CONCEPTUAL, EXTRANEOUS OR OTHER?: A CONFIGURATIVE REVIEW CONCERNING MATHEMATICAL METAPHORS IN EMPIRICAL EDUCATION RESEARCH

Abstract

In this review, metaphor is seen as an essential phenomenon for conceptualisation in mathematical teaching and learning. The aim of this review is to discern some patterns in concept use concerning metaphor, within 25 empirical mathematics education research papers, guided by two perspectives on metaphor; conceptual metaphor and extraneous metaphor. The analysis found explicit, vague or absent theoretical definitions of the metaphor concept as well as several metaphor examples, generating three categories of concept use; conceptual metaphor, extraneous metaphor, and potentially conceptual/extraneous. The review stresses the need for a theoretical and empirical distinction between different types of metaphors, in order to sharpen the analytical tools available to researchers and, by extension, to more clearly highlight the specific function of metaphor in teachers' classroom work.

Keywords: configurative review, metaphor, conceptual metaphor theory, mathematics education

Introduction

In the last few decades, the role of metaphors for human cognition, abstract thought, communication and culture, has been made prominent in a number of disciplines (Gibbs Jr., 2008). In the groundbreaking work *Metaphors we live by* (Lakoff & Johnson, 1980) our whole conceptual system is described as metaphoric; *Conceptual metaphor theory* (CMT in the following)—has been highly influential in metaphor research (Tendahl & Gibbs Jr., 2008) and the role of metaphor has been highlighted in the comprehension and communication of abstract social concepts (Winter & Yoshimi, 2020), concept formation (Borghi et al., 2017), and several other domains (Gibbs Jr., 2017).

An early interpretation of the metaphor concept from a mathematics education perspective has been a linguistic one, where metaphor serves a function within a *mathematics register* (Halliday, 1974, 1978; Pimm, 1987, 1995) where one borrows words like "face", "degree", and "natural", from everyday English, and uses them metaphorically, in order to be fully able to communicate about mathematics (Pimm, 1987). Presmeg (1998) suggested the imagery that accompany metaphors as "essential components in the representation of mathematical constructs for an individual" (p. 26). Metaphors are seen as representations, as they describe or stand for "one aspect of one's experience by another" (Kaput, 1991, p. 53).

Sfard (1997) underlined metaphors as tools of conceptualisation, their constitutive power of new conceptual structures, and how a metaphor brings concepts into being. Three years later, a breaking-point was reached, when Lakoff and Núñez (2000) published their book *Where Mathematics Comes From*, in which they claimed that all mathematical concepts are embodied and constituted through metaphorical reasoning. This widened perspective on metaphor has left a significant impact on the mathematics education community (Schiralli & Sinclair, 2003) but not without its fair share of criticism and debate (see Wagner, 2013; Schlimm, 2013), for instance regarding the issue of metaphors' role in creating mathematical entities (see Parsons & Brown, 2003). In addition, it has been noted that CMT "offers a high-level analogy to formulate a theory" (Tall, 2014, p. 88) about the constitution and ontology of mathematical concepts (Abrahamson, et al., 2016), while it gives us few clues as to the role of metaphor in the context of teaching and learning mathematics. Nevertheless, many scholars have throughout the recent decades acknowledged the potential of metaphorical reasoning in mathematical work (Soto-Andrade, 2014, 2018) while they have done this from different theoretical vantage points.

One differentiation is the potentially constitutive role of metaphor; if metaphor is a "genesis of mathematical ideas" (Núñez, 2008, p. 340), as constitutive of mathematical concepts, or if it serves (merely) illustrative purposes, describing "an already existing notion" (Sfard, 2008, p. 344), forming conceptions in the mind of the mathematising individual. The former perspective is, as previously mentioned, termed conceptual metaphor (CM). The latter perspective is for the purpose of this text called *extraneous metaphor* (EM) (Lakoff & Núñez, 2000), and the meaning of these two perspectives will be further explicated under "Theoretical considerations".

The choice of concept and theoretical perspective on metaphor, has consequences for what can be said about the function of metaphors in mathematics instruction but also for its potential as an analytical concept within research. As a consequence, we researchers limit ourselves in what questions we see as possible to ask, but we also limit the possible insights that research can contribute. This review will search for discrepancies and describe some patterns in the shape of three categories of metaphor concept use in empirical mathematics research. The results highlight some inconsistencies, and point to the importance of distinguishing the nature of metaphors used in teaching and learning and in the research concerning this practice.

Aim and research questions

This review discerns patterns in the use of the concept of metaphor in empirical mathematics education research, how metaphor is theoretically defined and framed, and seeks to understand the imprint of the two views on metaphor, without validating the theoretical constructs as such. The data consists of 25 empirical papers focusing on teaching and learning situations where metaphors are framed as possible tools for teaching and learning. The question guiding this review can be formulated as follows:

- What patterns in concept use can be discerned in 25 empirical mathematics education research papers, concerning the two views on metaphor; conceptual metaphor and extraneous metaphor?

Theoretical considerations

A standard definition of metaphor as a linguistic expression is that it is a statement in the form of "x is a y." One example of a metaphor in this form is MY LIFE IS A ROLLER-COASTER RIDE, where a concrete source domain ("roller-coaster ride") is metaphorically mapped onto an abstract target domain ("my life"). Inherent is a comparison or an interaction, between the two domains which can be explicit or implicit in the expression, creating and understanding of the target domain; in this case perhaps the sense of ups and downs in life.

From the perspective of mathematics and mathematics education, two perspectives on metaphor can be discerned in the literature, defined here with the help of Schiralli and Sinclair's (2003) two domains of mathematics; *conceptual mathematics* and *ideational mathematics*. Conceptual mathematics concerns mathematics as a subject or discipline, a set of ideas, patterns and rules that are distributed, negotiated and renegotiated in a social, cultural and historical context. Mathematical concepts are a set of objects, shared in this public context. Metaphor, from this perspective, is interpreted as a CM, a mechanism that plays an important part in the formation of mathematical concepts, in the "genesis of mathematical ideas" (Núñez, 2008, p. 340). CM is a "grounded, inference-preserving cross-domain mapping" (Lakoff & Núñez, 2000, p. 6)—a cognitive structure that allows us to use a source domain grounded in human embodiment to reason about another, often more abstract, target domain. There are two types of CMs; *grounding metaphors* (GM) and *linking metaphors* (LM). GMs are basic, directly grounded ideas, such as adding objects to or taking away objects from a collection, sets as containers, or members of a set as objects in a container (Lakoff & Núñez, 2000). Table 1 describes a selection of entailments of the grounding metaphor of ARITHMETIC IS OBJECT COLLECTION.

Table 1
The grounding metaphor of Arithmetic is object collection

Source domain *Object collection*		Target domain *Arithmetic*
The size of the collection	→	The size of the numbers
Bigger	→	Greater
Smaller	→	Less
Putting collections together	→	Addition
Taking a smaller collection from a larger collection	→	Subtraction

Adapted from Lakoff & Núñez (2000), p. 55

LMs, on the other hand, yield sophisticated ideas and are mappings from within mathematics itself, from one mathematical domain to another. Examples include NUMBERS AS POINTS ON A LINE, GEOMETRICAL FIGURES AS ALGEBRAIC EQUATIONS or OPERATIONS ON CLASSES AS ALGEBRAIC OPERATIONS (Lakoff & Núñez, 2000).

The basis for this perspective on metaphor is *embodied cognition*, the notion that our bodily actions provide part of the fundamentals for language and thought (Adams, 2010; Gibbs Jr., 2005). An essential concept for the understanding of CM is *image schemas*, "recurring, dynamic patterns of our perceptual interactions and motor programs that give coherence and structure to our experience" (Johnson, 1987, p. xiv), reflected in language, expressing spatial relations, movement, speed, etc. *The Source-Path-Goal schema* structures our perception and is revealed through words such as from, to, against, etc. *The Container schema* structures our understanding of concepts such as in, outside, inside, etc. These, and similar cognitive schemas, form the basis for CMs that enable us to talk about abstract concepts as concrete objects. From this perspective, expressions such as "*Take* 2 *from* 5 and you have 3 *left*" [italics in original] (Lakoff & Núñez, 2000, p. 56), conveys our metaphorical ability to process abstract concepts as physical objects in front of us (THE OBJECT METAPHOR, Font, Godino et al., 2010).

Within the domain of ideational mathematics, the concepts generated by CMs are turned into conceptions represented in the mind of the mathematising individual (Schiralli & Sinclair, 2003). These are the individual's own sense-making constructions, via previous experiences and metaphorical mappings from other mathematical and non-mathematical domains. These mappings are EMs (Lakoff & Núñez, 2000). Only one example of an EM is offered by Lakoff and Núñez—the idea of a staircase image as a visualisation for a "step function"—nor is any definition of the concept given. Schiralli and Sinclair however, mentions EMs as metaphorical mappings "that are made to and from the ongoing experiences (including non-mathematical ones) of the mathematician" (p. 84) and further describes these types of metaphors as sometimes "idiosyncratic, that individual mathematicians or students may construct in their learning experiences" (p. 85).

In the writings of Pimm (1987, 1995) and van Dormolen (1991) we find the similar concept of *extra-mathematical metaphor*, a device whereby we interpret mathematical ideas and processes in terms of real-world events, everyday objects and processes (Pimm, 1987).

These types of metaphors function as an "initial means of gaining experience of a new phenomenon" (Pimm, 1995, s. 127), a means for the teaching of mathematical concepts (van Dormolen, 1991). Extra-mathematical metaphors can be established within the mathematical community or school mathematics, or be idiosyncratic, spontaneously generated. Examples include A LINEAR EQUATION IS A GEAR or a GRAPH IS A PICTURE (Pimm, 1987), a student using the metaphor of a ship sailing on the x axis to find the acute angle in a trigonometry task (Presmeg (1992) or the comparison of certain properties of linear graphs with the action of climbing up hills (Moschkovich, 1996). The concept of extra-mathematical metaphor is here seen as equal to and interchangeable with EM, and for this presentation, the latter term is chosen.

Lakoff and Núñez (2000) emphasise the difference between EMs and CMs, as the former "... can be eliminated without any substantive change in the conceptual structure of mathematics, whereas eliminating GMs or LMs would make much of the conceptual content of mathematics disappear" (p. 53). In a thought experiment we can imagine "eliminating" the CM of arithmetic is object collection, which would make arithmetic processes lose much of their meaning. By contrast, eliminating the LINES ARE HILLS METAPHOR (Moschkovich, 1996), would not entail any conceptual loss in the mathematical domain of functions. Idiosyncratic and individually constructed EMs are nonetheless a powerful part of sense-making (Presmeg, 2002; Schiralli & Sinclair, 2003).

To summarise; CMs are constitutive of mathematical ideas within the domain of conceptual mathematics, through embodied experiences as the genesis of mathematical concepts in a social, historical, cultural context. Without a CM, the mathematical concept cannot exist (Winter & Yoshimi, 2020). EMs, on the other hand, are conducive to conceptions, within the domain of ideational mathematics and functions as a pedagogical, illustrative tool. Table 2 presents an overview of the distinctions between the two concepts.

Methodology, method, and execution

The overarching methodology in this review is a qualitative meta-synthesis; "an interpretive integration of qualitative findings in primary research reports" (Sandelowski & Barroso, 2007, p. 199) that seeks to understand and explain phenomena. The method involves induction and interpretation of concepts from different studies (Noblit & Hare, 2011). The review is systematic in the sense that the literature search performed was detailed and based upon a focused purpose where the identified research papers were appraised and synthesised using a step-by-step methodology (Green et al., 2006). However, this paper is classified as a configurative narrative literature review, as the aim of the present undertaking is to establish the state of knowledge within a field, where the phenomena studied are seen as multifaceted in nature, and where the analysed sources draw on diverse methodologies and have different theoretical points of departure (Andrews et al., 2021; Levinsson & Prøitz, 2017). This type of review is a synthesis of existing works that discuss theory and context and that draw conclusions into a holistic interpretation contributed by the reviewers' own experience, existing theories and models (Dochy, 2006; Meglio & Risberg, 2011).

Table 2
Definition of conceptual metaphor and extraneous metaphor

Aspect	Conceptual metaphor	Extraneous metaphor
A. Definition	An inference-preserving cross-domain mapping, from a source domain to a target domain (Lakoff & Núñez, 2000).	"Attempt to explain or interpret mathematical ideas and processes in terms of real-world events, and such metaphors can involve everyday objects and processes" (Pimm, 1987, p. 95)
B. Origin	Based on image schemas which "help to build reasoning by means of conceptual inference projections, among them metaphorical ones" (Font, Bolite et al., 2010, p. 132).	"Made to and from the ongoing experiences (including non-mathematical ones) of the mathematician" (p. 84). "Idiosyncratic, that individual mathematicians or students may construct in their learning experiences" (p. 85).
C. Function	GMs yields basic, directly grounded ideas (Lakoff & Núñez, 2000), "conceptually grounded in firsthand embodied experiences" (Kilhamn, 2018, p. 146). LMs yields sophisticated ideas, conceptualising "one domain of mathematics in terms of another" (Lakoff & Núñez, 2000, p. 99).	EMs are "offered as an initial means of gaining experience of a new phenomenon" (Pimm, 1995, p. 127), and "for ease of understanding" (Lakoff & Núñez, 2000, p. 6).
D. Elimination	Eliminating CMs "would make much of the conceptual content of mathematics disappear" (Lakoff & Núñez, 2000, p. 53).	"Can be eliminated without any substantive change in the conceptual structure of mathematics" (Lakoff & Núñez, 2000, p. 53).
E. Ontology	Closely linked to an embodied approach to cognition and mathematics: "All mathematical content resides in embodied mathematical ideas." (Núñez, 2008, p. 357).	Generally not linked to specific ontology.
F. Domain of mathematics	Creates concepts within the domain of conceptual mathematics (Schiralli & Sinclair, 2003).	Creates conceptions within the domain of ideational mathematics (Schiralli & Sinclair, 2003).

Search strategies

No previous review could be found that examined and discerned patterns in the use of the concept of metaphor in empirical mathematics education research. In order to find relevant papers an EDS database search was conducted, searching for term "metaphor*" in the abstracts and keywords, complemented with "mathematics", "education". If "metaphor*" was not found in the abstract or in the keywords of the paper, the paper was excluded. Only English-language scientific papers from peer-reviewed journals were included. No time limits were set for the search. An initial search was conducted January 2019, and a second search was conducted June 2021, yielding 357 papers between 1981 and 2021.

Inclusion and exclusion process

In order for a paper to be eligible for this review, it needed to meet four criteria:

1. The "teaching and learning context criterion"
2. The "theoretical concepts criterion"
3. The "primary concept criterion"
4. The "metaphor example criterion"

First, the concept of metaphor needed to be framed within a context of teaching and learning of mathematics, where people, regardless of age and place within the educational system, communicated and interacted in an experiment setting, interview, or classroom/lecture hall, with others or with material artefacts, computer screens/multimedia tools or images/graphical representations or notations, with the purpose of teaching or learning mathematics.

Second, the papers needed to present a clear theoretical framework, making it possible to identify essential concepts and the basic structure of ideas, abstractions, relationships, and formal theories guiding the research activities in the papers (Lester, 2005).

Third, metaphor needed to be a *primary concept* in the paper. This meant that the concept of metaphor needed to be a central concept in the paper, with the help of which the authors presented their results, discussed their findings and drew their conclusions.

Fourth, the papers needed to present clear metaphor examples through speech, gestures, images, graphic elements, symbolic notations, and so on, which were possible to analyse based on the purpose and guiding question of this review.

The first criterion excluded 324 papers, leaving 33 papers. The second criterion excluded four of these. The third criterion excluded four papers and the fourth criterion excluded two papers. Also included were two papers implicated by some of the remaining 23 papers and satisfying the four criteria, producing 25 papers as the basis for this review.

Table 3
Papers selected for review

Abrahamson, D., Gutiérrez, J. F., & Baddorf, A. K. (2012)
Alibali, M. W., & Nathan, M. J. (2012)
Arzarello, F., Robutti, O., & Thomas, M. (2015)
Caglayan, G., & Olive, J. (2010)
Carreira, S. (2001)
Chiu, M. M. (2001)
Danesi, M. (2003)
Dawkins, P. C. (2012)
Figueiras, L., & Arcavi, A. (2014)
Font, V., Bolite, J., & Acevedo, J. I. (2010)
González, G. (2013)
Junius, P. (2008)
Keene, K. A. (2007)
King, B. & Smith, C. P. (2018)
Lai, M. Y. (2013)
Magana, A. J. (2014)
Moreno, R., & Mayer, R. E. (1999)
Moschkovich, J. N. (1996)
Oehrtman, M. (2009)
Presmeg, N. C. (1992)
Sáenz-Ludlow, A. (2004)
Sfard, A. (2000)
Viirman, O. (2015)
Vitale, J. M., Swart, M. I., & Black, J. B. (2014)
Wittmann, M. C., Flood, V. J., & Black, K. E. (2013)

Data extraction and analysis

The data analysis process was structured in 5 steps and guided by a number of analytical questions. The end result of this process discerned three categories of concept use; 1) *Conceptual metaphor*, 2) *Potentially conceptual/extraneous*, and 3) *Extraneous metaphor*.

Step 1: Identifying essential concepts

The first step had the purpose of understanding the papers' essential concepts, and thereby giving an initial framing of metaphor in each paper. This analysis was carried out by reading the entire papers and specifically delving into the sections that described what the studies labelled *theoretical perspective* (e.g. Chiu, 2001), *theoretical framework* (e.g. Font et al., 2010), background (Magana, 2013) or *research framework*, the basic structure of ideas (i.e., abstractions and relationships) that served as the basis for the respective studies (Lester, 2005). One analytical question was posed in this initial part of the process:

a. What are the essential concepts (ideas, abstractions, and relationships), and how do these concepts frame and relate to metaphor?

Step 2: Identifying theoretical definition of metaphor

The second step concerned the papers' theoretical definitions of metaphor; to what extent the analysed papers delimited and defined the concept of metaphor. *Theoretical definition* is here defined as "the meaning we attach to the concept" (Shoemaker et al., 2004, Chapter 2, "Theoretical definitions" section). If there was no explicit theoretical definition of metaphor within the paper, for instance through direct quotations from other scholarly texts or reference to other researchers, then an implied definition was sought, through the author's further elaborations.

a. Is there an explicit theoretical definition of metaphor within the paper, for instance through direct quotations from other scholarly texts?

b. If this is true, how do the authors define the concept?

c. If no explicit definition exists, based on either the author's further elaboration or references to scholarly metaphor texts, what is the implied theoretical definition?

Step 3: Metaphor examples

The third step analysed how metaphor was made visible in the papers, the concrete manifestations of the theoretical concept.

a. What metaphor examples are presented in the papers' results?

Step 4: Metaphor elimination

The fourth step investigated what role the given metaphor examples could have for the mathematical concept in focus; if the metaphor functions as a genesis of mathematical ideas (Núñez, 2008), conducive to concepts, or as an illustrative tool, conducive to conceptions (Schiralli & Sinclair, 2003).

a. Can the metaphor examples given in the papers "be eliminated without any substantive change in the conceptual structure of mathematics"? (Lakoff & Núñez, 2000, p. 53)

Table 4
Categorisation example

	Example paper 1: Font, Bolite, & Acevedo (2010)	Example paper 2: González (2013)
Step 1: Identifying essential concepts	Embodied cognition, image schema, metaphor, CMT	Collective remembering, didactic memory, metaphor, prototypical images
Step 2: Identifying theoretical definition	”We interpret 'metaphor' as the comprehension of an object, thing, or domain in terms of another one. Metaphors create a conceptual relationship between an initial or source domain … and a final or target domain ... while properties are projected from the first to the second domain" (p. 132).	"Metaphor involves the seeing (and therefore the understanding) of one thing in terms of another … Extra-mathematical metaphors (Pimm, 1981) is a label for metaphors that connect mathematical concepts with objects or events from everyday life" (p. 399).
Step 3: Metaphor example	”Now, where would you put the maximum and the minimum?" (p. 139).	”Okay, the right angle that I'm talking about at the top is this B. H is the one that we have at the base here. The short side over here gives me the A. This one over here is D … Now, remembering what we did with this with the parachute guy thing" (p. 406).
Step 4: Metaphor elimination	The object metaphor (Font, Godino et al., 2010) is crucial to mathematical communication and conceptualisation: No elimination possible.	The parachute guy metaphor is not crucial to mathematical communication and conceptualisation: Elimination possible.
Step 5: Categorisation	Conceptual metaphor	Extraneous metaphor

Step 5: Categorisation

The fifth step in the analysis tied together the four previous parts of the result and illuminated the data from the two perspectives on metaphors. This generated three categories of concept use. This final part of the analysis was guided by the following questions:

a. How can the papers be categorised as they concern the two perspectives on metaphor, in light of the papers' conceptual domains, theoretical definitions, metaphor examples, and the issue of metaphor elimination?

In relation to this categorisation and analysis, an important clarification is in place: In this study, no assessment is made of whether the given metaphor examples can really be understood as metaphors. Carrying out such an analysis of each example would make this study far too comprehensive. The analysis is made based on the authors' metaphor definitions and examples. All of these papers frame a variety of verbal statements, images, gestures, diagrams, as metaphorical phenomena and in this text an attempt is made to categorize these phenomena from the two perspectives on metaphor. This study also does not seek to validate CMT as an idea or theory. The study thus becomes an overview of how empirical research over the years has interpreted the concept of metaphor, partly based on CMT as a dominant (albeit criticised) framework.

Results: Three categories of concept use

Conceptual metaphor

One of the seven papers in this first category presented an explicit definition of metaphor and showed through other ideas and references a clear connection to the embodiment of mathematics and CMT. Font et al. (2010) defined metaphor as

> The comprehension of an object, thing, or domain in terms of another one. Metaphors create a conceptual relationship between an initial or source domain (the one we are familiar with) and a final or target domain (the new or abstract one), while properties are projected from the first to the second domain. (Font et al., 2010, p. 132)

The metaphor examples consisted of verbal statements that referred to image schemas, for example the use of the object schema and related metaphors (Font et al., 2010; Núñez, 2000; Lakoff & Núñez, 2000). One example was the teachers' and students' verbal use of the MOTION METAPHOR: "this shows us that the graph is going to pass through here and here" (Font et al., 2010, p. 145). The informants in these studies thus spoke metaphorically of abstract mathematical objects as physical phenomena in a material reality.

Six papers lacked explicit definitions of the metaphor concept (i.e., Arzarello et al., 2015; Alibali & Nathan, 2012; Danesi, 2003; Keene, 2007; Vitale et al., 2014; Wittmann et al., 2014), although a majority of them linked their texts to CMT through further reasoning. What mainly connected these six papers to this category were the metaphor examples and the issue of elimination. The examples were based on image schemas or other CMT-related ideas, for instance talking about mathematical objects as physical objects, using a language of motion, saying "we get the v on the other side" (Wittmann et al., 2014, p. 177), through grasping and selecting gestures expressing the conceptual metaphor ARITHMETIC IS OBJECT COLLECTION (Arzarello et al., 2015), and linear equations, miming a removal gesture, expressing the metaphor of taking objects away from an equation (Alibali & Nathan, 2012).

Hand gestures on a computer screen depicting geometric shapes were said to represent novel grounding metaphors (Vitale et al., 2014), and diagrams representing the metaphors TIME IS A POINT ON A LINE and TIME IS A QUANTITY, related to the CM of TIME PASSING IS MOTION (Lakoff, 1993), emphasised the semiotic aspects of mathematical conceptualisation (Danesi, 2003). The metaphor TIME AS UNIDIMENSIONAL SPACE and THE FICTIVE MOTION METAPHOR were also verbally manifested, when a student reasoned that "time never stops" (Keene, 2007, p. 234) to create a constant to depict a skateboarder's journey in a mathematical problem concerning functions.

Potentially conceptual/extraneous

The second category consisted of seven papers where the authors only in some cases explicitly defined metaphor and the ones that did, did so in line with CM. Some papers lacked framing in line with CMT, but the metaphor examples given can be found in CMT-related literature, and two papers consisted of both potential CMs and EMs, hence creating this in-between category.

Chiu (2001) used the following definition: ”People reason metaphorically by projecting a source situation’s (e.g., motion) goals, entities, relations, and actions onto the target situation (e.g., polygon) to create new target goals, entities, relations, and actions” (p. 94) and one of the metaphor examples conveyed the metaphorical notion of mathematical objects as physical objects: ”I haven’t gone all the way up yet so it’d be negative 30” (Chiu, 2001, p. 102). Another metaphor was however seen as extraneous. The metaphor of ARITHMETIC IS SOCIAL TRANSACTION—“I lost 70, so I had to make up for that” (p. 110)—is in this context seen as possible to eliminate without conceptual repercussions within the domain of arithmetic.

Carreira (2001) defined metaphor “as a correspondence between two conceptual domains. It consists of a mechanism that allows us to understand one domain in terms of another, usually more familiar or closer to our daily experiences” (p. 264), and further elaborated the arguments in accordance with, among others Lakoff and Johnson (1980) and Black (1962). The metaphor concerned diagrams and symbolic notations, for instance a COUNTERBALANCE METAPHOR in a task concerning a utility function: “If we just increase the amount of beer, the amount of wine will immediately decrease” (Carreira, 2001, p. 272). This reasoning is said to reveal an “underlying metaphor and how the idea of counterbalance gives meaning to mathematical aspects of the utility model” (p. 272) and has been put forward as an important metaphor for the teaching of inequalities and linear equations (Boero et al., 2001; Vlassis, 2002), and related to the GM of ARITHMETIC IS OBJECT COLLECTION and its extensions (Lakoff & Núñez, 2000). THE MOUNTAIN METAPHOR in this paper was seen as extraneous as it was possible to eliminate, and spontaneously generated during student talk about the shape of a line in a graph: ”Look, I have this mountain. This inside area tells me the maximum height” (Carreira, 2001, p. 278).

Junius (2008) defined metaphors as “cross-domain mappings in which abstract mathematical ideas (target) are understood in terms of concrete, familiar domain (source)” (p. 988), and emphasised metaphors’ a priori function that creates new conceptual structures. The metaphor examples were for instance THE LINE OF SIGHT METAPHOR in Euclidean and non-Euclidean geometry, expressed by a student: “In marching band [sic], the line is straight in front of you when you only can see the person directly in front of you and you cannot see any marchers before that” (fig. 1, p. 991). Although this metaphor was grounded in embodied experiences, it could be seen as extraneous as it is idiosyncratic and possible to eliminate.

Magana (2014) defined metaphor as a "cross-domain mapping" (p. 369) and made no further theoretical discernments of the concept. The metaphor example was a logarithmic scale, with powers of ten represented as points on a line. This type of visual metaphor of scale could be seen as related to the number line metaphor, an important conceptual tool, connected to the embodiment of mathematics.

Three papers offered no theoretical definition, did not relate their reasoning to embodied cognition or CMT in any substantial matter, while their metaphor examples could be seen as powerful GMs and LMs such as ARITHMETIC IS OBJECT COLLECTION, ARITHMETIC IS OBJECT CONSTRUCTION, NUMBERS ARE POINT ON A LINE (Lakoff & Núñez, 2000) and THE BALANCE METAPHOR (Boero et al., 2001; Vlassis, 2002). In Caglayan and Olive (2010), the metaphor examples were the talk of and images of cups and tiles representing unknowns and knowns in a linear equation, presented on a computer screen. Another metaphor on a computer screen was THE NUMBER LINE METAPHOR (Moreno & Mayer, 1999) in a study concerning multimedia supported learning of arithmetic operations. Sáenz-Ludlow (2004) framed metaphor as an icon in the Peircean theory of signs (referencing Peirce, 1906, 1976). The participating students used a splitting metaphor while working on arithmetical computations: "80 plus 15 equals 95. I split the 15 into 5, 5, and 5. Then 95 plus 5 equals 100" (Sáenz-Ludlow, 2004, p. 44). The authors noted the "resemblance between the meaning of the word split in the physical world (separation of an object into parts) and the numerical world (separation a [sic] number into units)" (p. 45).

Oehrtman (2009) relied, without explicit definition, on Black's (1962) theories on metaphor and models. The study concerned a number of students' reasoning and ideas about limit concepts, categorised according to a number of so-called metaphor clusters. One of the starting points for the study was THE BASIC METAPHOR OF INFINITY (BMI), which, according to Lakoff and Núñez (2000), is a central CM for all cases of actual infinity. Oehrtman (2009) showed the student use a number of metaphors not in line with formal definitions. For instance, within THE APPROXIMATION METAPHOR cluster, "limits are viewed as a process of estimating some quantity with various degrees of accuracy" (Oehrtman, 2002, p. 148). A student reasoning within this cluster can be exemplified with the following quotation:

> In fact the power series for sin x will approximate a value infinitely close to the value of sin x and even a remainder can be calculated … The power series of sin x continues forever

> depending on how close you want your value to come to the value of sin x … The remainder is designed to show how much a power series deviates from the value of a function at a particular point ... the power series or polynomial for sin x is an approximation of its value that can be as close of a value as you want it to be. (Oehrtman, 2009, p. 415)

Although technically incorrect, these and similar metaphors in the study were not necessarily to the detriment of student understanding. They are in this context seen as closely related to established conceptual metaphors (such as the BMI), while they have idiosyncratic elements and show discrepancies with formal definitions (Oehrtman, 2009).

Extraneous metaphor

This category consisted of eleven papers, of which three papers supplied distinct theoretical definitions. These three papers all supplied examples of EM possible to eliminate.

King and Smith (2018) presented their study within an embodied cognition conceptual domain while at the same time defining metaphors as extraneous, and metaphorical mappings that are made "to and from the ongoing experiences (including non-mathematical ones) of the mathematician" (Schiralli & Sinclair, 2003, p. 84). The metaphor examples (technically similes, in this case) were utterances by students: "It's like a soccer player screaming, goal!" (p. 586), "He was like an airplane" (p. 586) and "It's like the statue of liberty" (p. 586).

González (2013) defined metaphors as phenomena "that connect mathematical concepts with objects or events from everyday life" (p. 47), and further elaborated a semiotic and linguistic view of metaphor as a mnemonic for students. The metaphor example concerned a "parachute guy" drawn by the teacher on the whiteboard as a device for the finding of angles in a triangle:

> The right angle that I'm talking about at the top is this B. H is the one that we have at the base here. The short side over here gives me the A. This one over here is D … This one's 2.8. This one's 9.2. Now, remembering what we did with this with the parachute guy thing. What … how did it all work? (González, 2013, p. 406)

Lai (2013) indeed referenced Lakoff (1994) and defined metaphor "as a correspondence between two conceptual domains" (Lai, 2013, p. 32) with a further elaboration referencing Carreira's (2001) definition (see above). Still, metaphor was in Lai predominantly framed in relation to linguistic concepts such as register and mathematics as language, hence not embracing the embodied nature of mathematics. The metaphor examples (technically analogies) were provided verbally by the teacher, in the process of constructing a mathematical meaning for the concept of similarity:

> When you take a photograph … and that man in the photograph is so small … the image still looks like the man. It is because the angles … remain unchanged. For example, a pupil with circular face, the image of his photograph still has a circular face. If his face becomes sharp, you will say that the image is not [him]. It isn't like [him]. In a convex/concave mirror, the image is not like [anyone]. It is because the angles are changed. [The object] is short but the image becomes lengthened [in a convex/concave]. (Lai, 2013, p. 36–37)

Nine papers did not offer any explicit definition of the metaphor concept. One (i.e. Figueiras & Arcavi, 2014) acknowledged the embodied experiences in mathematical conceptualisation, while framing metaphors as essential in making "sense of new (mathematical) ideas or objects in terms of already often non-mathematical) existing ones" (p. 126). The authors argued for the central role metaphors play in doing mathematics, while they did not fully accept the notion that "all mathematics grows out of metaphors" (p. 126). The metaphors were manifested through visual language and haptic gestures when a teacher metaphorically described ellipses to blind students as "The headlights of a car, when looking down … illuminating the road" (p. 128). Although this metaphorical expression can be understood as embodied and as powerful tools, it is interpreted in this context as extraneous, mainly based on the elimination criterion.

Abrahamson et al. (2012) described metaphors as "conducive to meaningful engagement in mathematical tasks" (p. 56) and provided several examples of EMs. One was THE SNAKE METAPHOR, where a student tried to find a pattern in a combinatorial task: "Well, they're really L-shaped, and well I kinda like made, kinda like a pattern like, it's like a snake, it's going 'dun dun dun' … and then it's going swivel around, and like that" (p. 62). Other papers within this section of this category were Moschkovich (1996), Presmeg (1992) and Dawkins (2012). Metaphor examples included THE LINES ARE HILLS METAPHOR; "Do you think this one … is steeper than this one … If you had to climb up this hill … would it be harder?" (Moschkovich, 1996, p. 259–260), THE WATER LEVEL METAPHOR; "It's like a ship sailing: can't sail that way really … There, that, sort of ... it can" (Presmeg, 1992, pp. 599–

600), and THE PARTY METAPHOR in a study concerning sequence convergence; "If you want to find a party, see where everyone is at" (Dawkins, 2012, p. 335), "How many terms make a party?" (p. 335) or "*K* is that last term where, that's the last straggler and everything after this *K* are all in that neighborhood or in the party" [italics in original] (p. 337).

Viirman (2015) utilised a commognitive analysis and framed the use of metaphorical expressions as explanation routines. Examples included: "If we have a vector that lies in this plane, then nothing will happen – if you draw with a piece of lipstick on the bathroom mirror then the mirror image is itself, more or less" (p. 1172) and "So if the function f is bounded there, then the function cannot take off towards infinity. There is a *floor and a ceiling* [emphasis added], such that the graph of the function stays between the floor and the ceiling" (p. 1172).

Finally, in Sfard (2000) a verbal metaphor was described as a point of departure in the construction process of a mathematical object. THE METAPHOR OF CONSISTENCY presented in the paper, where a student compared the life-span of a collection of batteries, interpreting a graph, could in the light of the elimination criterion be seen as extraneous. According to Sfard (2000), the metaphor used was drawn from childrens' everyday experience, where one knows "that a consistent person is one whose actions are in agreement with one another and whose future behavior is more predictable than that of the person who lacks consistency" (p. 314).

Discussion

This review aimed to discern patterns in the use of the concept of metaphor in empirical mathematics education research. It had the purpose of understanding in what different ways metaphor was theoretically defined and framed in 25 empirical papers and sought to describe the imprint of two views on metaphor, in this research. Apart from (in very few cases) the pattern of a distinct positioning in line with CM on one hand, and EM on the other, this section will discuss the absence or vagueness and/or the unclear alignment of the theoretical definitions with the metaphor examples. These patterns will be discussed in relation to the strengths and weaknesses of the categorisation in itself, and the (in some cases unnecessary) impact (and misuse) of CMT.

The first pattern is the issue of the few cases of explicit theoretical definitions. Only one paper (Font el al., 2010) was categorised as conceptual metaphor fully in line with CMT and embodied cognition, as the definition lined up with the broader references and metaphor examples. Three related papers (Arzarello et al., 2014; Alibali & Nathan, 2012; Vitale et al., 2014) did not explicitly define metaphor but supplied relevant reasoning, concepts and metaphor examples for this category. It is noteworthy that despite CMT's imprint in research, only a few papers got a distinct CMT-framing, aligning the theoretical definitions, framework and metaphor examples. On the other side of the metaphor spectrum we found two papers (González, 2013; King & Smith, 2018) that provided EM definitions and met with further reasoning and metaphor examples the necessary criteria for this category. The fact that so few papers aligned their concepts and metaphor examples in such explicit ways, may be due to the (too) strict categorisation this review is based on. However, this can also be seen as a call for stringency in concept use, for researchers to supply distinct theoretical and operational definitions of central concepts.

Another reason for this lack of explicitness may be due to the fact that the term and concept of extra-mathematical metaphor—which in this context was interpreted as equivalent to the chronologically later concept of EM—was conceptualised by David Pimm in the 1980s and is thus older and was later challenged by the CM framing. EM is by no means a widespread term and concept, as the reasoning regarding it is brief and found in a commentary by Schiralli and Sinclair (2003) on Lakoff and Núñez's (2000) book, and in a book review (Presmeg, 2002). Lakoff and Núñez similarly use the term only briefly in one paragraph. The research community seems content in some cases with incorporating all metaphor concepts with CMT without further investigating possible conceptual distinctions. This can be seen as an effect of the CMT hegemony, which has made it difficult to interpret the concept of metaphor in alternative ways. But as Presmeg and Schiralli and Sinclair point out, EMs can be powerful in mathematical thinking, and a distinction needs to be made between the different concepts of metaphor. EM as an analytical concept, therefore, deserves a place in the mathematics education research discourse.

The second pattern is the consequences of absent or vague definitions and the unclear alignment between theoretical definitions and metaphor examples. First, some of the metaphor examples shows how a paper with little reasoning in accordance with CMT or embodied cognition, still gives metaphor examples that share properties with established CMs. Granted, there are other established theoretical framings of metaphors within research

—for instance the perspective on mathematics and metaphors in a semiotic activity (Sáenz-Ludlow, 2004)—but, again, CMT in ways has seized the metaphorical discourse. It is, however, not an end in itself to place all powerful metaphor examples in a CMT category as in this review. This rather shows the need to employ alternative perspectives on metaphor in the context of mathematics education research.

Second, there are examples of papers (i.e. Carreira, 2001; Junius, 2008) where the framing is in line with CMT, while the metaphor examples can rather be categorised as extraneous, as the metaphors are idiosyncratic and possible to eliminate. Other researchers (i.e. Lai, 2013) combine in their reasoning the concept of metaphor based on older Lakovian texts with the reasoning by Pimm (1981, 1987) with the consequence that a certain ambiguity in concept use arises. There is also a tendency to limit Lakoff and Núñez's (2000) description of metaphors as a means towards meaningful engagement in mathematical tasks.

This shows that papers written at the beginning of the millennium or that use metaphor texts from the 80s and 90s tend to point to metaphors as extraneous. Again, this shows the imprint that Lakovian notion on metaphor has had in education research, while some researchers steer away from CMT, or use certain concepts in a vague manner. Here we find an intention to emphasise the power of metaphors, while not fully subscribing to an embodied cognition framework, avoiding a controversial discussion concerning the nature of mathematical concepts, or delving into a quarrel about the validity of CMT. In this review, there is but one paper that mentions this controversy as they make visible—although not fully ascribing to—the view that "all mathematics grows out of metaphors" (Figueiras & Arcavi, 2014, p. 126). In this context, such a description is unusual, as a clear majority of the papers often takes their starting point in Lakovian reasoning without presenting their own position.

The third issue within the second pattern concerns four papers (e.g. Dawkins, 2012; Moschkovich, 1996; Sfard, 2000; Viirman, 2015) with no theoretical definitions, no or few references to known metaphor texts, but where the metaphor examples presented make the papers possible to categorise as extraneous. The concept of metaphor is in these papers (with one exception) interpreted almost intuitively and the meaning of the concept is almost taken for granted. This may be due to the fact that the concept of metaphor in some of these papers coincide with other key concepts, and the main focus of the studies are elsewhere.

Standing out is Sfard (2000), a paper lacking definitions or references to other metaphor texts, while arguing for the role of metaphor in a discursive practice and its important role in the creation of mathematical objects. The metaphor examples, together with Sfard's own interpretation, were categorised as extraneous. Here, the categorisation in this review fails to cover a concept that could be labelled "the Sfardian CM," which is inherently something other than the Lakovian CM. Both concepts attach equal importance to metaphorical thinking and the role of metaphor in the origin, constitution and maintenance of mathematical concepts and objects.

Conclusion

Although this text was not intended to evaluate CMT as a theory, but only to examine, among other things, its imprint in research over the past decades, a commentary on the issue is in place. In all, a broad picture of the metaphor concept is presented in the data. Visual representations, graphical notations, verbal idiosyncratic examples, established explanatory models, gestures, seem to (according to the analyzed papers) entail some sort of metaphorical reasoning or projection. In this authors view, the described width of the concept simultaneously enriches and dilutes it. Apart from overt linguistic expressions in the form of "x is a y", the data gives examples of dead metaphors where utterances such as "it" and "go past" frames a mathematical object as a physical object, hence pointing towards conceptual metaphor (see Chiu, 2001). Likewise, there are examples of hand gestures referencing conceptual metaphors concerning arithmetic (see Arzarello et al., 2015). In all, conceptual metaphor seems to devour all kinds of phenomena.

This width, makes mathematical metaphor fuzzy, not clearly pointing to "the individual instances of the class of empirical objects to which it refers" (Blumer, 1969, p. 91) and it is some cases not distinguishable from other related classes of objects. Do the papers share their object of study only with multiple incarnations and operational instances? Alternatively, do these papers study different objects and phenomena, casually denoting them the same way? How the general metaphor concept can be separated from the concept of mathematical metaphor as well as from dead objectifying metaphors requires further discussion. Further, although EM as a concept is not widely used in the literature, this presentation shows that the concept is needed to distinguish it from CM (and from the general metaphor concept). In research about the role of metaphors in mathematical thinking, teaching and learning, a clear distinction should be made between these concepts and the domains on which they operate.

As previously noted, it is not an end in itself to categorize all types of metaphorical phenomena as related to CMT. What has become clear in this study, however, is that much of the empirical mathematical education research concerning metaphor over the past 40 years has in one way or another touched on, referred to, been grounded in, or distanced itself from, CMT. Despite the criticism CMT applied to mathematics and mathematics education has received—for example issues concerning language and culture (see Ernest, 2010) or its shortcomings in explaining metaphorical phenomena in the encounter between mathematizing individuals in problem-solving situations (see Abrahamsson, et al., 2016)—few other theoretical understandings of the concept of metaphor has of yet made any significant impression in empirical research.

References

*Abrahamson, D., Gutiérrez, J. F., & Baddorf, A. K. (2012). Try to see it my way: The discursive function of idiosyncratic mathematical metaphor. *Mathematical Thinking and Learning, 14*(1), 55–80.

Abrahamson, D., Sánchez–García, R. & Smyth, C. (2016). Metaphors are projected constraints on action: An ecological dynamics view on learning across the disciplines. In C.-K. Looi, J. L. Polman, U. Cress, & P. Reimann (Eds.), *Transforming learning, empowering learners, Proceedings of the International Conference of the Learning Sciences (ICLS 2016),* Vol. 1 (pp. 314-321). International Society of the Learning Sciences.

Adams, F. (2010). Embodied cognition. *Phenomenology and the Cognitive Sciences, 2010*(9), 619–628.

*Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*(2), 247–286.

Andrews, P., Bødtker Sunde, P., Nosrati, M., Petersson, J., Rosenqvist, E., Sayers, J., & Xenofontos, C. (2021). Computational estimation and mathematics education: A narrative literature review. *Journal of Mathematics Education*, *14*(1), 6–27.

*Arzarello, F., Robutti, O., & Thomas, M. (2015). Growth point and gestures: Looking inside mathematical meanings. *Educational Studies in Mathematics, 90*, 19–37.

Black, M. (1962). *Models and metaphor: Studies in language and philosophy*. Cornell University Press.

Blumer, H. (1969). *Symbolic interactionism: Perspective and method*. University of California Press.

Boero, P., Bazzini, L., & Garuti, R. (2001, July). Metaphors in teaching and learning mathematics: A case study concerning inequalities. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education* (pp. 185–192). Utrecht.

Borghi, A. M., Binkofski, F., Castelfranchi, C., Cimatti, F., Scorolli, C., & Tummolini, L. (2017). The challenge of abstract concepts. *Psychological Bulletin, 143*(3), 263–292.

*Caglayan, G., & Olive, J. (2010). Eighth grade students' representations of linear equations based on a cups and tiles model. *Educational Studies in Mathematics, 2010*(74), 143–162.

*Carreira, S. (2001). Where there's a model, there's a metaphor: Metaphorical thinking in students' understanding of a mathematical model. *Mathematical Thinking and Learning, 3*(4), 261–287.

*Chiu, M. M. (2001). Using metaphors to understand and solve arithmetic problems: Novices and experts working with negative numbers. *Mathematical Thinking and Learning, 3*(2–3), 93–124.

*Danesi, M. (2003). Conceptual metaphor theory and the teaching of mathematics: Findings of a pilot project. *Semiotica, 2003*(145), 71–83.

*Dawkins, P. C. (2012). Metaphor as a possible pathway to more formal understanding of the definition of sequence convergence. *Journal of Mathematical Behavior, 31*(2012), 331–343.

Dochy, F. (2006). A guide for writing scholarly articles or reviews for the Educational Research Review, 4(1–2). Retrieved from: https://www.researchgate.net/publication/284044504_A_guide_for_writing_scholarly_articles_or_reviews_for_the_Educational_Research_Review

Ely, R. (2011). Envisioning the infinite by projecting finite properties. *Journal of Mathematical Behavior, 2011*(30), 1–18.

Ernest, P. (2010). Mathematics and metaphor: A Response to Elizabeth Mowat & Brent Davis. *Complicity: An International Journal of Complexity and Education, 7*(1), 98–104.

*Figueiras, L., & Arcavi, A. (2014). A touch of mathematics: Coming to our senses by observing the visually impaired. *ZDM Mathematics Education, 2014*(46), 123–133.

*Font, V., Bolite, J., & Acevedo, J. I. (2010). Metaphors in mathematics classrooms: Analyzing the dynamic process of teaching and learning of graph functions. *Educational Studies in Mathematics, 2010*(75), 131–152.

Font, V., Godino, J. D., Núria, P., & Acevedo, J. I. (2010). The object metaphor and synecdoche in mathematics classroom discourse. *For the Learning of Mathematics*, *30*(1), 15–19.

Gibbs, R. W., Jr. (2005). *Embodiment and cognitive science*. Cambridge University Press.

Gibbs, R. W., Jr. (Ed.) (2008). *The Cambridge handbook of metaphor and thought*. Cambridge University Press.

Gibbs, R. W., Jr. (2017). *Metaphor wars: Conceptual metaphors in human life*. Cambridge University Press.

Goldin, G. A. (1992). On developing a unified model for the psychology of mathematical learning and problem solving. Plenary address. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16th annual meeting of the International Group for the Psychology of Mathematics Education*, Vol. 3 (pp. 235–261). Durham.

*González, G. (2013). A geometry teacher's use of a metaphor in relation to a prototypical image to help students remember a set of theorems. *Journal of Mathematical Behavior, 2013*(32), 397–414.

Green, B. N., Johnson, C. D., & Adams, A. (2006). Writing narrative literature reviews for peer-reviewed journals: Secrets of the trade. *Journal of Chiropractic Medicine, 5*(3), 101–117.

Halliday, M. A. K. (1974). Some aspects of sociolinguistics. In *UNESCO, Interactions between linguistics and mathematical education. Final report of the symposium sponsored by UNESCO, CEDO, and ICMI* (pp. 64–73). Nairobi.

Halliday, M. A. K. (1978). Sociolinguistics aspects of mathematical education. In M. Halliday (Ed.), *Language as social semiotic: The social interpretation of language and meaning* (pp. 195–204). University Park Press.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason*. University of Chicago Press.

*Junius, P. (2008). A case example of insect gymnastics: How is non-Euclidean geometry learned? *International Journal of Mathematical Education in Science and Technology*, *39*(8), 987–1002.

Kaput, J. J. (1991). Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 53–74). Kluwer.

*Keene, K. A. (2007). A characterization of dynamic reasoning: Reasoning with time as parameter. *Journal of Mathematical Behavior, 26*(2007), 230–246.

*King, B. & Smith, C. P. (2018). Mixed-reality learning environments: What happens when you move from a laboratory to a classroom? *International Journal of Research in Education and Science (IJRES)*, *4*(2), 577–594.

*Lai, M. Y. (2013). Constructing meanings of mathematical registers using metaphorical reasoning and models. *Mathematics Teacher Education and Development, 15*(1), 29–47.

Lakoff, G. (1987). *Women, fire, and dangerous things*. The University of Chicago Press.

Lakoff, G. (1993). The contemporary theory of metaphor. In: A. Ortony (Ed.), *Metaphor and thought* (pp. 202–251). Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. The University of Chicago Press.

Lakoff, G., & Núñez, R. E. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In L. D. English (Ed.), *Mathematical reasoning: Metaphor, analogies, and images* (pp. 21–89). Routledge.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Lester, F. K., Jr. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *ZDM Mathematics Education, 37*(6), 457–467.

Levinsson, M. & Prøitz, T. S. (2017). The (Non-)use of configurative reviews in education. *Education Inquiry*, *8*(3), 209–231.

*Magana, A. J. (2014). Learning strategies and multimedia techniques for scaffolding size and scale cognition. *Computers & Education, 72*(2014), 367–377.

Meglio, O. & Risberg, A. (2011). The (mis)measurement of M&A performance: A systematic narrative literature review. *Scandinavian Journal of Management*, *27*(2011), 418–433.

*Moreno, R., & Mayer, R. E. (1999). Multimedia-supported metaphors for meaning making in mathematics. *Cognition and Instruction, 17*(3), 215–248.

*Moschkovich, J. N. (1996). Moving up and getting steeper: Negotiating shared descriptions of linear graphs. *The Journal of the Learning Sciences, 5*(3), 239–277.

Noblit, G. W., & Hare, R. D. (2011). *Meta-Ethnography*. [PDF version].

Núñez, R. E. (2008). Conceptual metaphor, human cognition, and the nature of mathematics. In R. W. Gibbs Jr. (Ed.), *The Cambridge handbook of metaphor and thought* (pp. 339–362). Cambridge University Press.

Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*(1/3), 45–65.

Oehrtman, M. (2002). *Collapsing dimensions, physical limitation, and other student metaphors for limit concepts: An instrumentalist investigation into calculus students' spontaneous reasoning* (Publication No. AAT 3099508) [Doctoral dissertation, University of Texas at Austin, TX.]. ProQuest Dissertation & Theses database.

*Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, *40*(4), pp. 396–426.

Parsons, G. G., & Brown, J. R. (2003). Platonism, metaphor, and mathematics. *Dialogue XLIII*(2004), 47–66.

Peirce, C. S. (1906). Prolegomena to an apology for pragmaticism. In J. Hoopes (Ed.), *Peirce on signs* (pp. 249–252). The University of North Carolina Press.

Peirce, C. S. (1976). *The new elements of mathematics, Vol. 4, Mathematical philosophy* (C. Eisele, Vol. Ed.). Mouton.

Pimm, D. (1981). Metaphor and analogy in mathematics. *For the Learning of Mathematics*, *1*(3), 47–50.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. Routledge.

Pimm, D. (1995). *Symbols and meanings in school mathematics.* Routledge.

*Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics, 23*(6), 595–610.

Presmeg, N. C. (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behaviour, 1*(1), 25–32.

Presmeg, N. C. (2002, January). Mathematical idea analysis: A science of embodied mathematics [Review of the book *Where mathematics comes from. How the embodied mind brings mathematics into being*, by G. Lakoff & R. E. Núñez]. *Journal for Research in Mathematics Education*, *33*(1), pp. 59–63.

*Sáenz-Ludlow, A. (2004). Metaphor and numerical diagrams in the arithmetical activity of a fourth-grade class. *Journal for Research in Mathematics Education, 35*(1), 34–56.

Sáenz-Ludlow, A., & Kadunz, G. (2016). Constructing knowledge seen as a semiotic activity. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for learning* *mathematics: How to describe the construction, visualisation, and communication of mathematical concepts* (pp. 1–21). Sense Publishers.

Sandelowski, M., & Barroso, J. (2007). *Handbook for synthesizing qualitative research.* Springer.

Schiralli, M. & Sinclair, N. (2003). A constructive response to 'Where mathematics comes from'. *Educational Studies in Mathematics, 52*(2003), 79–91.

Schlimm, D. (2013). Conceptual metaphors and mathematical practice: On cognitive studies of historical developments in mathematics. *Topics in Cognitive Science*, *5*(2013), 283–298.

Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 339–372). Erlbaum.

*Sfard, A. (2000). Steering (Dis)course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects. *Journal for Research in Mathematics Education, 31*(3), 296–327.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Shoemaker, P. J., Tankard, J., & Lasorsa, D. L. (2004). *How to build social science theories*. SAGE Publications.

Soto-Andrade, J. (2014). Metaphors in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 447–453). Springer.

Soto-Andrade, J. (2018). Enactive metaphorising in the learning of mathematics. In G. Kaiser G., Forgasz H., Graven M., Kuzniak A., Simmt E. & Xu B. (Eds.), *Invited Lectures from the 13th International Congress on Mathematical Education* (pp. 619–638). Springer.

Tall, D. (2014). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press.

Tendahl, M., & Gibbs, R. W. Jr. (2008). Complementary perspectives on metaphor: Cognitive linguistics and relevance theory. *Journal of Pragmatics, 40*(2008), 1823–1864.

van Dormolen, J. (1991). Metaphors mediating the teaching and understanding of mathematics. In A. J. Bishop, S. Mellin-Olsen & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through learning* (pp. 89–106). Springer.

*Viirman, O. (2015). Explanation, motivation and question posing routines in university mathematics teachers' pedagogical discourse: A commognitive analysis. *Journal of Mathematical Education in Science and Technology, 46*(8), 1165–1181.

*Vitale, J. M., Swart, M. I., & Black, J. B. (2014). Integrating intuitive and novel grounded concepts in a dynamic geometry learning environment. *Computers & Education, 2014*(72), 231–248.

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, *49*(2002), 341–359.

Wagner, R. (2013). A historically and philosophically informed approach to mathematical metaphors. *International Studies in the Philosophy of Science*, *27*(2), 109–135.

Winter, B., & Yoshimi, J. (2020). Metaphor and the philosophical implications of embodied mathematics. *Frontiers in Psychology, 11*, Article 569487.

*Wittmann, M. C., Flood, V. J., & Black, K. E. (2013). Algebraic manipulation as motion within a landscape. *Educational Studies in Mathematics, 82*(2), 169–181.