On Herglotz-Nevanlinna functions in several variables

Mitja Nedic

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Abstract
In this thesis, we investigate different aspects of the class of Herglotz-Nevanlinna functions in several variables. These are holomorphic functions on the poly-upper half-plane having non-negative imaginary part. Our results are presented in the four research articles A1 - A4, which are included in this thesis.

Articles A1 and A2 establish a characterization of Herglotz-Nevanlinna functions in terms of an integral representation formula. The case of functions of two complex variables is presented in article A1, while the general case is treated in article A2, where different symmetry properties of Herglotz-Nevanlinna functions are also discussed.

Article A3 discusses, in detail, the convex combination problem for Herglotz-Nevanlinna functions. This problem asks us to relate the representing parameters of different Herglotz-Nevanlinna functions under the assumption that these functions are related in a very particular way involving the convex combination of several independent variables. A related class of boundary measures is also discussed.

Article A4 investigates the properties of Nevanlinna measures with respect to restrictions to coordinate orthogonal hyperplanes and the geometry of the support. A related class of measures on the unit poly-torus is also considered.

Furthermore, this thesis is supplemented by three additional publications concerning Herglotz-Nevanlinna functions in one variable, related topics and applications.

Article B1 concerns a particular class of convolution operators on the space of distributions that generalizes the well-studied class of passive operators. Article B2 introduces the class of quasi-Herglotz functions and discusses their integral representations, boundary values and sum-rules, as well as their applications in connection with convex optimization. Finally, the summary book-chapter C1 provides a general overview of the applications of Herglotz-Nevanlinna functions in electromagnetics.

Keywords: *Herglotz-Nevanlinna functions, several complex variables, integral representations, Nevanlinna measures.*

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Sammanfattning

I denna avhandling undersöker vi de olika aspekterna av klassen av Herglotz-Nevanlinnafunktioner i flera variabler. Dessa är holomorfa funktioner definierade i det polyövre halvplanet som har en icke-negativ imaginärdel. Våra resultat presenteras i de fyra vetenskapliga artiklarna A1 – A4 som inkluderas i avhandlingen.

Artiklar A1 och A2 sätter upp en karakterisering av Herglotz-Nevnlinnafunktioner genom en integralframställning. Fallet där vi behandlar funktioner av två komplexa variabler presenteras i artikeln A1 medan det generella fallet diskuteras i artikeln A2. Där diskuteras även egenskaper av den symmetriska utvidgningen av Herglotz-Nevnlinnafunktioner.

I artikeln A3 presenteras en detaljerad undersökning av det konvexa kombinationsproblemet för Herglotz-Nevanlinnafunktioner. Detta problem frågar efter ett samband mellan representationsparametrar av olika Herglotz-Nevanlinnafunktioner under antagandet att dessa funktioner kan förknippas med varandra genom en konvex kombination av flera oberoende variabler. En relaterad klass av randmått undersöks också.

Artikeln A4 undersöker egenskaper av Nevanlinnamått med avseende på deras restriktioner till koordinatortogonala hyperplan och geometrin av stödet. En relaterad klass av mått i enhets polytorusen diskuteras också.

Vidare kompletteras avhandlingen med ytterligare tre publikationer som handlar om Herglotz-Nevanlinnafunktioner i en variabler, relaterade ämnena och deras tillämpningar.

Artikeln B1 behandlar en viss klass av faltningsoperatorer definierade på rummet av distributioner som generalisera den välkända klassen av passiva operatorer. Artikeln B2 introducerar klassen av quasi-Herglotzfunktioner och presenterar deras integralframställningar, randvärdena och summaregler så som deras tillämpningar i sambandet med konvex optimering. Till slut förser sammanfattningsbokkapitel C1 oss en överblick över tillämpningar av Herglotz-Nevanlinnafunktioner inom elektromagnetisk teori.

List of Publications

The following publications, referred to in the text by their designations in this list, are included in this thesis.

A1: *A. Luger and M. Nedic,*
A characterization of Herlgotz-Nevanlinna functions in two variables via integral representations,
manuscript, final version published in Ark. Math. 55 (2017), nr. 1, 199–216. DOI of published version: 10.4310/ARKIV.2017.v55.n1.a10

A2: *A. Luger and M. Nedic,*
Herglotz-Nevanlinna functions in several variables,
manuscript, final version to appear in J. Math. Anal. Appl. DOI of on-line published verison: 10.1016/j.jmaa.2018.11.072

A3: *M. Nedic,*
A subclass of boundary measures and the convex combination problem for Herglotz-Nevanlinna functions in several variables,
manuscript, under review.

A4: *A. Luger and M. Nedic,*
Geometric properties of measures related to holomorphic functions having positive imaginary or real part,
manuscript, submitted.

B1: *M. Nedic,*
On pseudo-passive causal operators of slow growth,
manuscript, submitted.

B2: *Y. Ivanenko, M. Nedic, M. Gustafsson, B. L. G. Jonsson, A. Luger and S. Nordebo,*
Quasi-Herglotz functions and convex optimization,
manuscript, submitted.

C1: *M. Nedic, C. Ehrenborg, Y. Ivanenko, A. Ludvig-Osipov, S. Nordebo, A. Luger, B. L. G. Jonsson, D. Sjöberg and M. Gustafsson,*

Herglotz functions and applications in electromagnetics, manuscript, final version to appear in "Advances for mathematical methods in electromagnetics", edited by K. Kobayashi and P. Smith, 2019.

A preprint of article A1 is available on arXiv under the identifier 1605.06232. Preprints of article A2 are available under the title *An integral representation for Herglotz-Nevanlinna functions in several variables* in DiVA under the identifier `urn:nbn:se:su:diva-138166` and on arXiv under the identifier 1705.10562. Preprints of articles A3, A4 and B1 are available on arXiv under the identifiers 1711.01102, 1812.00627 and 1811.10258, respectively. A preprint of article B2 is available as a Technical report in DiVA at Linnæus University under the identifier `urn:nbn:se:lnu:diva-78933`. Preprints of articles A1 and A2 were also included in the author's licentiate thesis.

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General introduction

This chapter provides an overview of the basic concepts, definitions and theorems of the theory of Herglotz-Nevanlinna functions.

We begin with a popular-scientific description of the theory in Section 1, followed by a formal presentation of the concept of holomorphicity in Section 2. Herglotz-Nevanlinna function are defined in Section 3 where their representation theorems are also reviewed. Finally, in Section 4, we collect a detailed overview of the different publications included in this thesis.

1 Popular-scientific abstract

Holomorphic functions with a non-negative imaginary part

A major topic of study in the field of mathematical analysis is the concept of a *derivative* which, one can say, is used to describe the rate of change of an object of study. These objects of study are often taken to be *functions*. A function is a way of assigning to any admissible input one precisely determined output. The collection of all possible inputs is called the *domain* of the function and the collection of all possible outputs is called its *range*.

Common inputs of the functions that we study are *numbers*. In particular, we are interested in, what are called, *complex numbers*. These can be obtained by considering ordered pairs (x, y) of real numbers. The first element of the pair is called the *real part* of a complex number, while the second element of the pair is called the *imaginary part*. It is commonplace to write a complex number as $x + \mathfrak{i}\, y$ instead of (x, y), where the symbol $\mathfrak{i}$ is called the *imaginary unit*. The set of all complex number is denoted by the symbol $\mathbb{C}$.

If a function of one, or several, complex numbers is such that the output of the function is a complex number with non-negative imaginary part, we say, for short, that the function has a *non-negative imaginary part*. Furthermore, a function whose input is a complex number is called *holomorphic* if its derivative can always be described by one or several uniquely determined complex numbers. A holomorphic function with non-negative imaginary part is called a *Herglotz-Nevanlinna function*.

In this thesis, we study Herglotz-Nevanlinna functions and their properties. In particular, we are interested in describing all such functions in terms of a collection of parameters. We are then interested in knowing as much as possible about these parameters, as well as being able to relate, when possible, the parameters of different Herglotz-Nevanlinna functions to one another.

Holomorfa funktioner som har en icke-negativ imaginärdel

En stor del av studier inom matematisk analys kan förknippas med konceptet av *derivatan.* Detta koncept träder fram när ett objekt som undersökes genomgår en förändring. Objekten som undersökes i vårt sammanhang kallas för *funktioner.* En funktion kan beskrivas som ett sätt eller en regel som tilldelar till varje input en entydigt bestämd output. Mängden av alla möjliga input kallas för funktionens *domän* medan mängden av alla möjliga output kallas för funktionens *bild.*

Vanligtvis tar vi *tal* som input av funktioner som undersökes. Speciellt är vi intresserade i *de komplexa talen.* Varje komplext tal kan beskrivas som ett ordnad par (x,y) av reella tal. Parets första element kallas för komplexa talets *reelldel* och parets andra element kallas för talets *imaginärdel.* Det är vanligt att ett komplex tal skrivs som $x+\mathfrak{i}\,y$ istället för (x,y) där symbolen $\mathfrak{i}$ kallas för den imaginära enheten. Mängden av alla komplexa tal betecknas med symbolen $\mathbb{C}$.

För enkelhets skull säger vi att en funktion av en eller flera komplexa variabler vars output är ett komplext tal med icke-negativ imaginärdel är en funktion med en *icke-negativ imaginär del.* Vidare kallas en en funktion vars input är komplexa tal för en *holomorf* funktion om dess derivata alltid kan beskrivas med en eller flera entydigtbestämda komplexa tal. En holomorf funktion med en icke-negativ imaginärdel kallas för en *Herglotz-Nevanlinnafunktion.*

I denna avhandling undersöker vi Herglotz-Nevanlinnafunktioner och deras egenskaper. Särskild är vi intresserade i beskrivningar av denna funktionsklass med hjälp av en samling av olika slags parametrar. Vi är även intresserade att veta så mycket som möjligt om dessa parametrar så som möjligheten att relatera parametrar av olika Herglotz-Nevanlinnafunktioner.

2 The concept of holomorphicity

The field of complex analysis revolves around the concept of holomorphicity [2; 5; 7; 12; 13; 15]. This concept concerning functions defined on some domain in $\mathbb{C}^n$ is incredibly strong as it implies a multitude of wide and farreaching consequences. The formal definition of a holomorphic map $f\colon \Omega \to \mathbb{C}^m$, where $\Omega \subseteq \mathbb{C}^n$ and $m, n \in \mathbb{N}$, is built up in several steps. We start first with the case $m = n = 1$, *i.e.* holomorphic function of one complex variable.

A function $f\colon \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$, is said to have a *complex derivative* at a point $z_0 \in \Omega$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, we denote the above limit by $f'(z_0)$. The function f is then called *holomorphic* on an open set $U \subseteq \Omega$ if its complex derivative exists at every point $z_0 \in U$, and f is called holomorphic on some (possibly) non-open set V if it is holomorphic on some open set U containing V. The set of all holomorphic functions on Ω will be denoted by $\mathcal{O}(\Omega)$.

Remark 2.1. The letter $\mathcal{O}$ used to denote the set of holomorphic functions in used in honour of the Japanese mathematician Kiyoshi Oka [16].

Example 2.2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) := z^2$. In order to check that this function has a complex derivative at any point $z_0 \in \mathbb{C}$, we may write any nearby point z as $z = z_0 + h$ for $h \in \mathbb{C}$ and calculate that

$$\begin{aligned}\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h \to 0} \frac{(z_0 + h)^2 - z_0^2}{h} = \lim_{h \to 0} \frac{2z_0 h + h^2}{h} = 2z_0.\end{aligned}$$

Thus, the complex derivative f' of the function f is equal to $f'(z) = 2z$ for any $z \in \mathbb{C}$, and we conclude that f is a holomorphic function on $\mathbb{C}$. $\lozenge$

Example 2.3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \overline{z}$. Proceeding as in the previous example, we calculate that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}.$$

This limit does not exist, as it holds, when $h = u \in \mathbb{R}$, that

$$\lim_{h \to 0} \frac{\overline{h}}{h} = \lim_{u \to 0} \frac{u}{u} = 1$$

and, when $h = \mathrm{i}\, v \in \mathrm{i}\mathbb{R}$, that

$$\lim_{h \to 0} \frac{\overline{h}}{h} = \lim_{u \to 0} \frac{-\mathrm{i}\, v}{\mathrm{i}\, v} = -1.$$

Thus, we conclude that, at any point $z_0 \in \mathbb{C}$, the function f does not have a complex derivative. Therefore, it is not a holomorphic function on any open set $\Omega \subseteq \mathbb{C}$. $\lozenge$

In several variables, when $n \in \mathbb{N}$ but $m = 1$, *i.e.* we still have a scalar valued function, we define that $f\colon \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}^n$, is *holomorphic* on Ω if, for any point $\vec{\xi} = (\xi_1, \xi_2, \dots, \xi_n) \in \Omega$ and any index $j \in \{1, 2, \dots, n\}$, it holds that the function

$$z \mapsto f(\xi_1, \dots, \xi_{j-1}, z, \xi_{j+1}, \dots, \xi_n)$$

is holomorphic, as a function of one complex variable, on the set

$$\Omega_j(\vec{\xi}) := \{z \in \mathbb{C} \mid (\xi_1, \dots, \xi_{j-1}, z, \xi_{j+1}, \dots, \xi_n) \in \Omega\}.$$

Example 2.4. Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be defined as $f(z_1, z_2) := z_1 + \overline{z_2}$. Then, for any point $(\xi_1, \xi_2) \in \mathbb{C}^2$, it holds that the function

$$g_1\colon z \mapsto f(z, \xi_2) = z + \overline{\xi_2}$$

is holomorphic on $\mathbb{C}$, as it holds, for any point $z_0 \in \mathbb{C}$, that

$$\lim_{h \to 0} \frac{g_1(z_0 + h) - g_1(z_0)}{h} = \lim_{h \to 0} \frac{z + h + \overline{\xi_2} - z - \overline{\xi_2}}{h} = 1.$$

On the other hand, for the function

$$g_2\colon z \mapsto f(\xi_1, z) = \xi_1 + \overline{z},$$

it holds, for any point $z_0 \in \mathbb{C}$, that the limit

$$\lim_{h \to 0} \frac{g_2(z_0 + h) - g_2(z_0)}{h} = \lim_{h \to 0} \frac{\xi_1 + \overline{z} + \overline{h} - \xi_1 - \overline{z}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}$$

does not exist, as we find ourselves in the same situation as in Example 2.3. As such, we conclude that the function f is not holomorphic on $\mathbb{C}^2$. $\Diamond$

Finally, if both m and n are allowed to be arbitrary, *i.e.* we have a vector-valued function of several variables, we say that $f\colon \Omega \to \mathbb{C}^m$, where $\Omega \subseteq \mathbb{C}^n$, is *holomorphic* on Ω if for every $k \in \{1, 2, \dots, m\}$ the k-th coordinate function of f is holomorphic as a scalar valued function of several variables.

Example 2.5. Let $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ be defined as $f(z_1, z_2) := (z_1 + z_2^2, z_1 - \overline{z_2})$. Given what we have already calculated in Examples 2.2, 2.3 and 2.4, we conclude that the first coordinate function of f is holomorphic, while the second one is not. Therefore, the function f is not holomorphic. $\Diamond$

One can easily check that for any two holomorphic functions $f, g\colon \Omega \to \mathbb{C}^m$, where $\Omega \subseteq \mathbb{C}^n$, it holds that any linear combination of f and g yields another holomorphic function. If $m = 1$, then the product of f and g is also a holomorphic function, as well as the quotient of f by g assuming g has no zeros in Ω. Finally, under certain compatibility conditions, *e.g.* $m = n = 1$ and $f(\Omega) \subseteq \mathrm{Dom}(g)$, the composition $g \circ f$ also constitutes a holomorphic function.

The property of holomorphicity has many other characterizations, a few of which we collect below. A function $f\colon \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}^n$, is holomorphic on Ω if and only if

- it is, at any point $\vec{z}_0 \in \Omega$, equal to some convergent power series, *cf.* [12, Thm. 1.17],
- it holds that
$$\frac{\partial f}{\partial \overline{z_j}}(\vec{\xi}) = 0$$
for any index $j \in \{1, 2, \ldots, n\}$ and any point $\vec{\xi} \in \Omega$, *cf.* [7, Def. 2.1.1] and [12, pg. 5].

Additionally, in the case of one complex variable, one important characterizetion of holomorphicity is given by Morera's theorem, *cf.* [2, pg. 122] and [13, Thm. 10.17], which states that a continuous function $f\colon \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$, is holomorphic on Ω if and only if the integral

$$\oint_\Gamma f(z)\mathrm{d}z = 0,$$

for all coherently oriented triangles $\Gamma \subseteq \mathrm{int}(\Omega)$.

We also collect some of the most important implications of the concept of holomorphicity. For a holomorphic function $f\colon \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}^n$, it holds that:

- the functions $\mathrm{Re}[f]$ and $\mathrm{Im}[f]$ are pluriharmonic, *cf.* [5, Lem. 1.8.7],
- the function f is smooth and all of its partial derivatives are also holomorphic functions on Ω, *cf.* [12, Cor. 1.5],
- the function f, assuming it is non-constant, is an open mapping, *cf.* [12, Thm. 1.21].

Remark 2.6. One can say that the existence of a complex derivative is *infinitely* better than the existence of a real derivative. For a function $f\colon \mathbb{R} \to \mathbb{R}$, the existence of its real derivative f' yields nothing about higher order derivatives. However, as we have noted above, for a function $f\colon \mathbb{C} \to \mathbb{C}$, the existence of its complex derivative f' yields automatically the existence all higher order derivatives.

3 Herglotz-Nevanlinna functions

In this thesis, we will, for the most part, concern ourselves with the following class of holomorphic functions on the poly-upper half-plane $\mathbb{C}^{+n} := \{\vec{z} \in \mathbb{C}^n \mid \forall\, j = 1,2,\ldots,n : \mathrm{Im}[z_j] > 0\}$.

Definition 3.1. A function $q\colon \mathbb{C}^{+n} \to \mathbb{C}$ is called a *Herglotz-Nevanlinna function* if it is holomoprhic with non-negative imaginary part.

Example 3.2. Let $q(z) := z - \frac{1}{z+\mathrm{i}+1}$ for $z \in \mathbb{C}^+$. In order to show that q is a Herglotz-Nevanlinna function, we are required to check that it is holomoprhic on $\mathbb{C}^+$ and has non-negative imaginary part.

First, in order to check that the function q is holomorphic on $\mathbb{C}^+$, take $z_0 \in \mathbb{C}^+$ arbitrary and assume that $h \in \mathbb{C}$ is so small (in absolute value) so that $z_0 + h \in \mathbb{C}^+$. If that is the case, we calculate that

$$\begin{aligned}\lim_{h\to 0}\frac{q(z_0+h)-q(z_0)}{h} &= \lim_{h\to 0}\frac{1}{h}\left(z_0+h-\frac{1}{z_0+h+\mathrm{i}+1}-z_0+\frac{1}{z_0+\mathrm{i}+1}\right)\\ &= \lim_{h\to 0}\left(1+\frac{1}{(z_0+\mathrm{i}+1)(z_0+h+\mathrm{i}+1)}\right) = 1+\frac{1}{(z_0+\mathrm{i}+1)^2},\end{aligned}$$

yielding that the function q is, indeed, holomorphic on $\mathbb{C}^+$.

In order to check that also $\mathrm{Im}[q] \geq 0$, we write $z = x + \mathrm{i}\,y \in \mathbb{C}^+$ and calculate that

$$\begin{aligned}\mathrm{Im}[q(x+\mathrm{i}\,y)] &= \mathrm{Im}[x+\mathrm{i}\,y] - \mathrm{Im}\left[\frac{1}{x+\mathrm{i}\,y+\mathrm{i}+1}\right]\\ &= y - \frac{\mathrm{Im}[x+1-\mathrm{i}\,y-\mathrm{i}]}{|x+1+(y+1)\mathrm{i}|^2} = y + \frac{y+1}{(x+1)^2+(y+1)^2} \geq 0. \qquad (1)\end{aligned}$$

Thus, we have shown that the function q is, in fact, a Herglotz-Nevanlinna function. ◊

Example 3.3. Let $q(z_1,z_2) = \frac{z_2}{1-z_1 z_2}$. Firstly, if $(\xi_1,\xi_2) \in \mathbb{C}^{+2}$, then the functions

$$z \mapsto \frac{\xi_2}{1-z\xi_2} \quad \text{and} \quad z \mapsto \frac{z}{1-\xi_1 z}$$

are both holomorphic on $\mathbb{C}^+$, as they are quotients of holomorphic functions with no zeros in $\mathbb{C}^+$.

Furthermore, by writing $z_j = x_j + \mathrm{i}\,y_j$, we calculate directly that

$$\mathrm{Im}[q(x_1+\mathrm{i}\,y_1, x_2+\mathrm{i}\,y_2)] = \mathrm{Im}\left[\frac{x_2+\mathrm{i}\,y_2}{1-(x_1+\mathrm{i}\,y_1)(x_2+\mathrm{i}\,y_2)}\right]$$

$$= \frac{\mathrm{Im}[(x_2 + \mathrm{i}\, y_2)(1-(x_1 - \mathrm{i}\, y_1)(x_2 - \mathrm{i}\, y_2))]}{(1 - x_1 x_2 + y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2}$$

$$= \frac{y_1(x_2^2 + y_2^2) + y_2}{(1 - x_1 x_2 + y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} \geq 0.$$

Thus, we conclude that q is, indeed, a Herglotz-Nevanlinna function in two variables. $\Diamond$

The great power of Herglotz-Nevanlinna functions manifests itself via their different representations, which we now summarize.

3.1. Integral representation An integral representation of a function is a way of writing the value of a function at a certain point in terms of an integral expression and possibly some polynomial term. The integration area is usually the distinguished boundary of the domain of definition of the function, while the integrand often consist of a fixed function called the integration kernel, a collection of representation parameters and possibly some additional terms.

The most famous and fundamental integral representations are Cauchy's integral formula which holds for all holomorphic functions, *cf.* [2, Thm. 6, pg. 119], [12, Thm. 1.3] and [13, Thm. 10.15], and its generalization to $\mathcal{C}^1$-functions [7, Thm. 1.2.1]. In fact, Cauchy's formula serves as the basic ingredient for the following integral representation theorem for Herglotz-Nevanlinna functions in one variable, *cf.* [6, Thm. 2.2] and [8, pg. 2].

Theorem 3.4. *A holomorphic function $q\colon \mathbb{C}^+ \to \mathbb{C}$ is a Herglotz-Nevanlinna function if and only if it can be written, for any $z \in \mathbb{C}^+$, as*

$$q(z) = a + b\,z + \frac{1}{\pi}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) \mathrm{d}\mu(t), \tag{2}$$

where $a \in \mathbb{R}$, $b \geq 0$ and μ is a positive Borel measure on $\mathbb{R}$ satisfying the growth condition

$$\int_{\mathbb{R}} \frac{1}{1+t^2} \mathrm{d}\mu(t) < \infty.$$

Furthermore, it holds that the representing parameters a, b and μ are unique for a given function and can, in fact, be written explicitly in terms of the function q [6; 8]. More precisely, it holds that

$$a = \mathrm{Re}[q(\mathrm{i})], \tag{3}$$

the number b is given by the non-tangential limit of the function q at infinity, *i.e.*

$$b = \lim_{z \,\hat{\to}\, \infty} \frac{q(z)}{z}, \tag{4}$$

and the measure μ is given by the Stieltjes inversion formula, *i.e.* for any $\mathcal{C}^1$-function $\varphi\colon \mathbb{R} \to \mathbb{R}$ for which there exists a constant $C \in \mathbb{R}$ such that $|\varphi(x)| \leq C(1+x^2)^{-1}$, it holds that

$$\lim_{y \to 0^+} \int_{\mathbb{R}} \varphi(x) \mathrm{Im}[q(x + \mathrm{i}\, y)] \mathrm{d}x = \int_{\mathbb{R}} \varphi(t) \mathrm{d}\mu(t). \tag{5}$$

Integral representations of Herglotz-Nevanlinna functions in several variables are also known. The first results in this direction were obtained by Vladimirov in [14, Thm. 3.] and [15, Sec. 16.6]. Both results are considered in the more general setting of tubular domains over a cone, but neither result provides a characterization of Herglotz-Nevanlinna functions in several variables as is done by Theorem 3.4 for the case of Herglotz-Nevanlinna functions of one variable.

A complete characterization of Herglotz-Nevanlinna functions in several variables via a corresponding integral representation was first presented in [A1, Thm. 3.1] for the case of two complex variables, while the general case was first presented in [A2, Thm. 4.1].

3.2. Operator representation The concepts of Hilbert spaces, linear relations, self-adjointness and resolvents, see *e.g.* [4], are fundamental concepts in the field of functional analysis, and with the help of operator representations of holomorphic functions, we may connect the fields of complex analysis and functional analysis.

In the setting of Herglotz-Nevanlinna functions, an operator representation of a function is a way of writing the value of a function at a certain point primarily in terms of the resolvent of a self-adjoint linear relation on some Hilbert space, and possibly some constant term.

For Herglotz-Nevanlinna functions, the following operator representation theorem holds, *cf.* [10, Thm. 2.2] and [9, Sec. 1].

Theorem 3.5. *A holomorphic function $q\colon \mathbb{C}^+ \to \mathbb{C}$ is a Herglotz-Nevanlinna function if and only if there exists a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$, a self-adjoint linear relation A in $\mathcal{H}$, an element $v \in \mathcal{H}$ and a number $z_0 \in \mathbb{C}^+$ such that*

$$q(z) = \overline{q(z_0)} + (z - \overline{z_0})\langle (I + (z - z_0)(A - z)^{-1})v, v\rangle_{\mathcal{H}}.$$

Furthermore, Herglotz-Nevanlinna functions in one variable may also be characterized via the positive semi-definiteness of a certain kernel function, *cf.* [1, Thm. 1.3] and [11]. This result has been successfully extended to the case $n = 2$ [1, Thm. 1.5]. However, when considering Herglotz-Nevanlinna functions of three or more variables, one need an assumption on the growth of the function. This is made precise in [1, Thms. 1.6-1.9].

3.3. Representation via distributions A subclass of Herglotz-Nevanlinna functions in one variable that is of particular interest is the class of *symmetric Herglotz-Nevanlinna functions*. These are Herglotz-Nevanlinna functions satisfying the additional constraint that

$$-\overline{q(z)} = q(-\overline{z}) \tag{6}$$

for all $z \in \mathbb{C}^+$.

This has some immediate consequences with respect to Theorem 3.4. Firstly, the number a, which is given by equality (3), has to be zero, since the symmetry condition (6) implies that any symmetric Herglotz-Nevanlinna function takes only purely imaginary values along the imaginary axis. Furthermore, the symmetry condition (6) also implies that

$$\mathrm{Im}[q(x + \mathrm{i}\,y)] = \mathrm{Im}[q(-x + \mathrm{i}\,y)]$$

for any point $x + \mathrm{i}\,y \in \mathbb{C}^+$, yielding, via the Stieltjes inversion formula (5), that

$$\int_{\mathbb{R}} \varphi(t)\mathrm{d}\mu(t) = \int_{\mathbb{R}} \varphi(-t)\mathrm{d}\mu(t)$$

for any function φ as in formula (5). As such, we conclude that the representing measure of any symmetric Herglotz-Nevanlinna function is an even measure.

This information allows us to simplify representation (2) for symmetric Herglotz-Nevanlinna functions, yielding that

$$\begin{aligned} q(z) = b\,z + \lim_{R\to\infty} \frac{1}{\pi}\int_{-R}^{R} \frac{1}{t-z}\mathrm{d}\mu(t) &= b\,z + \lim_{R\to\infty} \frac{1}{\pi}\int_{-R}^{R} \frac{z}{t^2 - z^2}\mathrm{d}\mu(t) \\ &= b\,z - \frac{c}{z} + \frac{1}{\pi}\int_{(0,\infty)} \frac{2z}{t^2 - z^2}\mathrm{d}\mu(t), \end{aligned}$$

where the number b and measure μ are as in representation (2) and $c := \pi^{-1}\mu(\{0\})$.

This representation of symmetric Herglotz-Nevanlinna functions is related to a certain class of distributions using the following procedure, *cf.* [18, Sec. 10.5]. For any symmetric Herglotz-Nevanlinna function q, the function $s \mapsto -\mathrm{i}\,q(\mathrm{i}\,s)$ will be a will be a holomorphic function on the right half-plane with non-negative real part and taking only real values along the positive real line. Furthermore, its growth at ∞, away from the imaginary axis, will be, at most, linear due to formula (4). Therefore, we may consider the distribution

$$Y := \mathcal{L}^{-1}(s \mapsto -\mathrm{i}\,q(\mathrm{i}\,s)), \tag{7}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform, *cf.* [18, Sec. 8.4]. One can then show that this distribution Y will be a Schwartz distribution with $\operatorname{supp}(Y) \subseteq [0,\infty)$, satisfying the condition that

$$\operatorname{Re}\left[\int_{-\infty}^{t} \overline{\varphi(\tau)}(Y * \varphi)(\tau)\mathrm{d}\tau\right] \geq 0 \tag{8}$$

for all $t \in \mathbb{R}$ and all test functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R},\mathbb{C})$, *cf.* [18, Thm. 10.6-1]. In fact, the converse statement also holds, leading to the following theorem, *cf.* [18, Thm. 10.4-1] and [18, Thm. 10.6-1].

Theorem 3.6. *For any symmetric Herglotz-Nevanlinna function, the distribution Y defined by formula* (7) *is a Schwartz distribution with* $\operatorname{supp}(Y) \subseteq [0,\infty)$ *satisfying condition* (8). *Conversely, for any Schwartz distribution Y with* $\operatorname{supp}(Y) \subseteq [0,\infty)$ *satisfying condition* (8), *the function*

$$z \mapsto \mathrm{i}\,\mathcal{L}(Y)(-\mathrm{i}\,z)$$

will be a symmetric Herglotz-Nevanlinna function.

Distributions that satisfy different possible generalization of condition (8) have also been considered. The case when when the test function φ is vector-valued and Y is matrix with distributional entries has been considered *e.g.* in [15; 17]. On the other hand, one may preserve the forms of φ and Y, but change, instead, the right hand-side of condition (8). Such a generalization has recently been considered in [B1].

4 Overview of included publications

The main part of this thesis consists of the research articles A1 - A4 which investigates different aspects of the theory of Herglotz-Nevanlinna functions in several variables. Furthermore, the thesis is supplemented by three additional publications concerning Herglotz-Nevanlinna functions in one variable and related topics, namely the research articles B1 and B2 and the summary book-chapter C1. We now give a comprehensive overview of these publications.

A1: A characterization of Herglotz-Nevanlinna functions in two variables via integral representations

In article A1, we establish a characterization of Herglotz-Nevanlinna functions in two variables via an integral representation that is considered to be a two-variable analogue of representation (2), *cf.* [A1, Thm. 3.1]. In particular, we show that any Herglotz-Nevanlinna function in two variables can be uniquely represented by a real number a, two non-negative number b_1 and b_2 and a positive Borel measure μ satisfying two conditions.

Furthermore, we are able to conclude that the representing measure of a Herglotz-Nevanlinna function in two variables cannot be a finite unless it is trivial, *cf.* [A1, Prop. 4.3], and that any point in $\mathbb{R}^2$ is a zero-set of any representing measure, *cf.* [A1, Prop. 4.4].

In the published version of Article A1, the following typo has been discovered:

- In formula defining the kernel function K on the bottom of page 202, the term
$$\frac{\mathfrak{i}}{(1+t_1^2)(1+t_2^2)}$$
should be subtracted from the expression, not added.

A2: Herglotz-Nevanlinna functions in several variables

Article A2 investigates various aspects of Herglotz-Nevanlinna functions in several variables. Firstly, we extend the integral representation results that was obtain for the two-variable case in article A1 to the general case of Herglotz-Nevanlinna functions in several variables, *cf.* [A2, Thm. 4.1]. This gives a characterization of any Herglotz-Nevanlinna function in n variables in terms of a real number a, n non-negative numbers $b_1, b_2, \ldots, b_n$ and a positive Borel measure μ satisfying two conditions.

We are also able to establish the equivalence between four alternative descriptions of the class of representing measures, *cf.* [A2, Thm. 5.1], as well as investigate, in detail, the properties of the symmetric extension of a Herglotz-Nevanlinna function to $(\mathbb{C} \setminus \mathbb{R})^n$. More precisely, we investigate the symmetries of the kernel function K_n [A2, Prop. 6.1], of Cauchy-type functions [A2, Prop. 6.5], and of Herglotz-Nevanlinna functions in several variables [A2, Prop. 6.7]. We determine also the variable (non-)dependence of the symmetric extension of a Herglotz-Nevanlinna function, *cf.* [A2, Prop. 6.9].

Finally, we establish, contrary to the one-variable case, that the symmetric and analytic extensions of a Herglotz-Nevanlinna function in several variables cannot coincide unless its representing measure is trivial, *cf.* [A2, Cor. 6.10].

A3: A subclass of boundary measures and the convex combination problem for Herglotz-Nevanlinna functions in several variables

In article A3, we focus on the convex combination problem for Herglotz-Nevanlinna functions. This problem asks us to relate the representing parameters of two Herglotz-Nevanlinna functions q and $\widetilde{q}$, where q is a function of one variable and $\widetilde{q}$ a function of n variables, in the case where the function $\widetilde{q}$ can be obtained form the function q by replacing its input with a convex combination of n independent variables, *i.e.*

$$\widetilde{q}(z_1, z_2, \ldots, z_n) = q(k_1 z_1 + k_2 z_2 + \ldots + k_n z_n),$$

where $k_j > 0$ for all $j = 1, 2, \ldots, n$ and $k_1 + k_2 + \ldots + k_n = 1$, *cf.* [A3, Sec. 1].

In this direction, we first obtain a result that characterizes all representing measures of Herglotz-Nevanlinna functions in two variables of a particular form [A3, Thm. 3.3]. Afterwards, we provide a complete answer to the convex combination problem [A3, Thm. 4.2], with some special cases being stated separately, as they are of interest in their own right, *cf.* [A3, Cor. 4.3], [A3, Cor. 4.5] and [A3, Cor. 4.7]. The solution of the convex combination problem is dependent on a detailed study of the integration kernel K_n under a certain linear transformation, which is presented in [A3, Lem. 4.9], [A3, Lem. 4.10] and [A3, Lem. 4.11]. Additional corollaries of [A3, Thm. 4.2] describing an alternative form of the integral representation of the function $\widetilde{q}$, the properties of its representing measure and the solution to the convex combination problem where some (but not all) of the coefficients are trivial are formulated in [A3, Cor. 4.13], [A3, Cor. 4.14] and [A3, Cor. 4.15], respectively. Finally, the solution of the convex combination problem is extended to cover the the case of a general non-trivial non-negative linear combination in [A3, Cor. 5.1].

A4: Geometric properties of measures related to holomorphic functions having positive imaginary or real part

Article A4 primarily discusses the geometric properties of the class of representing measures of Herglotz-Nevanlinna functions, which we call *Nevanlinna measures*. One of the main results in this direction is the complete description of the restriction of any Nevanlinna measure to a coordinate-orthogonal hyperplane in $\mathbb{R}^n$ [A4, Thm. 3.4]. Furthermore, we show that a non-trivial Nevanlinna measure cannot be finite [A4, Prop. 3.3] and describe, in detail, how the integral representation of a Herglotz-Nevanlinna function changes should we remove the mass along a coordinate-orthogonal hyperplane from its representing measure [A4, Cor. 3.5]. Furthermore, we establish for all Nevanlinna measures that the mass of an affine subspace of codimenesion ≥ 2 is always trivial [A4, Cor. 3.7], as well us a particular relation between the variable-dependence of a Herglotz-Nevanlinna function and the mass of a coordinate orthogonal hyperplane of its representing measure [A4, Cor. 3.8].

The second and third main results are two theorems that identify certain subsets of $\mathbb{R}^n$ with the property that the only Nevanlinna measure whose support is contained in such a set is the trivial measure, *cf.* [A4, Thm. 3.10] and [A4, Thm. 3.16]. These results are further extended via applying particular biholomorphisms of the poly-upper half-plane, *cf.* [A4, Cor. 3.19] and [A4, Cor. 3.20].

The final main result is a theorem showing that the support of a non-trivial Nevanlinna measure must intersect the union of any collection of n coordinate-orthogonal strips [A4, Thm. 3.24], where n is the dimension of the ambient space.

Finally, we translate the aforementioned results for Nevanlinna measure to the case of measures with vanishing mixed Fourier coefficients on the unit poly-torus, *cf.* [A4, Thm. 4.1], [A4, Cor. 4.3], [A4, Cor. 4.4], [A4, Cor. 4.5] and [A4, Cor. 4.8].

B1: Holomorphic function representations of pseudo-passive causal operators of slow growth

Article B1 considers one possible generalization of the classical theory of passive operators, namely the titular class of pseudo-passive causal operators of slow growth [B1, Def. 2.8].

For every such operator, we show that the Laplace transform of its defining distribution exists and is a holomorphic function on the right half-plane of a vary particular type [B1, Thm. 3.2]. Furthermore, we also investigate the relations between the different components of the definition of pseudo-passive causal operators of slow growth [B1, Thm. 4.1]. Finally, a related class of operators is also considered, *cf.* [B1, Sec. 5].

B2: Quasi-Herglotz functions and convex optimization

Article B2 considers the class of quasi-Herglotz functions and its applications related to convex optimization.

The class of quasi Herglotz functions is defined as the real vector space consisting of all differences of Herglotz-Nevanlinna functions in one variable, *cf.* [B2, Sec. 2(b)]. Their integral representation and boundary values are discussed in [B2, Sec. 2(c)] and [B2, Sec. 2.(d)], respectively. An extension of the sum-rule identities for Herglotz-Nevanlinna functions, *cf.* [3, Thm. 4.1], to quasi-Herglotz functions is given by [B2, Thm. 3.1].

In [B2, Sec. 4], we discuss approximation and optimization theory based on quasi-Herglotz functions, while numerical examples that demonstrate the modelling of non-passive gain metamaterials is presented in [B2, Sec. 5].

C1: Herglotz functions and applications in electromagnetics

The book-chapter C1 gives an overview of some applications of Herglotz-Nevanlinna functions, there called *Herglotz functions*, in the field of electromagnetics.

Firstly, we give a summary of the basic properties of Herglotz-Nevanlinna functions in one variable [C1, Sec. 1.2] and review the connection between symmetric Herglotz-Nevanlinna functions and passive systems [C1, Sec. 1.3]. Afterwards, in [C1, Sec. 1.4], a summary of the applications of the, so-called, sum-rule identities for Herglotz-Nevanlinna functions for deriving physical bounds on passive systems is presented. Finally, it is reviewed how convex optimization based on the integral representation (2) of Herglotz-Nevanlinna functions can be used to identify and approximate passive systems [C1, Sec. 1.5].

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