

# THE FULLNESS AXIOM AND EXACT COMPLETIONS OF HOMOTOPY CATEGORIES

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ABSTRACT. We use a category-theoretic formulation of Aczel's Fullness Axiom from Constructive Set Theory to derive the local cartesian closure of an exact completion. As an application, we prove that such a formulation is valid in the homotopy category of any model category satisfying mild requirements, thus obtaining in particular the local cartesian closure of the exact completion of topological spaces and homotopy classes of maps. Under a type-theoretic reading, these results provide a general motivation for the local cartesian closure of the category of setoids. However, results and proofs are formulated solely in the language of categories, and no knowledge of type theory or constructive set theory is required on the reader's part.

## 1. INTRODUCTION

In the paper that generalises the exact completion construction to an arbitrary category with *weak* finite limits (where uniqueness of the universal arrow is dropped), Carboni and Vitale advocated a deeper study of that construction applied to homotopy categories [8]. These categories, indeed, form a large class of natural examples of categories with weak finite limits, in the sense that they do not arise as projective covers of finitely complete categories. A first step in this direction was made by Gran and Vitale in [12], where they provide a complete characterisation of those ex/wlex completions that produce a pretopos, and apply this result to show that the exact completion of the category of topological spaces and homotopy classes of maps is indeed a pretopos. However, the problem of determining whether it is also locally cartesian closed is explicitly left open.

The author has given a complete characterisation of locally cartesian closed ex/wlex completions in [9]. That characterisation is however not much suited to the study of the ex/wlex completion of homotopy categories, when instead a formulation in terms of the original model category  $\mathbb{M}$  would be preferable. The present paper provides a condition ensuring the local cartesian closure of  $(\text{Ho}\mathbb{M})_{\text{ex}}$  for a large class of model categories. Somewhat surprisingly, this condition turns out to be what Carboni and Rosolini named weak local cartesian closure in [7], that is, simply existence of weak dependent products.

The homotopy quotient of a weak dependent product in  $\mathbb{M}$ , indeed, turns out to be a *dependent full diagram* in  $\text{Ho}\mathbb{M}$ . The latter is a generalisation to arbitrary categories with weak finite limits of a concept introduced in [10] to prove the local cartesian closure of the exact completion of a well-pointed category with finite products and weak equalisers. The universal property of these diagrams is inspired by Aczel's Fullness Axiom from the constructive set theory CZF [1, 2]. The Fullness Axiom is a collection principle asserting the existence of what Aczel calls *full sets*, that is, sets containing enough total relations (a.k.a. multi-valued functions). This axiom is strictly weaker than the Power Set Axiom

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(and it is regarded as a predicative principle) but strong enough, in particular, to entail the Exponentiation Axiom, which asserts that functions between two sets form a set.

We prove that also in the general case of an exact category  $\mathbb{E}$  with enough projectives  $\mathbb{P}$ , i.e. for any ex/wlex completion, existence of dependent full diagrams in  $\mathbb{P}$  is enough to derive the local cartesian closure of  $\mathbb{E}$ . That this should be possible follows from the observation, due to Erik Palmgren, that arrows out of a weak product may be understood as multi-valued functions on the actual product. Indeed, recall that weak products in  $\mathbb{P}$  are projective covers of products in  $\mathbb{E}$ . Hence, given a weak product  $W \rightarrow X \times Y$  and an arrow  $W \rightarrow Z$ , the relation  $R$  arising from the image factorisation of  $W \rightarrow X \times Y \times Z$  can be seen as a family indexed by  $X$  of multi-valued functions from  $Y$  to  $Z$ . These are single-valued if and only if  $W \rightarrow Z$  induces an arrow  $X \times Y \rightarrow Z$ , in which case  $R$  is just the graph of  $X \times Y \rightarrow Z$ . In particular, if  $X \times Y$  was already projective for  $X$  and  $Y$  projectives, any family of multi-valued functions would have an associated family of single-valued functions.

In the case of ex/lex completions, that is, when projectives are closed under finite limits, focusing on families of single-valued functions is thus enough to characterise local cartesian closure in terms of a property of the projectives: these families are the same thing as the arrows *determined by projections* from [7], where such a characterisation is proved. In the more general case of ex/wlex completions however, expecting to obtain a characterisation of arrows in the exact completion using families of single-valued functions amounts to assuming an internal choice principle, which is indeed equivalent to the closure of the projectives under finite limits [9]. Dependent full diagrams allow us to overcome the limitations of single-valued functions. Indeed, they endow the internal logic of  $\mathbb{E}$  with implication and universal quantification, which then may be used to construct an exponential  $B^A$  in  $\mathbb{E}$  extracting from all multi-valued functions only those which are functional with respect to the equivalence relation in  $B$ .

In order to prove that dependent full diagrams are homotopy quotients of weak dependent products, we exploit the concepts of *path category* and *weak homotopy  $\Pi$ -type*, recently introduced by van den Berg and Moerdijk in [5]. A path category is a slight strengthening of Brown's fibration category. In particular, the category of fibrant objects  $\mathbb{M}_f$  is a path category as soon as all the objects in the model category  $\mathbb{M}$  are cofibrant. Weak homotopy  $\Pi$ -types in a path category  $\mathbb{C}$  are what van den Berg and Moerdijk use to derive the local cartesian closure of  $(\text{Ho } \mathbb{C})_{\text{ex}}$ , the homotopy exact completion of  $\mathbb{C}$ . We show that if  $\mathbb{M}$  is right proper, then weak homotopy  $\Pi$ -types arise as fibrant replacements of weak dependent products, so that a weak dependent product in  $\mathbb{M}$  gives rise to a weak homotopy  $\Pi$ -type in  $\mathbb{M}_f$ . Furthermore, in the same way as pullbacks along fibrations enjoy the additional universal property of homotopy pullbacks, also weak homotopy  $\Pi$ -types enjoy an additional universal property up to homotopy. As it turns out, once we look at this universal property in the homotopy category, we obtain exactly the property of dependent full diagrams.

Under a type-theoretic reading, the results in the present paper provide a general motivation for the local cartesian closure of the category of setoids in Martin-Löf type theory. Indeed, the category of contexts of Martin-Löf type theory is a path category [11, 4], and  $\Pi$ -types endow it with weak homotopy  $\Pi$ -types. More generally, we obtain a more elementary proof of the local cartesian closure of a homotopy exact completion. It should be noted that, under the reading of arrows out of weak products as multi-valued functions, single-valued functions in a homotopy category  $\text{Ho } \mathbb{C}$  appear “homotopy-irrelevant” arrows. Indeed, an arrow  $k$  out of a homotopy limit, say a homotopy pullback of  $f: X \rightarrow Y$  and  $g: Z \rightarrow Y$ , induce an arrow out of the actual pullback of  $f$  and  $g$  if and only if its values only depend on pairs  $(x, z)$  and not on the homotopy witnessing  $f(x) \simeq g(z)$ . The analogy

with the role of homotopy-irrelevant fibrations in the argument for the local cartesian closure of  $\text{Ho } \mathbb{C}_{\text{ex}}$  from van den Berg and Moerdijk [5] may be worth further investigation. Indeed, in type-theoretic terminology, these are the proof-irrelevant setoid families whose importance has been stressed by Palmgren [14].

Furthermore, the observation that dependent full diagrams naturally arise as homotopy quotients of weak dependent products shows that existence of the former is not just a particular feature of the category of types in Martin-Löf type theory, the only example in [10]. On the contrary, it provides a large class of examples of categories with weak finite limits and dependent full diagrams. In particular, we obtain the local cartesian closure of the exact completion of the category of spaces and homotopy classes of maps, thus answering a question left open in [12].

The main part of the paper is devoted to the proof that existence of dependent full diagrams imply the local cartesian closure of the exact completion. In order to simplify the presentation, we decided to split the argument in its two main steps. In Section 2, after a brief recap on ex/wlex completions, we define a non-indexed version of a full diagram in  $\mathbb{P}$  and, assuming that  $\mathbb{E}$  (or, equivalently,  $\mathbb{P}$ ) has the needed structure for implication and universal quantification, we construct from it an exponential in  $\mathbb{E}$ . Section 3 contains the definition of the more general dependent full diagrams and the proof that their existence gives rise both to right adjoints to inverse images and to non-indexed full diagrams. Finally, Section 4 covers the case of homotopy categories.

## 2. FULL DIAGRAMS

In this section and the next one  $\mathbb{E}$  denotes an exact category with enough projectives and  $\mathbb{P}$  a projective cover of it. This is equivalent to say that  $\mathbb{P}$  is a category with weak finite limits and  $\mathbb{E}$  is the exact completion of it [8]. Regular epis are denoted with a triangle head, like in  $A \twoheadrightarrow B$ , while hook arrows  $A \hookrightarrow B$  denote monos. The last six letters of the alphabet denote objects in  $\mathbb{P}$ . A square of the form

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & Y \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

is  $\mathbb{P}$ -covering if  $X, Y$  are objects of  $\mathbb{P}$  and the universal arrow  $X \rightarrow A \times_B Y$  is a regular epi. In this case we also say that  $\bar{f}$  covers  $f$ .

Recall that a weak limit in  $\mathbb{P}$  can be computed as a projective cover of a limit in  $\mathbb{E}$ . In a similar fashion, one obtains the notion of *weak exponential* in  $\mathbb{P}$  from exponentials in  $\mathbb{E}$ : given two projectives  $X$  and  $Y$ , take a cover  $W \twoheadrightarrow Y^X$  and a weak product  $V \twoheadrightarrow W \times X$ . Composing the latter with  $W \times X \twoheadrightarrow Y^X \times X \rightarrow Y$  yields an arrow in  $\mathbb{P}$  which is weakly terminal (in  $\mathbb{P}$ ) among arrows *determined by projections*, i.e. those that in  $\mathbb{E}$  are of the form  $V' \twoheadrightarrow U \times X \rightarrow Y$ , for  $U, V' \in \mathbb{P}$ .

The universal property of weak exponentials (and its strengthening considered in [7]) is not suited for the weak limits case, since it only applies to arrows determined by projections. This problem is discussed in [9], where a complete solution is also presented. Here we only sketch it to the extent that is needed to introduce the approach inspired by Aczel's Fullness Axiom.

A cartesian closed category with pullbacks has, in particular, right adjoints to inverse images along product projections:

$$\text{Sub}_{\mathbb{E}}(Y) \xleftarrow[\text{(\_)} \times X]{\begin{array}{c} \forall_X \\ \top \end{array}} \text{Sub}_{\mathbb{E}}(Y \times X).$$

When  $X$  and  $Y$  are projective,  $\text{Sub}_{\mathbb{E}}(Y \times X)$  is order isomorphic to the order reflection of  $\mathbb{P}/(Y, X)$ , the category of spans over  $X$  and  $Y$ , and similarly  $\text{Sub}_{\mathbb{E}}(Y) \cong (\mathbb{P}/Y)_{\text{po}}$ . The adjunction above then induces an adjunction between  $(\mathbb{P}/Y)_{\text{po}}$  and  $(\mathbb{P}/(Y, X))_{\text{po}}$ , where the function  $(\mathbb{P}/Y)_{\text{po}} \rightarrow (\mathbb{P}/(Y, X))_{\text{po}}$  is the one induced by the weak product in  $\mathbb{P}$ . Hence a necessary condition for the cartesian closure of  $\mathbb{E}$  is that the “weak product functor” in  $\mathbb{P}$  is left adjoint. However, in order to obtain such a right adjoint from (weakly) universal properties that apply only to arrows determined by projections (like the one of weak exponentials), a necessary assumption is the closure of projectives under finite products [9].

Therefore we need diagrams with a more general (weak) universal property. To this aim, let us look more carefully at arrows  $V \rightarrow Y$  out of a weak product  $V \rightarrow U \times X$ . Under the isomorphism  $(\mathbb{P}/(U, X, Y))_{\text{po}} \rightarrow \text{Sub}_{\mathbb{E}}(U \times X \times Y)$ , objects in  $\mathbb{P}/(U, X, Y)$  whose domain  $V$  is a weak product of  $U$  and  $X$  are mapped into relations  $R \hookrightarrow U \times X \times Y$  which are total on  $U$  and  $X$  or, in other terms, families of total relations (i.e. multi-valued functions) from  $X$  to  $Y$  indexed by  $U$ . Moreover,  $V \rightarrow Y$  is determined by projections precisely when the corresponding relation  $R \hookrightarrow U \times X \times Y$  is the graph of the arrow  $U \times X \rightarrow Y$  induced by  $V \rightarrow Y$ .

This observation suggests that, in order to have a suitable universal property with respect to arbitrary arrows  $V' \rightarrow Y$  out of a weak product  $V' \rightarrow U \times X$ , we should look for some property of closure with respect to (families of) total relations from  $X$  to  $Y$ . A promising notion, that indeed proves to be useful, is that of a *full set*. This concept was introduced in the context of Constructive Set Theory by Peter Aczel in [1] in order to provide a simpler formulation of the Subset Collection Axiom, an axiom schema actually. A set  $f$  is full for two sets  $a$  and  $b$  if it consists of multi-valued functions from  $a$  to  $b$  and, for every multi-valued function  $r$  from  $a$  to  $b$ , there is  $s \in f$  such that  $s \subseteq r$ . The *Fullness Axiom* states that for any two sets there is a full set. This assumption implies in particular that the class of functions between two sets form a set, i.e. the so-called Exponentiation Axiom. The next definition provides a categorical formulation of the concept of full set in the context of a category with weak finite limits.

**Definition 2.1.** Let  $X, Y$  be two objects in a category with weak finite limits. A diagram

$$\begin{array}{ccc} U & \longleftarrow V & \longrightarrow X \\ & & \downarrow \\ & & Y \end{array}$$

where the row is a weak product, is *full for total pseudo-relations from  $X$  to  $Y$*  (or simply *full*) if for any other such diagram there are an arrow  $U' \rightarrow U$ , a weak pullback  $U' \leftarrow P \rightarrow V$  of  $U' \rightarrow U \leftarrow V$  and an arrow  $P \rightarrow V'$  such that the diagram below commutes.

$$\begin{array}{ccccc} & & U' & \xleftarrow{\dots} & P & \xrightarrow{\dots} & V' & & \\ & & \swarrow & & \searrow & & \swarrow & & \\ & & U & \longleftarrow & V & \longrightarrow & X & & \\ & & & & \downarrow & & & & \\ & & & & Y & & & & \end{array}$$

- Remark 2.2.** (1) If  $V \rightarrow U \times X \times Y$  is a full diagram and  $U \leftarrow W \rightarrow X$  is another weak product, then  $W \rightarrow U \times X \times Y$  is also a full diagram.
- (2) Regarding  $\mathbb{P}$  as a subcategory of  $\mathbb{E}$ , the last diagram in Definition 2.1 can be rewritten as

$$\begin{array}{ccccc} & & \triangle & & \\ & & \curvearrowright & & \\ U' \times X & \longleftarrow & P & \longrightarrow & V' \\ \downarrow & & \downarrow & & \downarrow \\ U \times X & \longleftarrow & V & \longrightarrow & Y \end{array}$$

where the left-hand square is  $\mathbb{P}$ -covering. In particular, if  $U' \times X$  is projective there is  $U' \times X \rightarrow V$  in  $\mathbb{P}/(U, X, Y)$ .

The observation that a limit is the minimal weak limit, in the sense that it is a retract of any weak limit on the same diagram, implies that a weak exponential is minimal among full diagrams in the sense, specified by the following lemma, that weak exponentials (in  $\mathbb{P}$ ) are those full diagrams whose weak product is in fact a product. This is analogous to the observation that single-valued functions are minimal among multi-valued ones.

**Lemma 2.3.** *Suppose that projective objects are closed under binary products. Then in  $\mathbb{P}$  weak exponentials are full diagrams and any full diagram gives rise to a weak exponential.*

*Proof.* The fact that a weak exponential is in particular a full diagram is an immediate application of the fact that a product is a retract of any weak product. For the converse, consider a full diagram of two objects  $X$  and  $Y$ . The weak evaluation  $U \times X \rightarrow Y$  is obtained from a full diagram precomposing  $V \rightarrow Y$  with a section of  $V \rightarrow U \times X$ . Given any arrow  $U' \times X \rightarrow Y$ , the universal property yields an arrow  $U' \rightarrow U$  and a commutative diagram

$$\begin{array}{ccccc} U' \times X & \longleftarrow & P & \longrightarrow & U' \times X \\ \downarrow & & \downarrow & & \downarrow \\ U \times X & \longleftarrow & V & \longrightarrow & Y \end{array}$$

where the left-hand square is covering. The statement now follows from the fact that the arrow  $P \rightarrow U' \times X$  is a split epi and the section can be chosen so that the diagram below commutes.

$$\begin{array}{ccc} U' \times X & \hookrightarrow & P \\ \downarrow & & \downarrow \\ U \times X & \hookrightarrow & V. \end{array} \quad \square$$

Hence a category with finite limits has full diagrams if and only if it has weak exponentials. The next lemma shows that, whenever the internal logic of  $\mathbb{E}$  (equivalently, of  $\mathbb{P}$ ) supports implication and universal quantification, the left-to-right implication always holds.

**Lemma 2.4.** *Suppose that  $\mathbb{E}$  has right adjoints to inverse images. If  $\mathbb{P}$  has full diagrams, then for any  $X$  in  $\mathbb{P}$  and  $B$  in  $\mathbb{E}$  there are an object  $W$  in  $\mathbb{P}$  and an arrow  $W \times X \rightarrow B$  which are weakly terminal with respect to objects  $Z$  in  $\mathbb{P}$  and arrows  $Z \times X \rightarrow B$ .*

*Proof.* Let  $Y \rightarrow B$  be a cover, and take a full diagram  $V \rightarrow U \times X \times Y$ . The idea is to extract from  $U$  (codes of) functional relations. Let  $\gamma: I \hookrightarrow U \times X \times B$  be the image of  $V \rightarrow U \times X \times B$ ,  $I_1 \hookrightarrow I \times I$  the kernel pair of  $I \rightarrow U \times X$  and denote with  $\varphi: F \hookrightarrow U$  the subobject defined by

$$u : U \mid (\forall x : X)(\forall y, y' : B) \gamma(u, x, y) \wedge \gamma(u, x, y') \Rightarrow y = y'.$$

In other words, given an arrow  $g: A \rightarrow U$  consider the diagram

$$\begin{array}{ccccc}
 K & \longrightarrow & I_1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & L & \longrightarrow & I & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 L & \longrightarrow & I & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & A \times X & \xrightarrow{g \times X} & U \times X & 
 \end{array}$$

where all the squares are pullback. Then

- (1)  $g$  factors through  $\varphi$  if and only if  $L \rightarrow I \rightarrow B$  coequalises  $K \rightrightarrows L$ .

Taking  $g = \varphi$  then yields an arrow  $e: F \times X \rightarrow B$ . We will show that  $e$  satisfies the required universal property, It then follows easily that  $W \times X \rightarrow B$  satisfies it as well for any cover  $W \rightarrow F$ . In particular,  $\mathbb{P}$  will have weak exponentials.

Given  $f: Z \times X \rightarrow B$  with  $Z \in \mathbb{P}$ , take a cover  $V' \rightarrow Z \times X$ . We then have an arrow  $g: Z \rightarrow U$  and a commuting diagram

$$\begin{array}{ccccc}
 Z \times X & \xrightarrow{\quad} & Z \times X & & \\
 \downarrow & \swarrow P & \downarrow & \downarrow & \downarrow \\
 U \times X & \xrightarrow{\quad} & V & \longrightarrow & Y \longrightarrow B \\
 & \swarrow & \downarrow & \downarrow & \downarrow \\
 & & V & \longrightarrow & Y \longrightarrow B
 \end{array}$$

where the left-hand square is covering. In order to obtain an arrow  $h: Z \rightarrow F$  such that  $\varphi h = g$  and the lower triangle in the diagram below commutes, it is enough to show that the back square below commutes.

$$\begin{array}{ccccc}
 Z \times_U I & \xrightarrow{\quad} & I & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & F \times_U I & \longrightarrow & I & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 Z \times X & \xrightarrow{\quad} & F \times X & \longrightarrow & B \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & F \times X & \longrightarrow & B & 
 \end{array}$$

In this case, existence of  $h$  will follow from (1), and the equation  $\varphi h = g$  will imply that the upper triangle in (3) commutes, the lower triangle will then commute as well. To see that the back square in (3) commutes notice that, since the left-hand square in (2) is covering, there is  $P \rightarrow Z \times_U I$ . This arrow together with commutativity of the two front squares in (2) then yields the claim.  $\square$

**Theorem 2.5.** *Suppose that  $\mathbb{E}$  has right adjoints to inverse images. If  $\mathbb{P}$  has full diagrams, then  $\mathbb{E}$  is cartesian closed.*

*Proof.* We prove that every projective in  $\mathbb{P}$  is exponentiable in  $\mathbb{E}$ . In light of Lemma 4.5 in [9], this is enough to conclude the cartesian closure of  $\mathbb{E}$ .

Let  $X \in \mathbb{P}$  and  $B \in \mathbb{E}$  and let  $W \in \mathbb{P}$  and  $W \times X \rightarrow B$  be given by Lemma 2.4. The exponential  $B^X$  will be the quotient by a suitable equivalence relation of the object  $W$ . The kernel pair of  $\langle e, \text{pr}_X \rangle: W \times X \rightarrow B \times X$  factors through  $W \times W \times \Delta_X$  via an arrow  $k: K \hookrightarrow W \times W \times X$ . Let  $\rho := \forall_X k: R \hookrightarrow W \times W$ , then

- (4)  $f: U \rightarrow W \times W$  factors through  $\rho$  if and only if  $e(f_1 \times X) = e(f_2 \times X)$ .

Hence  $\rho$  is an equivalence relation, let  $W \rightarrow B^X$  denote the quotient. There is also an arrow  $ev: B^X \times X \rightarrow B$ , since  $e: W \times X \rightarrow B$  coequalises the kernel pair of  $\langle e, \text{pr}_X \rangle$ .

Let now  $f: A \times X \rightarrow B$  and  $A_1 \rightrightarrows A_0 \xrightarrow{a} A$  be exact with  $A_0 \in \mathbb{P}$ . The weak universal property of  $W$  ensures the existence of an arrow  $f': A_0 \rightarrow W$  such that  $e(f' \times X) = f(a \times X)$ . Since the diagram below commutes

$$\begin{array}{ccccc} A_1 \times X & \rightrightarrows & A_0 \times X & \rightarrow & W \times X \\ \downarrow & & \downarrow & & \downarrow \\ A_0 \times X & \rightrightarrows & A \times X & & \\ \downarrow & & \searrow & & \downarrow \\ W \times X & \longrightarrow & & & B \times X, \end{array}$$

the arrow  $A_1 \hookrightarrow A_0 \times A_0 \rightarrow B^X \times B^X$  factors through  $\rho$ . This in turn implies that  $A_0 \rightarrow B^X$  coequalises  $A_1 \rightrightarrows A_0$ , thus yielding an arrow  $\hat{f}: A \rightarrow B^X$ . The equation  $ev(\hat{f} \times X) = f$  is immediate.

For uniqueness, let  $g: A \rightarrow B^X$  be such that  $ev(g \times X) = f$ , and denote with  $g': A_0 \rightarrow W$  the arrow given by projectivity of  $A_0$ . We have that  $e(g' \times X) = ev((pg') \times X) = ev((ga) \times X) = f(a \times X) = e(f' \times X)$ , hence  $\langle f', g' \rangle: A_0 \rightarrow W \times W$  factors through  $\rho$  and so  $g = \hat{f}$ .  $\square$

Below we collect together the results in this section.

**Corollary 2.6.** *Suppose that  $\mathbb{E}$  has right adjoints to inverse images, and consider the following.*

- (1)  $\mathbb{P}$  has full diagrams.
- (2)  $\mathbb{E}$  is cartesian closed.
- (3)  $\mathbb{P}$  has weak exponentials.

We have  $1 \Rightarrow 2 \Rightarrow 3$ . If projective objects are closed under binary products, then  $3 \Rightarrow 1$ .

### 3. DEPENDENT FULL DIAGRAMS

We now define an indexed version of full diagrams, whose existence will endow the internal logic of  $\mathbb{P}$  (hence of  $\mathbb{E}$ ) with implication and universal quantification.

**Definition 3.1.** Let  $g: Y \rightarrow X$  and  $f: X \rightarrow J$  be two arrows in a category with weak finite limits. A *dependent full diagram over  $f, g$*  is a diagram

$$\begin{array}{ccccc} Y & \longleftarrow & V & \longrightarrow & U \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & J \end{array}$$

where the square is a weak pullback, and such that, for every other such diagram there are an arrow  $U' \rightarrow U$ , a weak pullback  $V \leftarrow P \rightarrow U'$  of  $V \rightarrow U \leftarrow U'$  and an arrow  $P \rightarrow V'$  making the diagram below commute.

$$\begin{array}{ccccc} & & V' & \longleftarrow & P & \longrightarrow & U' \\ & & \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & \longrightarrow & U & \longrightarrow & J \\ & \searrow & \downarrow & & \downarrow & & \\ & & X & \longrightarrow & J & & \end{array}$$

The same observations as in Remark 2.2 apply, mutatis mutandis, to dependent full diagrams. Furthermore, it is not difficult to see that dependent full diagrams generalise full families of pseudo-relations from [10], in the sense that the two notions coincide in well-pointed categories with finite products and weak equalisers.

As for the non-indexed case, as soon as projectives are closed under pullback, we may regard weak dependent products in  $\mathbb{P}$  as those dependent full diagrams whose weak pullback is a pullback.

**Lemma 3.2.** *Suppose that projective objects are closed under pullback. Then in  $\mathbb{P}$  weak dependent products are dependent full diagrams and any dependent full diagram gives rise to a weak dependent products.*

**Lemma 3.3.** *If  $\mathbb{P}$  has dependent full diagrams, then it has full diagrams.*

*Proof.* A full diagram for two objects  $X$  and  $Y$  can be obtained as a dependent full diagram over  $V \rightarrow U \rightarrow T$ , where  $T$  is weakly terminal,  $U$  is a weak product of  $X$  and  $T$  and  $V$  is a weak product of  $U$  and  $Y$ .  $\square$

**Lemma 3.4.**  *$\mathbb{P}$  has dependent full diagrams if and only if every slice of  $\mathbb{P}$  has them.*

**Lemma 3.5.** *If  $\mathbb{P}$  has dependent full diagrams, then  $\mathbb{E}$  has right adjoints to inverse images.*

*Proof.* It is easy to see that dependent full diagrams allow us to define, for every  $f: X \rightarrow Y$  in  $\mathbb{P}$ , functors  $\forall_f^w: (\mathbb{P}/X)_{\text{po}} \rightarrow (\mathbb{P}/Y)_{\text{po}}$  and that their universal property implies  $f^* \dashv \forall_f^w$ . Hence the isomorphisms  $(\mathbb{P}/X)_{\text{po}} \cong \text{Sub}_{\mathbb{E}}(X)$  ensure that inverse images between projectives are left adjoint. Using descent we may now lift these adjunctions to inverse images between arbitrary objects. Indeed, whenever  $\bar{f}: X \rightarrow Y$  covers  $f: A \rightarrow B$  we obtain a diagram

$$\begin{array}{ccc} \text{Sub}(X) & \xrightleftharpoons[\bar{f}^*]{\forall_{\bar{f}}} & \text{Sub}(Y) \\ \exists_p \downarrow \dashv \uparrow p^* & & \exists_q \downarrow \dashv \uparrow q^* \\ \text{Sub}(A) & \xleftarrow{f^*} & \text{Sub}(B) \end{array}$$

where the vertical adjunctions are monadic, the square of left-adjoints commutes (i.e.  $f^* \exists_q = \exists_p \bar{f}^*$ ) and  $f^*$  preserves regular epis. Hence Theorem 3 from Section 3.7 in [3] ensures that  $\forall_{\bar{f}}$  induces a functor  $\forall_f: \text{Sub}(A) \rightarrow \text{Sub}(B)$  which is right adjoint to  $f^*$ .  $\square$

**Theorem 3.6.** *If  $\mathbb{P}$  has dependent full diagrams, then  $\mathbb{E}$  is locally cartesian closed.*

*Proof.* It only remains to put together the previous results. Lemma 3.5 ensures that  $\mathbb{E}$  has right adjoints to inverse images, whereas Lemmas 3.3 and 3.4 imply that  $\mathbb{P}/X$  has full diagrams whenever  $X$  is in  $\mathbb{P}$ . Hence Theorem 2.5 yields the cartesian closure of  $\mathbb{E}/X$ . The general statement follows now from descent, using the fact that

$$\mathbb{E}/X \xleftarrow[\Sigma_p]{p^*} \mathbb{E}/A$$

is monadic when  $p: X \rightarrow A$  is a cover and applying again a theorem from Section 3.7 in [3].  $\square$

We may again collect together the results of this section.

**Corollary 3.7.** *Consider the following.*

- (1)  $\mathbb{P}$  has dependent full diagrams.
- (2)  $\mathbb{E}$  is locally cartesian closed.
- (3)  $\mathbb{P}$  has weak dependent products.

*We have  $1 \Rightarrow 2 \Rightarrow 3$ . If projective objects are closed under pullback, then  $3 \Rightarrow 1$ .*



4. FULL DIAGRAMS IN HOMOTOPY CATEGORIES

In this section we show that, under mild assumptions on a model category  $\mathbb{M}$ , the homotopy category  $\text{Ho } \mathbb{M}$  has dependent full diagrams if  $\mathbb{M}$  has weak dependent products. Using well-known results, this implies that the exact completion of the homotopy categories on spaces and CW-complexes yields locally cartesian closed pretoposes.

Fibrations, cofibrations and weak equivalences are denoted as  $\rightarrow$ ,  $\twoheadrightarrow$  and  $\xrightarrow{\sim}$ , respectively. A path object of an object  $A$  is denoted as  $PA$ , and a fibrewise path object over a fibration  $p: A \twoheadrightarrow B$  as  $P_p A$ . Since we shall not be concerned here with cylinder objects, we say that two arrows  $f, g: C \rightarrow A$  in  $\mathbb{M}$  are homotopic, written  $f \simeq g$ , if they are right homotopic (i.e. homotopic with respect to the path object  $PA$ ). Similarly, we will write  $f \simeq_p g$  to mean that they are fibrewise right homotopic over  $p$ .

We consider model categories  $\mathbb{M}$  where every object is cofibrant. In such a context, the homotopy category  $\text{Ho } \mathbb{M}$  is equivalent to the category obtained by quotienting the full subcategory  $\mathbb{M}_f$  of  $\mathbb{M}$  on fibrant objects by the homotopy relation [13]. Moreover,  $\mathbb{M}_f$  is a category of fibrant objects in the sense of Brown [6] where every acyclic fibration has a section and where weak equivalences and homotopy equivalences coincide. A category of fibrant objects satisfying these additional properties is called a *path category* by van den Berg and Moerdijk [5]. More explicitly, a path category may be axiomatised as follows (cf. [5]). It has a terminal object and two classes of distinguished arrows closed under isomorphism: weak equivalences and fibrations. The former are also closed under 2-out-of-6, the latter under pullback and terminal arrows. Their intersection is also closed under pullback, and pullbacks along fibrations exist. Finally, any trivial fibration has a section.

**Definition 4.1** ([5], Definition 5.2). Let  $g: B \twoheadrightarrow A$  and  $f: A \twoheadrightarrow I$  be two fibrations in a path category  $\mathbb{C}$ . A commuting diagram

$$\begin{array}{ccccc} B & \xleftarrow{e} & U \times_I A & \twoheadrightarrow & U \\ & \searrow & \downarrow & & \downarrow u \\ & & A & \twoheadrightarrow & I \end{array}$$

is a *homotopy weak dependent product* of  $f$  and  $g$  if for every other such diagram  $u': U' \twoheadrightarrow I$ ,  $e': U' \times_I A \rightarrow B$ , there is  $k: u' \rightarrow u$  over  $I$  such that  $e(k \times A) \simeq_g e'$ .

Homotopy weak dependent products are called weak homotopy  $\Pi$ -types in [5]. As observed in [5], the arrow  $u'$  need not be a fibration.

When the path category is  $\mathbb{M}_f$ , weak homotopy dependent products arise as fibrant replacements of weak dependent products in  $\mathbb{M}$ . To prove this fact we need the following result, which is a reformulation of Theorem 2.38 from [5].

**Theorem 4.2.** *Let  $\mathbb{M}$  be a model category and let  $B$  be a cofibrant object. Every commuting square*

$$(5) \quad \begin{array}{ccc} A & \xrightarrow{k} & C \\ f \downarrow & \lrcorner & \downarrow g \\ B & \xrightarrow{l} & D \end{array}$$

has a *homotopy diagonal filler*, i.e. an arrow  $d: B \rightarrow C$  such that  $gd = l$  and  $df \simeq_g k$ . Moreover, such a filler is unique up to fibrewise homotopy over  $g$ .

*Proof.* We will first show that every commuting square (5) has a lower filler, i.e. an arrow  $d$  such that  $gd = l$ . This fact, in turn, allows us to obtain a homotopy witnessing the

fact that the previously constructed lower filler is in fact a homotopy diagonal filler, and another homotopy witnessing its uniqueness.

Consider a factorisation of  $\langle f, k \rangle: A \rightarrow B \times_D C$  into an acyclic cofibration  $c: A \xrightarrow{\sim} E$  followed by a fibration  $p: E \rightarrow B \times_D C$ . From 2-out-of-3 we obtain that  $\pi_1 p: E \rightarrow B$  is an acyclic fibration and, since  $B$  is cofibrant, it has a section  $s: B \xrightarrow{\sim} E$ . But then  $d := \pi_2 p s: B \rightarrow C$  is the required lower filler, as  $gd = l\pi_1 p s = l$ .

Therefore every commuting square (5) has a lower filler. In particular, we obtain a homotopy  $s\pi_1 p \simeq_{\pi_1 p} id_X$  as a lower filler in

$$\begin{array}{ccc} B & \longrightarrow & P_{\pi_1 p} E \\ \downarrow \wr & & \downarrow \wr \\ E & \xrightarrow{(s\pi_1 p, id_X)} & E \times_B E, \end{array}$$

where the top horizontal arrow is  $s$  followed by reflexivity. Hence  $df = \pi_2 p s \pi_1 p c \simeq_g \pi_2 p c = k$ .

Finally, given another homotopy diagonal filler  $d'$ , the homotopy witnessing  $d \simeq_g d'$  is obtained as a lower filler in

$$\begin{array}{ccc} A & \longrightarrow & P_g C \\ \downarrow f \wr & & \downarrow \wr \\ B & \xrightarrow{(d, d')} & C \times_D C, \end{array}$$

where the top horizontal arrow is the concatenation  $df \simeq_g k \simeq_g d'f$ .  $\square$

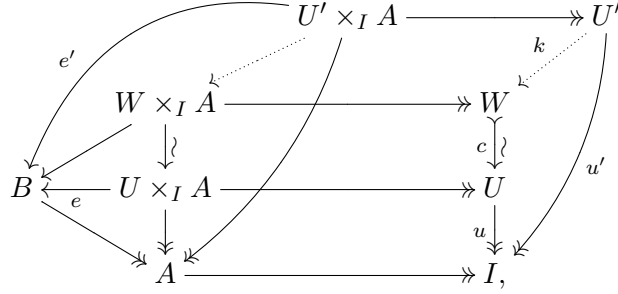
**Remark 4.3.** The argument used in the previous proof can be adapted to work in a path category, thus simplifying the proof of Theorem 2.38 in [5]. To this aim, it is enough to observe that the existence of a lower filler for squares (5) allows us to conclude that  $f \simeq_p f'$  implies  $gf \simeq_q gf'$  for every commuting diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \searrow f' & \downarrow p & & \downarrow q \\ & & D & \longrightarrow & E. \end{array}$$

**Corollary 4.4.** *Let  $\mathbb{M}$  be a right proper model category where every object is cofibrant. If  $\mathbb{M}$  has weak dependent products, then  $\mathbb{M}_f$  has homotopy weak dependent products for every pair of composable fibrations.*

*Proof.* Let  $f: A \rightarrow I$  and  $g: B \rightarrow A$  be two fibrations in  $\mathbb{M}_f$  and let  $w: W \rightarrow I$ ,  $d: W \times_I A \rightarrow B$  be a weak dependent product of them. Factor  $w$  as an acyclic cofibration  $c: W \xrightarrow{\sim} U$  followed by a fibration  $u: U \rightarrow I$ . Since  $\mathbb{M}$  is right proper,  $W \times_I A \rightarrow U \times_I A$  is also a weak equivalence, hence we obtain  $e: U \times_I A \rightarrow B$  as homotopy diagonal filler.

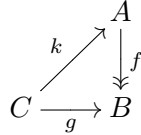
The required universal property is depicted in the diagram below



where  $e(ck \times A) \simeq_g e'$  since  $e$  is just a homotopy diagonal filler. □

It turns out that homotopy weak dependent products enjoy also another universal property with respect to certain homotopy diagrams. This is proved below in Lemma 4.9 and it is a consequence of the following result.

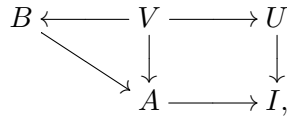
**Proposition 4.5** ([5], Proposition 2.31). *Let  $\mathbb{C}$  be a path category and let*



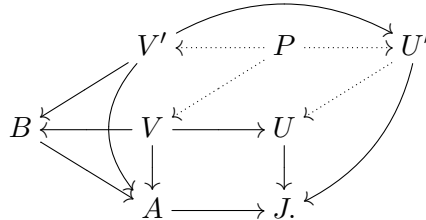
be a diagram that commutes up to homotopy. Then there is  $k': C \rightarrow A$  such that  $k' \simeq k$  and  $f k' = g$ .

**Remark 4.6.** Proposition 4.5 has the important consequence that pullbacks in  $\mathbb{M}_f$  along fibrations are homotopy pullbacks and so are mapped to weak pullbacks in  $\text{Ho } \mathbb{M}$ .

**Definition 4.7.** Let  $f: A \rightarrow I$  and  $g: B \rightarrow A$  be two arrows in a path category  $\mathbb{C}$ . A diagram



that commutes up to homotopy and where the square is a homotopy pullback, is *homotopy full over  $f, g$*  if for every other such diagram there are an arrow  $U' \rightarrow U$ , a homotopy pullback  $V \leftarrow P \rightarrow U'$  of  $V \rightarrow U \leftarrow U'$  and an arrow  $P \rightarrow V'$  making the diagram below commute up to homotopy.



**Remark 4.8.** Since  $\text{Ho } \mathbb{C}$  is the quotient of  $\mathbb{C}$  by the homotopy relation (cf. Theorem 2.16 in [5]), the image in  $\text{Ho } \mathbb{C}$  of a homotopy full diagram over  $f, g$  is a full diagram over  $[f], [g]$ .

**Lemma 4.9.** *Let  $\mathbb{C}$  be a path category and let  $f: A \rightarrow I$  and  $g: B \rightarrow A$  be two fibrations. A homotopy weak dependent product of  $f$  and  $g$  is a homotopy full diagram over  $f, g$ .*

*Proof.* Let  $u: U \twoheadrightarrow I$ ,  $e: U \times_I A \rightarrow B$  be a homotopy weak dependent product of  $f$  and  $g$ . Remark 4.6 implies that  $U \times_I A$  is a homotopy pullback.

Let now

$$\begin{array}{ccccc} B & \xleftarrow{e'} & V' & \xrightarrow{v_1} & U' \\ & \searrow & \downarrow v_2 & & \downarrow u' \\ & & A & \twoheadrightarrow & I, \end{array}$$

be commutative up to homotopy and such that the square is a homotopy pullback. Hence there is an arrow  $\psi: U' \times_I A \rightarrow V'$  such that  $v_1\psi \simeq \pi'_1$  and  $v_2\psi \simeq \pi'_2$ . In particular, the diagram below commutes up to homotopy

$$(6) \quad \begin{array}{ccccc} U' \times_I A & \xrightarrow{\psi} & V' & \xrightarrow{e'} & B \\ & \searrow \pi'_2 & & \swarrow & \\ & & A & & \end{array}$$

and Proposition 4.5 implies that there is  $h: U' \times_I A \rightarrow B$  that makes the above triangle commute and which is homotopic to  $e'\psi$ .

The universal property of the homotopy weak dependent product then yields an arrow  $U' \rightarrow U$  such that everything in the diagram below commutes strictly except for the two top-left triangles with edge  $h$ , which only commute up to homotopy.

$$(7) \quad \begin{array}{ccccc} & & V' & \xleftarrow{\psi} & U' \times_I A & \twoheadrightarrow & U' \\ & \swarrow & \nearrow h & & \nearrow & \searrow & \\ B & \twoheadrightarrow & U \times_I A & \twoheadrightarrow & U & \twoheadrightarrow & I \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & A & \twoheadrightarrow & I. & & \end{array}$$

As required, the square with two dotted sides is a homotopy pullback and the diagram below commutes up to homotopy.

$$\begin{array}{ccccc} & & V' & \xleftarrow{\psi} & U' \times_I A & \twoheadrightarrow & U' \\ & \swarrow & \nearrow & & \nearrow & \searrow & \\ B & \twoheadrightarrow & U \times_I A & \twoheadrightarrow & U & \twoheadrightarrow & I \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & A & \twoheadrightarrow & I. & & \end{array}$$

□

**Theorem 4.10.** *Let  $\mathbb{M}$  be a right proper model category where every object is cofibrant. If  $\mathbb{M}$  has weak dependent products, then  $\text{Ho } \mathbb{M}$  has dependent full diagrams and, in turn,  $(\text{Ho } \mathbb{M})_{\text{ex}}$  is locally cartesian closed.*

*Proof.* Lemma 4.9 and Remark 4.8 yield a full diagram over  $[f], [g]$  whenever  $f$  and  $g$  are both fibrations. Since arrows in  $\mathbb{M}$  factor as weak equivalences and fibrations, this is enough to conclude that  $\text{Ho } \mathbb{M}$  has dependent full diagrams. The last statement is an application of Theorem 3.6. □

As an application of the above result, consider the two standard model structure on the category of topological spaces by Quillen [15] and by Strøm [16], which we denote by  $\text{Top}_Q$  and  $\text{Top}_S$ , respectively. The latter is right proper and every space is cofibrant. Furthermore, Carboni and Rosolini showed that it has weak dependent products [7].

Therefore  $(\text{Ho Top}_S)_{\text{ex}}$  is not only a pretopos, as proved in [12], but also locally cartesian closed. This answers a question left open by Gran and Vitale in [12]. In addition, although in  $\text{Top}_Q$  the cofibrant objects are just the CW-complexes,  $\text{Top}_Q$  is Quillen equivalent to simplicial sets with the Quillen model structure. This latter category does satisfy the hypothesis of our theorem, therefore  $(\text{Ho Top}_Q)_{\text{ex}}$  is locally cartesian closed too.

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