<http://www.diva-portal.org>

Postprint

This is the accepted version of a chapter published in *Feferman on foundations: Logic, Mathematics, Philosophy*.

Citation for the original published chapter:

Westerståhl, D. (2017)

Sameness

In: Gerhard Jäger, Wilfried Sieg (ed.), *Feferman on foundations: Logic, Mathematics, Philosophy* (pp. 449-467). Berlin: Springer

Outstanding Contributions to Logic

https://doi.org/10.1007/978-3-319-63334-3_16

N.B. When citing this work, cite the original published chapter.

Permanent link to this version:

<http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-162325>

Sameness*

Dag Westerståhl
Department of Philosophy
Stockholm University

Abstract

I attempt an explication of what it means for an operation across domains to be the *same* on all domains, an issue that Feferman (1999) took to be central for a successful delimitation of the *logical* operations. Some properties that seem strongly related to sameness are examined, notably isomorphism invariance, and sameness under extensions of the domain. The conclusion is that although no precise criterion can satisfy all intuitions about sameness, combining the two properties just mentioned yields a reasonably robust and useful explication of sameness across domains.

Keywords: logical constants, isomorphism invariance, extension, generalized quantifiers

2010 Mathematics Subject Classification: 03B65, 03C80, 91F20

1 Introduction

In three fairly recent papers, Sol Feferman discussed the question of *logicality*, of what makes some symbols logical and others not.¹ Driven by a “personal feeling that the logical operations do not go beyond those represented in FOL” (Feferman, 2010), he explored different ways of characterizing logicality, starting with his analysis and critique of the Tarski-Sher thesis. Tarski’s invariance criterion, from Tarski (1986), was that logical notions are those which are *permutation invariant*; a property I will call PERM. Sher (1991) strengthened the requirement to *isomorphism invariance* (ISOM).² While granting that ISOM is

*Sol Feferman was an inspiration to me for most of my adult life and, for almost as long, a friend. I’m delighted to have been asked to contribute to this volume in his honor. As to the paper, thanks to Lauri Hella and Jouko Väänänen for helpful conversations on the model-theoretic parts, and to Stanley Peters who saved me from a rather embarrassing mistake (remaining ones are mine, of course). I also want to thank the two extremely encouraging and patient editors of this volume.

¹Feferman (1999, 2010, 2015).

²Thereby avoiding to treat as logical, for example, an operator which behaves as the universal quantifier on universes containing the number 0, and as the existential quantifier on other universes. Also, Sher restricted attention to operations of type level at most 2, and Feferman in general follows this restriction.

a necessary feature of logical constants, Feferman argued that it is far from sufficient, and suggested several stronger criteria that essentially narrow down the range of logical constants to those definable in FOL.

In this note I focus on just *one* part of his critique of the Tarski-Sher thesis. In Feferman (1999), he thought of this as “perhaps the strongest reason for rejecting the Tarski-Sher thesis”, and formulated it as follows: “No natural explanation is given by it of what constitutes the *same* logical operation over arbitrary basic domains”.

Let us call this criterion SAME. For a basic domain M , let $\langle M \rangle$ be a typed universe over M , relative to some suitable type system. We are considering *operations across domains*, i.e. operations O that with each M associate an object $O_M \in \langle M \rangle$ of a given type.

(SAME) O satisfies SAME if O_M is the *same* operation on each M .

The question, of course, is what “same operation” should mean, and indeed if it is possible to give it a “natural explanation”. The notion might seem inherently vague. There are, however, a number of fairly strong intuitions about SAME, so let us start with these.

2 Some intuitions

Feferman used an example from McGee of an operation across domains that clearly does *not* satisfy SAME: “a logical connective which acts like disjunction when the size of the domain is an even successor cardinal, like conjunction when the size of the domain is an odd successor cardinal, and like a biconditional at limits” (McGee, 1996):577. There are plenty of other, and simpler, examples in the same vein, for example, a quantifier which acts like $\forall$ on universes with at least 100 elements, and as $\exists$ on smaller universes. Thus:

(I1) There are numerous examples of operations across universes that behave differently on different universes, and therefore do not satisfy SAME.

From the examples illustrating (I1) it seems uncontroversial that ISOM does *not* imply SAME. Perhaps a slightly more debatable question is if the implication in the other direction holds. But if h is a bijection from M to M' , and hence lifts to a bijection from each M_τ to M'_τ (see section 3), it seems to me like a very reasonable requirement of sameness that an operation O_M should map along h to the corresponding operation on M'_τ , that is, that $h(O_M) = O_{h(M)}$, which is precisely what ISOM says. So I think the following intuition is quite strong too.

(I2) Operations satisfying SAME also satisfy ISOM.

Note that (I2) does not derive from the standard thought that ISOM captures *topic neutrality*, and therefore is a necessary feature of logicity. Rather, we are simply using the idea that isomorphic models have, in the strictest sense, *the same structure*.

Our next intuition about sameness is something practically everyone would agree to:

- (I3) The (interpretations of the) usual first-order symbols $\forall, \exists, \neg, \wedge, \vee, \rightarrow$ satisfy SAME.

There is a simple but important lesson to be drawn from (I1) and (I3):

- (1) The class of SAME operations is not closed under definability (even first-order definability).

Now one might think this already disqualifies the SAME notion from playing any significant part in characterizing logicity. Indeed, the role of definability for the notion of logicity is itself an interesting and intricate issue, discussed in McGee (1996), Feferman (1999, 2010), Bonnay (2008), and recently, from a general model-theoretic perspective, in Bonnay and Engström (2016).

But that issue is not the topic of the present note. My aim is just to see what sense can be made of SAME. I conclude from (1) that *if* there is a precise version of this idea, then its role for the notion of logicity would at most be as a requirement on *primitive* logical constants, not defined ones.

Before moving on, we should note that Feferman (1999) proposed to implement SAME (at least partly) by strengthening ISOM to HOM: invariance under *homomorphisms*. He proved that the operations definable in first-order logic *without equality* are precisely those definable from *monadic* operations across domains satisfying HOM. Granted that HOM is accepted, this of course contradicts (1).

Later, however, Feferman abandoned HOM as a criterion of logicity, not (at least not explicitly) for reasons related to SAME, but rather because (a) equality does not satisfy HOM, and (b) the restriction to monadic types seems rather arbitrary, and once you admit operations of non-monadic types, a host of non-first-order operations become definable; indeed, as Bonnay (2008) showed, the HOM operations of arbitrary type are exactly those definable in $L_{\infty\infty}$ without equality (which, as Feferman points out, for a language with finitely many predicate symbols, has essentially the same expressive power as full $L_{\infty\infty}$).

It also seems to me that HOM has a serious problem with SAME. It requires invariance even when you shrink the universe (homomorphically) to a single point. But, just to take an example, couldn't it be that the action of an operation across domains, say, on some set arguments, essentially depends on whether these are *disjoint* or not? Such dependencies will get lost if the domain is shrunk enough. It would at least require some independent argument, I think, to disqualify such operations from the start, which is what HOM does. For these reasons, I will ignore HOM in what follows, and maintain that (1) is a feature of the notion of sameness we are after.

3 Types and quantifiers

I restrict attention, in much of what follows, to the operations on domains usually called (generalized or Lindström) *quantifiers*. This is because intuitions, and facts, are simpler for quantifiers, or more generally for operations of type level ≤ 2 . Eventually, of course, one would like to be able to deal with operations of all types.

The type system used in Feferman (1999), let us call it TFT, is (with slight notational differences) as follows: e, t are basic *type symbols*, complex types have the form $\langle \sigma_1 \dots \sigma_n, \tau \rangle$. Given a domain M , $M_e = M$, $M_t = \{0, 1\}$, and $M_{\langle \sigma_1 \dots \sigma_n, \tau \rangle} = M_\tau^{M_{\sigma_1} \times \dots \times M_{\sigma_n}}$. Finally, $\langle M \rangle = \bigcup_\tau M_\tau$, where τ varies over the set of type symbols.

For many purposes, *relational* types are somewhat simpler to deal with. Every type symbol τ in TFT can be uniquely written as

$$\tau = \langle \sigma_{11} \dots \sigma_{1k_1}, \dots, \langle \sigma_{n1} \dots \sigma_{nk_n}, \tau_0 \rangle \dots \rangle$$

where τ_0 is either e or t . Following van Benthem (1989), call τ *individual* if $\tau_0 = e$ and *Boolean* if $\tau_0 = t$. Boolean types are relational but their arguments need not be; cf. $\langle \langle e, e \rangle, t \rangle$. Define the *strictly relational* types of TFT inductively as follows:

τ is *strictly relational* iff τ is either basic or of the form $\langle \sigma_1 \dots \sigma_n, t \rangle$, where each σ_i is strictly relational. (e is included for convenience here.)

For these types, a simpler *relational type system*, call it RT, is the following: the only basic type symbol is e , and complex type symbols have the form $\langle \sigma_1, \dots, \sigma_n \rangle$. Given M , $M_e = M$, and $M_{\langle \sigma_1, \dots, \sigma_n \rangle} = \mathcal{P}(M_{\sigma_1} \times \dots \times M_{\sigma_n})$.³ Then it is easy to verify that RT is isomorphic to the strictly relational part of TFT, and that operations across domains in this part of TFT transfer in an obvious way to operations across domains in RT, and vice versa.

Finally, for (generalized) quantifiers, there is the widely used type notation from Lindström (1966); we can extend it slightly to include 0-ary relations (i.e. truth values) as arguments, so that propositional connectives are quantifiers too. Thus, a *quantifier of type* $\langle k_1, \dots, k_n \rangle$ (where each $k_i \geq 0$) is an operation Q across domains such that, for each M , Q_M is an n -ary relation between relations over M , of arities $k_1, \dots, k_n$, respectively.⁴ For example, if Q is of type $\langle 1, 0, 3 \rangle$, then Q_M relates a subset of M , a truth value, and a subset of M^3 . A binary propositional connective like $\wedge$ has type $\langle 0, 0 \rangle$; here we can think of $\wedge_M$ as the usual truth function (which is independent of M).⁵ A quantifier of type

³Stipulating that the cartesian product of the empty sequence $()$ of sets is $\{\emptyset\}$, we have $M_{()} = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$, so $()$ corresponds to the type t in TFT.

⁴If π_m is the TFT type symbol $\langle e^m, t \rangle$ of m -ary relations between individuals (and $\pi_0 = t$), type $\langle k_1, \dots, k_n \rangle$ corresponds to $\langle \pi_{k_1} \dots \pi_{k_n}, t \rangle$. Similarly in RT, it corresponds to $((e^{k_1}), \dots, (e^{k_n}))$.

⁵I am using the same notation for the conjunction symbol and its interpretation (its corresponding quantifier), and similarly later on for other quantifier symbols; this should not generate confusion.

$\langle k_1, \dots, k_n \rangle$ is (*pure*) *monadic*, if each k_i is ≤ 1 (is $= 1$).

In familiar notation, $L(Q, Q', \dots)$ is first-order logic (FOL) with added quantifiers $Q, Q', \dots$: for each quantifier one adds a corresponding variable-binding operator of the same type, and extends the syntax and the definition of satisfaction accordingly; details can be found in any textbook or survey paper dealing with generalized quantifiers. For logics L and L' , $L \leq L'$ means that every L -sentence is equivalent to (has the same models as) some L' -sentence, $L \equiv L'$ means that $L \leq L'$ and $L' \leq L$, and $L < L'$ means that $L \leq L'$ and $L' \not\leq L$.

In logic, one generally requires $Q, Q', \dots$ to satisfy ISOM, which has the effect that $L(Q, Q', \dots)$ -sentences preserve their truth value across isomorphic models. This is of course consonant with intuition (I2).

4 Extension and a clash of intuitions

A serious candidate for a precise notion of sameness is the property of *Extension* (EXT), introduced by van Benthem.⁶ For quantifiers it is the following condition:

A quantifier Q of type $\langle k_1, \dots, k_n \rangle$ satisfies EXT if, whenever $M \subseteq M'$ and $R_i \subseteq M^{k_i}$ for $1 \leq i \leq n$, $Q_M(R_1, \dots, R_n) \Leftrightarrow Q_{M'}(R_1, \dots, R_n)$.

Indeed, this generalizes to all types in RT:

(EXT) An operator O across domains of type $(\sigma_1, \dots, \sigma_n)$ is EXT if $M \subseteq M'$ implies that $O_M = O_{M'} \upharpoonright M$ (where $O_{M'} \upharpoonright M = O_{M'} \cap (M_{\sigma_1} \times \dots \times M_{\sigma_n})$).

The reason this works is that types in RT are *monotone* in the following sense:

(2) If τ is an RT-type and $M \subseteq M'$, then $\mathcal{M}_\tau \subseteq \mathcal{M}'_\tau$.

(2) is in stark contrast to TFT types:⁷

(3) For most TFT-types τ , if M is a proper subset of M' , then $\mathcal{M}_\tau \cap \mathcal{M}'_\tau = \emptyset$.

⁶van Benthem (1984) made EXT part of the property of *conservativity* for type $\langle 1, 1 \rangle$ quantifiers. In van Benthem (1986) it was isolated as a separate property, applicable to all types, and given the natural label ‘extension’. van Benthem (1989) calls it ‘context insensitivity’.

⁷Call a TFT-type symbol τ *truth-functional* if it does not contain e , and *extended truth-functional* if it is either truth-functional or of the form

$$\langle \sigma_{11} \dots \sigma_{1k_1}, \dots, \langle \sigma_{n1} \dots \sigma_{nk_n}, e \rangle \dots \rangle$$

where each σ_{ij} is truth-functional. Westerståhl (2012) shows that the monotone types in TFT are exactly the extended truth-functional ones, and that all other TFT-types τ are *disjoint* in the sense that $M \neq M'$ implies $\mathcal{M}_\tau \cap \mathcal{M}'_\tau = \emptyset$.

One way to make TFT-types monotone is to allow *partial* functions rather than just total ones. But this raises other issues for a formulation of EXT; the situation is further discussed in Westerståhl (2012).

This is the main reason it is more problematic to define a natural version of EXT for arbitrary TFT-types.

As far as I know, van Benthem did not explicitly connect EXT to the SAME idea; that connection is discussed extensively in Peters and Westerståhl (2006), Ch. 3.4–5. But surely EXT is in itself a very strong notion of sameness across domains. It says that only the elements of M belonging to some tuple in some R_i matter for the truth value of the statement $Q_M(R_1, \dots, R_n)$. Extending the universe beyond that has no effect: the operation remains the ‘same’. More precisely, for $R \subseteq M^k$, let $field(R) = \{a \in M : \exists(a_1, \dots, a_k) \in R \exists j a = a_j\}$, and let $field(R_1, \dots, R_n) = \bigcup_{i=1}^n field(R_i)$. Then EXT for Q is equivalent to:

$$\text{If } field(R_1, \dots, R_n) \subseteq M, M', \text{ then } Q_M(R_1, \dots, R_n) \Leftrightarrow Q_{M'}(R_1, \dots, R_n).$$

Here is a different formulation of essentially the same notion. The quantifiers Q of type $\langle k_1, \dots, k_n \rangle$ we are interested in are usually *definable* in standard set theories, like ZF. That is, Q can be introduced by a ZF-formula $\psi(M, \bar{R})$ with at most the free variables $M, \bar{R}$, a *definition* of Q , such that

$$\forall M \neq \emptyset \forall R_1 \subseteq M^{k_1} \dots \forall R_n \subseteq M^{k_n} (Q_M(R_1, \dots, R_n) \leftrightarrow \psi(M, R_1, \dots, R_n))$$

Then:

- (4) A definable quantifier Q is EXT iff it has a definition in which the variable M does not occur.

Proof. If Q has such a definition, it is immediate that EXT holds. Conversely, suppose Q is EXT, and defined by $\psi(M, \bar{R})$. We have

$$Q_M(\bar{R}) \Leftrightarrow Q_{field(\bar{R})}(\bar{R})$$

Thus, $Q_M(\bar{R}) \Leftrightarrow \psi(field(\bar{R}), \bar{R})$, and the right-hand side can be written as a ZF-formula where at most the variables $\bar{R}$ occur. $\square$

Thus, the following is a strong intuition about sameness:

- (I4) EXT implies SAME.

But now we have a clash of intuitions: (I2) and (I4) are incompatible! For if both were true, it would follow that EXT implies ISOM, which is manifestly not the case. Many EXT quantifiers, such as the so-called Montagovian individuals $(I_a)_M(B) \Leftrightarrow a \in B$, and quantified noun phrase denotations like $(most^{student})_M(B) \Leftrightarrow |student \cap B| > |student - B|$, where a is a *fixed* individual and *student* a *fixed* set, are EXT but not ISOM.

It is not surprising — in fact, I think it is inevitable — that some intuitions about sameness clash. In the case of (I2) and (I4), we have already seen that they are based on quite different ideas about sameness. I will come back to the question of what we should conclude from this, but for now, in this and the next section, I take a closer look at the status of EXT in the literature. To begin, here are some sample facts :

- (5) a. The propositional connectives are EXT.
- b. The quantifiers $\exists_{\geq n}$ are EXT (where $(\exists_{\geq n})_M(B) \Leftrightarrow |B| \geq n$), and more generally, the quantifiers $\exists_{\geq \kappa}$ for any cardinal κ .
- c. The universal quantifier $\forall$ is not EXT.
- d. The trivial quantifiers **0** and **1** (in each type) are EXT.⁸
- e. The class of EXT quantifiers (in any type) is closed under conjunction, disjunction, and (outer) negation.
- f. The *Rescher quantifier* Q^R and the *Chang quantifier* Q^C are not EXT (where we have $(Q^R)_M(B) \Leftrightarrow |B| > |M-B|$, and $(Q^C)_M(B) \Leftrightarrow |B| = |M|$).⁹
- g. Under EXT, PERM is equivalent to ISOM, for all operations across domains of relational type.

As to (5-a), note that we are talking about propositional negation. *Predicate negation* is of type $\langle\langle e, t \rangle, \langle e, t \rangle\rangle$, which is not relational. Together with (I3); (5-c) and (5-f) indicate that the converse of (I4) fails, at least if these quantifiers are taken to satisfy SAME. Regarding (5-e), it seems a rather reasonable assumption that the class of SAME quantifiers (in any type) is also closed under boolean operations. But note that certain types of quantifiers also have a natural notion of *inner* negation. For example, in type $\langle 1, 1 \rangle$, we define $(Q\neg)_M(A, B) \Leftrightarrow Q_M(A, M - B)$. Clearly, EXT is *not* preserved under inner negation. In general, the class of EXT quantifiers is not closed under (first-order) definability, since e.g. $\forall$ is definable from $\exists$ and $\neg$. (5-g), finally, requires a small (but straightforward) argument, not given here.

I discuss EXT in connection with other quantifiers from mathematical logic in the next section, but let me end this section with some remarks on the role of EXT in formal semantics for natural language. *Noun phrases*, like *Mary*, *the tallest man on earth*, *most students*, *every professor*, can be taken to denote type $\langle 1 \rangle$ quantifiers, and *determiners*, like *every*, *some*, *most*, *at most ten*, *infinitely many*, to denote type $\langle 1, 1 \rangle$ quantifiers. For example,

$$most_M(A, B) \Leftrightarrow |A \cap B| > |A - B|$$

and thus, for all M ,

$$\begin{aligned} \llbracket most \rrbracket &= most \\ \llbracket most\ students \rrbracket_M(B) &\Leftrightarrow most_M(\llbracket student \rrbracket, B) \end{aligned}$$

⁸E.g. in type $\langle 1, 2 \rangle$, $\neg \mathbf{0}_M(A, R)$ for all $A \subseteq M$ and $R \subseteq M^2$, and $\mathbf{1}_M(A, R)$ for all $A \subseteq M$ and $R \subseteq M^2$.

⁹In the literature, the label ‘Rescher quantifier’ is often used for a stronger quantifier of type $\langle 1, 1 \rangle$, let us call it *more*, where $more_M(A, B) \Leftrightarrow |A| > |B|$ (which, incidentally, is EXT). (For example, the survey article Mundici (1985) uses ‘ Q^R ’ for *more*.) My usage here is historically more accurate, since precisely the type $\langle 1 \rangle$ quantifier Q^R was introduced in the abstract Rescher (1962). Rescher also mentioned the relativization of Q^R , which is the quantifier *most*, defined by $most_M(A, B) \Leftrightarrow |A \cap B| > |A - B|$; *most* is strictly stronger than Q^R , but strictly weaker than *more*; see, for example, Peters and Westerståhl (2006), Ch. 14.3.

Now it seems to be a linguistic fact that *all* determiner denotations, i.e. *all* type $\langle 1, 1 \rangle$ quantifiers that interpret natural language determiners, are EXT. An explanation for this is that determiners serve to *restrict the domain of quantification* to the first (noun) argument, and this enforces EXT.¹⁰

In fact, the EXT property is ubiquitous in natural language. The non-monadic quantifiers that feature in the interpretation of linguistic constructions such as reciprocals, branching, cumulative quantification, possessives, etc., are all EXT; see Peters and Westerståhl (2006) for examples and discussion. Likewise, almost all type $\langle 1 \rangle$ noun phrase denotations satisfy EXT; in particular, all denotations of *restricted* noun phrases of the form [determiner + noun], when the noun has a fixed interpretation (*most students*, *every professor*).

Indeed, the only exceptions seem to be certain noun phrases containing a word, such as the English *thing*, which can be taken to denote the universe. Thus, the denotations of *everything*, *most things*, *all but five things*, are not EXT.¹¹ Since *every*, *most*, *all but five*, are all EXT, it seems that the ‘blame’ for this state of affairs must be placed on words like *thing*; more on this in section 6 below.

5 Mathematical quantifiers and Ext

A quick survey of quantifiers discussed in model theory — for example, as presented in Mundici (1985) — reveals that many of them are EXT, but some are not. We shall look at just a few examples. The interest of these quantifiers lies in the model-theoretic properties of the corresponding logics, such as varying degrees of compactness, Löwenheim-Skolem properties, etc. But here the task is just to assess the status of EXT in a model-theoretic context.

We have already encountered Q^R and Q^C as non-EXT quantifiers. Cardinality quantifiers ($\exists_{\geq \aleph_\alpha}$ and variants), and cardinality comparison quantifiers like the *Härtig quantifier* ($I_M(A, B) \Leftrightarrow |A| = |B|$) and *more_M* ($A, B) \Leftrightarrow |A| > |B|$) (see note 9), are EXT.

The *Magidor-Malitz quantifiers* Q_κ^n , defined, for $R \subseteq M^n$, by

$$(Q_\kappa^n)_M(R) \Leftrightarrow \exists X \subseteq M(|X| = \kappa \ \& \ X^n \subseteq R),$$

are EXT, and so are the related *Ramsey quantifiers*, a typical variant of which is defined as follows: for Q of type $\langle 1, 1 \rangle$, $A \subseteq M$, and $R \subseteq M^n$, let

$$(Ram^n(Q))_M(A, R) \Leftrightarrow \exists X \subseteq A(Q_M(A, X) \ \& \ X^n - Id_X^n \subseteq R)$$

where $Id_X^n = \{(a, \dots, a) : (a, \dots, a) \in X^n\}$. That is, if Q is EXT, so is $Ram^n(Q)$.

¹⁰More precisely, it means that determiner denotations are *relativizations* of type $\langle 1 \rangle$ quantifiers (see section 5), which, for type $\langle 1, 1 \rangle$ quantifiers, can easily be shown to be equivalent to saying that they satisfy EXT and *conservativity*:

(CONSERV) For all M and all $A, B \subseteq M$, $Q_M(A, B) \Leftrightarrow Q_M(A, A \cap B)$.

¹¹These denotations are $\forall$, Q^R , and $(\exists_{=5})\neg$, respectively. On the other hand, *something*, or *at least three things*, also contain *thing*, but have EXT denotations.

Now consider *partially ordered* quantifiers, i.e. quantifiers that express non-linearly ordered prefixes of $\forall x$ and $\exists y$. For most purposes, it suffices to consider the *Henkin prefix*

$$\left\{ \begin{array}{l} \forall x \exists x' \\ \forall y \exists y' \end{array} \right\} \psi(x, x', y, y')$$

This expresses the type (4) *Henkin quantifier* Q^H , defined by

$$(Q^H)_M(R) \Leftrightarrow \exists f, g: M \rightarrow M \forall a, b \in M R(a, f(a), b, g(b))$$

Q^H does *not* satisfy EXT: indeed, $(Q^H)_M(R)$ and $M \subsetneq M'$ implies $\neg(Q^H)_{M'}(R)$. But this is ‘made up for’ by the fact that Q^H *relativizes*.

To explain this, recall that for any quantifier Q of type $\langle k_1, \dots, k_n \rangle$, the *relativization* Q^{rel} of Q is the type $\langle 1, k_1, \dots, k_n \rangle$ quantifier defined by

$$(Q^{rel})_M(A, R_1, \dots, R_n) \Leftrightarrow Q_A(R_1 \upharpoonright A, \dots, R_n \upharpoonright A)$$

A quantifier is *relativized* if it is equal to Q^{rel} for some Q , and we say that Q *relativizes* if Q^{rel} is definable in $L(Q)$. (Note that it is always the case that Q is definable in $L(Q^{rel})$.) Now, Q^H relativizes, since it is easy to check that $(Q^H)^{rel}$ is expressible by the following $L(Q^H)$ -sentence:

$$Q^H xx'yy'(Px \wedge Py \rightarrow Px' \wedge Py' \wedge Rxx'yy')$$

We will get back to what this has to do with EXT, but first we consider another group of familiar quantifiers.

Order quantifiers express properties of a (usually linear) order relation R , for example, that R is a well-order, or is dense and contains a countable dense subset, or has cofinality ω , or is isomorphic to some given ordered set. Other order quantifiers compare two order relations, R and S , saying, for example, that they are isomorphic.

Two things are striking (from our perspective) about the discussion of these quantifiers in the literature. *First*, R is sometimes taken to be an ordering of the *universe*, and sometimes just an ordering of its *field*. In the former case, the quantifier is not EXT; in the latter, it usually is. *Second*, not much attention is paid to this difference, apparently since it usually doesn’t matter for the relevant model-theoretic facts which version is chosen. Let us see what the general situation is.

Given any quantifier Q of type $\langle k_1, \dots, k_n \rangle$, we obtain a ‘universal version’ Q^{uni} of the same type:

$$(Q^{uni})_M(R_1, \dots, R_n) \Leftrightarrow Q_M(R_1, \dots, R_n) \ \& \ field(R_1, \dots, R_n) = M$$

Q^{uni} is definable from Q , since $field(\overline{R})$ is defined by the first-order formula $\varphi_{\overline{R}}(x)$:

$$(6) \quad \varphi_{\overline{R}}(x) := \bigvee_{i=1}^n \exists x_1 \dots \exists x_{k_i} (P_i x_1 \dots x_{k_i} \wedge \bigvee_{j=1}^{k_i} x = x_j)$$

Now, Q^{uni} is practically *never* EXT:

(7) If Q^{uni} is not the trivial quantifier $\mathbf{0}$, then Q^{uni} is not EXT.

Proof. If $Q^{uni} \neq \mathbf{0}$, there is $\mathcal{M} = (M, \bar{R})$ such that $Q_M(\bar{R})$ and $field(\bar{R}) = M$. Let $M' = M \cup \{a\}$, where $a \notin M$. Then we have that $(Q^{uni})_M(\bar{R})$ holds but not $(Q^{uni})_{M'}(\bar{R})$, so Q^{uni} does not satisfy EXT. $\square$

In the cases mentioned above, the (non-EXT) order quantifiers saying *of the universe* that R is a well-order of it, or an order of a given order type, or dense with a countable dense subset, or has cofinality ω , all have the form Q^{uni} , where Q is EXT and says the same thing about the field of R . Similarly for the isomorphism quantifier.

The following general facts hold about EXT and the expressive power of Q , Q^{rel} , and Q^{uni} .

Proposition 1

Let L be a logic and Q a quantifier of type $\langle k_1, \dots, k_n \rangle$.¹²

- (a) Relativized quantifiers satisfy EXT.
- (b) EXT quantifiers relativize. Hence, if Q is EXT, then Q is definable in L iff Q^{rel} is definable in L .
- (c) If Q is EXT, then $L((Q^{uni})^{rel}) \equiv L(Q)$. Moreover, if L relativizes,¹³ then Q is definable in L iff Q^{uni} is definable in L .

Proof. (a): This is immediate from the definition of relativization.

(b): If Q is EXT, then, with $\bar{R} = R_1, \dots, R_n$: $(Q^{rel})_M(A, \bar{R}) \Leftrightarrow Q_A(\bar{R} \upharpoonright A) \Leftrightarrow Q_M(\bar{R} \upharpoonright A)$ (by EXT). This condition is expressed in $L(Q)$ by the sentence (where $\bar{x}_i = x_{i1} \dots x_{ik_i}$, A interprets P , and R_i interprets P_i , $1 \leq i \leq n$):

$$Q\bar{x}_1, \dots, \bar{x}_n((P_1\bar{x}_1 \wedge \bigwedge_{j=1}^{k_1} Px_{1j}), \dots, (P_n\bar{x}_n \wedge \bigwedge_{j=1}^{k_n} Px_{nj}))$$

(c): We first observe that, for $A \subseteq M$,

$$\begin{aligned} (Q^{uni})_M^{rel}(A, \bar{R}) &\Leftrightarrow (Q^{uni})_A(\bar{R} \upharpoonright A) \\ &\Leftrightarrow Q_A(\bar{R} \upharpoonright A) \& field(\bar{R} \upharpoonright A) = A \\ &\Leftrightarrow Q_M(\bar{R} \upharpoonright A) \& field(\bar{R} \upharpoonright A) = A \quad (\text{by EXT}) \end{aligned}$$

This shows that $L((Q^{uni})^{rel}) \leq L(Q)$. Moreover, for the special case $A = field(\bar{R})$, we obtain

$$Q_M(\bar{R}) \Leftrightarrow (Q^{uni})_M^{rel}(field(\bar{R}), \bar{R})$$

¹²We assume that L extends FOL and is closed under substitution of formulas for predicate letters.

¹³If $L = L(Q_1, Q_2, \dots)$, then L relativizes if each Q_i^{rel} is definable in L . Similarly for other logics; for example, FOL, $L_{\omega_1\omega}$, and $L_{\omega_1\omega_1}$ relativize, but not e.g. $L_{\omega\omega_1}$.

which shows that $L(Q) \leq L((Q^{uni})^{rel})$.

The second claim in (c) follows from the first: The left-to-right direction holds trivially because Q^{uni} is definable in $L(Q)$, and the other direction follows using this easy folklore fact:

(8) If L relativizes, then Q is definable in L iff Q^{rel} is definable in L .

□

By (a) of the proposition, for quantifiers like Q^H that are not EXT but still relativize, we can, if required, ‘pretend’ they are EXT, by using their relativizations instead, which do satisfy EXT and have the same expressive power.¹⁴

This doesn’t work for order quantifiers of the form Q^{uni} , since these quantifiers *usually do not relativize*, even though Q is EXT and hence relativizes.¹⁵ But by (c) of the proposition, Q and Q^{uni} still do not differ with respect to definability in many common logics. For example, the quantifier ‘ R is a well-order of the universe’ is definable in $L_{\omega_1\omega_1}$, by conjoining the order axioms with the sentence

$$\neg \exists x_0 \dots x_n \dots (\bigwedge_{n < \omega} P x_{n+1} x_n)$$

Since $L_{\omega_1\omega_1}$ relativizes, it follows that the stronger quantifier ‘ R is a well-order of its field’ is also definable in $L_{\omega_1\omega_1}$.¹⁶

Finally, let us get back to the (pure) monadic quantifiers. If Q is of type $\langle 1, \dots, 1 \rangle$, then

$$(Q^{uni})_M(A_1, \dots, A_n) \Leftrightarrow Q_M(A_1, \dots, A_n) \ \& \ A_1 \cup \dots \cup A_n = M$$

Although Proposition 1 still applies, part (c) has no interest, since in general, Q^{uni} has no interesting connection to Q . For example, $\forall^{uni} = \exists^{uni} = (Q^R)^{uni} = \forall$, $(\neg \exists)^{uni} = (\neg Q^R)^{uni} = \mathbf{0}$. Similarly, $(all^{uni})_M(A, B) \Leftrightarrow B = M$. The latter quantifier is neither EXT nor CONSERV (note 10), and in a natural language (or any other?) context there seems to be no reason to consider it.

Summing up then, it seems fair to say that failure of EXT usually plays no significant role for generalized quantifiers studied in mathematical logic. Many of these quantifiers already satisfy EXT. Among the ones that don’t, some,

¹⁴ Q^H essentially involves the whole universe, which is reflected in the fact that $(Q^H)^{uni} = Q^H$.

¹⁵I have not attempted a general statement and proof of this fact here, but intuitively it should be fairly clear. Consider, for example, WO^{uni} , where $WO_M(R)$ says that R is a well-ordering of its field. Let η be the order type of the rationals, and let $\mathcal{M} = (M, A, R)$ be a model where R is a linear order of M with the order type $\eta + \omega + \eta$ and A is the subset corresponding to ω , and let $\mathcal{M}' = (M', A', R')$ be a similar model, but where R' has the order type $\eta + \omega + \omega^* + \omega + \eta$, and A' corresponds to $\omega + \omega^* + \omega$. Then $(WO^{uni})_M^{rel}(A, R)$ holds and $(WO^{uni})_{M'}^{rel}(A', R')$ fails, but $\mathcal{M}$ and $\mathcal{M}'$ cannot be distinguished in $L(WO^{uni})$. Indeed, over $\mathcal{M}'$, $L(WO^{uni})$ is equivalent to FOL, since there are no definable well-orders of the universe in $\mathcal{M}'$.

¹⁶Of course, in this case, this is immediate from a slight adjustment of the order axioms; the point here was to bring out the general situation.

like the Henkin quantifier, relativize, and thus it doesn't matter if one uses them or their relativized versions which do satisfy EXT, since they have the same expressive power. As to the group of order quantifiers, the (non-EXT) version describing properties of orders of the whole universe is usually strictly weaker than the (EXT) version where only the field of the relation is ordered. Here we have seen that, as regards definability in logics that themselves allow relativization, the difference between the two versions disappears. Of course, this does not show that this difference is irrelevant to other model-theoretic properties of these quantifiers that have been studied. Still, my claim would be that *usually*, when the weaker version is used, this is because it slightly simplifies certain arguments, but these arguments can easily be adapted to the stronger version as well.

Remark: I am not aware of any study of this issue in the literature. Restricting attention to the property of compactness, here are three examples: (1) The well-order quantifier WO (note 15) is such that neither $L(WO)$ nor $L(WO^{uni})$ is compact. (2) The cofinality quantifier Q_{ω}^{cf} , says of a linear order that it has cofinality ω in its field. Shelah (1975) showed that $L(Q_{\omega}^{cf})$ is fully compact, but, for example, the proof in Väänänen (2011) uses $(Q_{\omega}^{cf})^{uni}$ instead. But the result about $L((Q_{\omega}^{cf})^{uni})$ is only apparently weaker, since the proof is easily adjusted to deal with the stronger quantifier. (3) Let $(Q^D)_M(R) \Leftrightarrow R$ is a dense linear ordering of its field with a countable dense subset. Then $L(Q^D)$ is countably compact, and hence, so is $L((Q^D)^{uni})$. Indeed, the same proof of compactness can be given in both cases, by translation into stationary logic $L(aa)$, which is countably compact; see Mundici (1985).¹⁷ *End of remark.*

The situation is similar for monadic quantifiers, even though the version which requires the quantifier to 'act' on the whole universe is usually totally uninteresting. Here, the only non-EXT monadic quantifiers in the model-theoretic literature that I know of are of type $\langle 1 \rangle$, notably Q^R , Q^C , and of course $\forall$ (and variants such as $(\exists_{\geq n})\neg$, etc.). $\forall$ relativizes (as does its variants), so one can just as well use $\forall^{rel} = all$. But Q^R and Q^C do not. Further, neither $L(Q^R)$ nor $L(Q^C)$ is compact; in each there is a sentence with arbitrarily large finite models but no infinite model. Both logics satisfy an upward Löwenheim-Skolem theorem, hence none of them can characterize $(\omega, <)$, but it is not hard to show that this structure can be characterized in both $L((Q^R)^{rel}) = L(most)$ and $L((Q^C)^{rel})$.¹⁸

¹⁷However, Jouko Väänänen pointed out an exception (*p.c.*): the quantifier (not mentioned in Mundici (1985)) $(Q^{card})_M(R) \Leftrightarrow R$ is a $|field(R)|$ -like linear ordering of its field, that is, each proper initial segment has cardinality $< |field(R)|$. Then the sentence saying that R is discrete with a least element, and that for all a , $(Q^{card})_M(R \upharpoonright \{b : bRa\})$, characterizes $(\omega, <)$, so $L(Q^{card})$ is not compact. On the other hand, it can be shown with techniques that go back to Vaught and Fuhrken that $L((Q^{card})^{uni})$ is (countably) compact. So my claim is indeed tentative and does not hold across the board; it may be still of some interest to find sufficient conditions for when it does.

¹⁸In other respects, $L(Q^R)$ and $L(Q^C)$ are quite different: $L(Q^C)$ reduces to FOL on finite models, whereas Barwise and Cooper (1981) showed that $FOL < L(Q^R) < L(most)$ even on finite models (illustrating the need for type $\langle 1, 1 \rangle$ quantifiers in natural language semantics).

6 ‘thing’

I said that the only thing that seems to lead to violations of EXT in a natural language context are words like the English ‘thing’. This word arguably denotes an operation across domains, which we may call a *unary predicate*, of relational type (e), defined by

$$thing_M = M \text{ for all } M$$

We note that *thing* itself is PERM and EXT (and hence ISOM, by (5-g)). Indeed, let the unary predicate *empty* be defined by $empty_M = \emptyset$ for all M . Then we have:

- (9) The only unary predicates satisfying PERM and EXT are *thing* and *empty*.

This is a special case of a more general fact.

Proposition 2 ((Westerståhl, 1985))

The operations O across domains of relational type (e^n) that satisfy PERM and EXT are exactly the ones that are first-order definable without parameters by Boolean combinations of formulas of the form $x_i = x_j$, $1 \leq i, j \leq n$.

A typical example of type (e^3) would be

$$O_M(R) = \{(a, b, c) \in M^3 : a = b \text{ or } b \neq c\}$$

which is defined in pure first-order logic with identity by the formula

$$x_1 = x_2 \vee \neg x_2 = x_3$$

Getting back to the unary case, if we count *empty* as trivial, we can say that *thing* is the only non-trivial *logical* unary predicate.¹⁹ Even though it ‘refers to the universe’ in an obvious sense, it satisfies EXT and therefore, at least according to (I4), SAME.

In natural language semantics, it appears that all instances of non-EXT operations across domains arise from replacing a fixed predicate, such as *students* in *most students*, with *thing*: *most things* = Q^R . Curiously, *thing* itself is EXT, as is of course *most*. This kind of replacement doesn’t always destroy EXT: for example, from *at least ten students* we get *at least ten things* = $\exists_{\geq 10}$. It can be shown that preservation of EXT holds precisely when the determiner denotation in question is *symmetric*, in the sense that $Q_M(A, B)$ implies $Q_M(B, A)$ for all M and all $A, B \subseteq M$.²⁰

More generally, if we in a relativized quantifier of type $\langle 1, k_1, \dots, k_n \rangle$ (which is always EXT) replace the first argument by *thing*, we obtain the original quantifier (which may or may not be EXT).

¹⁹Barwise and Cooper (1981) use *thing* as a logical constant in their formal language.

²⁰Provided Q is CONSERV and EXT (see Peters and Westerståhl (2006), Ch. 6.1).

7 Taking stock

After these observations about EXT, we can get back to SAME. As we saw in section 2, the strongest intuitive cases of non-SAME quantifiers are those where Q ‘behaves differently’ on different M . Here are some more examples. Define (for all $M \neq \emptyset$)

$$\forall_{\text{even}_M} = \begin{cases} \forall_M & \text{if } |M| \text{ is even} \\ \exists_M & \text{otherwise} \end{cases}$$

In other words (for all $A \subseteq M$),

$$\forall_{\text{even}_M}(A) \Leftrightarrow \begin{cases} A = M & \text{if } |M| \text{ is even} \\ A \neq \emptyset & \text{otherwise} \end{cases}$$

This seems like a clear case: $\forall_{\text{even}}$ does *not* satisfy SAME. But what about the following two test cases (restricting attention to finite universes)?

$$\text{test1}_M(A) \Leftrightarrow \begin{cases} |A| \geq (|M| + 2)/2 & \text{if } |M| \text{ is even} \\ |A| \geq (|M| + 1)/2 & \text{if } |M| \text{ is odd} \end{cases}$$

$$\text{test2}_M(A) \Leftrightarrow \begin{cases} |A| \geq (|M| + 2)/2 & \text{if } |M| \text{ is even} \\ |A| \geq (|M| - 1)/2 & \text{if } |M| \text{ is odd} \end{cases}$$

There is a strong inclination to treat test1 and test2 on a par: either both satisfy SAME or both don’t; the difference between them seems minimal. And guided by the clear case of $\forall_{\text{even}}$, we may think the latter option is most reasonable.

But now we have another clash of intuitions. We have so far assumed that Q^R satisfies SAME, but, in fact,

$$\text{test1} = Q^R$$

(on finite universes)! So what shall we do: say that test1 but not test2 satisfies SAME, or that both do (but $\forall_{\text{even}}$ does not), or that none of them does?

We will see below that, for quantifiers *of this particular kind*, there is in fact a principled way to count test1 and $\forall$, but not test2 or $\forall_{\text{even}}$, among the quantifiers that behave ‘the same’ on all universes. However, for arbitrary quantifiers and more generally for operations across universes of arbitrary types, I think the example just reinforces the conclusion we have already drawn: there simply is no precise demarcation of the quantifiers, let alone the general operations across domains, that accords with *all* of our intuitions about sameness.

One might object that there could be a *vague* criterion that fits *most* of our intuitions: there would then be clear cases of operations that satisfy the criterion, and clear cases that don’t, but there would also be *borderline cases* where we don’t know exactly what to say. This may well be the case. But, first, so far there is no specification of such a criterion that goes beyond the basic intuitions we have discussed in this note. Second, and more importantly, our goal was precisely to *replace* this vague notion by an exact criterion. My

contention, then, is that it is not possible, in this way, to be faithful to all of those intuitions.

Certainly, this is not something I have demonstrated; presumably, it is not something that *could* be demonstrated. But cases analogous to *test1* and *test2* can be constructed *ad libitum*. More generally, along the lines above one can generate sequences $Q_1, \dots, Q_n$ of quantifiers such that the difference between each Q_i and Q_{i+1} seems so small that they should not differ with respect to sameness, but where Q_1 is a familiar quantifier normally thought of as being the same across different domains, whereas Q_n is an ‘artificial’ quantifier without that property. Similarly, we saw that the intuitions behind (I2) concerning ISOM and the intuitions behind (I4) concerning EXT, if taken to hold unrestrictedly, are at odds with each other.

8 A proposal

The conclusion just reached may seem disappointing. But we can end on a more positive note. The situation is precisely of a kind, I think, where a Carnapian *explication* is called for. In the present case, here is a proposal:

- Define: SAME = ISOM + EXT

In other words, stipulate that the conjunction of one necessary — on *one* intuition — and one sufficient — on *another* intuition — condition for sameness together constitute a necessary *and* sufficient condition.²¹ ISOM says that the operation across domains is preserved under structure-preserving maps. EXT says that the part of the domain which is outside the arguments of the operation never matters. Let’s require both.

This means, in effect, that we keep (I2) but reject (I4). It also means that, for example, $\forall$ does not satisfy SAME. As I have argued, effects of this kind cannot be avoided.

A proposal like this is not right or wrong, but more or less *fruitful*. To evaluate it, one needs to look at its consequences, which is essentially what I have done in this note. First, it entails that sameness is not closed under (first-order) definability, but we already accepted that. It is closed, however, under the usual Boolean operations.

Second, many operations, in particular quantifiers, that seem to be defined in the same way across different universes will not satisfy our precise version of

²¹As we noted, the proposal that SAME = PERM + EXT is equivalent. We can also formulate the proposal more succinctly: say that an n -ary operation O across domains of relational type is *closed under injections* if for all M , all relations R_i over M of suitable type, and all injections π from M to M' ,

$$(INJ) \quad O_M(R_1, \dots, R_n) \Leftrightarrow O_{M'}(\pi(R_1), \dots, \pi(R_n))$$

It is easy to check that $INJ \equiv ISOM + EXT$. However, the ideas behind ISOM and EXT are quite different, so they might as well be kept separate.

SAME. But we found reasons to believe that this is inevitable. What matters instead is whether the proposal has unacceptable consequences for familiar operations. Since sameness is not closed under definability anyway, we may be satisfied if these familiar operations are *definable* (in some suitable sense) from SAME operations. This seems indeed to hold in most cases. In particular, as regards the quantifiers in mathematical logic we looked at, all of them are ISOM and many are EXT, and most of the non-EXT ones are either definable from EXT quantifiers, or can be replaced without loss by their EXT versions (their relativizations).

This is even clearer for natural language quantifiers: the only non-EXT ones that occur seem to be obtained from EXT quantifiers (such as determiner denotations) and the unary (and ISOM and EXT) predicate *thing*. On the other hand, many noun phrase denotations and some determiner denotations are not ISOM. Examples of latter are possessives like *Mary's* or *most students'*. But it seems correct to say in these cases that, as determiner denotations, these are indeed *not* SAME. The (standard) truth conditions for *Q students' A are B* are

$$\begin{aligned} & Poss(Q, \llbracket \text{student} \rrbracket, R)_M \Leftrightarrow \\ & Q_M(\{a \in \llbracket \text{student} \rrbracket : \exists b \in A R(a, b)\}, \{a : R_a \subseteq B\}) \end{aligned}$$

where R is a possessive relation and $R_a = \{b : R(a, b)\}$.²² If we, as is reasonable, take $Q' = Poss(Q, \llbracket \text{student} \rrbracket, R)$ to be the denotation of *Q students'*, then Q' is EXT (provided Q is CONSERV and EXT), but not ISOM (even if Q is). As long as $\llbracket \text{student} \rrbracket$ and R are fixed, $Poss(Q, \llbracket \text{student} \rrbracket, R)$ is not the same across different domains. But if they are taken as variable arguments, we obtain an ISOM and EXT type $\langle 1, 2, 1, 1 \rangle$ quantifier Q'' defined by $Q''(C, R, A, B) \Leftrightarrow Poss(Q, C, R)(A, B)$.

It could perhaps be argued that the type $\langle 1 \rangle$ Rescher quantifier Q^R and the Chang quantifier Q^C , which are ISOM but not EXT and do not relativize, are particularly problematic cases for our proposal, since the intuition that these are the same on each domain may seem quite strong. These quantifiers have somewhat different logical properties than their relativizations, not with respect to compactness but with respect to being able to characterize the natural number ordering. So we cannot always use the relativizations instead.

Note that Q^R and Q^C are *monotone increasing* type $\langle 1 \rangle$ quantifiers, in the sense that $(Q^R)_M(A)$ and $A \subseteq A' \subseteq M$ implies $(Q^R)_M(A')$, and similarly for Q^C . For these quantifiers, we can, if we want, (under a few extra assumptions) replace EXT by a weaker requirement of sameness.

8.1 Smooth quantifiers

Consider ISOM and monotone increasing type $\langle 1 \rangle$ quantifiers over *finite* universes. (This reduces Q^C to the universal quantifier.) For these, we know exactly which ones relativize:

²²This is a special case of the more general truth conditions for possessives discussed at length in Peters and Westerståhl (2013).

Proposition 3 (Westerståhl (1991), Kolaitis and Väänänen (1995))

A quantifier of the above kind relativizes if and only if it is first-order definable.

Also, we know exactly which ones are EXT. The following is easy to show with standard methods from e.g. van Benthem (1984) or Peters and Westerståhl (2006).

Proposition 4

A quantifier of the above kind is EXT if and only if it is either the trivial quantifier $\mathbf{0}$ or $\exists_{\geq k}$ for some $k \geq 0$.

There are uncountably many non-first-order definable, and hence non-EXT, quantifiers of this kind. Is there a reasonable Carnapian explication of which of these nonetheless ‘behave the same’ on all (finite) universes?

Such a property has in fact been discussed in the literature. The idea is that when the universe is extended with *just one* new element, the behavior must not change too much. More precisely, since Q is monotone increasing and ISOM, for each M there is a *minimum size* m , depending only on $|M|$, such that $Q_M(A) \Leftrightarrow |A| \geq m$. Thus Q is determined by a function $f: N \rightarrow N$ such that $f(n) \leq n + 1$; we let $f(|M|) = n + 1$ when $Q_M = \emptyset$. Now, the requirement on Q , i.e. on f , is that for all n , $f(n) \leq f(n + 1) \leq f(n) + 1$. In other words, when one element is added to M , the minimum size either stays the same or increases with 1.

This property was called *smoothness* in Väänänen and Westerståhl (2002), who showed that smooth quantifiers are well-behaved in various logical ways, but it was already introduced (under different names) in van Benthem (1984, 1986), where one idea was that smooth quantifiers have a particularly uniform behavior that might be a reason to classify them as *logical*. My point here is that smoothness seems a quite reasonable requirement of *sameness* across different universes. When a new element is added, the quantifier is not required to stay *exactly* the same — that only holds, among the quantifiers discussed here, for $\mathbf{0}$ and $\exists_{\geq k}$ — but it is allowed to change ‘as little as possible’ (as much as the universes changes).

We can now check that $Q^R = \text{test1}$ from section 7 is indeed smooth, as is $\forall$, whereas test2 and $\forall\text{even}$ are not: the minimum level of test2 jumps *two* steps when we go, for example, from a universe with $2k - 1$ elements to one with $2k$ elements. Similarly, the level for $\forall\text{even}$ jumps two steps from a universe with $2k$ elements to one with $2k + 1$ elements.²³

Thus, if desired, we may use a different stipulation of the SAME property for this particular class of quantifiers: ISOM + SMOOTH, instead of the much more restrictive ISOM + EXT. However, there seems to be no way to extend this idea to quantifiers of arbitrary type.

²³We may note that, although test2 and $\forall\text{even}$ appear to be very different quantifiers, and intuitively the latter seems clearly non-SAME, they have essentially the same ‘jumping behavior’.

9 Conclusion

The main conclusion of our discussion is not really surprising: the idea of being the same operation on different domains is indeed vague, even when we grant — as is motivated by fairly clear examples — that this property is not closed under definability. However, there are strong intuitions about sameness, which can be cashed out in precise constraints. I have focused on two: isomorphism invariance (ISOM) and ‘extension invariance’ (EXT). The seemingly natural idea that the former is necessary and the latter sufficient for sameness cannot be upheld, but I suggested that the conjunction of the two is a fairly successful *explication* of sameness. It is a stipulation that allows almost all quantifiers used in practice, both in model theory and in natural language semantics, to be *definable* from SAME operations. The stipulation does not conform to every intuition about sameness, but this goal cannot be achieved anyway. I also suggested that, for the special case of monotone type $\langle 1 \rangle$ quantifiers over finite domains, the weaker constraint ISOM + SMOOTH does a better job of accounting for sameness intuitions.

The observations in this note were motivated by the identification in Feferman (1999) of sameness across domains as a crucial ingredient in the notion of *logical operations*. The delineation of logicity is the more important, and more difficult, task. But I do think that the notion of sameness has some independent interest. Also, since (at least in my view) logicity *is* closed under (suitably specified) definability, we cannot expect sameness to be necessary for logicity. But it seems perfectly feasible to require logical operations of relational type to be definable from ISOM and EXT operations. This is manifestly true for generalized quantifiers: Each ISOM quantifier is definable from an ISOM and EXT quantifier, namely, its relativization.

References

- Barwise, J. and Cooper, R. (1981). Generalized quantifiers and natural language. *Linguistics and Philosophy*, **4**, 159–219.
- Bonnay, D. (2008). Logicity and invariance. *Bulletin of Symbolic Logic*, **14:1**, 29–68.
- Bonnay, D. and Engström, F. (2016). Invariance and definability, with and without equality. *Notre Dame Journal of Formal Logic*, **58**, 00–00.
- Feferman, S. (1999). Logic, logics and logicism. *Notre Dame Journal of Formal Logic*, **40**, 31–54.
- Feferman, S. (2010). Set-theoretical invariance criteria for logicity. *Notre Dame Journal of Formal Logic*, **51**, 3–20.
- Feferman, S. (2015). Which quantifiers are logical? a combined semantical and inferential criterion. In A. Torza, editor, *Quantifiers, Quantifiers and Quantifiers*, pages 19–30. Springer, Berlin.

- Kolaitis, P. and Väänänen, J. (1995). Generalized quantifiers and pebble games on finite structures. *Annals of Pure and Applied Logic*, **74**, 23–75.
- Lindström, P. (1966). First order predicate logic with generalized quantifiers. *Theoria*, **32**, 175–71.
- McGee, V. (1996). Logical operations. *Journal of Philosophical Logic*, **25**, 567–80.
- Mundici, D. (1985). Other quantifiers: an overview. In J. Barwise and S. Feferman, editors, *Model-theoretic Logics*, pages 211–233. Springer-Verlag, Berlin.
- Peters, S. and Westerståhl, D. (2006). *Quantifiers in Language and Logic*. Oxford University Press, Oxford.
- Peters, S. and Westerståhl, D. (2013). The semantics of possessives. *Language*, **89**(4), 713–759.
- Rescher, N. (1962). Plurality-quantification (abstract). *Journal of Symbolic Logic*, **27**, 373–374.
- Shelah, S. (1975). Generalized quantifiers and compact logic. *Transactions of the American Mathematical Society*, **204**, 342–364.
- Sher, G. (1991). *The Bounds of Logic*. The MIT Press, Cambridge.
- Tarski, A. (1986). What are logical notions? *History and Philosophy of Logic*, **7**, 145–154.
- Väänänen, J. (2011). *Models and Games*. Cambridge University Press, Cambridge.
- Väänänen, J. and Westerståhl, D. (2002). On the expressive power of monotone natural language quantifiers over finite models. *Journal of Philosophical Logic*, **31**, 327–358.
- van Benthem, J. (1984). Questions about quantifiers. *Journal of Symbolic Logic*, **49**, 443–466.
- van Benthem, J. (1986). *Essays in Logical Semantics*. Kluwer, Dordrecht.
- van Benthem, J. (1989). Logical constants across varying types. *Notre Dame Journal of Formal Logic*, pages 315–342.
- Westerståhl, D. (1985). Logical constants in quantifier languages. *Linguistics and Philosophy*, **8**, 387–413.
- Westerståhl, D. (1991). Relativization of quantifiers in finite models. In J. van der Does and J. van Eijck, editors, *Generalized Quantifier Theory and Applications*, pages 187–205. ILLC, Amsterdam. Also in *Quantifiers: Logic, Models and Computation* (same editors), CSLI Publications, Stanford, 1996, pages 375–383.
- Westerståhl, D. (2012). Constant operators: partial quantifiers. In S. Larsson and L. Borin, editors, *From Quantification to Conversation*, pages 11–35. College Publications, London.