Around minimal Hilbert series problems for graded algebras

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Abstract

The Hilbert series of a graded algebra is an invariant that encodes the dimension of the algebra's graded components. It can be seen as a tool for measuring the size of a graded algebra. This gives rise to the idea of algebras with a "minimal Hilbert series", among the algebras within a certain family.

Let $A$ be a graded algebra defined as the quotient of a polynomial ring by a homogeneous ideal. We say that $A$ has the strong Lefschetz property if there is a linear form $L$ such that multiplication by any power of $L$ has maximal rank. Equivalently, the quotient of $A/(L^d)$ should have the smallest possible Hilbert series, for all $d$. According to a result by Richard P. Stanley from 1980, every monomial complete intersection in characteristic zero has the strong Lefschetz property. In the first and second paper of this thesis we study the analogue problem for positive characteristic. The main results of the two papers, combined with previous results by David Cook II, gives a complete classification of the monomial complete intersections in positive characteristic with the strong Lefschetz property.

In 1985 Ralf Fröberg conjectured a formula for the minimal Hilbert series of a polynomial ring modulo an ideal generated by homogeneous polynomials, given the number of variables, the number of generators of the ideal and their degrees. The conjecture remains an open problem, although it has been proved in a few cases. The questions studied in the third and fourth paper are inspired by this conjecture. In the third paper we search for the minimal Hilbert series of the quotient of an exterior algebra by a principal ideal. If the principal ideal is generated by an element of even degree, the Hilbert series is known by a result of Guillermo Moreno-Socías and Jan Snellman from 2002. In the third paper we give a lower bound for the series, in the case the generator has odd degree.

Instead of defining our algebra as a quotient, we may consider the subalgebra generated by certain elements. Given positive numbers $u$ and $d$, which set of $u$ homogeneous polynomials of degree $d$ generates a subalgebra with minimal Hilbert series? This problem was suggested by Mats Boij and Aldo Conca in a paper from 2018. In the fourth paper we focus on the first nontrivial case, which is subalgebras generated by elements of degree two. We conjecture that an algebra with minimal Hilbert series is generated by an initial segment in the lexicographic or reverse lexicographic monomial ordering.

In the fifth paper we shift focus from Hilbert series to another invariant, namely the Betti numbers. The object of study are ideals $I$ with the property that all powers $I^k$ have a linear resolution. Such ideals are said to have linear powers. The main result is that the Betti numbers of $A/I^k$, if $I$ is an ideal with linear powers, satisfy certain linear relations. When $A/I$ has low Krull dimension, little extra information is needed in order to compute the Betti numbers explicitly.

Keywords: Graded algebras, Hilbert series, Lefschetz properties, Exterior algebra, Linear resolution.