

Eigenvalue variation under moving mixed Dirichlet–Neumann boundary conditions and applications

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Abstract

We deal with the sharp asymptotic behaviour of eigenvalues of elliptic operators with varying mixed Dirichlet–Neumann boundary conditions. In case of simple eigenvalues, we compute explicitly the constant appearing in front of the expansion’s leading term. This allows inferring some remarkable consequences for Aharonov–Bohm eigenvalues when the singular part of the operator has two coalescing poles.

Keywords. Mixed boundary conditions, asymptotics of eigenvalues, Aharonov–Bohm eigenvalues.

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1 Introduction and main results

The present paper deals with elliptic operators with varying mixed Dirichlet–Neumann boundary conditions and their spectral stability under varying of the Dirichlet and Neumann boundary regions. More precisely, we study the behaviour of eigenvalues under a homogeneous Neumann condition on a portion of the boundary concentrating at a point and a homogeneous Dirichlet boundary condition on the complement.

Let Ω be a bounded open set in \mathbb{R}^2 having the following properties:

$$\Omega \text{ is Lipschitz,} \tag{1}$$

$$\text{there exists } \varepsilon_0 > 0 \text{ such that } \Gamma_{\varepsilon_0} := [-\varepsilon_0, \varepsilon_0] \times \{0\} \subset \partial\Omega. \tag{2}$$

We consider the eigenvalue problem for the Dirichlet Laplacian on the domain Ω

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We denote by $(\lambda_j)_{j \geq 1}$ the eigenvalues of Problem (3), arranged in non-decreasing order and counted with multiplicities.

For each $\varepsilon \in (0, \varepsilon_0]$, we also consider the following eigenvalue problem with mixed boundary conditions:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \setminus \Gamma_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_\varepsilon, \end{cases} \tag{4}$$

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with $\Gamma_\varepsilon := [-\varepsilon, \varepsilon] \times \{0\}$. We denote by $(\lambda_j(\varepsilon))_{j \geq 1}$ the eigenvalues of Problem (4), arranged in non-decreasing order and counted with multiplicities.

A rigorous weak formulation of the eigenvalue problems described above can be given as follows. For $\varepsilon \in (0, \varepsilon_0]$, we define

$$\mathcal{Q}_\varepsilon = \{u \in H^1(\Omega); \chi_{\partial\Omega \setminus \Gamma_\varepsilon} \gamma_0 u = 0 \text{ in } L^2(\partial\Omega)\},$$

where γ_0 is the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$, which is a continuous linear mapping (see for instance [21, Definition 13.2]) and $\chi_{\partial\Omega \setminus \Gamma_\varepsilon}$ is the indicator function of $\partial\Omega \setminus \Gamma_\varepsilon$ in $\partial\Omega$. Furthermore, we define the quadratic form q on $H^1(\Omega)$ by

$$q(u) := \int_{\Omega} |\nabla u|^2 dx. \quad (5)$$

Let us denote by q_0 the restriction of q to $H_0^1(\Omega)$ and by q_ε the restriction of q to \mathcal{Q}_ε . The sequences $(\lambda_j)_{j \geq 1}$ and $(\lambda_j(\varepsilon))_{j \geq 1}$ for $\varepsilon \in (0, \varepsilon_0]$ can then be defined by the min-max principle:

$$\lambda_j := \min_{\substack{\mathcal{E} \subset H_0^1(\Omega) \\ \dim(\mathcal{E})=j}} \max_{u \in \mathcal{E}} \frac{q(u)}{\|u\|^2} \quad (6)$$

and

$$\lambda_j(\varepsilon) := \min_{\substack{\mathcal{E} \subset \mathcal{Q}_\varepsilon \\ \dim(\mathcal{E})=j}} \max_{u \in \mathcal{E}} \frac{q(u)}{\|u\|^2}, \quad (7)$$

where

$$\|u\|^2 = \int_{\Omega} u^2(x) dx.$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ (see e.g. [21, Lemma 18.4]), the eigenvalues of q_0 , defined by Equation (6), and those of q_ε , defined by Equation (7), are of finite multiplicity, and form sequences tending to $+\infty$.

Remark 1.1. *Let us note that, absent any indication to the contrary, we denote by $L^2(\Omega)$, $H^1(\Omega)$ and similar notation the corresponding real Hilbert spaces of real-valued functions. In the present paper, complex valued functions only occur in connection with Aharonov-Bohm eigenvalues, in Sections 1.1 and 3.*

Remark 1.2. *Let us fix ε_1 and ε_2 in $(0, \varepsilon_0]$ such that $\varepsilon_1 > \varepsilon_2$. Since $H_0^1(\Omega) \subset \mathcal{Q}_{\varepsilon_2} \subset \mathcal{Q}_{\varepsilon_1}$, the definitions given by Formulas (6) and (7) immediately imply that $\lambda_j(\varepsilon_1) \leq \lambda_j(\varepsilon_2) \leq \lambda_j$ for each integer $j \geq 1$. The function $(0, \varepsilon_0] \ni \varepsilon \mapsto \lambda_j(\varepsilon)$ is therefore non-increasing and bounded by λ_j for each integer $j \geq 1$.*

Remark 1.3. *For the sake of simplicity, in the present paper we assume that the domain Ω satisfies assumption (2), i.e. that $\partial\Omega$ is straight in a neighborhood of 0. We observe that, since we are in dimension 2, this assumption is not restrictive. Indeed, starting from a general sufficiently regular domain Ω , a conformal transformation leads us to consider a new domain satisfying (2), see e.g. [10]. The counterpart is the appearance of a conformal weight in the new eigenvalue problem; however, if Ω is sufficiently regular, the weighted problem presents no additional difficulties.*

The purpose of the present paper is to study the eigenvalue function $\varepsilon \mapsto \lambda_j(\varepsilon)$ as $\varepsilon \rightarrow 0^+$. The continuity of this map as well as some asymptotic expansions were obtained in [12]. Here we mean to provide some explicit characterization of the leading terms in the expansion given in [12] and of the limit profiles arising from blowing-up of eigenfunctions.

Spectral stability and asymptotic expansion of the eigenvalue variation in a somehow complementary setting were obtained in [3]; indeed, if we consider the eigenvalue problem under homogeneous Dirichlet boundary conditions on a vanishing portion of a straight part of the boundary, Neumann conditions on the complement in the straight part and Dirichlet conditions elsewhere,

by a reflection the problem becomes equivalent to the one studied in [3], i.e. a Dirichlet eigenvalue problem in a domain with a small segment removed.

Related spectral stability results were discussed in [8, Section 4] for the first eigenvalue under mixed Dirichlet-Neumann boundary conditions on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 3$), both for vanishing Dirichlet boundary portion and for vanishing Neumann boundary portion.

We also mention that some regularity results for solutions to second-order elliptic problems with mixed Dirichlet-Neumann type boundary conditions were obtained in [13, 20], see also the references therein, whereas asymptotic expansions at Dirichlet-Neumann boundary junctions were derived in [10].

Let us assume that

$$\lambda_N \text{ (i.e. the } N\text{-th eigenvalue of } q_0\text{) is simple.} \quad (8)$$

Let u_N be a normalized eigenfunction associated to λ_N , i.e. u_N satisfies

$$\begin{cases} -\Delta u_N = \lambda_N u_N, & \text{in } \Omega, \\ u_N = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} u_N^2(x) dx = 1. \end{cases} \quad (9)$$

It is known (see [12]) that, under assumption (8), the rate of the convergence $\lambda_N^\varepsilon \rightarrow \lambda_N$ is strongly related to the order of vanishing of the Dirichlet eigenfunction u_N at 0. Moreover u_N has an integer order of vanishing $k \geq 1$ at $0 \in \partial\Omega$ and there exists $\beta \in \mathbb{R} \setminus \{0\}$ such that

$$r^{-k} u_N(r \cos t, r \sin t) \rightarrow \beta \sin(kt) \text{ as } r \rightarrow 0 \text{ in } C^{1,\tau}([0, \pi]) \quad (10)$$

for any $\tau \in (0, 1)$, see e.g. [9, Theorem 1.1].

Our main results provide sharp asymptotic estimates with explicit coefficients for the eigenvalue variation $\lambda_N - \lambda_N(\varepsilon)$ as $\varepsilon \rightarrow 0^+$ under assumption (8) (Theorem 1.4), as well as an explicit representation in elliptic coordinates of the limit blow-up profile for the corresponding eigenfunction u_N^ε (Theorem 1.5).

Theorem 1.4. *Let Ω be a bounded open set in \mathbb{R}^2 satisfying (1) and (2). Let $N \geq 1$ be such that the N -th eigenvalue λ_N of q_0 on Ω is simple with associated eigenfunctions having in 0 a zero of order k with k as in (10). For $\varepsilon \in (0, \varepsilon_0)$, let $\lambda_N(\varepsilon)$ be the N -th eigenvalue of q_ε on Ω . Then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\lambda_N - \lambda_N(\varepsilon)}{\varepsilon^{2k}} = \beta^2 \frac{k\pi}{2^{2k-1}} \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor}^2$$

with $\beta \neq 0$ being as in (10).

Theorem 1.5. *Let Ω be a bounded open set in \mathbb{R}^2 satisfying (1) and (2). Let $N \geq 1$ be such that the N -th eigenvalue λ_N of q_0 on Ω is simple with associated eigenfunctions having in 0 a zero of order k with k as in (10). For $\varepsilon \in [0, \varepsilon_0)$, let $\lambda_N(\varepsilon)$ be the N -th eigenvalue of q_ε on Ω and u_N^ε be an associated eigenfunction satisfying $\int_{\Omega} |u_N^\varepsilon|^2 dx = 1$ and $\int_{\Omega} u_N^\varepsilon u_N dx \geq 0$. Then*

$$\varepsilon^{-k} u_N^\varepsilon(\varepsilon x) \rightarrow \beta(\psi_k + W_k \circ F^{-1}) \text{ as } \varepsilon \rightarrow 0^+$$

in $H_{\text{loc}}^1(\overline{\mathbb{R}_+^2})$, a.e. and in $C_{\text{loc}}^2(\overline{\mathbb{R}_+^2} \setminus \{(1,0), (-1,0)\})$, where β is as in (10),

$$\psi_k(r \cos t, r \sin t) = r^k \sin(kt), \quad \text{for } t \in [0, \pi] \text{ and } r > 0, \quad (11)$$

$$F(\xi, \eta) = (\cosh(\xi) \cos(\eta), \sinh(\xi) \sin(\eta)), \quad \text{for } \xi \geq 0, \eta \in [0, 2\pi), \quad (12)$$

and

$$W_k(\xi, \eta) = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{j} \exp(-(k-2j)\xi) \sin((k-2j)\eta). \quad (13)$$

Actually, the fact that $\lim_{\varepsilon \rightarrow 0^+} \frac{\lambda_N - \lambda_N(\varepsilon)}{\varepsilon^{2k}}$ is finite and different from zero and the convergence of $\varepsilon^{-k} u_N^\varepsilon(\varepsilon x)$ to some nontrivial profile was proved in the paper [12], with a quite implicit description of the limits. The original contribution of the present paper relies in the explicit characterization of the leading term of the expansion provided by [12] and in its applications to Aharonov–Bohm operators, see Section 1.1. The key tool allowing us to write explicitly the coefficients of the expansion consists in the use of elliptic coordinates, which turn out to be more suitable to our problem than radial ones, see Section 2.

1.1 Applications to Aharonov–Bohm operators

The present work is in part motivated by the study of Aharonov–Bohm eigenvalues. Let us review their definition and some previous results. For any point $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, we define the Aharonov–Bohm potential of circulation $1/2$ by

$$\mathbf{A}_{\mathbf{a}}(x) = \frac{1}{2} \left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right).$$

Let us consider an open and bounded open set $\widehat{\Omega}$ with Lipschitz boundary, such that $0 \in \widehat{\Omega}$. For better readability, we denote by \mathcal{H} the complex Hilbert space of complex-valued functions $L^2(\widehat{\Omega}, \mathbb{C})$, equipped with the scalar product defined, for all $u, v \in \mathcal{H}$, by

$$\langle u, v \rangle := \int_{\widehat{\Omega}} u \bar{v} dx.$$

We define, for $\mathbf{a} \in \widehat{\Omega}$,

$$\mathcal{Q}_{\mathbf{a}}^{AB} := \left\{ u \in H_0^1(\widehat{\Omega}, \mathbb{C}) ; \frac{|u|}{|x - \mathbf{a}|} \in L^2(\widehat{\Omega}) \right\}, \quad (14)$$

the quadratic form $q_{\mathbf{a}}^{AB}$ on $\mathcal{Q}_{\mathbf{a}}^{AB}$ by

$$q_{\mathbf{a}}^{AB}(u) := \int_{\widehat{\Omega}} |(i\nabla + \mathbf{A}_{\mathbf{a}})u|^2 dx, \quad (15)$$

and the sequence of eigenvalues $(\lambda_j^{AB}(\mathbf{a}))_{j \geq 1}$ by the min-max principle

$$\lambda_j^{AB}(\mathbf{a}) := \min_{\substack{\mathcal{E} \subset \mathcal{Q}_{\mathbf{a}}^{AB} \\ \dim(\mathcal{E})=j}} \max_{u \in \mathcal{E}} \frac{q_{\mathbf{a}}^{AB}(u)}{\|u\|^2}. \quad (16)$$

It follows from the definition in Equation (14) that $\mathcal{Q}_{\mathbf{a}}^{AB}$ is compactly embedded in \mathcal{H} . The above eigenvalues are therefore of finite multiplicity and $\lambda_j^{AB}(\mathbf{a}) \rightarrow +\infty$ as $j \rightarrow +\infty$.

Remark 1.6. *Let us note that, as shown in [16, Lemma 2.1], $\mathcal{Q}_{\mathbf{a}}^{AB}$ is the completion of the set of smooth functions supported in $\widehat{\Omega} \setminus \{\mathbf{a}\}$ for the norm $\|\cdot\|_{\mathbf{a}}$ defined by*

$$\|u\|_{\mathbf{a}}^2 = \|u\|^2 + q_{\mathbf{a}}^{AB}(u).$$

Let us point out that functions in $\mathcal{Q}_{\mathbf{a}}^{AB}$ satisfy a Dirichlet boundary condition, which is not the case in [16]. However, this difference is unimportant for the compact embedding.

Remark 1.7. *We could also consider the Friedrichs extension of the differential operator*

$$(i\nabla + \mathbf{A}_{\mathbf{a}})^*(i\nabla + \mathbf{A}_{\mathbf{a}})u = -\Delta u + 2i\mathbf{A}_{\mathbf{a}} \cdot \nabla u + |\mathbf{A}_{\mathbf{a}}|^2 u$$

acting on functions $u \in C_c^\infty(\widehat{\Omega} \setminus \{\mathbf{a}\}, \mathbb{C})$. As shown for instance in [15, Section I] or [7, Section 2]), this defines a positive and self-adjoint operator with compact resolvent, whose eigenvalues, counted with multiplicities, are $(\lambda_j^{AB}(\mathbf{a}))_{j \geq 1}$. It is called the Aharonov–Bohm operator of pole \mathbf{a} and circulation $1/2$.

In recent years, several authors have studied the dependence of eigenvalues on the position of the pole. It has been established in [7, Theorem 1.1], that the functions $\mathbf{a} \mapsto \lambda_j^{AB}(\mathbf{a})$ are continuous in $\bar{\Omega}$. In [1, 2], the two first authors obtained the precise rate of convergence $\lambda_j^{AB}(\mathbf{a}) \rightarrow \lambda_j^{AB}(0)$ as \mathbf{a} converges to the interior point 0 for simple eigenvalues. In order to state the most complete result, given in [2, Theorem 1.2], we consider an L^2 -normalized eigenfunction u_N^0 of q_0^{AB} associated with the eigenvalue $\lambda_N^{AB}(0)$. We additionally assume that $\lambda_N^{AB}(0)$ is simple. From [11, Section 7] it follows that there exists an odd positive integer k and a non-zero complex number β_0 such that, up to a rotation of the coordinate axes,

$$r^{-\frac{k}{2}} u_N^0(r \cos t, r \sin t) \rightarrow \beta_0 e^{i\frac{k}{2}t} \sin\left(\frac{k}{2}t\right) \text{ in } C^{1,\tau}([0, 2\pi], \mathbb{C})$$

as $r \rightarrow 0^+$, for all $\tau \in (0, 1)$. The integer k has a simple geometric interpretation: it is the number of nodal lines of the function u_N^0 which meet at 0. We say that u_N^0 has a zero of order $k/2$ in 0. Our coordinate axes are chosen in such a way that one of these nodal lines is tangent to the positive x_1 semi-axis.

Theorem 1.8. *Let us define $\mathbf{a}_\varepsilon := \varepsilon(\cos(\alpha), \sin(\alpha))$, with $\varepsilon > 0$. We have, as $\varepsilon \rightarrow 0^+$,*

$$\lambda_N^{AB}(\mathbf{a}_\varepsilon) = \lambda_N^{AB}(0) - \frac{k\pi\beta_0^2}{2^{2k-1}} \left(\frac{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2 \cos(k\alpha)\varepsilon^k + o(\varepsilon^k).$$

Remark 1.9. *The expansion in [1, 2] involves a constant depending on k , defined as the minimal energy in a Dirichlet-type problem. We compute this constant in Appendix A in order to obtain the more explicit result in Theorem 1.8.*

Let us now consider, for any $\varepsilon > 0$, an Aharonov-Bohm potential with two poles $(\varepsilon, 0)$ and $(-\varepsilon, 0)$, of fluxes respectively $1/2$ and $-1/2$:

$$\mathbf{A}_\varepsilon := \mathbf{A}_{(\varepsilon,0)} - \mathbf{A}_{(-\varepsilon,0)}.$$

As in the case of one pole, we define the vector space $\mathcal{Q}_\varepsilon^{AB}$ by

$$\mathcal{Q}_\varepsilon^{AB} := \left\{ u \in H_0^1(\hat{\Omega}, \mathbb{C}) ; \frac{|u|}{|x \pm \varepsilon \mathbf{e}|} \in L^2(\hat{\Omega}) \right\}, \quad (17)$$

where $\mathbf{e} = (1, 0)$, the quadratic form q_ε^{AB} on $\mathcal{Q}_\varepsilon^{AB}$ by

$$q_\varepsilon^{AB}(u) := \int_{\hat{\Omega}} |(i\nabla + \mathbf{A}_\varepsilon)u|^2 dx, \quad (18)$$

and the sequence of eigenvalue $(\lambda_j^{AB}(\varepsilon))_{j \geq 1}$ by the min-max principle

$$\lambda_j^{AB}(\varepsilon) := \min_{\substack{\mathcal{E} \subset \mathcal{Q}_\varepsilon^{AB} \\ \dim(\mathcal{E})=j}} \max_{u \in \mathcal{E}} \frac{q_\varepsilon^{AB}(u)}{\|u\|^2}. \quad (19)$$

It follows from [15, Corollary 3.5] that, for any $j \geq 1$, $\lambda_j^{AB}(\varepsilon)$ converges to the j -th eigenvalue of the Laplacian in $\hat{\Omega}$ as $\varepsilon \rightarrow 0^+$. In [4, 3] the authors obtained in some cases a sharp rate of convergence. In order to state the result, let us introduce some notation. We denote by \hat{q}_0 the quadratic form on $H_0^1(\hat{\Omega})$ defined by Equation (5), replacing Ω with $\hat{\Omega}$, and we denote by $(\hat{\lambda}_j)_{j \geq 1}$ the sequence of eigenvalues defined by Equation (6), replacing Ω with $\hat{\Omega}$ and q with \hat{q}_0 . We fix an integer $N \geq 1$ and assume that $\hat{\lambda}_N$ is a simple eigenvalue. We denote by \hat{u}_N an associated eigenfunction, normalized in $L^2(\hat{\Omega})$.

Theorem 1.10. [4, Theorem 1.2] If $\widehat{u}_N(0) \neq 0$, we have, as $\varepsilon \rightarrow 0$,

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{2\pi}{|\log(\varepsilon)|} \widehat{u}_N^2(0) + o\left(\frac{1}{|\log(\varepsilon)|}\right).$$

In the case $\widehat{u}_N(0) = 0$, it is well known that there exist $k \in \mathbb{N} \setminus \{0\}$, $\widehat{\beta} \in \mathbb{R} \setminus \{0\}$ and $\alpha \in [0, \pi)$ such that

$$r^{-k} \widehat{u}_N(r \cos t, r \sin t) \rightarrow \widehat{\beta} \sin(\alpha - kt) \text{ in } C^{1,\tau}([0, 2\pi], \mathbb{C})$$

as $r \rightarrow 0^+$ for all $\tau \in (0, 1)$. In particular, there is a nodal line whose tangent makes the angle α/k with the positive x_1 semi-axis.

Let us assume that

$$\widehat{\Omega} \text{ is symmetric with respect to the } x_1\text{-axis.}$$

Since $\widehat{\lambda}_N$ is simple, \widehat{u}_N is either even or odd in the variable x_2 and α is either $\pi/2$ or 0 accordingly.

Theorem 1.11. [3, Theorem 1.16] If \widehat{u}_N is even in x_2 , which corresponds to $\alpha = \pi/2$, we have, as $\varepsilon \rightarrow 0^+$,

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{k\pi\widehat{\beta}^2}{4^{k-1}} \left(\frac{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2 \varepsilon^{2k} + o(\varepsilon^{2k}).$$

Remark 1.12. The statements in [3] contain a constant C_k which we put in a simpler form in Appendix A, in order to obtain Theorem 1.11.

As a corollary of Theorem 1.4, we prove in Section 3 the following result, which complements the previous theorem.

Theorem 1.13. If \widehat{u}_N is odd in x_2 , which corresponds to $\alpha = 0$, we have, as $\varepsilon \rightarrow 0^+$,

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N - \frac{k\pi\widehat{\beta}^2}{4^{k-1}} \left(\frac{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2 \varepsilon^{2k} + o(\varepsilon^{2k}).$$

Remark 1.14. As discussed in Section 3, the assumption that $\widehat{\lambda}_N$ is simple can be slightly relaxed, admitting, in some cases, also double eigenvalues.

2 Sharp asymptotics for the eigenvalue variation

As already mentioned in the introduction, some asymptotic expansions for the eigenvalue variation $\lambda_N - \lambda_N(\varepsilon)$ were derived in [12]. Let us first recall the results from [12] which are the starting of our analysis.

Let $s := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geq 1 \text{ or } x_1 \leq -1\}$. We denote as \mathcal{Q} the completion of $C_c^\infty(\mathbb{R}_+^2 \setminus s)$ under the norm $(\int_{\mathbb{R}_+^2} |\nabla u|^2 dx)^{1/2}$. From the Hardy type inequality proved in [14] and a change of gauge, it follows that functions in \mathcal{Q} satisfy the Hardy type inequalities

$$\frac{1}{4} \int_{\mathbb{R}_+^2} \frac{|\varphi(x)|^2}{|x - \mathbf{e}|^2} dx \leq \int_{\mathbb{R}_+^2} |\nabla \varphi(x)|^2 dx, \quad \text{for all } \varphi \in \mathcal{Q}, \quad (20)$$

and

$$\frac{1}{4} \int_{\mathbb{R}_+^2} \frac{|\varphi(x)|^2}{|x + \mathbf{e}|^2} dx \leq \int_{\mathbb{R}_+^2} |\nabla \varphi(x)|^2 dx, \quad \text{for all } \varphi \in \mathcal{Q}, \quad (21)$$

where $\mathbf{e} = (1, 0)$. Inequalities (20) and (21) allow characterizing \mathcal{Q} as the following concrete functional space:

$$\mathcal{Q} = \left\{ u \in L_{\text{loc}}^1(\mathbb{R}_+^2) : \nabla u \in L^2(\mathbb{R}_+^2), \frac{u}{|x \pm \mathbf{e}|} \in L^2(\mathbb{R}_+^2), \text{ and } u = 0 \text{ on } s \right\}.$$

We refer to the paper [12], where the following theorem can be found as a particular case of more general results.

Theorem 2.1 ([12]). *Let Ω be a bounded open set in \mathbb{R}^2 satisfying (1) and (2). Let $N \geq 1$ be such that the N -th eigenvalue λ_N of q_0 on Ω is simple with associated eigenfunction u_N having in 0 a zero of order k with k as in (10). For $\varepsilon \in (0, \varepsilon_0)$, let $\lambda_N(\varepsilon)$ be the N -th eigenvalue of q_ε on Ω and u_N^ε be its associated eigenfunction, normalized to satisfy $\int_\Omega |u_N^\varepsilon|^2 dx = 1$ and $\int_\Omega u_N^\varepsilon u_N dx \geq 0$. Then, as $\varepsilon \rightarrow 0^+$,*

$$\frac{\lambda_N - \lambda_N(\varepsilon)}{\varepsilon^{2k}} \rightarrow -\beta^2 \int_{-1}^1 \frac{\partial w_k}{\partial x_2} w_k dx_1, \quad (22)$$

$$\varepsilon^{-k} u_N^\varepsilon(\varepsilon x) \rightarrow \beta(\psi_k + w_k) \quad \text{in } H_{\text{loc}}^1(\overline{\mathbb{R}_+^2}), \quad (23)$$

with $\beta \neq 0$ being as in (10), ψ_k being defined in (11), and w_k being the unique \mathcal{Q} -weak solution to the problem

$$\begin{cases} -\Delta w_k = 0, & \text{in } \mathbb{R}_+^2, \\ w_k = 0, & \text{on } s, \\ \frac{\partial w_k}{\partial \nu} = -\frac{\partial \psi_k}{\partial \nu}, & \text{on } \Gamma_1. \end{cases} \quad (24)$$

Convergence (22) can be obtained combining [12, Equation (4.6)] for simple eigenvalues, [12, Equation (3.4)] together with [12, Lemma 3.3]. As well, (23) is given by [12, Equation (2.3)], which is a consequence of [12, Theorem 5.2], [12, Equation (4.10)], [12, Lemma 3.3].

We remark that in [12] the author describes the limit profile w_k solving (24) with polar coordinates. On the contrary, our contribution relies essentially on the use of elliptic coordinates in place of polar ones. This allows us to compute explicitly the right hand side of (22), thus obtaining the following result.

Proposition 2.2. *For any positive integer k ,*

$$\int_{-1}^1 \frac{\partial w_k}{\partial x_2} w_k dx_1 = -\frac{k\pi}{2^{2k-1}} \left(\frac{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2.$$

The proof of Proposition 2.2 relies in an explicit construction of the limit profile w_k , using a parametrization of the upper half-plane \mathbb{R}_+^2 by elliptic coordinates, a finite trigonometric expansion, and the simplification of a sum involving binomial coefficients.

2.1 Computation of the limit profile w_k

Let us first compute w_k . By uniqueness, any function in the functional space \mathcal{Q} that satisfies all the conditions of Problem (24) is equal to w_k . In order to find such a function, we use the elliptic coordinates (ξ, η) defined by

$$\begin{cases} x_1 = \cosh(\xi) \cos(\eta), \\ x_2 = \sinh(\xi) \sin(\eta). \end{cases} \quad (25)$$

More precisely, we consider the function $F : (\xi, \eta) \mapsto (x_1, x_2)$ defined by the equations (25). It is a C^∞ diffeomorphism from $D := (0, +\infty) \times (0, \pi)$ to \mathbb{R}_+^2 . We note that F is actually a conformal mapping. Indeed, if we define the complex variables $z := x_1 + ix_2$ and $\zeta := \xi + i\eta$, we have $z = \cosh(\zeta)$, which proves the claim since \cosh is an entire function. Let us denote by $h(\xi, \eta)$ the scale factor associated with F , expressed in elliptic coordinates. We have

$$h(\xi, \eta) = |\cosh'(\zeta)| = |\sinh(\zeta)| = |\sinh(\xi) \cos(\eta) + i \cosh(\xi) \sin(\eta)| = \sqrt{\cosh^2(\xi) - \cos^2(\eta)}.$$

For any function $u \in \mathcal{H}$, let us define $U := u \circ F$. From the fact the F is conformal, it follows that $|\nabla U|$ is in $L^2(D)$ with

$$\int_D |\nabla U|^2 d\xi d\eta = \int_{\mathbb{R}_+^2} |\nabla u|^2 dx.$$

We also have

$$\frac{\partial u}{\partial \nu}(x) = -\frac{1}{h(0, \eta)} \frac{\partial U}{\partial \xi}(0, \eta) \quad (26)$$

for any $x \in \Gamma_1$, where $\eta \in (0, \pi)$ satisfies $x = F(0, \eta) = (\cos(\eta), 0)$. Furthermore, U is harmonic in D if, and only if, u is harmonic in \mathbb{R}_+^2 .

We now give an explicit formula for $w_k \circ F$.

Proposition 2.3. *For any positive integer k , $w_k \circ F = W_k$, where W_k is defined in (13).*

Proof. Let us begin by computing the function $\Psi_k := \psi_k \circ F$. We have $\psi_k(x) = \text{Im}(z^k)$, so that $\Psi_k(\xi, \eta) = \text{Im}((\cosh(\zeta))^k)$, where the complex variables z and ζ are defined as above. Using the binomial theorem, we find

$$\Psi_k(\xi, \eta) = \text{Im} \left(\frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} e^{(k-2j)\zeta} \right) = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} e^{(k-2j)\xi} \sin((k-2j)\eta).$$

This can be written

$$\Psi_k(\xi, \eta) = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{j} \sinh((k-2j)\xi) \sin((k-2j)\eta)$$

by grouping terms of the sum in pairs, starting from opposite extremities. In particular, for all $\eta \in (0, \pi)$,

$$\frac{\partial \Psi_k}{\partial \xi}(0, \eta) = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j} \sin((k-2j)\eta).$$

We now define

$$V(\xi, \eta) = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{j} e^{-(k-2j)\xi} \sin((k-2j)\eta).$$

The function $|\nabla V|$ is in $L^2(D)$ and, for all $\eta \in (0, \pi)$,

$$\frac{\partial V}{\partial \xi}(0, \eta) = -\frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j} \sin((k-2j)\eta).$$

Additionally, V vanishes on half-lines defined by $\eta = 0$ and $\eta = \pi$, which are the lower and upper boundary of D , respectively, and are mapped to $\mathbb{R} \times \{0\} \setminus \Gamma_1$ by F . It can be checked directly that $V \circ F^{-1} \in \mathcal{Q}$. Finally, V is harmonic in D , since it is a linear combination of functions of the type $(\xi, \eta) \mapsto e^{\pm n\xi} e^{\pm in\eta}$, which are harmonic. We conclude that $V \circ F^{-1}$ is a solution of Problem (24), and therefore $V = w_k \circ F$ by uniqueness. \square

Proof of Theorem 1.5. Theorem 1.5 follows combining Theorem 2.1 and Proposition 2.3. \square

Corollary 2.4. *For any positive integer $k \geq 1$,*

$$\int_{-1}^1 \frac{\partial w_k}{\partial x_2} w_k dx_2 = -\frac{\pi}{2^{2k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j}^2. \quad (27)$$

Proof. Using (13), a direct computation gives

$$\nabla W_k(\xi, \eta) = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j} e^{-(k-2j)\xi} (-\sin((k-2j)\eta), \cos((k-2j)\eta)).$$

Recalling (26), we perform a standard change of variables in the left-hand side of (27) to elliptic coordinates and this yields the thesis. \square

2.2 Simplification of the sum

We now prove the following result.

Lemma 2.5. *For every integer $k \geq 1$,*

$$\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j}^2 = k \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor}.$$

Proof. We will use repeatedly the two following properties of binomial coefficients. First, the *Vandermonde identity*: for any non-negative integers m, n and r ,

$$\sum_{j=0}^r \binom{m}{j} \binom{n}{r-j} = \binom{m+n}{r}; \quad (28)$$

and second, the elementary identity

$$\binom{n}{r} = \frac{n}{r} \binom{n-1}{r-1} \quad (29)$$

with n and r positive integers.

Let us now fix an integer $k \geq 1$. To simplify the notation, we write

$$s := \left\lfloor \frac{k-1}{2} \right\rfloor \quad \text{and} \quad S := \sum_{j=0}^s (k-2j) \binom{k}{j}^2.$$

Next, we remark that

$$S = S_0 - \frac{2}{k} S_1 - \frac{2}{k} S_2,$$

with

$$S_0 := \sum_{j=0}^s k \binom{k}{j}^2, \quad S_1 := \sum_{j=0}^s j(k-j) \binom{k}{j}^2, \quad S_2 := \sum_{j=0}^s j^2 \binom{k}{j}^2.$$

Let us compute the previous sums when $k = 2p + 1$, with p a non-negative integer. We first have

$$\frac{S_0}{k} = \frac{1}{2} \sum_{j=0}^k \binom{k}{j}^2 = \frac{1}{2} \binom{2k}{k},$$

where the last equality is a special case of identity (28). We then find

$$\begin{aligned} S_1 &= \frac{1}{2} \sum_{j=0}^k j \binom{k}{j} (k-j) \binom{k}{k-j} = \frac{k^2}{2} \sum_{j=1}^{k-1} \binom{k-1}{j-1} \binom{k-1}{k-j-1} \\ &= \frac{k^2}{2} \sum_{\ell=0}^{k-2} \binom{k-1}{\ell} \binom{k-1}{k-2-\ell} = \frac{k^2}{2} \binom{2k-2}{k-2} \end{aligned}$$

by applying Identity (29) followed by (28). Finally, Identity (29) implies

$$\begin{aligned} S_2 &= k^2 \sum_{j=1}^p \binom{k-1}{j-1}^2 = k^2 \sum_{\ell=0}^{p-1} \binom{k-1}{\ell}^2 \\ &= \frac{k^2}{2} \left(\sum_{\ell=0}^{k-1} \binom{k-1}{\ell}^2 - \binom{k-1}{p}^2 \right) = \frac{k^2}{2} \binom{2k-2}{k-1} - \frac{k^2}{2} \binom{k-1}{p}^2. \end{aligned}$$

We obtain

$$\begin{aligned} S &= \frac{k}{2} \binom{2k}{k} - k \binom{2k-2}{k-2} - k \binom{2k-2}{k-1} + k \binom{k-1}{p}^2 \\ &= \frac{k}{2} \binom{2k}{k} - k \binom{2k-1}{k-1} + k \binom{k-1}{p}^2 = k \binom{k-1}{p}^2 \end{aligned}$$

where the second equality follows from Pascal's identity and the third from Identity (29).

Let us now treat the case $k = 2p$, with p a positive integer. In a similar way as before, we find

$$S_0 = \frac{k}{2} \left(\sum_{j=0}^k \binom{k}{j}^2 - \binom{k}{p}^2 \right) = \frac{k}{2} \binom{2k}{k} - \frac{k}{2} \binom{k}{p}^2,$$

$$S_1 = \frac{1}{2} \left(\sum_{j=0}^k j \binom{k}{j} (k-j) \binom{k}{k-j} - p^2 \binom{k}{p}^2 \right) = \frac{k^2}{2} \binom{2k-2}{k-2} - \frac{k^2}{8} \binom{k}{p}^2$$

and

$$S_2 = k^2 \sum_{j=0}^{p-2} \binom{k-1}{j}^2 = \frac{k^2}{2} \left(\sum_{j=0}^{k-1} \binom{k-1}{j}^2 - 2 \binom{k-1}{p-1}^2 \right) = \frac{k^2}{2} \binom{2k-2}{k-1} - k^2 \binom{k-1}{p-1}^2.$$

We finally obtain, after simplifications,

$$S = k \binom{k-1}{p-1}^2.$$

This completes the proof of Lemma 2.5. □

Proof of Proposition 2.2. It follows from Corollary 2.4 and Lemma 2.5. □

Combining the above results, we can now prove our main theorem.

Proof of Theorem 1.4. Theorem 1.4 follows from the combination of Theorem 2.1 and Proposition 2.2. □

3 Asymptotic estimates for Aharonov–Bohm eigenvalues

3.1 Symmetry for the Aharonov–Bohm operator

As in Section 1.1, we assume $\widehat{\Omega} \subset \mathbb{R}^2$ to be a bounded open set with a Lipschitz boundary, such that $0 \in \widehat{\Omega}$. We additionally assume that $\widehat{\Omega}$ is symmetric with respect to the x_1 -axis and that $\Omega := \widehat{\Omega} \cap \mathbb{R}_+^2$ also has a Lipschitz boundary. According to [18, Theorem VIII.15], there exists a unique Friedrichs extension H_ε of the quadratic form q_ε^{AB} , that is to say a self-adjoint operator whose domain $\mathcal{D}(H_\varepsilon)$ is contained in $\mathcal{Q}_\varepsilon^{AB}$ and which satisfies

$$\langle H_\varepsilon u, v \rangle = q_\varepsilon^{AB}(u, v) = \int_{\widehat{\Omega}} (i\nabla + \mathbf{A}_\varepsilon)u \cdot \overline{(i\nabla + \mathbf{A}_\varepsilon)v} dx \quad \text{for all } u, v \in \mathcal{D}(H_\varepsilon),$$

where we are denoting by q_ε^{AB} both the quadratic form defined in (18) and the associated bilinear form. We recall in this section the results proved in [3] concerning the properties of H_ε , in particular the effect of the symmetry of the domain on its spectrum. Since most of the proofs in the present section reduce to a series of standard verifications, we generally only give an indication of them. We use gauge functions Φ_ε , for $\varepsilon \in (0, \varepsilon_0]$, whose existence is guaranteed by the following result. In the sequel we denote as σ the reflection through the x_1 -axis, i.e. $\sigma(x_1, x_2) = (x_1, -x_2)$.

Lemma 3.1. *For each $\varepsilon > 0$, there exists a function Φ_ε in $C^\infty(\mathbb{R}^2 \setminus \Gamma_\varepsilon)$ satisfying*

- (i) $\Phi_\varepsilon \circ \sigma = \overline{\Phi_\varepsilon}$ in $\mathbb{R}^2 \setminus \Gamma_\varepsilon$;
- (ii) $|\Phi_\varepsilon| = 1$ in $\mathbb{R}^2 \setminus \Gamma_\varepsilon$;
- (iii) $(i\nabla + \mathbf{A}_\varepsilon)\Phi_\varepsilon = 0$ in $\mathbb{R}^2 \setminus \Gamma_\varepsilon$;
- (iv) $\Phi_\varepsilon = 1$ on $(\mathbb{R} \times \{0\}) \setminus \Gamma_\varepsilon$ and $\lim_{\delta \rightarrow 0^+} \Phi_\varepsilon(t, \pm\delta) = \pm i$ for every $t \in (-\varepsilon, \varepsilon)$.

We define the anti-unitary operators K_ε and Σ^c by $K_\varepsilon u := \Phi_\varepsilon^2 \bar{u}$ and $\Sigma^c u := \bar{u} \circ \sigma$. The subspace $\mathcal{D}(H_\varepsilon) \subset \mathcal{H}$ is preserved by K_ε and Σ^c . The operators K_ε , Σ^c and H_ε mutually commute. In particular, we can define the following subsets

$$\begin{aligned} \mathcal{H}_{K,\varepsilon} &:= \{u \in \mathcal{H} : K_\varepsilon u = u\}; \\ \mathcal{D}(H_{K,\varepsilon}) &:= \{u \in \mathcal{D}(H_\varepsilon) : K_\varepsilon u = u\}. \end{aligned}$$

The scalar product $\langle \cdot, \cdot \rangle$ gives $\mathcal{H}_{K,\varepsilon}$ the structure of a real Hilbert space. As suggested by the notation, we define $H_{K,\varepsilon}$ as the restriction of H_ε to $\mathcal{D}(H_{K,\varepsilon})$. It is a positive self-adjoint operator on $\mathcal{H}_{K,\varepsilon}$ of domain $\mathcal{D}(H_{K,\varepsilon})$, with compact resolvent. It has the same eigenvalues as H_ε , with the same multiplicities. The fact that K and Σ^c commute ensures that $\mathcal{H}_{K,\varepsilon}$ and $\mathcal{D}(H_{K,\varepsilon})$ are Σ^c -invariant. We can therefore define

$$\begin{aligned} \mathcal{H}_{K,\varepsilon}^s &:= \{u \in \mathcal{H}_{K,\varepsilon} : \Sigma^c u = u\}; \\ \mathcal{D}(H_{K,\varepsilon}^s) &:= \{u \in \mathcal{D}(H_{K,\varepsilon}) : \Sigma^c u = u\}; \\ \mathcal{H}_{K,\varepsilon}^a &:= \{u \in \mathcal{H}_{K,\varepsilon} : \Sigma^c u = -u\}; \\ \mathcal{D}(H_{K,\varepsilon}^a) &:= \{u \in \mathcal{D}(H_{K,\varepsilon}) : \Sigma^c u = -u\}. \end{aligned}$$

We have the following orthogonal decomposition of $\mathcal{H}_{K,\varepsilon}$ into spaces of symmetric and antisymmetric functions:

$$\mathcal{H}_{K,\varepsilon} = \mathcal{H}_{K,\varepsilon}^s \oplus \mathcal{H}_{K,\varepsilon}^a. \quad (30)$$

We also define $H_{K,\varepsilon}^s$ and $H_{K,\varepsilon}^a$ as the restrictions of $H_{K,\varepsilon}$ to $\mathcal{D}(H_{K,\varepsilon}^s)$ and $\mathcal{D}(H_{K,\varepsilon}^a)$ respectively. The operator $H_{K,\varepsilon}^s$ is positive and self-adjoint on $\mathcal{H}_{K,\varepsilon}^s$ of domain $\mathcal{D}(H_{K,\varepsilon}^s)$, with compact resolvent. Similar conclusions hold for $\mathcal{H}_{K,\varepsilon}^a$. Decomposition (30) implies the following result.

Lemma 3.2. *The spectrum of $H_{K,\varepsilon}$ is the union of the spectra of $H_{K,\varepsilon}^s$ and $H_{K,\varepsilon}^a$, counted with multiplicities.*

Remark 3.3. *Let us note that we can give an alternative description of the spectra of $H_{K,\varepsilon}^s$ and $H_{K,\varepsilon}^a$. One can check that they are the spectra of the quadratic form q_ε^{AB} restricted to $\mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^s$ and $\mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^a$ respectively. These spectra can therefore be obtained by the min-max principle.*

3.2 Isospectrality

In this subsection, we establish an isospectrality result between Aharonov-Bohm eigenvalue problems with symmetry and Laplacian eigenvalue problems with mixed boundary conditions, in the spirit of [6].

To this aim, we define an additional family of eigenvalue problems, similar to Problems (3) and (4). With the notation $\partial\Omega_+ := \partial\Omega \cap \mathbb{R}_+^2$ and $\partial\Omega_0 := \partial\Omega \cap (\mathbb{R} \times \{0\})$, we consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega_+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega_0, \end{cases} \quad (31)$$

We denote by $(\mu_j)_{j \geq 1}$ the eigenvalues of Problem (31). We also consider, for each $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega_+ \cup \Gamma_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega_0 \setminus \Gamma_\varepsilon. \end{cases} \quad (32)$$

and denote by $(\mu_j(\varepsilon))_{j \geq 1}$ the corresponding eigenvalues. In order to give a rigorous definitions, we use a weak formulation. We define

$$\mathcal{R}_0 = \{u \in H^1(\Omega); \chi_{\partial\Omega_+} \gamma_0 u = 0 \text{ in } L^2(\partial\Omega)\},$$

and, for $\varepsilon \in (0, \varepsilon_0]$,

$$\mathcal{R}_\varepsilon = \{u \in H^1(\Omega); \chi_{\partial\Omega_+ \cup \Gamma_\varepsilon} \gamma_0 u = 0 \text{ in } L^2(\partial\Omega)\}.$$

We denote by r_0 and r_ε the restriction of the quadratic form q , defined in Equation (5), to \mathcal{R}_0 and \mathcal{R}_ε respectively. We then define $(\mu_j)_{j \geq 1}$ and $(\mu_j(\varepsilon))_{j \geq 1}$ as, respectively, the eigenvalues of the quadratic forms r_0 and r_ε ; they are obtained by the min-max principle.

Remark 3.4. *We can give another interpretation of the eigenvalues $(\mu_j)_{j \geq 1}$ and $(\lambda_j)_{j \geq 1}$. Using the unitary operator $\Sigma : u \mapsto u \circ \sigma$, we obtain a orthogonal decomposition of $L^2(\widehat{\Omega})$ into symmetric and antisymmetric functions:*

$$L^2(\widehat{\Omega}) = \ker(I - \Sigma) \oplus \ker(I + \Sigma). \quad (33)$$

This decomposition is preserved by the action of the Dirichlet Laplacian $-\widehat{\Delta}$, and we can therefore define $-\Delta^s$ (resp. $-\Delta^a$) as the restriction of $-\widehat{\Delta}$ to symmetric (resp. antisymmetric) functions in the domain of $-\widehat{\Delta}$. One can then check that $(\mu_j)_{j \geq 1}$ is the spectrum of $-\Delta^s$ and $(\lambda_j)_{j \geq 1}$ is the spectrum of $-\Delta^a$.

It remains to connect the eigenvalues of Problems (32) and (4) to the eigenvalues of H_ε . To this end, we define the following linear operator, which performs a gauge transformation:

$$\begin{aligned} U_\varepsilon : \mathcal{H} &\rightarrow L^2(\Omega, \mathbb{C}) \\ u &\mapsto \sqrt{2} \overline{\Phi}_\varepsilon u|_\Omega. \end{aligned}$$

We recall that $L^2(\Omega)$ denotes the real Hilbert space of real-valued L^2 functions in Ω . We have the following result.

Lemma 3.5. *The operator U_ε satisfies the following properties:*

- (i) $U_\varepsilon(\mathcal{H}_{K,\varepsilon}) \subset L^2(\Omega)$ and $U_\varepsilon(\mathcal{Q}_\varepsilon^{AB}) \subset H^1(\Omega, \mathbb{C})$;
- (ii) U_ε induces a real-unitary bijective map from $\mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^s$ to \mathcal{R}_ε such that $q_\varepsilon^{AB}(u) = q(U_\varepsilon u)$ for all $u \in \mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^s$;
- (iii) U_ε induces a real-unitary bijective map from $\mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^a$ to \mathcal{Q}_ε such that $q_\varepsilon^{AB}(u) = q(U_\varepsilon u)$ for all $u \in \mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^a$.

Proof. If $u \in \mathcal{H}_{K,\varepsilon}$, then $u = \Phi_\varepsilon^2 \bar{u}$, so that $\overline{\Phi}_\varepsilon u = \overline{\Phi_\varepsilon^2 \bar{u}} = \Phi_\varepsilon^2 u$, that is to say $\overline{\Phi}_\varepsilon u$ is real-valued. This proves the first half of (i). For the second half, let us assume that $u \in \mathcal{Q}_\varepsilon^{AB}$. Using the definition of $\mathcal{Q}_\varepsilon^{AB}$, given in Equation (17), and Property (iii) of Lemma 3.1, we find the following identity, in the sense of distributions in Ω :

$$\nabla(\overline{\Phi}_\varepsilon u) = \overline{\Phi}_\varepsilon \nabla u + \nabla(\overline{\Phi}_\varepsilon) u = \overline{\Phi}_\varepsilon (\nabla - iA_\varepsilon) u \quad \text{in } \Omega.$$

This proves that $\overline{\Phi}_\varepsilon u|_\Omega \in H^1(\Omega, \mathbb{C})$ and that

$$\int_\Omega |(\nabla - iA_\varepsilon) u|^2 dx = \int_\Omega |\nabla(\overline{\Phi}_\varepsilon u)|^2 dx.$$

Let us now additionally assume that $u \in \mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^s$. Since $\Sigma^c u = u$, Property (i) of Lemma 3.1 implies that $(\bar{\Phi}_\varepsilon u) \circ \sigma = \Phi_\varepsilon \bar{u}$. Therefore,

$$\int_{\widehat{\Omega}} |u|^2 dx = 2 \int_{\Omega} |\bar{\Phi}_\varepsilon u|^2 dx = \int_{\Omega} |U_\varepsilon u|^2 dx.$$

Furthermore, Property (iv) of Lemma 3.1 and the equation $\Sigma^c u = u$ imply that u vanishes on Γ_ε , hence $U_\varepsilon u \in \mathcal{R}_\varepsilon$. This implies that $\bar{\Phi}_\varepsilon u \in H^1(\widehat{\Omega})$ and

$$\int_{\widehat{\Omega}} |(\nabla - iA_\varepsilon) u|^2 dx = \int_{\widehat{\Omega}} |\nabla (\bar{\Phi}_\varepsilon u)|^2 dx = 2 \int_{\Omega} |\nabla (\bar{\Phi}_\varepsilon u)|^2 dx = \int_{\Omega} |\nabla (U_\varepsilon u)|^2 dx.$$

We conclude that the mapping $U_\varepsilon : \mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^s \rightarrow \mathcal{R}_\varepsilon$ is well-defined, real-unitary, and that $q_\varepsilon^{AB}(u) = q(U_\varepsilon u)$. To show that the mapping is bijective, we consider the operator V_ε defined in the following way: given $v \in L^2(\Omega)$, we denote by \tilde{v} its extension by symmetry to $\widehat{\Omega}$ and we set

$$V_\varepsilon v := \frac{1}{\sqrt{2}} \Phi_\varepsilon \tilde{v}.$$

It can be checked, in a way similar to what has been done for U_ε , that V_ε induces the inverse of U_ε , from \mathcal{R}_ε to $\mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^s$. This proves (ii). The proof of (iii) is similar, the difference being that we must check that $\bar{\Phi}_\varepsilon u$ vanishes on $(\mathbb{R} \times \{0\}) \setminus \Gamma_\varepsilon$ when $u \in \mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^a$. \square

Corollary 3.6. *The spectra of $H_{K,\varepsilon}^s$ and $H_{K,\varepsilon}^a$ are $(\mu_j(\varepsilon))_{j \geq 1}$ and $(\lambda_j(\varepsilon))_{j \geq 1}$ respectively.*

3.3 Eigenvalues variations

Let us first state some auxiliary results, which we prove in Appendix B.

Proposition 3.7. *For all $N \in \mathbb{N}^*$, $\mu_N(\varepsilon) \rightarrow \mu_N$ as $\varepsilon \rightarrow 0$.*

Proposition 3.8. *Let μ_N be a simple eigenvalue of $-\Delta^s$ (see Remark 3.4) and u_N be an associated eigenfunction, normalized in $L^2(\widehat{\Omega})$. If $u_N(0) \neq 0$, then*

$$\mu_N(\varepsilon) = \mu_N + \frac{2\pi}{|\log(\varepsilon)|} u_N^2(0) + o\left(\frac{1}{|\log(\varepsilon)|}\right) \quad \text{as } \varepsilon \rightarrow 0.$$

If

$$r^{-k} u_N(r \cos t, r \sin t) \rightarrow \widehat{\beta} \cos(kt) \quad \text{in } C^{1,\tau}([0, \pi], \mathbb{R})$$

as $r \rightarrow 0^+$ for all $\tau \in (0, 1)$, with $k \in \mathbb{N}^*$ and $\widehat{\beta} \in \mathbb{R} \setminus \{0\}$, then

$$\mu_N(\varepsilon) = \mu_N + \frac{k\pi \widehat{\beta}^2}{4^{k-1}} \left(\frac{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2 \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \rightarrow 0.$$

We now prove Theorem 1.13. Since \widehat{u}_N is odd in x_2 , $\widehat{\lambda}_N$ belongs to the spectrum of $-\Delta^a$. Since $\widehat{\lambda}_N$ is simple, it does not belong to the spectrum of $-\Delta^s$, according to the orthogonal decomposition (33). It follows from Remark 3.4 that there exists $K \in \mathbb{N}^*$ such that $\widehat{\lambda}_N = \lambda_K$ and that λ_K is a simple eigenvalue of q_0 in Ω . By continuity, $\lambda_K(\varepsilon) \rightarrow \lambda_K$ as $\varepsilon \rightarrow 0^+$.

From Corollary 3.6, Proposition 3.7 and the fact that $\widehat{\lambda}_N$ is simple, it follows that there exists $\varepsilon_1 > 0$ such that $\lambda_N^{AB}(\varepsilon) = \lambda_K(\varepsilon)$ for every $\varepsilon \in (0, \varepsilon_1)$. The conclusion of Theorem 1.13 follows from Theorem 1.4, using the fact that λ_K is simple. Let us note that the eigenfunction \widehat{u}_N in Theorem 1.13 is normalized in $L^2(\widehat{\Omega})$, while the eigenfunction u_N in Theorem 1.4 is normalized in $L^2(\Omega)$. We therefore have to apply Theorem 1.4 with $\beta = \sqrt{2} \widehat{\beta}$ to obtain the correct result.

We can use the results of the preceding sections to study some multiple eigenvalues. Let $\widehat{\lambda}_N$ be an eigenvalue of $-\Delta$ on $\widehat{\Omega}$, possibly multiple. We define

$$N_0 := \min \left\{ M \in \mathbb{N}^* ; \widehat{\lambda}_M = \widehat{\lambda}_N \right\} \quad \text{and} \quad N_1 := \max \left\{ M \in \mathbb{N}^* ; \widehat{\lambda}_M = \widehat{\lambda}_N \right\}.$$

According to Remark 3.4, there exists $K \in \mathbb{N}^*$ such that $\widehat{\lambda}_N = \lambda_K$ or there exists $L \in \mathbb{N}^*$ such that $\widehat{\lambda}_N = \mu_L$.

Proposition 3.9. *Let us assume that $\widehat{\lambda}_N = \lambda_K$ with $K \in \mathbb{N}^*$ and that λ_K is a simple eigenvalue of q_0 . Let us denote by u_K an associated normalized eigenfunction for q_0 , and let us assume that*

$$r^{-k} u_K(r \cos t, r \sin t) \rightarrow \beta \sin(kt) \text{ in } C^{1,\tau}([0, \pi], \mathbb{R})$$

as $r \rightarrow 0^+$ for all $\tau \in (0, 1)$, with $k \in \mathbb{N}^*$ and $\beta \in \mathbb{R} \setminus \{0\}$. Then

$$\lambda_{N_0}^{AB}(\varepsilon) = \widehat{\lambda}_N - \frac{k\pi\beta^2}{2^{2k-1}} \left(\frac{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2 \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. Let us set $m := N_1 - N_0 + 1$, the multiplicity of $\widehat{\lambda}_N$. If $m = 1$, the conclusion follows from Theorem 1.13. We therefore assume $m \geq 2$ in the rest of the proof. Remark 3.4 and the fact that λ_K is simple imply that there exists $L \in \mathbb{N}^*$ such that $\mu_L = \mu_{L+1} = \dots = \mu_{L+m-2} = \widehat{\lambda}_N$. From Proposition 3.7, we deduce that there exists $\varepsilon_1 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_1)$,

$$\{\lambda_{N_0}^{AB}(\varepsilon); \lambda_{N_0+1}^{AB}(\varepsilon), \dots, \lambda_{N_1}^{AB}(\varepsilon)\} = \{\lambda_K(\varepsilon), \mu_L(\varepsilon), \dots, \mu_{L+m-2}(\varepsilon)\}.$$

The function $\varepsilon \mapsto \lambda_K(\varepsilon)$ is non-increasing, and the function $\varepsilon \mapsto \mu_j(\varepsilon)$ is non-decreasing for every $j \in \{L, \dots, L+m-2\}$, therefore $\mu_j(\varepsilon) \geq \mu_j = \widehat{\lambda}_N = \lambda_K \geq \lambda_K(\varepsilon)$. In particular $\lambda_{N_0}^{AB}(\varepsilon) = \lambda_K(\varepsilon)$ for every $\varepsilon \in (0, \varepsilon_1)$. The conclusion follows from Theorem 1.4. \square

Proposition 3.10. *Let us assume that $\widehat{\lambda}_N = \mu_L$ with $L \in \mathbb{N}^*$ and that μ_L is a simple eigenvalue of $-\Delta^s$. Let us denote by u_L an associated eigenfunction for $-\Delta^s$, normalized in $L^2(\widehat{\Omega})$. If $u_L(0) \neq 0$, then*

$$\lambda_{N_1}^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{2\pi}{|\log(\varepsilon)|} u_L^2(0) + o\left(\frac{1}{|\log(\varepsilon)|}\right) \quad \text{as } \varepsilon \rightarrow 0.$$

If

$$r^{-k} u_L(r \cos t, r \sin t) \rightarrow \widehat{\beta} \cos(kt) \text{ in } C^{1,\tau}([0, \pi], \mathbb{R})$$

as $r \rightarrow 0^+$ for all $\tau \in (0, 1)$, with $k \in \mathbb{N}^*$ and $\widehat{\beta} \in \mathbb{R} \setminus \{0\}$, then

$$\lambda_{N_1}^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{k\pi\widehat{\beta}^2}{4^{k-1}} \left(\frac{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2 \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. In a similar way as in the proof of Proposition 3.9, we show that there exists $\varepsilon_1 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_1)$, $\lambda_{N_1}^{AB}(\varepsilon) = \mu_L(\varepsilon)$. The conclusion then follows from Proposition 3.8. \square

3.4 Example: the square

As an application of the preceding results, let us study the first four eigenvalues of the Dirichlet Laplacian for the square

$$\widehat{\Omega} := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} < \frac{\pi}{2} \right\}. \quad (34)$$

The open set $\widehat{\Omega}$ is symmetric with respect to the x_1 -axis. We define $\Omega := \widehat{\Omega} \cap \mathbb{R}_+^2$. We denote by $(\widehat{\lambda}_j)_{j \geq 1}$ the eigenvalues of the Dirichlet Laplacian on the square $\widehat{\Omega}$ and, for $\varepsilon \in (0, \pi/2)$, we consider the Aharonov-Bohm eigenvalues $(\lambda_j^{AB}(\varepsilon))_{j \geq 1}$ defined in Section 1.1.

It is well known that the eigenvalues of the Dirichlet Laplacian on $\widehat{\Omega}$ are

$$\widehat{\lambda}_{m,n} := m^2 + n^2,$$

with m and n positive integers, and that an associated orthonormal family of eigenfunctions is given by

$$u_{m,n}(x_1, x_2) = \frac{2}{\pi} f_m(x_1) f_n(x_2),$$

where

$$f_k(x) = \begin{cases} \sin(kx), & \text{if } k \text{ is even,} \\ \cos(kx), & \text{if } k \text{ is odd.} \end{cases}$$

Proposition 3.11. *Let us assume that $\widehat{\lambda}_N$ is simple. Then $\widehat{\lambda}_N = \widehat{\lambda}_{m,m} = 2m^2$ for some positive integer m , and $\widehat{\lambda}_N$ cannot be written in any other way as a sum of squares of positive integers. Then we have, as $\varepsilon \rightarrow 0^+$,*

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{8}{\pi |\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right)$$

if m is odd and

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N - \frac{m^4}{2\pi} \varepsilon^4 + o(\varepsilon^4)$$

if m is even.

Proof. In the case where m is odd, an associated eigenfunction, normalized in $L^2(\widehat{\Omega})$, is

$$u_{m,m}(x_1, x_2) = \frac{2}{\pi} \cos(mx_1) \cos(mx_2).$$

The first asymptotic expansion then follows from Theorem 1.10.

In the case where m is even, an associated eigenfunction, normalized in $L^2(\widehat{\Omega})$, is

$$u_{m,m}(x_1, x_2) = \frac{2}{\pi} \sin(mx_1) \sin(mx_2).$$

Then $\widehat{\lambda}_N = \lambda_K$, where λ_K is a simple eigenvalue of q_0 . Furthermore,

$$r^{-2} u_{m,m}(r \cos t, r \sin t) \rightarrow \frac{m^2}{\pi} \sin(2t) \text{ in } C^{1,\tau}([0, \pi], \mathbb{R})$$

as $r \rightarrow 0^+$ for all $\tau \in (0, 1)$. An application of Proposition 3.9, taking care of normalizing in $L^2(\Omega)$, gives the second asymptotic expansion. \square

Proposition 3.12. *Assume that $\widehat{\lambda}_N = \widehat{\lambda}_{m,n} = m^2 + n^2$ with m even and n odd, and that $\widehat{\lambda}_N$ has no other representation as a sum of two squares of positive integers, up to the exchange of m and n . Then $\widehat{\lambda}_N$ has multiplicity two; up to replacing N with $N - 1$, we can assume that $\lambda_N = \widehat{\lambda}_{N+1}$. Then, as $\varepsilon \rightarrow 0^+$,*

$$\begin{aligned} \lambda_N^{AB}(\varepsilon) &= \widehat{\lambda}_N - \frac{4m^2}{\pi} \varepsilon^2 + o(\varepsilon^2); \\ \lambda_{N+1}^{AB}(\varepsilon) &= \widehat{\lambda}_N + \frac{4m^2}{\pi} \varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

Proof. The associated eigenfunctions

$$u_{m,n}(x_1, x_2) = \frac{2}{\pi} \sin(mx_1) \cos(nx_2)$$

and

$$u_{n,m}(x_1, x_2) = \frac{2}{\pi} \cos(nx_1) \sin(mx_2).$$

are normalized in $L^2(\widehat{\Omega})$ and respectively symmetric and antisymmetric in the variable x_2 . It follows that $\widehat{\lambda}_N = \mu_L = \lambda_K$, where μ_L is a simple eigenvalue of r_0 and λ_K a simple eigenvalue of q_0 . Furthermore,

$$r^{-1}u_{m,n}(r \cos t, r \sin t) \rightarrow \frac{2m}{\pi} \cos(t) \text{ in } C^{1,\tau}([0, \pi], \mathbb{R})$$

and

$$r^{-1}u_{n,m}(r \cos t, r \sin t) \rightarrow \frac{2m}{\pi} \sin(t) \text{ in } C^{1,\tau}([0, \pi], \mathbb{R})$$

as $r \rightarrow 0^+$ for all $\tau \in (0, 1)$. The asymptotic expansions then follow from Propositions 3.10 and 3.9 \square

Remark 3.13. *It is a well known fact from number theory that the number of representation of an integer as the sum of two squares, counted with changes of sign, exchanges of terms and including zero terms, can be computed from its prime factorization. To be more specific, let us consider λ a positive integer and let us denote by $r(\lambda)$ the number of representations. The integer λ has a unique prime factorization, which can be written*

$$\lambda = 2^\alpha \prod_{s=1}^S p_s^{\beta_s} \prod_{t=1}^T q_t^{\gamma_t},$$

where the p_s 's are primes of the form $4k+1$ and the q_t 's primes of the form $4k+3$. Then $r(\lambda) = 0$ as soon as one of the γ_t 's is odd. If all γ_t 's are even,

$$r(\lambda) = 4 \prod_{s=1}^S (1 + \beta_s),$$

see [19, Theorem 4.12]. If $\widehat{\lambda}_N$ satisfies the hypotheses of Proposition 3.11, $\widehat{\lambda}_N$ is even and $r(\widehat{\lambda}_N) = 4$. From this and the above formula, we deduce that

$$\widehat{\lambda}_N = 2^{2\alpha'+1} \left(\prod_{t=1}^T q_t^{\gamma'_t} \right)^2.$$

If $\widehat{\lambda}_N$ satisfies the hypotheses of Proposition 3.12, $\widehat{\lambda}_N$ is odd and $r(\widehat{\lambda}_N) = 8$. From this and the above formula, we deduce that

$$\widehat{\lambda}_N = p_1 \left(\prod_{t=1}^T q_t^{\gamma'_t} \right)^2.$$

Remark 3.14. *We note that if $\widehat{\lambda}_N$ is even, in any representation $\widehat{\lambda}_N = m^2 + n^2$, m and n have the same parity. Therefore, if $n \neq m$, $\widehat{\lambda}_N$ cannot be a simple eigenvalue either of r_0 or of q_0 . On the other hand, if $\widehat{\lambda}_N$ is odd, in any representation $\widehat{\lambda}_N = m^2 + n^2$, m and n have the opposite parity. Therefore, as soon as $r(\widehat{\lambda}_N) > 8$, $\widehat{\lambda}_N$ cannot be a simple eigenvalue either of r_0 or of q_0 . The cases described in Propositions 3.11 and 3.12 are thus the only ones in which we can apply the results of Section 3.3 for the square.*

The first four eigenvalues of the Dirichlet Laplacian on the square $\widehat{\Omega}$ satisfy the assumptions of either Proposition 3.11 or Proposition 3.12, so we can apply the previous results to derive the following asymptotic expansions of the Aharonov-Bohm eigenvalues $\lambda_j^{AB}(\varepsilon)$ for $j = 1, 2, 3, 4$.

Corollary 3.15. *Let $\lambda_j^{AB}(\varepsilon)$ be the Aharonov-Bohm eigenvalues defined in (18)–(19) with $\widehat{\Omega}$ being the square defined in (34). Then we have, as $\varepsilon \rightarrow 0^+$,*

$$\begin{aligned}\lambda_1^{AB}(\varepsilon) &= 2 + \frac{8}{\pi} \frac{1}{|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right); \\ \lambda_2^{AB}(\varepsilon) &= 5 - \frac{16}{\pi} \varepsilon^2 + o(\varepsilon^2); \\ \lambda_3^{AB}(\varepsilon) &= 5 + \frac{16}{\pi} \varepsilon^2 + o(\varepsilon^2); \\ \lambda_4^{AB}(\varepsilon) &= 8 - \frac{8}{\pi} \varepsilon^4 + o(\varepsilon^4).\end{aligned}$$

3.5 Example: the disk

Let $(r, t) \in [0, 1] \times [0, 2\pi)$ be the polar coordinates of the disk. It is well known that the eigenvalues of the Dirichlet Laplacian on the disk are given by the sequences

$$\{j_{0,k}^2\}_{k \geq 1} \cup \{j_{n,k}^2\}_{n,k \geq 1},$$

where $j_{n,k}$ denotes the k -th zero of the Bessel function J_n for $n \geq 0$, $k \geq 1$. We recall that $j_{n,k} = j_{n',k'}$ if, and only if, $n = n'$ and $k = k'$ (see [22, Section 15.28]). The first set is therefore made of simple eigenvalues; their eigenfunctions are given by the Bessel functions

$$u_{0,k}(r \cos t, r \sin t) := \sqrt{\frac{1}{\pi} \frac{1}{|J_0'(j_{0,k})|}} J_0(j_{0,k}r) \quad \text{for } k \geq 1. \quad (35)$$

The second set is made of double eigenvalues whose eigenfunctions are spanned by

$$u_{n,k}^s(r \cos t, r \sin t) := \sqrt{\frac{2}{\pi} \frac{1}{|J_n'(j_{n,k})|}} J_n(j_{n,k}r) \cos nt, \quad (36)$$

$$u_{n,k}^a(r \cos t, r \sin t) := \sqrt{\frac{2}{\pi} \frac{1}{|J_n'(j_{n,k})|}} J_n(j_{n,k}r) \sin nt, \quad (37)$$

for $n, k \geq 1$. We stress that these eigenfunctions have L^2 -norm equal to 1 on the disk. It is convenient to recall (see [22, Chapter III]) even that for any $n \in \mathbb{N} \cup \{0\}$

$$J_n(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{1}{2}z\right)^{n+2k}}{k! \Gamma(n+k+1)}. \quad (38)$$

We denote by $(\widehat{\lambda}_j)_{j \geq 1}$ the eigenvalues of the Dirichlet Laplacian on the disk and, for $\varepsilon \in (0, 1/2)$, we consider the Aharonov-Bohm eigenvalues $(\lambda_j^{AB}(\varepsilon))_{j \geq 1}$ defined in Section 1.1.

Proposition 3.16. *If $\widehat{\lambda}_N$ is simple, there exists an integer $k \geq 1$ such that $\widehat{\lambda}_N = j_{0,k}^2$. Then*

$$\lambda_N^{AB}(\varepsilon) = j_{0,k}^2 + \frac{2}{|J_0'(j_{0,k})|^2} \frac{1}{|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right) \quad (39)$$

as $\varepsilon \rightarrow 0^+$. *If $\widehat{\lambda}_N$ is double, there exist integers $n \geq 1$ and $k \geq 1$ such that $\widehat{\lambda}_N = j_{n,k}^2$. Up to replacing N by $N-1$, we can assume that $\widehat{\lambda}_N = \widehat{\lambda}_{N+1}$. Then, as $\varepsilon \rightarrow 0^+$,*

$$\lambda_N^{AB}(\varepsilon) = j_{n,k}^2 - \frac{2nj_{n,k}^{2n}}{(n!)^2 4^{2n-1} |J_n'(j_{n,k})|^2} \left(\frac{n-1}{\lfloor \frac{n-1}{2} \rfloor} \right)^2 \varepsilon^{2n} + o(\varepsilon^{2n}), \quad (40)$$

$$\lambda_{N+1}^{AB}(\varepsilon) = j_{n,k}^2 + \frac{2nj_{n,k}^{2n}}{(n!)^2 4^{2n-1} |J_n'(j_{n,k})|^2} \left(\frac{n-1}{\lfloor \frac{n-1}{2} \rfloor} \right)^2 \varepsilon^{2n} + o(\varepsilon^{2n}). \quad (41)$$

Proof. We first consider the case where the eigenvalue $\widehat{\lambda}_N = j_{0,k}^2$ is simple; then an associated eigenfunction, normalized in the disk, is $u_{0,k}$ defined by Equation (35). It follows from Equation (38) that

$$u_{0,k}(0) = \sqrt{\frac{1}{\pi}} \frac{1}{|J'_0(j_{0,k})|} > 0.$$

Theorem 1.10 gives us the asymptotic expansion (39).

We then consider the case where $\widehat{\lambda}_N$ is double, with $\widehat{\lambda}_N = \widehat{\lambda}_{N+1} = j_{n,k}^2$, $n, k \geq 1$. We note that $j_{n,k}^2$ is a simple eigenvalue of q_0 , and that the restriction of $\sqrt{2}u_{n,k}^a$ to the upper half-disk is an associated normalized eigenfunction. It follows from Equation (38) that

$$r^{-n}u_{n,k}^a(r \cos t, r \sin t) \rightarrow \sqrt{\frac{2}{\pi}} \frac{1}{|J'_n(j_{n,k})|} \frac{1}{\Gamma(n+1)} \left(\frac{j_{n,k}}{2}\right)^n \sin nt \quad \text{in } C^{1,\tau}([0, \pi], \mathbb{R})$$

as $r \rightarrow 0^+$. The asymptotic expansion (40) then follows from Proposition 3.9. In a similar way, $j_{n,k}^2$ is a simple eigenvalue of $-\Delta^s$, and $u_{n,k}^s$ is an associated normalized eigenfunction. It follows from Equation (38) that

$$r^{-n}u_{n,k}^s(r \cos t, r \sin t) \rightarrow \sqrt{\frac{2}{\pi}} \frac{1}{|J'_n(j_{n,k})|} \frac{1}{\Gamma(n+1)} \left(\frac{j_{n,k}}{2}\right)^n \cos nt \quad \text{in } C^{1,\tau}([0, \pi], \mathbb{R})$$

as $r \rightarrow 0^+$. The asymptotic expansion (41) then follows from the second case of Proposition 3.10. \square

Additionally, there exist relations between the zeros of Bessel functions (to this aim we refer to [22, Chapter XV.22]): in particular, the positive zeros of the Bessel function J_n are interlaced with those of the Bessel function J_{n+1} and by Porter's Theorem there is an odd number of zeros of J_{n+2} between two consecutive zeros of J_n . Then, we have,

$$0 < j_{0,1} < j_{1,1} < j_{2,1} < j_{0,2} < j_{1,2} < \dots$$

and hence, since $j_{3,1} > j_{2,1}$, the first three zeros of Bessel functions are, in order,

$$0 < j_{0,1} < j_{1,1} < j_{2,1}.$$

Combining this information with Proposition 3.16, we find for example the following asymptotic expansions for the first few Aharonov-Bohm eigenvalues $\lambda_j^{AB}(\varepsilon)$ on the disk as $\varepsilon \rightarrow 0^+$:

$$\begin{aligned} \lambda_1^{AB}(\varepsilon) &= j_{0,1}^2 + \frac{2}{|J'_0(j_{0,1})|^2} \frac{1}{|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right), \\ \lambda_2^{AB}(\varepsilon) &= j_{1,1}^2 - \frac{1}{2} \frac{j_{1,1}^2}{|J'_1(j_{1,1})|^2} \varepsilon^2 + o(\varepsilon^2), \\ \lambda_3^{AB}(\varepsilon) &= j_{1,1}^2 + \frac{1}{2} \frac{j_{1,1}^2}{|J'_1(j_{1,1})|^2} \varepsilon^2 + o(\varepsilon^2), \\ \lambda_4^{AB}(\varepsilon) &= j_{2,1}^2 - \frac{1}{64} \frac{j_{2,1}^4}{|J'_2(j_{2,1})|^2} \varepsilon^4 + o(\varepsilon^4), \\ \lambda_5^{AB}(\varepsilon) &= j_{2,1}^2 + \frac{1}{64} \frac{j_{2,1}^4}{|J'_2(j_{2,1})|^2} \varepsilon^4 + o(\varepsilon^4). \end{aligned}$$

A Computation of the constants

A.1 The Neumann-Dirichlet case

In the present section, we use the above results to compute the quantities appearing in [1, Section 4]. In order to avoid a conflict of notation with the present paper, for any odd positive integer k , we denote here by ψ'_k , \mathbf{m}'_k and w'_k what is denoted in [1] by ψ_k , \mathbf{m}_k and w_k respectively.

As in [1], we use the notation

$$s_0 := \{(x'_1, 0) ; x'_1 \geq 0\};$$

and

$$s := \{(x'_1, 0) ; x'_1 \geq 1\}.$$

We now define the mapping $G : \mathbb{R}_+^2 \rightarrow \mathbb{R}^2 \setminus s_0$ by

$$G(x) := (x_1^2 - x_2^2, 2x_1x_2).$$

The mapping is conformal; indeed, if for $x \in \mathbb{R}_+^2$ we write $z := x_1 + ix_2$ and $z' := x'_1 + ix'_2$, with $x' = (x'_1, x'_2) := G(x)$, we have $z' = z^2$. The scale factor associated with G is $h(x) = 2|z| = 2|x|$. Let u' be a function in $H^1(\mathbb{R}^2 \setminus s_0)$ and $u := u' \circ G$. Since G is conformal, $|\nabla u|$ is in $L^2(\mathbb{R}_+^2)$, with

$$\int_{\mathbb{R}_+^2} |\nabla u|^2 dx = \int_{\mathbb{R}^2 \setminus s_0} |\nabla u'|^2 dx'.$$

Furthermore, for any x' in the segment $(0, 1) \times \{0\}$, which we write as $x' = (x'_1, 0)$, we have

$$\frac{\partial u'}{\partial \nu_+}(x') = -\frac{1}{2\sqrt{x'_1}} \frac{\partial u}{\partial x_2}(\sqrt{x'_1}, 0) \quad \text{and} \quad \frac{\partial u'}{\partial \nu_-}(x') = -\frac{1}{2\sqrt{x'_1}} \frac{\partial u}{\partial x_2}(-\sqrt{x'_1}, 0),$$

where $\frac{\partial u'}{\partial \nu_+}(x')$ and $\frac{\partial u'}{\partial \nu_-}(x')$ denote the normal derivative at x' respectively from above and from below. We also note that u is harmonic in \mathbb{R}_+^2 if, and only if, u' is harmonic in $\mathbb{R}^2 \setminus s_0$.

Let us now denote by \tilde{w}'_k the extension by reflexion to $\mathbb{R}^2 \setminus s_0$ of w'_k , originally defined on \mathbb{R}_+^2 . We recall that w'_k is the unique finite energy solution to the problem

$$\begin{cases} -\Delta w'_k = 0, & \text{in } \mathbb{R}_+^2, \\ w'_k = 0, & \text{on } s, \\ \frac{\partial w'_k}{\partial \nu} = -\frac{\partial \psi'_k}{\partial \nu}, & \text{on } \partial \mathbb{R}_+^2 \setminus s, \end{cases}$$

where $\psi'_k(r \cos t, r \sin t) = r^{k/2} \sin(\frac{k}{2}t)$.

Lemma A.1. *For any odd positive integer k , $w_k = \tilde{w}'_k \circ G$.*

Proof. Let us write $v := \tilde{w}'_k \circ G$. By uniqueness, it is enough to prove that v solves (24). From the remarks at the beginning of the present section, it follows that v is harmonic in \mathbb{R}_+^2 . Let us now show that $\psi_k := \psi'_k \circ G$. Indeed, for $x' \in \mathbb{R}^2 \setminus s_0$, $\psi'_k(x') = \text{Im}((z')^{k/2})$, and therefore $f(x) = \text{Im}((z^2)^{k/2}) = \text{Im}(z^k) = \psi_k(x)$, where $x' = G(x)$, z and z' are defined as above, and where we use the determination of the square root on $\mathbb{C} \setminus s_0$ defined by G^{-1} . From this and the previous remarks, it follows that v satisfies the boundary conditions of Problem (24). \square

As in [1] we define

$$\mathbf{m}'_k = -\frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla w'_k|^2 dx$$

and

$$\mathbf{m}_k = -\frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla w_k|^2 dx.$$

We note that the right hand side of (22) is equal to $-2\beta^2 \mathbf{m}_k$.

Corollary A.2. *For any odd positive integer k , $\mathbf{m}'_k = \frac{1}{2} \mathbf{m}_k$.*

Proof. We have

$$\mathbf{m}'_k = -\frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla w'_k|^2 dx' = -\frac{1}{4} \int_{\mathbb{R}^2 \setminus s_0} |\nabla \tilde{w}'_k|^2 dx'.$$

Using Lemma A.1 and the conformal invariance of the L^2 -norm of the gradient, we find

$$\int_{\mathbb{R}^2 \setminus s_0} |\nabla \tilde{w}'_k|^2 dx' = \int_{\mathbb{R}_+^2} |\nabla w_k|^2 dx = -2\mathbf{m}_k. \quad \square$$

In particular, Corollary A.2 and Proposition 2.2 imply that

$$\mathbf{m}'_k = -\frac{k\pi}{4 \cdot 2^{2k-1}} \left(\binom{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2,$$

thus proving, in view of [1, Theorem 1.2], the explicit constant appearing in the asymptotic expansion of Theorem 1.8.

A.2 The u -capacities of segments

In this last section, we simplify the constant C_k occurring in [3, Lemma 2.3].

Proposition A.3. *For any positive integer k ,*

$$C_k = \frac{k}{4^{k-1}} \left(\binom{k-1}{\lfloor \frac{k-1}{2} \rfloor} \right)^2.$$

Proof. According to Equation (22) in [3, Lemma 2.3],

$$C_k = \sum_{j=1}^k j |A_{j,k}|^2,$$

where $A_{j,k}$ is the j -th cosine Fourier coefficient of the function $\eta \mapsto (\cos \eta)^k$. To be more explicit, let us expand $(\cos \eta)^k$ into a trigonometric polynomial. We write

$$(\cos \eta)^k = \left(\frac{e^{i\eta} + e^{-i\eta}}{2} \right)^k = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} e^{(k-2j)i\eta}.$$

By grouping the terms of the sum in pairs starting from opposite extremities, we find

$$(\cos \eta)^k = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{j} \cos((k-2j)\eta) + c_k$$

where

$$c_k = 0 \text{ if } k = 2p + 1 \quad \text{and} \quad c_k = \frac{1}{2^k} \binom{k}{p} \text{ if } k = 2p.$$

It follows that

$$C_k = \frac{1}{4^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j}^2$$

and we conclude using Lemma 2.5. □

Proposition A.3 and [3, Theorem 1.16] provide the the explicit constant appearing in the asymptotic expansion of Theorem 1.11.

B Auxiliary results for eigenvalues variations

This section is dedicated to the proof of Propositions 3.7 and 3.8. In order to make a connection to the results of [3], which we use, let us present an alternative characterization of the eigenvalues $(\mu_j)_{j \geq 1}$ and $(\mu_j(\varepsilon))_{j \geq 1}$. We define

$$\widehat{\mathcal{Q}}_\varepsilon^s := \left\{ u \in H_0^1(\widehat{\Omega} \setminus \Gamma_\varepsilon) : u \circ \sigma = u \right\},$$

and we denote by $\widehat{q}_\varepsilon^s$ the restriction of \widehat{q}_0 (see the paragraph preceding Theorem 1.10 for the notation) to $\widehat{\mathcal{Q}}_\varepsilon^s$. One can then check that we obtain the eigenvalues $(\mu_j(\varepsilon))_{j \geq 1}$ from $\widehat{q}_\varepsilon^s$ by the min-max principle. In the same way, we define

$$\widehat{\mathcal{Q}}^s := \left\{ u \in H_0^1(\widehat{\Omega}) : u \circ \sigma = u \right\},$$

we denote by \widehat{q}^s the restriction of the quadratic form \widehat{q}_0 , and one can check that we obtain the eigenvalues $(\mu_j)_{j \geq 1}$ from \widehat{q}^s by the min-max principle. Let us note that $-\Delta^s$, defined in Remark 3.4 as a self-adjoint operator in $\ker(I - \Sigma)$, is the Friedrichs extension of \widehat{q}^s . We denote by $-\Delta_\varepsilon^s$ the Friedrichs extension of $\widehat{q}_\varepsilon^s$, which is also a self-adjoint operator in $\ker(I - \Sigma)$.

Let us first prove Proposition 3.7. Since $\mu_N(\varepsilon) \geq \mu_N$ for all $\varepsilon \in (0, \varepsilon_0]$ and since $\varepsilon \mapsto \mu_N(\varepsilon)$ is non-decreasing, we have existence of $\mu_N^* := \lim_{\varepsilon \rightarrow 0^+} \mu_N(\varepsilon)$, with $\mu_N^* \geq \mu_N$. It only remains to show that $\mu_N^* \leq \mu_N$. In order to do this, let us note that the space

$$\mathcal{D}^s := \left\{ u \in C_c^\infty(\widehat{\Omega} \setminus \{0\}) : u = u \circ \sigma \right\}$$

is dense in $\ker(I - \Sigma)$. Indeed, the space $C_c^\infty(\widehat{\Omega} \setminus \{0\})$ is dense in $L^2(\widehat{\Omega})$, since $\{0\}$ has measure 0. Therefore, if we fix $u \in \ker(I - \Sigma)$, there exists a sequence $(\varphi_n)_{n \geq 1}$ of elements of $C_c^\infty(\widehat{\Omega} \setminus \{0\})$ converging to u in $L^2(\widehat{\Omega})$. We now set $\widetilde{\varphi}_n := 1/2(\varphi_n + \varphi_n \circ \sigma)$. We have $\widetilde{\varphi}_n \in \mathcal{D}^s$ for every integer $n \geq 1$. Since $u = 1/2(u + u \circ \sigma)$, we have the inequality

$$\|\widetilde{\varphi}_n - u\|_{L^2(\widehat{\Omega})} \leq \frac{1}{2} \|\varphi_n - u\|_{L^2(\widehat{\Omega})} + \frac{1}{2} \|\varphi_n \circ \sigma - u \circ \sigma\|_{L^2(\widehat{\Omega})} = \|\varphi_n - u\|_{L^2(\widehat{\Omega})},$$

and this implies that the sequence $(\widetilde{\varphi}_n)_{n \geq 1}$ converges to u in $\ker(I - \Sigma)$.

According to the min-max characterization of eigenvalues and the previous density result,

$$\mu_N = \inf_{\substack{\mathcal{E} \subset \mathcal{D}^s \\ \dim(\mathcal{E})=N}} \max_{u \in \mathcal{E}} \frac{\widehat{q}_0(u)}{\|u\|^2}.$$

Let us now fix $\delta > 0$ and an N -dimensional subspace $\mathcal{E}_\delta \subset \mathcal{D}^s$ such that

$$\max_{u \in \mathcal{E}_\delta} \frac{\widehat{q}_0(u)}{\|u\|^2} \leq \mu_N + \delta.$$

There exists $\varepsilon_1 > 0$ such that $\mathcal{E}_\delta \subset \widehat{\mathcal{Q}}_\varepsilon^s$ for every $\varepsilon \in (0, \varepsilon_1]$. This implies that, for every $\varepsilon \in (0, \varepsilon_1]$,

$$\mu_N(\varepsilon) = \min_{\substack{\mathcal{E} \subset \widehat{\mathcal{Q}}_\varepsilon^s \\ \dim(\mathcal{E})=N}} \max_{u \in \mathcal{E}} \frac{\widehat{q}_\varepsilon^s(u)}{\|u\|^2} \leq \max_{u \in \mathcal{E}_\delta} \frac{\widehat{q}_0(u)}{\|u\|^2} \leq \mu_N + \delta.$$

Passing to the limit, we obtain first $\mu_N^* \leq \mu_N + \delta$, and then $\mu_N^* \leq \mu_N$, concluding the proof.

Let us finally prove Proposition 3.8. We recall that, as a corollary of Theorem 1.10 in [3], taking into account Proposition A.3 we have the following result.

Proposition B.1. *Let $\widehat{\lambda}_N$ be a simple eigenvalue of $-\widehat{\Delta}$ and u_N an associated eigenfunction normalized in $L^2(\widehat{\Omega})$. Let us assume that $u_N \in \widehat{\mathcal{Q}}^s$. For $\varepsilon > 0$ small, we denote as $\widehat{\lambda}_N(\varepsilon)$ the N -th eigenvalue of the Dirichlet Laplacian in $\widehat{\Omega} \setminus \Gamma_\varepsilon$. If $u_N(0) \neq 0$, then*

$$\widehat{\lambda}_N(\varepsilon) = \widehat{\lambda}_N + \frac{2\pi}{|\log(\varepsilon)|} u_N(0)^2 + o\left(\frac{1}{|\log(\varepsilon)|}\right).$$

If

$$r^{-k} u_N(r \cos t, r \sin t) \rightarrow \widehat{\beta} \cos(kt) \text{ in } C^{1,\tau}([0, \pi], \mathbb{R})$$

as $r \rightarrow 0^+$ for all $\tau \in (0, 1)$, with $k \in \mathbb{N}^*$ and $\widehat{\beta} \in \mathbb{R} \setminus \{0\}$, then

$$\widehat{\lambda}_N(\varepsilon) = \widehat{\lambda}_N + \frac{k\pi\widehat{\beta}^2}{4^{k-1}} \left(\begin{matrix} k-1 \\ \lfloor \frac{k-1}{2} \rfloor \end{matrix} \right)^2 \varepsilon^{2k} + o(\varepsilon^{2k}).$$

Let us note that if the hypotheses of Proposition B.1 are satisfied, $\widehat{\lambda}_N$ is a simple eigenvalue of $-\Delta^s$ and u_N an associated eigenfunction. But the converse is not true. Indeed, we have seen in Section 3.5, in the case of $\widehat{\lambda}_3$ for the unit disk that $\widehat{\lambda}_N$ can be simple for $-\Delta^s$ without being simple for $-\widehat{\Delta}$. Proposition B.1 is therefore weaker than Proposition 3.8. However, the proof of Theorem 1.10 in [3] can be adapted to prove Proposition 3.8. Let us sketch the changes to be made. The proof in [3] mainly relies on Theorem 1.4 of [3], and uses the u -capacity and the associated potential defined in [3, Equations (6), (7), and (8)]. The following Lemma gives an alternative expression when both u and the compact set K are symmetric; it follows easily from Steiner symmetrization arguments.

Lemma B.2. *If $u \in \widehat{\mathcal{Q}}^s$ and $K \subset \widehat{\Omega}$ is a compact set such that $\sigma(K) = K$, then*

$$\text{Cap}_{\widehat{\Omega}}(K, u) = \min \left\{ \widehat{q}^s(V) : V \in \widehat{\mathcal{Q}}^s \text{ and } u - V \in H_0^1(\widehat{\Omega} \setminus K) \right\}$$

and the potential $V_{K,u}$ attaining the above minimum belongs to $\widehat{\mathcal{Q}}^s$.

Our proof of Proposition 3.8 relies on the following analog to [3, Theorem 1.4].

Proposition B.3. *Let μ_L be a simple eigenvalue of $-\Delta^s$ and u_L an associated eigenfunction, normalized in $L^2(\widehat{\Omega})$. Then*

$$\mu_L(\varepsilon) = \mu_L + \text{Cap}_{\widehat{\Omega}}(\Gamma_\varepsilon, u_L) + o(\text{Cap}_{\widehat{\Omega}}(\Gamma_\varepsilon, u_L)).$$

In order to prove Proposition B.3, we note that Lemma B.2 implies in particular that $u_L - V_{\Gamma_\varepsilon, u_L}$ is the orthogonal projection of u_L on $H_0^1(\widehat{\Omega} \setminus \Gamma_\varepsilon) \cap \widehat{\mathcal{Q}}^s$ and $\text{Cap}_{\widehat{\Omega}}(\Gamma_\varepsilon, u_L)$ the square of the distance of u_L from $H_0^1(\widehat{\Omega} \setminus \Gamma_\varepsilon) \cap \widehat{\mathcal{Q}}^s$, both defined with respect to the scalar product induced by \widehat{q}^s on $\widehat{\mathcal{Q}}^s$. We also note that we can use the estimates of $V_{\Gamma_\varepsilon, u_L}$ given in Lemma A.1 and Corollary A.2 of [3]. We can therefore repeat step by step the proof of Theorem 1.4 in Appendix A of [3], replacing $L^2(\widehat{\Omega})$ by $\ker(I - \Sigma)$, $H_0^1(\widehat{\Omega})$ with $\widehat{\mathcal{Q}}^s$, $H_0^1(\widehat{\Omega} \setminus \Gamma_\varepsilon)$ by $H_0^1(\widehat{\Omega} \setminus \Gamma_\varepsilon) \cap \widehat{\mathcal{Q}}^s$, \widehat{q} and \widehat{q}_ε by \widehat{q}^s and $\widehat{q}_\varepsilon^s$, $-\widehat{\Delta}$ and $-\widehat{\Delta}_\varepsilon$ by $-\Delta^s$ and $-\Delta_\varepsilon^s$, $\widehat{\lambda}_N$ by μ_L and $u_N \in H_0^1(\widehat{\Omega})$ by $u_L \in \widehat{\mathcal{Q}}^s$. We obtain Proposition B.3. The estimates of $\text{Cap}_{\widehat{\Omega}}(\Gamma_\varepsilon, u)$ proved in [3, Section 2] then give us Proposition 3.8.

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