Exact completion and type-theoretic structures

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Abstract

This thesis consists of four papers and is a contribution to the study of representations of extensional properties in intensional type theories using, mainly, the language and tools from category theory. Our main focus is on exact completions of categories with weak finite limits as a category-theoretic description of the setoid construction in Martin-Löf's intensional type theory.

Paper I, which is joint work with Erik Palmgren, provides sufficient conditions for such an exact completion to produce a model of the system CETCS (Constructive Elementary Theory of the Category of Sets), a finite axiomatisation of the theory of well-pointed locally cartesian closed pretoposes with a natural numbers object and enough projectives. In particular, we use a condition inspired by Aczel's set-theoretic Fullness Axiom to obtain the local cartesian closure of an exact completion. As an application, we obtain a simple uniform proof that the category of setoids is a model of CETCS.

Paper II was prompted by the discovery of an overlooked issue in the characterisationof local cartesian closure for exact completions due to Carboni and Rosolini. In this paper we clarify the problem, show that their characterisation is still valid when the base category has finite limits, and provide a complete solution in the general case of a category with weak finite limits.

In paper III we generalise the approach used in paper I to obtain the local cartesian closure of an exact completion to arbitrary categories with finite limits. We then show how this condition inspired by the Fullness Axiom naturally arises in several homotopy categories and apply this result to obtain the local cartesian closure of the exact completion of the homotopy category of spaces, thus answering a question left open by Marino Gran and Enrico Vitale.

Finally, in paper IV we abandon the pure category-theoretic approach and instead present a type-theoretic construction, formalised in Coq, of W-types in the category of setoids from dependent W-types in the underlying intensional theory. In particular, contrary to previous approaches, this construction does not require the assumption of Uniqueness of Identity Proofs nor recursion into a type universe.

Keywords: *exact completion, type theory, setoid, weak limits, cartesian closure, inductive types.*

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Sammanfattning

Denna avhandling består av fyra artiklar och är ett bidrag till studiet av representationer av extensionella egenskaper i intensionella typteorier, mestadels genom att använda kategoriteorins språk och verktyg. Vårt huvudfokus är på exakta kompletteringar av kategorier med svaga ändliga gränser, som en kategoriteoretisk beskrivning av setoid-konstruktionen i Martin-Löfs intensionella typteori.

Artikel I, som är ett gemensamt arbete med Erik Palmgren, ger tillräckliga villkor för att en sådan exakt komplettering skall ge en modell av systemet CETCS (Constructive Elementary Theory of the Category of Sets), en ändlig axiomatisering av teorin för väl-punktade lokalt kartesiskt slutna pretoposar med naturligt-tal-objekt och försedda med tillräckligt många projektiva objekt. I synnerhet använder vi ett villkor, inspirerat av Aczels mängdteoretiska Fullhetsaxiom, för lokal kartesisk slutenhet hos en exakt komplettering. Som en tillämpning, erhåller vi ett enkelt likformigt bevis av att kategorin av setoider är en modell för CETCS.

Artikel II föranleddes av upptäckten av ett förbisett problem i Carboni and Rosolinis karakterisering av lokal kartesisk slutenhet för exakta kompletteringar. I denna artikel klargör vi problemet, visar att deras karakterisering fortfar att gälla när baskategorin har ändliga gränser, och ger en fullständig lösning i det allmäna fallet av en kategori med svaga ändliga gränser.

I artikel III generaliserar vi tillvägagångssättet från artikel I för erhålla lokal kartesisk tillslutning för godtyckliga kategorier med ändliga gränser. Vi visar sedan hur detta villkor, inspirerat av Fullhetsaxiomet, uppkommer naturligt i flera homotopikategorier, och tillämpar detta resultat för att erhålla lokal kartesisk slutenhet av homotopikategorin för topologiska rum, vilket besvarar en öppen fråga av Marino Gran och Enrico Vitale.

Avslutningsvis, i artikel IV, överger vi de rent kategoriteoretiska metoderna och presenterar istället en typteoretisk konstruktion, formaliserad i Coq, av W-typer i kategorin av setoider med hjälp av beroende W-typer i den underliggande intensionella teorin. I motsats till tidigare ansatser, kräver denna konstruktion inte något antagande om Unikhet av identitetsbevis eller rekursion in i ett typuniversum.

List of papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

I. **Exact completion and constructive theories of sets**
J. Emmenegger and Erik Palmgren. Submitted. 2017

II. **On the local cartesian closure of exact completions**
J. Emmenegger. Submitted. 2018

III. **The Fullness Axiom and exact completions of homotopy categories**
J. Emmenegger. 2018

IV. **W-types in setoids**
J. Emmenegger. 2018

Contents

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First and foremost I wish to express my deep gratitude to Erik Palmgren for his guidance during these past years. In our regular meetings he always had some new stimulating questions, a small subset of which are addressed in the present thesis, and never avoided a request of clarification, no matter how silly my questions could be. He has been very generous in sharing ideas, suggesting possible approaches and directing my often awkward and rough attempts towards some reference that could help me to refine (or drop) an idea. I learned a lot from him and his vast knowledge of mathematical logic represents for me a model and incentive.

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I was lucky enough to find a vibrant and stimulating environment in the Stockholm logic group. I feel honoured for having been part of it and I am grateful to all its members, past and present, for giving rise to the structure that distinguishes it from a mere set of people.

Introduction

This thesis consists of four papers and is a contribution to the study of representations of extensional properties in intensional type theories using, mainly, the language and tools from category theory. Our main focus is on exact completions as a category-theoretic description of the setoid construction in intensional Martin-Löf type theory, where by setoid we mean a type together with a type-valued equivalence relation. The relevance of the setoid construction in type-theory, and of quotient completions more generally, stems from the desire to conciliate the computationally useful properties of intensional type theory, decidability of type-checking above all, with mathematically useful properties and constructs such as function extensionality and quotients.

Exact completions are constructions that freely add quotients of equivalence relation to a given category $\mathbb{C}$. Several variants of exact completions are available in the literature, varying according to the properties satisfied by the base category $\mathbb{C}$. In particular, when $\mathbb{C}$ has finite limits we speak of ex/lex completion, whereas if $\mathbb{C}$ only has weak finite limits, where a universal arrow is not required to be unique, we speak of ex/wlex completion. Ex/lex completions have been widely studied and have a robust theory, whereas this is not the case for ex/wlex completions which, nevertheless, are more relevant from a type-theoretic perspective. Indeed, we may regard the category of setoids Std as the exact completion of another category which has products but, in general, only weak equalisers. The main subject of our investigations are thus ex/wlex completions which we shall refer to simply as exact completions.

1 Overview of the content

The next sections of the present introduction provide a brief description of the category-theoretic preliminaries relevant for our treatment. Our presentation is necessarily incomplete and we rather tried to convey the main ideas and to provide some intuition for the concepts involved.

We then begin our investigation in paper I, *Exact completion and constructive theories of sets*, where we give conditions for an exact completion to produce a model of the Constructive Elementary Theory of the Category of Sets (CETCS). This theory provides a finite axiomatisation of the theory of well-pointed locally cartesian closed pretoposes with enough projectives and a natural numbers object and was introduced by Palmgren with the aim of formalising the set theory used by Bishop [Pal12a]. The name of course draws from Lawvere's Elementary Theory of the Category of Sets, of which CETCS may be seen as a constructive version. As an application of such conditions we then obtain a simple proof that Std is a model of CETCS.

The main technical contribution of paper I is the formulation of a condition which ensures the local cartesian closure of certain exact completions. These are the completions relevant for models of CETCS, that is, of well-pointed categories with products and weak equalisers. This condition was inspired by Aczel's Fullness Axiom in CZF [Acz78], and we named it closure for pseudo relations. We formulated it because we were not able to apply Carboni and Rosolini's characterisation of locally cartesian closed exact completions in terms of a property of the projectives [CR00].

A thorough analysis of their argument, indeed, revealed a mistake in the proof. This discovery led to the results in paper II, *On the local cartesian closure of exact completions*. Here we prove that Carboni and Rosolini's argument goes through if and only if the projectives are closed under pullback, so that their characterisation only applies to ex/lex completions and leaves open the weak limits case. We then solve the problem in this more general case, proving a complete characterisation of (local) cartesian closure as a property of the projectives. This characterisation reduces to Carboni and Rosolini's one in the case of ex/lex completions. In addition, it may be used in conjunction with the results from paper I to obtain a complete characterisation of models of CETCS in terms of subcategories of choice objects (*i.e.* projective objects).

From a type-theoretic perspective, however, the condition of closure for pseudo relations formulated in paper I is more natural than the characterisation from paper II, since the former directly arises from certain diagrams involving Π-types. The original aim of Paper III, *The Fullness Axiom and exact completions of homotopy categories*, was to make this observation more precise. Here we begin generalising closure for pseudo relations to arbitrary categories with only weak finite limits, and proving that it implies the local cartesian closure of the corresponding exact completion.

We then focus on homotopy categories, which in general have products but only weak equalisers, of certain Quillen model categories (right proper and every object cofibrant). The full subcategory $\mathbb{M}_f$ on fibrant objects in such a Quillen model category $\mathbb{M}$ is a path category, a strengthening of Brown's category with fibrant objects recently introduced by van den Berg and Moerdijk [BM18]. We first observe that weak dependent products in $\mathbb{M}$ endow $\mathbb{M}_f$ with what van den Berg and Moerdijk call homotopy weak dependent products. We then prove that existence of homotopy weak dependent products imply, for any path category $\mathbb{C}$, closure for pseudo relations of its homotopy category $\mathrm{Ho}\,\mathbb{C}$.

As an immediate application we obtain the local cartesian closure of the exact completion of several homotopy categories, including the category of topological spaces and homotopy classes of maps, thus answering a question left open by Gran and Vitale [GV98]. Furthermore, we achieve the original aim of paper III observing that the category **Ctx** of contexts of Martin-Löf type theory is a path category (this is a result by van den Berg [Ber18] which expands previous work by Gambino and Garner [GG08] and others) with homotopy weak dependent products (thanks to $\prod$-types), and that its homotopy category is essentially the category of types whose exact completion is Std.

Finally, we turn our attention to W-types or, in category-theoretic terminology, initial algebras for polynomial endofunctors. Paper IV, *W-types in setoids*, describes a construction of W-types for setoids which uses dependent W-types in Martin-Löf type theory, that is, families of mutually defined W-types [PS89]. Unlike the others, this paper is entirely type-theoretic and the argument therein has been formalised in Coq. The basic observation is that, in order to construct "extensional" W-types as quotients of W-types, one needs to use inductively defined predicates over W-types. As shown in the paper, this task can be naturally achieved using dependent W-types.

Previously known constructions require UIP (due to van den Berg [Ber05]) or use recursion into a universe (due to Palmgren and adapted by Bressan to Minimalist type theory [Bre15]). Hence the former is not valid in a fully intensional theory, whereas the latter is not portable to a simple category-theoretic setting, like models of CETCS. The advantage of our argument is that it fulfils both the above requirements. Indeed, we plan to generalise it to van den Berg and Moerdijk's homotopy exact completion, *i.e.* the exact completion of the homotopy category of a path category [BM18], and to the more general quotient completions of doctrines introduced by Maietti and Rosolini [MR13b, MR13a]. It should also be noted that, by applying a result of Gambino and Hyland [GH04], we actually obtain dependent W-types in the extensional model from dependent W-types in the intensional theory.

2 Exact categories

Exact categories were introduced by Michael Barr in [Bar71] as an answer to the question "What is missing from an additive category to make it abelian?". Informally, exact categories are categories where we can take quotients of equivalence relations.

Let $\mathbb{E}$ be a category with finite limits. An *equivalence relation* on an object A in $\mathbb{E}$ is a monic arrow $r\colon R \hookrightarrow A \times A$ together with arrows ρ, σ and τ witnessing reflexivity symmetry and transitivity, that is to say, making the diagrams below commute:

$$\begin{array}{ccc} & & R \\ & \overset{\rho}{\nearrow} & \big\downarrow r \\ A & \xrightarrow[\Delta_A]{} & A \times A \end{array} \qquad \begin{array}{ccc} & & R \\ & \overset{\sigma}{\nearrow} & \big\downarrow r \\ R & \xrightarrow[\langle r_2, r_1 \rangle]{} & A \times A \end{array} \qquad \begin{array}{ccc} & & R \\ & \overset{\tau}{\nearrow} & \big\downarrow r \\ R \times_A R & \xrightarrow[\langle r_1 p_1, r_2 p_2 \rangle]{} & A \times A, \end{array} \tag{1}$$

where $\Delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle$ is the diagonal, $r_1, r_2\colon R \to A$ denote the first and second component of r and $R \xleftarrow{p_1} R \times_A R \xrightarrow{p_2} R$ is a pullback of $R \xrightarrow{r_2} A \xleftarrow{r_1} R$. Any arrow $f\colon A \to B$ gives rise to an equivalence relation $k := \langle k_1, k_2 \rangle\colon A \times_B A \hookrightarrow A \times A$ on A obtained by pulling back f along itself:

$$\begin{array}{ccc} A \times_B A & \xrightarrow{k_2} & A \\ {\scriptstyle k_1}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B. \end{array}$$

A relation $r\colon R \hookrightarrow A \times A$ whose two components fit into a pullback square as above is called a *kernel pair* of f. Intuitively, two elements in A are related in the kernel pair if and only if they are mapped to equal elements by f.

In order to define what a quotient $q\colon A \to Q$ of an equivalence relation $r\colon R \hookrightarrow A \times A$ is, we need to make sure that the definition captures the two main properties of a quotient:

a) that two elements in A have the same image Q if and only if they are related, and

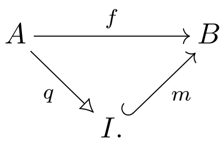
b) that a function $f\colon A \to B$ preserving the relation, *i.e.* mapping related elements to equal elements, factors uniquely through q.

Property (a) amounts to say that r is a kernel pair of q, and property (b) to say that q is a coequaliser of $r_1, r_2\colon R \rightrightarrows A$. Hence we say that a diagram $R \rightrightarrows A \to Q$ is *exact* if it is a coequaliser diagram and $R \to A \times A$

is a kernel pair of $A \to Q$.[1] We denote regular epis, *i.e.* arrows arising as coequalisers, by triangle head arrows $\twoheadrightarrow$. In paper I these arrows are also called quotients.

Definition 2.1. A category with finite limits is *exact* if every equivalence relation fits into an exact diagram and regular epis are stable under pullback.

In an exact category we can factor every arrow as a regular epi followed by a mono:



Such a triangle is called an *image factorisation* of f, and the arrow m is called *image* of f: it is the minimal subobject of B which f factors through. The cover $A \twoheadrightarrow I$ is the coequaliser of the kernel pair of f, and m is obtained from the universal property of q. Hence I may be understood as the quotient of A by the equivalence relation which identifies two elements if they have the same image in B. This construction should then be familiar from, say, the first isomorphism theorem.

In this sense, it should not be surprising that the image of a regular epi is always an isomorphism. In general, an arrow with this property is called cover, or extremal epi. More precisely, f is a *cover* if whenever $f = mg$ with m monic, then m is in fact an iso. It is then clear that, in an exact category, covers and regular epis coincide: we shall use the two words interchangeably.

Example 2.2.

1. Let Set be the category of sets in (a model of) CZF, *i.e.* Constructive Zermelo-Fraenkel set theory [Acz78, AR01]. In this category the monos are injective functions, every subset $S \subseteq A$ gives rise to a canonical mono $S \hookrightarrow A$, namely the inclusion, and every other mono $m\colon M \hookrightarrow A$ such that $m(M) = S$ is isomorphic to the inclusion. Modulo this distinction between monos and subsets, equivalence relations $R \hookrightarrow A \times A$ are usual set-theoretic equivalence relations. Regular epis are precisely the surjective functions and the kernel pair of $f\colon A \to B$ consists of those pairs $a, a' \in A$ with the same image in B. It is then immediate to verify that Set is an exact category.

[1] This notion of exact diagram coincide, in an abelian category, with the exactness of the short exact sequence $0 \to R \to A \oplus A \to Q \to 0$

Of course, also the category of sets in a model of ZF is exact, and in fact every topos is exact.

2. The category of abelian groups is exact and similar considerations as above apply. More generally, for a ring R the category of R-modules is exact. In fact both categories are abelian. The category of groups and the category of rings (either with or without unit) are examples of exact categories which are not abelian.

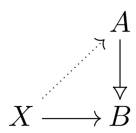
3. The category of setoids Std in Martin-Löf type theory is exact, and the proof of this fact has been formalised by Erik Palmgren in Coq [Pal12b]. This example is presented in more details in the last section of paper I.

4. The category of topological spaces is not exact since regular epis are not stable under pullback. Other non-examples are the category of small categories and the category of posets.

Henceforth $\mathbb{E}$ shall denote an exact category.

3 Projective covers and weak limits

An object X in $\mathbb{E}$ is called *projective* if, for any diagram $f\colon X \to B \twoheadleftarrow A : g$ there is $X \to A$, called a *lift* of f along g, making the triangle



commute.[2] In fact, since covers are stable under pullbacks, this is equivalent to require that every cover $f\colon A \twoheadrightarrow X$ splits, *i.e.* that there is $s\colon X \to A$ such that $fs = id_X$. Such an arrow s is called a section of f.

Definition 3.1. A *projective cover* $\mathbb{P}$ of $\mathbb{E}$ is a full subcategory such that

a) every object in $\mathbb{P}$ is projective in $\mathbb{E}$, and

b) every object A in $\mathbb{E}$ is covered by an object X in $\mathbb{P}$, *i.e.* there is a cover $X \twoheadrightarrow A$.

$\mathbb{E}$ has *enough projectives* if it has a projective cover.

[2]These are usually called regular projectives, since the arrow $A \to B$ is required to be a regular epi.

As the following examples indicate, projectivity incarnates as a choice property in categories of sets, while in categories of algebraic objects is tightly related to the existence of free objects.

Example 3.2.

1. In Set regular epis and surjections coincide, hence a set S is projective if it is a choice set, *i.e.* if every surjection with codomain S has a section. In fact, the Axiom of Choice (AC) is equivalent to the statement that every object is projective. Adding AC to CZF makes it equivalent to ZFC, in the sense that they prove the same theorems. Hence assuming AC makes Set the usual category of sets which, in particular, is a projective cover of itself.

   There is a weaker choice principle in set theory, compatible with intuitionistic logic, known as the Presentation Axiom (PA). This was introduced by Peter Aczel, and postulates that every set is the surjective image of a choice set [Acz78, AR01]. Assuming PA entails that Set has enough projectives.

2. Free (abelian) groups and free modules form projective covers of the respective categories. This follows essentially from the fact that in these categories regular epis coincide with surjective homomorphisms and that the forgetful functor into sets has a left adjoint. Of course to conclude that free objects are projective one usually assumes AC in the set theory, but PA suffices as well. Indeed, whenever we have an adjunction $F \dashv U$ where the forgetful functor $U \colon \mathbb{E} \to \mathsf{Set}$ is conservative and preserves regular epis, if $\mathbb{P}$ is a projective cover of Set then $F(\mathbb{P})$ is a projective cover of $\mathbb{E}$.

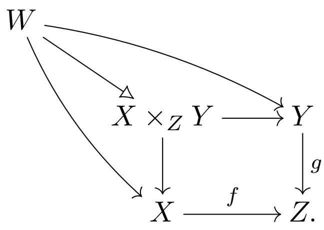
3. In Std, setoids whose equivalence relation is the identity type form a projective cover. This is essentially due to the distributivity of $\prod$-types over $\sum$-types, also known as the type-theoretic axiom of choice, and the fact that the identity type, being inductively defined, is the minimal reflexive relation on a type. For more details on this example we refer to the last section of paper I, and in particular to Remark 6.2 and the paragraph preceding it.

We shall use the last letters of the alphabet to denote objects in a projective cover $\mathbb{P}$, and the first letters to denote any object in $\mathbb{E}$.

A projective cover $\mathbb{P}$ is not in general closed under limits, but it does posses *weak finite limits*. These are the same as usual limits, but with uniqueness of the universal arrow dropped. So for example a weak product of X and Y is a diagram $X \leftarrow W \rightarrow Y$ such that for any other diagram

$X \leftarrow Z \rightarrow Y$ there is a (not necessarily unique) arrow $Z \rightarrow W$ making the two triangles commute.

Weak limits in $\mathbb{P}$ are computed taking covers of limits in $\mathbb{E}$. Let us consider the case of weak pullbacks in $\mathbb{P}$: given $f\colon X \rightarrow Z \leftarrow Y : g$ in $\mathbb{P}$ we can form the pullback $X \times_Z Y$ in $\mathbb{E}$. Property (b) from 3.1 then ensures the existence of a cover $W \twoheadrightarrow X \times_Z Y$ and fullness of $\mathbb{P}$ in $\mathbb{E}$ yields a commuting square in $\mathbb{P}$, the outer one in



(2)

Given two arrows $h\colon U \rightarrow X$ and $k\colon U \rightarrow Y$ in $\mathbb{P}$ such that $fh = gk$, we have $\langle h, k\rangle\colon U \rightarrow X \times_Z Y$. Property a) from 3.1 then yields an arrow $U \rightarrow W$ with the required properties.

The following are examples of weak limits.

Example 3.3.

1. Of course a limit is also a weak limit. For a simple example that the converse fails, consider two inhabited sets X and Y in Set. Any diagram $X \leftarrow X \times 2 \times Y \rightarrow Y$ where the two arrows are the obvious projections and 2 is a set with two elements, is a weak product of X and Y: for a pair of functions $f\colon U \rightarrow X$ and $g\colon U \rightarrow Y$, there are 2^U many different functions $U \rightarrow X \times 2 \times Y$ whose first and third component are f and g, respectively.

2. Let $\mathrm{Ho}(\mathsf{Top})$ be the category of topological spaces and homotopy classes of continuous maps. Homotopy pullbacks in Top are weak pullbacks in $\mathrm{Ho}(\mathsf{Top})$. These can be described as follows: let f and g as in diagram (2), the homotopy pullback $X \times^h_Z Y$ is given by

$$\left\{ (x, y, h) \in X \times Y \times Z^{[0,1]} \;\middle|\; h(0) = f(x) \text{ and } h(1) = g(y) \right\}$$

together with the two projections. We may then have two maps $k, k'\colon U \rightarrow X \times^h_Z Y$ which coincide up to homotopy on X and Y but such that the two paths in $Z^{[0,1]}$ are not homotopic. This happens for example when $f = g$ are the universal cover of the circle $\mathbb{R} \rightarrow S^1$ and $k, k'\colon \mathbb{R} \rightarrow \mathbb{R} \times^h_{S^1} \mathbb{R}$ both have the identity on $\mathbb{R}$ as the first two components, but as third component k has the constant path at x while k' has the loop at x with winding number 1.

3. Let Type be the category of small types in Martin-Löf type theory. More details on this category are in section 6 of paper I, after Remark 6.2. Here a weak pullback of two functions f and g as in diagram (2) is given by the type

$$\sum_{(x,y):X\times Y} f(x) =_Z g(y)$$

together with the two projections π_X and π_Y. Suppose that two functions k, k' from a type U into the weak pullback coincide on X and Y, that is, we have

$$\varphi : \prod_{u:U} \pi_X k(u) =_X \pi_X k'(u) \quad \text{and} \quad \psi : \prod_{u:U} \pi_Y k(u) =_Y \pi_Y k'(u).$$

In order to conclude that k and k' are in fact the same function we would need to prove that also the two proofs $h(u) : f\pi_X k(u) =_Z g\pi_Y k(u)$ and $h'(u) : f\pi_X k'(u) =_Z g\pi_Y k'(u)$ are equal over $\varphi(u)$ and $\psi(u)$. This amounts to show that the two proofs of $f\pi_X k(u) =_Z g\pi_Y k'(u)$ obtained by transitivity applied to $f\varphi(u)$ and $h'(u)$ on one side and $h(u)$ and $g\psi(u)$ on the other are themselves equal, *i.e.*

$$\mathsf{tra}(f\varphi(u), h'(u)) =_{f\pi_X k(u) =_Z g\pi_Y k'(u)} \mathsf{tra}(h(u), g\psi(u)),$$

where $\mathsf{tra} : a =_X b \to b =_X c \to a =_X c$ denotes a proof of transitivity. This is possible if we know Z to be an h-set, that is, a type where any two proofs of an identity $p, p' : a =_X b$ are themselves equal: $p =_{a =_X b} p'$ [UFP13]. But it is otherwise not the case in general. In fact, Remark 6.3 in paper I shows that Type has pullbacks if and only if the underlying type theory proves that all small types are h-sets.

Remark 3.4. It appears from the above examples that a weak limit is, very roughly, a limit with additional information. As in the first example, this information may be totally irrelevant: this happens when the category also possesses limits, and in this case it easy to see that a limit is a retract of any weak limit of the same diagram. But it may also happen that this additional information is essential: a pullback in Top does not give rise to a (weak) pullback in $\mathrm{Ho}(\mathsf{Top})$ since in the latter category commutativity holds up to homotopy, and so a weak pullback must keep track of the homotopy witnessing commutativity. Similarly, in the third example, the weak pullback must contain a proof of the commutativity of the square. In this sense, we may informally understand weak limits as "proof-relevant limits". Let us also point out that, as soon as this “proof” is irrelevant, *e.g.* when the space Z has trivial loop space or the type Z is an h-set, the weak limit is in fact a limit.

The similarity of the last two examples is no coincidence. Indeed, thanks to the insights of Homotopy Type Theory, we see that h-sets correspond to spaces where every loop is contractible. Hence weak limits fail to be limits for the same reason, that there are spaces where not every loop contracts to a point (actually, in type theory the correct formulation is to say that there might be types which are not h-sets, but for our treatment this caveat is irrelevant).

Before proceeding, we find useful to introduce some, mostly standard, terminology. When we need to regard a weak limit in $\mathbb{P}$ as a diagram in $\mathbb{E}$, we call it a *quasi limit*: this is to emphasize the fact that a weak limit has in fact no universal property in $\mathbb{E}$. A quasi pullback in $\mathbb{E}$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

where the right-hand arrow is a cover (hence so is the left-hand one) is called a *covering square*. Finally, a pair of parallel arrows $R \rightrightarrows X$ in a category with weak finite limits is a *pseudo equivalence relation* if it has arrows witnessing reflexivity, symmetry and transitivity as in (1) (but where the pullback is replaced by a weak pullback). The two parallel arrows are not required to be jointly monic.

4 The exact completion

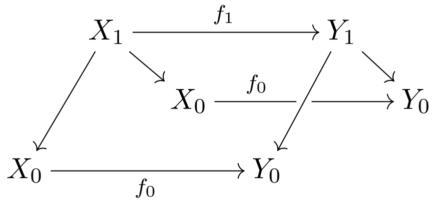
The main result which sets the ground for our investigation is the proof by Carboni and Vitale that every category with weak finite limits $\mathbb{C}$ is (equivalent to) a projective cover of an exact category $\mathbb{C}_{\mathrm{ex}}$, known as the *exact completion* of $\mathbb{C}$. Roughly speaking, this is possible because, given an exact category with enough projectives $\mathbb{E}$, any projective cover $\mathbb{P}$ contains all the information to recover the exact category $\mathbb{E}$ (up to equivalence). Furthermore, this reconstruction only depends on $\mathbb{P}$ having weak finite limits.

Indeed, every object in $\mathbb{E}$ is a quotient of a pseudo equivalence relation in $\mathbb{P}$

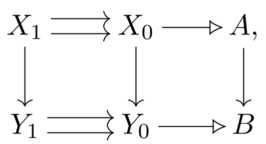
$$X_1 \rightrightarrows X_0 \twoheadrightarrow A, \tag{3}$$

where the cover $X_0 \twoheadrightarrow A$ is given by property b) in 3.1, and $X_1 \rightrightarrows X_0$ is its quasi kernel pair, *i.e.* $X_1 \rightrightarrows X_0$ is obtained as a cover of the actual kernel pair $X_0 \times_A X_0 \rightrightarrows X_0$. In this case we call a diagram as in (3) a *quasi exact diagram.*

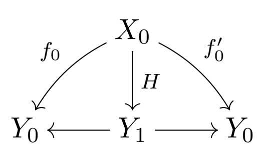
Arrows in $\mathbb{E}$ can be recovered from morphisms of pseudo equivalence relations in $\mathbb{P}$, where a morphism from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$ is a pair of arrows f_0, f_1 making



commute. The correspondence goes as follows: given $A \to B$ we obtain $X_0 \to Y_0$ and $X_1 \to Y_1$ as lifts using property a) in 3.1 and the fact that $Y_1 \twoheadrightarrow Y_0 \times_B Y_0$ is a cover. Conversely, using the universal property of the coequaliser (3) we see that a morphism from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$ induces $A \to B$ making the diagram below commute.



Notice that two morphisms of pseudo equivalence relations $(f_0, f_1), (f_0', f_1')$ from $(X_1 \rightrightarrows X_0)$ to $(Y_1 \rightrightarrows Y_0)$ induce the same arrow $A \to B$ if and only if $qf_0 = qf_0'$, where $q \colon Y_0 \twoheadrightarrow B$. But since the bottom row in the above diagram is quasi exact, this happens exactly when $(f_0, f_1), (f_0', f_1')$ are Y_1-related, *i.e.* when there is $H \colon X_0 \to Y_1$ making the two triangles below commute.

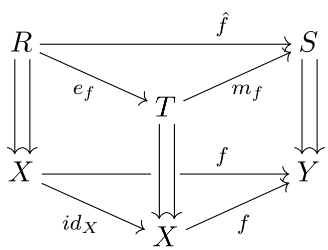


Hence this correspondence establishes an equivalence between $\mathbb{E}$ and the category $\mathbb{P}_{\text{ex}}$ defined as follows. Objects of $\mathbb{P}_{\text{ex}}$ are pseudo equivalence relations in $\mathbb{P}$, its arrows are morphisms of pseudo equivalence relations and two arrows from $(X_1 \rightrightarrows X_0)$ to $(Y_1 \rightrightarrows Y_0)$ are identified if they are Y_1-related.

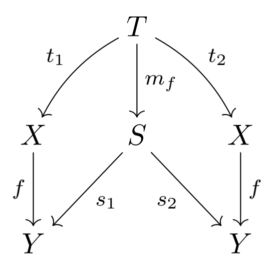
Furthermore, the above construction of $\mathbb{P}_{\text{ex}}$ from $\mathbb{P}$ can be performed from any category $\mathbb{C}$ with weak finite limits and it always produces an exact category: the *exact completion* $\mathbb{C}_{\text{ex}}$. There is also a fully faithful functor $\Gamma \colon \mathbb{C} \to \mathbb{C}_{\text{ex}}$ which maps an object X of $\mathbb{P}$ into $id_X, id_X \colon X \rightrightarrows X$

and an arrow f into the pair f, f, and whose image is a projective cover of $\mathbb{C}_{\mathrm{ex}}$.

To see this, observe that any arrow $(f, \hat{f})\colon (R \rightrightarrows X) \to (S \rightrightarrows Y)$ in $\mathbb{C}_{\mathrm{ex}}$ can be factored as

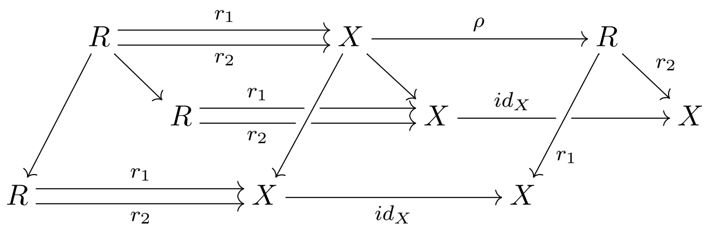


where the right-hand square is the weak limit



and the arrow $e_f\colon R \to T$ is obtained from its weak universal property. In particular, two arrows h and k into X are T-related if and only if fh and fk are S-related, hence $(f, m_f)\colon (T \rightrightarrows X) \to (S \rightrightarrows Y)$ is monic in $\mathbb{C}_{\mathrm{ex}}$. Unfolding the definitions shows that $(id_X, e_f)\colon (R \rightrightarrows X) \to (T \rightrightarrows X)$ is the coequaliser in $\mathbb{C}_{\mathrm{ex}}$ of the pair of arrows given by $(t_1, \rho t_1)$ and $(t_2, \rho t_2)$ from ΓT into $R \rightrightarrows X$, where $\rho\colon X \to R$ is an arrow witnessing reflexivity of r_1, r_2.

Hence the construction above produces an image factorisation for every arrow in $\mathbb{C}_{\mathrm{ex}}$ and, furthermore, it provides canonical forms for monos and regular epis in $\mathbb{C}_{\mathrm{ex}}$: the former are isomorphic to arrows which, seen as diagrams in $\mathbb{C}$, are weak limits, the latter are isomorphic to arrows whose underlying arrow in $\mathbb{C}$ is an identity. Using this characterisation of regular epis, it is easy to prove that $\mathbb{C}_{\mathrm{ex}}$ is indeed exact and that the image of Γ is a projective cover of $\mathbb{C}_{\mathrm{ex}}$. In particular, quasi exact diagrams in $\mathbb{C}_{\text{ex}}$ over an object $R \rightrightarrows X$ in $\mathbb{C}_{\text{ex}}$ are (up to isomorphism) of the form



and, more generally, coequalisers of equivalence relations are computed just replacing the pseudo equivalence part of an object in $\mathbb{C}_{\text{ex}}$ and taking the identity as underlying arrow of the regular epi.

We can collect these facts into the following theorem.

Theorem 4.1 (Carboni–Vitale, [CV98])**.** *Every exact category with enough projectives is the exact completion of any of its projective covers. Conversely, every category with weak finite limits is a projective cover of an exact category, namely its exact completion.*

In fact, if $\mathbb{C}$ has finite limits, $\mathbb{C}_{\text{ex}}$ is the free exact category on $\mathbb{C}$, in the precise sense that the forgetful 2-functor from the 2-category of exact categories and exact functors into the 2-category of finitely complete categories and cartesian functors (also known as left exact categories and left exact functors) has a left biadjoint which, on objects, is the exact completion construction.

As shown by Carboni and Vitale, in the weak limits case it is not possible to collect categories with weak finite limits into a 2-category in order to extend the exact completion construction to a left biadjoint. However, also in this case the exact completion possesses a universal property which exhibits it as a free construction. Say that a functor $F\colon \mathbb{C} \to \mathbb{E}$ from a category with weak finite limits $\mathbb{C}$ into an exact category $\mathbb{E}$ is *left covering* if it maps weak limits into quasi limits. Then $\Gamma\colon \mathbb{C} \to \mathbb{C}_{\text{ex}}$ is left covering and precomposing with it induces an equivalence between exact functors $\mathbb{C}_{\text{ex}} \to \mathbb{E}$ and left covering functors $\mathbb{C} \to \mathbb{E}$.

Henceforth we denote with $\mathbb{E}$ an exact category with enough projectives and with $\mathbb{P}$ a projective cover of it. We shall also refer to $\mathbb{E}$ as an exact completion.

Let us point out that an exact category with enough projectives may have different projective covers, and so it may be the exact completion of different categories with weak finite limits. The full subcategory of *all* projectives is however uniquely determined and, of course, it is a projective cover. This category can be obtained from any other projective cover $\mathbb{P}$ applying a construction known under several names: *splitting of idempotents*, *Karoubi envelope* or *Cauchy completion*. In particular, two categories with weak finite limits have equivalent exact completions if and only if they have equivalent splitting of idempotents. Since we are assuming a projective cover $\mathbb{P}$ to be given, we denote as $\overline{\mathbb{P}}$ the full subcategory of $\mathbb{E}$ on all projectives. In this case, $\overline{\mathbb{P}}$ may be described as the closure of $\mathbb{P}$ under retracts in $\mathbb{E}$, *i.e.* objects of $\overline{\mathbb{P}}$ are retracts of objects in $\mathbb{P}$.

5 Slices of exact completions as descent data

If X is a projective in $\mathbb{P}$ then $\mathbb{P}/X$ is a projective cover of $\mathbb{E}/X$, so that every slice of an exact completion over a projective is an exact completion too. In fact, the forgetful functor $\mathbb{E}/X \to \mathbb{E}$ is exact and preserves and reflects projectives. Furthermore, we can also recover all other slices of $\mathbb{E}$ in terms of slices over projectives and, in turn, in terms of slices of $\mathbb{P}$. The basic idea is that exact diagrams allow us to describe slices as categories of algebras for a suitable monad or, in equivalent terms, as categories of *descent data* for a regular epi. This means, intuitively, that for any regular epi $p\colon U \twoheadrightarrow I$, arrows into I are arrows $b\colon B \to U$ together with a proof that b "respects" the equivalence relation given by the kernel pair of p. By respecting the equivalence relation we mean, slightly abusing the internal language of $\mathbb{E}$, that for $u, u' : U$ such that $p(u) = p(u')$ the two fibres B_u and $B_{u'}$ are isomorphic.

More precisely, given an object I in $\mathbb{E}$ and an exact diagram $R \rightrightarrows U \overset{p}{\twoheadrightarrow} I$ with U in $\mathbb{P}$, define an endofunctor F_p on $\mathbb{E}/U$ which maps $b\colon B \to U$ into $p_2(p_1^* b)\colon B \times_U R \to U$ as in the diagram below

$$\begin{array}{ccccc} B \times_U R & \xrightarrow{p_1^* b} & R & \xrightarrow{p_2} & U \\ \downarrow & & \downarrow{\scriptstyle p_1} & & \downarrow \\ B & \xrightarrow{b} & U & \longrightarrow & I. \end{array} \tag{4}$$

It is possible to show that this endofunctor is in fact a monad. An algebra for this monad is thus a commutative triangle

$$\begin{array}{ccc} B \times_U R & \xrightarrow{\theta} & B \\ & {\scriptstyle F_p b}\searrow \quad \swarrow{\scriptstyle b} & \\ & U & \end{array} \tag{5}$$

which may understood as a family of transport functions $\theta_{uu'}\colon B_u \to B_{u'}$ for every $u \sim_R u'$. The monad axioms ensure $\theta_{uu} = id$ and $\theta_{uu'} \circ \theta_{u'u''} =$

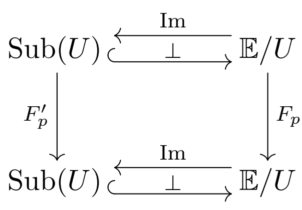
$\theta_{uu''}$. So in particular $\theta_{uu'} \circ \theta_{u'u} = id$, *i.e.* θ is in fact a family of isomorphisms. Such an algebra is a *descent datum* for b over the cover p. The category $\mathcal{D}es_p$ of descent data over p is the Eilenberg–Moore category of algebras for F_p. The slice $\mathbb{E}/I$ is equivalent to $\mathcal{D}es_p$, see for example Appendix C in [JM95]. Informally, the reason is that a family $A \to I$, when pulled back along p, is a family $B \to U$ whose fibres B_u and $B_{u'}$ are equal (and equal to $A_{p(u)}$) whenever $u \sim_R u'$. Conversely, any family $B \to U$ together with a descent datum $u \sim_R u' \vdash \theta_{uu'} : B_u \cong B_{u'}$ induces a family over I obtained identifying two fibres B_u and $B_{u'}$ according to the isomorphism $\theta_{uu'}$.

The above can be very briefly stated as saying that the adjunction

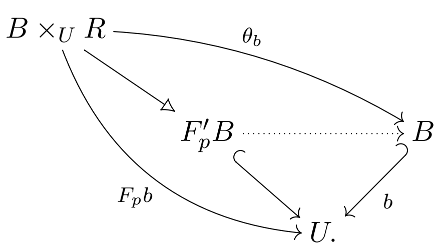
$$\Sigma_p \colon \mathbb{E}/U \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbb{E}/I : p^* \tag{6}$$

is monadic, as (4) makes clear that F_p is nothing but the composite $p^*\Sigma_p$. This enables us to use known results about monadic adjunctions (see for example chapter 3 in [BW85]) to transfer properties of $\mathbb{E}/U$ to $\mathbb{E}/I$, most notably cartesian closure in paper II.

Furthermore, the monad F_p restricts to subobjects



by taking the image factorisation of $F_p b$ for a subobject $b \colon B \hookrightarrow U$. Notice that, because of the commutativity of (5), when b is monic there is at most one F_p-algebra, *i.e.* one descent datum, on b. This algebra θ_b in turn induces a unique F'_p-algebra on b by taking the image factorisation of θ_b, and every F'_p-algebra arises in this way:



That is to say, the category of descent data of F'_p is a reflective subcategory of $\mathcal{D}es_p$, and so of $\mathbb{E}/I$. It should not be surprising at this point that descent data of F'_p are equivalent to $\mathrm{Sub}(I)$ [CR00] or, equivalently, that the restriction of (6) to subobjects

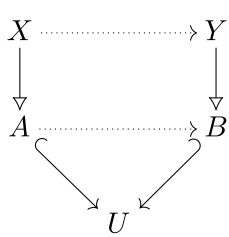
$$\exists_p \colon \mathrm{Sub}(U) \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathrm{Sub}(I) : p^*$$

is also a monadic adjunction. Informally this tells us that when predicates over I are those predicates φ over U which respects the equivalence relation, *i.e.* such that $u \sim_R u'$, $\varphi(u) \vdash \varphi(u')$. This facts provides us with a characterisation of the internal logic of $\mathbb{E}$ in terms of the logic over projective objects. In the next section we describe some of the features of the latter which, in particular, include a strong existential quantifier.

6 The logic of projective objects

Recall that the *poset reflection* $\mathbb{C}_{\mathrm{po}}$ of a category $\mathbb{C}$ is the poset whose elements are the objects of $\mathbb{C}$, such that $A \leq B$ if there is an arrow $A \to B$, and where two objects are identified if $A \leq B$ and $B \leq A$. In fact, it is the left biadjoint to the inclusion of the 2-category of posets into the 2-category of categories.

For every projective U, there is an isomorphism between the poset of subobjects $\mathrm{Sub}(U)$ in $\mathbb{E}$ and the order reflection of the slice $\mathbb{P}/U$: for every subobject there is a cover with domain in $\mathbb{P}$, conversely every arrow from a projective into U may be factored through a mono with codomain U, and this correspondence easily extends to arrows over U using the lifting property of projectives in one direction, and the universal property of coequalisers in the other one, as depicted below.



The above isomorphism can be generalised to an n-fold product of projectives. Let $U_1, \dots, U_n$ be projective objects and denote with $\mathbb{P}/(U_1, \dots, U_n)$ the category of spans in $\mathbb{P}$ over $U_1, \dots, U_n$. More precisely, this is the comma category $\Delta \downarrow U_1, \dots, U_n$: objects are families of arrows $(f_i \colon X \to U_i)_{1 \leq i \leq n}$, and arrows from $(f_i)_i$ to $(g_i \colon Y \to U_i)_i$ are arrows $h \colon X \to Y$ in $\mathbb{P}$ such that $g_i h = f_i$ for every $i = 1, \dots, n$. Then the poset of subobjects on $U_1 \times \dots \times U_n$ is isomorphic to the order reflection of

$\mathbb{P}/(U_1, \ldots, U_n)$:

$$\mathrm{Sub}(U_1 \times \cdots \times U_n) \cong (\mathbb{P}/U_1, \ldots, U_n)_{\mathrm{po}}. \tag{7}$$

Notice that this isomorphism also holds when $n = 0$, *i.e.* $\mathrm{Sub}(\mathbf{1}) \cong \mathbb{P}_{\mathrm{po}}$.

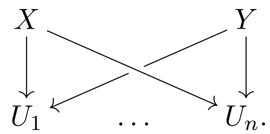
The standard interpretation of regular logic in the posets of subobjects, when restricted to finite products of projective objects, is thus isomorphic to the interpretation into the poset reflection. The latter is a bit less known than the standard with subobjects, so we recall it here.

This interpretation was first considered by Lawvere in [Law96] where he named the above poset reflection *Curry-Läuchli adjoint*, and applied it to in the case of a locally cartesian closed category. In this case the interpretation may be understood as the proposition-as-types interpretation of first order logic into Martin-Löf type theory, followed by the interpretation of the latter into a locally cartesian closed category [See84, Hof94]. In particular, this interpretation has a strong existential quantifier, in the sense that a proof of an existential statements explicitly contains a witness.

Because of this feature, Palmgren called it *categorical BHK interpretation* in [Pal04], where the more general case of a quasi-cartesian category is considered, *i.e.* a category with finite products and weak equalisers. Presubobjects have also been studied by Grandis in [Gra97, Gra00] with the name of weak subobjects or variations, although not primarily from a logical point of view. Maietti and Rosolini then named this interpretation *logic of weak subobjects* in their work on quotient completions of elementary doctrines, which generalises various constructions, including the exact completion, from a logical perspective [MR13b, MR13a, MR16]. They also consider this interpretation for a quasi-cartesian category, but we shall present it without assuming that projectives are closed under products.

As it is customary to consider monos instead of subobjects, we shall work with spans instead of elements in $(\mathbb{P}/U_1, \ldots, U_n)_{\mathrm{po}}$. We say that a span $(f_i \colon X \to U_i)_i$ is a *presubobject* over $U_1, \ldots, U_n$, we refer to the order relation $\leq$ in the poset reflection as the *presubobject order*, and similarly we say that two presubobjects are *equivalent* if they are equal in the poset reflection.

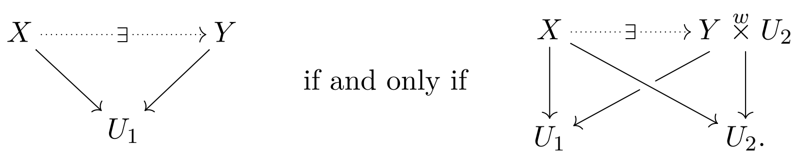
The top element in $(\mathbb{P}/U_1, \ldots, U_n)_{\mathrm{po}}$ is any span whose domain is a weak product of the U_i's, which for convenience we denote as $U_1 \overset{w}{\times} \ldots \overset{w}{\times} U_n$, and whose legs are the weak product projections. The meet of two spans $(f_i \colon X \to U_i)_i$ and $(g_i \colon Y \to U_i)_i$ is the span $(Z \to U_i)_i$ obtained as weak limit of



For weakening and the existential quantifier notice that, since weak limits are unique up to presubobject equivalence, taking a weak limit along some $(f_i)_i$ as above induces an order preserving function $(\mathbb{P}/U_1,\dots,U_n)_{\mathrm{po}} \to (\mathbb{P}/X)_{\mathrm{po}}$ with left adjoint the postcomposition with $(f_i)_i$. In case $(f_i)_i$ is given by the first $n-1$ projections of the weak product $U_1 \overset{w}{\times} \dots \overset{w}{\times} U_n$ we obtain the required adjunction

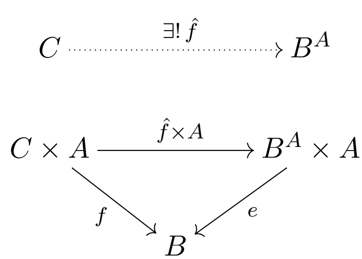
$$(\mathbb{P}/U_1,\dots,U_n)_{\mathrm{po}} \underset{\exists_{U_n}}{\overset{\longleftarrow}{\underset{\longrightarrow}{\top}}} (\mathbb{P}/U_1,\dots,U_{n-1})_{\mathrm{po}}$$

observing that $(\mathbb{P}/U_1,\dots,U_n)_{\mathrm{po}} \cong (\mathbb{P}/U_1 \overset{w}{\times} \dots \overset{w}{\times} U_n)_{\mathrm{po}}$. This adjunction is probably best understood regarding the right adjoint as weak product with U_n and the left adjoint, *i.e.* the existential quantifier, as simply dropping the last leg of a span over $U_1,\dots,U_n$. For $n=2$, $f=(f_i\colon X\to U_i)_{i=1,2}$ and $g\colon Y\to U_1$ we have $(\exists_{U_2} f) \le g \Leftrightarrow f \le (g \overset{w}{\times} U_2)$ or, diagrammatically



7 Cartesian closed exact completions

A cartesian closed category is a category where each set of arrows between two objects is represented by an object of the category, together with an arrow which is best understood as evaluation. More precisely, an *exponential* of two objects A and B is an object, usually denoted B^A, together with an arrow $e\colon B^A\times A\to B$ such that, for any arrow $f\colon C\times A\to B$ there is a unique $\hat{f}\colon C\to B^A$ which makes the triangle below commute.



Example 7.1.

1. In the category of sets, exponentials are exactly sets of functions, that is, sets of total and single-valued binary relations.

2. In setoids, exponentials are formed using the $\prod$-type and its elimination rule. For setoids $A = (A_0, \sim_A)$ and $B = (B_0, \sim_B)$ and a function term $f\colon A_0 \to B_0$, define

$$ext_{A,B}(f) \coloneqq \prod_{a,a' : A_0} a \sim_A a' \to f(a) \sim_B f(a').$$

   Then the exponential of A and B is the type

$$\sum_{f : A_0 \to B_0} ext_{A,B}(f)$$

   together with the equivalence relation

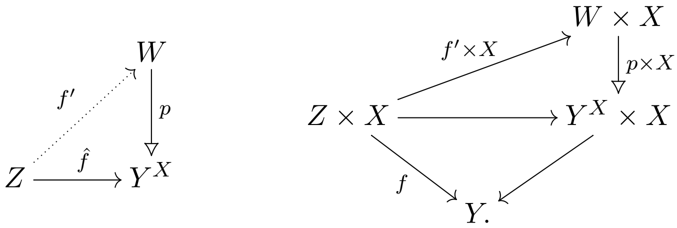
$$(f, e_f) \sim (g, g_f) \coloneqq \prod_{a : A_0} f(a) \sim_B g(a).$$

3. In algebraic categories, such as categories of modules, groups etc. exponentials are usually obtained imposing a pointwise algebraic structure on the set of homomorphisms.

4. The category of topological spaces is famously not cartesian closed. Various fixes have been proposed either restricting the category, *e.g.* to compactly generated Hausdorff spaces, or enlarging it, *e.g.* to equilogical spaces. The latter category may be described as a reflective subcategory of the exact completion of Top [CR00].

As for limits, dropping uniqueness of $\hat{f}$ yields the notion of *weak exponential*. The analogy can be brought further: in an exact category with enough projectives, covering an exponential in $\mathbb{E}$ with a projective in $\mathbb{P}$ yields a weak exponential in $\mathbb{P}$. We shall explain this construction in two steps, first assuming that $\mathbb{P}$ has binary products and then for the general case.

Given two projectives X and Y we can form their exponential in $\mathbb{E}$ and then cover it with a projective W, the universal property of exponentials together with the lifting property of projectives yield the weak universal property of weak exponentials as shown in the diagram below:



Example 7.2. The category Type has weak exponentials given by the non-dependent function type $A_0 \to B_0$ and its elimination. We can embed the latter in Std as a free setoid $(A_0 \to B_0, =_{A_0 \to B_0})$ and obtain the canonical function into the exponential of the free setoids $(A_0, =_{A_0})$ and $(B_0, =_{B_0})$

$$(A_0 \to B_0, =_{A_0 \to B_0}) \longrightarrow (\textstyle\sum_{f : A_0 \to B_0} ext_{A_0, B_0}(f), \sim).$$

This function is clearly surjective (so a regular epi in Std), and it splits if and only if $f \sim g \to f =_{A_0 \to B_0} g$, that is, if and only if function extensionality holds.

In case $\mathbb{P}$ is not closed under binary products the arrow $W \times X \to Y$ is not in $\mathbb{P}$, but this can be fixed taking a cover $V \twoheadrightarrow W \times X$. We refer to the composite $V \twoheadrightarrow W \times X \to Y$ as *weak evaluation*. Let now $V' \twoheadrightarrow Z \times X$ and $V' \to Y$ be given. In order to obtain $\hat{f} \colon Z \to Y^X$ and consequently $f' \colon Z \to W$ as above, we need to know that $V' \to Y$ factors through the product $Z \times X$. Hence the weak evaluation factors through the product in $\mathbb{E}$ and, in addition, its weak universal property is only with respect to arrows $V' \to Y$ which also factor through the product in $\mathbb{E}$. This property can be stated without any reference to products in the category $\mathbb{E}$.

Let $X \leftarrow V \to Y$ be a weak product (or a weak pullback), say that an arrow $f \colon V \to Z$ is *determined by projections* if it coequalises any pair of arrows coequalised by the two weak product projections p_1 and p_2, that is, if

$$p_1 h = p_1 k \quad \text{and} \quad p_2 h = p_2 k \quad \text{implies} \quad fh = fk.$$

An arrow is determined by projections if and only if it coequalises the (quasi) kernel pair of $V \twoheadrightarrow X \times Y$ if and only if it induces an arrow $X \times Y \to Z$.

Example 7.3. The following examples continue those in 3.3.

1. A function $f \colon X \times 2 \times Y \to Z$ in Set is determined by projections precisely when its values do not depend on elements in 2. In this

case this happens exactly when it factors through $X \times Y$. As it shall be clear from the next two examples this really is a trivial case (or rather a degenerate one), due to the fact that the product and the weak product live in the same category.

2. A function $k\colon X \times^h_Z Y \to U$ in Ho(Top) is determined by projections when its values do not depend on the homotopy witnessing $f(x) \simeq g(y)$, that is, when $k(x,y,h) \simeq k(x,y,h')$ for all such triples. This amounts to say that it induces a function from the pullback, which lives in $(\mathrm{Ho}\,\mathsf{Top})_{\mathrm{ex}}$: the pullback is not (some embedding of) $X \times_Z Y$ but the pseudo equivalence relation $\sim$ on $X \times^h_Z Y$ such that $(x,y,h) \sim (x',y',h')$ if and only if $x \simeq x'$ and $y \simeq y'$.

3. Let $V := \sum_{(x,y):X\times Y} f(x) =_Z g(y)$ be a weak pullback in Type. A function term $k : V \to U$ is determined by projections when we can prove

$$\prod_{v,v':V} \pi_X(v) =_X \pi_X(v') \to \pi_Y(v) =_Y \pi_Y(v') \to k(v) =_U k(v'),$$

that is, when values of k only depend on the X and Y components. Again, this is equivalent to say that, once mapped into the category of setoids, k induces a function from the pullback in Std. If this is the case, the induced function is k together with the above proof term witnessing that k is determined by projections.

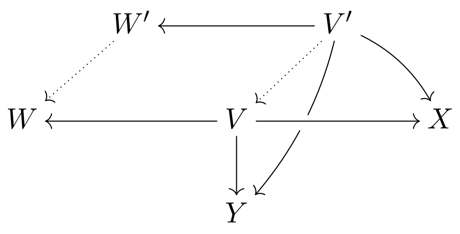
Carrying on the intuition of weak limits as proof-relevant limits, we may understand arrows determined by projections as proof-irrelevant arrows.

Definition 7.4. A *weak exponential* in $\mathbb{P}$ of two objects X and Y is a diagram

$$\begin{array}{ccccc} W & \longleftarrow & V & \longrightarrow & X \\ & & \big\downarrow & & \\ & & Y & & \end{array}$$

where the top row is a weak product and the arrow $V \to Y$ is determined by projections, which in addition is weakly terminal among diagrams of this form in the sense that if $W' \leftarrow V' \to X$ is a weak product and

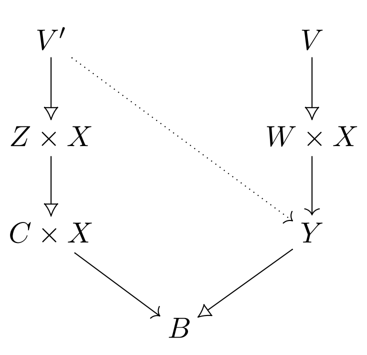
$V' \to Y$ is determined by projections, there are dotted arrows as below



making the diagram commute.

We have seen that if $\mathbb{E}$ is cartesian closed, then we can compute weak exponentials in $\mathbb{P}$ in the same way as we do for limits: taking projective covers of exponentials in $\mathbb{E}$. Unfortunately the similarity ends here: contrary to what happens with limits, an exponential of A and B cannot be constructed as a quotient of a (weak) exponential of projectives X and Y alone, where $X \twoheadrightarrow A$ and $Y \twoheadrightarrow B$. Very briefly, the reason behind this fact lies in the universal property of weak exponentials, which only holds for arrows determined by projections.

As a simple indication of what fails, suppose that we were to obtain the exponential of a projective X and an object B as a quotient of a weak exponential W of X and Y, where $Y \twoheadrightarrow B$. Given an object C and an arrow $C \times X \to B$, the obvious approach would be to take a cover $Z \twoheadrightarrow C$, try to use the weak universal property of W to obtain an arrow $Z \to W$, and then use the latter to induce an arrow from C to the quotient of W which is the candidate exponential. Now, the weak universal property of W only holds in $\mathbb{P}$, hence we need to first consider a weak product $V' \twoheadrightarrow Z \times X$, and then lift the composite $V' \to B$ to an arrow $V' \to Y$ in $\mathbb{P}$. The situation is depicted below.



Recalling the definition of weak exponential in $\mathbb{P}$, we see that we would obtain arrows $Z \to W$ and $V' \to V$ making the required diagram commute if (and, in fact, only if) the lift $V' \to Y$ is determined by projections.

But this is not necessarily the case: from our understanding of arrows determined by projections as proof irrelevant functions, it may be the case that the quotient $Y \twoheadrightarrow B$ forgets some piece of information thus making a proof relevant arrow into Y a proof irrelevant one into B.

In fact there is a very simple example of a "proof irrelevant" arrow with a "proof relevant lift": the identity arrow id_V on a weak product $V \twoheadrightarrow X \times Y$ is determined by projections if and only if the latter cover splits, if and only if $X \times Y$ is projective. As a concrete example consider again the homotopy kernel pair of the universal cover of the circle $f\colon \mathbb{R} \to S^1$ from Example 2 in 3.3 and 7.3, but now embedded into the exact completion $(\mathrm{Ho}\,\mathsf{Top})_{\mathrm{ex}}$. There is a canonical cover $q\colon \Gamma(\mathbb{R} \times^h_{S^1} \mathbb{R}) \twoheadrightarrow \Gamma\mathbb{R} \times_{\Gamma S^1} \Gamma\mathbb{R}$, whose codomain is the kernel pair of Γf in $(\mathrm{Ho}\,\mathsf{Top})_{\mathrm{ex}}$ which has been described in general in Example 2 in 7.3. The cover q is clearly determined by projections, since it trivially factors through the pullback. However, the identity on $\Gamma(\mathbb{R} \times^h_{S^1} \mathbb{R})$ is a lift of q along q itself, and it is clearly not determined by projections since loops in S^1 are not all contractible.

Paper II contains a more rigorous and thorough argument which, in particular, shows that exponentials are obtained as quotients of weak exponentials if and only if projectives are closed under binary products, and an analogous relation exists between local cartesian closure and closure of the projectives under pullback. Notice however that the condition only requires *all* projectives to be closed under binary products (resp. pullbacks), *i.e.* it is only a property of the projective cover $\overline{\mathbb{P}}$. For the two examples we have been considering in the present introduction $\mathrm{Ho}(\mathsf{Top})$ and Type, we do not know whether their splitting of idempotents are closed under pullbacks or not but, if this is the case, then Carboni and Rosolini's characterisation of local cartesian closure would still apply. We only observe that, as argued in Example 3.3 and Remark 3.4, the closure of a projective cover under products or pullbacks can be seen as a proof irrelevance principle. For example, it translates to UIP in type theory and to the contractibility of arbitrary loops in spaces. The above question can then be phrased as asking how much proof irrelevance is introduced by splitting idempotents. However, we have left this question for future investigations and instead preferred to focus on the problem of characterising local cartesian closure in the most general case, without further assumptions on the projective cover. This has led to the complete characterisation contained in paper II and to a, perhaps more useful, sufficient condition in paper III.

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