Marginal permutation invariant covariance matrices with applications to linear models

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Abstract

The goal of the present paper is to perform a comprehensive study of the covariance structures in balanced linear models containing random factors which are invariant with respect to marginal permutations of the random factors. We shall focus on model formulation and interpretation rather than the estimation of parameters. It is proven that permutation invariance implies a specific structure for the covariance matrices. Useful results are obtained for the spectra of permutation invariant covariance matrices. In particular, the reparameterization of random effects, i.e., imposing certain constraints, will be considered. There are many possibilities to choose reparameterization constraints in a linear model, however not every reparameterization keeps permutation invariance. The question is if there are natural restrictions on the random effects in a given model, i.e., such reparameterizations which are defined by the covariance structure of the corresponding factor. Examining relationships between the reparameterization conditions applied to the random factors of the models and the spectrum of the corresponding covariance matrices when permutation invariance is assumed, restrictions on the spectrum of the covariance matrix are obtained which lead to “sum-to-zero” reparameterization of the corresponding factor.

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1. Introduction

A linear model represents a relationship between a continuous response variable $Y$ and one or more predictor variables. These variables are often called factors and they may be either continuous or categorical. In general, any linear model can be presented as

$$Y = X\hat{\beta} + \varepsilon,$$

(1.1)

where, in the one-dimensional case, $Y$ is an $n$-vector of observations, $X$ is an $n \times p$ known design matrix, $\hat{\beta}$ is a $p$-vector associated with effects of interest, fixed or random, and $\varepsilon$ is an $n$-vector of random errors, which are often assumed to be normally distributed. In the case of a mixed linear model, we can rewrite (1.1) more specifically

$$Y = X_1\beta + X_2\xi + \varepsilon,$$

(1.2)

where, $Y$ is an $n$-vector of observations, $X_1: n \times p_1$ and $X_2: n \times p_2$ are known design matrices, $\beta$ is a $p_1$-vector associated with fixed effects, $\xi' = (\xi'_1, \ldots, \xi'_k)$ is a $p_2$-vector associated with random effects, the vectors $\xi_i, i = 1, \ldots, k$, are called random factors, and $\varepsilon$ is an $n$-vector of random errors. We assume that $\xi$ and $\varepsilon$ are independently distributed, $\varepsilon \sim (0, \Sigma)$, $\xi \sim (0, D(\xi))$, and $D(\xi) = \text{Diag}(D(\xi_1), \ldots, D(\xi_k))$, where $D(\xi_i)$ is the covariance matrix of $\xi_i, i = 1, \ldots, k$.

The notation $\sim (\mu, \Psi)$ stands for distributed with mean $\mu$ and covariance matrix $\Psi$.

An increasing interest in modeling and analysis of complex covariance structures (including high-order interactions) can be noticed of late. Decomposable and graphical models can serve as typical examples. On the other hand, to represent complex models adequately one needs a lot of parameters and this complicates the statistical identification of such models. Invariance with respect to some group of transformations imposes additional structure on the model and reduces thereby the number of parameters to be estimated.

In many practical applications, for example, in psychometric and medical research, the assumption about interchangeability of levels of factors may be both reasonable and convenient. This motivates the use of the concept of invariance and for the interpretation so called marginal permutations play a key role.

The idea of invariance is based on an assumption that there exists a process which has generated data and which leads to the presence of symmetry (invariance) in data (see for example, [11,19]). In this case, it is natural to model data so that arbitrary permutations of factor levels do not affect inference. In particular, it is assumed that an arbitrary permutation of levels of a factor must not affect the covariance matrix of that factor which means that the covariance matrix must exhibit some structure (pattern). Furthermore, when a structure exists, incorporating this covariance pattern in the analysis will generally lead to more efficient inference. For some details and examples, see [22,21,5,11], etc.

Perlman [14] discussed and summarized results related to group symmetry models (see [1,2,6]) in multivariate analysis, i.e., linear models for which the covariance matrix $D(Y)$ of observations $Y$ is assumed to satisfy certain symmetry restrictions. Examples of group symmetry models are circular block symmetry, dihedral block symmetry and complete block symmetry models (for details see [14,11–13], for example). Arnold [3] studied problems concerned with patterned covariance matrices and generalized the intraclass correlation model of Wilks [22]. These types of structured matrices arise when we have factors with interchangeable random levels. Dawid [4] discussed the symmetry approach in the case of structured data layouts. Invariance under the symmetry group of arbitrary permutations was considered. His study is based on the works of Andersson [1], Speed [16], Speed and Bailey [17]. The work is devoted to the understanding of symmetry by means
of either association schemes or groups of transformations. According to Dawid [4] symmetry
possessed by exchangeability is the key to its usefulness in order to specify relevant hypotheses.
A crucial definition in this paper is the following one.

**Definition 1.1.** Let \( Q \) be an arbitrary element of a group \( \mathcal{G} \) of one-to-one transformations. The
covariance matrix \( D(\xi) \) of a factor \( \xi \) is called invariant with respect to \( \mathcal{G} \) if
\( D(\xi) = D(Q \xi) \) which is the same condition as \( D(\xi) = Q D(\xi) Q' \).

The main objective of this paper is to study and extend the understanding of covariance struc-
tures in \( K \)-way tables via invariance. Let us start by considering the observations \( Y i_{g \cdots i_1} \), i.e., we
have a model for the observation which consists of \( g \) factors ([18]). Suppose it is natural to assume
that the covariance matrices of these factors are invariant under permutations. For example, the
covariance matrix of \( Y i_{g \cdots i_1} \) will not depend on which value \( i_1 \) takes. This means that if we permute
the levels within a factor, for example, the values of \( i_1 \), the covariance structure of the model will
not change. A similar property should hold for the other factors. In a \( K \)-way table it is natural
to consider so called *marginal permutation invariance*, i.e., each level within a factor can be
permuted without any changes in the covariance structure of the model. As a consequence all
combinations of factor levels will be present and it means that we have a balanced model. It does
not make sense to suppose that if we interchange levels between various factors that this should
not affect the covariance structure of the model.

Let \( P^{(h)} \) denote the permutation that interchanges levels of a factor \( \xi^{(h)}, h = \ldots, g \). Thus,
invariance in \( K \)-way tables means that if we permute the values of any of the indices \( i_1, \ldots, i_g \)
and the other are held fixed, the covariance matrix of \( Y i_{g \cdots i_1} \) will not change.

**Theorem 1.1.** In the case of \( K \)-way tables the structure of the permutation matrix \( P_g \) of the
observations \( Y \) equals
\[
P_g = P^{(g)} \otimes \cdots \otimes P^{(1)},
\]
where \( \otimes \) denotes the Kronecker product and \( P^{(k)} \) are permutation matrices, \( k = 1, \ldots, g \).

For a proof of the theorem see [10].

**Definition 1.2.** The matrix \( P_h = P^{(k_h)} \otimes \cdots \otimes P^{(k_1)}, \{ k_1, \ldots, k_h \} \subseteq \{ 1, \ldots, g \}, \) is called a mar-
ginal permutation matrix.

Theorem 1.1 implies that the \( K \)-way table leads to a natural group of transformations, namely,
the group of marginal permutations.

Finally we note that one should distinguish between two types of permutation invariance: full
permutation invariance, i.e., the covariance matrix of observations \( D(Y) \) is permutation invariant
implying permutation invariance for all factors in a model, and partial permutation invariance,
i.e., some factors in the model have permutation invariant covariance matrices which imply that
the covariance matrix of the observations \( Y \) will consist of patterned blocks (blocks with a special
structure).

2. Permutation invariant covariance matrices

This section is dedicated to the study of permutation invariant covariance matrices which arise
from \( K \)-way tables, i.e., certain types of patterned (structured) matrices which are generated via
statements about invariance. We prove that permutation invariance implies a specific structure for the covariance matrices which in [10] was termed self similar or fractal. We also present a number of spectral properties of the invariant covariance matrices.

Let $P(h)$ denote the permutation that interchanges levels of factor $\xi(h)$ representing the main effects, $h = 1, \ldots, g,$ and let $\gamma(s)$ represent a factor of $s$-order interaction effects among factors $\xi(1), \ldots, \xi(g), s = 2, \ldots, g.$ Let $n_h$ be the number of sampled levels of factor $\xi(h)$, then $\gamma(s)$ is an $N$-vector with $N = n_{h_1} \cdots n_{h_s}$ components, where $\{ h_1, \ldots, h_s \} \subseteq \{ 1, \ldots, g \}.$ The covariance matrix of the factor $\gamma(s)$ is denoted by $\Sigma_s$. We number the components of $\gamma(s)$ lexicographically.

We are mainly going to deal with interaction effects. The reason for this is that results for the main effects follow immediately from those about interaction effects.

Let us define the following matrix:

$$J_{n_i}^{v_i} = \begin{cases} I_{n_i}, & \text{if } v_i = 0, \\ J_{n_i}, & \text{if } v_i = 1, \end{cases}$$

(2.1)

where $I_{n_i}$ is the identity matrix of order $n_i$, $J_{n_i}$ is an $n_i \times n_i$ matrix with all elements equal to 1, $i = 1, \ldots, s$.

**Theorem 2.1.** The covariance matrix $\Sigma_1: n_1 \times n_1$ of the factor $\xi$ is invariant with respect to all permutations $P_1$ ($P_1$-invariant), iff it has the following structure:

$$\Sigma_1 = \sum_{v_1=0}^{1} c_{v_1} J_{n_1}^{v_1},$$

(2.2)

where $c_0$ and $c_1$ are constants, the matrices $J_{n_1}^{v_1}, v_1 \in \{0, 1\}$, are given by (2.1).

See [9] for a proof.

It turns out that invariance under marginal permutations $P_s$ also implies certain patterns for the covariance matrices of factors which represent interaction effects. The next result reveals the structure of the invariant covariance matrix of the factor representing second-order interaction effects.

**Theorem 2.2.** The matrix $\Sigma_2: n_s n_1 \times n_2 n_1$ is invariant with respect to all marginal permutations $P_2$, given by Definition 1.2, iff it has the following structure:

$$\Sigma_2 = \sum_{v_2=0}^{1} \sum_{v_1=0}^{1} c_{v_2 v_1} J_{n_2}^{v_2} \otimes J_{n_1}^{v_1},$$

(2.3)

where $c_{v_2 v_1}$ are constants, $v_1 \in \{0, 1\}$ and $v_2 \in \{0, 1\}$.

**Proof.** Let $N = n_2 n_1$. It is clear that we can write $\Sigma_2$ as

$$\Sigma_2 = \sum_{k=1}^{N} \sum_{l=1}^{N} \sigma_{kl} e_k e'_l,$$

(2.4)

where $e_k, e_l$ are the $k$th and the $l$th columns of the identity matrix $I_N$, respectively. Moreover, observe that we can express $\Sigma_2$ in the following way:

$$\Sigma_2 = \sum_{i_2, j_2=1}^{n_2} \sum_{i_1, j_1=1}^{n_1} \sigma_{(i_2 i_1)(j_2 j_1)} (e_{2, j_2} \otimes e_{1, j_1})(e'_{2, j_2} \otimes e'_{1, j_1})$$

(2.5)
The condition \( P_{(i)} \) first component where \( i \) and \( h \) are associated with the corresponding factor levels of \( \xi(h) \), \( h = 1, 2 \), and \( \sigma_{(i)(j_1,j_2)} = \text{Cov}(y_{(i)}, y_{(j_1,j_2)}) \) is the element of \( \Sigma_2 \) in the \( k \)th row and the \( l \)th column,

\[
k = (i_2 - 1)n_1 + i_1, \\
l = (j_2 - 1)n_1 + j_1,
\]

and

\[
e_k = e_{2,i_2} \otimes e_{1,i_1}, \\
e'_k = e'_{2,i_2} \otimes e'_{1,i_1}.
\]

The condition \( P_2 \Sigma_2 P'_2 = \Sigma_2 \), for all \( P_2 \), given by Definition 1.2, means that

\[
\Sigma_2 = \sum_{i_2,j_2=1}^{n_2} \sum_{i_1,j_1=1}^{n_1} \sigma_{(i_2,i_1)(j_2,j_1)} \left( P^{(2)}(e_{2,i_2}e'_{2,j_2}) P^{(2)'}, P^{(1)}(e_{1,i_1}e'_{1,i_1}) P^{(1)'} \right) \\
= \sum_{i_2} \sum_{i_1,j_1} \sigma_{(i_2,i_1)(j_2,j_1)} \left( P^{(2)}(e_{2,i_2}e'_{2,j_2}) P^{(2)'}, P^{(1)}(e_{1,i_1}e'_{1,i_1}) P^{(1)'} \right) \\
+ \sum_{i_2 \neq j_2} \sum_{i_1,j_1} \sigma_{(i_2,i_1)(j_2,j_1)} \left( P^{(2)}(e_{2,i_2}e'_{2,j_2}) P^{(2)'}, P^{(1)}(e_{1,i_1}e'_{1,i_1}) P^{(1)'} \right).
\]

Each component \( P^{(h)} \) of the Kronecker product \( P^{(2)} \otimes P^{(1)} \) acts on the components of \( y^{(2)} \) which are associated with the corresponding factor levels of \( \xi^{(h)} \), \( h = 1, 2 \). Thus, with respect to the first component \( \left( P^{(2)}(e_{2,i_2}e'_{2,j_2}) P^{(2)'} \right) \), the invariance of \( \Sigma_2 \) implies that in (2.11) we may define constants

\[
\sigma_{1(i_1)(j_1)} = \sigma_{(i_2,i_1)(j_2,j_1)} = \sigma_{(i_2,i_1)(j_2,j_1)}, \quad \forall i_2, j_2; \quad \forall i_1, j_1, \\
\sigma_{2(i_1)(j_1)} = \sigma_{(i_2,i_1)(j_2,j_1)} = \sigma_{(i_2,i_1)(j_2,j_1)}, \quad \forall i_2 \neq j_2, i_2 \neq j_2; \quad \forall i_1, j_1,
\]

where \( i_1, j_1, i'_1, j'_1 = 1, \ldots, n_1, i_2, j_2, i'_2, j'_2 = 1, \ldots, n_2 \).

Thus, (2.11) becomes

\[
\Sigma_2 = \sum_{i_1,j_1} \sigma_{1(i_1)(j_1)} I_{n_2} \otimes \left( P^{(1)}(e_{1,i_1}e'_{1,i_1}) P^{(1)'} \right) \\
+ \sum_{i_1,j_1} \sigma_{2(i_1)(j_1)} (J_{n_2} - I_{n_2}) \otimes \left( P^{(1)}(e_{1,i_1}e'_{1,i_1}) P^{(1)'} \right)
\]

and

\[
\Sigma_2 = \sum_{i_1} \sigma_{1(i_1)(i_1)} I_{n_2} \otimes \left( P^{(1)}(e_{1,i_1}e'_{1,i_1}) P^{(1)'} \right) + \sum_{i_1 \neq j_1} \sigma_{1(i_1)(j_1)} I_{n_2} \otimes \left( P^{(1)}(e_{1,i_1}e'_{1,j_1}) P^{(1)'} \right) \\
+ \sum_{i_1} \sigma_{2(i_1)(i_1)} (J_{n_2} - I_{n_2}) \otimes \left( P^{(1)}(e_{1,i_1}e'_{1,i_1}) P^{(1)'} \right)
\]
The invariance of $\Sigma_2$ with respect to \( P(1)(e_{1,i_1}e'_{1,j_1})P(1)' \) implies that in (2.13) we may again define constants

\[
\begin{align*}
\tau_1 &= \sigma_{1(i_1)(i_1)}, \quad \forall i_1, \\
\tau_2 &= \sigma_{1(i_1)(j_1)}, \quad \forall i_1 \neq j_1, \\
\tau_3 &= \sigma_{2(i_1)(i_1)}, \quad \forall i_1, \\
\tau_4 &= \sigma_{2(i_1)(j_1)}, \quad \forall i_1 \neq j_1.
\end{align*}
\]

Hence, we have the following structure for $\Sigma_2$:

\[
\Sigma_2 = \tau_1 I_{n_2} \otimes I_{n_1} + \tau_2 I_{n_2} \otimes (J_{n_1} - I_{n_1})
+ \tau_3 (J_{n_2} - I_{n_2}) \otimes I_{n_1} + \tau_4 (J_{n_2} - I_{n_2}) \otimes (J_{n_1} - I_{n_1}).
\]

After some regrouping of terms in (2.14), $\Sigma_2$ can be written as

\[
\begin{align*}
\Sigma_2 &= (\tau_1 - \tau_2 - \tau_3 + \tau_4) I_{n_2} \otimes I_{n_1} + (\tau_2 - \tau_4) I_{n_2} \otimes J_{n_1} \\
&\quad + (\tau_3 - \tau_4) J_{n_2} \otimes I_{n_1} + \tau_4 J_{n_2} \otimes J_{n_1} \\
&= c_{00} I_{n_2} \otimes I_{n_1} + c_{01} I_{n_2} \otimes J_{n_1} + c_{10} J_{n_2} \otimes I_{n_1} + c_{11} J_{n_2} \otimes J_{n_1}.
\end{align*}
\]

Thus we have shown that permutation invariance implies (2.3). For the converse we note that the structure of $\Sigma_2$ given in (2.3) implies that $\Sigma_2$ is invariant with respect to all marginal permutations $P_2$. □

Observe that in Theorem 2.2

\[
\begin{align*}
\tau_1 &= \text{Cov}(\gamma_{ij}^{(2)}), \quad \forall i, \\
\tau_3 &= \text{Cov}(\gamma_{ij}^{(2)}, \gamma_{ij'}^{(2)}) \text{ if } i \neq i', \\
\tau_2 &= \text{Cov}(\gamma_{ij}^{(2)}, \gamma_{ij'}^{(2)}) \text{ if } j \neq j', \\
\tau_4 &= \text{Cov}(\gamma_{ij}^{(2)}, \gamma_{ij'}^{(2)}) \text{ if } i \neq i', j \neq j'.
\end{align*}
\]

The following auxiliary result is needed.

**Lemma 2.1.** Any covariance matrix $\Sigma_s$ can be expressed as

\[
\Sigma_s = \sum_{i_s \ldots i_1, j_s \ldots j_1} \sigma_{(i_s \ldots i_2,j_s \ldots j_2)(i_s,i_1)(j_s,j_1)} (e_{s,i_s}e'_{s,j_s}) \otimes \cdots \otimes (e_{2,i_2}e'_{2,j_2}) \otimes (e_{1,i_1}e'_{1,j_1}),
\]

where

\[
\sigma_{(i_s \ldots i_2,j_s \ldots j_2)} = \sigma_{kl},
\]

with

\[
\begin{align*}
k &= \sum_{h=1}^{s-1} (i_{s-h+1} - 1)n_{s-h} \cdots n_2 \cdot n_1 + i_1, \\
l &= \sum_{h=1}^{s-1} (j_{s-h+1} - 1)n_{s-h} \cdots n_2 \cdot n_1 + j_1,
\end{align*}
\]

and $e_{h,i_h}$ is the $i_h$th column of the identity matrix $I_{n_h}$; $i_h, j_h = 1, \ldots, n_h, h = 1, \ldots, s$. 

\[
+ \sum_{i_1 \neq j_1} \sigma_{2(i_1)(j_1)} (J_{n_2} - I_{n_2}) \otimes \left( P(1)(e_{1,i_1}e'_{1,j_1})P(1)'. \right)
\]

\[
\text{(2.13)}
\]
Proof. It is obvious that $\Sigma_s$ can be written in the following way:

$$\Sigma_s = \sum_{k,l=1}^{N_s} \sigma_{kl}(e_ke_l'),$$

(2.19)

where $N_s = n_{h_1}n_{h_2} \cdots n_{h_s}n_{h_1}$, $\{h_1, \ldots, h_s\} \subseteq \{1, \ldots, g\}$. If $s = 1$, then $k = i_1$, $l = j_1$. The case $s = 2$ is covered in (2.7) and (2.8). Suppose formula (2.17) is true for $\Sigma_{s-1}$: $N_{s-1} \times N_{s-1}$, then we shall show that it is also true for $\Sigma_s$. However, the $s - 1$ factors can be viewed as one factor with the index given via $i_1, \ldots, i_{s-1}$:

$$s-1 \sum_{h=2} (i_{s-h+1} - 1)n_{s-h} \cdots n_1 + i_1.$$  

(2.20)

The index of the $s$ factor is given by $i_s$. By using the formula (2.7) for two factors we obtain

$$k = (i_s - 1)n_{s-1} \cdots n_1 + \left(\sum_{h=2} (i_{s-h+1} - 1)n_{s-h} \cdots n_1 + i_1\right)$$

$$= \sum_{h=1}^{s-1} (i_{s-h+1} - 1)n_{s-h} \cdots n_1 + i_1,$$

(2.21)

and thus the formula is proved via induction. Analogously, (2.18) can be established. Eq. (2.16) follows from the fact that instead of $e_k$ in (2.19) we may use the following expression:

$$e_{s,i_s} \otimes \cdots \otimes e_{2,i_2} \otimes e_{1,i_1}.$$  

Similarly, we may write $e_l$ in (2.19) as

$$e_{s,j_s} \otimes \cdots \otimes e_{2,j_2} \otimes e_{1,j_1}.$$  

□

In the next theorem we extend the result of Theorem 2.2 to the $s$-factor case.

**Theorem 2.3.** The covariance matrix $\Sigma_s$ of factor $\gamma^{(s)}$ representing $s$-order interaction effects is invariant with respect to all marginal permutations $P_s$, iff it has the following structure:

$$\Sigma_s = \sum_{v_2=0}^{1} \cdots \sum_{v_1=0}^{1} c_{v_2 \cdots v_2 v_1} J_{v_2}^{v_2} \otimes \cdots \otimes J_{v_1}^{v_1} \otimes J_{n_2}^{n_2} \otimes J_{n_1}^{n_1},$$

(2.22)

where $J_{n_h}^{v_h}$ is given by (2.1), $h = 1, \ldots, s$, and $c_{v_2 \cdots v_2 v_1}$ are constants.

Proof. First we prove necessity. Using the result of Lemma 2.1 we can write $\Sigma_s$ as

$$\Sigma_s = \sum_{i_s, \ldots, i_1}^{i_s, \ldots, i_1} \sigma_{i_s \cdots i_1}(e_{s,i_s} \otimes \cdots \otimes e_{2,i_2} \otimes e_{1,i_1})(e_{s,j_s} \otimes \cdots \otimes e_{2,j_2} \otimes e_{1,j_1})'$$

$$= \sum_{i_s, \ldots, i_1}^{i_s, \ldots, i_1} \sigma_{i_s \cdots i_1}(e_{s,i_s} e_{s,j_s}') \otimes \cdots \otimes (e_{2,i_2} e_{2,j_2}') \otimes (e_{1,i_1} e_{1,j_1}').$$
If $\Sigma_s$ is $P_s$-invariant, i.e., $P_s \Sigma_s P'_s = \Sigma_s$, then

$$
\Sigma_s = \sum_{i_s, \ldots, i_1} \sigma(i_s \ldots i_1)(j_s \ldots j_1) \left( P^{(h_s)}(e_{s,i_s} e'_{s,j_s}) P^{(h_s)'} \right) \otimes \cdots \otimes \left( P^{(h_1)}(e_{1,i_1} e'_{1,j_1}) P^{(h_1)'} \right). 
$$

(2.23)

Once again induction is used. We have shown that the theorem is true for $s = 1$ and $s = 2$, i.e., the invariance with respect to the marginal permutations $P_1$ and $P_2$ implies a specific pattern for the covariance matrix. Suppose the theorem is true for $s - 1$. Rewrite the condition (2.23) as

$$
\Sigma_s = \sum_{i_s} \left( P^{(h_s)} e_{s,i_s} e'_{s,i_s} P^{(h_s)'} \right) 
\otimes \left[ \sum_{i_s-1, \ldots, i_1} \sigma(i_s \ldots i_1)(i_s \ldots j_1) \left( P^{(h_{s-1})} e_{s-1,i_{s-1}} e'_{s-1,j_{s-1}} P^{(h_{s-1})}' \right) 
\otimes \cdots \otimes \left( P^{(h_1)}(e_{1,i_1} e'_{1,j_1}) P^{(h_1)'} \right) \right] 
+ \sum_{i_s \neq j_s} \left( P^{(h_s)} e_{s,i_s} e'_{s,j_s} P^{(h_s)'} \right) 
\otimes \left[ \sum_{i_s-1, \ldots, i_1} \sigma(i_s \ldots i_1)(j_s \ldots j_1) \left( P^{(h_{s-1})} e_{s-1,i_{s-1}} e'_{s-1,j_{s-1}} P^{(h_{s-1})}' \right) 
\otimes \cdots \otimes \left( P^{(h_1)}(e_{1,i_1} e'_{1,j_1}) P^{(h_1)'} \right) \right].
$$

(2.24)

The invariance with respect to $P_{s-1}$, i.e., the induction assumption, implies the next structure of $\Sigma_s$:

$$
\Sigma_s = \sum_{i_s} \left( P^{(h_s)} e_{s,i_s} e'_{s,i_s} P^{(h_s)'} \right) \otimes \left[ \sum_{v_{s-1}=0}^{1} \cdots \sum_{v_1=0}^{1} c_{v_{s-1} \ldots v_1(i_s)} J^{v_{s-1}}_{R_{s-1}} \otimes \cdots \otimes J^{v_1}_{R_1} \right] 
+ \sum_{i_s \neq j_s} \left( P^{(h_s)} e_{s,i_s} e'_{s,j_s} P^{(h_s)'} \right) \otimes \left[ \sum_{v_{s-1}=0}^{1} \cdots \sum_{v_1=0}^{1} c_{v_{s-1} \ldots v_1(i_s,j_s)} J^{v_{s-1}}_{R_{s-1}} \otimes \cdots \otimes J^{v_1}_{R_1} \right],
$$

(2.25)

where the constants $c_{v_{s-1} \ldots v_1(i_s)}$ and $c_{v_{s-1} \ldots v_1(i_s,j_s)}$ in (2.25) are linear combinations of elements of $\Sigma_s$. In order for $\Sigma_s$ to be invariant with respect to $P_s$ the coefficients $c_{v_{s-1} \ldots v_1(i_s)}$ must take the same values for all $i_s$, and $c_{v_{s-1} \ldots v_1(i_s,j_s)}$ must take the same values for all $i_s$ and $j_s$, $i_s \neq j_s$. Therefore, we may write

$$
\Sigma_s = I_{n_s} \otimes \left[ \sum_{v_{s-1}=0}^{1} \cdots \sum_{v_1=0}^{1} c^{v_{s-1}}_{v_{s-1} \ldots v_1} J^{v_{s-1}}_{R_{s-1}} \otimes \cdots \otimes J^{v_1}_{R_1} \right].
$$
\[ + (J_n - I_n) \otimes \left[ \sum_{\nu_1=0}^{1} \sum_{v_1=0}^{1} c_{v_1}^{\nu_1} J_{n-1}^{\nu_1} \otimes \cdots \otimes J_1^{v_1} \right], \]

which can be rewritten as
\[ \Sigma_s = \sum_{\nu_s=0}^{1} \cdots \sum_{\nu_1=0}^{1} c_{\nu_s} \cdots c_{\nu_1} J_{n_s}^{\nu_s} \otimes \cdots \otimes J_{n_1}^{\nu_1}, \]

and establishes necessity. The sufficiency follows immediately due to the structure of \( \Sigma_s \) in (2.22). \( \square \)

In formula (2.22) the matrices \( J_{n_s}^{\nu_s} \otimes \cdots \otimes J_{n_1}^{\nu_1}, \nu_h \in \{0, 1\}, h = 1, \ldots, s, \) act as bases vectors in the space of permutation invariant matrices. Depending on the index \( \nu_h \) we have either \( I_{n_h} \) (\( \nu_h = 0 \)) or \( J_{n_h} \) (\( \nu_h = 1 \)). With the help of the \( J_{n_h} \)-matrices the covariances within and between factors are specified, i.e., the off-diagonal elements in \( \Sigma_s \) are specified. Therefore, it is of interest to collect all bases which are built up with one \( J_{n_h} \) matrix, two \( J_{n_h} \) matrices, etc. This way of presenting Theorem 2.3 is given in the next corollary.

**Corollary 2.1.** The \( P_s \)-invariant covariance matrix \( \Sigma_s \) with structure given in (2.22) can be expressed as a linear combination of components \( I_{n_h} \) and \( (J_{n_h} - I_{n_h}) \) in the following way
\[ \Sigma_s = \sum_{\nu_s=0}^{1} \cdots \sum_{\nu_1=0}^{1} d_{\nu_s} \otimes \cdots \otimes (J_{n_s} - I_{n_s})^{\nu_1}, \]

where the index function \( a(k) \) is defined as
\[ a(0) = 1, \quad a(k) = a(k-1) + \binom{s}{k} = \sum_{i=0}^{k} \binom{s}{i}, \quad k = 1, 2, \ldots, s, \]

and \( c_i, i = 1, \ldots, 2^s, \) are constants.

Since the matrices \( I_{n_h} \) and \( J_{n_h} - I_{n_h}, h = 1, \ldots, s, \) comprise different nonzero elements, the next corollary is also of interest. Let us first introduce the following operator.

**Definition 2.1.** For matrices \( A_1, \ldots, A_s \) the operator \( \otimes \) represents the product
\[ \bigotimes_{h=s}^1 A_h = A_s \otimes A_{s-1} \otimes \cdots \otimes A_1. \] \hspace{1cm} (2.26)

**Corollary 2.2.** The \( P_s \)-invariant covariance matrix \( \Sigma_s \) with structure given in (2.22) can be expressed as a linear combination of components \( I_{n_h} \) and \( (J_{n_h} - I_{n_h}) \) in the following way:
\[ \Sigma_s = \sum_{\nu_s=0}^{1} \cdots \sum_{\nu_1=0}^{1} d_{\nu_s} \otimes \cdots \otimes (J_{n_s} - I_{n_s})^{\nu_1}, \]
where symbol $\otimes$ denotes the Kronecker product, defined in (2.26), $d_i$ are constants ($i = 1, \ldots, 2^n$) and

$$k = \sum_{h=1}^{s} \nu_h \cdot 2^{h-1} + 1. \quad (2.27)$$

It is worth to notice that Theorem 2.3 does not show the explicit form of the invariant covariance matrix $\Sigma_3$. In general, the structure of $\Sigma_3$ is rather complicated. In practical data analysis, the second- and third-order interaction terms are often of main interest. One can present the covariance matrix $\Sigma_3$ as a function of the four parameters $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ of Theorem 2.2. These parameters will also appear in the next theorem, but the main object is to focus on the pattern of $\Sigma_2$ (see [8]).

**Theorem 2.4.** If the covariance matrix $\Sigma_2: n_2 n_1 \times n_2 n_1$ is invariant with respect to all marginal permutations $P_2$ of factor $\gamma^{(2)}$ levels, then it has the following structure:

$$\Sigma_2 = I_{n_2} \otimes \Sigma_1^{(1)} + (J_{n_2} - I_{n_2}) \otimes \Sigma_1^{(2)}, \quad (2.28)$$

where

$$\Sigma_1^{(1)} = (\tau_1 - \tau_2) I_{n_1} + \tau_2 J_{n_1}, \quad \text{(2.29)}$$

$$\Sigma_1^{(2)} = (\tau_3 - \tau_4) I_{n_1} + \tau_4 J_{n_1} \quad \text{(2.30)}$$

and parameters $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ are defined in (2.15), with $i, i' = 1, \ldots, n_2$, $j, j' = 1, \ldots, n_1$.

For the factor $\gamma^{(3)}$ representing third-order interaction effects, the permutation invariant covariance matrix $\Sigma_3$ can be constructed recursively in the following way.

Firstly, let

$$\tau_1 = \text{Cov}(\gamma^{(3)}_{ijk}, \gamma^{(3)}_{ijk}), \quad \tau_5 = \text{Cov}(\gamma^{(3)}_{ijk}, \gamma^{(3)}_{i'jk}), \quad i \neq i',$$

$$\tau_2 = \text{Cov}(\gamma^{(3)}_{ijk}, \gamma^{(3)}_{ijk}), \quad k \neq k', \quad \tau_6 = \text{Cov}(\gamma^{(3)}_{ijk}, \gamma^{(3)}_{i'j'k}), \quad i \neq i', k \neq k',$$

$$\tau_3 = \text{Cov}(\gamma^{(3)}_{ijk}, \gamma^{(3)}_{ijk}), \quad j \neq j', \quad \tau_7 = \text{Cov}(\gamma^{(3)}_{ijk}, \gamma^{(3)}_{i'j'k}), \quad i \neq i', j \neq j',$$

$$\tau_4 = \text{Cov}(\gamma^{(3)}_{ijk}, \gamma^{(3)}_{ijk}), \quad j \neq j', k \neq k', \quad \tau_8 = \text{Cov}(\gamma^{(3)}_{ijk}, \gamma^{(3)}_{i'j'k}), \quad i \neq i', j \neq j', k \neq k',$$

(2.31)

where $i, i' = 1, \ldots, n_3$, $j, j' = 1, \ldots, n_2$, $k, k' = 1, \ldots, n_1$, and construct

$$\Sigma_1^{(1)} = I_{n_1} \tau_1 + (J_{n_1} - I_{n_1}) \tau_2, \quad \Sigma_1^{(3)} = I_{n_1} \tau_5 + (J_{n_1} - I_{n_1}) \tau_6, \quad (2.32)$$

$$\Sigma_1^{(2)} = I_{n_1} \tau_3 + (J_{n_1} - I_{n_1}) \tau_4, \quad \Sigma_1^{(4)} = I_{n_1} \tau_7 + (J_{n_1} - I_{n_1}) \tau_8.$$

Secondly, define

$$\Sigma_2^{(1)} = I_{n_2} \otimes \Sigma_1^{(1)} + (J_{n_2} - I_{n_2}) \otimes \Sigma_1^{(2)}, \quad (2.33)$$

$$\Sigma_2^{(2)} = I_{n_2} \otimes \Sigma_1^{(3)} + (J_{n_2} - I_{n_2}) \otimes \Sigma_1^{(4)} \quad (2.34)$$

and then the following theorem can be stated.
Theorem 2.5. If the covariance matrix $\Sigma_3$ of factor $\gamma^{(3)}$ is $P_3$-invariant, then it has the following structure:

$$\Sigma_3 = I_{n_3} \otimes \Sigma_2^{(1)} + (J_{n_3} - I_{n_3}) \otimes \Sigma_2^{(2)},$$

where the matrices $\Sigma_2^{(1)}$ and $\Sigma_2^{(2)}$ are defined by (2.31)–(2.34).

Alternatively, one may write $\Sigma_3$ according to Corollary 2.1 with explicitly given coefficients $c_k, k = 1, \ldots, 2^s$, as presented in the next result.

Theorem 2.6. The $P_3$-invariant covariance matrix $\Sigma_3$ can be expressed as

$$\Sigma_3 = I_{n_3} \otimes [I_{n_2} \otimes [(\tau_1 - \tau_2 - \tau_3 + \tau_4 - \tau_5 + \tau_6 + \tau_7 - \tau_8)I_{n_1} + (\tau_2 - \tau_4 - \tau_6 + \tau_8)J_{n_1} + (\tau_3 - \tau_4 - \tau_7 + \tau_8)I_{n_1} + (\tau_4 - \tau_8)J_{n_1}]] + J_{n_2} \otimes [((\tau_1 - \tau_2 - \tau_3 + \tau_4 - \tau_5 + \tau_6 - \tau_8)I_{n_1} + (\tau_2 - \tau_4 - \tau_6 + \tau_8)J_{n_1} + (\tau_3 - \tau_4 - \tau_7 + \tau_8)I_{n_1} + (\tau_4 - \tau_8)J_{n_1}] + J_{n_2} \otimes [((\tau_7 - \tau_8)I_{n_1} + \tau_8 J_{n_1})],$$

where the parameters $\tau_1, \ldots, \tau_8$ are defined in (2.31).

The way of constructing the $P_3$-invariant covariance matrix $\Sigma_3$ can be generalized to an arbitrary number of factors.

Theorem 2.7. The covariance matrix $\Sigma_s$ given by (2.22), can be written in a recursive form as

$$\Sigma_s = I_{n_s} \otimes \Sigma_{s-1}^{(1)} + (J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}^{(2)},$$

(2.36)

where

$$\Sigma_{s-1}^{(i_h)} = \tau_h, h = 1, \ldots, 2^s,$$

$$\Sigma_{s}^{(i_k)} = I_{n_k} \otimes \Sigma_{s-1}^{(2i_{k+1})} + (J_{n_k} - I_{n_k}) \otimes \Sigma_{s-1}^{(2i_k)},$$

$$i_k = 1, \ldots, 2^{s-k}, \quad k = 1, \ldots, s - 1,$$

(2.37)

(2.38)

and the constants $\tau_h$ are covariances between the components of $\gamma^{(s)}$, defined similarly to those in (2.31).

Proof. It is clear that formula (2.36) is valid for $s = 1$:

$$\Sigma_1 = I_{n_1} \tau_1 + (J_{n_1} - I_{n_1}) \tau_2.$$

In this case, observe that according to (2.37) $\tau_1 = \Sigma_0^{(1)}$ and $\tau_2 = \Sigma_0^{(2)}$.

For $s = 2$, according to (2.29)–(2.30),

$$\Sigma_2 = I_{n_2} \otimes (I_{n_1} \tau_1 + (J_{n_1} - I_{n_1}) \tau_2) + (J_{n_2} - I_{n_2}) \otimes (I_{n_1} \tau_3 + (J_{n_1} - I_{n_1}) \tau_4).$$

In this case, formula (2.37) gives $\Sigma_0^{(h)} = \tau_h, h = 1, \ldots, 4$, and (2.38) implies

$$\Sigma_1^{(1)} = I_{n_1} \tau_1 + (J_{n_1} - I_{n_1}) \tau_2, \quad \Sigma_1^{(2)} = I_{n_1} \tau_3 + (J_{n_1} - I_{n_1}) \tau_4.$$

Assume now, that the statement of the theorem is true for $\Sigma_{s-1}$. Let us show that then it is also true for $\Sigma_s$. According to Corollary 2.1 we can write $\Sigma_s$ as
\[ \Sigma_s = c_a(0) I_{n_s} \otimes \cdots \otimes I_{n_1} \\
+ c_a(0)+1 I_{n_s} \otimes \cdots \otimes I_{n_2} \otimes J_{n_1} + \cdots + c_a(1) I_{n_s} \otimes I_{n_{s-1}} \otimes \cdots \otimes I_{n_1} \\
+ c_a(1)+1 I_{n_s} \otimes \cdots \otimes I_{n_3} \otimes J_{n_2} \otimes J_{n_1} + \cdots + c_a(2) I_{n_s} \otimes J_{n_{s-1}} \otimes J_{n_{s-2}} \otimes \cdots \otimes I_{n_1} \\
+ \cdots + \\
+ c_a(s-2)+1 I_{n_s} \otimes J_{n_{s-1}} \otimes \cdots \otimes J_{n_1} + \cdots + c_a(s-1) I_{n_s} \otimes \cdots \otimes J_{n_2} \otimes I_{n_1} \\
+ c_a(s) I_{n_s} \otimes J_{n_{s-1}} \otimes \cdots \otimes J_{n_1}. \]

Split this expression into two parts (each consisting of \(2^{s-1}\) terms): one will consist of all terms with \(I_{n_s}\) on the first place of the Kronecker product and the other group with \(J_{n_s}\) on the first place:

\[ \Sigma_s = I_{n_s} \otimes \left[ c_{i_1} I_{n_{s-1}} \otimes \cdots \otimes I_{n_1} + c_{i_2} I_{n_{s-1}} \otimes \cdots \otimes I_{n_2} \otimes J_{n_1} + \cdots + c_{i_p} I_{n_{s-1}} \otimes \cdots \otimes J_{n_1} \right] \\
+ J_{n_s} \otimes \left[ c_{i_{p+1}} I_{n_{s-1}} \otimes \cdots \otimes I_{n_1} + c_{i_{p+2}} I_{n_{s-1}} \otimes \cdots \otimes I_{n_2} \otimes J_{n_1} + \cdots + c_{i_{2p}} I_{n_{s-1}} \otimes \cdots \otimes J_{n_1} \right] \\
+ I_{n_s} \otimes \left[ c_{i_{p+1}} I_{n_{s-1}} \otimes \cdots \otimes I_{n_1} + c_{i_{p+2}} I_{n_{s-1}} \otimes \cdots \otimes I_{n_2} \otimes J_{n_1} + \cdots + c_{i_{2p}} I_{n_{s-1}} \otimes \cdots \otimes J_{n_1} \right] \\
= I_{n_s} \otimes [ (c_{i_1} - c_{i_{p+1}}) I_{n_{s-1}} \otimes \cdots \otimes I_{n_1} + (c_{i_2} - c_{i_{p+2}}) I_{n_{s-1}} \otimes \cdots \otimes I_{n_2} \otimes J_{n_1} + \cdots + (c_{i_p} - c_{i_{2p}}) I_{n_{s-1}} \otimes \cdots \otimes J_{n_1} ] \\
+ (J_{n_s} - I_{n_s}) \otimes [ c_{i_{p+1}} I_{n_{s-1}} \otimes \cdots \otimes I_{n_1} + c_{i_{p+2}} I_{n_{s-1}} \otimes \cdots \otimes I_{n_2} \otimes J_{n_1} + \cdots + c_{i_{2p}} I_{n_{s-1}} \otimes \cdots \otimes J_{n_1} ] . \tag{2.39} \]

In (2.39) we have \(2^{s-1}\) bases vectors which give, according to Corollary 2.1, the expression for some \(\Sigma_{s-1}^{(1)}\):

\[ \Sigma_{s-1}^{(1)} = \sum_{v_{s-1}=0}^{1} \cdots \sum_{v_1=0}^{1} c_{v_{s-1} \cdots v_1} J_{n_{s-1}}^{v_{s-1}} \otimes \cdots \otimes J_{n_1}^{v_1} . \tag{2.41} \]

In (2.40) we also have \(2^{s-1}\) bases vectors which give the expression for some \(\Sigma_{s-1}^{(2)}\) (according to Corollary 2.1):

\[ \Sigma_{s-1}^{(2)} = \sum_{v_{s-1}=0}^{1} \cdots \sum_{v_1=0}^{1} c'_{v_{s-1} \cdots v_1} J_{n_{s-1}}^{v_{s-1}} \otimes \cdots \otimes J_{n_1}^{v_1} . \tag{2.42} \]

The induction step implies for the covariance matrices \(\Sigma_{s-1}^{(h)}\), \(h = 1, 2\), defined in (2.41) and (2.42), the following structure:
\[
\begin{align*}
\Sigma_{s-1}^{(1)} &= I_{n_{s-1}} \otimes \Sigma_{s-2}^{(1)} + (J_{n_{s-1}} - I_{n_{s-1}}) \otimes \Sigma_{s-2}^{(2)}, \\
\Sigma_{s-1}^{(2)} &= I_{n_{s-1}} \otimes \Sigma_{s-2}^{(3)} + (J_{n_{s-1}} - I_{n_{s-1}}) \otimes \Sigma_{s-2}^{(4)}.
\end{align*}
\tag{2.43}
\]

Therefore, for the matrix \( \Sigma_s \) we have
\[
\begin{align*}
\Sigma_s &= I_{n_s} \otimes \left[ I_{n_{s-1}} \otimes \Sigma_{s-2}^{(1)} + (J_{n_{s-1}} - I_{n_{s-1}}) \otimes \Sigma_{s-2}^{(2)} \right] \\
&\quad + (J_{n_s} - I_{n_s}) \otimes \left[ I_{n_{s-1}} \otimes \Sigma_{s-2}^{(3)} + (J_{n_{s-1}} - I_{n_{s-1}}) \otimes \Sigma_{s-2}^{(4)} \right] \\
&= I_{n_s} \otimes \Sigma_{s-1}^{(1)} + (J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}^{(2)}. \quad \Box
\end{align*}
\tag{2.44}
\]

It makes sense to obtain the expression for \( \Sigma_s \) in terms of its elements \( \tau_1, \ldots, \tau_p, \) \( p = 2^s \), which completely define the pattern of \( \Sigma_s \).

**Theorem 2.8.** The \( P_s \)-invariant covariance matrix \( \Sigma_s \) is defined by \( 2^s \) parameters as
\[
\Sigma_s(\tau_1, \ldots, \tau_p) = I_{n_s} \otimes \Sigma_{s-1}(\tau_1, \ldots, \tau_{p/2}) + (J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}(\tau_{p/2+1}, \ldots, \tau_p),
\tag{2.45}
\]
where \( p = 2^s \) and \( \Sigma_h(x_1, \ldots, x_r) \) is a function of \( r = 2^h \) arguments \( x_1, \ldots, x_r, \) \( h = 1, \ldots, s \).

**Proof.** Define the covariance matrix \( \Sigma_s \) as a function of \( p \) arguments \( x_1, \ldots, x_p \):
\[
\Sigma_s(x_1, \ldots, x_p) = f(x_1, \ldots, x_p) = I_{n_s} \otimes \Sigma_{s-1}(x_1, \ldots, x_{p/2}) + (J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}(x_{p/2+1}, \ldots, x_p),
\]
where \( p = 2^s \). Next, according to (2.36)–(2.38),
\[
\begin{align*}
\Sigma_0(\tau_h) &= \tau_h, \quad h = 1, 2, \\
\Sigma_1(\tau_1, \tau_2) &= I_{n_1} \otimes \Sigma_0(\tau_1) + (J_{n_1} - I_{n_1}) \otimes \Sigma_0(\tau_2), \\
\Sigma_2(\tau_1, \tau_2, \tau_3, \tau_4) &= I_{n_2} \otimes \Sigma_1(\tau_1, \tau_2) + (J_{n_2} - I_{n_2}) \otimes \Sigma_1(\tau_3, \tau_4), \\
\Sigma_3(\tau_1, \ldots, \tau_8) &= I_{n_3} \otimes \Sigma_2(\tau_1, \tau_2, \tau_3, \tau_4) + (J_{n_3} - I_{n_3}) \otimes \Sigma_2(\tau_5, \tau_6, \tau_7, \tau_8).
\end{align*}
\]
Assume now, that the statement of the theorem is true for \( s - 1 \). Then, according to (2.36) we have
\[
\Sigma_s(\tau_1, \ldots, \tau_p) = I_{n_s} \otimes \Sigma_{s-1}^{(1)} + (J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}^{(2)},
\tag{2.46}
\]
and applying the induction step to \( \Sigma_{s-1}^{(1)} \) we may write
\[
\Sigma_{s-1}^{(1)}(\tau_1, \ldots, \tau_{p/2}) = I_{n_{s-1}} \otimes \Sigma_{s-2}(\tau_1, \ldots, \tau_{p/4}) + (J_{n_{s-1}} - I_{n_{s-1}}) \otimes \Sigma_{s-2}(\tau_{p/4+1}, \ldots, \tau_{p/2}).
\tag{2.47}
\]

Since the matrix \( \Sigma_s \) in (2.36) is defined by \( 2^s \) parameters \( \tau_1, \ldots, \tau_p \), and \( \Sigma_{s-1}^{(1)} \) is a function of \( p/2 = 2^{s-1} \) parameters, it follows that \( \Sigma_{s-1}^{(2)} \) must be a function of \( \tau_{p/2+1}, \ldots, \tau_p \). Hence, we can write \( \Sigma_s \) as
\[
\Sigma_s(\tau_1, \ldots, \tau_p) = I_{n_s} \otimes \Sigma_{s-1}(\tau_1, \ldots, \tau_{p/2}) + (J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}(\tau_{p/2+1}, \ldots, \tau_p). \quad \Box
\tag{2.48}
\]
3. Spectrum and eigenvectors of the invariant covariance matrix

In the present section we study the spectrum and eigenvectors of the permutation invariant covariance matrix $\Sigma_s$, i.e., we shall provide the insight into the structure of the eigenvalues and eigenvectors of such patterned matrices. However note that we also would have defined the eigenstructure of $D(Y)$. One observes that $D(Y)$ consists of commuting terms where each term connected to a factor. Because of the commutativity the eigenstructure of $D(Y)$ can immediately be obtained (for details see [10]).

In the next theorem the spectrum of $\Sigma_s$, given in Theorem 2.7, is presented.

**Theorem 3.1.** Let the covariance matrix $\Sigma_s$ be defined as in (2.36)–(2.38). Let $\lambda_i^{(1)}$ and $\lambda_i^{(2)}$ be eigenvalues of $\Sigma_{s-1}^{(1)}$ and $\Sigma_{s-1}^{(2)}$, respectively, $i = 1, \ldots, r$, and $r = n_1 \cdots n_{s-1}$. Then the spectrum of $\Sigma_s$ consists of eigenvalues of the form $\lambda_i^{(1)} + (n_s - 1)\lambda_i^{(2)}$, each of multiplicity 1, and of eigenvalues of the form $\lambda_i^{(1)} - \lambda_i^{(2)}$, each of multiplicity $n_s - 1$.

**Proof.** The matrices $I_{n_s}$ and $J_{n_s}$ commute, and the construction of $\Sigma_{s-1}^{(1)}$ and $\Sigma_{s-1}^{(2)}$ in (2.38) implies that they also commute. Hence, $I_{n_s} \otimes \Sigma_{s-1}^{(1)}$ and $(J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}^{(2)}$ are normal matrices which commute. There exists an orthogonal matrix $\Gamma = \Gamma_2 \otimes \Gamma_1$ such that

$$\Gamma \Sigma_s \Gamma' = A = (\Gamma_2 \otimes \Gamma_1) \left(I_{n_s} \otimes \Sigma_{s-1}^{(1)} + (J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}^{(2)}\right) (\Gamma_2 \otimes \Gamma_1')$$

$$= I_{n_s} \otimes A_{s-1}^{(1)} + \text{Diag}\{n_s - 1, -I_{n_s-1}\} \otimes A_{s-1}^{(2)}, \quad (3.1)$$

where

$$\Gamma_k = \left(n_k^{-1/2} I_{n_k} : H_k\right), \quad H_k' I_{n_k} = 0, \quad H_k' H_k = I_k, \quad k = 1, 2, \quad (3.2)$$

and $A_{s-1}^{(l)}$ is a diagonal matrix with the eigenvalues of $\Sigma_{s-1}^{(l)}$ on the main diagonal, i.e., $A_{s-1}^{(l)} = \text{Diag}\{\lambda_1^{(l)}, \ldots, \lambda_r^{(l)}\}, l = 1, 2$.

Thus, the spectrum of $\Sigma_s$ satisfies the following: $\lambda_1^{(1)} - \lambda_1^{(2)}, \ldots, \lambda_r^{(1)} - \lambda_r^{(2)}$ each of multiplicity $(n_s - 1)$ and $\lambda_1^{(1)} + (n_s - 1)\lambda_1^{(2)}, \ldots, \lambda_r^{(1)} + (n_s - 1)\lambda_r^{(2)}$ each of multiplicity 1. $\Box$

A straightforward consequence of the theorem is the following.

**Corollary 3.1.** The determinant of $\Sigma_s$ is given by

$$|\Sigma_s| = \left|A_{s-1}^{(1)} + (n_s - 1)A_{s-1}^{(2)}\right| \cdot \left|A_{s-1}^{(1)} - A_{s-1}^{(2)}\right|^{n_s-1}, \quad (3.3)$$

where $A_{s-1}^{(k)}, k = 1, 2$, is defined in the proof of Theorem 3.1.

The next step is to obtain expressions for the eigenvalues of $\Sigma_s$ by means of the parameters $\tau$’s used in Theorem 2.7. The multiplicities of the corresponding eigenvalues will also be given. As an example, we first consider the cases $s = 1, 2, 3$.

Define the following binary variables:

$$\alpha_h \in \{0, h\}; \quad h = 1, \ldots, s. \quad (3.4)$$
If $s = 1$, the matrix $\Sigma_1$ is defined by two parameters $\tau_1, \tau_2$ and has two distinct eigenvalues given by

$$\lambda_{\alpha_1} = \tau_1 + (n_{\alpha_1} - 1)\tau_2,$$

$$n_{\alpha_1} = \begin{cases} 0, & \text{if } \alpha_1 = 0, \\ n_1, & \text{if } \alpha_1 = 1. \end{cases}$$

The multiplicity $m(\lambda_{\alpha_1})$ of $\lambda_{\alpha_1}$ equals

$$m(\lambda_{\alpha_1}) = \begin{cases} n_1 - 1, & \text{if } \alpha_1 = 0, \\ 1, & \text{if } \alpha_1 = 1. \end{cases}$$

For $s = 2$, the matrix $\Sigma_2$ is defined by four parameters $\tau_1, \ldots, \tau_4$, and there are four distinct eigenvalues given by the following formula:

$$\begin{align*}
\lambda_{\alpha_2, \alpha_1} &= [\tau_1 + (n_{\alpha_1} - 1)\tau_2] + (n_{\alpha_2} - 1)[\tau_3 + (n_{\alpha_1} - 1)\tau_4], \\
n_{\alpha_i} &= \begin{cases} 0, & \text{if } \alpha_i = 0, \quad i = 1, 2, \\ n_i, & \text{if } \alpha_i = i, \quad i = 1, 2. \end{cases}
\end{align*}$$

The multiplicity $m(\lambda_{\alpha_2, \alpha_1})$ of $\lambda_{\alpha_2, \alpha_1}$ is the following

$$m(\lambda_{\alpha_2, \alpha_1}) = \begin{cases} (n_2 - 1)(n_1 - 1), & \text{if } \alpha_2 = 0, \quad \alpha_1 = 0, \\
n_2 - 1, & \text{if } \alpha_2 = 0, \quad \alpha_1 = 1, \\
n_1 - 1, & \text{if } \alpha_2 = 2, \quad \alpha_1 = 0, \\
1, & \text{if } \alpha_2 = 2, \quad \alpha_1 = 1. \end{cases}$$

If $s = 3$, then the covariance matrix $\Sigma_3$ is defined by eight parameters $\tau_1, \ldots, \tau_8$ and it has eight distinct eigenvalues:

$$\begin{align*}
\lambda_{\alpha_3, \alpha_2, \alpha_1} &= [\tau_1 + (n_{\alpha_1} - 1)\tau_2 + (n_{\alpha_2} - 1)(\tau_3 + (n_{\alpha_1} - 1)\tau_4)] \\
&\quad + (n_{\alpha_3} - 1)[\tau_5 + (n_{\alpha_1} - 1)\tau_6 + (n_{\alpha_2} - 1)(\tau_7 + (n_{\alpha_1} - 1)\tau_8)], \\
n_{\alpha_i} &= \begin{cases} 0, & \text{if } \alpha_i = 0, \quad i = 1, 2, 3, \\
n_i, & \text{if } \alpha_i = i, \quad i = 1, 2, 3. \end{cases}
\end{align*}$$

The multiplicity $m(\lambda_{\alpha_3, \alpha_2, \alpha_1})$ of $\lambda_{\alpha_3, \alpha_2, \alpha_1}$ is the following:

$$m(\lambda_{\alpha_3, \alpha_2, \alpha_1}) = \begin{cases} (n_3 - 1)(n_2 - 1)(n_1 - 1), & \text{if } \alpha_3 = 0, \quad \alpha_2 = 0, \quad \alpha_1 = 0, \\
(n_3 - 1)(n_2 - 1), & \text{if } \alpha_3 = 0, \quad \alpha_2 = 0, \quad \alpha_1 = 1, \\
(n_3 - 1)(n_1 - 1), & \text{if } \alpha_3 = 0, \quad \alpha_2 = 2, \quad \alpha_1 = 0, \\
(n_2 - 1)(n_1 - 1), & \text{if } \alpha_3 = 3, \quad \alpha_2 = 0, \quad \alpha_1 = 0, \\
n_3 - 1, & \text{if } \alpha_3 = 0, \quad \alpha_2 = 2, \quad \alpha_1 = 1, \\
n_2 - 1, & \text{if } \alpha_3 = 3, \quad \alpha_2 = 0, \quad \alpha_1 = 1, \\
n_1 - 1, & \text{if } \alpha_3 = 3, \quad \alpha_2 = 2, \quad \alpha_1 = 0, \\
1, & \text{if } \alpha_3 = 3, \quad \alpha_2 = 2, \quad \alpha_1 = 1. \end{cases}$$

Define now the eigenvalue $\lambda_{x_1, \ldots, x_p}(x_1, \ldots, x_p)$ as a function of the parameters $\tau_1, \ldots, \tau_p$, $p = 2^s$, given in Theorem 2.7.

**Theorem 3.2.** The eigenvalues of $\Sigma_s$ can be presented in the following recursive way:

$$\begin{align*}
\lambda_{x_1, \ldots, x_1}(\tau_1, \ldots, \tau_p) &= \lambda_{x_{s-1}, \ldots, x_1}(\tau_1, \ldots, \tau_{p/2}) \\
&\quad + (n_{x_s} - 1)\lambda_{x_{s-1}, \ldots, x_1}(\tau_{p/2+1}, \ldots, \tau_p),
\end{align*}$$

(3.8)
where
\[ n_{\alpha_s} = \begin{cases} 
0, & \text{if } \alpha_s = 0, \\
n_s, & \text{if } \alpha_s = s.
\end{cases} \]  
(3.9)

**Proof.** According to (2.45) in Theorem 2.8 we can write
\[ \Sigma_s(\tau_1, \ldots, \tau_p) = I_{n_s} \otimes \Sigma_{s-1}(\tau_1, \ldots, \tau_{p/2}) + (J_{n_s} - I_{n_s}) \otimes \Sigma_{s-1}(\tau_{p/2+1}, \ldots, \tau_p). \]
Let \( \Gamma'_{n_i} \Gamma_{n_i} = I_{n_i}, \)
\[ \Gamma'_{n_i} J_{n_i} \Gamma_{n_i} = \begin{pmatrix} n_i & 0 \\
0 & 0_{n_i-1} \end{pmatrix}, \]  
(3.10)
and
\[ \Gamma'_{s-1} \Sigma_{s-1}(\tau_1, \ldots, \tau_h) \Gamma_{s-1} = A_{s-1}(\tau_1, \ldots, \tau_h), \]
where \( 0_{n_i-1}: (n_i - 1) \times (n_i - 1) \) is the matrix with all elements equal to zero, \( A_{s-1}(\tau_1, \ldots, \tau_h) \) is the diagonal matrix with eigenvalues \( \lambda_{\alpha_{s-1}, \ldots, \alpha_1}(\tau_1, \ldots, \tau_h) \) of \( \Sigma_{s-1}(\tau_1, \ldots, \tau_h) \) on the main diagonal, \( i = 1, \ldots, s \). Hence, \( A_s = \Gamma'_s \Sigma_s \Gamma_s \) becomes
\[ A_s(\tau_1, \ldots, \tau_p) = I_{n_s} \otimes A_{s-1}(\tau_1, \ldots, \tau_{p/2}) + \left[ \Gamma'_s(J_{n_s} - I_{n_s}) \Gamma_s \right] \otimes A_{s-1}(\tau_{p/2+1}, \ldots, \tau_p) = I_{n_s} \otimes A_{s-1}(\tau_1, \ldots, \tau_{p/2}) + \left( n_s - 1 \ 0 \\
0 \ -I_{n_s-1} \right) \otimes A_{s-1}(\tau_{p/2+1}, \ldots, \tau_p), \]  
(3.11)
which completes the proof of the theorem. \( \Box \)

Define now auxiliary binary variables \( \delta_h, h = 1, \ldots, s, \) as
\[ \delta_h = \begin{cases} 
0, & \text{if } \alpha_h = 0, \\
1, & \text{if } \alpha_h = s.
\end{cases} \]  
(3.12)
According to Theorem 2.3 the structure of \( \Sigma_s \) turned out to be a linear combination of \( 2^s \) linearly independent terms each of which is the Kronecker product of \( s \) matrices which either equal \( I_h \) or \( J_h, h = 1, \ldots, s. \)

Searle and Henderson [15] studied properties of the covariance matrix of the form
\[ V_s = \sum_{i=0}^{1} \theta_i(J_{s}^{i_s} \otimes J_{s-1}^{i_{s-1}} \otimes \cdots \otimes J_1^{i_1}), \]  
(3.13)
where \( i \) is a multipartite number, some of the \( \theta_i \) equal \( \sigma^2 \) and others are zero. They obtained results concerning the spectrum, determinant and the inverse of such a covariance matrix. For example, if we use the results of Searle and Henderson [15] we may state the following theorem.

**Theorem 3.3.** If the covariance matrix \( \Sigma_s \) has the structure
\[ \Sigma_s = \sum_{v_s=0}^{1} \ldots \sum_{v_1=0}^{1} c_{v_s \cdots v_2 v_1} J_{n_s}^{v_s} \otimes \cdots \otimes J_{n_2}^{v_2} \otimes J_{n_1}^{v_1}, \]
then its eigenvalues are given by
\[ \lambda_{\alpha_s,\ldots,\alpha_1} = \sum_{v_s=0}^{1} \cdots \sum_{v_1=0}^{1} \sum_{i=1}^{s} \tau_i \prod_{i=1}^{s} n_i^{\delta_i}, \] (3.14)
and the multiplicities of the corresponding eigenvalues are the following:
\[ m(\lambda_{\alpha_s,\ldots,\alpha_1}) = \prod_{h=1}^{s} (n_h - 1)^{1-\delta_h}, \] (3.15)
where \( \delta_h \) is given in (3.12).

The following theorem specifies the structure of the eigenvector \( w(\lambda_{\alpha_s,\ldots,\alpha_1}) \) which corresponds to the eigenvalue \( \lambda_{\alpha_s,\ldots,\alpha_1} \) in Theorem 3.3.

**Theorem 3.4.** The eigenvector \( w(\lambda_{\alpha_s,\ldots,\alpha_1}) \) corresponding to \( \lambda_{\alpha_s,\ldots,\alpha_1} \) in (3.14) of \( \Sigma_s \), given by (2.36), equals
\[ w(\lambda_{\alpha_s,\ldots,\alpha_1}) = \bigotimes_{i=s}^{1} v_{n_i}^{1-\delta_i}, \] (3.16)
where \( \delta_h \) is defined in (3.12), \( \alpha_h \) is given by (3.4), and the \( n_h \)-vectors \( v_{n_h} \) satisfy \( v_{n_h}' 1_{n_h} = 0 \) and \( v_{n_h}^0 = 1_{n_h}, \ h = 1, \ldots, s. \)

**Proof.** The proof follows immediately from the structure of \( \Sigma_s \) in (2.36) and properties of the Kronecker product. \( \square \)

In the next theorem we shall extend the results presented in Searle and Henderson [15] by also allowing for nonzero covariances.

**Theorem 3.5.** All eigenvalues of the matrix \( \Sigma_s \) given by (2.36) can be obtained by the following formula:
\[ \lambda_{\alpha_s,\ldots,\alpha_1} = \sum_{v_s=0}^{1} \cdots \sum_{v_1=0}^{1} \tau_k \prod_{i=1}^{s} (n_{\alpha_i} - 1)^{v_i}, \] (3.17)
where the index \( \alpha_h \) is given by (3.4) and
\[ k = \sum_{h=1}^{s} v_h \cdot 2^{h-1} + 1. \] (3.18)

**Proof.** To prove the statement of the theorem induction is used. It is easy to see from (3.5) and (3.6) that the formula (3.17) is true for \( s = 1 \) and \( s = 2. \) Suppose the formula is true for \( s - 1. \) According to Theorem 3.2 we may write
\[ \lambda_{\alpha_s,\ldots,\alpha_1}(\tau_1, \ldots, \tau_p) = \lambda_{\alpha_{s-1},\ldots,\alpha_1}(\tau_1, \ldots, \tau_{p/2}) + (n_{\alpha_s} - 1)\lambda_{\alpha_{s-1},\ldots,\alpha_1}(\tau_{p/2+1}, \ldots, \tau_p). \]
Applying the induction step, we may rewrite the expression for $\lambda_{\alpha_s, \ldots, \alpha_1}$ as follows:

$$
\lambda_{\alpha_s, \ldots, \alpha_1}(\tau_1, \ldots, \tau_p) = \sum_{v_{s-1}=0}^{1} \cdots \sum_{v_1=0}^{1} \tau_{k_1} \prod_{i=1}^{s-1} (n_{\alpha_i} - 1)^{v_i} + (n_{\alpha_s} - 1) \sum_{v_{s-1}=0}^{1} \cdots \sum_{v_1=0}^{1} \tau_{k_1+p/2} \prod_{i=1}^{s-1} (n_{\alpha_i} - 1)^{v_i},
$$

(3.19)

where the index $k_1 = 1, \ldots, p/2$ can be expressed as

$$
k_1 = \sum_{h=1}^{s-1} v_h \cdot 2^{h-1} + 1 = \sum_{h=1}^{s-1} v_h \cdot 2^{h-1} + v_s \cdot 2^{s-1} + 1 = \sum_{h=1}^{s} v_h \cdot 2^{h-1} + 1, \quad \text{assuming } v_s = 0.
$$

(3.20)

It is clear, that $k_1$ in (3.20) equals $k$ in (3.18). It is also easy to see, that $k_1 + p/2$ equals $k$ in (3.18) when $v_s = 1$:

$$
k_1 + p/2 = k_1 + 2^{s-1} = \sum_{h=1}^{s-1} v_h \cdot 2^{h-1} + 1 + v_s \cdot 2^{s-1} = \sum_{h=1}^{s} v_h \cdot 2^{h-1} + 1.
$$

(3.21)

Finally, summarizing the expressions for $k_1$ and $k_1 + p/2$, we have in (3.19)

$$
\lambda_{\alpha_{s-1}, \ldots, \alpha_1} = \sum_{v_{s-1}=0}^{1} \cdots \sum_{v_1=0}^{1} \tau_{k} \prod_{i=1}^{s} (n_{\alpha_i} - 1)^{v_i},
$$

where

$$
k = \sum_{h=1}^{s} v_h \cdot 2^{h-1} + 1. \quad \Box
$$

Another way of summarizing eigenvalues of $\Sigma_s$ in a compact form is given in the next theorem.

**Theorem 3.6.** The eigenvalues of $\Sigma_s$ can be found by the following formula:

$$
\lambda_{\alpha_s, \ldots, \alpha_1} = \bigotimes_{i=s}^{1} \left( \frac{1}{n_{\alpha_i} - 1} \right) \tau,
$$

(3.22)

where $\bigotimes$ is defined in (2.26), $\tau = (\tau_1, \ldots, \tau_p)'$ with components as defined in Theorem 2.7 and $p = 2^s$.

**Proof.** To prove the statement of the theorem induction is used. For $s = 1$

$$
\lambda_{\alpha_1} = \left( \frac{1}{n_{\alpha_1} - 1} \right)' \left( \frac{\tau_1}{\tau_2} \right) = \tau_1 + (n_{\alpha_1} - 1)\tau_2.
$$
For \( s = 2 \) we have

\[
\lambda_{\alpha_2, \alpha_1} = \bigotimes_{i=2}^s \left( \begin{array}{c} 1 \\ n_{\alpha_i} - 1 \end{array} \right)^\dagger \left( \begin{array}{c} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{array} \right) = \left( \begin{array}{c} 1 : n_{\alpha_2} - 1 \\ \vdots \\ \tau_2 \\ \tau_3 \end{array} \right) \otimes \left( \begin{array}{c} 1 : n_{\alpha_1} - 1 \\ \tau_2 \\ \tau_3 \end{array} \right)
\]

\[
= \left( \begin{array}{c} 1 \\ n_{\alpha_1} - 1 \\ n_{\alpha_2} - 1 \\ (n_{\alpha_2} - 1)(n_{\alpha_1} - 1) \end{array} \right)^\dagger \left( \begin{array}{c} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{array} \right),
\]

which gives (3.6). Assume that the statement of the theorem is true for \( s - 1 \). Then, based on (3.8),

\[
\lambda_{\alpha_s, \ldots, \alpha_1}(\tau_1, \ldots, \tau_p) = \lambda_{\alpha_{s-1}, \ldots, \alpha_1}(\tau_1, \ldots, \tau_{p/2}) + (n_{\alpha_s} - 1)\lambda_{\alpha_{s-1}, \ldots, \alpha_1}(\tau_{p/2+1}, \ldots, \tau_p)
\]

and we can write

\[
\lambda_{\alpha_s, \ldots, \alpha_1}(\tau_1, \ldots, \tau_p) = \bigotimes_{i=s-1}^s \left( \begin{array}{c} 1 \\ n_{\alpha_i} - 1 \end{array} \right)^\dagger \left( \begin{array}{c} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{p/2} \end{array} \right)
\]

\[
+ (n_{\alpha_s} - 1) \bigotimes_{i=s-1}^s \left( \begin{array}{c} 1 \\ n_{\alpha_i} - 1 \end{array} \right)^\dagger \left( \begin{array}{c} \tau_{p/2+1} \\ \vdots \\ \tau_p \end{array} \right).
\]

Hence,

\[
\lambda_{\alpha_s, \ldots, \alpha_1}(\tau_1, \ldots, \tau_p) = \bigotimes_{i=s}^s \left( \begin{array}{c} 1 \\ n_{\alpha_i} - 1 \end{array} \right)^\dagger \left( \begin{array}{c} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_p \end{array} \right). \quad \square
\]

As pointed out in Searle and Henderson [15], the structure (2.22) allows us to write eigenvalues of \( \Sigma_s \) as

\[
\left( \begin{array}{c} \lambda_{0,0,\ldots,0} \\ \vdots \\ \lambda_{s,s-1,\ldots,1} \end{array} \right) = \bigotimes_{i=s}^s \left( \begin{array}{c} 1 \\ 0 \end{array} \right)^\dagger \left( \begin{array}{c} c_{00\ldots0} \\ \vdots \\ c_{11\ldots1} \end{array} \right).
\]

Using this result and taking the inverse of the Kronecker product in (3.23),

\[
\left( \begin{array}{c} 1 \\ 1 \\ n_i \end{array} \right)^{-1} = \left( \begin{array}{cc} -\frac{1}{n_i} & 0 \\ \frac{1}{n_i} & \frac{1}{n_i} \end{array} \right).
\]
we can rewrite $\Sigma_s$ in (2.22) via its spectrum using the following relation:
\[
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}
= \frac{1}{n_i} \left( \frac{1}{n_i} \right)
\begin{pmatrix}
\lambda_{s,0} & \ldots & \lambda_{s,s-1}
\end{pmatrix}
\left( \begin{pmatrix}
\ldots & \lambda_{0,0} & \ldots
\end{pmatrix}
\right).
\]

As an example, using the structure given by (2.35) and the relations in (3.26), the covariance matrix $\Sigma_3$ may be expressed via its spectrum as
\[
\Sigma_3 = I_{n_3} \otimes \left[ I_{n_2} \otimes \left[ \frac{1}{n_1} \left( \lambda_{0,0,0} I_{n_1} - \frac{1}{n_1} \left( \lambda_{0,0,0} - \lambda_{0,0,1} \right) J_{n_1} \right) \right] \right]
- \frac{1}{n_2} J_{n_2} \otimes \left[ \frac{1}{n_1} \left( \lambda_{0,0,0} - \lambda_{0,2,0} \right) I_{n_1} - \frac{1}{n_1} \left( \lambda_{0,0,0} - \lambda_{0,0,1} - \lambda_{0,2,0} + \lambda_{0,2,1} \right) J_{n_1} \right]
- \frac{1}{n_2} J_{n_2} \otimes \left[ \frac{1}{n_1} \left( \lambda_{0,0,0} - \lambda_{3,0,0} \right) I_{n_1} - \frac{1}{n_1} \left( \lambda_{0,0,0} - \lambda_{0,0,1} + \lambda_{3,0,0} + \lambda_{3,2,0} \right) J_{n_1} \right]
- \frac{1}{n_2} J_{n_2} \otimes \left[ \frac{1}{n_1} \left( \lambda_{0,0,0} - \lambda_{0,0,1} - \lambda_{0,2,0} + \lambda_{0,2,1} - \lambda_{3,0,0} + \lambda_{3,0,1} + \lambda_{3,2,0} - \lambda_{3,2,1} \right) J_{n_1} \right].
\]

4. Reparameterization constraints and permutation invariance in linear models

The structure of a statistical linear model is identified by the design of the experiment via the design matrix $X$ and the nature of the factors which are involved in the model. The random factors in their turn are characterized by their covariance matrices. In order to formulate correctly a linear model we should give explicit interpretations of all components of the model. Under reparameterization we mean imposing certain constraints on the random factors. The most commonly used constraints are “sum-to-zero” and “set-to-zero” constraints: if $\xi = (\xi_1, \ldots, \xi_l)'$, then the “sum-to-zero” constraint for $\xi$ equals $\sum_{i=1}^l \xi_i = 0$ and the “set-to-zero” constraint for $\xi$, for example, equals $\xi_l = 0$.

Differently reparameterized linear models may have considerably different interpretations. While the consequences of putting constraints are well known for fixed factors, new possibilities and questions arise in linear models with random factors. In particular, there is no unified and comprehensive approach how to handle the interactions between random and fixed factors. They are sometimes considered to be independent and sometimes restricted to sum zero over the levels of fixed factor (see, for example, [7, 20]).

Singularity of the covariance matrix of the random factor implies that there is dependence among factor levels and the question is if there are natural restrictions on this random factor.

In this section we shall demonstrate that permutation invariance of the singular covariance matrix can results in classical “sum-to-zero” reparameterization constraints of this factor. We shall show that it is possible to express classical “sum-to-zero” reparameterization conditions
through the spectrum of the covariance matrix. This approach attracts because the covariance matrix is a well understood quantity which describes a basic property of a random factor. To put conditions on the random factor via its covariance matrix is natural and makes inference more efficient.

The next theorem is a modification of a result presented in [9].

**Theorem 4.1.** Let \( \xi = (\xi_1, \ldots, \xi_n)' \) be a factor of main effects, and let \( \xi_i \neq \xi_j \) a.s., \( i \neq j \). Let \( E(\xi) = 0 \) and assume that \( \Sigma_1 \) is \( P_1 \)-invariant, i.e. \( P_1 \Sigma_1 P_1' = \Sigma_1 \). Then the following conditions are equivalent:

(i) \( 1_{n_1}' \xi = 0 \) a.s.

(ii) \( \Sigma_1 = \frac{1}{n_1} (I_{n_1} - \frac{1}{n_1} J_{n_1}) \), where \( \tau_1 = D(\xi_i), i = 1, \ldots, n_1 \).

(iii) \( \Sigma_1 \) is singular.

**Proof.** See [9]. \( \square \)

It is interesting to see that in the case of permutation invariance the singularity of the covariance matrix \( D(\xi) \) is equivalent to the condition that \( 1_{n_1}' \xi = 0 \), and vice versa. In general, this is obviously not the case. Note that \( 1_{n_1} \) is the eigenvector corresponding to the eigenvalue zero of multiplicity 1.

The situation with the factors representing \( s \)-order interaction effects is more complicated. The singularity of the \( P_s \)-invariant covariance matrix of \( \gamma^{(s)} \) does not, in general, imply the classical “sum-to-zero” reparameterization of \( \gamma^{(s)} \). Further, we shall demonstrate that the “sum-to-zero” reparameterization condition for \( \gamma^{(s)} \) has a clear interpretation via the spectrum of \( \Sigma_2 \).

### 4.1. The two factor case

We shall show that any of the following classical “sum-to-zero” reparameterization conditions for \( \gamma^{(2)} \)

(a) \( \sum_i^{n_2} \gamma^{(2)}_{ij} = 0 \) a.s., for all \( j \),

(b) \( \sum_j^{n_1} \gamma^{(2)}_{ij} = 0 \) a.s., for all \( i \),

(c) \( \sum_i^{n_2} \gamma^{(2)}_{ij} = 0 \) a.s., for all \( j \), and \( \sum_j^{n_1} \gamma^{(2)}_{ij} = 0 \) a.s., for all \( i \),

can be formulated as specific restrictions on the eigenvalues of the invariant covariance matrix \( \Sigma_2 \). It is obvious that if \( \gamma^{(2)} \) is reparameterized according to the conditions (a), (b) or (c), then its covariance matrix is singular. From singularity of \( \Sigma_2 \), in general, conditions (a), (b) or (c) do not follow immediately.

The next result provides conditions under which the spectrum of the \( P_2 \)-invariant covariance matrix \( \Sigma_2 \) leads to the classical “sum-to-zero” reparameterization for \( \gamma^{(2)} \).

**Theorem 4.2.** Let the factor \( \gamma^{(2)} \) with the covariance matrix \( \Sigma_2 \) represent interaction effects of two factors. Assume \( \gamma^{(2)}_{ij} \neq \gamma^{(2)}_{kj} \) for all \( j \), and \( \gamma^{(2)}_{ij} \neq \gamma^{(2)}_{is} \) for all \( i \), a.s. Let \( E(\gamma^{(2)}) = 0 \) and let \( \Sigma_2 \) be \( P_2 \)-invariant. Let \( \lambda_{0,0}, \lambda_{0,1}, \lambda_{2,0}, \lambda_{2,1} \) be distinct eigenvalues of \( \Sigma_2 \) defined in (3.6). Then the following conditions hold:
(i) $\sum_{i}^{n_2} \gamma_{ij}^{(2)} = 0 \ a.s., \ \forall j, \ \text{iff} \ \lambda_{2,0} = \lambda_{2,1} = 0.$

(ii) $\sum_{j}^{n_1} \gamma_{ij}^{(2)} = 0 \ a.s., \ \forall i, \ \text{iff} \ \lambda_{0,1} = \lambda_{2,1} = 0.$

(iii) $\sum_{i}^{n_2} \gamma_{ij}^{(2)} = 0 \ a.s., \ \forall j, \ \text{and} \ \sum_{j}^{n_1} \gamma_{ij}^{(2)} = 0 \ a.s., \ \forall i, \ \text{iff} \ \lambda_{0,1} = \lambda_{2,0} = \lambda_{2,1} = 0.$

**Proof.** See [8]. □

A straightforward consequence of the theorem is the next result.

**Corollary 4.1.** If the covariance matrix $\Sigma_2$ of $\gamma^{(2)}$ is $P_2$-invariant, then a “sum-to-zero” reparameterizations of $\gamma^{(2)}$ imply the following specific structures for $\Sigma_2$:

(i) $\sum_{i}^{n_2} \gamma_{ij}^{(2)} = 0 \ a.s., \ \forall j, \ \text{iff}$

$$\Sigma_2 = \frac{n_2}{n_2 - 1} \left( I_{n_2} - \frac{1}{n_2} J_{n_2} \right) \otimes ((\tau_1 - \tau_2)I_{n_1} + \tau_2 J_{n_1}).$$

(4.1)

(ii) $\sum_{j}^{n_1} \gamma_{ij}^{(2)} = 0 \ a.s., \ \forall i, \ \text{iff}$

$$\Sigma_2 = \frac{n_1}{n_1 - 1} ((\tau_1 - \tau_3)I_{n_2} + \tau_3 J_{n_2}) \otimes \left( I_{n_1} - \frac{1}{n_1} J_{n_1} \right).$$

(4.2)

(iii) $\sum_{i}^{n_2} \gamma_{ij}^{(2)} = 0 \ a.s., \ \forall j, \ \text{and} \ \sum_{j}^{n_1} \gamma_{ij}^{(2)} = 0 \ a.s., \ \forall i, \ \text{iff}$

$$\Sigma_2 = \frac{n_2 n_1 \tau_1}{(n_2 - 1)(n_1 - 1)} \left( I_{n_2} - \frac{1}{n_2} J_{n_2} \right) \otimes \left( I_{n_1} - \frac{1}{n_1} J_{n_1} \right).$$

(4.3)

The parameters $\tau_1, \tau_2, \tau_3$ in conditions (i)–(iii) are defined in Theorem 2.4.

**Proof.** See [8]. □

4.2. The three factor case

The goal of the next theorem is to give the constraints on the spectrum of the singular $P_3$-invariant covariance matrix $\Sigma_3$ which result in classical “sum-to-zero” reparameterizations for $\gamma^{(3)}$. In the given context, under classical “sum-to-zero” reparameterization conditions for $\gamma^{(3)}$ we mean the following conditions:

$$\sum_{i} \gamma_{ijk}^{(3)} = 0 \ a.s., \ \forall j, k, \ \sum_{j} \gamma_{ijk}^{(3)} = 0 \ a.s., \ \forall i, k, \ \sum_{k} \gamma_{ijk}^{(3)} = 0 \ a.s., \ \forall i, j.$$ 

**Theorem 4.3.** Let factor $\gamma^{(3)}$ represent the third-order interaction effects. Assume $\gamma_{ijk}^{(3)} \neq \gamma_{i'jk}^{(3)}$ a.s. for all $i \neq i', j$ and $k$, $\gamma_{ijk}^{(3)} \neq \gamma_{ij'k}^{(3)}$ a.s. for all $j \neq j', i$ and $k$, and $\gamma_{ijk}^{(3)} \neq \gamma_{ijk'}^{(3)}$ a.s. for all
Further, notice that give by (3.26), and solving equations in (4.6), we get that

\[ \frac{k}{k'} \neq k', \, i \text{ and } j. \] Let \( E(\gamma(3)) = 0 \) and let \( \Sigma_3 = D(\gamma(3)) \) be \( P_3 \)-invariant. Let \( \lambda_{0,0,0}, \ldots, \lambda_{3,2,1} \) be the eigenvalues of \( \Sigma_3 \) as defined in (3.7). Then the following conditions hold:

(i) \( \sum_i \gamma_{ijk}^{(3)} = 0 \) a.s., \( \forall j, k, \) iff \( \lambda_{3,0,0} = \lambda_{3,0,1} = \lambda_{3,2,0} = \lambda_{3,2,1} = 0. \)

(ii) \( \sum_j \gamma_{ijk}^{(3)} = 0 \) a.s., \( \forall i, k, \) iff \( \lambda_{0,2,0} = \lambda_{0,2,1} = \lambda_{3,2,0} = \lambda_{3,2,1} = 0. \)

(iii) \( \sum_k \gamma_{ijk}^{(3)} = 0 \) a.s., \( \forall i, j, \) iff \( \lambda_{0,0,1} = \lambda_{0,2,1} = \lambda_{3,0,1} = \lambda_{3,2,1} = 0. \)

**Proof.** Proof of (i). First, rewrite the condition \( \sum_i \gamma_{ijk}^{(3)} = 0, \forall j, k, \) as

\[ (I'_{n_3} \otimes I_{n_2} \otimes I_{n_1}) \gamma^{(3)} = 0. \] (4.4)

Further, notice that

\[ 0 = D[(I'_{n_3} \otimes I_{n_2} \otimes I_{n_1}) \gamma^{(3)}] = (I'_{n_3} \otimes I_{n_2} \otimes I_{n_1}) \Sigma_3 (I_{n_3} \otimes I_{n_2} \otimes I_{n_1}) \]

and, according to the definition of \( \Sigma_3 \) in (2.22), this implies

\[ \sum_{v_3=0}^{1} \sum_{v_1=0}^{1} (c_{0v_3v_1} n_3 + c_{1v_3v_1} n_3^2) J_{n_3}^{v_3} \otimes J_{n_1}^{v_1} = 0. \] (4.5)

In (4.5), the matrices \( J_{n_2}^{v_2} \otimes J_{n_1}^{v_1}, \, v_1 \in \{0, 1\} \) and \( v_2 \in \{0, 1\} \), are linearly independent, i.e., all coefficients in (4.5) equal zero:

\[ c_{000} + c_{100} n_3 = 0, \quad c_{010} + c_{110} n_3 = 0, \quad c_{001} + c_{101} n_3 = 0, \quad c_{011} + c_{111} n_3 = 0. \] (4.6)

Using the relationship among coefficients \( c_{000}, \ldots, c_{111} \) and the eigenvalues \( \lambda_{0,0,0}, \ldots, \lambda_{3,2,1} \) given in (3.26), and solving equations in (4.6), we get that

\[ \lambda_{3,0,0} = \lambda_{3,0,1} = \lambda_{3,2,0} = \lambda_{3,2,1} = 0. \]

Suppose now, \( \lambda_{3,0,0} = \lambda_{3,0,1} = \lambda_{3,2,0} = \lambda_{3,2,1} = 0. \) From (3.27) it follows that

\[ \Sigma_3 = \left[ I_{n_3} - \frac{1}{n_3} \right] \otimes \left[ I_{n_2} \otimes \left[ \lambda_{0,0,0} I_{n_1} - \frac{1}{n_1} [\lambda_{0,0,0} - \lambda_{0,0,1}] J_{n_1} \right] \right. \\
\left. - \frac{1}{n_2} J_{n_2} \otimes \left[ (\lambda_{0,0,0} - \lambda_{0,2,0}) I_{n_1} - \frac{1}{n_1} [\lambda_{0,0,0} - \lambda_{0,0,1} - \lambda_{0,2,0} + \lambda_{0,2,1}] J_{n_1} \right] \right]. \] (4.7)

Let \( U = I_{n_3} \otimes I_{n_2} \otimes I_{n_1}. \) The expectation \( E(\gamma(3)) = 0 \) implies \( E(U' \gamma^{(3)}) = 0, \) and we have that

\[ U' \Sigma_3 U = 0. \]

Thus, \( U' \gamma^{(3)} = 0 \) a.s. what implies \( \sum_i \gamma_{ijk}^{(3)} = 0, \) for all \( j \) and \( k. \) This completes the proof of (i).

Condition (ii) is proved in a similar way. The condition \( \sum_j \gamma_{ijk}^{(3)} = 0, \) for all \( i \) and \( k, \) implies that \( (I_{n_3} \otimes I'_{n_2} \otimes I_{n_1}) \gamma^{(3)} = 0, \) which in turn leads to

\[ 0 = D[(I_{n_3} \otimes I'_{n_2} \otimes I_{n_1}) \gamma^{(3)}] = (I_{n_3} \otimes I'_{n_2} \otimes I_{n_1}) \Sigma_3 (I_{n_3} \otimes I_{n_2} \otimes I_{n_1}). \]

Consequently, using the form of \( \Sigma_3 \) given by (2.22), we get

\[ \sum_{v_3=0}^{1} \sum_{v_1=0}^{1} (c_{v_30v_1} n_2 + c_{v_31v_1} n_2^2) J_{n_3}^{v_3} \otimes J_{n_1}^{v_1} = 0. \] (4.8)
Further, observe that in (4.8) the matrices $J_{n_3}^{v_3} \otimes J_{n_1}^{v_1}$, $v_1 \in \{0, 1\}$ and $v_3 \in \{0, 1\}$, are linearly independent and hence all coefficients in (4.8) equal zero. Thus, we obtain the following system of equations:

$$
c_{000} + c_{010} n_2 = 0, \quad c_{100} + c_{110} n_2 = 0, \\
c_{001} + c_{011} n_2 = 0, \quad c_{101} + c_{111} n_2 = 0. \quad (4.9)
$$

Once again, relations between $c_{000}, \ldots, c_{111}$ and the eigenvalues $\lambda_{0,0,0}, \ldots, \lambda_{3,2,1}$ in (3.26) applied to (4.9), result in

$$
\lambda_{0,2,0} = \lambda_{0,2,1} = \lambda_{3,2,0} = \lambda_{3,2,1} = 0. \quad (4.10)
$$

Let us show now that the condition $\lambda_{0,2,0} = \lambda_{0,2,1} = \lambda_{3,2,0} = \lambda_{3,2,1} = 0$ implies $\sum_j Y_{ijk}^{(3)} = 0$, for all $i$ and $k$. Notice that if $\lambda_{0,2,0} = \lambda_{0,2,1} = \lambda_{3,2,0} = \lambda_{3,2,1} = 0$, then from (3.27) it follows that

$$
\Sigma_3 = I_{n_3} \otimes \left[ I_{n_2} - \frac{1}{n_2} J_{n_2} \right] \otimes \left[ \lambda_{0,0,0} I_{n_1} - \frac{1}{n_1} [\lambda_{0,0,0} - \lambda_{0,0,1}] J_{n_1} \right] \\
- \frac{1}{n_3} J_{n_3} \otimes \left[ I_{n_2} - \frac{1}{n_2} J_{n_2} \right] \otimes \left[ \lambda_{0,0,0} - \lambda_{3,0,0} \right] J_{n_1} \\
- \frac{1}{n_1} [\lambda_{0,0,0} - \lambda_{0,0,1} - \lambda_{3,0,0} + \lambda_{3,0,1}] J_{n_1}. \quad (4.11)
$$

Let $U = I_{n_3} \otimes 1_{n_2} \otimes I_{n_1}$. Now $E(\gamma^{(3)}) = 0$ implies $E(U'\gamma^{(3)}) = 0$, and because of the structure of $\Sigma_3$ in (4.11)

$$
U' \Sigma_3 U = 0.
$$

Thus, $U' \gamma^{(3)} = 0$ a.s., which implies $\sum_j Y_{ijk}^{(3)} = 0$, for all $i$ and $k$, and the proof of (ii) is complete. The condition (iii) follows immediately from conditions (i) and (ii). □

The next corollary follows from the proof of Theorem 4.3.

**Corollary 4.2**

(i) $\sum_i \sum_j Y_{ijk}^{(3)} = 0$ a.s., $\forall k$, iff $\lambda_{3,2,0} = \lambda_{3,2,1} = 0$.

(ii) $\sum_j \sum_k Y_{ijk}^{(3)} = 0$ a.s., $\forall i$, iff $\lambda_{0,2,1} = \lambda_{3,2,1} = 0$.

(iii) $\sum_i \sum_k Y_{ijk}^{(3)} = 0$ a.s., $\forall j$, iff $\lambda_{3,0,1} = \lambda_{3,2,1} = 0$.

(iv) $\sum_i \sum_j \sum_k Y_{ijk}^{(3)} = 0$ a.s. iff $\lambda_{3,2,1} = 0$.

**4.3. The s-factor case**

In the present section, we shall extend the results presented in the previous sections. We shall show what kind of reparameterizations are natural for $s$-order interactions in the case of permutation invariance.
Let
\[ L_h = I_{n_2} \otimes \cdots \otimes I_{n_{h+1}} \otimes 1_{n_h} \otimes I_{n_{h-1}} \otimes \cdots \otimes I_{n_1}. \] (4.12)

The “sum-to-zero” condition for \( \gamma^{(s)} \) over one index \( i_h, \ h = 1, \ldots, s \),
\[ \sum_{i_h} \gamma^{(s)}_{i_1 \cdots i_j} = 0 \text{ a.s. for all } i_j \neq i_h, \] (4.13)

can be expressed in matrix notation as
\[ L_h' \gamma^{(s)} = 0 \text{ a.s.} \] (4.14)

The next theorem demonstrates relationships between “sum-to-zero” reparameterizations for the factor \( \gamma^{(s)} \), \( E(\gamma^{(s)}) = 0 \), and restrictions on the spectrum of the permutation invariant covariance matrix \( \Sigma_s \) of \( \gamma^{(s)} \).

**Theorem 4.4.** Let the matrix \( L_h \) be given by (4.12). For any \( h \in \{1, \ldots, s\} \), the condition \( L_h' \gamma^{(s)} = 0 \text{ a.s.} \) holds if and only if the covariance matrix \( \Sigma_s \) of \( \gamma^{(s)} \) has the following restriction on its spectrum \( \lambda_{0,0,\ldots,0}, \ldots, \lambda_{s,s-1,\ldots,1} \):
\[ \lambda_{\alpha_1,\ldots,\alpha_l (\alpha_l = h)} = 0 \text{ a.s.}, \] (4.15)

where the indices \( \alpha_1, \ldots, \alpha_s \) are defined in (3.4).

**Proof.** We show first that for each \( h \in \{1, \ldots, s\} \) the condition \( L_h' \gamma^{(s)} = 0 \) leads to certain restrictions on the spectrum of \( \Sigma_s \), namely, \( \lambda_{\alpha_1,\ldots,\alpha_l (\alpha_l = h)} = 0 \), where the indices \( \alpha_1, \ldots, \alpha_s \) are defined in (3.4). Condition (4.14) implies \( D(L_h' \gamma^{(s)}) = 0 \), which results in
\[ L_h' \Sigma_s L_h = 0. \] (4.16)

Taking into account the structure of \( \Sigma_s \) given in Theorem 2.3,
\[ \Sigma_s = \sum_{v_1=0}^{1} \cdots \sum_{v_1=0}^{1} c_{v_1,\ldots,v_2,v_1} J_{n_2}^{v_2} \otimes \cdots \otimes J_{n_1}^{v_1}, \] (4.17)

the expression in (4.16) is identical to
\[ \sum_{v_1=0}^{1} \cdots \sum_{v_1=0}^{1} c_{v_1,\ldots,v_2,v_1} J_{n_2}^{v_2} \otimes \cdots \otimes J_{n_{h+1}}^{v_{h+1}} \otimes (1_{n_h} J_{n_h}^{v_h} 1_{n_h}) \otimes J_{n_{h-1}}^{v_{h-1}} \otimes \cdots \otimes J_{n_1}^{v_1} = 0, \] (4.18)

which in turn can be written
\[ \sum_{v_1=0}^{1} \cdots \sum_{v_1=0}^{1} (c_{v_1,\ldots,v_1 (v_1 = 0) n_h} + c_{v_1,\ldots,v_1 (v_1 = 1) n_h}) \prod_{i=1}^{n_h} J_{n_i}^{v_i} = 0. \] (4.19)

In (4.19) a linear combination of linearly independent matrices equals zero. Thus, all coefficients in (4.19) should be equal to zero:
\[ c_{v_1,\ldots,v_1 (v_1 = 0) n_h} + c_{v_1,\ldots,v_1 (v_1 = 1) n_h} = 0, \text{ for all } v_k, \ k \neq h, \ k = 1, \ldots, s. \] (4.20)

The system of linear equations in (4.20) can be expressed in matrix notation as follows:
\[ (I_2 \otimes \cdots \otimes I_2 \otimes (1 : n_h) \otimes I_2 \otimes \cdots \otimes I_2)c = 0, \] (4.21)
where the components of \( c = (c_{00,\ldots,0}, \ldots, c_{11,\ldots,1})' \) are ordered lexicographically. Now, using the relations between \( c_{00,\ldots,0} \), \( c_{11,\ldots,1} \) and the eigenvalues \( \lambda_{0,0,\ldots,0}, \ldots, \lambda_{s,s-1,\ldots,1} \) of \( \Sigma_s \) in (3.26), replace the vector of coefficients \( c \) in (4.21) by \( \lambda = (\lambda_{0,0,\ldots,0} , \ldots, \lambda_{s,s-1,\ldots,1})' \). Thus, we obtain
\[
(I_2 \otimes \cdots \otimes I_2 \otimes (1 : n_h) \otimes I_2 \otimes \cdots \otimes I_2) \bigotimes_{i=s}^1 \left( \begin{array}{cc}
1 & 0 \\
- \frac{1}{n_i} & \frac{1}{n_i} 
\end{array} \right) \lambda = 0. 
\]

(4.22)

A basic property of the Kronecker product allows us to rewrite (4.22) as
\[
(T_i \otimes \cdots \otimes T_{h+1} \otimes (0 : 1) \otimes T_{h-1} \otimes \cdots \otimes T_1) \lambda = 0,
\]
where
\[
T_i = \left( \begin{array}{cc}
1 & 0 \\
- \frac{1}{n_i} & \frac{1}{n_i} 
\end{array} \right), \quad i = 1, \ldots, s.
\]

(4.24)

It is obvious that \(T_i^{-1}, i = 1, \ldots, s\), exists (\(|T_i| \neq 0\)), and
\[
T_i^{-1} = \left( \begin{array}{cc}
1 & 0 \\
1 & n_i 
\end{array} \right).
\]

(4.25)

Define then
\[
M = T_s \otimes \cdots \otimes T_{h+1} \otimes I_{n_h} \otimes T_{h-1} \otimes \cdots \otimes T_1
\]
with
\[
M^{-1} = T_s^{-1} \otimes \cdots \otimes T_{h+1}^{-1} \otimes I_{n_h} \otimes T_{h-1}^{-1} \otimes \cdots \otimes T_1^{-1},
\]

(4.27)

Pre-multiplying (4.23) by \(M^{-1}\) leads to
\[
(I_2 \otimes \cdots \otimes I_2 \otimes (0 : 1) \otimes I_2 \otimes \cdots \otimes I_2) \lambda = 0
\]
which can be written as
\[
(I_{2^s-h} \otimes (0 : 1) \otimes I_{2^h-1}) \lambda = 0,
\]
or
\[
(I_{2^s-h} \otimes (0_{2^{h-1}} : I_{2^{h-1}})) \lambda = 0.
\]

(4.29)

From (4.29) it follows that
\[
\lambda_{0, \ldots, h, \ldots, 0} = \cdots = \lambda_{s, \ldots, h, \ldots, 1} = 0.
\]

(4.30)

Let us now show that imposing restrictions on the spectrum of the covariance matrix \(\Sigma_s\) of the factor \(\gamma^{(s)}\) implies a certain “sum-to-zero” reparameterization for \(\gamma^{(s)}\). Define
\[
R = (I_2 \otimes \cdots \otimes I_2 \otimes (1 : n_h) \otimes I_2 \otimes \cdots \otimes I_2)
\]
and
\[
S = (I_2 \otimes \cdots \otimes I_2 \otimes (0 : 1) \otimes I_2 \otimes \cdots \otimes I_2).
\]

(4.31)

(4.32)

Then, because of the relations between the vectors \(c\) and \(\lambda\) in (3.23), condition \(S\lambda = 0\) implies that
\[
(I_2 \otimes \cdots \otimes I_2 \otimes (0 : 1) \otimes I_2 \otimes \cdots \otimes I_2) \bigotimes_{i=s}^1 \left( \begin{array}{cc}
1 & 0 \\
1 & n_i 
\end{array} \right) c = 0.
\]

(4.33)
Once again, a basic property of the Kronecker product allows us to rewrite (4.33) as
\[
(T^{-1}_s \otimes \cdots \otimes T^{-1}_{h+1} \otimes (1 : n_h) \otimes T^{-1}_{h-1} \otimes \cdots \otimes T^{-1}_1)c = 0.
\] (4.34)

Pre-multiplying (4.34) by \(M\), given by (4.26), implies
\[
(I_2 \otimes \cdots \otimes I_2 \otimes (1 : n_h) \otimes I_2 \otimes \cdots \otimes I_2)c \equiv Rc = 0
\] (4.35)
or
\[
(I_{2-h} \otimes (1 : n_h) \otimes I_{2h-1})c = 0.
\] (4.36)

As noted before, the matrix equation \(Rc = 0\) represents a system of linear equations given in (4.20). Thus, this is equivalent to (4.19). It is easy to see that (4.19) can be written as (4.18) which, in turn, implies (4.16), i.e., \(L'_h \Sigma_s L_s = 0\). Hence, \(E(L'_h \gamma^{(s)}) = 0\) and \(D(L'_h \gamma^{(s)}) = 0\), it follows that \(L'_h \gamma^{(s)} = 0\) a.s. This completes the proof of the theorem. □

Let the matrix \(L_{hk}^s\) be a Kronecker product of \(s\) matrices where the \(h_i\)th component is an \(n_{h_i}\)-vector \(1_{n_{h_i}} \left\{ h_1, \ldots, h_k \right\} \subseteq \{1, \ldots, s\}, i = 1, \ldots, k, k = 1, \ldots, s\) and the others \(s-k\) components are identity matrices \(I_{n_r}, r = 1, \ldots, s, r \neq h_i\). For example, in the case of 3 factors \(L_{31}^3 = I_{n_3} \otimes I_{n_2} \otimes 1_{n_1}\), in the case of 4 factors \(L_{31}^4 = I_{n_4} \otimes I_{n_3} \otimes I_{n_2} \otimes I_{n_1}\).

In general, \(L_{hk}^s = I_{n_h} \otimes \cdots \otimes I_{n_{h_k-1}} \otimes 1_{n_{h_k}} \otimes I_{n_{h_k-1}} \otimes \cdots \otimes 1_{n_1} \otimes \cdots \otimes I_{n_1}\). (4.37)

The “sum-to-zero” condition for \(\gamma^{(s)}\), when summing over \(k\) indices \(i_{h_1}, \ldots, i_{h_k}, k = 1, \ldots, s\),
\[
\sum_{i_{h_1}, \ldots, i_{h_k}} \gamma_{i_{h_2}\ldots i_{h_k}}^{(s)} = 0 \text{ a.s., for all } i_r \notin \{i_{h_1}, \ldots, i_{h_k}\}
\]
can be expressed in matrix notation as
\[
(L_{hk}^s)^\prime \gamma^{(s)} = 0 \text{ a.s.}
\] (4.38)

We now inquire into the conditions under which (4.38) holds. In light of the proof of Theorem 4.4 we can generalize the results of the previous theorem.

**Theorem 4.5.** Let \(L_{hk}^s\) be given by (4.37). The condition \((L_{hk}^s)^\prime \gamma^{(s)} = 0 \text{ a.s.}\) holds if and only if the covariance matrix \(\Sigma_s\) of \(\gamma^{(s)}\) has the following restriction on its spectrum \(\lambda_{0,0,\ldots,0}, \ldots, \lambda_{s,s-1,\ldots,1}\):
\[
\lambda_{\alpha_1,\ldots,\alpha_k(a_{h_i} = h_i)} = 0 \text{ a.s., } i = 1, \ldots, k,
\] (4.39)
where the indices \(\alpha_1, \ldots, \alpha_s\) are defined in (3.4).

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