leftarrow V \rightarrow X$ and arrow $V' \rightarrow Y_0$ which preserves projections, there are arrows $W' \rightarrow W$ and $V' \rightarrow V$ making everything commute.

Remarks (1), (3) and (4) in Remark 4.10 apply, mutatis mutandis, to generalised weak exponentials as well.

Remark 4.11. If \mathbb{P} has generalised weak simple products, then it has generalised weak exponentials too. Indeed, given X and a pseudo equivalence relation $Y_1 \rightrightarrows Y_0$, take a weak product U_0 of T , X and Y_0 , where T is weakly terminal, and define a pseudo equivalence relation $U_1 \rightrightarrows U_0$ as a weak limit of

$$\begin{array}{ccccc} & U_0 & & Y_1 & & U_0 \\ & \downarrow & & \swarrow & & \searrow \\ & Y_0 & & & & Y_0. \end{array}$$

A straightforward computation shows that a generalised weak simple product of $T \leftarrow U_0 \rightarrow X$ with respect to $U_1 \rightrightarrows U_0$ is a generalised weak exponential of X and $Y_1 \rightrightarrows Y_0$.

Proposition 4.12. *If \mathbb{E} is cartesian closed, then \mathbb{P} has generalised weak simple products.*

Proof. Let $Y_1 \rightrightarrows Y_0$ be a pseudo equivalence relation. Any span $J \leftarrow Y_0 \rightarrow X$ whose legs coequalise $Y_1 \rightrightarrows Y_0$ induces an arrow $B \rightarrow J \times X$ from the quotient $Y_0 \rightarrow B$ of $Y_1 \rightrightarrows Y_0$. Let $W \rightarrow J$ be the composite of the simple product $\Pi_X B \rightarrow J$ with a \mathbb{P} -cover $W \rightarrow \Pi_X B$. The weak product $W \leftarrow V \rightarrow X$ and the arrow $V \rightarrow Y_0$ are obtained from a \mathbb{P} -covering square

$$\begin{array}{ccc} V & \longrightarrow & W \times X \\ \downarrow & & \downarrow \\ Y_0 & \longrightarrow & B. \end{array}$$

Since $Y_1 \rightrightarrows Y_0$ is the kernel pair of $Y_0 \rightarrow B$, it is clear that $V \rightarrow Y_0$ preserves projections.

The required universal property follows from that one of the simple product once we recall that, for any $W' \rightarrow J$ and weak product $W' \leftarrow V' \rightarrow X$, an arrow $V' \rightarrow Y_0$ over J and X which preserves projections induces an arrow $W' \times X \rightarrow B$ over $J \times X$. \square

4.3. The characterisation. We are now in a position to formulate and prove the general characterisation of (locally) cartesian closed exact completion in terms of projectives.

Theorem 4.13. *\mathbb{E} is cartesian closed if and only if \mathbb{P} has generalised weak simple products.*

Proof. Thanks to Lemma 4.5, it is enough to construct an exponential of X and B with $X \in \mathbb{P}$. Let $Y_1 \rightrightarrows Y_0 \rightarrow B$ be quasi-exact with $Y_0, Y_1 \in \mathbb{P}$, let W and $V \rightarrow Y_0$ be a generalised weak exponential of X and $Y_1 \rightrightarrows Y_0$ and denote with $w: W \times X \rightarrow B$ the arrow induced by $V \rightarrow Y_0$ on the quotients.

The kernel pair of $\langle w, \text{pr}_X \rangle: W \times X \rightarrow B \times X$ factors through $W \times W \times \Delta_X$ via an arrow $k: K \hookrightarrow W \times W \times X$. Item (2) in Remark 4.10 and Lemma 4.3 ensure that the functor $(_) \times X: \text{Sub}(W \times W) \rightarrow \text{Sub}(W \times W \times X)$ has a right adjoint \forall_X . Define $r := \forall_X k: R \hookrightarrow W \times W$, then

(3) $t: T \rightarrow W \times W$ factors through r if and only if $w(t_1 \times X) = w(t_2 \times X)$.

Hence r is an equivalence relation, let $q: W \rightarrow E$ denote its quotient. It easy to see, using the counit of the adjunction $(_) \times X \dashv \forall_X$, that the arrow $w: W \times X \rightarrow B$ coequalises $r_1 \times X$ and $r_2 \times X$. Hence there is a (unique) arrow $e: E \times X \rightarrow B$ such that $e(q \times X) = w$.

Let now $f: C \times X \rightarrow B$, $Z_1 \rightrightarrows Z_0 \rightarrow C$ be quasi-exact with $Z_0, Z_1 \in \mathbb{P}$ and denote with $z: Z_0 \rightarrow C$ the cover. Any \mathbb{P} -cover $V' \rightarrow Y_0$ of $f(z \times X): Z_0 \times X \rightarrow B$ preserves projections, so the weak universal property of W ensures the existence of an arrow $f': Z_0 \rightarrow W$ such that $w(f' \times X) = f(z \times X)$. Since the diagram below commutes

$$\begin{array}{ccccc}
 Z_1 \times X & \longrightarrow & Z_0 \times X & \xrightarrow{f' \times X} & W \times X \\
 \downarrow & & \downarrow & & \downarrow w \\
 Z_0 \times X & \longrightarrow & C \times X & \xrightarrow{f} & B \\
 \downarrow f' \times X & & & & \downarrow w \\
 W \times X & \xrightarrow{w} & & & B,
 \end{array}$$

the arrow $Z_1 \rightarrow Z_0 \times Z_0 \rightarrow W \times W$ factors through r because of (3). This in turn implies that $Z_0 \rightarrow E$ coequalises $Z_1 \rightrightarrows Z_0$, thus yielding an arrow $\hat{f}: C \rightarrow E$. The equation $e(\hat{f} \times X) = f$ follows immediately once we precompose the two sides with $z \times X$.

For uniqueness, let $g: C \rightarrow E$ be such that $e(g \times X) = f$, and denote with $g': Z_0 \rightarrow W$ the arrow given by projectivity of Z_0 . We have that $w(g' \times X) = e((gg') \times X) = e((gz) \times X) = f(z \times X) = e(f' \times X)$, hence $\langle f', g' \rangle: Z_0 \rightarrow W \times W$ factors through r and so $g = \hat{f}$. \square

In order to treat local cartesian closure, Carboni and Rosolini defined a category with weak finite limits to be weakly locally cartesian closed if every slice has weak simple products [7]. However, in light of the preceding discussion, we prefer to reformulate it as follows. Notice first that the the notion of equality makes sense also for weak pullbacks, so that we can also refer to those arrows out of a weak pullback which preserve projections with respect to a given pseudo equivalence relation.

Definition 4.14. Let $y_1, y_2: Y_1 \rightrightarrows Y_0$ be a pseudo equivalence relation and let $Y_0 \xrightarrow{g} X \xrightarrow{f} J$ be a pair of arrows such that f coequalises y_1, y_2 . A *generalised weak dependent product of g along f with respect to y_1, y_2* consists of a commutative diagram

$$\begin{array}{ccccc}
 Y_0 & \longleftarrow & V & \longrightarrow & W \\
 & \searrow & \downarrow & & \downarrow \\
 & & X & \longrightarrow & J
 \end{array}$$

where the square is a weak pullback and $V \rightarrow Y_0$ preserves projections with respect to y_1, y_2 , and such that for any other commutative diagram

$$\begin{array}{ccccc}
 Y_0 & \longleftarrow & V' & \longrightarrow & W' \\
 & \searrow & \downarrow & & \downarrow \\
 & & X & \longrightarrow & J
 \end{array}$$

where the square is a weak pullback and $V' \rightarrow Y_0$ preserves projections with respect to y_1, y_2 , there are $W' \rightarrow W$ and $V' \rightarrow V$ making everything commute.

A category \mathbb{P} with weak finite limits is *weakly locally cartesian closed* if it has generalised weak dependent products.

Remark 4.15. It is possible to make similar observation for generalised weak dependent products as those in Remark 4.10. In particular, if \mathbb{P} has generalised weak dependent products, then it has right adjoints to weak pullback along any arrow.

However we will only give details for showing that \mathbb{P} has generalised weak dependent products if and only if every slice of \mathbb{P} has generalised weak simple products. One direction is obvious, for the other one it is only a matter to construct a suitable pseudo equivalence relation: Given a span $J \leftarrow Y_0 \rightarrow X$ over a projective U whose arrows coequalise a pseudo equivalence relation $Y_1 \rightrightarrows Y_0$ also over U , let $J \leftarrow V \rightarrow X$ be a weak pullback and consider a pseudo equivalence relation $Z_1 \rightrightarrows Z_0$ defined from the following weak limit diagrams

$$\begin{array}{ccc}
 & Z_0 & \\
 & \swarrow \quad \searrow & \\
 V & & Y_0 \\
 \downarrow & \swarrow \quad \searrow & \downarrow \\
 J & & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 & Z_1 & \\
 & \swarrow \quad \searrow & \\
 Z_0 & & Y_1 & & Z_0 \\
 \downarrow & \swarrow \quad \searrow & \downarrow & \swarrow \quad \searrow & \downarrow \\
 Y_0 & & V & & Y_0
 \end{array}$$

Then a generalised weak dependent product for $Z_1 \rightrightarrows Z_0 \rightarrow V \rightarrow J$ is easily seen to be a generalised weak simple product of $J \leftarrow Y_0 \rightarrow X$ with respect to $Y_1 \rightrightarrows Y_0$ in \mathbb{P}/U .

Theorem 4.16. *\mathbb{E} is locally cartesian closed if and only if \mathbb{P} is weakly locally cartesian closed.*

Proof. For every $X \in \mathbb{P}$, \mathbb{P}/X is a projective cover of \mathbb{E}/X , hence Remark 4.15 and Theorem 4.13 yields the cartesian closure of \mathbb{E}/X . The general statement follows now from descent, for example applying Theorem 3 in Section 3.7 of [3] to the diagram

$$\begin{array}{ccc}
 \mathbb{E}/I & \xrightarrow{(_) \times_I a} & \mathbb{E}/I \\
 \uparrow \Sigma_p \dashv p^* & & \uparrow \Sigma_p \dashv p^* \\
 \mathbb{E}/J & \xrightarrow{(_) \times_J p^* a} & \mathbb{E}/J \\
 & \leftarrow \perp & \\
 & (_)^{p^* a} &
 \end{array}$$

where $a: A \rightarrow I$ is an object in \mathbb{E}/I and $p: J \rightarrow I$ is a \mathbb{P} -cover. □

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REFERENCES

- [1] P. Aczel. The type theoretic interpretation of constructive set theory. In A. MacIntyre, L. Pacholski, and J. Paris, editors, *Logic Colloquium '77*, volume 96 of *Studies in Logic and the Foundations of Mathematics*, pages 55–66. North-Holland, Amsterdam, 1978.
- [2] P. Aczel and M. Rathjen. Notes on Constructive Set Theory. Technical Report 40, Mittag-Leffler Institute, The Swedish Royal Academy of Sciences, 2001.
- [3] M. Barr and C. Wells. *Toposes, Triples and Theories*. Springer-Verlag, New York, 1985.
- [4] L. Birkedal, A. Carboni, G. Rosolini, and D. S. Scott. Type theory via exact categories (extended abstract). In *Proceedings of the 13th Annual IEEE Symposium on Logic in Computer Science LICS '98*, pages 188–198. IEEE Computer Society Press, 1998.
- [5] A. Carboni. Some free constructions in realizability and proof theory. *Journal of Pure and Applied Algebra*, 103(2):117–148, 1995.

- [6] A. Carboni and R. Celia Magno. The free exact category on a left exact one. *Journal of the Australian Mathematical Society*, 33(3):295–301, 1982.
- [7] A. Carboni and G. Rosolini. Locally cartesian closed exact completions. *Journal of Pure and Applied Algebra*, 154(1-3):103–116, 2000.
- [8] A. Carboni and E.M. Vitale. Regular and exact completions. *Journal of Pure and Applied Algebra*, 125(1-3):79–116, 1998.
- [9] J. Emmenegger. Aczel’s Fullness Axiom and homotopy categories. In preparation.
- [10] J. Emmenegger and E. Palmgren. Exact completion and constructive theories of sets. Submitted. arXiv:1710.10685, 2017.
- [11] W. Lawvere. An elementary theory of the category of sets. *Proceedings of the National Academy of Science of the U.S.A.*, 52:1506–1511, 1965.
- [12] W. Lawvere and C. McLarty. An elementary theory of the category of sets (long version) with commentary. *Reprints in Theory and Applications of Categories*, 11:1–35, 2005.
- [13] nLab. General discussions: internally projective objects. <https://nforum.ncatlab.org/discussion/4342/internally-projective-objects/>.
- [14] nLab. Presentation axiom. <https://ncatlab.org/nlab/show/presentation+axiom/>.
- [15] E. Palmgren. Constructivist and structuralist foundations: Bishop’s and Lawvere’s theories of sets. *Annals of Pure and Applied Logic*, 163(10):1384–1399, 2012.
- [16] J. Rosicky. Cartesian closed exact completions. *Journal of Pure and Applied Algebra*, 142(3):261–270, 1999.