Approximations of Integral Equations for Wave Scattering

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Abstract

Wave scattering is the phenomenon in which a wave field interacts with physical objects. An incoming wave is scattered at the surface of the object and a scattered wave is produced. Common practical cases are acoustic, electromagnetic and elastic wave scattering. The numerical simulation of the scattering process is important, for example, in noise control, antenna design, prediction of radar cross sections and nondestructive testing.

Important classes of numerical methods for accurate simulation of scattering are based on integral representations of the wave fields and these representations require the knowledge of potentials on the surfaces of the scattering objects. The potential is typically computed by a numerical approximation of an integral equation that is defined on the surface. We first develop such numerical methods in time domain for the scalar wave equation. The efficiency of the techniques are improved by analytic quadrature and in some cases by local approximation of the potential.

Most scattering simulations are done for harmonic or single frequency waves. In the electromagnetic case the corresponding integral equation method is called the method of moments. This numerical approximation is computationally very costly for high frequency waves. A simplification is suggested by physical optics, which directly gives an approximation of the potential without the solution of an integral equation. Physical optics is however only accurate for very high frequencies.

In this thesis we improve the accuracy in the physical optics approximation of scalar waves by basing the computation of the potential on the theory of radiation boundary conditions. This theory describes the local coupling of derivatives in the wave field and if it is applied at the surface of the scattering object it generates an expression for the unknown potential. The full wave field is then computed as for other integral equation methods.

The new numerical techniques are analyzed mathematically and their efficiency is established in a sequence of numerical experiments. The new on surface radiation conditions give, for example, substantial improvement in the estimation of the scattered waves in the acoustic case. This numerical experiment corresponds to radar cross-section estimation in the electromagnetic case.

Keywords: Integral equations, Marching on in time, On surface radiation condition, Physical Optics.
Contents

1 Introduction 3
  1.1 Mathematical Formulation of Wave Propagation . . . . . . . . . . . . 3
  1.2 Numerical Methods for Wave Propagation . . . . . . . . . . . . . . 9
  1.3 Acceleration of Integral Equation Methods . . . . . . . . . . . . . . 10
  1.4 Computational Complexity . . . . . . . . . . . . . . . . . . . . . . 12
  1.5 Outline and Main Results . . . . . . . . . . . . . . . . . . . . . . . 13
  1.6 Published Papers and the GEMS Project . . . . . . . . . . . . . . . 16

2 Time Domain Integral Equation method 17
  2.1 The Scalar Wave Equation . . . . . . . . . . . . . . . . . . . . . . 18
  2.2 Basis Functions in Space and Time . . . . . . . . . . . . . . . . . 20
  2.3 Variational Formulation, Dirichlet Case . . . . . . . . . . . . . . . 21
  2.4 Variational Formulation, Neumann Case . . . . . . . . . . . . . . . 22
  2.5 Point Representation on Triangle Plane . . . . . . . . . . . . . . . 23
  2.6 Integrals Over Time . . . . . . . . . . . . . . . . . . . . . . . . . 25
  2.7 Dirichlet Discretization . . . . . . . . . . . . . . . . . . . . . . . . 27
  2.8 Neumann Discretization . . . . . . . . . . . . . . . . . . . . . . . . 28
  2.9 Quadrature . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 31
  2.10 The Marching On In Time Method . . . . . . . . . . . . . . . . . . 45
  2.11 Stabilization . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 50
  2.12 Numerical Examples . . . . . . . . . . . . . . . . . . . . . . . . . . 53
  2.13 On Surface Radiation Conditions . . . . . . . . . . . . . . . . . . . 65

3 Analytical Solutions for Objects with Constant Curvature 71
  3.1 Analytical Expressions for the Potential on a Circular Cylinder With
    Dirichlet Boundary Condition . . . . . . . . . . . . . . . . . . . . . 71
  3.2 Analytical Expressions for the Potential on a Circular Cylinder with
    Neumann Boundary Condition . . . . . . . . . . . . . . . . . . . . . 73
  3.3 Analytical Expressions for the Potential on a Sphere With Dirichlet
    Boundary Condition . . . . . . . . . . . . . . . . . . . . . . . . . . . 77
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.4</td>
<td>Analytical Expressions for the Potential on a Sphere With Neumann Boundary Conditions</td>
<td>78</td>
</tr>
<tr>
<td>3.5</td>
<td>Analytical Solution for the Surface Current in Maxwell’s Equation on a Sphere with PEC Boundary Condition</td>
<td>80</td>
</tr>
<tr>
<td>3.6</td>
<td>The Electromagnetic Operator $T$</td>
<td>83</td>
</tr>
<tr>
<td>4</td>
<td>On Surface Radiation Conditions</td>
<td>85</td>
</tr>
<tr>
<td>4.1</td>
<td>Physical Optics Approximation</td>
<td>85</td>
</tr>
<tr>
<td>4.2</td>
<td>PO for a PEC Object in Electromagnetics</td>
<td>88</td>
</tr>
<tr>
<td>4.3</td>
<td>Approximation of Logarithmic Derivatives</td>
<td>89</td>
</tr>
<tr>
<td>4.4</td>
<td>On Surface Radiation Conditions</td>
<td>99</td>
</tr>
<tr>
<td>4.5</td>
<td>Higdon Type OSRC</td>
<td>103</td>
</tr>
<tr>
<td>4.6</td>
<td>The Inverse OSRC Operator</td>
<td>104</td>
</tr>
<tr>
<td>4.7</td>
<td>An Implicit Second Order OSRC</td>
<td>106</td>
</tr>
<tr>
<td>4.8</td>
<td>Regularization of the Improved OSRC</td>
<td>107</td>
</tr>
<tr>
<td>4.9</td>
<td>An OSRC in Electromagnetics</td>
<td>109</td>
</tr>
<tr>
<td>4.10</td>
<td>Engquist-Majda on Slowly Varying Functions</td>
<td>111</td>
</tr>
<tr>
<td>4.11</td>
<td>Generalized Physical Optics (GPO)</td>
<td>111</td>
</tr>
<tr>
<td>5</td>
<td>Wave Scattering</td>
<td>127</td>
</tr>
<tr>
<td>5.1</td>
<td>Restriction of the Integration Domain</td>
<td>127</td>
</tr>
<tr>
<td>5.2</td>
<td>Determination of the Shadow Boundary</td>
<td>130</td>
</tr>
<tr>
<td>5.3</td>
<td>Radar Cross Section Calculations</td>
<td>132</td>
</tr>
<tr>
<td>5.4</td>
<td>Error in Far-field Computations With OSRC</td>
<td>132</td>
</tr>
<tr>
<td>5.5</td>
<td>Numerical computations of the far-field</td>
<td>135</td>
</tr>
<tr>
<td>A</td>
<td>The GEMS project</td>
<td>141</td>
</tr>
<tr>
<td>B</td>
<td>Differential Geometry</td>
<td>145</td>
</tr>
<tr>
<td>C</td>
<td>Sobolev Spaces</td>
<td>147</td>
</tr>
<tr>
<td>D</td>
<td>Conforming Basis Functions</td>
<td>149</td>
</tr>
<tr>
<td>E</td>
<td>List of Symbols and Abbreviations</td>
<td>151</td>
</tr>
<tr>
<td></td>
<td>Bibliography</td>
<td>153</td>
</tr>
</tbody>
</table>
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Chapter 1

Introduction

1.1 Mathematical Formulation of Wave Propagation

In his classical text on the theory of sound [60], Lord Rayleigh writes that the understanding of wave phenomena can be divided in three steps: the production of the wave, the propagation of the wave including scattering, and the reception of the wave. The first person who studied musical waves was Pythagoras who discovered that a shorter string produced a higher frequency. For an excellent background of the theory of sound, see [60]. From these early examples, the theory for physical and mathematical understanding of wave phenomena have evolved to a mature state, see e.g. [45], [67], [73].

Figure 1.1: Left, scattering by an object $\Omega$ with incoming wave $u^{inc}$ and scattered wave $u^{sc}$. Right, scattering surface of a loudspeaker.
CHAPTER 1. INTRODUCTION

In electromagnetic problems the production and reception of the electromagnetic waves are studied in antenna theory. An important computational challenge is to produce antenna diagrams for different sources and antennas to describe far-field wave propagation. Other important computational tasks are the prediction of radar cross section, RCS, and the study of electromagnetic compatibility. In this thesis, we will concentrate on the wave scattering part of the wave propagation, where an incoming wave field $u^{\text{inc}}$ is assumed to propagate in an exterior domain and illuminate a scattering object $\Omega$ to generate a scattered wave field $u^{\text{sc}}$, see Figure 1.1. The wave produces a potential on the boundary of the object. Once the potential is computed, the scattered field can be obtained from the potential. We will focus on approximative numerical methods for computing this potential.

This thesis will mainly focus on acoustic wave propagation, but the theory presented can also be applied to electromagnetic and elastic waves. The rest of this chapter will briefly introduce the different equations for wave propagation and the numerical methods to solve these equations, both for the differential equation and the integral equation formulations. We will also discuss some techniques to make the numerical solution of the integral equations faster by introducing the on surface radiation condition (OSRC) method. The introduction ends with an outline of the thesis, which also presents the main results.

Acoustic Wave Scattering

There are many applications involving acoustic waves. Some examples are noise control, underwater acoustics and nondestructive testing. The acoustic wave equation is the simplest model of a wave propagating in some medium. Together with boundary condition and initial value, the equation for the exterior problem is given by

$$\partial_t^2 u(\mathbf{r}, t) - c^2 \Delta u(\mathbf{r}, t), \quad \text{in } \Omega' \times (0, \infty), \quad (1.1)$$

$$Bu(\mathbf{r}, t) = f(\mathbf{r}, t), \quad \text{on } \Gamma \times (0, \infty), \quad (1.2)$$

$$u(\mathbf{r}, 0) = g_0(\mathbf{r}), \quad (1.3)$$

$$\partial_t u(\mathbf{r}, 0) = g_1(\mathbf{r}), \quad (1.4)$$

where $\Gamma$ is the boundary of $\Omega$, see Figure 1.1. The boundary condition $Bu = f$ can, for example, be Dirichlet, where $u = 0$, or Neumann, where $\partial_n u = 0$. We could also impose an impedance condition $\partial_n u - \lambda \partial_t u = f$, with $\Re \lambda > 0$.

The model for the scattering problem consists of an incoming wave $u^{\text{inc}}$ that propagates in a medium $\Omega'$ and interacts with some scatterers $\Omega$ with boundary $\Gamma$. The interaction produces a scattered wave $u^{\text{sc}}$ in the exterior. The total physical wave is the sum of the incoming and the scattered parts of the wave,

$$u^{\text{tot}} = u^{\text{inc}} + u^{\text{sc}}. \quad (1.6)$$
The incoming wave is a solution to the wave equation in $\mathbb{R}^d$, and the scattered wave is a solution in the exterior domain $\Omega^d = \mathbb{R}^d \setminus \Omega$,

$$
\Delta u^{inc}(r, t) - \frac{1}{c^2} \partial_t^2 u^{inc}(r, t) = 0, \quad r \in \mathbb{R}^d, \quad (1.7)
$$

$$
\Delta u^{sc}(r, t) - \frac{1}{c^2} \partial_t^2 u^{sc}(r, t) = 0, \quad r \in \mathbb{R}^d \setminus \Omega. \quad (1.8)
$$

The scatterers have a specified boundary condition. For acoustic problems, the scatterers can be Dirichlet (sound soft), where

$$
u^{tot}(r, t) = 0, \quad r \in \Gamma, \quad (1.9)$$

or Neumann (sound hard), where the normal derivative vanishes,

$$\partial_n u^{tot}(r, t) = 0, \quad r \in \Gamma. \quad (1.10)$$

We can also impose an impedance condition on the scatterer,

$$\partial_n u^{tot}(r, t) - \lambda \partial_t u^{tot}(r, t) = f(r, t), \quad r \in \Gamma, \quad (1.11)$$

with the restriction $\Re \lambda > 0$ and that $f$ is continuous. These boundary conditions couple the incoming and scattered field, in order to get a well-defined model.

A very important special case of the wave equation is when the incoming field is a time-harmonic wave, with a given wave number $k = \omega/c$, such that the wave can be written as $u(r, t) = \hat{u}(r)e^{-i\omega t}$. The wave equation is then reduced to Helmholtz equation, with Sommerfeld radiation condition

$$
\Delta \hat{u}^{inc}(r) + k^2 \hat{u}^{inc}(r) = 0, \quad r \in \mathbb{R}^d, \quad (1.12)
$$

$$
\Delta \hat{u}^{sc}(r) + k^2 \hat{u}^{sc}(r) = 0, \quad r \in \mathbb{R}^d \setminus \Omega, \quad (1.13)
$$

$$
\lim_{r \to \infty} r^{\frac{d-1}{2}} (\partial_r \hat{u}^{sc}(r) - ikr^{sc}(r)) = 0, \quad r = |r|, \quad (1.14)
$$

where $| \cdot |$ denotes the Euclidian norm. The $\hat{u}$ will hereafter be replaced by $u$, when we are solving time-harmonic problems. The Green’s function for Helmholtz equation is

$$
G(r, r') = \frac{i}{4} H_0^{(1)}(kr - r'), \quad \text{in } 2D, \quad (1.15)
$$

$$
G(r, r') = \frac{e^{ik|r-r'|}}{4\pi|r-r'|}, \quad \text{in } 3D, \quad (1.16)
$$

and these Green’s functions correspond to the solution of the Helmholtz equation with a Dirac $\delta$-function source at $r'$. The solution to Helmholtz equation can be represented as an integral over the boundary $\Gamma$ of the scatterer,

$$
\int_{\Gamma} (\partial_{\nu} G(r, r')) u^{tot}(r') - G(r, r') \left( \partial_n u^{tot}(r') \right) d\Gamma' = u^{sc}(r), \quad r \in \mathbb{R}^d \setminus \Omega, \quad (1.17)
$$

$$
\int_{\Gamma} (\partial_{\nu} G(r, r')) u^{tot}(r') - G(r, r') \left( \partial_n u^{tot}(r') \right) d\Gamma' = -u^{inc}(r), \quad r \in \Gamma, \quad (1.18)
$$
CHAPTER 1. INTRODUCTION

where the normal derivative \( \partial_n u(r) \) is defined as \( \hat{n}(r) \cdot \nabla_r u(r) \), where \( \hat{n} \) is the normal in the direction from \( \Omega \) into \( \Omega' \). Once we have computed the potentials \( u^{\text{tot}} \) and \( \nabla_n u^{\text{tot}} \) on the surface \( \Gamma \), we can use the integral formulation (1.18) to obtain the scattered field in the exterior of the scatterer.

For a scatterer with a Dirichlet boundary, the total field vanishes, \( u^{\text{tot}} = 0 \) and the integrand is reduced to one term and we get a Fredholm integral equation of the first kind for the unknown quantity \( \partial_n u(r) \) on the boundary \( \Gamma \).

For a Neumann boundary, the normal derivative of the total field vanishes, \( \partial_n u^{\text{tot}} = 0 \) and the integrand is reduced also in this case. The integral equation may not be suited for direct solution by numerical integral methods, but there are other integral equation formulations, which are Fredholm integral equations of the second kind, that are known to have a better properties.

The first step in a numerical computation is to find the potential \( \partial_n u^{\text{tot}} \) or \( u^{\text{tot}} \) for a Dirichlet or Neumann scatterer respectively. This can be done by solving equation (1.17). Another way to find an approximation to the potential is to locally approximate the scatterer with an osculating object for which we know the potential analytically. For sufficiently large frequencies, the potential only depends on a small neighborhood of the scatterer which is well represented by the simpler scatterer. This is a property we will focus on.

A time domain integral equation (TDIE) is obtained by taking the inverse Fourier transform of the time-harmonic integral formulation. In 3D, we get the Kirchhoff’s integral formulation in a source free medium,

\[
\begin{align*}
  u(r, t) &= \frac{1}{4\pi} \int_{\Gamma} \left\{ \frac{1}{R} \partial_n u^* - \partial_n \left( \frac{1}{R} \right) u^* + \frac{1}{cR} (\partial_n R)(\partial_t u^*) \right\} d\Gamma', \quad (1.19)
\end{align*}
\]

there \( u^*(r', t) = u(r', t - \frac{1}{c}R) \) is the retarded potential and \( R = |r - r'| \). The boundary conditions can essentially be chosen as in the time-harmonic case. A variational formulation of the TDIE 1.19 can be used to construct a time-marching scheme, which is commonly called Marching On in Time (MOT).

Electromagnetic Wave Scattering

Electromagnetic scattering problems arise in many applications, such as radar cross section computations, antenna design and electromagnetic compatibility. The wave field is given by Maxwell’s equations

\[
\begin{align*}
  \nabla \times \mathbf{E}(r, t) + \partial_t \mathbf{B}(r, t) &= 0, \quad \text{in } \Omega \times (0, \infty), \quad (1.20) \\
  \nabla \times \mathbf{H}(r, t) - \partial_t \mathbf{D}(r, t) &= \mathbf{J}_e, \quad \text{in } \Omega \times (0, \infty), \quad (1.21) \\
  \nabla \cdot \mathbf{D}(r, t) &= \rho_e, \quad \text{in } \Omega \times (0, \infty), \quad (1.22) \\
  \nabla \cdot \mathbf{B}(r, t) &= 0, \quad \text{in } \Omega \times (0, \infty), \quad (1.23)
\end{align*}
\]

for the electric field \( \mathbf{E} \) and the magnetic field \( \mathbf{H} \), with \( \mathbf{B} = \mu \mathbf{H} \) and \( \mathbf{D} = \varepsilon \mathbf{E} \). The initial conditions are \( \mathbf{E}(r, t) = \mathbf{H}(r, t) = 0 \) when \( t \leq 0 \). Finally, \( \mathbf{J}_e \) is the electric
1.1. MATHEMATICAL FORMULATION OF WAVE PROPAGATION

current and $\rho_e$ is the electric charge in an isotropic medium which is defined by the electric permittivity $\epsilon$ and the magnetic permeability $\mu$.

Introducing the electromagnetic potentials yields

$$
\mathbf{E} = -\nabla \Phi - \partial_t \mathbf{A},
\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A},
$$

where $\Phi$ and $A$ solves the Helmholtz equation,

$$
\Delta \Phi + k^2 \Phi = -\frac{\rho_e}{\epsilon},
\Delta A + k^2 A = -\mu J_e,
$$

with the integral representation

$$
\Phi(r) = \frac{1}{\epsilon} \int G(r,r') \rho_e(r') dr',
\quad
A(r) = \mu \int G(r,r') J_e(r') dr'.
$$

The model for the scattering problem consists of an incoming electric and magnetic field $\mathbf{E}^{inc}$ and $\mathbf{H}^{inc}$ that is defined in $\mathbb{R}^d$. The physical electromagnetic field is $\mathbf{E}^{tot}$ and $\mathbf{H}^{tot}$. The incoming fields illuminate the scatterer $\Omega$ and produce a scattered field $\mathbf{E}^{sc} = \mathbf{E}^{tot} - \mathbf{E}^{inc}$ and $\mathbf{H}^{sc} = \mathbf{H}^{tot} - \mathbf{H}^{inc}$, which is defined in the exterior of the scatterer, $\mathbb{R}^d \setminus \bar{\Omega}$.

Introducing the distribution

$$
\tilde{\mathbf{E}} = \begin{cases} 
-\mathbf{E}^{inc}, & \text{in } \Omega, \\
\mathbf{E}^{sc}, & \text{in } \mathbb{R}^d \setminus \bar{\Omega},
\end{cases}
$$

leads to the following version of Maxwell’s equations: the distribution form of the source free Maxwell’s equations becomes [43], [55], [68],

$$
\nabla \times \tilde{\mathbf{E}} + \partial_t \tilde{\mathbf{B}} = [\hat{n} \times \tilde{\mathbf{E}}]_{\Gamma},
\nabla \times \tilde{\mathbf{H}} - \partial_t \tilde{\mathbf{D}} = [\hat{n} \times \tilde{\mathbf{H}}]_{\Gamma},
\nabla \cdot \tilde{\mathbf{D}} = [\hat{n} \cdot \tilde{\mathbf{D}}]_{\Gamma},
\nabla \cdot \tilde{\mathbf{B}} = [\hat{n} \cdot \tilde{\mathbf{B}}]_{\Gamma},
$$

where $[f] = f^{ext}|_{\Gamma} - f^{int}|_{\Gamma}$.

In order to close the system, we need to specify some boundary condition at $\Gamma = \partial \Omega$. The simplest b.c. is the perfect electric conductor, where $[\hat{n} \times \tilde{\mathbf{E}}] = 0$ and $[\hat{n} \cdot \tilde{\mathbf{B}}] = 0$. We also define $\tilde{\mathbf{J}} = [\hat{n} \times \tilde{\mathbf{H}}]_{\Gamma}$ and $\tilde{\rho}_s = [\hat{n} \cdot \tilde{\mathbf{D}}]_{\Gamma}$. 

A very important special case of electromagnetic scattering is when the waves are time-harmonic, \( \mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{-i\omega t} \),

\[
\begin{align*}
\nabla \times \mathbf{E} - i\omega \mathbf{B} &= 0, \\
\nabla \times \mathbf{H} + i\omega \mathbf{D} &= \mathbf{J}_s, \\
\n\nabla \cdot \mathbf{D} &= \rho_s, \\
\n\nabla \cdot \mathbf{B} &= 0.
\end{align*}
\]

The Silver-Müller radiation condition \([55]\) \((\hat{n} = \frac{\mathbf{r}}{|\mathbf{r}|})\),

\[
\left|\sqrt{\varepsilon}\mathbf{E}^{sc} + \sqrt{\mu}\hat{n} \times \mathbf{H}^{sc}\right| \leq c |\mathbf{r}|^2,
\]

(1.26)
is imposed in order to get a unique solution.

The solution to the time-harmonic Maxwell equations can be expressed in scalar and vector potentials, which for the electric conductor case are \([41]\),

\[
\begin{align*}
\mathbf{E} &= -\nabla \Phi + i\omega \mathbf{A}, \\
\mathbf{H} &= \frac{1}{\mu} \nabla \times \mathbf{A},
\end{align*}
\]

where \(\Phi\) and \(A\) solve the Helmholtz equation,

\[
\begin{align*}
\nabla \Phi + k^2 \Phi &= -\frac{\rho_s}{\varepsilon}, \\
\nabla A + k^2 A &= -\mu \mathbf{J}_s,
\end{align*}
\]

(1.27)

(1.28)

with the integral representation

\[
\begin{align*}
\Phi(\mathbf{r}) &= \frac{1}{\varepsilon} \int G(\mathbf{r}, \mathbf{r}') \rho_s(\mathbf{r}') d\Gamma', \\
\mathbf{A}(\mathbf{r}) &= \mu \int G(\mathbf{r}, \mathbf{r}') \mathbf{J}_s(\mathbf{r}') d\Gamma'.
\end{align*}
\]

The current \(\mathbf{J}_s\) and the current density \(\rho_s\) are coupled by the continuity equation \(i\omega \rho_s = \nabla \cdot \mathbf{J}_s\). This means that once we find an expression for \(\mathbf{J}_s\), then we can compute all other quantities, \(\rho_s, \mathbf{A}, \Phi, \mathbf{E}\) and \(\mathbf{H}\).

**Elastic Scattering**

The propagation of elastic waves in a solid is expressed in terms of the displacement vector \(\mathbf{u}(\mathbf{x}, t)\) and the stress tensor \(\mathbf{T}(\mathbf{x}, t)\), such that the stress in direction \(\mathbf{v}\) is

\[
\mathbf{T}^{(v)} = \mathbf{v} \cdot (\lambda (\nabla \cdot \mathbf{u}) I + \mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T)).
\]

For isotropic elastic source-free solids, the displacement equation of motion reads

\[
(\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} + \rho \omega^2 \mathbf{u} = 0,
\]
where $\lambda$ and $\mu$ are the Lamé constants which depend on the material properties of the solid. The displacement equation has the Green’s function [57]

$$G(|r - r'|) = \frac{1}{\rho \omega^2} \left( k_p G(k_s |r - r'|) + \nabla \left( G(k_p |r - r'|) - G(k_s |r - r'|) \right) \nabla' \right),$$

where $G$ is the Green’s function (1.15) in 2D and (1.16) in 3D.

One simple example of a scattering problem is an incoming P-wave that illuminates a cavity that has the shape of a circular cylinder. The boundary condition is then that the radial components of the stress tensor vanish. The solution is derived in [56]. Another example [56] is a rigid inclusion in the form of a circular cylinder.

### 1.2 Numerical Methods for Wave Propagation

A scattering problem can be solved either by using the differential equation formulation, or by using integral formulations based on the fundamental solution. Both these formulations can be done in time domain or in frequency domain.

#### Differential Equation Methods

The differential equation (1.8) and (1.26) in acoustics and electromagnetics respectively can be discretized with finite differences in time domain. The discretization of Maxwell’s equations is called the FDTD scheme [75], which uses a staggered grid in order to obtain both a time and a memory efficient scheme. Another time domain method is a finite volume scheme [26], which is the first in a family of methods, called discontinuous Galerkin (DG). These methods have become increasingly popular in recent years, see for instance [38]. It is also possible to use a FEM discretization [27]. The methods mentioned here can also be applied in frequency domain.

#### Integral Equation Methods

The frequency domain integral formulations (1.18) and (1.24), (1.25) in acoustics and electromagnetics respectively are Fredholm integral equations of the first kind, [36]. These integral equations can be discretized in essentially three different ways: the Nyström method, the collocation method and the Galerkin method. All three methods end up with a full system of equations. In the case of integral equations of the first kind, the condition number of the system matrix will be large. As a consequence, an iterative method like GMRES [64] needs many iterations to converge. It is therefore often convenient to rewrite the integral equations to Fredholm equations of the second kind, [36], resulting in a system matrix with a small condition number, which leads to fewer iterations.

The Nyström method is the simplest discretization. The integral is computed numerically using, for instance, the trapezoidal rule. The solution is obtained in the discretization points of the numerical integration.
In the collocation method, the solution is expressed using basis functions. The solution is the coefficients to the basis functions.

The Galerkin method is the method most commonly used and has the most developed theory. The variational formulations for both acoustic and electromagnetic scattering problems are described in [55].

An alternative method is to use time domain integral equation (TDIE) [59], which contains retarded potentials. An analysis of appropriate spaces and corresponding variational formulations can be found in [68]. The variational formulation leads to a time-stepping scheme, commonly called marching on in time (MOT), where the potential depends on the incoming field and potential at earlier times. Such a time-stepping scheme was implemented and is described in the author’s licentiate thesis [7].

1.3 Acceleration of Integral Equation Methods

In many applications, in particular with complex geometries and/or high frequencies, the number of unknowns is very large and it takes too long time and is too memory consuming to solve the system of equations. There exist many different ways to get an approximate solution of the integral equations. One approach is to use the fast multipole method, FMM, which was originally derived by Greengard and Rokhlin in [34] to use in particle simulations. The method introduces a low rank approximation of the far-field effect and reduces the computational cost for one iteration step from $O(M^2)$ to $O(M \log M)$ where $M$ is the number of particles. The FMM has been adapted to integral equations for Helmholtz equation in 2D, [61]. The far-field approximation can be diagonalized [63], [62], in order to decrease the complexity even further for the far-field.

A similar acceleration technique has been developed by Michielssen [31] for time domain integral equations, called the plane wave time domain, (PWTD) method.

An alternative to FMM is to accelerate the computation using FFT. In recent years, Bruno et al. [19], [18], have developed a new method by using a partition of unity. The idea is to split the scatterer into several subdomains using partition of unity. The adjacent interactions are computed directly. The non-adjacent interactions are accelerated using FFT on each partition. The same idea has been used for scattering problems by penetrable bodies based on the Lippmann-Schwinger equation in 2D [17] and 3D [42].

Geometrical Optics and Physical Optics

The FMM and FFT techniques give in the limit of increased refinement the true wave equation solution. There are also methods that are only accurate for high frequencies. When the wave number tends to infinity and it is computationally too costly to numerically resolve the waves. Instead it is more convenient to express the solution to the wave equation in terms of the phase $\phi$ and amplitude $A$. The
so called WKB-form \([30]\) is

\[
  u(t, r) = e^{i\omega\phi(t, r)} \sum_{k=0}^{\infty} A_k(t, r)(i\omega)^{-k}.
\]  

(1.29)

Inserting the WKB-form in the Helmholtz equation we obtain a WKB-form for which we can identify the coefficients. The leading terms satisfy the eikonal equation for the phase \(\phi\),

\[
  \partial_\phi \pm c|\nabla \phi| = 0.
\]

Next term satisfies a transport equation for the coefficient \(A_0\),

\[
  (A_0)_t + c\frac{\nabla \phi \cdot \nabla A_0}{|\nabla \phi|} + \frac{c^2 \delta \phi - \partial_{tt} \phi}{2c|\nabla \phi|} A_0 = 0.
\]  

(1.30)

The WKB-form does not include diffraction effects, which have an expansion in fractional powers of \(\omega\).

An alternative formulation of geometrical optics is ray tracing, where the Hamiltonian \(H(x, p) = c(x)|p|\) is introduced. We obtain the system of ODEs \([30]\)

\[
  \frac{dx}{dt} = \nabla_p H(x, p) = c(x)\frac{p}{|p|},
\]  

(1.31)

\[
  \frac{dp}{dt} = -\nabla_x H(x, p) = -|p|\nabla c(x).
\]  

(1.32)

The geometrical theory of diffraction (GTD) adds diffraction effects to the geometrical optics by adding half order terms to the WKB-form (1.29),

\[
  u(t, r) = e^{i\omega\phi(t, r)} \sum_{k=0}^{\infty} A_k(t, r)(i\omega)^{-k} + e^{i\omega\phi_d(t, r)} \sum_{k=0}^{\infty} B_k(t, r)(i\omega)^{-k-1/2}.
\]

The physical optics (PO) approximation is obtained from locally approximating the scatterer by an infinite plane. This is a good approximation for high frequencies.

In acoustics with a Dirichlet scatterer, the PO approximation is

\[
  \partial_n u^{tot} = \begin{cases} 2\partial_n u^{inc}, & \text{on } \Gamma_{lit}, \\ 0, & \text{on } \Gamma_{shad}. \end{cases}
\]

in both the cylindrical case and in 3D. The PO approximation for a Neumann scatterer reads

\[
  u^{tot} = \begin{cases} 2u^{inc}, & \text{on } \Gamma_{lit}, \\ 0, & \text{on } \Gamma_{shad}. \end{cases}
\]
CHAPTER 1. INTRODUCTION

In 3D electromagnetics where the scatterer is an electric conductor, the PO approximation for the surface current is

\[
J_s = \begin{cases} 
2\hat{n} \times H^{inc}, & \text{on }\Gamma_{lit}, \\
0, & \text{on }\Gamma_{shad}.
\end{cases}
\]

The physical theory of diffraction (PTD) was developed by Ufimtsev [70, 71, 72]. The method adds edge waves to physical optics.

**On Surface Radiation Conditions (OSRC)**

Radiation boundary conditions are used to limit the computational domain of a partial differential equation when the original physical domain is infinite. The scattering problem is a typical example. Radiation boundary conditions are also called artificial, absorbing, non-reflecting or far-field boundary conditions in the literature. There exist several different ways to define radiation boundary conditions. Examples are Bayliss-Turkel [11] and Engquist-Majda [29] radiation boundary conditions. These conditions are described Chapter 4. In 1987 Kriegmann [49] suggested to use radiation boundary conditions directly on the scatterer. In this way it is possible to express the potential directly from the incoming field. The OSRC condition in acoustics is a Dirichlet-to-Neumann operator

\[
\partial_nu^{sc} = OSRC[u^{sc}].
\]

The scattered field can be obtained by using the OSRC in the potential,

\[
u^{sc} = - \int G\partial_nu^{tot}d\Gamma' = - \int G(\partial_nu^{inc} - OSRC[u^{inc}])d\Gamma', \tag{1.33}
\]

for a Dirichlet scatterer. The corresponding formula for a Neumann scatterer is

\[
u^{sc} = \int \partial_nGu^{tot}d\Gamma' = \int \partial_nG(u^{inc} - OSRC^{-1}[\partial_nu^{inc}])d\Gamma'. \tag{1.34}
\]

A second approach is to use microlocal analysis [52] to approximate the potential. This has been done by X. Antoine [6]. The main advantage in 3D with this approach is that it allows the scatterer to have different curvatures in different directions. A third approach to derive approximations to the OSRC is to use the analytical solution of some simple scatterers. The general scatterer are then approximated locally by a circle or a sphere.

1.4 Computational Complexity

When FDTD is used for an equidistant mesh in 3D with \(N\) discretization points in each direction and \(N\) discretization points in time, the computational complexity is \(O(N^4)\). In the method of moments the corresponding discretization needs
1.5. OUTLINE AND MAIN RESULTS

$N^2$ discretization points to mesh the two dimensional surface of the scatterer. We get a full matrix system to solve which requires $O \left((N^2)^3\right)$ operations. By using an iterative solver with a suitable preconditioner, the amount of work can be reduced to $O \left((N^2)^2\right)$. The complexity can be further reduced, by using FMM. This yields $O \left(N^2 \log N\right)$ operations, to get the surface current. The constant is however quite large. The OSRC technique only requires $O(N^2)$ operations with a small constant in the order estimate. When we are interested in the monostatic RCS (the backscattered far-field in acoustics), then each computation yields the RCS for one direction. In the bistatic RCS, the complexity for method of moments is multiplied by the number of different directions in the RCS. In this case the complexities of the two methods are comparable.

In FDTD, the number of discretization points per wave length increases with frequency [48] and (of course) the accuracy requirement. In MoM, the number of points per wave length is independent of the frequency and therefore only depends on the accuracy requirement. This gives further advantage for the integral equation methods.

1.5 Outline and Main Results

Chapter 2 is the only chapter, which is devoted to the time domain integral formulations of wave equations. The numerical approximation is based on the Galerkin method and the resulting integrals are solved analytically whenever possible.

In Chapter 3, the analytical formulas for the potential are stated for objects with constant curvature in 2D and 3D. For the acoustic problem we consider Dirichlet (sound soft) and Neumann (sound hard) materials in the scatterer. For the electromagnetic problem we consider a perfect conductor (PEC) material in the scatterer. The analytical expressions of the various potentials all have the logarithmic derivative of the Hankel function as a common theme. For the circular cylinder we obtain the expressions

$$\left(\partial_n u^{sc}\right)_m = \frac{\partial_n H_{m}^{(1)}(kR)}{H_{m}^{(1)}(kR)} \hat{u}^{sc}_m,$$

for the Dirichlet problem, and

$$\hat{u}^{sc}_m = \frac{H_{m}^{(1)}(kR)}{\partial_n H_{m}^{(1)}(kR)} (\partial_n u^{sc})_m,$$

for the Neumann problem. The parameter $m$ is the Fourier dual for the angle relative to the incoming wave. In the case of a sphere, the Hankel functions are replaced by the spherical Hankel functions $h_{m}^{(1)}$.

The electromagnetic scattering problem is somewhat more difficult. The electric surface current $J_e$ is split into a rotational part and a gradient part. The rotational
part is related to the Dirichlet case for a sphere in acoustics and the gradient part is related to the Neumann case.

A plot of the potential for the two-dimensional acoustic problem and Dirichlet boundary conditions is given in Figure 1.2.

![Figure 1.2: Real (solid line) and imaginary part (dashed line) of the potential $\partial_n u^{\text{tot}}$ as function of the angle $\phi$ of a circular cylinder with Dirichlet boundary condition and wave number $k = 20$ and 80. Angle $\phi = 0$ corresponds to normal incidence.](image)

These analytical formulas are used in Chapter 4 to derive OSRCs. Even if the new conditions are based on expressions for objects with constant curvature they apply by localization to general geometries. The chapter also contains analysis of the new condition as well as a study of PO and conditions based on Engquist-Majda radiation boundary conditions. The accuracy of different OSRCs for the two-dimensional acoustic problem and Dirichlet boundary conditions can be seen in Figure 1.3. The methods are defined in chapter 4.

In chapter 5 the OSRCs are applied to the computation of the scattered wave field. The numerical results of different formulations are compared. These results are also compared to the exact analytical solution when that is known. If the analytical solution is not known the OSRC results are compared to computations based on MoM. An example of a calculation of the bistatic radar cross section of an elliptic scatterer is given in Figure 1.4. The plot shows the strength of the scattered wave in the direction given by the lower axes in the figure from an incoming wave at 0 degree.

In summary the major contributions in this thesis are

- Development of a numerical technique for the time domain integral formulation of acoustic scattering in three space dimensions, with a new stabilization process and on surface radiation conditions.
1.5. OUTLINE AND MAIN RESULTS

Figure 1.3: Absolute error of potential for a circular cylinder with Dirichlet boundary with wavenumber $k = 20$ and 80 for different approximations of the potential.

Figure 1.4: Absolute error of far-field computations for different approximations on an elliptic cylinder with Dirichlet boundary condition and with wave number $k = 20$ and 80.

- Derivation and analysis of new on surface radiation conditions from localization of solutions to objects with constant curvature.

- Improving the standard physical optics technique by higher accurate on surface radiation conditions.
1.6 Published Papers and the GEMS Project

This thesis is partially based on material from the licentiate thesis [7] and the contribution to the WAVES05 conference at Brown university [8] which will appear in *Journal of computational and applied mathematics*. These references are listed below:


2. WAVES05, 'On Surface Radiation Conditions for High Frequency Wave Scattering' ([8] in the reference list)

The research in this thesis was performed within the GEMS project, but it represents my own contribution. The acronym GEMS stands for General Electro-Magnetic Solver, which was a research and software project based on collaboration between academic and industrial partners, see Appendix A. My collaborative research in GEMS involved programming a block version of the GMRES iterative solver and is not a part of this thesis. In GEMS, I have also programmed a fast multipole accelerator for Helmholtz equation in 2D.
Chapter 2

Time Domain Integral Equation method

In this chapter we will derive integral representations of the scattered field for the acoustic equation. In section 2.1, the Kirchhoff formula is introduced for the solution of the scalar wave equation. The Kirchhoff formula is used to get an integral representation that couples the incoming and the scattered field on the boundary of the scatterer. The coupling depends on the material properties of the scatterer. When we have a sound soft scatterer, we obtain a Dirichlet boundary condition. If we have a sound hard scatterer, then we get a Neumann boundary condition.

The variational formulations described in this chapter have been derived by Bamberger and Ha Duong in [9], [10]. They first derive the variational formulations for Helmholtz equations for one frequency. This formulation is shown to yield a unique solution for the corresponding Helmholtz problem. The formulations for the wave equation are obtained by using Parseval’s identity. We will give a short review of the derivation of the variational formulation in the Dirichlet case. A more thorough derivation is given in [9] for the Dirichlet case and in [10] for the Neumann case. In section 2.2, we introduce basis functions in time and space. In sections 2.3 and 2.4 we discuss the variational formulations for the Dirichlet and Neumann cases, respectively. In section 2.5 we introduce notation for representing points on different planes. We define a $K$-gradient and show how it is related to the “normal” gradient. In section 2.6 time is eliminated in our variational formulations, by integration. In sections 2.7 and 2.8, we face the consequences of eliminating the time dependence for the Dirichlet and Neumann case. In section 2.8, we derive integrals, which we evaluate over triangles, circle sectors, circles and circle segments in section 2.9. The marching on in time (MOT) method and the related matrix structures are presented in section 2.10. In section 2.11 we introduce a new stabilization technique and in section 2.12 numerical examples are given. We develop on surface radiation conditions to reduce the computational cost for high frequency problems in section 2.13.
CHAPTER 2. TIME DOMAIN INTEGRAL EQUATION METHOD

2.1 The Scalar Wave Equation

Consider the 3D wave equation for the pressure $u$ with given sound speed $c$,

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = -g(r, t), \quad (2.1)$$

$$u(r, t) = 0, \quad t \leq 0, \quad (2.2)$$

where $r = (x, y, z)$ is the spatial coordinate. Let $\Omega$ be a closed domain bounded by a regular surface $\Gamma$ and let $\Omega' = \mathbb{R}^3 \setminus \Omega$ be the exterior domain. Suppose that $u$ is a scalar function which has two continuous derivatives in $\Omega$ and $\Gamma$. Using the fundamental solution of the wave equation yields the Kirchhoff formula [66]

$$4\pi u(r, t) = \int_{\Omega} \frac{1}{R} g^* dv' + \int_{\Gamma} \left\{ \frac{1}{R} \frac{\partial u^*}{\partial n} - \frac{\partial}{\partial n} \left( \frac{1}{R} \frac{\partial u^*}{\partial t} \right) \right\} d\Gamma', \quad (2.3)$$

where

$$g^*(r', t) = g(r', t - R/c), \quad R = |r - r'|, \quad (2.4)$$

and $n$ is the outwards normal.

The field can be divided into an incoming part $u^{inc}$ and a scattered part $u^{sc}$. The total field $u^{tot}$ is the sum of the two parts. For a given incoming field $u^{inc}(r, t)$, we want to compute the scattered field in $\Omega' \times \mathbb{R}^+$. The equations for $u^{inc}$ and $u^{sc}$ are

$$\nabla^2 u^{inc} - \frac{1}{c^2} \frac{\partial^2 u^{inc}}{\partial t^2} = -g(r, t), \quad \text{in} \ \mathbb{R}^3 \times \mathbb{R}^+, \quad (2.5)$$

$$\nabla^2 u^{sc} - \frac{1}{c^2} \frac{\partial^2 u^{sc}}{\partial t^2} = 0, \quad \text{in} \ \Omega' \times \mathbb{R}^+. \quad (2.6)$$

Define the function $\tilde{u}(r, t)$ in $\mathbb{R}^3 \times \mathbb{R}$

$$\tilde{u} = \begin{cases} -u^{inc}, & \text{in} \ \Omega \times \mathbb{R}^+, \\ u^{sc}, & \text{in} \ \Omega' \times \mathbb{R}^+. \end{cases} \quad (2.7)$$

The equation for $\tilde{u}$ away from $\Gamma$ is

$$\tilde{u} = \frac{1}{4\pi} \int_{\Gamma} \left\{ \frac{1}{R} \left[ \frac{\partial \tilde{u}^*}{\partial n} \right] - \frac{\partial}{\partial n} \left( \frac{1}{R} \right) \tilde{u}^* + \frac{1}{cR} \frac{\partial}{\partial n} \frac{\partial}{\partial t} \tilde{u}^* \right\} d\Gamma', \quad (2.8)$$

where $[\tilde{u}] = \tilde{u}^{int} - \tilde{u}^{ext}$ and $\tilde{u}^{int}, \tilde{u}^{ext}$ are the solutions to the interior and exterior problem respectively. To get a unique solution to this problem, we need a boundary condition on $\Gamma$. There are at least three possible boundary conditions, namely the Dirichlet, Neumann and Robin boundary condition. The Dirichlet and the Neumann boundary condition correspond to a sound-soft and sound-hard object, respectively. The Robin boundary condition corresponds to an object that is neither sound-soft or sound-hard.
2.1. THE SCALAR WAVE EQUATION

Dirichlet Problem

Consider a Dirichlet problem that has $u^{tot} = 0$ on the boundary. This is equivalent to $[\tilde{u}] = 0$ on the boundary and the integral equation can be written

$$\tilde{u} = P^D \left[ \frac{\partial \tilde{u}}{\partial n} \right] \triangleq \frac{1}{4\pi} \int_{\Gamma} \frac{1}{R} \left[ \frac{\partial \tilde{u}^*}{\partial n} \right] d\Gamma',$$

or equivalently, with $J = \left[ \frac{\partial \tilde{u}}{\partial n} \right]$,

$$-u^{inc}(r, t) = P^D (J)(r, t), \quad \forall (r, t) \in \Gamma \times \mathbb{R} \quad (2.10)$$

$$u^{sc}(r, t) = P^D (J)(r, t), \quad \forall (r, t) \in \Omega' \times \mathbb{R}. \quad (2.11)$$

A solution to the Dirichlet problem consists of two steps. We want to find a solution of equation (2.10). This can be done by multiplying with test functions $J^t$ and solve the system of equations in order to get the potential $J$. Let $V^1(r)$ be the space of linear functions in space and $W^0(t)$ be the space of constant functions in time. We obtain the variational formulation 1.

Variational formulation 1 (Dirichlet). Find $J \in V^1(r) \times W^0(t)$ such that

$$-\iint u^{inc} J^t d\Gamma dt = \iint P^D (J^t) J^t d\Gamma dt, \quad \forall J^t \in V^1(r) \times W^0(t). \quad (2.12)$$

The potential can then be used to compute the scattered field $u^{sc}$ outside the scatterer in equation (2.11).

Neumann Problem

Consider a Neumann problem that has $\frac{\partial u^{tot}}{\partial n} = 0$ on the boundary. This is equivalent to $[\frac{\partial \tilde{u}}{\partial n}] = 0$ on the boundary and the integral equation can be written

$$\tilde{u} = P^N ([\tilde{u}]) \triangleq \frac{1}{4\pi} \int_{\Gamma} -\frac{\partial}{\partial n} \left( \frac{1}{R} \right) [\tilde{u}^*] + \frac{1}{\varepsilon R} \frac{\partial R}{\partial n} \frac{\partial}{\partial t} [\tilde{u}^*] d\Gamma',$$

or equivalently

$$-u^{inc}(r, t) = P^N ([\tilde{u}]) (r, t), \quad \forall (r, t) \in \Gamma \times \mathbb{R} \quad (2.14)$$

$$u^{sc}(r, t) = P^N ([\tilde{u}]) (r, t), \quad \forall (r, t) \in \Omega' \times \mathbb{R}. \quad (2.15)$$

A solution to the Neumann problem consists of two steps. The solution to equation (2.14) yields $[\tilde{u}]$. A variational formulation of the Neumann problem can be found in [24]. This can be used to compute the scattered field $u^{sc}$ outside the scatterer in equation (2.15).
Robin Problem

When the scatterer surface is neither Dirichlet nor Neumann, we can have a Robin boundary condition on $\Gamma$. For a given $\alpha$,

$$\frac{\partial u_{\text{tot}}}{\partial n} + \alpha u_{\text{tot}} = -f, \quad \text{on } \Gamma.$$ \hspace{1cm} (2.16)

If $J = [\frac{\partial \tilde{u}}{\partial n}]$ and $M = [\tilde{u}]$, then the general problem can be written

$$\tilde{u} = P^D(J) + P^N(M), \quad \text{on } \Gamma.$$ \hspace{1cm} (2.17)

$$J + \alpha M = f, \quad \text{on } \Gamma.$$ \hspace{1cm} (2.18)

There is also an impedance formulation of the problem, which can be found in [35].

2.2 Basis Functions in Space and Time

Our goal is to find a solution to the wave equation, that can be written in some basis functions

$$J(r,t) = \sum_{m,l} J_{ml} \Phi_m(r) \tilde{\Psi}_l(t),$$ \hspace{1cm} (2.19)

where $\Phi_m(r)$ are spatial basis functions and $\tilde{\Psi}_l(t)$ are basis functions in time. The scatterer $\Gamma$ is triangulated. Introduce linear spatial nodal elements $\Phi_j(r)$ on the triangulation. The spatial elements are defined as the piecewise linear function that satisfies

$$\Phi_j(r) = \begin{cases} 1, & r = r_j, \\ 0, & r = r_i, \quad i \neq j. \end{cases}$$ \hspace{1cm} (2.20)

For a certain triangle $K$, we have three local spatial basis function as we denote $\Phi^K_j$, with local indices $j = 1, 2, 3$. Let $r^K_j$ be the nodes of $K$. Then the point $r \in K$ can be parameterized

$$r(x,y) = r^K_1 + x(r^K_2 - r^K_1) + y(r^K_3 - r^K_1), \quad x, y \geq 0, \quad x + y \leq 1.$$ \hspace{1cm} (2.21)

The local spatial basis functions are defined as

$$\Phi^K_1(r(x,y)) = 1 - x - y,$$ \hspace{1cm} (2.22)

$$\Phi^K_2(r(x,y)) = x,$$ \hspace{1cm} (2.23)

$$\Phi^K_3(r(x,y)) = y.$$ \hspace{1cm} (2.24)

We define the space

$$V^K_0(r) = \{\text{Linear combinations of } \Phi^K_j(r)\}.$$ \hspace{1cm} (2.25)
2.3. VARIATIONAL FORMULATION, DIRICHLET CASE

When we have a physical coordinate \( r \) and want to calculate the spatial basis function, we need to get \( x \) and \( y \). This can be done by solving the system (with \( r_j = r_j^K \) and \( r_{ij} = r_i - r_j \))

\[
\begin{pmatrix}
    r_{21}^T r_{21} & r_{31}^T r_{21} \\
    r_{31}^T r_{31} & r_{31}^T r_{31}
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix} = \begin{pmatrix}
    (r - r_1) \cdot r_{21} \\
    (r - r_1) \cdot r_{31}
\end{pmatrix}.
\]

(2.26)

This system has a unique solution as long as the triangle edges are non parallel.

There are different ways of choosing the basis functions in time. In [74], Weile, Shanker and Michielssen use BLIF basis functions. The BLIF functions are several time steps wide, which leads to an implicit solver. For the Dirichlet problem, we will use constant elements in time. The time basis function are defined as

\[
\Psi_k(t) = \begin{cases} 
1, & t \in [(k-1)\Delta t, k\Delta t), \\
0, & \text{otherwise.}
\end{cases}
\]

(2.27)

For the Neumann problem, we need more regular basis functions, and we use the basis functions \( \int_{-\infty}^{t} \Psi_k(\tau)d\tau \).

We define the finite dimensional spaces

\[
W_0^h(t) = \{\text{Linear combinations of } \Psi_k(t)\},
\]

(2.28)

\[
W_1^h(t) = \{\text{Linear combinations of } \int_{-\infty}^{t} \Psi_k(\tau)d\tau\}.
\]

(2.29)

2.3 Variational Formulation, Dirichlet Case

The variational formulation for the Dirichlet problem was proposed by Bamberger and Ha Duong [9]. When deriving a variational formulation for the Dirichlet problem, we first consider the case of a single frequency \( k \), with \( \Im k > 0 \). Define the single layer potential

\[
(S_k \phi)(r) = \frac{1}{4\pi} \int_{\Gamma} \frac{e^{ik|r-r'|}}{|r-r'|} \phi(r') d\Gamma'.
\]

(2.30)

The Dirichlet problem for a fixed frequency \( k \) is

\[
S_k \phi = g,
\]

(2.31)

where \( \phi \) is the jump of \( \frac{\partial u}{\partial n} \) over the boundary and \( g = -u^{inc} \). In [9] it is shown that the variational equation that admits a unique solution for the fixed frequency \( k \), with \( \Im k > 0 \) is:

**Variational formulation 2** (Dirichlet problem, Helmholtz equation). Suppose that \( g \in H^{1/2}(\Gamma) \). Then the variational formulation for the Helmholtz Dirichlet problem is to find \( \phi \in H^{-1/2}(\Gamma) \) such that

\[
<\psi, -ikS_k \phi > = <\psi, -ikg >, \quad \forall \psi \in H^{-1/2}(\Gamma).
\]

(2.32)
CHAPTER 2. TIME DOMAIN INTEGRAL EQUATION METHOD

The corresponding retarded potential to the single layer potential (2.30) is
\[
(S\phi)(t, r) = \frac{1}{4\pi} \int_\Gamma \frac{\phi(t - R/c, r')}{R} d\Gamma'.
\] (2.33)

The Parseval identity yields the variational formulation for the time dependent problem:

**Variational formulation 3** (Dirichlet problem). Suppose that \(u^{inc} \in H^{3/2}_{\omega/2}(\mathbb{R}_+, H^{1/2}(\Gamma))\). The variational formulation for the Dirichlet problem is to find \(J \in H^1_{\omega/2}(\mathbb{R}_+, H^{-1/2}(\Gamma))\) such that
\[
\int\int\int e^{-\omega t} \frac{\partial}{\partial t} J(t - R/c, r') d\Gamma' J'(t, r) d\Gamma dt = -\int\int e^{-\omega t} \frac{\partial}{\partial t} u^{inc}(t, r) J'(t, r) d\Gamma' dt, \quad \forall J' \in H^1_{\omega/2}(\mathbb{R}_+, H^{-1/2}(\Gamma)).
\] (2.34)

We search for solutions in the finite dimensional space \(V_h^1(r) \times W_0^0(t)\) and the basis function representation
\[
J(r, t) = \sum_{m,l} J_{ml} \Phi_m(r) \Psi_l(t) \in V_h^1(r) \times W_0^0(t),
\] (2.35)
\[
J'(r, t) = \Phi_j(r) \Psi_k(t) \in V_h^1(r) \times W_0^0(t),
\] (2.36)
yields the discrete variational formulation

**Variational formulation 4** (Dirichlet problem). Find the coefficients \(J_{ml}\) of \(J \in V_h^1(r) \times W_0^0(t)\) in (2.35) such that
\[
\sum_{m,l} J_{ml} \int\int \Phi_j(r) \Psi_m(r') \frac{\partial}{\partial t} \Psi_l(t - R/c, r') dt d\Gamma' = -\int\int e^{-\omega t} \frac{\partial}{\partial t} u^{inc}(t, r) \Phi_j(r) \Psi_k(t) dt d\Gamma', \quad \forall \Phi_j(r) \Psi_k(t) \in V_h^1 \times W_0^0(t),
\] (2.37)

2.4 Variational Formulation, Neumann Case

The variational formulation for the Neumann problem was proposed by Bamberger and Ha Duong [10]. Following approximately the same procedure as for the Dirichlet problem, we get the variational problem

**Variational formulation 5** (Neumann problem). Suppose that \(\frac{\partial u^{inc}}{\partial n} \in H^{3}_{\omega/2}(\mathbb{R}_+, H^{-1/2}(\Gamma))\). The variational formulation for the Neumann prob-
2.5. POINT REPRESENTATION ON TRIANGLE PLANE

The aim is to find \( J \in H^2_{\omega/2}(\mathbb{R}^+, H^{1/2}(\Gamma)) \) such that

\[
\int \int \int e^{-\omega t} \frac{n \cdot n'}{4\pi R} \frac{\partial^2}{\partial t^2} J(t - R/c, r') \frac{\partial}{\partial t} J(t, r) + e^{-\omega t} \nabla' \cdot (r - R/c, r') \cdot \nabla \frac{\partial}{\partial t} J(t, r)
\]

\[
d\Gamma' d\Gamma dt
\]

\[
= - \int \int e^{-\omega t} \frac{\partial}{\partial n} u_{inc}(t, r) \frac{\partial}{\partial t} J(t, r) d\Gamma dt, \quad \forall J \in H^2_{\omega/2}(\mathbb{R}^+, H^{1/2}(\Gamma)).
\]

This variational formulation requires the time basis functions to be more regular. Using the basis functions

\[
J(r, t) = \sum_{m,l} J_{ml} \Phi_m(r) \int_{-\infty}^{t} \Psi_l(t) dt \in V^1_h(r) \times W^1_{\Delta t},
\]

(2.39)

\[
J'(r, t) = \Phi_j(r) \int_{-\infty}^{t} \Psi_k(t) dt \in V^1_h(r) \times W^1_{\Delta t},
\]

(2.40)

yield the discrete variational formulation

**Variational formulation 6** (Neumann problem). Find the coefficients \( J_{ml} \) of \( J \in V^1_h \times W^1_{\Delta t} \) in (2.39) such that

\[
\sum_{m,l} J_{ml} \int \frac{n \cdot n'}{4\pi R} \int e^{-\omega t} \Psi_k(t, r) \frac{\partial}{\partial t} \Psi_l(t - R/c, r') dt + \frac{n \times \nabla \Phi_j(r)}{4\pi R} \cdot n' \times \nabla (\Phi_m(r')) \int e^{-\omega t} \Psi_k(t) dt \int_{-\infty}^{t} \Psi_l(t - R/c) dr dt d\Gamma'
\]

\[
= - \int \int e^{-\omega t} u_{inc}(t, r) \Phi_j(r) \Psi_k(t) dt d\Gamma, \quad \forall \Phi_j(r) \int_{-\infty}^{t} \Psi_k(t) dt \in V^1_h \times W^1_{\Delta t}.
\]

(2.41)

### 2.5 Point Representation on Triangle Plane

In order to obtain a useful variational formulation for the discretized problems, we need to find the domain of integration which is a strip over a triangle. To express points on the triangle plane, different basis for each triangle are used, such that the third component of the point in the triangle plane is zero. We also need to evaluate gradients on the triangles in the triangle plane basis.

A point \( r \) at an arbitrary triangle \( K \) in 3D with nodes \( r_1, r_2 \) and \( r_3 \) is parameterized according to

\[
r = r_1 + \alpha r_{21} + \beta r_{31}, \quad \alpha \in [0, 1], \quad \beta \in [0, 1 - \alpha].
\]

(2.42)
CHAPTER 2. TIME DOMAIN INTEGRAL EQUATION METHOD

It is assumed that \( r_{21} = r_2 - r_1 \) and \( r_{31} = r_3 - r_1 \) are non-parallel. A basis for this triangle plane is

\[
e^K_1 = \frac{r_{21}}{|r_{21}|}, \quad e^K_2 = \frac{r_{31} - (r_{31} \cdot e^K_1)e^K_1}{|r_{31} - (r_{31} \cdot e^K_1)e^K_1|}, \quad e^K_3 = e^K_1 \times e^K_2.
\]

(2.43)

(2.44)

(2.45)

The triangles in the scatterer are numbered s.t. \( e^K_3 \) is equal to the outwards normal \( n \). We define the coordinate transformation:

**Definition 1** (Coordinate transformation). The representation of a point in the triangle plane basis is written as

\[
(x_1, x_2, x_3)_K = x_1 e^K_1 + x_2 e^K_2 + x_3 e^K_3.
\]

(2.46)

**Definition 2** (K-plane). We say that a point \( r \) is in the K-plane iff

\[
r = (x_1, x_2, 0)_K.
\]

(2.47)

for some parameters \( x_1 \) and \( x_2 \).

The point \( r = r_1 + \alpha r_{21} + \beta r_{31} \) on the triangle can now be written as

\[
r = r_1 + (\alpha |r_{21}| + \beta (r_{31} \cdot e^K_1), \beta |r_{31} - (r_{31} \cdot e^K_1)e^K_1|, 0)_K.
\]

(2.48)

Now we define the \( K \)-gradient

**Definition 3** (K-gradient). Suppose that \( r = r_1 + (x_1, x_2, x_3)_K \). Then the \( K \)-gradient of \( \Phi(r) \) is defined as

\[
\nabla_K \Phi(r) = \left( \frac{\partial \Phi(r)}{\partial x_1}, \frac{\partial \Phi(r)}{\partial x_2}, \frac{\partial \Phi(r)}{\partial x_3} \right)_K.
\]

(2.49)

**Lemma 1** (K-gradient in \( \alpha \) and \( \beta \)). Suppose we have the triangle representation \( r = r_1 + \alpha r_{21} + \beta r_{31} \), where \( r \) is in the K-plane. Then the \( K \)-gradient is

\[
\nabla_K \Phi(r) = \left( \frac{1}{|r_{21}|} \frac{\partial \Phi}{\partial \alpha}, \frac{1}{|r_{31} - (r_{31} \cdot e^K_1)e^K_1|} \left( \frac{\partial \Phi}{\partial \beta} - \frac{(r_{31} \cdot e^K_1) \partial \Phi}{|r_{21}|} \right), 0 \right)_K.
\]

(2.50)

**Proof.** We use the chain rule

\[
\nabla_K \Phi(r) = \left( \frac{\partial \Phi}{\partial \alpha} \frac{\partial \alpha}{\partial x_1} + \frac{\partial \Phi}{\partial \beta} \frac{\partial \beta}{\partial x_1} + \frac{\partial \Phi}{\partial \alpha} \frac{\partial \alpha}{\partial x_2} + \frac{\partial \Phi}{\partial \beta} \frac{\partial \beta}{\partial x_2}, 0 \right)_K.
\]

(2.51)
Rewriting the parameterization as
\[
\alpha = \frac{1}{|\mathbf{r}_{21}|} \left( x_1 - \frac{x_2 (\mathbf{r}_{31} \cdot \mathbf{e}_K^1)}{|\mathbf{r}_{31} - (\mathbf{r}_{31} \cdot \mathbf{e}_K^1)\mathbf{e}_K^1|} \right) \quad (2.52)
\]
\[
\beta = \frac{x_2}{|\mathbf{r}_{31} - (\mathbf{r}_{31} \cdot \mathbf{e}_K^1)\mathbf{e}_K^1|} \quad (2.53)
\]
yields the derivatives
\[
\frac{\partial \alpha}{\partial x_1} = \frac{1}{|\mathbf{r}_{21}|}, \quad (2.54)
\frac{\partial \beta}{\partial x_1} = 0, \quad (2.55)
\frac{\partial \alpha}{\partial x_2} = \frac{- (\mathbf{r}_{31} \cdot \mathbf{e}_K^1)}{|\mathbf{r}_{21}| |\mathbf{r}_{31} - (\mathbf{r}_{31} \cdot \mathbf{e}_K^1)\mathbf{e}_K^1|} \quad (2.56)
\frac{\partial \beta}{\partial x_2} = \frac{1}{|\mathbf{r}_{31} - (\mathbf{r}_{31} \cdot \mathbf{e}_K^1)\mathbf{e}_K^1|} \quad (2.57)
\]
By inserting the derivatives in the chain rule we obtain the lemma.

The cross product of the gradient can be written in $K$-plane coordinates.

**Lemma 2** (Cross product transformation). Suppose $\mathbf{n} = \mathbf{e}_3^K$ and $\mathbf{n}_K = (0, 0, 1)_K$. Then
\[
\mathbf{n} \times \nabla \Phi(\mathbf{r}) = (\mathbf{n}_K \times \nabla_K \Phi(\mathbf{r}))_K.
\]

We get the gradient expression
\[
\mathbf{n} \times \nabla \Phi(\mathbf{r}) = (\mathbf{n}_K \times \nabla_K \Phi(\mathbf{r}))_K
\]
\[
= \left( -\frac{\partial \Phi}{\partial x_2} \frac{\partial \Phi}{\partial x_1} , 0 \right)_K
\]
\[
= \left( \frac{1}{|\mathbf{r}_{31} - (\mathbf{r}_{31} \cdot \mathbf{e}_K^1)\mathbf{e}_K^1|} \left( (\mathbf{r}_{31} \cdot \mathbf{e}_K^1) \frac{\partial \Phi}{\partial \alpha} - \frac{\partial \Phi}{\partial \beta} \right) , 1 \frac{\partial \Phi}{\partial \alpha} , 0 \right)_K \quad (2.58)
\]
where the coefficients in the expression can be precalculated. Observe that the spatial basis is linear and therefore the gradient is constant. Thus, we can move the gradient of the basis functions outside the integral.

### 2.6 Integrals Over Time

In the variational formulation of both the Dirichlet and Neumann cases, integrals over time are obtained, for which we can find analytical expressions expressed in
\[ R. \] Define the integrals
\[
I_1^\omega(R, k - l) = e^{k\omega t} \int e^{-\omega t} \psi_k(t) \psi_l(t - R/c) dt, \tag{2.59}
\]
\[
I_2^\omega(R, k - l) = e^{k\omega t} \int e^{-\omega t} \psi_k(t) \frac{\partial}{\partial t} \psi_l(t - R/c) dt, \tag{2.60}
\]
\[
I_3^\omega(R, k - l) = e^{k\omega t} \int e^{-\omega t} \psi_k(t) \int_{-\infty}^t \psi_l(t - R/c) dr dt, \tag{2.61}
\]
where \( \omega \geq 0 \). Introduce the interval \( I(u_1, u_2) \) such that
\[
R \in I(u_1, u_2) \iff u_1e \Delta t < R < u_2e \Delta t. \tag{2.62}
\]

If constant elements in time are used, we obtain the following integrals over time
\[
I_1^\omega(R, u) = \frac{1}{\omega} \begin{cases} 
\varepsilon^{\omega \Delta t} - e^{-\omega \Delta t} e^{-\frac{\Delta t}{c}}, & R \in I(u - 1, u) \\
\varepsilon^{(u + 1) \Delta t} e^{-\frac{\Delta t}{c}} - 1, & R \in I(u, u + 1) \\
0, & \text{otherwise},
\end{cases} \tag{2.63}
\]
\[
I_2^\omega(R, u) = \begin{cases} 
-\varepsilon^{\omega \Delta t} e^{-\frac{\Delta t}{c}}, & R \in I(u - 1, u) \\
\varepsilon^{(u + 1) \Delta t} e^{-\frac{\Delta t}{c}}, & R \in I(u, u + 1) \\
0, & \text{otherwise},
\end{cases} \tag{2.64}
\]
\[
I_3^\omega(R, u) = \frac{1}{\omega^2} \begin{cases} 
\omega \Delta t (e^{\omega \Delta t} - 1), & R \in I(0, u - 1) \\
(\omega u \Delta t + 1) e^{\omega \Delta t} - \omega \Delta t - \frac{\omega^2}{2} e^{\omega \Delta t} R - e^{\omega \Delta t} e^{-\frac{\Delta t}{c}}, & R \in I(u - 1, u) \\
\omega \Delta t (u + 1) - 1 + \frac{\omega}{2} R e^{(u + 1) \Delta t} e^{-\frac{\Delta t}{c}}, & R \in I(u, u + 1) \\
0, & \text{otherwise},
\end{cases} \tag{2.65}
\]
where \( I \) is defined in (2.62).

Taking the limit \( \omega \to 0 \) produces
\[
I_1^0(R, u) = \begin{cases} 
(1 - u) \Delta t + \frac{R}{c}, & R \in I(u - 1, u) \\
(1 + u) \Delta t - \frac{R}{c}, & R \in I(u, u + 1) \\
0, & \text{otherwise},
\end{cases} \tag{2.66}
\]
\[
I_2^0(R, u) = \begin{cases} 
-1, & R \in I(u - 1, u) \\
1, & R \in I(u, u + 1) \\
0, & \text{otherwise},
\end{cases} \tag{2.67}
\]
\[
I_3^0(R, u) = \begin{cases} 
\Delta t^2, & R \in I(0, u - 1) \\
\left(\frac{1}{2} + u - \frac{u^2}{2}\right) \Delta t^2 + (u - 1) \frac{R \Delta t}{c} - \frac{R^2}{2c}, & R \in I(u - 1, u) \\
\left(\frac{1}{2} + u + \frac{u^2}{2}\right) \Delta t^2 - (u + 1) \frac{R \Delta t}{c} + \frac{R^2}{2c}, & R \in I(u, u + 1) \\
0, & \text{otherwise},
\end{cases} \tag{2.68}
\]
where $\tilde{I}$ is defined in (2.62).

The integrations over time produce strips in space with a radius that depends on the difference in basis functions indices in time. Introduce $\delta$ such that

$$R = (u + \delta)c\Delta t.$$  (2.69)

The integrals $I_\omega$ are functions of $\delta$ (up to a factor in $\Delta t$ and $\omega$). In Figure 2.6, the integrals $I_\omega$ are presented as a function of $\delta$. The functions $I_\omega$ have been normalized to have the maximum height 1. (For the case when $\omega = 0$, this corresponds to $\Delta t = 1$). Observe that $I_\omega$ is nonzero for all negative $\delta$. This corresponds to an integration over all earlier potentials. It is interesting to see that for $\omega = 0$, the mass (or area) of $I_0^1$ and $I_0^2$ is symmetric and antisymmetric, respectively at $\delta = 0$. When $\omega$ is increased, the mass center moves to the left. This can be interpreted as the method is becoming less implicit.

![Figure 2.1: Time integral contribution. When the parameter $\omega$ is chosen positive, then the mass center of the contribution from the time integral moves to the left and the method becomes less implicit.](image)

2.7 Dirichlet Discretization

After discretizing the outer integral of variational problem 3 and introducing the integrals $I_\omega$, the Dirichlet integral equation becomes
Variational formulation 7 (Dirichlet problem). Find the coefficients $J_{m l}$ of $J \in V_h^1 \times W_0^0$ in (2.35) such that

$$\frac{1}{4\pi} \sum_{m,l,p} J_{m l} w_p \Phi_j(r_p) \int \frac{\Phi_m(r')}{R} I_2''(R,u) d\Omega' = - \int \int e^{-\omega t} \frac{\partial}{\partial t} u^{inc}(t + k \Delta t, r) \Phi_j(r) \Psi(t) dtd\Gamma, \quad \forall \Phi_j(r) \Psi_k(t) \in V_h^1 \times W_0^0. \quad (2.70)$$

where $R = |r' - r_p|$. In the assembly process, we need to evaluate the integral

$$J_2'' = \int \frac{\Phi_m(r')}{R} I_2''(R,u) dS, \quad (2.71)$$

where $S$ is a triangle, circle sector, circle segment or a circle, lying on the triangle $K' \subset \Gamma'$.

2.8 Neumann Discretization

To obtain the discretized Neumann formulation, the following lemmas are needed in order to write a useful discretization.

**Lemma 3** (Gradient chain rule). Suppose that $R = |r' - r|$. Then

$$\nabla' (\Phi(r') \Psi(\tau - R/c)) = \nabla' (\Phi(r')) \Psi(\tau - R/c) + \frac{\partial}{\partial \tau} \Psi(\tau - R/c) \frac{r - r'}{cR} \Phi(r'). \quad (2.72)$$

Proof. Let $\tau^* = \tau - R/c$. Now we have the chain rule

$$\nabla' (\Phi(r') \Psi(\tau^*)) = \nabla' (\Phi(r')) \Psi(\tau^*) + \Phi(r') \nabla' (\Psi(\tau^*)), \quad (2.73)$$

where

$$\nabla' (\Psi(\tau^*)) = \frac{\partial \Psi(\tau^*)}{\partial \tau} \frac{\partial}{\partial \tau} \cdot \nabla' \tau^* = \frac{\partial \Psi(\tau^*)}{\partial \tau} \cdot \frac{r - r'}{cR}. \quad (2.74)$$

Inserting (2.74) in (2.73) yields the lemma. □

**Lemma 4** (Derivative of integral). Suppose that $\Psi(t) = 0$ for $t \leq 0$. Then

$$\frac{\partial}{\partial t} \int_{-\infty}^{t} \Psi(\tau - R/c) d\tau = \Psi(t - R/c). \quad (2.75)$$
2.8. NEUMANN DISCRETIZATION

Lemma 5 (Cross product simplification). Suppose that \( n' \) is a normal to the \( K' \)-plane and that \( Pr \) is the projection of \( r \) onto the \( K' \)-plane. Let \( r' \in K' \)-plane. Then the following holds
\[
n' \times (r - r') = n' \times (Pr - r').
\] (2.76)

Proof. Since \( n' \) and \( r - Pr \) are parallel, it is true that
\[
n' \times (r - r') = n' \times (r - Pr + Pr - r') = 0 + n' \times (Pr - r').
\]

Lemma 6 (Combination of lemmas).
\[
n' \times \nabla'(\Phi(r') \int_t^{\tau} \Psi(\tau - R/c) d\tau) = n' \times \nabla'(\Phi(r')) \int_t^{\tau} \Psi(\tau - R/c) d\tau + n' \times (Pr - r') \Phi(r') \Psi(t - R/c) cR.
\] (2.77)

Using the lemmas and discretizing the outer integral, we get the integral equation for the Neumann integral equation

Variational formulation 8 (Neumann problem). Find the coefficients \( J_{ml} \) of \( J \in V^1_k \times W^1_{h\Delta t} \) in (2.39) such that
\[
\frac{1}{4\pi} \sum_{m,l,p} J_{ml} w_p (n \cdot n') \Phi_j (r_p) \int \frac{\Phi_m (r')}{R} I_2 (R, k - l) d\Gamma' + (n \times \nabla \Phi_j (r_p)) \cdot (n' \times \nabla' \Phi_m (r')) \int \frac{1}{R} I_3 (R, k - l) d\Gamma' \nonumber \nonumber + (n \times \nabla \Phi_j (r_p)) \cdot \left( \frac{(n' \times (Pr - r')) \Phi_m (r') cR}{R^2} I_2 (R, k - l) d\Gamma' \right) = - \int \frac{\partial}{\partial n} u^{inc} (t, r) \Phi_j (r) \Psi_k (t) d\Gamma',
\]
\( \forall \Phi_j (r) \int_{\Gamma'}^{t} \Psi_k (\tau) e^{V^1_k \times W^1_{h\Delta t}}. \) (2.78)

where \( R = |r' - r_p| \). Observe that \( n' \times \nabla' \Phi_m (r') \) can be moved outside the integral over \( \Gamma' \), since \( \Phi_m \) is linear.

In the assembly process, the following integrals
\[
J_i^1 = \int \frac{(n' \times (Pr - r')) \Phi_m (r')}{cR^2} I_2 (R, u) dS, \quad (2.79)
J_i^2 = \int \frac{\Phi_m (r')}{R} I_2 (R, u) dS, \quad (2.80)
J_i^3 = \int \frac{1}{R} I_3 (R, u) dS, \quad (2.81)
\]
have to be evaluated where \( S \) is either a triangle, circle sector, circle segment or a circle, lying on the triangle \( K' \subset \Gamma' \).
CHAPTER 2. TIME DOMAIN INTEGRAL EQUATION METHOD

Integrals $J_\omega^\nu$

After discretizing the integrals over $K$ and by using analytical evaluation of the time integrals, we are left with the integrals $J_\omega^\nu$ over $K'$. In order to simplify the integrals, we define points of integration

$r = Pr + (0, 0, ((r - Pr) \cdot e_3^{K'})_{K'})$, \hspace{1cm} (2.82)

$r' = Pr + (r'_1, r'_2, 0)_{K'}$ \hspace{1cm} (2.83)

where $Pr$ is the projection of $r$ onto the $K'$-plane and

$n' \times (Pr - r') = (r'_{2}, -r'_{1}, 0)_{K'}$, \hspace{1cm} (2.84)

$R = \sqrt{|r - Pr|^2 + |(r'_1, r'_2, 0)_{K'}|^2}$ \hspace{1cm} (2.85)

can now be computed.

Case When $\omega = 0$

In the case when $\omega = 0$, we obtain

$J_1^0 = d_0 \int \frac{(r_{2}, -r'_{1}, 0)_{K'}}{cR^2} \Phi_m(r')dS + d_1 \int \frac{(r_{2}, -r'_{1}, 0)_{K'}}{cR} \Phi_m(r')dS$, (2.86)

$J_2^0 = d_0 \int \frac{1}{R} \Phi_m(r')dS$, \hspace{1cm} (2.87)

$J_3^0 = d_0 \int \frac{1}{R} dS + d_1 \int dS + d_2 \int RdS$, \hspace{1cm} (2.88)

where $d_j, j = 0, 1, 2$ matches the coefficients in $I^p_\nu$, $p = 1, 2, 3$. Since there are three different basis functions $\Phi_m$ on each triangle, this is 18 different scalar integral evaluations. (Twelve for $J_1^0$ and three for $J_2^0$ and $J_3^0$, respectively.) In the Dirichlet case, only three different scalar integrals have to be evaluated. Most parts of these integrals are computed analytically. Some parts of the integrals are computed numerically. A detailed description of the integration is given in Section 2.9.

Case When $\omega > 0$

When $\omega > 0$, the following integrals are obtained:

$J_1^\omega = d_0 \int \frac{(r_{2}, -r'_{1}, 0)_{K'}}{cR^2} \Phi_m(r')dS + d_1 \int \frac{e^{-\frac{\pi R}{cR}}(r_{2}, -r'_{1}, 0)_{K'}}{cR^2} \Phi_m(r')dS$, (2.89)

$J_2^\omega = d_0 \int \frac{e^{-\frac{\pi R}{cR}}}{R} \Phi_m(r')dS$, \hspace{1cm} (2.90)

$J_3^\omega = d_0 \int \frac{1}{R} dS + d_1 \int dS + d_2 \int \frac{e^{-\frac{\pi R}{cR}}}{R} dS$, \hspace{1cm} (2.91)
where \( d_j, j = 0, 1, 2 \) matches the coefficients in \( I_p^\omega \), \( p = 1, 2, 3 \). There are the same number of scalar integral evaluations as in the case \( \omega = 0 \). The terms containing \( e^{-\omega R} \) are evaluated numerically except for some special cases which are explained in Section 2.9. The other terms appears also in the case when \( \omega = 0 \).

### 2.9 Quadrature

We use the variational formulations proposed by Bamberger and Ha Doung in [9], [10] for a Dirichlet and a Neumann scatterer. Those variational formulations resulted in integrals \( J_p^\omega, p = 1, \ldots, 3 \), (2.79)- (2.81) after discretizing the integral over the triangles \( K \) and integrating in time analytically. These integrals also apply to the Kirchhoff variational formulation 1 of the Dirichlet problem. In this section, we will show how the remaining integrals are evaluated over triangles, circle sectors, circles and circle segments, lying in the \( K' \)-plane.

#### Integration of a Triangle

In section 2.9 the parameterization of a triangle \( K'' \) is given in local coordinates, as well as the representation of the spatial basis function. The rest of section 2.9 treats the computation of the integrals \( J_p^\omega \) in the two cases \( \omega = 0 \) and \( \omega > 0 \), respectively. The integration is done in three steps. The first step is to compute integrals analytically and in some cases also numerically. Some of the integrals become infinite for certain locations of origo relative to the triangle. These locations can be avoided by reordering the nodes in the triangle. In the second step the computed integrals are combined to evaluate (2.126)-(2.128) when \( \omega = 0 \). In the third step, the integrals obtained in the second step is finally combined to get the integrals \( J_p^0 \). The case \( \omega > 0 \) is also treated, in a similar manner.

#### Local Coordinates on a Triangle

Consider a triangle \( K'' \) with corners \( r_4, r_5 \) and \( r_6 \), where \( r_4 \) is closest to the point \( P \) in the triangle plane of \( K' \). \( K'' \) lie in the same plane as the triangle \( K' \) with corners \( r_1, r_2 \) and \( r_3 \). Denote \( r_{ij} = r_i - r_j \). The points \( r'' \) on \( K'' \) are then parameterized on both \( K' \) and \( K'' \) by

\[
\begin{align*}
r'' &= r_4 + \alpha r_{54} + \beta r_{64}, \quad \alpha, \beta \geq 0, \quad \alpha + \beta \leq 1, \\
r'' &= r_1 + x r_{21} + y r_{31},
\end{align*}
\]

(2.92) \hspace{1cm} (2.93)

where \( x \) and \( y \) depends on \( \alpha \) and \( \beta \). In the local variables \( \alpha \) and \( \beta \), a general integral can be written as

\[
\int f(r'') dK'' = 2|K''| \int_0^1 \int_0^{1-\alpha} f(r''(\alpha, \beta)) d\beta d\alpha.
\]

(2.94)
In the calculation of the spatial basis function, the parameters \( x \) and \( y \) need to be expressed in \( \alpha \) and \( \beta \). This produces the system
\[
\begin{pmatrix}
\mathbf{r}_2^T \mathbf{r}_{21} & \mathbf{r}_3^T \mathbf{r}_{21} \\
\mathbf{r}_2^T \mathbf{r}_{31} & \mathbf{r}_3^T \mathbf{r}_{31}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
\mathbf{r}_2^T \mathbf{r}_{21} \\
\mathbf{r}_3^T \mathbf{r}_{31}
\end{pmatrix}
+ \alpha \begin{pmatrix}
\mathbf{r}_2^T \mathbf{r}_{21} \\
\mathbf{r}_3^T \mathbf{r}_{31}
\end{pmatrix}
+ \beta \begin{pmatrix}
\mathbf{r}_6^T \mathbf{r}_{21} \\
\mathbf{r}_6^T \mathbf{r}_{31}
\end{pmatrix}.
\] (2.95)

Solving this system yields the local spatial basis functions
\[
\Phi_{jk''}(\mathbf{r}(\alpha, \beta)) = a_0 + a_1 \alpha + a_2 \beta.
\] (2.96)

Observe that the \( a \)'s differ for the different \( j \)'s. The matrix inverse depends only on the coordinates of triangle \( K \) and can be precalculated before the assembly process.

**Case \( \omega = 0 \)**

The goal is to evaluate the integrals \( J^0_p, p = 1, 2, 3 \), introduced in (2.86)-(2.88). To do this, we first need to compute some integrals analytically but also some numerically, which are combined to obtain middle-step integrals. These middle-step integrals are then combined to evaluate \( J^0_p, p = 1, 2, 3 \).

**Analytically Evaluated Integrals**

We want to evaluate the integrals \( I_{R40}, I_{R41}, I_{R60}, I_{R61}, I_{R10}, I_{R11} \), defined as
\[
I_{R4j} = \int_0^1 \alpha^j \sqrt{|\mathbf{r} - \mathbf{P}|^2 + |\mathbf{r}_4 + \alpha \mathbf{r}_{54}|^2} d\alpha,
\] (2.97)
\[
I_{R6j} = \int_0^1 \alpha^j \sqrt{|\mathbf{r} - \mathbf{P}|^2 + |\mathbf{r}_6 + \alpha \mathbf{r}_{56}|^2} d\alpha,
\] (2.98)
\[
I_{R1j} = \int_0^1 \alpha^j R_1(\alpha) d\alpha,
\] (2.99)

where
\[
R_1(\alpha) = \sqrt{(|\mathbf{r} - \mathbf{P}|^2 + |\mathbf{r}_4 + \alpha \mathbf{r}_{54}|^2)|\mathbf{r}_{64}|^2 - ((\mathbf{r}_4 + \alpha \mathbf{r}_{54})^T \mathbf{r}_{64})^2}.
\] (2.100)

To do this, the integrals
\[
I_R(\mathbf{b}_0, \mathbf{b}_1, n) = \int_0^1 \alpha^n |\mathbf{b}_0 + \alpha \mathbf{b}_1| d\alpha,
\] (2.101)
are required. In the case $n = 0, 1$, an analytical solution can be computed,

$$
I_R(b_0, b_1, 0) = A_1 + A_2 \log A_3, \tag{2.102}
$$

$$
I_R(b_0, b_1, 1) = B_1 + B_2 \log B_3, \tag{2.103}
$$

$$
A_1 = \frac{|b_0 + b_1||b_2|^2 + b_0^T b_1|b_0 + b_1| - b_0^T b_1|b_0|}{2|b_1|^2}, \tag{2.104}
$$

$$
A_2 = \frac{|b_0|^2|b_1|^2 - (b_0^T b_1)^2}{2|b_1|^3}, \tag{2.105}
$$

$$
A_3 = \frac{(b_0 + b_1)^T b_1 + |b_0 + b_1||b_1|}{b_0^T b_1 + |b_0||b_1|}, \tag{2.106}
$$

$$
B_1 = \frac{|b_0 + b_1|^3}{3|b_1|^2} - \frac{b_0^T b_1}{|b_1|^2} A_1, \tag{2.107}
$$

$$
B_2 = -\frac{b_0^T b_1}{|b_1|^2} A_2, \tag{2.108}
$$

$$
B_3 = A_3, \tag{2.109}
$$

and

$$
I_{R10} = \frac{t_3 + t_5 (t_3 - t_2)}{2t_1^3} + \frac{(t_1^2(|r - Pr|^2 + |r_4|^2) + tr_4^T r_{64} - t_3 r_4^T r_{54})t_5 r_{64}|2}{2t_1^4}. \tag{2.110}
$$

$$
I_{R11} = \frac{t_3^2 - t_2^2}{3t_1^4} - \frac{t_5}{2t_1^4} \cdot (t_1^2 t_3 + t_4 t_5 (t_3 - t_2)) + |r_{64}|^2((|r - Pr|^2 + |r_4|^2)t_1^2 + r_4^T r_{64} t_4 - r_4^T r_{54} t_5)) \tag{2.111}
$$

where the constants $t_j$ are defined as

$$
t_1 = \sqrt{|r_{54}|^2 |r_{64}|^2 - (r_4^T r_{64})^2}, \tag{2.112}
$$

$$
t_2 = \sqrt{(|r - Pr|^2 + |r_4|^2)|r_{64}|^2 - (r_4^T r_{64})^2}, \tag{2.113}
$$

$$
t_3 = \sqrt{(|r - Pr|^2 + |r_5|^2)|r_{64}|^2 - (r_4^T r_{64})^2}, \tag{2.114}
$$

$$
t_4 = r_4^T r_{54} \cdot r_4^T r_{64} - r_4^T r_{64} \cdot |r_{54}|^2, \tag{2.115}
$$

$$
t_5 = r_4^T r_{54} \cdot |r_{64}|^2 - r_4^T r_{64} \cdot r_4^T r_{64}, \tag{2.116}
$$

$$
t_6 = \log \left(\frac{|r_4^T r_{54} |r_{64}|^2 - r_4^T r_{64} \cdot r_4^T r_{64}}{t_1 t_2 + t_5}\right). \tag{2.117}
$$

Next, define

$$
I_R^* (|b_0|, |b_1|, b_0 + b_1, b_0^T b_1, (b_0 + b_1)^T b_1, n) = I_R(b_0, b_1, n). \tag{2.118}
$$
and evaluate

\[ I_{R40} = I_2^R(\sqrt{|r - P r|^2 + |r_4|^2}, |r_{54}|, \sqrt{|r - P r|^2 + |r_5|^2}, r_4^T r_{54}, r_4^2 r_{54}^2, 0), \quad (2.119) \]
\[ I_{R41} = I_2^R(\sqrt{|r - P r|^2 + |r_4|^2}, |r_{54}|, \sqrt{|r - P r|^2 + |r_5|^2}, r_4^T r_{54}, r_4^2 r_{54}^2, 1), \quad (2.120) \]
\[ I_{R60} = I_2^R(\sqrt{|r - P r|^2 + |r_6|^2}, |r_{56}|, \sqrt{|r - P r|^2 + |r_5|^2}, r_6^T r_{56}, r_6^2 r_{56}^2, 0), \quad (2.121) \]
\[ I_{R61} = I_2^R(\sqrt{|r - P r|^2 + |r_6|^2}, |r_{56}|, \sqrt{|r - P r|^2 + |r_5|^2}, r_6^T r_{56}, r_6^2 r_{56}^2, 1). \quad (2.122) \]

Numerical Integrals

The integrals to be evaluated numerically are \( I_{log,1,j} \), \( j = 0, 1, 2 \), \( I_{log,2,j} \), \( j = 0, 1 \), \( I_{at,j} \), \( j = 0, 1, 2 \) defined as

\[ I_{log,1,j} = \frac{1}{r_{64}|^2} \int_0^1 \alpha^j \log\left( \frac{(r_6 + \alpha r_{56})^T r_{64} + |r_6 + \alpha r_{56}| |r_{64}|}{(r_4 + \alpha r_{54})^T r_{64} + |r_4 + \alpha r_{54}| |r_{64}|} \right) d\alpha, \quad (2.123) \]
\[ I_{log,2,j} = \int_0^1 \alpha^j \log\left( \frac{|r_6 + \alpha r_{56}|^2}{|r_4 + \alpha r_{54}|^2} \right) d\alpha, \quad (2.124) \]
\[ I_{at,j} = \int_0^1 \alpha^j \frac{\arctan\left( \frac{(r_6 + \alpha r_{56})^T r_{64}}{R_1(\alpha)} \right) - \arctan\left( \frac{(r_4 + \alpha r_{54})^T r_{64}}{R_1(\alpha)} \right)}{R_1(\alpha)} d\alpha. \quad (2.125) \]

These integrals are evaluated using numerical integration, where an adaptive Romberg method is used.

Forbidden Domains and Node Reordering

Some of the integrals become infinite when origo of the triangle plane is in the wrong place. When \( |r - P r| = 0 \), the integral \( I_{R4j} \) becomes infinite if \( r_4 + \theta r_{54} = 0 \), for \( \theta \geq 0 \). The same happens for \( I_{R6j} \) when \( r_6 + \theta r_{56} = 0 \), for \( \theta \geq 0 \). The integral \( I_{at,j} \) is infinite whenever \( R_1(\alpha) = 0 \). This yields the forbidden strip \( r_4 + \alpha r_{54} + \theta r_{64} = 0, \alpha \in [0, 1] \) and \( \theta \in \mathbb{R} \). All other forbidden points are covered by those three cases. The forbidden strip and the two forbidden lines are indicated in figure 2.2.

In order to avoid the forbidden domains, the nodes in the triangle are reordered. The goal is to order the nodes such that origo is to the left of the shaded strip in figure 2.2. To find out how to order the nodes, we compute cross products \( s_j \), which gives information about the angles between \(-r_m\) and \( r_j - r_m \), for \( j = 1, 2, 3 \). If two angles \( \phi_{j1} \) and \( \phi_{j2} \) (or equivalently \( s_{j1} \) and \( s_{j2} \)) have different signs, then we know that we are on one out of two domains restricted by two lines \( r_{j1} - r_m \) and \( r_{j2} - r_m \). To exclude the wrong domain, we take one more cross product, to see on which side of \( r_{j2} - r_{j1} \), the origo is. The last check depends on the node orientation, which is also computed. A pseudo code is listed in algorithm 1.
Compute \( r_m = (r_4 + r_5 + r_6)/3 \);
Compute \( \text{orient} = r_{54} \times r_{64} \);
Compute \( s_4 = (-r_m) \times (r_4 - r_m) \);
Compute \( s_5 = (-r_m) \times (r_5 - r_m) \);
Compute \( s_6 = (-r_m) \times (r_6 - r_m) \):
\[ \text{if } s_{45} s_{56} < 0 \text{ and } \text{orient} \cdot (r_4 - r_5) < 0 \text{ then} \]
\[ \text{swap}(r_5, r_6); \]
\[ \text{else} \]
\[ s_{56} s_{64} < 0 \text{ and } \text{orient} \cdot (r_5 - r_6) < 0 \]
\[ \text{end} \]
\[ \text{swap}(r_4, r_5); \]

\[ \text{Algorithm 1: Triangle node reordering} \]

**Middle Step of Integration, Nonsingular Case**

Here we evaluate the integrals

\[
\text{Int}_1(n,m) = \int_0^1 \int_0^{1-\alpha} \frac{\alpha^n \beta^m}{R} d\beta d\alpha, \quad (2.126)
\]

\[
\text{Int}_2(n,m) = \int_0^1 \int_0^{1-\alpha} \frac{\alpha^n \beta^m}{R^2} d\beta d\alpha, \quad (2.127)
\]

\[
\text{Int}_R = \int_0^1 \int_0^{1-\alpha} R d\beta d\alpha. \quad (2.128)
\]
The interesting tuples $\{n, m\}$ are $(0, 0), (1, 0), (0, 1), (1, 1), (2, 0)$ and $(0, 2)$. These integrals can be computed using the analytical and numerical integrals evaluated previously.

\[
\begin{align*}
\text{Int}_1^{(0, 0)} & = I_{\log 1, 0}, \\
\text{Int}_1^{(1, 0)} & = I_{\log 1, 1}, \\
\text{Int}_1^{(0, 1)} & = \frac{I_{R60} - I_{R40} - r_i^T r_{64} I_{\log 1, 0} - r_j^T r_{64} I_{\log 1, 1}}{|r_{64}|^2}, \\
\text{Int}_1^{(1, 1)} & = \frac{I_{R61} - I_{R41} - r_i^T r_{64} I_{\log 1, 1} - r_j^T r_{64} I_{\log 1, 2}}{|r_{64}|^2}, \\
\text{Int}_1^{(2, 0)} & = I_{\log 1, 2}, \\
\text{Int}_1^{(0, 2)} & = \frac{1}{2|r_{64}|^4} \left( (3(r_i^T r_{64})^2 - |r_{54}|^2 |r_{64}|^2) \cdot I_{\log 1, 2} \\
& + (6(r_i^T r_{64})(r_i^T r_{64}) - 2r_i^T r_{54} |r_{64}|^2) \cdot I_{\log 1, 1} \\
& + (3(r_j^T r_{64})^2 - |r_{64}|^2 |r_{64}|^2) \cdot I_{\log 1, 0} \\
& + (r_j^T r_{64} - 4r_i^T r_{64}) \cdot I_{R60} + 3r_j^T r_{64} \cdot I_{R40} \\
& + (r_i^T r_{64} - 4r_i^T r_{64}) \cdot I_{R61} + 3r_i^T r_{64} \cdot I_{R41} \right). \tag{2.134}
\end{align*}
\]

\[
\begin{align*}
\text{Int}_2^{(0, 0)} & = I_{at, 0}, \\
\text{Int}_2^{(1, 0)} & = I_{at, 1}, \\
\text{Int}_2^{(0, 1)} & = \frac{\frac{1}{2} I_{\log 2, 0} - r_i^T r_{64} \cdot I_{at, 0} - r_j^T r_{64} \cdot I_{at, 1}}{|r_{64}|^2}, \\
\text{Int}_2^{(1, 1)} & = \frac{\frac{1}{2} I_{\log 2, 1} - r_i^T r_{64} \cdot I_{at, 1} - r_j^T r_{64} \cdot I_{at, 2}}{|r_{64}|^2}, \\
\text{Int}_2^{(2, 0)} & = I_{at, 0}, \\
\text{Int}_2^{(0, 2)} & = \frac{1}{|r_{64}|^4} \left( \frac{1}{2} |r_{64}|^2 - r_i^T r_{64} \cdot I_{\log 2, 0} - r_j^T r_{64} \cdot I_{\log 2, 1} \\
& - ((|r - Pr|^2 + |r_j|^2) |r_{64}|^2 - 2(r_i^T r_{64})^2) \cdot I_{at, 0} \\
& - (2r_i^T r_{54} |r_{64}|^2 - 4r_j^T r_{64} T_{54} r_{64}) \cdot I_{at, 1} \\
& - (|r_{54}|^2 |r_{64}|^2 - 2(r_i^T r_{64})^2) \cdot I_{at, 2} \right). \tag{2.140}
\end{align*}
\]

\[
\begin{align*}
\text{Int}_R & = \frac{1}{2|r_{64}|^2} \left( r_i^T r_{64} \cdot I_{R60} + r_j^T r_{64} \cdot I_{R61} \\
& - r_i^T r_{64} \cdot I_{R40} + r_j^T r_{64} \cdot I_{R41} \\
& + t_2 \cdot I_{in10} + 2t_5 \cdot I_{in11} + t_2^2 \cdot I_{in12} \right). \tag{2.141}
\end{align*}
\]
2.9. QUADRATURE

Middle Step of Integration, Singular Case

In the singular case, \(|r - Pr|^2 + |r_4|^2 = 0\). We evaluate the same integrals as in the non-singular case,

\[
\begin{align*}
\text{Int}_1(n, m) &= \int_0^1 \int_0^{\alpha} \frac{\alpha^n \beta^m}{R} \, d\alpha, \\
\text{Int}_2(n, m) &= \int_0^1 \int_0^{\alpha} \frac{\alpha^n \beta^m}{R^2} \, d\alpha, \\
\text{Int}_R &= \int_0^1 \int_0^{\alpha} R \, d\alpha,
\end{align*}
\]

where \(R = |\alpha r_{54} + \beta r_{64}|\). The integrands (2.142) and (2.143) become singular as \(\alpha, \beta \to 0\). The interesting tuples \((n, m)\) are \((0, 0), (1, 0), (0, 1), (1, 1), (2, 0)\) and \((0, 2)\). To get rid of the singularity, we perform a Duffy transform, \(\alpha = \xi_1 \xi_2\) and \(\beta = \xi_1 (1 - \xi_2)\). The Jacobian contribution is \(\xi_1\). Define \(R_s\) and we get

\[
\begin{align*}
R_s(\xi_2) &= |r_{64} + \xi_2 r_{56}|, \\
R(\alpha, \beta) &= \xi_1 R_s(\xi_2), \\
R_{1s} &= \sqrt{|r_6|^2 |r_{56}|^2 - (r_6^T r_{56})^2}. \tag{2.147}
\end{align*}
\]

Consider

\[
\begin{align*}
\text{Int}_p(n, m) &= \int_0^1 \int_0^{\alpha} \frac{\alpha^n \beta^m}{R^p} \, d\alpha, \\
&= \int_0^1 \xi_1^{1+n+m-p} d\xi_1 \cdot \int_0^1 \frac{\xi_2^m (1 - \xi_2)^m}{R_s^p(\xi_2)} d\xi_2 \\
&= \frac{1}{2 + n + m - p} \int_0^1 \xi_2^m (1 - \xi_2)^m \frac{R_s^p(\xi_2)}{R_s^p(\xi_2)} d\xi_2. \tag{2.148}
\end{align*}
\]

This is valid in the case \(2 + n + m > p\). Otherwise the integral is improper. Hence, \(\text{Int}_2(0, 0)\) is an improper integral. This integral is not needed, as we will see in the last step of integration. Define

\[
\begin{align*}
\text{Int}_{p_0}(n) &= \int_0^1 \frac{\xi_2^n}{R_s^p(\xi_2)} \, d\xi_2, \tag{2.149}
\end{align*}
\]
which can be evaluated analytically,

\[
\begin{align*}
\text{Int}_1^1(0) &= \frac{1}{|r_{56}|} \log \left( \frac{r_5^T r_{56} + |r_5||r_{56}|}{r_6^T r_{56} + |r_6||r_{56}|} \right), \\
\text{Int}_1^1(1) &= \frac{|r_5| - |r_6|}{|r_{56}|^2} - \frac{r_5^T r_{56}}{|r_{56}|^2} \text{Int}_1^1(0), \\
\text{Int}_1^1(2) &= \frac{1}{2|r_{56}|^2} \left( (|r_{56}|^2 - 3r_6^T r_{56})|r_5| + 3r_6^T r_{56}|r_6| \\
&\quad + (3r_6^T r_{56})^2 - |r_6|^2|r_{56}|^2 \text{Int}_1^1(0) \right), \\
\text{Int}_2^2(0) &= \frac{\arctan \left( \frac{r_5^T r_{56}}{R_{1s}} \right) - \arctan \left( \frac{r_6^T r_{56}}{R_{1s}} \right)}{R_{1s}}, \\
\text{Int}_2^2(1) &= \frac{\log \left( |r_5|/|r_6| \right) - r_6^T r_{56} \cdot \text{Int}_2^2(0)}{|r_{56}|^2}, \\
\text{Int}_2^2(2) &= \frac{1}{|r_{56}|^4} \left( |r_{56}|^2 - 2r_6^T r_{56} \log \left( |r_5|/|r_6| \right) \\
&\quad + (2r_6^T r_{56})^2 - |r_6|^2|r_{56}|^2 \text{Int}_2^2(0) \right).
\end{align*}
\]

The required integrals can be evaluated

\[
\begin{align*}
\text{Int}_1(0, 0) &= \text{Int}_1^1(0), \\
\text{Int}_1(1, 0) &= \frac{1}{2} \text{Int}_1^1(0), \\
\text{Int}_1(0, 1) &= \frac{1}{2} (\text{Int}_1^1(0) - \text{Int}_1^1(1)), \\
\text{Int}_1(1, 1) &= \frac{1}{3} (\text{Int}_1^1(1) - \text{Int}_1^1(2)), \\
\text{Int}_1(2, 0) &= \frac{1}{3} \text{Int}_1^1(0), \\
\text{Int}_1(0, 2) &= \frac{1}{3} (\text{Int}_1^1(0) - 2\text{Int}_1^1(1) + \text{Int}_1^1(2)), \\
\text{Int}_2(0, 0) &= [\text{Improper}], \\
\text{Int}_2(1, 0) &= \text{Int}_2^2(1), \\
\text{Int}_2(0, 1) &= \text{Int}_2^2(0) - \text{Int}_2^2(1), \\
\text{Int}_2(1, 1) &= \frac{1}{2} (\text{Int}_2^2(1) - \text{Int}_2^2(2)), \\
\text{Int}_2(2, 0) &= \frac{1}{2} \text{Int}_2^2(2), \\
\text{Int}_2(0, 2) &= \frac{1}{2} (\text{Int}_2^2(0) - 2\text{Int}_2^2(1) + \text{Int}_2^2(2)).
\end{align*}
\]
2.9. QUADRATURE

The integral $\text{Int}_R$ is evaluated from

$$\text{Int}_R = I_R(r_{64}, r_{56}, 0),$$

(2.168)

with $I_R$ defined in the nonsingular case.

**Last Step of the Integration**

The last step of the integration is to combine the integrals (2.126)-(2.128) in the middle step, in order to evaluate the integrals $J^0_p$. At this point, it does not matter if the integrand was singular in the middle step of integration. When computing $J^0_1$, we evaluate

$$J^0_1(p, j) = \int \frac{r'_j}{cR} \Phi_m(r')dS, \quad j = 1, 2, \quad p = 1, 2,$$

(2.169)

where

$$r'_j = r_{4,j} + \alpha r_{54,j} + \beta r_{64,j},$$

$$\Phi_m(r') = a_0 + a_1 \alpha + a_2 \beta.$$  

(2.170, 2.171)

The integrals $J^0_1(p, j)$ and $J^0_2(p, j)$ are linear combinations of the integrals $\text{Int}_p(n, m)$, computed for both the nonsingular and singular cases. It should be noted that in the singular case $r_{4,j} = 0$, and we get no contribution from $\text{Int}_p(0, 0)$ which is an improper integral. The integral $J^0_3$ is a linear combination of $\text{Int}_p(n, m)$ and $\text{Int}_R$.

**Case $\omega > 0$**

Our goal is to evaluate the integrals $J^\omega_p$, $p = 1, 2, 3$, introduced earlier in (2.89)-(2.91). We focus on

$$\int \frac{r'_j}{cR} e^{-\frac{\pi \omega}{cR}} \Phi_m(r')dS, \quad \int \frac{1}{R} e^{-\frac{\pi \omega}{cR}} \Phi_m(r')dS$$

since the other integrals are either special cases of those above or evaluated in the case when $\omega = 0$. We have not found any analytical expressions for these integrals, and numerical quadrature is used. As a first step we evaluate the integrals

$$\int_0^1 \int_0^{1-\alpha} \frac{1, \alpha, \beta, \alpha \beta, \alpha^2, \beta^2}{R^2} e^{-\frac{\pi \omega}{cR}d3\alpha}, \quad \int_0^1 \int_0^{1-\alpha} \frac{1, \alpha, \beta}{R} e^{-\frac{\pi \omega}{cR}d3\alpha},$$

numerically. Those integrals are then combined to get the required integrals, for different $r'_j$ and $\Phi_m(r')$. In the singular case, we perform the Duffy transform, $\alpha = \xi_1 \xi_2$ and $\beta = \xi_1(1 - \xi_2)$. When $\omega > 0$, the integrals do not decouple, which leads to numerical integration over a square. This is done by integrating over two triangles with the same algorithm as in the nonsingular case.
CHAPTER 2. TIME DOMAIN INTEGRAL EQUATION METHOD

Integration of a Circle Sector

Local Coordinates on a Circle Sector

We want to integrate over a circle sector in the $K'$ triangle plane with center in $P_r$. The circle sector has radius $v_2$ and angles between $\phi_1$ and $\phi_2$ in the $K'$ triangle plane. We have the parameterization

$$ r'(v, \phi) = P_r + v \cos \phi e_{1}' + v \sin \phi e_{2}' $$

where

$$ v \in [0, v_2], \quad \phi \in [\phi_1, \phi_2], $$

and $e_{j}'$ are the basis functions for the $K'$ triangle plane. The integral to be computed is

$$ \int f(r') dr' = \int_0^{v_2} \int_{\phi_1}^{\phi_2} v f(r'(v, \phi)) d\phi dv. $$

We need to express $x$ and $y$ that appears in $\Phi_{K'}^m$ in $\alpha$ and $\beta$ by solving the system (2.26), with $r = r'(v, \phi)$. The spatial basis function can then be computed

$$ \Phi_{K'}^m(r'(v, \phi)) = a_0 + a_1 v \cos \phi + a_2 v \sin \phi, $$

following the same procedure as in the triangle case.

Elimination of $\phi$

It should be observed that $R = \sqrt{|r - P_r|^2 + v^2}$ is independent of $\phi$. This means that $\phi$ only appears in $r_j'$ and $\Phi_m(r')$, when computing the integrals $J_p^\omega$. Let $f(R)$ be the part of $J_p^\omega$ that is independent of $\phi$ and consider the integrals

$$ \int v \Phi_m(r') f(R) d\phi dv = \int \left( \int v \Phi_m(r') d\phi \right) f(R) dv, $$

$$ \int v r_j' \Phi_m(r') f(R) d\phi dv = \int \left( \int v r_j' \Phi_m(r') d\phi \right) f(R) dv. $$

From (2.176) we obtain the integral

$$ \int v \Phi_m(r') d\phi = b_0 v + b_1 v^2, $$

where

$$ b_0 = (\phi_2 - \phi_1)a_0, $$

$$ b_1 = (\sin \phi_2 - \sin \phi_1)a_1 - (\cos \phi_2 - \cos \phi_1)a_2. $$
Since \( r'_1 = v \cos \phi \) and \( r'_2 = v \sin \phi \), we obtain the integrals

\[
\int vr'_1 \Phi_m(r') d\phi = b_{01} v^2 + b_{11} v^3, \\
\int vr'_2 \Phi_m(r') d\phi = b_{02} v^2 + b_{12} v^3,
\]

from (2.177) where

\[
b_{01} = (\sin \phi_2 - \sin \phi_1) a_0, \\
b_{11} = (t_1 + t_2) a_1 + t_3 a_2, \\
b_{02} = -(\cos \phi_2 - \cos \phi_1) a_0, \\
b_{12} = t_3 a_1 + (t_1 - t_2) a_2, \\
t_1 = \frac{1}{2} (\phi_2 - \phi_1), \\
t_2 = \frac{1}{2} (\cos \phi_2 \sin \phi_2 - \cos \phi_1 \sin \phi_1), \\
t_3 = -\frac{1}{2} (\cos^2 \phi_2 - \cos^2 \phi_1).
\]

Case \( \omega = 0 \)

When the integral over \( \phi \) has been eliminated, the integrals \( J_p^0 \) can be reduced to the cases

\[
Int_1(p) = \int_0^{v_2} \frac{v^p}{\sqrt{|r - Pr|^2 + v^2}} dv, \quad p = 1, 2, 3, \\
Int_2(p) = \int_0^{v_2} \frac{v^p}{|r - Pr|^2 + v^2} dv, \quad p = 2, 3, \\
Int_R = \int_0^{v_2} v \sqrt{|r - Pr|^2 + v^2} dv, \\
Int_0 = \int_0^{v_2} v dv.
\]
These can be computed analytically and we obtain

\[ \text{Int}_1(1) = \frac{v_2^2}{\sqrt{|r - Pr|^2 + v_2^2} + \sqrt{|r - Pr|^2}}, \quad (2.194) \]

\[ \text{Int}_1(2) = \frac{v_2^2}{2} \sqrt{|r - Pr|^2 + v_2^2} - \frac{|r - Pr|^2}{2} \log \left( \frac{v_2 + \sqrt{|r - Pr|^2 + v_2^2}}{|r - Pr|} \right), \quad (2.195) \]

\[ \text{Int}_1(3) = \frac{v_2^2}{3} \left( \sqrt{|r - Pr|^2 + v_2^2} - \frac{2|r - Pr|^2}{(\sqrt{|r - Pr|^2 + v_2^2} + |r - Pr|)} \right), \quad (2.196) \]

\[ \text{Int}_2(2) = v_2 - |r - Pr| \arctan \left( \frac{v_2}{d_2} \right), \quad (2.197) \]

\[ \text{Int}_2(3) = \frac{v_2^2}{2} - |r - Pr|^2 \log \left( \frac{\sqrt{|r - Pr|^2 + v_2^2}}{|r - Pr|} \right), \quad (2.198) \]

\[ \text{Int}_R = \left( \frac{\sqrt{|r - Pr|^2 + v_2^2}}{3} - |r - Pr|^3 \right), \quad (2.199) \]

\[ \text{Int}_0 = \frac{v_2^2}{2}. \quad (2.200) \]

In the singular case, when \(|r - Pr| = 0\), the integrals are

\[ \text{Int}_1(1) = v_2, \quad (2.201) \]

\[ \text{Int}_1(2) = \frac{v_2^2}{2}, \quad (2.202) \]

\[ \text{Int}_1(3) = \frac{v_2^3}{3}, \quad (2.203) \]

\[ \text{Int}_2(2) = v_2, \quad (2.204) \]

\[ \text{Int}_2(3) = \frac{v_2^2}{2}, \quad (2.205) \]

\[ \text{Int}_R = \frac{v_2^2}{3}, \quad (2.206) \]

\[ \text{Int}_0 = \frac{v_2^2}{2}. \quad (2.207) \]

The integrals \( J_0^p \) are linear combinations of the integrals above.

**Case \( \omega > 0 \)**

After eliminating the integral over \( \phi \), we focus on the integrals

\[ \text{Int}_1(p) = \int_0^{v_2} \frac{v^p e^{-\frac{\pi R}{R}}}{R^p} dv, \quad p = 1, 2, \quad (2.208) \]

\[ \text{Int}_2(p) = \int_0^{v_2} \frac{v^p e^{-\frac{\pi R}{R^2}}}{R^p} dv, \quad p = 2, 3. \quad (2.209) \]
All other integrals are evaluated for the case $\omega = 0$. We can compute
\[
\text{Int}_t(1) = -\frac{c}{\omega} \left( e^{-\frac{\omega}{2} \sqrt{|r - Pr|^2 + v^2}} - e^{-\frac{\omega}{2} |r - Pr|} \right)
\] (2.210)
analytically. The other integrals are computed numerically, using an adaptive Romberg method.

**Integration of a Circle**

This is a special case of the circle sector case with $\phi_1 = 0$ and $\phi_2 = 2\pi$. In the elimination of the integral over $\phi$, we get
\[
b_0 = 2\pi a_0, \quad b_{11} = \pi a_1, \quad b_{12} = \pi a_2, \quad b_1 = b_{01} = b_{02} = 0.
\]
Some of the integrals over the circle sector can therefore be omitted.

**Integration of a Circle Segment**

When the domain is a circle segment, we have two alternatives in evaluating the integral. The first alternative is to integrate the corresponding circle sector and subtract the corresponding triangle. The other alternative is to evaluate the integrals numerically, using Gaussian quadrature. The second approach is good for small angles. If the angle increases, then $R$ is not approximately constant and the Gaussian quadrature is no longer usable. Then we use the first approach.

If Gaussian quadrature is to be used, the integrand has to behave like a polynomial, i.e. the distance term has to be close to constant. The worst case is when $|r - Pr| = 0$. If $R$ is allowed to vary
\[
|r' - Pr| \in [(1 - \varepsilon)v, v],
\]
the maximum angle can be computed as
\[
\Delta\phi = 2 \arccos(1 - \varepsilon).
\]
For small $\varepsilon$, $1/|r' - Pr|$ can be considered to be constant. For instance, if $\varepsilon = 0.1$, angles up to $51.7^\circ$ are allowed. Numerical experiments show that it is sufficient to use 10 gauss points for $\alpha$ and 4 gauss points for $\beta$ to evaluate the integrals to machine precision, in the case when $|r - Pr| = 0$ and $\Delta\phi = 45^\circ$. When $\Delta\phi = 22.5^\circ$, it suffices with 6 gauss points for $\alpha$ and 2 gauss points for $\beta$ to get at least 14 digits accuracy. The rest of the chapter will treat the evaluation of the integral numerically.

**Local Coordinates on a Circle Segment**

We want to integrate over a circle segment in the $K'$ triangle plane with center in $Pr$ and radius $v$. Let $r'_\phi(\alpha)$ denote a point on the circle between the angles $\phi_1$ and
\( \phi_2 \) and \( r_f'(\alpha) \) denote a point on the line between the same angles. Introducing the notation

\[
\begin{align*}
c(\alpha) &= (1 - \alpha) \cos \phi_1 + \alpha \cos \phi_2, \\
s(\alpha) &= (1 - \alpha) \sin \phi_1 + \alpha \sin \phi_2, \\
\phi(\alpha) &= (1 - \alpha) \phi_1 + \alpha \phi_2,
\end{align*}
\]

(2.211) (2.212) (2.213)
yields the parameterization

\[
\begin{align*}
r_f'(\alpha) &= Pr + v \cos(\phi(\alpha))e_1^{K'} + v \sin(\phi(\alpha))e_2^{K'}, \\
r_f'(\alpha) &= Pr + vc(\alpha)e_1^{K'} + vs(\alpha)e_2^{K'}
\end{align*}
\]

(2.214) (2.215)
where \( v \) is the radius of the circle. The points on the circle segment can be parameterized

\[
r'(\alpha, \beta) = (1 - \beta)r_f'(\alpha) + \beta r_f'(\alpha).
\]

(2.216)
The integral to be computed is

\[
\int f(r')dr' = \int_0^1 \int_0^1 f(r'(\alpha, \beta)) \left| \frac{dr'(\alpha, \beta)}{d(\alpha, \beta)} \right| \ d\alpha d\beta.
\]

(2.217)
Integration over \( \alpha \) and \( \beta \) yields a Jacobian contribution

\[
\frac{dr'}{d(\alpha, \beta)} = v^2 (1 - \beta)(\phi_2 - \phi_1) [1 - c(\alpha) \cos \phi - s(\alpha) \sin \phi] +
+ v^2 \beta [(s_2 - s_1)(\cos \phi(\alpha) - c(\alpha)) - (c_2 - c_1)(\sin \phi(\alpha) - s(\alpha))],
\]

(2.218)
where \( c_j = \cos(\phi_j) \) and \( s_j = \sin(\phi_j) \).

For the case of a circle segment, the basis function can be written

\[
\Phi^K_j = a_0 + a_1 \beta + a_2 \alpha \beta + a_3 \cos \phi + a_4 \sin \phi + a_5 \cos \phi + a_6 \beta \sin \phi,
\]

(2.219)
where \( \Phi^K_j = \Phi^K_j(r'(\alpha, \beta)) \) and \( \phi = \phi(\alpha) \). Using the two representations of \( r' \), we derive a matrix system, as in (2.26), where the solution are the parameters \( x \) and \( y \). The system is

\[
\begin{pmatrix} r_{21} \cdot r_{21} & r_{31} \cdot r_{21} \\
r_{21} \cdot r_{31} & r_{31} \cdot r_{31} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_2 \\ b_3 \end{pmatrix},
\]

(2.220)
where

\[
\begin{align*}
c^0 &= (-r_1, r_{j1}), \\
c^1 &= \(e_1, r_{j1}), \quad (2.221) \\
c^2 &= \(e_2, r_{j1}), \quad (2.222) \\
b_j &= c^0_j + v[c^1_j \cos(\phi_1) + c^2_j \sin(\phi_1)]\beta \\
&\quad + v[c^1_j \cos(\phi_2) - \cos(\phi_1)] + c^2_j \sin(\phi_2) - \sin(\phi_1)]\alpha \beta \\
&\quad + v[c^1_j \cos(\phi) + vc^2_j \sin(\phi) - vc^2_j \beta \cos(\phi) - vc^2_j \beta \sin(\phi)].
\end{align*}
\]

(2.223) (2.224)
Solving this system of equations for $a_p$ with seven different right hand sides, yields the constants $a_{p, p = 0, 1, \ldots, 6}$, for the different spatial basis functions. The computation of $J_{p}^0$ and $J_{p}^\omega$ is now direct, using Gaussian quadrature.

2.10 The Marching On In Time Method

In this section we will discuss how to construct a computational algorithm out of the different variational formulations obtained in the previous chapters. Variational formulations containing retarded potentials have a coupling in space and time, where the solutions at a time depend on solutions at other times. In the case of constant elements in time, the solution only depends on solutions from the past. We can then step forward in time in an explicit way. If we are using higher order elements, the solution also depends on later time steps, which makes the scheme implicit. We will only consider the case with an explicit scheme.

In section 2.10, the matrix structure in the method is discussed. The method results in a lower triangular block matrix, which can be solved with forward substitution.

In section 2.10, the assembly procedure is explained. The integral over the spatial test functions are evaluated using numerical quadrature. For each quadrature point on each test triangle, a strip is obtained, with a radius depending on the difference in basis function index in time. Those strips cover parts of some triangles, and these triangle parts are detected.

Matrix Structure in MOT

The solution of the wave equation based on one of the variational formulations discussed earlier is done in two steps:

- Assembly of the matrix of the system
- Solving the system (or stepping in time)

For test functions $J_{j,k}(r,t) = \Phi_j(r)\tilde{\Psi}_k(t)$, defined in section 2.2, we get the system

$$\sum_{m,l} A_{j,m,k-l}J_{ml} = b_{jk}.$$  

For the constant time element case, when $k < l$ in our variational formulations, then $A_{j,m,k-l} = 0$. Ordering the vector $J_l$, $b_k$ and the matrix $A_u$, with $u = k - l$ s.t.

$$(J_l)_m = J_{ml}, \quad (b_k)_j = b_{jk}, \quad (A_u)_{j,m} = A_{j,m,u},$$

$$
\sum_{m,l}$$

Solving this system of equations for $a_p$ with seven different right hand sides, yields the constants $a_{p, p = 0, 1, \ldots, 6}$, for the different spatial basis functions. The computation of $J_{p}^0$ and $J_{p}^\omega$ is now direct, using Gaussian quadrature.
the following block matrix system

$$
\begin{pmatrix}
A_0 & A_1 & A_0 & A_2 & A_1 & A_0 & \cdots \\
A_1 & A_0 & A_1 & A_2 & A_1 & A_0 & \cdots \\
A_2 & A_1 & A_0 & A_2 & A_1 & A_0 & \cdots \\
A_3 & A_2 & A_1 & A_2 & A_1 & A_0 & \cdots \\
\vdots & A_3 & A_2 & A_1 & A_2 & A_1 & \cdots \\
\vdots & \vdots & A_3 & A_2 & A_1 & A_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & A_3 & A_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & A_3 & \cdots \\
\end{pmatrix}
= \begin{pmatrix}
J_1 \\
J_2 \\
J_3 \\
\vdots
\end{pmatrix} = \begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
\vdots
\end{pmatrix}
$$

(2.225)

is obtained. The system is solved with forward substitution,

$$
J_k = A_0^{-1} \left( b_k - \sum_{p=0}^{P} A_p J_{k-p} \right),
$$

(2.226)

where $J_q = 0$, if $q \leq 0$. The smallest sphere that includes the whole scatterer has diameter $\lesssim Pc\Delta t$. This diameter determines the number of terms in the sum.

**Assembly of Matrix Block $A_u$**

In the assembly process, we have to compute triple integrals, where two of the integrals are over the scatterer and one integral is over time. The time integral is computed analytically. The outer integration is done with a numerical integration method,

$$
\int \int f(r, r')dK'dK = \sum_p w_p \int f(r_p)dK',
$$

where $w_p$ depends on the quadrature formula, e.g. Gaussian quadrature on a triangle. For each triangle $K$, and for all the integration points $r_p$, we get a strip over some of the triangles $K'$ in the inner integral. The integrals over those strips are calculated and assembled into the matrix.
2.10. THE MARCHING ON IN TIME METHOD

![Diagram of triangles]

Figure 2.3: For each integration point on triangle $K$ (left) yields a strip over triangle $K'$ (right).

A pseudo code for the assembly process of the matrix blocks $A_u$ is

```plaintext
for all triangles $K$ do
    for all quadpoints $r$ on triangle $K$ do
        for all triangles $K'$ do
            First selection of admissible time basis differences;
            $u = k - l$, $p_1 \leq u \leq q_1$;
            for $u = p_1$ to $q_1$ do
                Find domain $D \subseteq K'$ that interact with quadpoint $r$ with current $u$;
                if domain $D \neq \emptyset$ then
                    Integrate over $D$;
                    Assemble the matrix with the integral value;
                end
            end
        end
    end
end
```

**Algorithm 2**: Assembly process of matrix blocks

**First Selection of Admissible Time Differences**

Since the basis functions have compact support, for fixed triangles $K$ and $K'$ and time $k\Delta t$, we only get contribution from finite number of time steps at times $l\Delta t$. When using constant time elements, we get a sharp condition for the admissible time basis difference $u = k - l$,

$$\min_{r \in K, r' \in K'} \frac{|r - r'|}{c\Delta t} - 1 < u < \max_{r \in K, r' \in K'} \frac{|r - r'|}{c\Delta t} + 1,$$
where $\Psi_k$ and $\Psi_l$ are the interacting basis functions in time. Let the midpoint $r_c^K$ of a triangle $K$ be the mean value of its corners. The triangle radius is defined as
\[
\text{rad}(K) = \max_j |r^K_j - r_c^K|,
\]
where $r^K_j$ are the nodes of the triangle $K$. To estimate the bounds on $u$, we can use the midpoint and radius of the triangles. The first selection of admissible time differences is
\[
\frac{|r^K_c - r_c^{K'}| - \text{rad}(K) - \text{rad}(K')}{c\Delta t} - 1 \leq u < \frac{|r^K_c - r_c^{K'}| + \text{rad}(K) + \text{rad}(K')}{c\Delta t} + 1.
\]
The radii $\text{rad}(K)$ can be precalculated before the assembly process.

**Find Domain on $K'$**

Let $K$ and $u = k - l$ be given. A point $r \in K$ interacts with points $r'$ lying within two spherical shells $v_{\text{min}} \leq |r - r'| \leq v_{\text{mid}}$ and $v_{\text{mid}} \leq |r - r'| \leq v_{\text{max}}$, s.t.
\[
\begin{align*}
v_{\text{min}} &= (u - 1)c\Delta t, \\
v_{\text{mid}} &= uc\Delta t, \\
v_{\text{max}} &= (u + 1)c\Delta t.
\end{align*}
\]

We want to find the domain on triangle $K'$ that intersects with this spherical shell. The triangle plane of $K'$ cuts out a circle from the sphere. The center is the projection $Pr$ of $r$ onto the $K'$-plane. The radius $Pv$ of the circle is calculated by
\[
Pv = \sqrt{v^2 - |r - Pr|^2},
\]
where $v \in [v_{\text{min}}, v_{\text{max}}]$, see figure 2.4. If $v^2 - |r - Pr|^2$ is negative, the sphere has not reached the triangle plane. We need the circles corresponding to $v_{\text{min}}$, $v_{\text{mid}}$ and $v_{\text{max}}$.

We get the integral of a strip over the triangle by calculating over two domains as indicated in figure 2.5. Each triangle pair $(K, K')$ and each time basis function difference $u$ requires three integrals where the domain is the intersection of a circle and a triangle. Each of the three integral domains are subdivided into triangle, circle, circle sector and circle segment domains, which can be computed, either exact or with numerical integration. The integration is discussed in Section 2.9.

**Circle Intersecting a Triangle**

We want to calculate the intersections between a circle and a triangle. The only time we can get an odd number of intersections is the case when the circle intersect the triangle in a corner and the case when a triangle side is a tangent to the circle. If we treat these as special cases, we can identify the intersected domain with the
2.10. THE MARCHING ON IN TIME METHOD

Figure 2.4: The projection of a sphere with center in \( r \in K \) yields a circle lying in the triangle plane of \( K' \).

Figure 2.5: Integration of a strip over triangle \( K' \)

number of intersection points and the number of triangle corners inside the circle. There are two different cases, when the projection \( Pr \) of the point \( r \in K \) are inside or outside the triangle \( K' \). Each of the cases can be divided in another eight different cases, depending on the number of triangle corners that is inside the circle and the number of intersections between the circle and triangle, see figure 2.6. All of the 16 cases can be constructed from triangles, circles and circle segments. The integrals arising in the different variational formulations are treated in Section 2.9.
CHAPTER 2. TIME DOMAIN INTEGRAL EQUATION METHOD

50

2.11 Stabilization

Our Marching On in Time method in this chapter is on the form

\[ A_0 J_n + A_1 J_{n-1} + \ldots + A_k J_{n-k} = b_n, \]

\[ J_p = 0, \quad p \leq 0. \]

(2.231)

where \( A_p \in \mathbb{R}^{N \times N} \), \( J_p, b_n \in \mathbb{R}^N \) and \( N \) is the number of nodes on the scatterer surface. Moreover we require \( b_p = 0 \), for \( p \leq 0 \). It is assumed that \( A_0 \) is nonsingular, otherwise the solution does not exist or is non-unique. When we solve the Dirichlet problem with the variational formulation 1 coming from the Kirchhoff formula, we have some eigenvalues close to \(-1\). If those eigenvalues leaves the unit circle due to numerical errors, the time marching scheme becomes unstable.

In [23], P.J. Davies and D.B. Duncan uses a time averaging scheme, with a filter \( J_n \rightarrow \frac{1}{4} (J_{n+1} + 2J_n + J_{n-1}) \). The scheme is listed in algorithm 3.

\[ \sigma_n = b_n - \sum_{p=2}^{k} A_p J_{n-p}; \]
\[ J^*_n = A_0^{-1} (\sigma_n - A_1 J^*_n); \]
\[ J_{n-1} = \frac{1}{4} (J^*_n + 2J^*_n - J_{n-1}); \]
\[ J^{**}_n = A_0^{-1} (\sigma_n - A_1 J_{n-1}); \]

Algorithm 3: Averaging scheme of Davies and Duncan

In algorithm 3, \( J^*_n \) and \( J^{**}_n \) are stored only temporarily. We will see that this averaging scheme is a natural choice. To do this we define what we mean with stability.
2.11. STABILIZATION

Definition 4. The algorithm (2.230) is stable iff

\[ \|J_n\| \leq C_1 \sum_{p=0}^{n} \|b_p\|. \tag{2.232} \]

where the constant \(C_1\) does not depend on \(n\).

Two different problem types can be distinguished:

- Finite object, with fixed number of matrix blocks \(k\) (for given \(\Delta t\)).
- Infinite object, with increasing matrix blocks \(k\) (for given \(\Delta t\)).

Here we only consider the case of a finite object.

Stability Analysis for a Finite Object

By writing (2.230) as a one step method, we can examine the stability.

\[ \tilde{J}_n = \tilde{A}_k \tilde{J}_{n-1} + \tilde{b}_n, \quad \tilde{J}_0 = 0, \tag{2.233} \]

\[ \tilde{A}_k = \begin{pmatrix} -A_0^{-1} & A_1 & \cdots & -A_{k-1} & -A_k \\ I & \ddots & & & I \\ & & \ddots \end{pmatrix}, \tag{2.234} \]

\[ \tilde{J}_n^T = (J_n, \ldots, J_{n-k}), \quad \tilde{b}_n^T = \left( (A_0^{-1}b_n)^T, 0, \ldots, 0 \right) . \tag{2.235} \]

Lemma 7. If the eigenvalues of \(\tilde{A}_k\) are strictly inside the unit circle, then the algorithm (2.230) is stable.

Proof. Since \(\tilde{J}_0 = 0\), we have

\[ \tilde{J}_n = \sum_{p=0}^{n} \tilde{A}_k^p \tilde{b}_{n-p} \]

\[ = [ \text{Jordan decomposition } \tilde{A}_k = \tilde{S} \tilde{K} \tilde{S}^{-1} ] \]

\[ = \sum_{p=0}^{n} \tilde{S} \tilde{K}^p \tilde{S}^{-1} \tilde{b}_{n-p} \tag{2.237} \]

If all eigenvalues of \(\tilde{A}_k\) are strictly inside the unit circle, then \(\|\tilde{S} \tilde{K}^p \tilde{S}^{-1}\| \leq C\) and \(\tilde{J}_n\) is bounded by

\[ \|\tilde{J}_n\| \leq C \sum_{p=0}^{n} \|\tilde{b}_{n-p}\|. \tag{2.238} \]
The constant $C$ depends on the largest eigenvalue and the size of the largest Jordan block, which both are independent of $n$. For the original problem, this yields

$$\|J_n\| \leq C_1 \sum_{p=0}^{n} \|b_p\|, \quad C_1 = \|A_0^{-1}\|C; \quad (2.239)$$

i.e. the constant $C_1$ does not depend on $n$. □

A stabilizing filter should have at least two properties

- Sufficiently high order in time, not to destroy the time order
- Increase the stability region of the scheme

Such a filter is

$$\tilde{J}_n \rightarrow \frac{1}{4} \left( \tilde{J}_{n+1} + 2\tilde{J}_n + \tilde{J}_{n-1} \right). \quad (2.240)$$

We can show the following

**Lemma 8.** The filter (2.240) applied on the scalar scheme $y_n = \lambda y_{n-1}$ is second order accurate in time and is stable for $\lambda$ such that $|\lambda + 1| < 2$.

**Proof.** The second order accuracy is shown by Taylor expansion of $y_{n\pm1}$. Applying the filter, yields

$$y_n \rightarrow \frac{1}{4} (y_{n+1} + 2y_n + y_{n-1}) = \left( \frac{\lambda + 1}{2} \right)^2 y_{n-1}. \quad (2.241)$$

This shows that the stability region is $\lambda$ such that $|\lambda + 1| < 2$. □

Using the filter yields the following modified algorithm

$$\tilde{J}_n = \left( \frac{\tilde{A}_k + 1}{2} \right)^2 \tilde{J}_{n-1} + \frac{1}{4} \left( \tilde{A}_k \tilde{b}_n + 2\tilde{b}_n + \tilde{b}_{n+1} \right), \quad \tilde{J}_{-1} = 0. \quad (2.242)$$

If we require $\tilde{b}_1 = 0$, then we can have the same initial condition as in the non-stabilized algorithm, i.e. $\tilde{J}_0 = 0$. In the case of solving the wave equation, we can delay the incident transient by $\Delta t$, to force $b_1 = 0$. Writing the filter on the $k$-step form yields algorithm 4.
2.12. NUMERICAL EXAMPLES

\[ \sigma_{n+1} = b_{n+1} - \sum_{p=2}^{k} A_p J_{n+1-p}; \]
\[ J_{n+1} = A_0^{-1} (\sigma_{n+1} - A_1 J^*_{n+1}); \]
\[ J_n = \frac{1}{2} (J_{n+1} + J^*_{n+1} + J_{n-1}); \]
\[ J_{n-1} = \frac{1}{2} (J^*_{n} + 2J_{n-1} + J_{n-2}); \]
\[ J_{n-p} = \frac{1}{2} (J_{n-p+1} + 2J_{n-p} + J_{n-p-1}), \quad p = 2, \ldots, k-1; \]
\[ J^*_n = A_0^{-1} (\sigma_{n+1} - A_1 J_n); \]

Algorithm 4: New averaging scheme

The algorithm updates the last \( k \) time-steps of \( J \) in each iteration. This means that \( J_{n-k} \) has an \( k \Delta t^2 \) error. In case of a wave propagation algorithm, \( k \) depends on the size of the scatterer as \( k \sim (\text{Size of scatterer})/c \Delta t \). The error of the filter increases with the object size. If we ignore all but the first smoothing step, then this scheme becomes identical to the scheme proposed by Davies and Duncan. This will actually be the most stable choice. In the numerical experiments we will only use one smoothing step.

2.12 Numerical Examples

Numerical Experiments On Kirchhoff Integral Equation

In this section, we will present numerical tests of the variational formulation 1. Two test cases are considered; a point source illuminating a plate and a plane wave illuminating a sphere. Those two cases have analytical solutions.

In the case of a point source illuminating a plate, we use the analytic solution from an infinite plane. In order to determine if the method in stable, we compute eigenvalues of the system matrix in the one-step method (2.233). We conclude that it is necessary to use the stabilization filter in algorithm 3. The stability of the method is related to the number of numerical integration points in the integral over the test functions in space. The method has first order of accuracy in time and second order in space in the computation of the potential on the surface. The computed scattered fields are compared with analytical solutions.

In the case of a plane wave illuminating a sphere, the computed scattered fields are compared with analytical solutions at three different observation points.

Test Case With a Plate

Consider a Dirichlet plate \((x, y) \in [-5, 5]^2, z = 0\) as in figure 2.7. We want to compute the potential \( J^{\text{CFL}}(r, t) \) on the plate for a point source in \( r_s = (0, 0, 2) \), for different CFL-numbers, where \( CFL = \frac{c}{h} \Delta t \) and \( h \) is the length of the largest edge.
on the scatterer. The point source produce an incoming wave

\[ u^{inc}(\mathbf{r}, t) = \frac{f^\infty((t - |\mathbf{r} - \mathbf{r}_s|/c_0 - t_0)/T)}{|\mathbf{r} - \mathbf{r}_s|} \]

(2.243)

where \( t_0 \) is a time delay and \( T \) determines the width of the pulse. The velocity is set to \( c_0 = \frac{1}{2} \). The potential can be used to compute the scattered field. In the case of an infinite plate, we can compute the exact scattered field by the method of images [45]. We compare the scattered field of our finite plate with the exact scattered field of the infinite plate. The potential on a \( 17 \times 17 \) grid, where \( T = 20 \) and \( t_0 = 30 \) is computed with a stabilized scheme and presented in figure 2.8. We see that the effect of the boundary is marginal in the interior of the plate. The scattered field at an observation point in \((0, 0, 0.25)\) is computed with a stabilized scheme and plotted in figure 2.9. From the leftmost subfigures (c) and (f) it is clear that the scheme is stable, since the error rapidly decreases when the incoming pulse has passed the plate.

\[ f^\infty(x) = \begin{cases} 
      e^{1-1/(1-|x|^2)}, & |x| < 1, \\
      0, & |x| \geq 1 
\end{cases} \]

(2.244)

Figure 2.7: Triangulated plate with \( 11 \times 11 \) nodes (left) and sphere with 92 nodes (right).

**Stability of a Dirichlet Plate**

In order to determine if a scheme is stable or not, we compute the eigenvalues of the corresponding one-step method given by (2.233). If the largest eigenvalues are outside the unit circle, then the scheme is unstable. We consider a \( 9 \times 9 \) grid discretization of the plate. We use a trapezoidal method for the outer integral (over the triangles \( K \)), with 6 and 10 points. In table 2.1, we present the largest
2.12. NUMERICAL EXAMPLES

(a) \( t = 22.1 \) s
(b) \( t = 33.1 \) s
(c) \( t = 44.2 \) s
(d) \( t = 55.2 \) s

Figure 2.8: Potential at different times for \( 17 \times 17 \) plate, with \( CFL = 0.5 \).

eigenvalues of the one-step method. The table shows that we get a more stable scheme using 10 quadrature points than using 6 points for the integrals over \( K \). This indicates that the number of required quadrature points increases as the CFL number decreases. When we use the stabilization filter the scheme becomes stable for smaller CFL numbers.

Order of Accuracy in Time of a Dirichlet Plate

In order to get first order of accuracy in time, we need to resolve space sufficiently good. Consider a Dirichlet plate \((x, y) \in [-5, 5], z = 0\). The plate is discretized with 121 and 289 nodes. We compute the potential \( J^{CFL}(r, t) \) on the plate for a point source in \( r_s = (0, 0, 2) \), for different CFL-numbers. The point source produces
CHAPTER 2. TIME DOMAIN INTEGRAL EQUATION METHOD

Figure 2.9: Scattered field for a 17 × 17 plate for CFL = 1 (top) and CFL = 0.5 (bottom). The scale is different for the first two columns.

<table>
<thead>
<tr>
<th>CFL</th>
<th>6 pts, no stab.</th>
<th>10 pts, no stab.</th>
<th>6 pts, stab.</th>
<th>10 pts, stab.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>-0.9988</td>
<td>-0.9994</td>
<td>0.6782</td>
<td>0.6752</td>
</tr>
<tr>
<td>0.500</td>
<td>-0.9992</td>
<td>-0.9991</td>
<td>0.8212</td>
<td>0.8194</td>
</tr>
<tr>
<td>0.250</td>
<td>-1.2189</td>
<td>-1.0618</td>
<td>0.9062</td>
<td>0.9052</td>
</tr>
<tr>
<td>0.125</td>
<td>-1.4454 ± 1.6727</td>
<td>-2.2671</td>
<td>-1.2997</td>
<td>0.6597 ± 0.6885</td>
</tr>
</tbody>
</table>

Table 2.1: Eigenvalues of the corresponding one-step method for different CFL-numbers, with and without stabilization filter. *) indicates that the scheme is unstable.

an incoming wave as in (2.243), with \( t_0 = 30 \) and \( T = 20 \). We use 10 quadrature points for the integrals over \( K \). The order of accuracy in time can be computed numerically by

\[
\text{Order of accuracy in time} \approx \log_2 \frac{\| J^{4\text{CFL}} - J^{2\text{CFL}} \|_{L^2((0,T_{\text{end}}),\Gamma)}}{\| J^{2\text{CFL}} - J^{\text{CFL}} \|_{L^2((0,T_{\text{end}}),\Gamma)}},
\]

(2.245)

The potential is computed until \( t = T_{\text{end}} \approx 44.19 \). This corresponds to 125 time steps on the 11 × 11 grid and 200 time steps on the 17 × 17 grid. The order of accuracy in time is computed and is presented in table 7.2 and we see that this is a first order scheme in time.
2.12. NUMERICAL EXAMPLES

<table>
<thead>
<tr>
<th></th>
<th>11 × 11 plate</th>
<th>Time order</th>
<th>17 × 17 plate</th>
<th>Time order</th>
</tr>
</thead>
<tbody>
<tr>
<td>CFL</td>
<td>$|J^{CFL} - J^{CFL}|_2$</td>
<td></td>
<td>CFL</td>
<td>$|J^{CFL} - J^{CFL}|_2$</td>
</tr>
<tr>
<td>1.000</td>
<td>0.2754</td>
<td>0.7309</td>
<td>1.000</td>
<td>0.1986</td>
</tr>
<tr>
<td>0.500</td>
<td>0.1659</td>
<td>0.9503</td>
<td>0.500</td>
<td>0.1074</td>
</tr>
<tr>
<td>0.250</td>
<td>0.0859</td>
<td>0.9948</td>
<td>0.250</td>
<td>0.0543</td>
</tr>
<tr>
<td>0.125</td>
<td>0.0431</td>
<td></td>
<td>0.125</td>
<td>0.0271</td>
</tr>
</tbody>
</table>

Table 2.2: Order of accuracy in time of Dirichlet 11 × 11 plate (left) and a 17 × 17 plate (right).

<table>
<thead>
<tr>
<th>grid</th>
<th>CFL</th>
<th>$|J^{CFL} - J^{CFL}|_2$</th>
<th>Spatial order</th>
</tr>
</thead>
<tbody>
<tr>
<td>9 × 9</td>
<td>0.81</td>
<td>0.004249</td>
<td></td>
</tr>
<tr>
<td>13 × 13</td>
<td>0.54</td>
<td>0.002303</td>
<td>1.5104</td>
</tr>
<tr>
<td>19 × 19</td>
<td>0.36</td>
<td>0.001001</td>
<td>2.0544</td>
</tr>
<tr>
<td>28 × 28</td>
<td>0.24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3: Spatial order of Dirichlet plate.

Order of Accuracy in Space of a Dirichlet Plate

We want to do computations to show that the scheme has at least second order of accuracy in space. We have already showed that the scheme is first order in time. The error in the potential is

$$\|J(\Delta t, h) - \tilde{J}\|_2 = C_1 \Delta t + C_2 h^p,$$  \hspace{1cm} (2.246)

where $\tilde{J}$ is the exact potential and $h$ is the largest edge of all triangles on the scatterer. In order to get at least second order in space, we compute

$$\text{Spatial order} \geq \log_2 \frac{\|J(\Delta t, h) - J(q^2 \Delta t, qh)\|_2}{\|J(q^2 \Delta t, qh) - J(q^4 \Delta t, q^2 h)\|_2},$$  \hspace{1cm} (2.247)

where $q$ is the mesh refinement ratio. Normally $q = \frac{1}{2}$, but this yields a too small CFL-number on the finest grid. In the computation, we use $q = \frac{3}{2}$. The conclusion from the result presented in table 2.3, is that the scheme is second order accurate in space. It would have been good to refine the mesh once more, but then the size of the problem becomes too large.
CHAPTER 2. TIME DOMAIN INTEGRAL EQUATION METHOD

Test Case With a Dirichlet Sphere

As another test, we use a Dirichlet sphere with 92 nodes and 180 triangles, where
the incoming field is a plane wave. The speed of sound has been changed to \( c_0 = 1 \).
The incoming field is

\[
\begin{align*}
    u^{\text{inc}} & = \begin{cases} 
        \sin(2\pi \gamma(r, t)) f^\infty \left( \frac{t}{10T} - 1 \right), & t < 10T, \\
        \sin(2\pi \gamma(r, t)), & t \geq 10T,
    \end{cases} \\
    \gamma(r, t) & = \frac{1}{T} \left( t - t_0 - \frac{k \cdot r}{c_0} \right),
\end{align*}
\]

(2.248)

(2.249)

where \( \hat{k} \) is the direction of the incoming wave, \( T = t_0 = 40 \) and \( \text{CFL} = \frac{1}{4} \). The
solution of this incoming wave is compared to the exact solution of a single fre-
quency, which can be found in [14]. The scattered field is presented in figure 2.10,
for three different observation points relative to the direction of the incoming field.
The solution matches the analytical solution for all observation points considered.

Numerical Experiments on Variational Formulation From FD

In this section, we will perform numerical tests on the variational formulations 3
and 5 in this chapter. Two test cases are considered; a point source illuminating a
plate and a plane wave illuminating a sphere. Those two cases has an analytical
solution.

The variational formulations has a non-negative parameter \( \omega \), which depends
on the incoming field. Numerical tests are made in the Dirichlet case that indicates
that it is sufficient to use \( \omega = 0 \) in both test cases. The parameter \( \omega = 0 \) yields a
stable scheme for the first 10000 time steps.

In the case of a point source illuminating a Dirichlet plate, the largest eigenvalues
are multiple eigenvalues at 1. Long time calculations show no signs of instabilities.
The method has first order of accuracy in time in the computation of the potential
of the surface, but the spatial order of accuracy seems to be super quadratic.

In the case of a plane wave illuminating a Dirichlet sphere, the computed
scattered fields are compared with the analytical solutions at three different ob-
servation points.

In the case of a plane wave illuminating a Neumann sphere, the method in
unstable. Some comments are made on the possible reasons for the instability.

Dirichlet Plate, With Various \( \omega \)

Consider a 9 \( \times \) 9 plate as in section 2.12, with the point source at \( \mathbf{r}_s = (0, 0, 2) \) in
(2.243). The parameters are \( c_0 = 1, T = 20, t_0 = 30 \). The continuous variational
formulations discussed in this chapter are continuous and coercive whenever a para-
meter \( \omega > 0 \), see [9]. In figure 2.11, the errors \( u^{\omega}_s - u^{\omega}_s^f \) and \( u^{\omega}_s - u^{\omega}_0 \) are presented
for different $\omega$. When $\omega$ increases, the difference $u_{\omega}^{sc} - u_0^{sc}$ increases. This indicates that the error increases with $\omega$ and the conclusion is that we want to take as small $\omega$ as possible. In addition, when $\omega > 0$, many of the integrals are computed numerically. This makes the computer program run much slower than in the case $\omega = 0$, where most integrals have an analytic expression. In practice, the method is only useful for $\omega = 0$. This restricts the incoming field, not to have exponential
growth.

\[ \frac{\omega}{\omega_{sc}} = 0 \]

\[ \frac{\omega}{\omega_{sc}} = 0.01 \]

\[ \frac{\omega}{\omega_{sc}} = 0.1 \]

\[ \frac{\omega}{\omega_{sc}} = 0.3 \]

Figure 2.11: Computation on a 9 × 9 plate, for various $\omega$.

**Stability of a Dirichlet Plate, With $\omega = 0$**

Consider the test case as in (2.12), with a discretized plate with 9 × 9 nodes. The integral over $K$ is computed with Gaussian quadrature over the triangle, [25]. For adjacent triangle pairs we use 7 Gauss points, for nonadjacent, we use 3 Gauss points. This is not the same quadrature as for the numerical tests of Kirchhoff integral equations. We do computations with CFL-number 1, 0.5, 0.25 and 0.125. The largest eigenvalue is a multiple eigenvalue of 1 (in 14 decimals). The multiplicity of the eigenvalues are listed in table 2.4. The long time behavior, 10000 time steps, of the error with CFL-number 0.5 is illustrated in figure 2.12. This corresponds to a case with an eigenvalue 1 with multiplicity $\geq 10$. There are no visible growth in the error in the first 10000 time steps. The scheme seems to be long time stable.
2.12. NUMERICAL EXAMPLES

<table>
<thead>
<tr>
<th>CFL</th>
<th>multiplicity of eigenvalue 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>$\geq 10$</td>
</tr>
<tr>
<td>0.500</td>
<td>$\geq 10$</td>
</tr>
<tr>
<td>0.250</td>
<td>4</td>
</tr>
<tr>
<td>0.125</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 2.4: Multiplicity of eigenvalue 1 on $9 \times 9$ plate for different CFL numbers.

Figure 2.12: Long time error of $u^{sc}$ on test case with a plate with $9 \times 9$ nodes and $CFL = 0.5$.

Stability of a Dirichlet Sphere, With $\omega = 0$

Consider a Dirichlet sphere with radius 5 and an incidence plane wave as in section 2.12. The parameters are set to $T = t_0 = 20$, $c_0 = 1$ and $\hat{k} = (0, 0, -1)$. Moreover, CFL = 0.5. The corresponding one-step method given by (2.233) has a triple eigenvalue at 1. In figure 2.13, we present the computed back scattered field after 10000 time steps. The dotted curve is the analytical solution, in [14]. The computed scattered field match the analytical and there is no indication that the scheme is unstable. We conclude that for practical computations, $\omega = 0$ is sufficient to get a stable scheme.

Time Order of a Dirichlet Plate, With $\omega = 0$

In order to obtain first order of accuracy in time, we need to resolve the space. Consider a Dirichlet plate $(x, y) \in [-5, 5], z = 0$. The plate is discretized with both 121 and 289 nodes. We want to compute the potential, $J^{CFL}(r, t)$, on the plate
for a point source at $r_s = (0, 0, 2)$, using different CFL-numbers. The point source produces an incoming wave as in (2.243), with $t_0 = 30$ and $T = 20$.

I take 125 time steps on the $11 \times 11$ plate and 200 time steps on the $17 \times 17$ grid. This corresponds to $T_{end} \approx 44.19$. The order of accuracy in time is computed and is presented in table 8.2. This is a first order scheme in time.

**Order of Accuracy in Space of a Dirichlet Plate, With $\omega = 0$**

In previous section, we concluded that the scheme is first order accurate in time. This is used to determine the order of accuracy in space. We proceed as in section
2.12. NUMERICAL EXAMPLES

<table>
<thead>
<tr>
<th>grid</th>
<th>CFL</th>
<th>$|J^{CFL} - J_{L^2}^{CFL}|$</th>
<th>Spatial order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9 \times 9$</td>
<td>0.81</td>
<td>0.010942</td>
<td>2.2466</td>
</tr>
<tr>
<td>$13 \times 13$</td>
<td>0.54</td>
<td>0.004400</td>
<td>2.3148</td>
</tr>
<tr>
<td>$19 \times 19$</td>
<td>0.36</td>
<td>0.001721</td>
<td></td>
</tr>
<tr>
<td>$28 \times 28$</td>
<td>0.24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.6: Order of accuracy in space of Dirichlet plate.

2.12. Results are presented in table 2.6 and the scheme appears to be more than second order in space.

**Dirichlet Sphere, With $\omega = 0$**

Here we consider a plane wave crossing a Dirichlet sphere with radius 5. The incoming field is a sine wave with a smooth transition at $t = 0$,

$$u^{inc}(r, t) = \begin{cases} 
\sin \left( \frac{2\pi t - t_0 - \hat{k} \cdot r/c_0}{T} \right) f_{\infty} \left( \frac{t}{10T} \right), & t < 10T, \\
\sin \left( \frac{2\pi t - t_0 - \hat{k} \cdot r/c_0}{T} \right), & t \geq 10T. 
\end{cases}$$

We compare the computed scattered field with the analytical solution in [14]. In figure 2.14, we compare the back scattered field, the field that is perpendicular to the incoming field and the field behind the sphere relative to the incoming field. We use $CFL = 0.25$ and pulse width $T = 40$. In figure 2.15, we compare the backscattered field for various $CFL$ numbers for pulse width $T = 5$. The energy in the scattered field decreases when $CFL$ is chosen too large.

**Instability of a Neumann Sphere, With $\omega = 0$**

Several test cases have been performed, to get a stable solution to the Neumann sphere. One possible explanation of the instability is that the the spatial basis functions are not sufficiently regular. According to the variational formulation 5 in this chapter, the basis functions in time should have two continuous derivatives in time. In our implementation, we only use basis functions with one continuous time derivative. This problem can be avoided if we use linear elements in time. However, this results in an implicit scheme.

Another possibility is that we get a problem when we change the order of integration when the integrand contains Dirac $\delta$-functions. This is the case for the integral $I_2$ in section 2.6. However, this integral is also used for the stable Dirichlet variational formulation.
A remark should be made, that if we remove the last term containing $I_1^p$ in the Neumann variational formulation 8 in this chapter, then the scheme is stable, but the computed potential does not match the analytical solution.

![Graphs showing scattered field for different angles](image)

(a) Backscattered field ($\Theta = 0^\circ$).

(b) Scattered field perpendicular to incoming field ($\Theta = 90^\circ$).

(c) Scattered field behind sphere, relative to incoming field ($\Theta = 180^\circ$).

Figure 2.14: Scattered field for a Dirichlet sphere, with pulse width $T = 40$. The dotted curves are the analytical solutions.
2.13. **ON SURFACE RADIATION CONDITIONS**

When solving an integral formulation of the wave equation (2.251) with the Marching On in Time method (MOT), described in this chapter, the computational cost of the $k$-step marching algorithm increases substantially with the size of the object (or as the frequency increases). In other words, MOT is a low to moderate frequency

---

**Figure 2.15**: Scattered field for a Dirichlet sphere, with pulse width $T = 5$. The dotted curves are the analytical solutions.

---

2.13 **On Surface Radiation Conditions**

When solving an integral formulation of the wave equation (2.251) with the Marching On in Time method (MOT), described in this chapter, the computational cost of the $k$-step marching algorithm increases substantially with the size of the object (or as the frequency increases). In other words, MOT is a low to moderate frequency
method. For high frequencies, the method is expensive to use. There are several ways of improving the computational complexity. In frequency domain, we have the fast multipole method. In time domain, Michielssen [31] has developed PWTD, using plane waves to reduce the cost for the matrix-vector multiplications in MOT. Existing high frequency approximations that are used in frequency domain are for instance physical optics (PO), e.g. by Edlund [28] and general theory of diffraction (GTD), by Keller [47]. These methods are only accurate approximations in the limit of high frequencies. We want to develop a high frequency approximation for MOT, by constructing a PDE for the scattered field on the surface of the scatterer. Then we can use the boundary condition to replace the scattered field with the incoming field on the surface. The goal is to express the scattered field as an integral of the incoming field over the surface of the scatterer. This approach is called On Surface Radiation Condition (OSRC). This has been done in frequency domain by G.A. Kriegsmann [49] and D.S. Jones [46].

The OSRC method can be outlined as follows,

- Express scattered field $u^{sc}$ in spherical coordinates and insert the field as a solution in the wave equation.
- We obtain a relation that couples $u^{sc}$, $\frac{\partial u^{sc}}{\partial t}$ and $\frac{\partial u^{sc}}{\partial n}$.
- Use the boundary condition of the scatterer to eliminate appropriate terms in the relation.
- Insert the relation in Kirchhoff formula for the scattered field, s.t. the scattered field on the surface is eliminated or is easily computed.

The resulting integral formula contains no global coupling over the surface. Instead we at most solve a local problem for each point on the surface.

We want to solve the scalar wave equation for the scattered field,

$$\nabla^2 u^{sc} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u^{sc} = 0, \quad \text{with } u^{sc} = 0, \text{ for } t \leq 0,$$

in the exterior of a scatterer. Write the solution in spherical coordinates, \[54\]

$$u^{sc}(R_0, \theta, \phi, t) = \sum_{i=1}^{\infty} \frac{f_i(t - R_0/c, \theta, \phi)}{R_0^i},$$

where $R_0$ is the distance to the center of the scatterer. By inserting the expansion in the wave equation (2.251), we get

$$\sum_{i=1}^{\infty} \frac{1}{R_0^{i+2}} \left( \frac{2i (f_{i+1})_t}{c} + i(i-1)(i) + \nabla^2_{\theta, \phi} f_i \right) = 0.$$  

By truncation and letting $R_0 \to \infty$, the approximate relation

$$(f_{i+1})_t = -\frac{c}{2} \left( (i-1)(i) + \frac{1}{i} \nabla^2_{\theta, \phi} f_i \right),$$
is solved (with $f_i(-\infty, \theta, \phi) = 0$)

$$f_{i+1}(t, \theta, \phi) = -\frac{c}{2} \int_{-\infty}^{t} (i - 1) f_i(\tau, \theta, \phi) + \frac{1}{i} \nabla_0^2 f_i(\tau, \theta, \phi) d\tau.$$ (2.255)

From (2.252), we derive the relation

$$\frac{\partial u^{sc}}{\partial R_0} = -\frac{1}{c} \frac{\partial u^{sc}}{\partial t} - \frac{u^{sc}}{R_0} - \frac{f_2(t - R_0/c, \theta, \phi)}{R_0^3} + O(R_0^{-4})$$

$$= -\frac{1}{c} \frac{\partial u^{sc}}{\partial t} - \frac{u^{sc}}{R_0} + \frac{c}{2 R_0^3} \int_{-\infty}^{t - R_0/c} \nabla_0^2 f_1(\tau, \theta, \phi) d\tau + O(R_0^{-4}).$$ (2.256)

From (2.255), we express $f_1$ in $u^{sc}$

$$f_1(t - R_0/c, \theta, \phi) = R_0 u^{sc}(R_0, \theta, \phi, t) + O(R_0^{-1}),$$ (2.257)

which yields

$$\frac{\partial u^{sc}}{\partial R_0} = -\frac{1}{c} \frac{\partial u^{sc}}{\partial t} - \frac{u^{sc}}{R_0} + \frac{c}{2 R_0^3} \int_{0}^{t} \nabla_0^2 u^{sc}(R_0, \theta, \phi, \tau) d\tau + O(R_0^{-3}).$$ (2.258)

In order to compute the scattered field, we need to couple the incoming and scattered field on the scatterer boundary. In the coupling, we need the normal derivative rather than the radial. By Jones, [46], we go to non-spherical coordinates by the substitutions

$$\frac{\partial u^{sc}}{\partial R_0} \rightarrow \frac{\partial u^{sc}}{\partial n},$$ (2.259)

$$\frac{1}{R_0} \rightarrow H(r), \quad \text{(Curvature at r)}.$$ (2.260)

$$\frac{1}{R_0^2} \nabla_0^2 u^{sc} \rightarrow \nabla_0^2 u^{sc},$$ (2.261)

$$\frac{\partial u^{sc}}{\partial n}(r, t) + \frac{1}{c} \frac{\partial u^{sc}}{\partial t}(r, t) + H(r) u^{sc}(r, t) = \frac{c}{2} \int_{0}^{t} \nabla_0^2 u^{sc}(r, \tau) d\tau.$$ (2.262)

The condition (2.262) is used together with the Kirchhoff formula (2.3) to derive a method to compute the scattered field for both a Dirichlet and a Neumann boundary condition on the surface. A program is implemented for a Dirichlet sphere.

**OSRC for Dirichlet Problems**

For the Dirichlet problem, we have the boundary condition

$$u^{inc} + u^{sc} = 0, \quad \frac{\partial}{\partial n}(u^{inc} + u^{sc}) = 0$$ (2.263)
on the boundary $\Gamma$. Together with the derived condition (2.262), we can write the Kirchhoff formula (2.3) (with $u^{inc} = u^{inc}(r', t - R/c)$)

$$u^{sc}(r, t) = \frac{1}{4\pi} \int_{\Gamma} K^1_D(R) \frac{\partial u^{inc}}{\partial t} + K^2_D(R, r') u^{inc}_{r'} + K^3_D[u^{inc}](R) d\Gamma',$$

(2.264)

$$K^1_D(R) = \frac{1}{cR} \left( 1 - \frac{\partial R}{\partial n} \right),$$

(2.265)

$$K^2_D(R, r') = \frac{\partial}{\partial n} \left( \frac{1}{R} \right) + \frac{1}{R} H(r'),$$

(2.266)

$$K^3_D[u^{inc}](R) = -\frac{c}{2R} \int_{t-R/c}^{t} \nabla^2_{\Gamma} u^{inc}(r', \tau) d\tau.$$  

(2.267)

We get a direct representation of the scattered field.

**OSRC for Neumann Problems**

For the Neumann problem, we have the boundary condition

$$\frac{\partial}{\partial n}(u^{inc} + u^{sc}) = 0,$$

(2.268)

on the boundary $\Gamma$. Together with the derived condition (2.262), the ODE

$$\frac{1}{c} \frac{\partial u^{sc}}{\partial t}(r, t) + H(r) u^{sc}(r, t) - \frac{c}{2} \int_{0}^{t} \nabla^2_{\Gamma} u^{sc}(r, \tau) d\tau = \frac{\partial u^{inc}}{\partial n}(r, t)$$

(2.269)

is derived. Solving this ODE for each point $r \in \Gamma$ yields $u^{sc}(r, t)$ on $\Gamma$. Next we eliminate the time derivative

$$\frac{1}{c} \frac{\partial u^{sc}}{\partial t}(r, t) = \frac{\partial u^{inc}}{\partial n}(r, t) - H(r) u^{sc}(r, t) + \frac{c}{2} \int_{0}^{t} \nabla^2_{\Gamma} u^{sc}(r, \tau) d\tau$$

(2.270)

which can be inserted in the Kirchhoff formula (2.3) and we get

$$u^{sc}(r, t) = \frac{1}{4\pi} \int_{\Gamma} K^1_N(R) \frac{\partial u^{inc}}{\partial n} + K^2_N(R, r') u^{sc}_{r'} + K^3_N[u^{sc}](R) d\Gamma',$$

(2.271)

$$K^1_N(R) = \frac{1}{R} \left( \frac{\partial R}{\partial n} - 1 \right),$$

(2.272)

$$K^2_N(R, r') = -\left( \frac{\partial}{\partial n} \left( \frac{1}{R} \right) + \frac{1}{R} \frac{\partial R}{\partial n} H(r') \right),$$

(2.273)

$$K^3_N[u^{sc}](R) = \frac{c}{2R} \frac{\partial R}{\partial n} \int_{t-R/c}^{t} \nabla^2_{\Gamma} u^{sc}(r', \tau) d\tau.$$  

(2.274)

A time stepping scheme is obtained, in which for each time step $k$
1. Solve the ODE in (2.269) to get $u^{sc}(r, k\Delta t)$ on the scatterer $\Gamma$.

2. Compute $K^N_{\gamma}[u^{sc}](R)$.

3. Compute the scattered field in the exterior, in (2.271).

**Dirichlet Test Case on Sphere**

As a simple test case, we have chosen a sphere with radius $R_0$. The sphere $\Gamma$ is parameterized by

\begin{align*}
x &= R_0 \cos \phi \sin \theta, \quad (2.275) \\
y &= R_0 \sin \phi \sin \theta, \quad (2.276) \\
z &= R_0 \cos \theta, \quad (2.277)
\end{align*}

where $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$. The sphere is discretized with a uniform mesh in $\theta$ and $\phi$, with $M\Delta \theta = \pi$, $N\Delta \phi = 2\pi$ and

\begin{align*}
\theta_i &= i\Delta \theta, \quad i = 1, \ldots, M - 1, \quad (2.278) \\
\phi_j &= j\Delta \phi, \quad j = 1, \ldots, N - 1. \quad (2.279)
\end{align*}

The curvature is constant
\[ H(r') = \frac{1}{R_0} \]

and the surface Laplace-Beltrami-operator is
\[ \nabla^2_{\Gamma,inc} = \frac{1}{R_0^2} \nabla^2_{\theta,inc} = \frac{1}{R_0^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u^{inc}}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u^{inc}}{\partial \phi^2} \right). \quad (2.280) \]

For a sphere, we have
\[ \sin \theta \nabla^2_{\theta,inc}(R_0, \theta, \phi) = 0, \quad \theta = 0, \pi. \quad (2.281) \]

Moreover $\sin \theta \nabla^2_{\theta,inc}$ is $2\pi$-periodic in $\phi$. Let $u_{i,j} = u(\theta_i, \phi_j)$ and the operator 2.280 can be discretized as

\begin{align*}
\sin \theta_i \nabla^2_{\theta,inc}u_{i,j} &= D_{0,i} \sin \theta_i D_{0,j} u_{i,j} + \frac{1}{\sin \theta_i} D_{+j} D_{-j} u_{i,j} + O \left( \Delta \theta^2 + \frac{\Delta \phi^2}{\sin \theta_i} \right), \\
D_{0,i} \sin \theta_i D_{0,j} u_{i,j} &= \sin \theta_{i+1/2} u_{i+1,j} - \sin \theta_{i-1/2} u_{i-1,j} + \sin \theta_{i-1/2} u_{i+1,j} + \sin \theta_{i+1/2} u_{i-1,j}, \\
D_{+j} D_{-j} u_{i,j} &= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta \phi^2}.
\end{align*}

The discretization error is large for $\theta$ close to 0 and $\pi$. But for those values of $\theta$, the expression $\sin \theta \nabla^2_{\theta,inc}(R_0, \theta, \phi)$ is vanishing, and we can hope that the error does not destroy the expected second order accuracy.
Numerical Experiments

We use the incoming field as in equation (2.243), with \( T = 5 \), \( t_0 = 10 \) and \( \hat{k} = (1, 0, 0) \). The sphere with radius \( R_0 = 5 \) is discretized with \( 21 \times 21 \) points in \( \phi \) and \( \theta \) and use \( \Delta t = \frac{1}{5} \). As a reference solution, we use the solution obtained by the Dirichlet MOT solver in this chapter with a sphere with 92 nodes and 180 triangles. The computed solutions is presented in figure 2.16. The OSRC solution somewhat resembles the solution obtained by the MOT solver.

Figure 2.16: OSRC solution vs MOT solution of the scattered field for different observation points \( \mathbf{r} \).
Upper left: \( \mathbf{r} = (0,0,10) \), Upper right: \( \mathbf{r} = (10,0,0) \),
Lower left: \( \mathbf{r} = (0,10,0) \), Lower right: \( \mathbf{r} = (-10,0,0) \)
Chapter 3

Analytical Solutions for Objects with Constant Curvature

The analytical representation of scattering problems are known for some scatterers of simple shapes. Such scatterers are circular and elliptical infinite cylinders in 2D and the sphere and spheroid in 3D. For problems with large wave number geometrical optics are a good approximation. This indicates that the main interactions on the surface are local and that one therefore can approximate the scatterer locally with a circle defined by the curvature in 2D and a sphere defined by the mean curvature in 3D. In this chapter we give analytical representation of the potentials for objects with constant curvature, [14]. A common theme for all scatterers with constant curvature is that the potential (or current in electromagnetics) can be expressed in terms of logarithmic derivatives of Hankel-functions. The logarithmic derivatives of the Hankel functions will later be used to construct several different On Surface Radiation Condition approximations. The potentials are also presented in different figures for the wave number 5, 20, 80 and 320. This gives us an understanding of the behaviour of the potential for different angles, relative the incoming field and can be used for comparison with numerical approximations.

3.1 Analytical Expressions for the Potential on a Circular Cylinder With Dirichlet Boundary Condition

The Dirichlet problem where the scatterer $\Gamma$ is a cylinder with radius $R$ will be used as a model problem for the presentation in 2D. Consider the case when the incoming field is a plane wave in the negative $x$-direction, $u^{inc}(r, \phi) = e^{-ikr\cos\phi}$. Using Fourier series in $\phi$ yields the Fourier components

\[ \hat{u}_{\phi}^{inc}(R) = (-i)^m J_m(kR), \]
\[ \hat{u}_{\phi}^{sc}(R) = (-i)^m J_m(kR). \]
CHAPTER 3. ANALYTICAL SOLUTIONS FOR OBJECTS WITH CONSTANT CURVATURE

The analytical scattered field for the Dirichlet problem is, [14],

\[ u^{sc}(r, \phi) = \sum_{m=0}^{\infty} \varepsilon_m \frac{H^{(1)}_m(\alpha m)}{H^{(1)}_m(\alpha R)} \hat{u}^{sc}_m(R) \cos m\phi. \] (3.1)

An analytical expression for the potential is

\[ \partial_n u^{sc}(R, \phi), \]

\[ \partial_n u^{sc}(R, \phi) = \sum_{m=0}^{\infty} \varepsilon_m \frac{\partial_n H^{(1)}_m(\alpha R)}{H^{(1)}_m(\alpha R)} \hat{u}^{sc}_m(R) \cos m\phi. \] (3.2)

The symbol of the Dirichlet-to-Neumann (DtN)-operator for the Dirichlet cylinder is

\[ (\partial_n u^{sc})_m = \frac{\partial_n H^{(1)}_m(\alpha R)}{H^{(1)}_m(\alpha R)} \hat{u}^{sc}_m. \] (3.3)

We are also interested in the arclength surface derivatives \( \partial_s = \frac{1}{R} \partial_\phi \). We have

\[ \partial_s^2 u^{sc}(R, \phi) = \sum_{m=0}^{\infty} \varepsilon_m \left( -\frac{m^2}{R^2} \right)^2 \hat{u}^{sc}_m(R) \cos m\phi, \]

\[ \partial_s^2 \partial_n u^{sc}(R, \phi) = \sum_{m=0}^{\infty} \varepsilon_m \left( -\frac{m^2}{R^2} \right)^2 \frac{\partial_n H^{(1)}_m(\alpha R)}{H^{(1)}_m(\alpha R)} \hat{u}^{sc}_m(R) \cos m\phi. \]

In Figure 3.1, the potential \( \partial_n u^{tot} \) is plotted for various wavenumber \( \alpha \). The amplitude of the potential increases with \( \alpha \). For high frequencies the potential on the shadowside (\( |\phi| > 90^\circ \)) is negligible.

**High Frequency Approximation**

When the wave number is large, it is convenient to use a asymptotic (or semi-asymptotic) representation of the potential \( \partial_n u^{tot} \). Such a representation was derived in 1955 [32] and is valid on the illuminated part of the circle (\( |\phi| < 90^\circ \)),

\[ \partial_n u^{tot} = \begin{cases} -2i k \cos \phi e^{-ikR \cos \phi} \left( 1 + \frac{1}{2kR \cos \phi} \frac{1 + 3 \sin^2 \phi}{2kR \cos \phi} \right) + \ldots, \\ -ik e^{i\pi/6} \sum_i D_i \frac{v_i(\tau)}{v_i(3\pi/2 - \alpha_i)} e^{i\nu_i(3\pi/2 + \phi)} + \ldots, \end{cases} \] (3.4)

where

\[ v_i = kR + e^{i\pi/3} \alpha_i \left( \frac{1}{2} kR \right)^{1/3} + \ldots, \]

\[ D_i = \frac{1}{\alpha_i(-\alpha_i)} \left( 1 + e^{i\pi/3} \alpha_i \left( \frac{1}{10} \right) kR^{-2/3} \right) + \ldots, \]
3.2 Analytical Expressions for the Potential on a Circular Cylinder with Neumann Boundary Condition

In the case of a Neumann cylinder, we will show that the expression of the scattered field is related to that of on Dirichlet cylinder. The analytic expression for the

where $\alpha_l$ is the $l$th zero of $Ai(-\alpha)$. The first part of the expression is the optics contribution to the surface field. The last part of the expression represents the creeping wave contribution. In the shadow part of the domain we only get contributions from the creeping wave part of the expression. In Figure 3.2 the contribution from the creeping waves is presented. The creeping wave contribution is larger for small wave number $k$.

3.2 Analytical Expressions for the Potential on a Circular Cylinder with Neumann Boundary Condition

In the case of a Neumann cylinder, we will show that the expression of the scattered field is related to that of on Dirichlet cylinder. The analytic expression for the
CHAPTER 3. ANALYTICAL SOLUTIONS FOR OBJECTS WITH CONSTANT CURVATURE

Figure 3.2: Real (solid line) and imaginary part (dashed line) of the creeping wave contribution to the potential \( \partial_n u^{\text{tot}} \) for \( k = 5, 20, 80 \) and 320 as function of the angle \( \phi \) of a circular cylinder with Dirichlet boundary condition.

The scattered field of a Neumann cylinder is, [14],

\[
  u^{sc}(r, \phi) = \sum_{m=0}^{\infty} \varepsilon_m \frac{H^{(1)}_m(kr)}{\partial_n H^{(1)}_m(kR)} \partial_n \hat{u}^{sc}_m(R) \cos m\phi.
\]

Using \( \partial_n \hat{u}^{sc}_m = (\hat{\partial}_n u^{sc})_m \) yields

\[
  u^{sc}(R, \phi) = \sum_{m=0}^{\infty} \varepsilon_m \frac{H^{(1)}_m(kR)}{\partial_n H^{(1)}_m(kR)} (\hat{\partial}_n u^{sc})_m \cos m\phi. \tag{3.5}
\]

The symbol of the DtN-operator for the Neumann cylinder is

\[
  \hat{u}^{sc}_m = \frac{H^{(1)}_m(kR)}{\partial_n H^{(1)}_m(kR)} (\hat{\partial}_n u^{sc})_m. \tag{3.6}
\]
The expression (3.6) is the same as expression (3.3). It is therefore sufficient to consider the Dirichlet case when deriving expressions using the symbol of the DtN-operator.

In Figure 3.3, the potential $u^{\text{tot}}$ is plotted for various wavenumber $k$. The amplitude of the potential is not increasing with $k$. For high frequencies, the potential diminishes on the shadowside, but not as fast as with Dirichlet boundary condition.

Figure 3.3: Real (solid line) and imaginary part (dashed line) of the potential $u^{\text{tot}}$ for various $k$ as function of the angle $\phi$ of a circular cylinder with Neumann boundary condition.
CHAPTER 3. ANALYTICAL SOLUTIONS FOR OBJECTS WITH
CONSTANT CURVATURE

High Frequency Approximation

For large wave numbers, we have a semi-asymptotic formula [32],

\[
\begin{align*}
    u^\text{tot} &= 2e^{-ikR\cos\phi} \left( 1 - \frac{2kR\cos^3\phi}{\Delta_l} \right) \left( 1 + \frac{1 + 3\sin^2\phi}{4k^2R^2\cos^6\phi} + \ldots \right) \\
    &+ \sum_l \bar{D}_l \frac{\bar{v}_l}{\bar{v}_l(i\pi/2 - \phi)} + e^{i\bar{v}_l(i\pi/2 + \phi)} \left( \frac{1 - e^{2\pi i\bar{v}_l}}{2\pi i\bar{v}_l} \right),
\end{align*}
\]

in the illuminated region, where \( \bar{v}_l \) where \( \bar{D}_l \sim 1/\beta_l \), and \( \beta_l \) is the \( l \)th zero of \( Ai'(-\beta) \). The first part of the expression for the potential is the contribution from the optics and the second part is the contribution from the creeping waves. In the shadow part of the cylinder, only the creeping wave appears. The creeping wave contributions are presented in Figure 3.4 for different values of \( k \).

Figure 3.4: Real (solid line) and imaginary part (dashed line) of the creeping wave contribution to the potential \( \partial_n u^\text{tot} \) for \( k = 5, 20, 80 \) and \( 320 \) as function of the angle \( \phi \) of a circular cylinder with Neumann boundary condition.
3.3. ANALYTICAL EXPRESSIONS FOR THE POTENTIAL ON A SPHERE WITH DIRICHLET BOUNDARY CONDITION

3.3 Analytical Expressions for the Potential on a Sphere With Dirichlet Boundary Condition

We will use the Dirichlet sphere with radius $R$ as a model problem in 3D. Consider the case with a plane wave incident in the direction of the negative $z$-axis. The incoming field can be represented in Legendre-Fourier series

$$u^{\text{inc}}(r, \theta, \phi) = \sum_{m=0}^{\infty} \hat{u}^{\text{inc}}_m(r) \mathbb{P}_m(\cos \theta),$$

$$\hat{u}^{\text{inc}}_m(r) = (-i)^m (2m+1) j_m(kr),$$

where $j_m(x) = \sqrt{\frac{2}{\pi x}} j_{m+1/2}(x)$ is the spherical Bessel function and $\mathbb{P}_m$ is the Legendre function. The scattered field is then represented by, [14],

$$u^{\text{sc}}(r, \theta, \phi) = \sum_{m=0}^{\infty} \hat{u}^{\text{sc}}_m(R) \mathbb{P}_m(\cos \theta),$$

$$\hat{u}^{\text{sc}}_m(R) = -(-i)^m (2m+1) j_m(kR),$$

where $h^{(1)}_m(x) = \sqrt{\frac{2}{\pi x}} H^{(1)}_{m+1/2}(x)$ is the spherical Hankel function. The analytical expression for the potential, $\partial_n u^{\text{sc}}(R, \theta, \phi)$ is

$$\partial_n u^{\text{sc}}(R, \theta, \phi) = \sum_{m=0}^{\infty} \frac{\partial_n h^{(1)}_m(kR)}{h^{(1)}_m(kR)} \hat{u}^{\text{sc}}_m(R) \mathbb{P}_m(\cos \theta).$$

The symbol of the DtN-operator for the Dirichlet sphere is

$$\hat{(\partial_n u^{\text{sc}})}_m = \frac{\partial_n h^{(1)}_m(kR)}{h^{(1)}_m(kR)} \hat{u}^{\text{sc}}_m.$$

(3.8)

The surface Laplacian, $\Delta_\Gamma$, in spherical coordinates is

$$\Delta_\Gamma u(R, \theta, \phi) = \frac{1}{R^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta u(R, \theta, \phi)) + \frac{1}{R^2 \sin^2 \theta} \partial_\phi^2 u(R, \theta, \phi).$$

Direct computation yields

$$\Delta_\Gamma \mathbb{P}_m(\cos \theta) = -\frac{m(m+1)}{R^2} \mathbb{P}_m(\cos \theta).$$

This yields the surface derivatives

$$\Delta_\Gamma^{\text{sc}} \mathbb{P}_m(\cos \theta) = \sum_{m=0}^{\infty} \left( -\frac{m(m+1)}{R^2} \right)^p \hat{u}^{\text{sc}}_m(R) \mathbb{P}_m(\cos \theta),$$

(3.9)

$$\Delta_\Gamma^{\text{sc}} \partial_n u^{\text{sc}}(R, \theta, \phi) = \sum_{m=0}^{\infty} \left( -\frac{m(m+1)}{R^2} \right)^p \frac{\partial_n h^{(1)}_m(kR)}{h^{(1)}_m(kR)} \hat{u}^{\text{sc}}_m(R) \mathbb{P}_m(\cos \theta).$$

(3.10)
In Figure 3.5, the potential $\partial_n u^{tot}$ is plotted for various wavenumber $k$. The amplitude of the potential increases with $k$. For high frequencies the potential on the shadowside ($|\phi| > 90^\circ$) is negligible.

![Figure 3.5](image)

Figure 3.5: Real (solid line) and imaginary part (dashed line) of the potential $\partial_n u^{tot}$ for various $k$ as function of the angle $\theta$ of a sphere with Dirichlet boundary condition.

### 3.4 Analytical Expressions for the Potential on a Sphere With Neumann Boundary Conditions

The analytic expression for the scattered field of a Neumann sphere in 3D is, [14],

$$u^{sc}(r, \theta, \phi) = \sum_{m=0}^{\infty} h_m^{(1)}(kr) \left( \partial_n \widehat{u^{sc}} \right)_m(R) P_m(\cos \theta),$$

$$\left( \partial_n \widehat{u^{sc}} \right)_m(r) = -(-i)^m (2m+1) \partial_n \bar{j}_m(kr).$$
3.4. ANALYTICAL EXPRESSIONS FOR THE POTENTIAL ON A SPHERE WITH NEUMANN BOUNDARY CONDITIONS

The symbol of the DtN-operator for the Neumann sphere is

$$\tilde{u}_m^{sc} = \frac{h_n^{(1)}(kR)}{\partial_n h_n^{(1)}(kR)} (\partial_n u_m^{sc})_m.$$  \hspace{1cm} (3.11)

The expression (3.11) is the same as expression (3.8). It is sufficient also in 3D to consider only the Dirichlet problem.

In Figure 3.6, the potential $u^{tot}$ is plotted for various wavenumber $k$. The amplitude of the potential is not increasing with $k$. For high frequencies, the potential diminishes on the shadowside, but not as fast as with Dirichlet boundary condition.

![Figure 3.6](image)

Figure 3.6: Real (solid line) and imaginary part (dashed line) of the potential $u^{tot}$ for various $k$ as function of the angle $\theta$ of a sphere with Neumann boundary condition.

The peak at $180^\circ$ is due to focusing of creeping waves.
3.5 Analytical Solution for the Surface Current in Maxwell’s Equation on a Sphere with PEC Boundary Condition

Consider the incoming plane wave along the negative z-axis,

\[ \mathbf{E}^{inc} = \hat{x}e^{-ikz}, \]
\[ \mathbf{H}^{inc} = -\hat{y}Y e^{-ikz}. \]

The analytical expression for the scattered field of Maxwell’s equation on a PEC sphere, in spherical coordinates is then, [14]

\[ E^{sc}_r = -i \frac{\cos \phi}{k^2 R^2} \sum_{m=1}^{\infty} (-i)^m (2m + 1)b_m \psi_m^{(1)}(x) \psi_m^{(1)}(x), \]  
\[ E^{sc}_\theta = \frac{\cos \phi}{k R} \sum_{m=1}^{\infty} C_m \left( a_m \psi_m^{(1)}(x) v_m^1(\theta) + ib_m \psi_m^{(1)}(x) v_m^2(\theta) \right), \]  
\[ E^{sc}_\phi = \sin \phi \frac{\cos \phi}{k R} \sum_{m=1}^{\infty} b_m \psi_m^{(1)}(x) v_m^2(\theta) + ib_m \psi_m^{(1)}(x) v_m^1(\theta), \]  
\[ H^{sc}_r = iY \sin \phi \frac{\cos \phi}{k^2 R^2} \sum_{m=1}^{\infty} (-i)^m (2m + 1)a_m \psi_m^{(1)}(x) v_m^1(\theta), \]  
\[ H^{sc}_\theta = Y \sin \phi \frac{\cos \phi}{k R} \sum_{m=1}^{\infty} \left( b_m \psi_m^{(1)}(x) v_m^2(\theta) + ia_m \psi_m^{(1)}(x) v_m^1(\theta) \right), \]  
\[ H^{sc}_\phi = \frac{Y \cos \phi}{k R} \sum_{m=1}^{\infty} C_m \left( b_m \psi_m^{(1)}(x) v_m^2(\theta) + ia_m \psi_m^{(1)}(x) v_m^1(\theta) \right), \]

where

\[ \psi_m(x) = x j_m(x), \]
\[ \psi_m^{(1)}(x) = x h_m^{(1)}(x), \]
\[ a_m = \frac{\psi_m(ka)}{\psi_m^{(1)}(ka)}, \]
\[ b_m = \frac{\psi_m^{(1)}(ka)}{\psi_m^{(1)}(ka)}, \]
\[ C_m = (-1)^m \frac{2m + 1}{m(m + 1)}, \]
\[ v_m^1(\theta) = \frac{\partial P^1_m(\cos \theta)}{\partial \theta}, \]
\[ v_m^2(\theta) = \frac{\partial P^1_m(\cos \theta)}{\partial \theta}. \]
3.5. **Analytical Solution for the Surface Current in Maxwell’s Equation on a Sphere with PEC Boundary Condition**

In Figure 3.7, the current $J_\theta$ is presented for various wavenumbers $k$. For high frequencies the potential on the shadowside ($|\phi| > 90^\circ$) is negligible, but increases close to $\theta = 180^\circ$. This is due to focusing effects of the creeping waves. In Figure 3.7, the current $J_\phi$ is presented for various wavenumbers $k$.

![Figure 3.7](image)

Figure 3.7: Real (solid line) and imaginary part (dashed line) of $T_1$ in the current $J_\theta = YT_1(\theta)\sin \phi$ for various $k$ as function of the angle $\theta$ on a PEC sphere.

We want to split the field in a divergence-conforming and a curl-conforming part, as is done in appendix D. We define

$$
\begin{align*}
G_m^c(\theta, \phi) &= \nabla \left( r \cos \phi \mathbb{P}^1_m(\cos \theta) \right), \\
R_m^c(\theta, \phi) &= \nabla \times (\hat{n} r \cos \phi \mathbb{P}^1_m(\cos \theta)), \\
G_m^s(\theta, \phi) &= \nabla \left( r \sin \phi \mathbb{P}^1_m(\cos \theta) \right), \\
R_m^s(\theta, \phi) &= \nabla \times (\hat{n} r \sin \phi \mathbb{P}^1_m(\cos \theta)).
\end{align*}
$$
CHAPTER 3. ANALYTICAL SOLUTIONS FOR OBJECTS WITH CONSTANT CURVATURE

We get the field

\begin{align*}
E_r &= \frac{1}{k^2 R^2} \sum_{m=1}^{\infty} (-i)^m (2m + 1) B_m G_{m,r}^c (\theta, \phi), \\
E_\theta &= \frac{1}{k R} \sum_{m=1}^{\infty} (-i)^m \frac{2m + 1}{m(m + 1)} (A_m R_m^s (\theta, \phi) + B_m G_{m,\theta}^c (\theta, \phi)), \\
E_\phi &= \frac{1}{k R} \sum_{m=1}^{\infty} (-i)^m \frac{2m + 1}{m(m + 1)} (A_m R_m^s (\theta, \phi) + B_m G_{m,\phi}^c (\theta, \phi)), \\
H_r &= -\frac{iY}{k^2 R^2} \sum_{m=1}^{\infty} (-i)^m (2m + 1) A_m G_{m,r}^s (\theta, \phi), \\
H_\theta &= -\frac{iY}{k R} \sum_{m=1}^{\infty} (-i)^m \frac{2m + 1}{m(m + 1)} (B_m R_m^c (\theta, \phi) + A_m G_{m,\theta}^s (\theta, \phi)), \\
H_\phi &= -\frac{iY}{k R} \sum_{m=1}^{\infty} (-i)^m \frac{2m + 1}{m(m + 1)} (B_m R_m^c (\theta, \phi) + A_m G_{m,\phi}^s (\theta, \phi)),
\end{align*}

Figure 3.8: Real (solid line) and imaginary part (dashed line) of $T_2$ in the current $J_\phi = Y T_2 (\theta) \cos \phi$ for various $k$ as function of the angle $\theta$ on a PEC sphere.
3.6. THE ELECTROMAGNETIC OPERATOR $T$

where

$$A_m = \begin{cases} \psi_m(kR), & \text{for } E = E^{inc}, \\ -a_m \varsigma_m^{(1)}(kR), & \text{for } E = E^{sc}, \end{cases}$$

and

$$A'_m = \begin{cases} \psi'_m(kR), & \text{for } E = E^{inc}, \\ -a_m \varsigma_m^{(1)'}(kR), & \text{for } E = E^{sc}, \end{cases}$$

and a similar definition of $B_m$ and $B'_m$, where $a_m$ is substituted for $b_m$. We have showed how to split the tangential part of the field in a diverge-conforming part with basis $G_m$ and a curl-conforming part with basis $R_m$.

High Frequency Approximation

The total magnetic field on the surface can be split into a reflected and a creeping wave part,

$$H_{\text{tot}} = H_{\text{refl}} + H_{\text{cr.w.}}.$$  

The reflected magnetic wave can be represented by an expansion in $kR$,

$$H_{\text{tot}}^\theta = 2H_{\text{inc}}^\theta \left( 1 + \frac{i \sin^2 \theta}{2kR \cos^2 \theta} + \frac{5 \sin^2 \theta - \sin^4 \theta}{2k^2 R^2 \cos^6 \theta} + \ldots \right), \quad \text{(3.18)}$$

$$H_{\text{tot}}^\phi = 2H_{\text{inc}}^\phi \left( 1 - \frac{i \sin^2 \theta}{2kR \cos^2 \theta} - \frac{9 \sin^2 \theta - \sin^4 \theta}{2k^2 R^2 \cos^6 \theta} + \ldots \right), \quad \text{(3.19)}$$

where the leading terms correspond to the physical optics approximation. The contribution from the creeping waves is given in [14].

3.6 The Electromagnetic Operator $T$

For electromagnetic scattering problems with a PEC object, the unknown quantity is the surface current, $J_s$. Schelkunoff equivalence principle [41],[65], yields

$$J_s = \mathbf{n} \times \mathbf{H^{tot}}. $$

In order to compute $J_s$, we need an expression for the scattered field $\mathbf{H^{sc}}$. In [5], the operator

$$T[\mathbf{E_{tan}^{sc}}] = -\mathbf{n} \times \mathbf{H^{sc}}, $$

is introduced. From the PEC condition we have $\mathbf{E_{tan}^{sc}} = -\mathbf{E_{tan}^{inc}}$ and the surface current can be computed,

$$J_s = \mathbf{n} \times \mathbf{H^{inc}} + T[\mathbf{E_{tan}^{inc}}].$$
CHAPTER 3. ANALYTICAL SOLUTIONS FOR OBJECTS WITH CONSTANT CURVATURE

Every vector field \( \mathbf{v} \), tangential to \( \Gamma \) can be written in the basis functions

\[
\mathbf{G}_l^m = \frac{1}{\sqrt{l(l+1)}} \text{grad}_\Gamma Y_l^m, \\
\mathbf{R}_l^m = \frac{1}{\sqrt{l(l+1)}} \text{curl}_\Gamma Y_l^m,
\]
such that,

\[
\mathbf{E}^{sc\tan} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \alpha_l^m \mathbf{G}_l^m + \beta_l^m \mathbf{R}_l^m.
\]

There is an explicit expression for the operator \( T \), [5],

\[
T[\mathbf{E}^{sc\tan}] = Y \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{\gamma_l} \alpha_l^m \mathbf{G}_l^m + \gamma_l \beta_l^m \mathbf{R}_l^m, \quad (3.20)
\]

\[
\gamma_l = \frac{1 + R \partial_n \log h_l^{(1)}(kR)}{ikR}, \quad (3.21)
\]

where \( Y = \sqrt{\frac{\mu}{\epsilon}} \). This is to be compared to the symbol of the DtN-operator for the Dirichlet sphere (3.8).

In the construction of On Surface Radiation Conditions, we can separate the rotational and gradient parts by taking certain surface derivatives. This separation is used e.g. in the construction of On Surface Radiation Conditions in electromagnetics. In [4], the following is computed,

\[
\text{grad}_\Gamma \text{div}_\Gamma (\alpha_l^m \mathbf{G}_l^m + \beta_l^m \mathbf{R}_l^m) = -\frac{l(l+1)}{R^2} \alpha_l^m \mathbf{G}_l^m, \quad (3.22)
\]

\[
-\text{curl}_\Gamma \text{curl}_\Gamma (\alpha_l^m \mathbf{G}_l^m + \beta_l^m \mathbf{R}_l^m) = -\frac{l(l+1)}{R^2} \beta_l^m \mathbf{R}_l^m. \quad (3.23)
\]

This is similar to the surface derivatives of the Legendre function, (3.9).

Remark 1. In the case of a PEC sphere, the operator \( T \) can be derived from (3.12)-(3.17).
Chapter 4

On Surface Radiation Conditions

Physical optics is a practically used high frequency approximation to generate the unknown potential or current on a scattering surface. The surface is locally assumed to be flat. In this chapter we apply the theory for radiation boundary conditions to achieve better approximations of the potential or current. The curvature of the surface plays a role. A novel technique for deriving on surface radiation conditions is the use of analytical solutions for objects with constant curvature, which was introduced in Chapter 3.

4.1 Physical Optics Approximation

For high frequencies or when the scatterer has small curvature, a common approximation is to approximate the illuminated region of the scatterer locally with infinite planes. We construct the scattered field from the incoming field by using the method of images. This approach gives an expression for the potential directly from the incoming field and we do not need to solve the integral equations arising e.g. in the method of moments. Consider an incoming plane wave
\[ u^{inc} = e^{-ik(x \cos(\phi) + y \sin(\phi))}, \]
hitting an infinite plane at \( x = 0 \). The method of images suggests the scattered field
\[ u^{sc} = C_1 e^{-ik(-x \cos(\phi) + y \sin(\phi))}, \]
where \( C_1 \) is chosen such that the boundary condition is fulfilled. A simple computation yields \( C_1 = -1 \) and 1 for a Dirichlet and Neumann plane, respectively. A direct computation yields the PO approximation in the plane \( x = 0 \),

\[
\begin{align*}
\begin{cases}
  u^{tot} = 0, \\
  \partial_n u^{tot} = 2\partial_n u^{inc},
\end{cases} & \text{ on a Dirichlet plane,} \quad (4.1) \\
\begin{cases}
  u^{tot} = 2u^{inc}, \\
  \partial_n u^{tot} = 0,
\end{cases} & \text{ on a Neumann plane.} \quad (4.2)
\end{align*}
\]
The relative error of the scattered part of the potential is presented in Figure 4.1 for both a Dirichlet and Neumann circular cylinder. The relative error is very large for the Neumann cylinder for angles slightly larger than 90°. This is due to the slow decay of the total potential on the shadow region in Figure 3.3. The relative error for a sphere is presented in Figure 4.2. The error in the shadow region is smaller in 3D than in 2D.

Figure 4.1: Relative error of the potential computed with physical optics, for a circular cylinder with wave number \( k = 5, 20, 80 \) and 320.

Figure 4.2: Relative error of the potential computed with physical optics, for a sphere with wave number \( k = 5, 20, 80 \) and 320.

In order to extend this to a general convex domain, we define the lit region
4.1. PHYSICAL OPTICS APPROXIMATION

which is the points on the scatterer $\Gamma$, which are illuminated by the incoming field. In other words with the incoming field direction $\mathbf{k} = (\cos(\phi), \sin(\phi))$ and normal $\mathbf{n}(y)$, the lit region is the points $y \in \Gamma_{y}^{\text{lit}} \subset \Gamma_{y}$ s.t.

$$\mathbf{k} \cdot \mathbf{n}(y) > 0.$$  

Figure 4.3: Illustration of methods of images and lit region.

For completion, we also define the shadow region $\Gamma_{y}^{\text{shad}}$, and the boundary between the lit and shadow regions, $\partial \Gamma_{y}^{\text{lit}}$,

$$\Gamma_{y}^{\text{shad}} = \Gamma_{y} \setminus \Gamma_{y}^{\text{lit}}, \quad \partial \Gamma_{y}^{\text{lit}} = \{ y \in \Gamma_{y} : \mathbf{k} \cdot \mathbf{n}(y) = 0 \}.$$  

Inserting these conditions for the total field in Integration formulation 1.17 and 1.18 in the introduction yields a direct expression for the scattered field. The
expression is presented in Integration formulation 1 and 2 for the scattering problem with Dirichlet and Neumann boundary, respectively. The Green’s function $G$ is defined in (1.15) and (1.16) for 2D and 3D respectively.

**Integral Formulation 1** (PO approximation of a Dirichlet problem).

$$u^{sc}(x) = -2 \int G(x,y)\partial_n u^{inc}(y)d\Gamma_{lit}$$ \hfill (4.3)

**Integral Formulation 2** (PO approximation of a Neumann problem).

$$u^{sc}(x) = 2 \int \partial_n G(x,y)u^{inc}(y)d\Gamma_{lit}$$ \hfill (4.4)

**Remark 2.** The physical optics condition in acoustics generalizes trivially to 3D.

### 4.2 PO for a PEC Object in Electromagnetics

A PEC boundary yields two boundary conditions,

- $-\hat{n} \times \hat{n} \times \mathbf{E}^{tot} = 0$, \hfill (4.5)
- $\hat{n} \cdot \mathbf{H}^{tot} = 0$. \hfill (4.6)

The unknown quantity in Maxwell’s equations is the surface current $\mathbf{J}_s$, which by Schelkunoff equivalence [41],[65] is $\mathbf{J}_s = \hat{n} \times \mathbf{H}^{tot}$. The physical optics approach yields a direct expression for this quantity. Consider the problem where an incoming magnetic field $\mathbf{H}^{inc} = \hat{h}e^{-ikm_r}$ illuminates an infinite plane at $z = 0$ and produces a scattered field $\mathbf{H}^{sc}$. The polarization is orthogonal to the direction of the field, $\hat{h} \cdot \hat{n} = 0$. The direction of the field is reflected in the normal direction, but remains the same in the tangential direction. Since the polarization is orthogonal to the direction, we also get reflection in the polarization and the scattered field becomes $\mathbf{H}^{sc} = (h_x, h_y, -h_z)e^{-ik(m_x, m_y, m_z)}r$. It holds that, with $\hat{n} = (0,0,1)$,

$$\hat{n} \cdot \mathbf{H}^{tot} = 0, \quad \hat{n} \times \mathbf{H}^{sc} = \hat{n} \times \mathbf{H}^{inc},$$

i.e. the PEC-condition is fulfilled and the PO-approximation becomes

$$\hat{n} \times \mathbf{H}^{tot} = \begin{cases} 2\hat{n} \times \mathbf{H}^{inc}, & \text{on } \Gamma_{lit}, \\ 0, & \text{on } \Gamma_{shad}. \end{cases}$$

This generalizes trivially to planes other than $z = 0$ (since everything is linear).

In Figure 4.4, the relative error of the scattered part of the current is presented as a function of $\theta$ for two different angles $\phi = 0^\circ$ and $90^\circ$. The error on the shadow side is larger when $\phi = 0^\circ$. 
4.3 APPROXIMATION OF LOGARITHMIC DERIVATIVES

The analytical solution to several different wave propagation problems contains logarithmic derivatives of either Hankel functions or spherical Hankel functions. The goal is to find an expression in closed form that is an approximation to the logarithmic derivatives. The closed expression can then be used to construct different conditions for the potential on the scatterer. In later sections the derivations of higher order generalized physical optics and several different OSRCs are presented, where the approximations obtained in this section are used. We will derive approximations to the different logarithmic derivatives, which are valid for the dual parameter \( m \), where \( m \leq M_1 < kR \). Thereafter, we will estimate the introduced errors on the parameter interval \( m > M_2 > kR \), as well as for \( M_1 \leq m < M_2 \).

The goal is to find an approximation for (3.3) and (3.8) for parameter values \( m < kR \), see Figure 4.5.

In order to find a new OSRC in 2D, we need an approximation of the logarithmic derivative \( \partial_n \log H_m^{(1)}(kR) \) on a closed form, where \( |m| < kR \). In 3D, we have the corresponding logarithmic derivative \( \partial_n \log h_m^{(1)}(kR) \), where \( h_m^{(1)} \) is the spherical Hankel function. We will derive an approximation for the logarithmic derivatives in two different ways. The first approach is to use a representation of the logarithmic derivatives in polar coordinates. In the second approach we derive exactly the same approximation, using certain formulas in [1].

Figure 4.4: Relative error of the current \( J \) computed with physical optics, for a sphere with wave number \( k = 5, 20, 80 \) and 320.
Derivation of Approximation to Logarithmic Derivatives Using Polar Coordinates

The Hankel function can be written in polar coordinates,

\[ H_{\nu}^{(1)}(x) = M_\nu(x)e^{i\theta_\nu(x)}, \]

so that

\[ \log H_{\nu}^{(1)}(x) = \log |M_\nu(x)| + i\theta_\nu(x). \]

We have the representation

\[ M_\nu(x) = \sqrt{J_\nu(x)^2 + Y_\nu(x)^2}, \]

\[ \theta_\nu(x) = \arctan \frac{Y_\nu(x)}{J_\nu(x)}. \]
4.3. APPROXIMATION OF LOGARITHMIC DERIVATIVES

which can be expanded (9.2.28 in [1]),

\[ M^2_\nu(x) = \frac{2}{\pi x} S_\nu(x), \]  
\[ S_\nu(x) = 1 + \sum_{p=1}^{\infty} a_p(\nu) \left( \frac{\nu}{x} \right)^{2p}, \]  
\[ a_p = \frac{(2p-1)!!(2p)!!}{(2p)!!} \prod_{q=1}^{p} \left( 1 - \frac{(q-\frac{1}{2})^2}{\nu^2} \right). \]

Moreover, the truncated expansion \( S_{\nu,n} = 1 + \sum_{p=1}^{n-1} a_p(\nu) \left( \frac{\nu}{x} \right)^{2p} \) error is bounded by the next term, i.e.

\[ |S_{\nu,n} - S_\nu| \leq |a_n(\nu)| \left( \frac{\nu}{x} \right)^{2n}, \]

provided that \( n \geq |\nu| - \frac{1}{2} \). The error bound is essential, since the coefficients \( a_p(\nu) \to \infty \) as \( p \to \infty \). If \( n \) becomes too large, the error term will start to increase.

We want to express the logarithmic derivatives of the Hankel function as the expansion \( S_\nu(x) \).

**Lemma 9.** The logarithmic derivative of the (spherical) Hankel function is

\[ \partial_x \left( \log H^{(1)}_\nu(x) \right) = -\frac{1}{2x} + \frac{1}{2} \frac{S'_\nu(x)}{S_\nu(x)} + i \frac{1}{S_\nu(x)}. \]

\[ \partial_x \left( \log h^{(1)}_{\nu-1/2}(x) \right) = -\frac{1}{x} + \frac{1}{2} \frac{S'_\nu(x)}{S_\nu(x)} + i \frac{1}{S_\nu(x)}. \]

**Proof.** The real parts follow directly from taking logarithmic derivatives of \( M_\nu \) and \( \sqrt{\pi x} M_\nu \) for the Hankel and spherical Hankel functions respectively. The imaginary part can be derived,

\[ \partial_x \theta_\nu(x) = \partial_x \arctan \frac{Y_\nu(x)}{J_\nu(x)} = J_\nu(x) \partial_x Y_\nu(x) - \partial_x J_\nu(x) Y_\nu(x) \]

\[ J_\nu^2(x) + Y_\nu^2(x) = \frac{1}{S_\nu(x)}, \]

using the recurrence relation 9.1.27 and the Wronskian 9.1.16 in [1],

\[ 2 \partial_x C_\nu(x) = C_{\nu-1}(x) - C_{\nu+1}(x), \]

\[ J_{\nu+1}(x) Y_\nu(x) - J_\nu(x) Y_{\nu+1}(x) = \frac{2}{\pi x}, \]

where \( C \) is either \( J \) or \( Y \).
In order to obtain an approximation of the logarithmic derivatives, we will approximate the coefficients \( a_p \), such that \( S_\nu \) is expressed in an appropriate closed form. We reformulate \( a_p \),

\[
a_p = \frac{(2p-1)!!}{(2p)!!} \prod_{q=1}^{p} \left( 1 - \frac{(q - \frac{1}{2})^2}{\nu^2} \right)
\]

\[
= \frac{(2p-1)!!}{(2p)!!} \left( 1 - \sum_{q=1}^{p} \frac{(q - \frac{1}{2})^2}{\nu^2} + \sum_{q_1, q_2 = 1}^{p} \frac{(q_1 - \frac{1}{2})^2(q_2 - \frac{1}{2})^2}{\nu^4} + \ldots \right).
\]

A first approximation \( \tilde{S}_\nu \approx S_\nu(x) \) is obtained by ignoring the \( \frac{1}{\nu^2} \)-terms in \( a_p \), for \( q \geq 1 \).

\[
\tilde{a}_p = \frac{(2p-1)!!}{(2p)!!},
\]

which yields the closed expression, with \( y = \frac{\nu^2}{x^2} \),

\[
\tilde{S}_\nu(x) = \frac{1}{\sqrt{1-y}}.
\]

Since \( y \) is defined in terms of \( \nu \), we can not expect to get a second order approximation in \( \frac{1}{x} \) because the number of terms needed in (4.8) depends on \( \nu \). In Figure 4.6 we present the relative error and the numerical order of the approximation \( S_n u(x) \) as a function of \( \frac{x}{R} \). When \( x \) is large, then \( \tilde{S}_\nu(x) \) is an almost second order approximation of \( S_\nu(x) \) if \( \frac{x}{R} \) is not close to one.

The approximation for the logarithmic derivatives becomes, with \( y = \frac{\nu^2}{x^2} \),

\[
\partial_x \log H_{\nu}^{(1)}(x) \approx -\frac{1}{2x} \frac{1}{1-y} + i \sqrt{1-y}, \quad (4.10)
\]

\[
\partial_x \log b_{\nu}^{(1)}(x) \approx -\frac{1}{2x} \frac{2-y}{1-y} + i \sqrt{1-y}. \quad (4.11)
\]

In 2D, we can identify \(-\frac{m^2}{R^2} \hat{u} = \partial^2_x \hat{u} \), which yields the Taylor expansion

\[
\partial_n u = \sum_{p=0}^{\infty} \left( -\frac{1}{2R} (-1)^p + ik \left( \frac{\frac{1}{2}}{p} \right) \right) \frac{\partial^2_{\nu} u}{k^{2p}}. \quad (4.12)
\]

In 3D, we identify \(-\frac{m(n+1)}{R^2} \hat{u} = (\Delta - \frac{1}{R^2}) u \), which yields the similar Taylor expansion

\[
\partial_n u = -\frac{u}{2R} + \sum_{p=0}^{\infty} \left( -\frac{1}{2R} (-1)^p + ik \left( \frac{\frac{1}{2}}{p} \right) \right) \frac{(\Delta R - \frac{1}{2R})^p u}{k^{2p}}. \quad (4.13)
\]
4.3. APPROXIMATION OF LOGARITHMIC DERIVATIVES

Figure 4.6: The approximation \( \tilde{S}_\nu(x) \) as function of \( \frac{\nu}{x} (= \sqrt{y}) \).
In subfigure a), the relative error \( \left| \frac{S_\nu(x) - \tilde{S}_\nu}{S_\nu} \right| \) is presented for \( x = 2.5, 5, 10 \) up to 640. The relative error decreases with \( x \).
In subfigure b), the numerical order \( \log_2 \left| \frac{S_\nu(x) - \tilde{S}_\nu}{S_\nu(2x) - \tilde{S}_\nu} \right| \) is presented for \( x = 2.5, 5, 10 \) up to 320. The order increases with \( x \).

Remark 3. The expression (4.13) can be approximated to second order by replacing \((m + 1/2)^2\) with \(m(m + 1)\), which are related to the Laplacian of the Legendre function.

Remark 4. Taking the first two terms in the expansion (4.12) yields 2DEM2. Similarly, taking the first two terms in (4.13) yields 3DEM2 (up to higher order terms).

In order to get a more accurate expression for the logarithmic Hankel function, the approximation of the expansion (4.8) of \( S_\nu(x) \) has to be improved. We approximate the product in the coefficient of the expansion (4.9) and keep \( y = -\frac{x^2}{2\pi} \) constant.

\[
\prod_{q=1}^{p} \left(1 - \frac{(q - \frac{1}{2})^2}{\nu^2}\right) = 1 + \frac{1}{x^2 y} \sum_{q=1}^{p} (q - \frac{1}{2})^2 + O \left( \frac{1}{x^2} \right),
\]
Lemma 10. Let \( f = f_0 + \frac{1}{2}f_1 + \frac{1}{x^2} f_2 \), with \( f_0 \neq 0 \). The polynomial approximation \( g \) of the inverse \( f \) has the expression \( g = \frac{1}{f_0} - \frac{4}{f_0^3} f_1 + \frac{8}{f_0^5} f_2 \), such that \( fg = 1 + O\left(\frac{1}{x^2}\right) \).

Proof. A direct computation yields

\[
fg = 1 + \frac{1}{x^3} \frac{f_1^3 - 2f_0 f_1 f_2}{f_0^2} + \frac{1}{x^3} \frac{f_1^2 f_2 - f_0 f_2^2}{f_0^2}.
\]

In certain cases, it is useful to have an expression for the inverse function \( \frac{1}{S_\nu(x)} \) as a polynomial. We will now construct a general third order inverse to a third order expression.

Lemma 11. The logarithmic derivative admits a third order approximation in \( x \),

\[
\frac{1}{S_\nu(x)} = \frac{1}{x(1+y)^{3/2}} \left( y(1+y)^3 + \frac{1}{8x^2} \left( 2 - 21y + 12y^2 \right) \right) + O\left(\frac{1}{x^4}\right).
\]
when \( y = \frac{\nu}{x} \) is constant,

\[
\partial_x \log H^{(1)}_{\nu}(x) = i\sqrt{1 - y^2} - \frac{1}{2x} \frac{1}{1 - y^2} + i \frac{1 + 4y^2}{8x^2(1 - y^2)^{3/2}} \quad (4.14)
\]

\[
\partial_x \log h^{(1)}_{\nu - 1/2}(x) = i\sqrt{1 - y^2} - \frac{1}{2x} \frac{2 - y^2}{1 - y^2} + i \frac{1 + 4y^2}{8x^2(1 - y^2)^{3/2}} \quad (4.15)
\]

\[
\frac{1 + x\partial_x \log h^{(1)}_{\nu - 1/2}(x)}{ix} = \sqrt{1 - y^2} + i \frac{1}{2x} \frac{y^2}{1 - y^2} + \frac{1 + 4y^2}{8x^2(1 - y^2)^{3/2}} \quad (4.16)
\]

**Proof.** The expression (4.14) is already proved. Expression (4.15) follows from that

\[
\partial_x \log h^{(1)}_{\nu - 1/2}(x) = \partial_x \log H^{(1)}_{\nu}(x) - \frac{1}{2x}.
\]

Finally, expression (4.16) follows from (4.15).

**Lemma 12.** The following third order approximations of the inverse expressions hold

\[
\frac{1}{\partial_x \log H^{(1)}_{\nu}(x)} = i\sqrt{1 - y^2} - \frac{1}{2x} \frac{1}{2x(1 - y^2)^2} + i \frac{3 + 4y^2}{8x^2(1 - y^2)^{3/2}} \quad (4.17)
\]

\[
\frac{1}{\partial_x \log h^{(1)}_{\nu - 1/2}(x)} = i\sqrt{1 - y^2} - \frac{1}{2x} \frac{2 - y^2}{2x(1 - y^2)^2} + i \frac{9 - 4y^2 + 2y^4}{8x^2(1 - y^2)^{3/2}} \quad (4.18)
\]

\[
\frac{1}{ix \partial_x \log h^{(1)}_{\nu - 1/2}(x)} = \frac{1}{\sqrt{1 - y^2}} - \frac{1}{2x} \frac{y^2}{2x(1 - y^2)^2} + \frac{1 + 4y^2 + 2y^4}{8x^2(1 - y^2)^{3/2}} \quad (4.19)
\]

**Proof.** This lemma is proved by using lemma 10.

**Alternative Derivation of the Approximation of the Logarithmic Derivative**

The approximation of the coefficients \( a_p(\nu) \) above, where we approximate a product with the identity, is not satisfactory. Therefore, we will derive the expression (4.10) in another way. Once we have (4.10), it is easy to compute (4.11), since \( \partial_x \log h^{(1)}_m(x) = -\frac{1}{2x} + \partial_x \log H^{(1)}_{m+1/2}(x) \). We use the formulas 9.3.15-22 in [1],
which slightly rewritten are

\[ H_\nu^{(1)}(\nu \sec \beta) = \sqrt{\frac{2}{\pi \nu \tan \beta}} (L - iM) e^{i\psi}, \]

\[ H_\nu^{(1)'}(\nu \sec \beta) = \sqrt{\frac{\sin 2\beta}{\pi \nu}} (iN - O) e^{i\psi}, \]

\[ L = \sum_{p=0}^{\infty} \frac{u_{2p}(i \cot \beta)}{\nu^{2p}}, \]

\[ M = -i \sum_{p=0}^{\infty} \frac{u_{2p+1}(i \cot \beta)}{\nu^{2p+1}}, \]

\[ N = \sum_{p=0}^{\infty} \frac{v_{2p}(i \cot \beta)}{\nu^{2p}}, \]

\[ O = i \sum_{p=0}^{\infty} \frac{v_{2p+1}(i \cot \beta)}{\nu^{2p+1}}, \]

where \( u_p \) and \( v_p \) are 3\( p \)th degree polynomials defined in 9.3.10 and 9.3.14 respectively in [1]. If we identify \( \beta = \arccos y \), where \( y = \frac{m}{2} \), we get

\[ \partial_x \log H_\nu^{(1)}(x) = \sqrt{\frac{\sin 2\beta \tan \beta}{2}} \frac{iN - O}{L - iM}. \]

We need the following identities,

**Lemma 13.** The following two identities hold for \( y \in [-1, 1] \),

\[ \sin 2 \arccos y = 2y \sqrt{1 - y^2} \]

\[ \tan \arccos y = \frac{\sqrt{1 - y^2}}{y} \]

**Proof.** Do the substitution \( y = \cos \phi, \phi \in [0, \pi] \). Then \( \sin \phi = \sqrt{1 - y^2} \) and

\[ \sin 2 \arccos y = \sin 2\phi = 2 \cos \phi \sin \phi = 2y \sqrt{1 - y^2} \]

and

\[ \tan \arccos y = \tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{\sqrt{1 - y^2}}{y} \]

\[ \Box \]

**Lemma 14.** The following expression holds for the logarithmic derivate of the Hankel function, valid for \( y = \frac{m}{2} \in [0, 1] \)

\[ \partial_x \log H_\nu^{(1)}(x) = \sqrt{\frac{\sin 2\beta \tan \beta}{2}} \frac{iN - O}{L - iM} \]
4.3. APPROXIMATION OF LOGARITHMIC DERIVATIVES

Approximating the rational function yields
\[ \partial_x \log H_m^{(1)}(x) = \frac{i\sqrt{1-y^2} - \frac{1}{2x(1-y^2)} + \mathcal{O} \left( \frac{1}{x^3} \right)}{1 - \frac{1 + 4y^2}{8x^2(1-y^2)^3} + \mathcal{O} \left( \frac{1}{x^3} \right)} \]
\[ = \left( \frac{i\sqrt{1-y^2} - \frac{1}{2x(1-y^2)}}{1 + \frac{1 + 4y^2}{8x^2(1-y^2)^3}} \right) \frac{1}{\mathcal{O} \left( \frac{1}{x^3} \right)} \]
\[ = i\sqrt{1-y^2} - \frac{1}{2x(1-y^2)} + i\frac{1 + 4y^2}{8x^2(1-y^2)^{3/2}} + \mathcal{O} \left( \frac{1}{x^3} \right), \]
where the \( \mathcal{O} \)-terms are functions of \( \frac{1}{\sqrt{1-y^2}} \) which tend to infinity as \( y \to 1 \). The expression (4.20) is exactly the same as (4.14).

Regularization of the Approximation

A second order approximation to the logarithmic derivative of the Hankel function \( \partial_x \log H_m^{(1)}(x) \) is
\[ F(x, y) = i\sqrt{1-y^2} - \frac{1}{2x(1-y^2)}, \quad (4.20) \]
where \( y = \frac{m}{x} \) is considered as independent of \( x \). The approximation has a singularity at \( y = 1 \), which causes many problems. We can regularize (4.20) with a second order correction,
\[ F_\delta(x, y) = \begin{cases} 
    i\sqrt{1-y^2} - \frac{1}{2x(1-y^2)} + \frac{1}{2x(1-y^2)^{3/2}}, & y \leq 1, \\
    i\sqrt{1-y^2} + \frac{1}{2x(1-y^2)^{3/2}}, & y > 1,
\end{cases} \quad (4.21) \]
where \( \delta \) is a small parameter independent of \( x \) and \( y \). This regularization will be used in the construction of a generalized PO approximation, but also in the analysis of the error contribution from the large values of the dual parameter \( m \). In the construction of the second order OSRCs, we can use the original approximation (4.20) without the regularization, since the approximation itself is regularized.

Error Analysis

We have found both a second and third order approximation \( F(x, y) \) of the logarithmic derivative of the Hankel function for the case where the dual parameter \( m < M_1 < kR \). Using the approximation \( F \) introduces an error, which can be divided in three parts, \( m = 0 \ldots M_1 - 1, m = M_1 \ldots M_2 - 1 \) and \( m = M_2 \ldots \infty \), where \( M_1 < kR \) and \( M_2 > kR \). If we denote \( \varepsilon_m = \partial_x \log H_m^{(1)}(x)\big|_{x=kR} - F \left( x, \frac{m}{x} \right) \), we obtain the error
\[ E = \sum_{m=0}^{M_1-1} \varepsilon_m e_m k \tilde{u}_m^{ae} \cos m\phi + \sum_{m=M_1}^{M_2-1} \varepsilon_m e_m k \tilde{u}_m^{ae} \cos m\phi + \sum_{m=M_2}^{\infty} \varepsilon_m e_m k \tilde{u}_m^{ae} \cos m\phi. \]
In the case of a circular cylinder, the Fourier coefficient \( \hat{u}_{sm}^{sc} = -(-i)^m J_m(kr) \) can be estimated when \( m \geq M_2 > kR \). Define the quotient \( \frac{m}{R} = \frac{1}{\cos \theta_m}, \theta_m \in [0, \frac{\pi}{2}] \). The Fourier coefficient can then be bounded by 9.1.63 in [1],

\[
|\hat{u}_{sm}^{sc}| \leq \left( \frac{\cos \theta_m}{1 + \sin \theta_m} e^{\sin \theta_m} \right)^m =: \varepsilon(\theta_m)^m.
\]

Since \( \varepsilon(\theta_m) \) is a decreasing function in \( \theta \), we have

\[
\varepsilon(\theta_m)^m \leq \varepsilon(\theta_{M_2})^m, \quad \text{for } m \geq M_2.
\]

**Lemma 15.** We get the estimate for the truncation error where \( M > kR \),

\[
\left| \sum_{m=M_2}^{\infty} 2 \frac{m}{R} \hat{u}_{sm}^{sc} \cos m\phi \right| \leq 2 M_2 \varepsilon(\theta_{M_2})^{M_2} \frac{R}{\left( 1 - \varepsilon(\theta_{M_2})^2 \right)^2}.
\]

Proof. We obtain with \( z = \varepsilon(\theta_{M_2}) \)

\[
\left| \sum_{|m| \geq M_2} 2 \frac{m}{R} \hat{u}_{sm}^{sc} \cos m\phi \right| \leq 2 \sum_{m=M_2}^{\infty} \frac{m}{R} z^m = 2 M_2 z^{M_2} \frac{1 - (1 - z)}{R(1 - z)^2} \left( 1 - \frac{M_2 - 1}{M_2} z \right),
\]

where \( 1 - \frac{M_2 - 1}{M_2} z \leq 1 \).

The errors are presented in Figure 4.7, in the case where the radius \( R = 1 \) and \( k = 5, 20, 80 \) and 320. For large wave numbers, the decay in error is rapid for increasing quotient \( \frac{m}{R} > 1 \).

When \( F \) is a second order approximation of the logarithmic derivative of the Hankel function, then we can bound \( F(x, y) < (1 + \delta)y \), when \( y^3 - y > 1 \) for \( x = kR \). Therefore, we can use Lemma 15 to get a bound on

\[
\sum_{m=M_2}^{\infty} 2k F(x, y) \hat{u}_{sm}^{sc} \cos (m\phi) \leq (1 + \delta) \frac{2 M_2 \varepsilon(\theta_{M_2})^{M_2}}{R(1 - \varepsilon(\theta_{M_2})^2)^2}.
\]

Numerical computations show that we can bound \( |\partial \log H_m^{(1)}(x)| < \frac{M_2}{x^3} \), and we can again use Lemma 15 to get a bound. For sufficiently large \( x = kR \), we can always find a \( M_2 \) such that the contributions from the dual parameter \( m \geq M \) is negligible.

The contributions from the midrange terms where \( M_1 < |m| < M_2 \) is hard to estimate since delicate cancellation plays an important role. These terms contribute to the creeping waves. An upper bound of the terms should thus contain the factor with exponential decay in terms of \( k \) that is typical for creeping waves, see [44], [50], [53].
4.4  On Surface Radiation Conditions

We will develop rational approximations of the logarithmic derivatives of the different Hankel functions $\partial_x \log H^{(1)}_m(x)$ and $\partial_x \log h^{(1)}_m(x)$. In [2], [3], Alpert, Greengard and Hagstrom found a rational approximation for a fixed $m$, using the zeros [22] of the Hankel function of real order. They also gave an algorithm of how to compute the coefficients in the rational expression. An alternative way of computing the coefficients in the framework of model reduction of linear dynamical systems is found in [51]. An alternative rational approximation is given in [20]. This approximation can be used in the design several different OSRCs.

Consider a scattering problem that is solved numerically with a time domain discretization of the wave equation or Maxwell’s equation for a field in the exterior of the scatterer. In order to restrict the computational domain, one needs to introduce absorbing boundary to restrict the computational domain. One of the first such conditions is the Engquist-Majda condition, [29] in 1977, which approximates the symbol of the wave equation on an infinite plane and for a circular cylinder. A second order Engquist-Majda condition for the scalar Helmholtz equation in polar coordinates $(r, \theta)$ is,

$$\partial_r u = \left( ik - \frac{1}{2r} \right) u + \frac{1}{2k^2} \left( ik + \frac{1}{r} \right) \frac{1}{r^2} \partial_\theta^2 u. \quad (4.24)$$

Another condition is the Bayliss-Turkel condition, [11] from 1980, which absorbs
waves in the radial direction, by annihilation of the first terms in the radial expansion of the scattered field. In spherical coordinates, the radial expansion for a solution to Helmholtz equation is

$$u(r, \theta, \phi) = e^{ikr} \sum_{j=0}^{\infty} \frac{u_j(\theta, \phi)}{r^{j+1}}.$$  (4.25)

The first $m$ terms are annihilated by the boundary operator

$$B_m = \prod_{l=1}^{m} \left( -ik + \partial_r + \frac{2l-1}{r} \right).$$

After elimination of $\partial_r^2$ in $B_2$, the second order Bayliss-Turkel condition becomes

$$\left( \frac{2}{r} - 2ik \right) \partial_r u = \left( 2k^2 + \frac{4ik}{r} - \frac{2}{r^2} \right) u + \frac{1}{r^2} \Delta_0 u,$$  

where $\frac{1}{r^2} \Delta_0$ is the surface Laplacian of a sphere. A similar ABC is obtained in 2D,

$$\left( \frac{2}{r} - 2ik \right) \partial_r u = \left( 2k^2 + \frac{3ik}{r} - \frac{3}{4r^2} \right) u + \frac{1}{r^2} \partial_\phi^2 u.$$  (4.26)

It should be noted that the Engquist-Majda conditions are equivalent to Bayliss-Turkel on a circular cylinder for an incoming plane wave.

In 1986 a Higdon type absorbing boundary condition was constructed, [39]. It is a condition that is absorbing in some arbitrary directions, as a generalization to the Engquist-Majda condition, which is absorbing in the normal direction. A condition that is perfectly absorbing at angles $\pm \phi_j$, $j = 1, \ldots, m$ is given by

$$\left( \prod_{j=1}^{m} \left( \partial_n - ik \cos \phi_j \right) \right) u = 0.$$  (4.27)

A reformulation of the Higdon type condition was proposed by Givoli and Neta in [33] in 2003. They introduced new variables in order to avoid high order derivatives. In 2004, Hagstrom and Warburton, [37] modified the Givoli and Neta ABC in order to get an even smaller reflection. The perfectly matched layer was introduced by Berenger [12] in 1994. This is one of the most common ways of truncating the computational domain but is not used to design an OSRC.

In Chapter 4.1, we have discussed how to directly approximate the potential with an expression in the known incoming field with the physical optics approximation. The relation is exact when the scatterer is an infinite plate. The aim with this chapter is to develop a similar approximation which has a small error for a cylinder in 2D and a sphere in 3D. Two-dimensional objects with non-constant curvature $\kappa$ are approximated locally with a circle with the same radius of curvature $R = \frac{1}{\kappa}$. A
4.4. **ON SURFACE RADIATION CONDITIONS**

A general 3D object has two curvatures in every point. We choose to use the sphere with mean curvature \( H \), with the radius of curvature \( R = \frac{1}{H} \).

One idea is to use an existing absorbing boundary condition to get a relation between \( \partial_n u_{sc} \) and \( u_{sc} \). This idea was first proposed by Kriegsmann et.al. [49] in 1987 for a cylinder in 2D. They used a similar annihilating operator as in the Bayliss-Turkel ABC. This was extended to 3D by Jones in [46]. He used a recursion formula for the coefficients in the expansion (4.25),

\[
2ik(j + 1)u_{j+1} = j(j + 1)u_j + \Delta_0 u_j,
\]

to get the radiation condition

\[
\partial_r u = \left( ik \frac{1}{r} \right) u + \frac{i}{2kr^2} \Delta_0 u.
\]

One approach, suggested by Calvo et.al. [20] uses the logarithmic derivative of Hankel-functions,

\[
\partial_r u = kLu,
\]

\[
L \left( \partial^2 \frac{1}{kr}, 1 \right) = \frac{1}{k} \partial r \log \left( H \sqrt{-\partial^2} \right) - (kr).
\]

The operator \( L \) can be approximated with a rational expression

\[
L \left( X, \frac{1}{kr} \right) = L_0 \left( \frac{1}{kr} \right) \prod_{i=1}^{N} \frac{1 + \alpha_{i,N} X}{1 + \beta_{i,N} X},
\]

\[
L_0 \left( \frac{1}{kr} \right) = \frac{1}{k} \partial_r \log H_0^{(1)}(kr).
\]

The coefficients are computed using least squares in a certain region of the parameter \( kr \).

The original artificial boundary conditions in 2D are expressed in terms of \( \partial_r \), \( \frac{1}{R} \) and \( \partial_\phi \), where \( R \) is the radius and \( \phi \) is the angle on a circular cylinder. In order to obtain a general ABC, the substitution

\[
\partial_r \rightarrow \partial_n, \quad \frac{1}{R} \rightarrow \kappa, \quad \frac{1}{R^2} \partial^2 \phi \rightarrow \partial^2 s.
\]

The corresponding ABC in 3D are expressed on a sphere, with radius \( R \) and the surface derivatives \( \Delta_0 \). The substitutions in order to obtain a general formula are
CHAPTER 4. ON SURFACE RADIATION CONDITIONS

<table>
<thead>
<tr>
<th>OSRC</th>
<th>Abbr.</th>
<th>$A_0$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Engquist-Majda 2D</td>
<td>$2DEM1$</td>
<td>$ik - \frac{k}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>2nd Engquist-Majda 2D</td>
<td>$2DEM2$</td>
<td>$ik - \frac{k}{2}$</td>
<td>$\frac{1}{2k^2}(ik + \kappa)$</td>
</tr>
<tr>
<td>1st Engquist-Majda 3D</td>
<td>$3DEM1$</td>
<td>$ik - \mathcal{H}$</td>
<td>0</td>
</tr>
<tr>
<td>2nd Engquist-Majda 3D</td>
<td>$3DEM2$</td>
<td>$ik - \mathcal{H}$</td>
<td>$\frac{1}{2k^2}(ik + \mathcal{H})$</td>
</tr>
<tr>
<td>1st Beyliss-Turkel 2D</td>
<td>$2DBT1$</td>
<td>$ik - \frac{k}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>2nd Beyliss-Turkel 2D</td>
<td>$2DBT2$</td>
<td>$\frac{1}{2k^2-2ik} \left(3ik\kappa + 2k^2 - \frac{3\kappa^2}{4}\right)$</td>
<td>$\frac{1}{2k^2-2ik}$</td>
</tr>
<tr>
<td>1st Beyliss-Turkel 3D</td>
<td>$2DBT1$</td>
<td>$ik - \mathcal{H}$</td>
<td>0</td>
</tr>
<tr>
<td>2nd Beyliss-Turkel 3D</td>
<td>$2DBT2$</td>
<td>$\frac{1}{2k^2-2ik} (k^2 + 2ik\mathcal{H} - \mathcal{H}^2)$</td>
<td>$\frac{1}{2k^2-2ik}$</td>
</tr>
</tbody>
</table>

Table 4.1: Different OSRC, $\partial_n u = A_0 u + A_2 \Delta \Gamma u$.

now

$$\frac{1}{R} \rightarrow \mathcal{H},$$

$$\frac{1}{R^2} \Delta_0 \rightarrow \Delta \Gamma,$$

where $\mathcal{H}$ is the mean curvature and $\Delta \Gamma$ is the surface Laplacian. See (B.11) in the appendix for a motivation that $\frac{1}{R}$ should be replaced by the mean curvature in 3D, at least when Helmholtz equation is used in the derivation.

Most ABC yields a relation that can be written

$$\partial_n u = OSRC[u] = \begin{cases} A_0 u + A_2 \partial^2_s u, & \text{in 2D}, \\ A_0 u + A_2 \Delta \Gamma u, & \text{in 3D}, \end{cases}$$

where $s$ is an arclength parameter on the 2D scatterer. Some examples are listed in Table 4.1.

When the OSRC-condition is applied to the scattered field on the surface with Dirichlet boundary conditions, we get

$$\partial_n u^s = OSRC[u^s] = -OSRC[u^{inc}],$$

where $u^s$ is the scattered field and $u^{inc}$ is the incident field.
and Integral formulation 1.17 and 1.18 in the introduction for a Dirichlet surface yields

**Integral Formulation 3** (OSRC approximation of a Dirichlet problem).

\[ u^{sc} = -\int G (\partial_{n}u^{inc} - OSRC[u^{inc}]) \, d\Gamma. \]  

(4.34)

In the Neumann case, the corresponding Integral formulation yields

**Integral Formulation 4** (OSRC approximation of a Neumann problem).

\[ u^{sc} = \int \partial_{n}G (u^{inc} - OSRC^{-1}[\partial_{n}u^{inc}]) \, d\Gamma. \]  

(4.35)

### 4.5 Higdon Type OSRC

The most common OSRCs arise from existing Absorbing Boundary Conditions for a circular boundary. These ABCs are accurate for waves in the outwards normal direction. For scattering problems this yields an accurate condition when the incoming fields are parallel to the normal. We develop a condition that is accurate for different incident angles, which arises from the PO condition (4.1). The Dirichlet PO conditions yields

\[ \partial_{n}u^{sc} = \begin{cases} \partial_{n}u^{inc} = -ik \cos(\phi)u^{inc}, & \text{on } \Gamma_{lit}^{y}, \\ -\partial_{n}u^{inc} = ik \cos(\phi)u^{inc}, & \text{on } \Gamma_{shad}^{y}, \end{cases} \]

which leads to the first order PO-OSRC

\[ \partial_{n}u^{sc} - ik|\cos(\phi)|u^{sc} = 0. \]  

(4.36)

**Remark 5.** The condition (4.36) can also be derived from the Neumann PO condition (4.2).

**Remark 6.** \( \cos \phi \) is to be interpreted as \(-\hat{n} \cdot \hat{m}\), where \( \hat{n} \) is the normal and \( \hat{m} \) is the direction of the incoming field.

A second order condition is obtained by applying the first order condition twice suggested by Higdon, [39], [40], and combining it with the cylindrical Helmholtz equation,

\[ (\partial_{r} - ik|\cos \phi|) (\partial_{r} - ik|\cos \phi|) u^{sc} = 0, \]

\[ \partial_{r}^{2}u^{sc} + \frac{1}{r}\partial_{r}u^{sc} + \frac{1}{r^{2}}\partial_{\phi}^{2}u^{sc} + k^{2}u^{sc} = 0. \]
Eliminating the $\partial^2_r u$ term, yields the condition

$$-\left(2ik|\cos \phi| + \frac{1}{r}\right)\partial_r u^{sc} = k^2 \left(1 + \cos^2(\phi)\right)u^{sc} + \frac{1}{r^2}\partial^2_\phi u^{sc}. $$

By replacing

$$\partial_r \rightarrow \partial_n, \quad \frac{1}{r^2}\partial^2_\phi \rightarrow \partial^2_s, \quad \frac{1}{r} \rightarrow \kappa, $$

we obtain the two 2D-conditions listed in Table 4.5. The 3D-conditions in the table are derived in a similar fashion, using the spherical Helmholtz equation,

$$(\partial_r - ik|\cos \phi|)(\partial_r - ik|\cos \phi|)u^{sc} = 0,$$

$$\partial^2_r u^{sc} + 2\frac{1}{r}\partial_r u^{sc} + \frac{1}{r}\Delta u^{sc} + k^2 u^{sc} = 0,$$

and replacing

$$\partial_r \rightarrow \partial_n, \quad \frac{1}{r} \rightarrow \mathcal{H},$$

where $\mathcal{H}$ is the mean curvature. The angle $\phi$ is the angle between the direction of the incoming field, $\hat{m}$, and the normal, $\hat{n}$, such that $\cos \phi = -\hat{n} \cdot \hat{m}$. The second order conditions in Table 4.5 are no good on flat planes in the direction of the incoming field, when $|\hat{n} \cdot \hat{m}|$ and the curvatures are vanishing.

The relative error of the potential of a circular cylinder is presented in Figure 4.8, when the wave number $k = 20, 80$. The incoming plane wave propagates along the negative $x$-axis. This approximation is not suited for a problem with a Neumann boundary. Solving the ODE as an initial value problem yields pure imaginary eigenvalues $\pm ik\sqrt{1 + \cos^2 \phi}$. The errors in the approximation will therefore be transported along the scatterer without any damping.

### 4.6 The Inverse OSRC Operator

The Engquist-Majda and Bayliss-Turkel types of OSRC are both well suited for scatterers with a Dirichlet boundary. When the scatterer has a Neumann boundary, there is a need to solve a tridiagonal system of equations. The Engquist-Majda OSRC can be reformulated in an inverse form,

$$u^{sc} = B_0u^{sc}_n + B_2u^{sc}_{nss},$$

such that the equation system is avoided.

Consider an OSRC-condition on the form

$$(B_0 + B_2\partial^2_s + \ldots )\partial_n u^{sc} = (A_0 + A_2\partial^2_s + \ldots )u^{sc}. $$

This yields the relation on the Fourier side,

$$k \frac{H^{(1)}_m(kR)}{H^{(1)}_m(kR)} = \frac{\sum_{p=0}^{\infty} A_{2p} \left(-\frac{m^2}{R^2}\right)^p}{\sum_{p=0}^{\infty} B_{2p} \left(-\frac{m^2}{R^2}\right)^p}. $$
4.6. THE INVERSE OSRC OPERATOR

Figure 4.8: Relative error of the potential $\partial_n u^{sc}$ computed with the Hidgon type OSRC, for a circular cylinder with a Dirichlet boundary.

<table>
<thead>
<tr>
<th>Abbr.</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>2DPO1</td>
<td>$\partial_n u^{sc} = ik</td>
</tr>
<tr>
<td>2DPO2</td>
<td>$-(2ik</td>
</tr>
<tr>
<td>3DPO1</td>
<td>$\partial_n u^{sc} = ik</td>
</tr>
<tr>
<td>3DPO2</td>
<td>$-(2ik</td>
</tr>
</tbody>
</table>

Table 4.2: PO-OSRC, with $u^{inc}(r) = e^{ik\hat{m} \cdot r}$.

Using the Taylor expansion (4.12) yields

$$\sum_{p,q=0}^{\infty} B_{2(p-q)} \left(-\frac{1}{2R}(-1)^q + ik \left(\frac{1}{q}\right)\right) \frac{1}{k^{2q}} \left(-\frac{m^2}{R^2}\right)^p = \sum_{p=0}^{\infty} A_{2p} \left(-\frac{m^2}{R^2}\right)^p.$$ 

In order to find a Padé expansion, we can identify the coefficients,

$$\begin{align*}
A_0 &= \left(ik - \frac{1}{2R}\right) B_0, \\
A_2 &= \left(ik + \frac{1}{R}\right) \frac{1}{2k^2} B_0 + \left(ik - \frac{1}{2R}\right) B_2.
\end{align*}$$

By choosing $B_0 = 1$ and $B_2 = 0$, we get the standard second order Engquist-Majda expression. This is useful when solving a Dirichlet problem. If we instead consider
a Neumann problem, we need to invert the expression. To avoid the inversion, we
can put \( B_0 = 1 \) and \( A_2 = 0 \). The condition becomes

\[
\left( ik - \frac{1}{2R} \right)^2 u^{sc} = \left( ik - \frac{1}{2R} \right) \partial_n u^{sc} - \frac{1}{2k^2} \left( ik + \frac{1}{R} \right) \partial_s^2 \partial_n u^{sc}.
\]

In 3D if we neglect the \( \frac{1}{4} \) in the Taylor expansion of the logarithmic derivative of the Hankel functions that only affect higher order terms we get,

\[
\left( ik - \frac{1}{R} \right)^2 u^{sc} = \left( ik - \frac{1}{R} \right) \partial_n u^{sc} - \frac{1}{2k^2} \left( ik + \frac{1}{R} \right) \Delta \Gamma \partial_n u^{sc}.
\]

The Neumann boundary condition \( \partial_n u^{tot} = 0 \) can now be used to get an explicit inverse OSRC operator.

There is an alternative way of deriving the inverse Engquist-Majda operator. We can also use the Taylor expansion of the inverse logarithmic derivative (4.17) and (4.18) for 2D and 3D, respectively. We will obtain exactly the same formulas as we did with the Padé approach.

4.7 An Implicit Second Order OSRC

In the derivation of the Engquist-Majda OSRC, we have used an approximation of the logarithmic derivative of certain Hankel functions (4.14) and (4.15). In those expressions, we treat \( x = kR \) and \( y = \frac{\nu}{x} \) as independent variables. We want to find an OSRC which is second order in \( kR \). This means that the errors in \( y \) have to be bounded. In Chapter 4.3, we discuss how the error in \( y \) increase with the angle from the incoming field, due to the fact that the approximation is bad for the grazing angle. The Engquist-Majda condition can be derived using Padé-2/0 on the symbol of the DtN-operator. The inverse Engquist-Majda condition (4.37) is obtained by taking the Padé-0/2 approximation. Taking higher order Padé-expansions corresponds to taking more surface derivatives, which is hard to compute on a general scatterer. In order to get a better condition, we take the Padé-2/2 approximation \( F^{2D}_{2/2}(x) \) of the logarithmic derivative \( F^{2D}(x) \) and obtain

\[
kF^{2D}_{2/2}(kR) = \frac{-\frac{1}{R} + \frac{ik}{R} - 2k^2 - \left( \frac{3}{R} - \frac{15}{4kR} \right) \partial_s^2}{\frac{1}{R} + 2ik + \left( \frac{1}{2R^2} + \frac{1}{4k^2} \right) \partial_s^2}, \quad (4.37)
\]

where \( -y^2 = -\frac{m^2}{R^2} \) and \( \partial_s^2 \), when transforming back from the Fourier space. The OSRC-scheme \( \partial_n u^{sc} = kF^{2D}_{2/2}(kR)u^{sc} \) becomes

\[
B_0 \partial_n u^{sc} + B_2 \partial_s^2 \partial_n u^{sc} = A_0 u^{sc} + A_2 \partial_s^2 u^{sc}, \quad (4.38)
\]
4.8. REGULARIZATION OF THE IMPROVED OSRC

where
\[ B_0 = \frac{2}{R} + 2ik, \quad B_2 = \frac{2}{k^2 R} + \frac{i}{2k^2}, \] \hspace{1cm} (4.39)
\[ A_0 = -\left( \frac{1}{R^2} - \frac{ik}{R} + 2k^2 \right), \quad A_2 = -\left( \frac{3}{2} - \frac{15i}{4kR} \right). \] \hspace{1cm} (4.40)

By using (4.11) instead of (4.10) yields the 3D condition
\[ B_0 \partial_n u + B_2 \Delta \Gamma \partial_n u = A_0 u + A_2 \Delta \Gamma u, \] \hspace{1cm} (4.41)
where
\[ B_0 = ik + \frac{1}{R}, \quad B_2 = \frac{1}{k^2 R} + \frac{i}{4k}, \] \hspace{1cm} (4.42)
\[ A_0 = -\left( k^2 + \frac{1}{R^2} \right), \quad A_2 = -\left( \frac{1}{2k^2 R^2} - \frac{7i}{4kR} + \frac{3}{4} \right). \] \hspace{1cm} (4.43)

**Figure 4.9:** Relative error of the potential \( \partial_n u^{sc} \) computed with the Implicit OSRC, for a circular cylinder with a Dirichlet boundary.

4.8 Regularization of the Improved OSRC

The improved OSRC-approximation was derived on a circle and sphere in 2D and 3D respectively. When we compute the potential on a general object, some errors are introduced, due to varying curvature. The introduced errors should not influence the potential far from the error. This means that if we consider the OSRC-approximation as an initial value problem, we would like the solution to be rapidly decaying. Consider the OSRC scheme
\[ B_0 u_n^{sc} + B_2 u_{nss}^{sc} = A_0 u^{sc} + A_2 u_{ss}^{sc}, \] \hspace{1cm} (4.44)
where $B_j$ and $A_j$ are defined in (4.39) and (4.40). The Dirichlet problem for a circular cylinder can be written as a second order boundary value problem for $v = u_n^{inc}$,

$$B_0 v + B_2 v_{ss} = -A_0 u^{inc} - A_2 u_{ss}^{inc} =: F(u^{inc}).$$  

Introducing a new variable $w = (v, v_s)^T$, yields the first order system

$$w_s = \left( \begin{array}{cc} 0 & 1 \\ -B_2^{-1} B_0 & 0 \end{array} \right) w + \left( \begin{array}{c} 0 \\ -B_2^{-1} F(u^{inc}) \end{array} \right) =: Aw + \tilde{F}.$$  

Diagonalizing $AS = S\Lambda$, with

$$S = \left( \begin{array}{cc} 1 & 1 \\ \lambda & -\lambda \end{array} \right),$$

$$\Lambda = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right).$$

The eigenvalues are, with $x = kR$,

$$\lambda = \sqrt{-B_2^{-1} B_0} = 2ik \sqrt{1 + 4\frac{x}{x^2}}.$$  

The real part of $\lambda$ is non-vanishing, i.e. $\Re \lambda_1 < 0$ and $\Re \lambda_2 > 0$. The positive real part of $\lambda_2$ can be interpreted as a decaying solution in the $-s$ direction. A Taylor expansion of $\lambda_1$ yields

$$\Re \lambda_1 = -\frac{3}{R} + \frac{267}{8k^2 R^3} + O \left( \frac{1}{k^4 R^5} \right).$$

The asymptotic $\frac{1}{R}$-dependence is very bad, since this corresponds to a global coupling in $u_n$ for e.g. an infinite plane. This is to be compared with the physical optics solution which is exact for this case and has no coupling in $u_n$. In order to reduce the coupling, we introduce a regularization term in $B_0$,

$$\tilde{B}_0 = (1 + i\epsilon) B_0,$$

which yields the asymptotic eigenvalue of the damped system

$$\Re \lambda_1 = -\frac{3}{R} - \epsilon k + O \left( \frac{1}{k^2 R^3} + \frac{\epsilon}{k R^2} + \frac{\epsilon^2}{R} + k \epsilon^3 \right).$$

If we have the explicit scheme e.g. second order Engquist-Majda,

$$u_n = E_0 u + E_2 u_{ss},$$
then the scheme with a regularization term becomes
\[(1 + i\varepsilon)B_0 u_n + B_2 u_{nss} = (A_0 + i\varepsilon E_0)u + (A_2 + i\varepsilon E_2)u_{ss}.\] (4.54)
A similar derivation for the Neumann problem yields an asymptotic expansion
\[
\lambda = -\frac{2}{\sqrt{3R}}, \quad \text{as } x \to \infty.
\] (4.55)
The regularization term now appears in \(A_0\),
\[
\tilde{A}_0 = (1 + i\varepsilon)A_0,
\] (4.56)
which yields the modified scheme
\[(1 + i\varepsilon)A_0 u + A_2 u_{ss} = (B_0 + i\varepsilon F_0)u_n + (B_2 + i\varepsilon F_2)u_{nss},\] (4.57)
where
\[u = F_0 u_n + F_2 u_{nss},\] (4.58)
is e.g. the inverse of the Engquist-Majda scheme.

In 3D, we consider the sphere as a model problem to investigate if the same approach works as in 2D. We have an incoming wave in the \(-z\)-direction. The potential remains constant in the \(\phi\)-direction (i.e. in the \(xy\)-space). This means that the surface Laplacian is a function in \(\theta\). If we choose \(s\) as the arclength parameter in the \(\theta\)-direction, then \(\Delta u = \partial_s^2 u\) and we are back in the 2D case (but with different coefficients \(A_j, B_j, E_j\) and \(F_j\)). The same computations as in 2D yield the eigenvalues for the corresponding schemes in 3D with a regularization term similar to the 2D schemes,
\[
\Re \lambda = \begin{cases} 
\frac{4}{\pi} - \varepsilon k, & \text{for the Dirichlet problem} \\
\frac{\pi}{\sqrt{3}} \sqrt{\frac{\varepsilon}{\sqrt{3}}} & \text{for the Neumann problem}
\end{cases}
\] (4.59)

4.9 An OSRC in Electromagnetics

In electromagnetics, we are interested in approximations of the expression (3.21),
\[
G \left( \frac{l(l+1)}{k^2 R^2} \right) = \gamma_l = \frac{1 + R \partial_n \log h_l^{(1)}(kR)}{ikR}.
\]
Inserting the expression (4.11), yields
\[
G(x) = \sqrt{1 + x} + \frac{1}{2ikR} \frac{x}{1 + x}.
\] (4.60)
CHAPTER 4. ON SURFACE RADIATION CONDITIONS

Takings the Padé-p/q approximation of $G$, yields $G_{pq}$,

$$G_{10}(x) = 1 + \frac{kR - i}{2k^2R} (k^2 x) = 1 + A_{10}k^2 x,$$

$$G_{01}(x) = \frac{1}{1 + \frac{k^2R}{2k^2R} (k^2 x)} = \frac{1}{1 + B_{01}k^2 x},$$

$$G_{11}(x) = \frac{1 + \frac{1}{k^2} \frac{\gamma}{1 - \frac{k^2}{k^2}} (k^2 x)}{1 + \frac{1}{k^2} \frac{\gamma}{1 - \frac{k^2}{k^2}} (k^2 x)} = 1 + A_{11}k^2 x \quad \text{(4.63)}$$

Let $x_G$ and $x_R$ be the symbol $\frac{R/(1 + \frac{k}{R})}{R}$ in equation (3.22) and (3.23), respectively. The simplest OSRC is obtained by approximating

$$\gamma_l = 1 + A_{10}k^2 x_R,$$

which yields the condition

$$\frac{1}{\gamma_l} = 1 + B_{01}k^2 x_G,$$

Replacing the scattered fields with the incoming yields

$$ZJ_s^{sc} = E_{tan}^{inc} + \frac{i - BR}{2k^3R} \left( \text{grad}_\Gamma \text{div}_\Gamma E_{tan}^{inc} + \text{curl}_\Gamma \text{curl}_\Gamma E_{tan}^{inc} \right).$$

An implicit OSRC can be obtained, observing that

$$-\text{curl}_\Gamma \text{curl}_\Gamma \text{grad}_\Gamma (\alpha_l^m G_l^m + \beta_l^m R_l^m) = 0,$$

$$-\text{grad}_\Gamma \text{div}_\Gamma \text{curl}_\Gamma (\alpha_l^m G_l^m + \beta_l^m R_l^m) = 0,$$

which yields that $x_R G_l^m = x_R R_l^m = 0$ and also that the mixed terms of the symbols are zero, $x_G x_R = x_R x_G = 0$. The electromagnetic operator (3.20) can now be approximated

$$ZJ_s^{sc} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left( \frac{1 + B_{11}x_G}{1 + A_{11}x_G} \alpha_l^m G_l^m + \frac{1 + A_{11}x_R}{1 + B_{11}x_R} \beta_l^m R_l^m \right).$$

Rearranging the terms and using the simplifications yields the OSRC-scheme

$$(1 + A_{11} \text{grad}_\Gamma - B_{11} \text{curl}_\Gamma \text{curl}_\Gamma) ZJ_s^{sc} = (1 + B_{11} \text{grad}_\Gamma - A_{11} \text{curl}_\Gamma \text{curl}_\Gamma) E_{tan}^{inc}$$

The current $J_s^{inc}$ could be obtained using a variational formulation, with basis functions described in appendix D.
4.10 Engquist-Majda on Slowly Varying Functions

When the wave-number is increasing, we need to resolve the data with more discretization points. This can be avoided by filtering out the incoming field. For a high frequency scattering problem on a convex object, Bruno et al. [16] suggested to use slowly varying functions, which arise when the incoming field is filtered out from the potential. For a Dirichlet problem, we introduce new variables $\tilde{u}^{sc}$ and $\tilde{u}_n^{sc}$ by

$$
\begin{align*}
  u^{sc} &= \tilde{u}^{sc}u^{inc}, \\
  \partial_n u^{sc} &= \tilde{u}_n^{sc}u^{inc}.
\end{align*}
$$

Taking derivatives yields with $q_j = \partial_j u^{inc}/u^{inc}$,

$$
\begin{align*}
  \partial^2_s u &= \partial^2_s (\tilde{u}u^{inc}) \\
  &= \partial^2_s \tilde{u}u^{inc} + 2\partial_s \tilde{u}\partial_s u^{inc} + \tilde{u}\partial^2_s u^{inc} \\
  &= (\partial^2_s \tilde{u} + 2q_1 \partial_s \tilde{u} + q_2 \tilde{u}) u^{inc}.
\end{align*}
$$

The filtered Engquist-Majda condition is

$$
\tilde{u}_n^{sc} = -A_0 - A_2 q_2.
$$

For a convex scatterer, the oscillations filtered out and the number of discretization points can be significantly reduced.

The slowly varying potential $\partial_n \tilde{u}^{tot}$ is presented in Figure 4.10. An interesting observation is that the real part of the PO-approximation vanishes. The Engquist-Majda OSRC resolves the real part for small angles, while the implicit OSRC is better for a larger range of angles.

4.11 Generalized Physical Optics (GPO)

In the derivation of the different OSRCs, we have used different truncated Taylor and Padé approximations of the logarithmic derivative of the Hankel function. In the derivation of a generalized PO approximation, we will use all terms in the Taylor expansions. We compute high order surface derivatives of the incoming field up to an order in $kR$. This means that we approximate each term rather than truncating the Taylor expansion.

Before we can show how to obtain higher order PO-approximations, we list a few lemmas. In the derivations, we need to identify Taylor expansions containing polynomials in the summation index with derivatives of the closed expression.

**Lemma 16.** Suppose that $f(z)$ has the Taylor expansion

$$
f(z) = \sum_{p=0}^{\infty} f_p z^p.
$$


Then it hold that
\[ \sum_{p=0}^{\infty} f_p p^j z^p = (z\partial_z)^j f(z). \]

The derived PO approximations will contain expressions of the form \( \frac{1}{\cos^2 \phi} \) that becomes unbounded as \( \phi \to \frac{\pi}{2} \). We will introduce a correction \( \cos^2 \phi \to \cos^2 \phi + \varepsilon \) to the approximations. Observing that the difference
\[ \frac{1}{\cos^2 \phi} - \frac{1}{\cos^2 \phi + \varepsilon} = \frac{\varepsilon}{\cos^4 \phi + \varepsilon \cos^2 \phi}, \]
indicates that \( \varepsilon = \delta \sin^2 \phi \), where \( \delta \) is a small constant is an appropriate choice to get an approximation of order \( p \).

The generalized PO approximation on a circular cylinder is derived for an incoming plane wave in the negative \( x \)-direction. In order to make the PO valid for other incident wave directions \( \hat{\mathbf{m}} \) in 2D, we can identify
\[
\begin{align*}
\hat{\mathbf{n}} \cdot \hat{\mathbf{m}} &= -\cos \phi, \quad (4.72) \\
|\hat{\mathbf{n}} \times \hat{\mathbf{m}}| &= \sin \phi, \quad (4.73) \\
\delta_{\text{lit}} &= \frac{1 + \text{sgn}(\cos \phi)}{2}, \quad (4.74)
\end{align*}
\]

where \( \hat{\mathbf{n}} \) is the outward normal. The indicator \( \delta_{\text{lit}} \) is 1 in the illuminated (lit) region of the scatterer and 0 otherwise. The only difference in 3D is that \( \phi \) is changed to \( \theta \).
A GPO Approximation in 2D for a Dirichlet Scatterer

Instead of adding the sum in equation (4.12) term by term, we collect all k’s of the highest powers and use the terms arising from that. We state the following lemma,

Lemma 17. Let $u^{inc} = e^{-ikr \cos \phi}$. The surface derivatives on the circular cylinder with radius $R$ are

$$\frac{\partial^2 p u^{inc}}{k^{2p}} = \left( -\sin^2(\phi) \right)^p + \frac{i}{kR} \cos(\phi) \left( 2p^2 - p \right) (-\sin^2(\phi))^{p-1} \right.$$ 

$$- \frac{1}{k^2 R^2} \left( 2p^4 - \frac{14}{3} p^3 + \frac{7}{2} p^2 - \frac{5}{6} \right) \left( -\sin^2(\phi) \right)^{p-1} \right.$$ 

$$- \frac{1}{k^2 R^2} \left( 2p^4 - 6p^3 + \frac{11}{2} p^2 - \frac{3}{2} \right) \left( -\sin^2(\phi) \right)^{p-2} + \mathcal{O}\left( \frac{1}{k^3 R^3} \right) u^{inc}. \right.$$

Proof. This is proved by induction. The expression is valid for $p = 1$. Assuming validity for a positive $p$ yields the expression for $p + 1$ by direct computation.

We can now derive a higher order PO approximation.

Lemma 18 (Second order PO-scheme in 2D). Assume that expression (4.10) is valid. A second order PO-scheme is then

$$\partial_n u^{sc} = \text{sgn} \cos(\phi) \partial_n u^{inc} + \frac{\delta_{\text{lit}}}{R \cos^2(\phi)} u^{inc}. \quad (4.75)$$

Proof. Define

$$F(x,y) = -\frac{1}{2x} \frac{1}{1+y} + i \sqrt{1+y} = \sum_{p=0}^{\infty} \left( -\frac{1}{2x} (-1)^p + i \left( \frac{1}{p} \right) \right) y^p,$$

and insert the two first terms for the derivative in lemma 17 in the expression. Using the Dirichlet boundary condition $u^{tot} = 0$ and the Taylor expansion (4.12) together with lemma 16 yields the second order approximation

$$\partial_n u^{sc} = -kF(kR, \frac{\partial^2 p u^{inc}}{k^{2p}}) u^{inc}$$

$$= - \left( F(kR, z) + \frac{i}{kR} \cos \phi \left( 2 \frac{d}{dz} z \frac{d}{dz} F(kR, z) - \frac{d}{dz} F(kR, z) \right) \right.$$ 

$$+ \mathcal{O}\left( \frac{1}{kR^2} \right) u^{inc},.$$
where $z = -\sin^2 \phi$. Taking the derivatives of the closed expression of $F(x, z)$ yields

$$\frac{\partial_n u^{sc}}{\partial_n u^{inc}} = \left(-\frac{1}{R} \frac{\partial}{\partial \phi} \frac{1}{\cos \phi} + \frac{1}{2R} \frac{\partial}{\partial \phi} \frac{1}{\cos \phi} + O \left(1 \frac{1}{R^2} \right) \right) u^{inc}.$$  

Writing $\delta_{lit} = \frac{1 + \text{sgn} \cos \phi}{2}$ and observing that $\partial_n u^{inc} = -ik \cos \phi u^{inc}$ completes the proof. \[ \square \]

The PO approximation can be generalized to third order by taking all terms in the derivative in lemma 17 and using the third order approximation (4.14). Identifying $\cos \phi = -\hat{n} \cdot \hat{m}$ and $\sin \phi = |\hat{n} \times \hat{m}|$ yields the third order PO approximation for a Dirichlet cylinder without a correction.

**Approximation 1** (Generalized 3rd order PO for a Dirichlet cylinder, without correction). Consider the incoming wave $u^{inc}(r) = e^{ik \hat{m} \cdot r}$, which propagates in the direction $\hat{m}$. A third order PO approximation of the potential is

$$\partial_n u^{tot} = \begin{cases} 
2 \partial_n u^{inc} + \frac{1}{R} \frac{\partial}{\partial \hat{n} \cdot \hat{m}} u^{inc} + \frac{1}{2R^2} \frac{\partial}{\partial \hat{n} \times \hat{m}} u^{inc}, & \text{on } \Gamma_{lit}, \\
0, & \text{on } \Gamma_{shad}.
\end{cases}$$

To obtain a useful approximation, we need to add a correction that removes the singularities at the grazing angle, $\phi = 90^\circ$. In the second order PO we use the correction $|\hat{n} \cdot \hat{m}|^2 \rightarrow |\hat{n} \cdot \hat{m}|^2 + \frac{1}{R^2} |\hat{n} \times \hat{m}|^2$. Furthermore, we introduce the curvature by the substitution $\frac{1}{R} \rightarrow \kappa$ and obtain

**Approximation 2** (Generalized 2nd order PO for a Dirichlet cylinder, with correction). Consider the incoming wave $u^{inc}(r) = e^{ik \hat{m} \cdot r}$, which propagates in the direction $\hat{m}$. A second order PO approximation of the potential is

$$\partial_n u^{tot} = \begin{cases} 
2 \partial_n u^{inc} + \frac{\kappa}{|\hat{n} \cdot \hat{m}|^2 + \frac{1}{R^2} |\hat{n} \times \hat{m}|^2} u^{inc}, & \text{on } \Gamma_{lit}, \\
0, & \text{on } \Gamma_{shad}.
\end{cases}$$

In the case of the third order approximation, we need to use a different correction in order to get the correct order. We use the corrections $|\hat{n} \cdot \hat{m}|^2 \rightarrow |\hat{n} \cdot \hat{m}|^2 + \frac{1}{R^2} |\hat{n} \times \hat{m}|^2$ and $|\hat{n} \cdot \hat{m}|^6 \rightarrow |\hat{n} \cdot \hat{m}|^6 + \frac{1}{R^2} (1 - |\hat{n} \cdot \hat{m}|^6)$.

**Approximation 3** (Generalized 3rd order PO for a Dirichlet cylinder, with correction). Consider the incoming wave $u^{inc}(r) = e^{ik \hat{m} \cdot r}$, which propagates in the direction $\hat{m}$. A third order PO approximation of the potential is

$$\partial_n u^{tot} = \begin{cases} 
2 \partial_n u^{inc} + \frac{\kappa}{|\hat{n} \cdot \hat{m}|^2 + \frac{1}{R^2} |\hat{n} \times \hat{m}|^2} u^{inc} + \frac{1 + 3|\hat{n} \cdot \hat{m}|^2 \kappa^2}{R^2 (|\hat{n} \cdot \hat{m}|^2 + \frac{1}{R^2} |\hat{n} \times \hat{m}|^2)} \partial_{\hat{n}} u^{inc}, & \text{on } \Gamma_{lit}, \\
0, & \text{on } \Gamma_{shad}.
\end{cases}$$

The leading error term will grow when we approach the grazing angle. The consequence of the growing error is that we need to increase the wave number
4.11. GENERALIZED PHYSICAL OPTICS (GPO)

to get accuracy close to the grazing angle. The higher order PO approximations coincide with the original PO-error when $\delta = \infty$. In Figure 4.11, we present the relative error of the generalized PO approximations when $kR = 20$ and 80. The parameter is chosen to be $\delta = 4$. The error for the grazing angle is larger for large wave numbers. We have to choose between high order (and keep $\delta$ independent of the wave number) or eliminating the error growth at the grazing angle by making $\delta$ a function of $kR$. This is a kind of Scylla-Charybdis situation, that if we choose to keep the order, then the error at the grazing angle increases and vice versa.

![Figure 4.11](image_url)

Figure 4.11: Relative error in potential using PO for a circular cylinder with Dirichlet boundary conditions, with the parameter $\delta = 4$.

A GPO Approximation in 2D for a Neumann Scatterer

In the derivation of a generalized PO approximation for a Neumann scatterer in 2D, we use the approximation (4.17) of inverse of the logarithmic derivative of the Hankel function, and its Taylor expansion. We will need high order surface derivatives of $\partial_n u^{inc}$.

**Lemma 19.** Let $u^{inc} = e^{-ikr \cos \phi}$. The surface derivatives of the normal derivative of the incoming field on the circular cylinder with radius $R$ are

$$
\frac{\partial^{2p} \partial_n u^{inc}}{k^{2p}} = \left( -ik \cos \phi (-\sin^2(\phi))^p + \frac{2p^2 + p}{R} (-\sin^2(\phi))^p + \frac{2p^2 - p}{R} (-\sin^2(\phi))^{p-1}
\right.
\left. + \frac{i \cos \phi}{kR^2} \left( 2p^4 - \frac{2}{3}p^3 - \frac{1}{2}p^2 + \frac{1}{6}p \right) (-\sin^2 \phi)^{p-1}
\right.
\left. + \frac{i \cos \phi}{kR^2} \left( 2p^4 - 6p^3 + \frac{11}{2}p^2 - \frac{3}{2}p \right) (-\sin^2 \phi)^{p-2} + O \left( \frac{1}{k^2 R^2} \right) \right) u^{inc}.
$$
CHAPTER 4. ON SURFACE RADIATION CONDITIONS

Proof. The lemma is proved by induction. $\square$

We use the third order approximation (4.17) of the inverse of the logarithmic derivative of the Hankel function, which we denote by $G(x, y)$,

$$G(x, y) = \frac{1}{i\sqrt{1+y}} - \frac{1}{2x(1+y)^2} + \frac{i}{8x^2(1+y)^{7/2}}$$

such that $u^m = -\frac{k}{k} G\left(kR, \frac{\partial}{\partial n}\right) \partial_n u^{inc}$ for a circular cylinder with Neumann boundary conditions. We obtain the GPO for a circular Neumann cylinder,

**Approximation 4** (Generalized 3rd order PO for a Neumann cylinder, without correction). Consider the incoming wave $u^{inc}(r) = e^{ikm \cdot r}$, which propagates in the direction $m$. A third order PO approximation of the potential is

$$u^{tot} = \begin{cases} 2u^{inc} + \frac{1}{k^2 \|n\times m\|^2} \partial_n u^{inc} - \frac{2+6\|n\times m\|^2}{k^2 \|n\times m\|^2} u^{inc}, & \text{on } \Gamma_{lit}, \\ 0, & \text{on } \Gamma_{shad}. \end{cases}$$

The GPO approximation without any correction gets unbounded as we reach the grazing angle. Therefore, we need to introduce a correction that does not destroy the order. An appropriate choice leads to the following approximation.

**Approximation 5** (Generalized 2nd order PO for a Neumann cylinder, with correction). Consider the incoming wave $u^{inc}(r) = e^{ikm \cdot r}$, which propagates in the direction $m$. A second order GPO approximation of the potential is

$$u^{tot} = \begin{cases} 2u^{inc} + \frac{k}{k^2 \|n\times m\|^2} \partial_n u^{inc} - \frac{2+6\|n\times m\|^2}{k^2 \|n\times m\|^2} u^{inc}, & \text{on } \Gamma_{lit}, \\ 0, & \text{on } \Gamma_{shad}. \end{cases}$$

A third order GPO approximation with a correction is

**Approximation 6** (Generalized 3rd order PO for a Neumann cylinder, with correction). Consider the incoming wave $u^{inc}(r) = e^{ikm \cdot r}$, which propagates in the direction $m$. A third order GPO approximation of the potential is

$$u^{tot} = \begin{cases} 2u^{inc} + \frac{k}{k^2 \|n\times m\|^2} \partial_n u^{inc} - \frac{(2+6\|n\times m\|^2)k^2}{k^2 \|n\times m\|^2} u^{inc}, & \text{on } \Gamma_{lit}, \\ 0, & \text{on } \Gamma_{shad}. \end{cases}$$

In Figure 4.12, we present the relative error of the potential when $kR = 20$ and $40$, for the parameter $\delta = 4$. 
A GPO Approximation in 3D for a Dirichlet Scatterer

In 3D, we consider an incoming plane wave along the negative z-axis, \( u^{inc} = e^{-ikx} = e^{-ikr \cos \theta} \). The surface derivatives will only depend on the \( \theta \)-angle (and not \( \phi \)). In the following lemma we compute the surface Laplacians with a correction for the discrepancy between the eigenvalues \( -\frac{m(m+1)}{k^2 R^2} \) of \( \Delta \) on the spherical surface and the parameter \( y = -\frac{(m+1/2)^2}{k^2 R^2} \) in the expression for the logarithmic derivative of the spherical Hankel function.

**Lemma 20.** Suppose that \( u^{inc} = e^{-ikR \cos \theta} \). The surface Laplacians with a correction term \( -\frac{1}{4R^2} \) are then

\[
\left( \Delta \Gamma - \frac{1}{4R^2} \right) u^{inc} = \left( -\sin^2 \theta \right)^p \cos \theta (-\sin^2 \theta)^{p-1} \left( -\frac{2p^4 - 8p^3 + p^2 - \frac{1}{12}p}{k^2 R^2} (-\sin^2 \theta)^{p-1} \right. \\
\left. -\frac{2p^4 - 4p^3 + 2p^2}{k^2 R^2} (-\sin^2 \theta)^{p-2} + O \left( \frac{1}{k^3 R^3} \right) \right) u^{inc}.
\]

**Proof.** The lemma is proved by induction. \( \square \)

This yields the condition

**Approximation 7** (Generalized 3rd order PO for a Dirichlet cylinder in 3D, without correction). Consider the incoming wave \( u^{inc}(r) = e^{ikm \cdot r} \), which propa-
ates in the direction $\hat{\mathbf{m}}$. A third order PO approximation of the potential is
\[
\partial_n u^{tot} = \begin{cases} 
2\partial_n u^{inc} + \frac{1+|\hat{n} \times \hat{m}|^2}{R} u^{inc} + \frac{1+|\hat{n} \times \hat{m}|^2}{k^2 R |\hat{n} \cdot \hat{m}|^2} \partial_n u^{inc}, & \text{on } \Gamma_{lit}, \\
0, & \text{on } \Gamma_{shad}.
\end{cases}
\]

The approximation is unbounded at the grazing angle. We introduce the same correction as in 2D to avoid the singularity but to keep the order of the approximation. For incoming plane waves in the direction $\hat{\mathbf{m}}$, we can identify
\[
\begin{align}
\hat{n} \cdot \hat{m} &= -\cos \theta, \\
|\hat{n} \times \hat{m}| &= \sin \theta,
\end{align}
\]
which yields the modified scheme
\[
\partial_n u = \left( i k |\hat{n} \cdot \hat{m}| - \frac{\delta_{lit} (1 + |\hat{n} \cdot \hat{m}|^2)}{2 |\hat{n} \times \hat{m}|^2 + R |\hat{n} \cdot \hat{m}|^2} \right) u.
\]

The updated Dirichlet PO-scheme in 3D becomes
\[
\partial_n u^{tot} = \begin{cases} 
2\partial_n u^{inc} + \frac{\eta (1+|\hat{n} \times \hat{m}|^2)}{R^2 |\hat{n} \times \hat{m}|^2 + |\hat{n} \cdot \hat{m}|^2} u^{inc}, & \text{in } \Gamma_{lit}, \\
0, & \text{in } \Gamma_{shad}.
\end{cases}
\]

A GPO Approximation in 3D for a Neumann Scatterer

In the following lemma we compute the surface Laplacians with a correction for the discrepancy between the eigenvalues $-\frac{m(m+1)}{k^2 R^2}$ of $\frac{\Delta}{R^2}$ on the spherical surface and the parameter $y = -\frac{(m+1/2)^2}{k^2 R^2}$ in the expression for the logarithmic derivative of the spherical Hankel function.

Lemma 21. Suppose that $u^{inc} = e^{-ik R \cos \theta}$. The surface Laplacians with a correction term $\frac{1}{k R^2}$ are then
\[
(\Delta_{\Gamma} - \frac{1}{k R^2})^p \partial_n u^{inc} = \begin{pmatrix}
- i k \cos \theta (-\sin^2 \theta)^p \\
+ \frac{\cos \theta}{R} (2p^2 + 2p) (-\sin^2 \theta)^p + \frac{\cos \theta}{R} 2p^2 (-\sin^2 \theta)^{p-1} \\
+ i \frac{\cos \theta}{k R^2} \left( 2p^4 + \frac{4}{3} p^3 - p^2 - \frac{1}{12} p \right) (-\sin^2 \theta)^{p-1} \\
+ i \frac{\cos \theta}{k R^2} (2p^4 - 4p^3 + 2p^2) (-\sin^2 \theta)^{p-2} \end{pmatrix} u^{inc} + O \left( \frac{1}{k^2 R^3} \right).
\]
4.11. GENERALIZED PHYSICAL OPTICS (GPO)  

Proof. The lemma is proved by induction. □

This yields the condition

**Approximation 8** (Generalized 3rd order PO for a Neumann cylinder in 3D, without correction). Consider the incoming wave \( u^{inc}(r) = e^{i \mathbf{k} \cdot \mathbf{r}} \), which propagates in the direction \( \hat{m} \). A third order PO approximation of the potential is

\[
\begin{align*}
    u^{tot} = \begin{cases}
        2u^{inc} + \frac{1+i \eta \eta^2}{k^2 R (n \cdot m)} \partial_n u^{inc} - \frac{6+2 \eta \eta^2}{k^2 R (n \cdot m)} u^{inc}, & \text{on } \Gamma_{it}, \\
        0, & \text{on } \Gamma_{shad}.
    \end{cases}
\end{align*}
\]

**A GPO Approximation in Electromagnetics**

In acoustics, we have developed a generalized PO approximation for both the Dirichlet case and the Neumann case in both 2D and 3D. For the electromagnetic case we assume that we know both the incoming direction, \( \hat{m} \), as well as the polarization, \( \hat{e} \) and \( \hat{h} \), such that \( \hat{e} \cdot \hat{m} = 0 \) and \( \hat{h} = \hat{m} \times \hat{e} \). Without loss of generality, we can assume that \( \hat{m} = (0,0,-1) \), \( \hat{e} = (1,0,0) \) and \( \hat{h} = (0,-1,0) \). This choice of parameters coincide with the model problem in chapter 3.5 for which we have an analytical solution. In order to get a similar expression for electromagnetic problems as for the acoustic problems, we need to represent the involved quantities in a divergence-conforming and a curl-conforming part, described in appendix D, such that \( \mathbf{E}_{\text{tan}} = \mathbf{E}_{\text{div}} + \mathbf{E}_{\text{curl}} \). We will use the electromagnetic operator, described in chapter 3.6. An approximation of \( \gamma_t \) in expression (3.21) was derived in chapter 4.3. We write the formulas again, slightly reformulated,

\[
F(x,y) = \gamma_t = \sqrt{1+y} - i \frac{y}{2x(1+y)} + \frac{1-4y}{8x^2(1+y)^{5/2}} = \sum_{p=0}^{\infty} f_p(x)y^p, \quad (4.81)
\]

\[
G(x,y) = \frac{1}{\gamma_t} = \frac{1}{\sqrt{1+y}} + i \frac{y}{2x(1+y)^2} - \frac{1-4y+2y^2}{8x^2(1+y)^{7/2}} = \sum_{p=0}^{\infty} g_p(x)y^p, \quad (4.82)
\]

where \( y = -\frac{4z}{k^2 R^2} \). The only Taylor-coefficients that are needed explicitly are

\[
\begin{align*}
    f_0 &= 1 + \frac{1}{8x^2}, \quad f_1 = \frac{1}{2} - \frac{i}{2x} - \frac{13}{16x^2}, \quad g_0 = 1 - \frac{1}{8x^2}, \quad g_1 = -\frac{1}{2} + \frac{i}{2x} + \frac{15}{16x^2}.
\end{align*}
\]

We define the operators

\[
\begin{align*}
    \mathbf{CC} &= -\mathbf{curl}_\Gamma \mathbf{curl}_\Gamma - \frac{1}{4k^2 R^2}, \quad \mathbf{GD} = \mathbf{grad}_\Gamma \mathbf{div}_\Gamma - \frac{1}{4k^2 R^2}.
\end{align*}
\]

From the formula (3.22) and (3.23) for the surface derivatives, we conclude that

\[
\begin{align*}
    F(kR, \mathbf{CC}) \mathbf{E}^{inc}_{\text{tan}} &= F(kR, \mathbf{CC}) \mathbf{E}^{inc}_{\text{curl}} + F\left(kR, -\frac{1}{4k^2 R^2}\right) \mathbf{E}^{inc}_{\text{div}}, \\
    G(kR, \mathbf{GD}) \mathbf{E}^{inc}_{\text{tan}} &= G(kR, \mathbf{GD}) \mathbf{E}^{inc}_{\text{div}} + G\left(kR, -\frac{1}{4k^2 R^2}\right) \mathbf{E}^{inc}_{\text{curl}}.
\end{align*}
\]
Lemma 22. The following approximations hold

\[ F \left( x, -\frac{1}{4x^2} \right) = 1 + O \left( \frac{1}{x^3} \right), \quad G \left( x, -\frac{1}{4x^2} \right) = 1 + O \left( \frac{1}{x^3} \right) \]

Proof. We have that \( y = -\frac{1}{4x^2} = -\frac{(l+1/2)^2}{4x^2} \), i.e. \( l = 0 \) or \( l = -1 \). Therefore, it is sufficient to realize that \( \partial_x \log h_{(1)}^{(0)}(x) = i - \frac{1}{x} \) for \( l = 0, -1 \). Then \( \gamma_l = \frac{1+z \partial_x \log h_{(1)}^{(0)}(x)}{ex} = 1 \) and we can conclude that the approximations to be proved are of the same order as the approximation of \( F \) and \( G \).

\[ \square \]

The electromagnetic operator admits the approximation

\[ ZJ^{sc} = F(kR, CC) E^{inc}_{div} + G(kR, GD) E^{inc}_{div}, \]

which together with lemma 22 yield the third order approximation

\[ ZJ^{sc} = -E^{inc}_{tan} + F(kR, CC) E^{inc}_{tan} + G(kR, GD) E^{inc}_{tan}. \] (4.84)

In order to get a generalized PO approximation, we use the Taylor expansion of \( F \) and \( G \), which contains high order surface-derivatives on the sphere.

We need the following lemma
Lemma 23. Suppose that $E_{\text{inc}}^{\text{tan}} = e^{-ikR\cos\theta}$. Then the following is valid for $p \geq 1$,

\[
\begin{align*}
\text{(CC)}^p E_{\text{inc}}^{\text{tan}} &= \left( -\frac{1}{ikR\cos\theta} (-\sin^2 \theta)^{p-1} - \frac{1}{k^2 R^2} (2p^2 - 2p) (-\sin^2 \theta)^{p-2} \\
&- \frac{1}{4k^2 R^2 \delta_{p,1}} \right) E_{\phi}^{\text{inc}} \hat{\phi} \\
&+ \left( (-\sin^2 \theta)^p - \frac{(2p^2 - 1) \cos \theta}{ikR} (-\sin^2 \theta)^{p-1} \\
&- \frac{1}{k^2 R^2} \left( \frac{2p^4 - \frac{8}{3} p^3 - p^2 + \frac{23}{12} p}{2p^4 - 4p^3 + 2p} \right) (-\sin^2 \theta)^{p-2} \right) E_{\phi}^{\text{inc}} \hat{\phi} + O \left( \frac{1}{k^3 R^3} \right),
\end{align*}
\]

\[
\begin{align*}
\text{(GD)}^p E_{\text{inc}}^{\text{tan}} &= \left( (-\sin^2 \theta)^p - \frac{2p^2 + 2p}{ikR \cos \theta} (-\sin^2 \theta)^p - \frac{2p^2 - 1}{ikR \cos \theta} (-\sin^2 \theta)^{p-1} \\
&- \frac{2p^4 + \frac{4}{3} p^3 - p^2 - \frac{1}{12} p}{k^2 R^2} \right) (-\sin^2 \theta)^{p-2} E_{\phi}^{\text{inc}} \hat{\phi} \\
&+ \left( -\frac{\cos \theta}{ikR} (-\sin^2 \theta)^{p-1} - \frac{2p^2}{k^2 R^2} (-\sin^2 \theta)^{p-1} \\
&- \frac{2p^2 - 2p}{k^2 R^2} \delta_{p,1} \right) E_{\phi}^{\text{inc}} \hat{\phi} + O \left( \frac{1}{k^3 R^3} \right),
\end{align*}
\]

where $\delta_{p,q}$ is the Kronecker $\delta$-function.

Proof. The expression can be computed explicitly for $p = 1$ and 2 and proved by induction for $p \geq 2$.

Lemma 24. Let $z = -\sin^2 \theta$ and $x = kR$. Then a third order approximation in $x$
is obtained,

\[
F(x, CC)E_{\tan}^{\text{inc}} = \left( 1 + \frac{i}{x} \frac{1}{(1 + |\cos \theta|) \cos \theta} + \frac{1}{x^2 \cos^3 \theta} \right) E_{\theta}^{\text{inc}} \hat{\theta} + \left( |\cos \theta| + \frac{i}{x} \frac{\delta_{lt} \sin^2 \theta}{\cos^2 \theta} + \frac{i \sgn(\cos \theta)}{x} \right) E_{\phi}^{\text{inc}} \hat{\phi} + \delta_{lt} \frac{1 + 4 \sin^2 \theta - \sin^4 \theta}{\cos^3 \theta} E_{\theta}^{\text{inc}} \hat{\theta} \]

\[
G(x, GD)E_{\tan}^{\text{inc}} = \left( \frac{1}{|\cos \theta|} - \frac{i}{x} \frac{\delta_{lt} \sin^2 \theta}{\cos^4 \theta} - \frac{i}{x \cos \theta (1 + |\cos \theta|)} \right) E_{\theta}^{\text{inc}} \hat{\theta} - \delta_{lt} \frac{1}{x^2 \cos^3 \theta (1 + |\cos \theta|)} E_{\phi}^{\text{inc}} \hat{\phi} + \left( 1 - \frac{i \sgn(\cos \theta)}{x} - \frac{1}{x^2} \frac{\delta_{lt} \sgn(\cos \theta) (1 + \cos \theta)}{(1 + |\cos \theta|) \cos \theta} \right) E_{\phi}^{\text{inc}} \hat{\phi}.
\]

**Proof.** Inserting the expressions from lemma 23 in the Taylor expansion and computing the closed expressions from the new expansions yield after many tedious computations the lemma. It should be remarked that the Kronecker-terms cancel the terms different from 1 in \( f_0 \) and \( g_0 \) up to the order of the approximation. \( \square \)

A third order GPO approximation derived from the expression 4.84 can now be stated,

**Theorem 1.** A third order generalized PO approximation admits the formula

\[
ZJ_s^{sc} = \left( \frac{1}{|\cos \theta|} + \frac{\delta_{lt} \sin^2 \theta}{x \cos^4 \theta} - \frac{\delta_{lt} (9 \sin^2 \theta - \sin^4 \theta)}{x^2 \cos^3 \theta} \right) E_{\theta}^{\text{inc}} \hat{\theta} + \left( |\cos \theta| - \frac{\delta_{lt} \sin^2 \theta}{x \cos^2 \theta} + \frac{\delta_{lt} (5 \sin^2 \theta - \sin^4 \theta)}{x^2 \cos^3 \theta} \right) E_{\phi}^{\text{inc}} \hat{\phi}
\]

**Lemma 25.** Suppose that the incoming field is

\[
E^{\text{inc}} = xe^{-ikz}, \quad H^{\text{inc}} = -ye^{-ikz}.
\]

The PO current

\[
J_{\text{PO}}^{sc} = \begin{cases} \hat{n} \times H^{\text{inc}}, & \text{on } \Gamma_{ltt}, \\ -\hat{n} \times H^{\text{inc}}, & \text{on } \Gamma_{shad}. \end{cases}
\]
has the following two equivalent expressions in spherical coordinates

\[ J_{\text{PO}}^\text{sc} = \begin{cases} 
- H^\text{inc}_\phi \hat{\theta} + H^\text{inc}_\theta \hat{\phi}, & \text{on } \Gamma_{\text{lit}}, \\
H^\text{inc}_\phi \hat{\theta} - H^\text{inc}_\theta \hat{\phi}, & \text{on } \Gamma_{\text{shad}}, 
\end{cases} \]

\[ J_{\text{PO}}^\text{sc} = Y \frac{1}{|\cos \theta|} E^\text{inc}_\theta \hat{\theta} + Y |\cos \theta| E^\text{inc}_\phi \hat{\phi}. \]

**Proof.** The representation in \( H \) is immediate by taking the cross product in the \((\hat{r}, \hat{\theta}, \hat{\phi})\)-coordinate system. The representation in the electric field is easily concluded from the spherical representation of the incoming field,

\[ E^\text{inc}_x = e^{-ikz}, \]
\[ E^\text{inc}_\theta = \cos \theta \cos \phi E^\text{inc}_x, \]
\[ E^\text{inc}_\phi = -\sin \phi E^\text{inc}_x, \]
\[ H^\text{inc}_\theta = -Y \cos \theta \sin \phi E^\text{inc}_x, \]
\[ H^\text{inc}_\phi = -Y \cos \phi E^\text{inc}_x, \]

which yields

\[ H^\text{inc}_\phi = -Y \frac{1}{|\cos \theta|} E^\text{inc}_\theta, \]
\[ H^\text{inc}_\theta = Y \cos \theta E^\text{inc}_\phi, \]

and the lemma follows from that \(|\cos \theta| = \text{sgn}(\cos \theta) \cos \theta. \]

**Theorem 2.** Consider an incoming field

\[ E^\text{inc} = \hat{x} e^{-ikz}, \]
\[ H^\text{inc} = -Y \hat{y} e^{-ikz}. \]

A second order physical optics approximation admits the formula

\[ J^\text{tot} = \begin{cases} 
2 \hat{n} \times H^\text{inc} - \frac{1}{ikR} \sin^2 \theta (H^\text{inc}_\phi \hat{\theta} + H^\text{inc}_\theta \hat{\phi}), & \text{in } \Gamma_{\text{lit}}, \\
0, & \text{in } \Gamma_{\text{shad}}, 
\end{cases} \]

where

\[ \hat{\theta} = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix}, \quad \hat{\phi} = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}. \]

In order to get the second order approximation of the current when the incoming field comes in another direction, and when the scatterer is not a sphere, we need to modify theorem 2. We replace the \( \frac{1}{ik} \) term with the mean curvature \( \mathcal{H} \) and use the following lemma to update the approximation.
Lemma 26. Consider the incoming plane wave,
\[ \mathbf{E}^{inc} = \hat{e} e^{i \mathbf{k} \mathbf{m} \cdot \mathbf{r}}, \]
\[ \mathbf{H}^{inc} = Y \hat{e} e^{i \mathbf{k} \mathbf{m} \cdot \mathbf{r}}, \]
where \( \hat{e} \cdot \mathbf{m} = 0 \) and \( \hat{h} = \mathbf{m} \times \hat{e} \). In the new coordinate system, we can replace the trigonometric functions in the second order PO approximation,
\[ E_x^{inc} = e^{i \mathbf{k} \mathbf{m} \cdot \mathbf{r}}, \]
\[ \cos \theta = -\hat{n} \cdot \hat{m}, \]
\[ \sin \theta = |\hat{n} \times \hat{m}|, \]
\[ \cos \phi = \frac{\hat{n} \cdot \hat{e}}{|\hat{n} \times \hat{m}|}, \]
\[ \sin \phi = -\frac{\hat{n} \cdot \hat{h}}{|\hat{n} \times \hat{m}|}. \]

Proof. In theorem 2, we have the vectors
\[ \mathbf{n} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \]
The trigonometric expressions follow by direct computation, using the vectors \( \mathbf{n}, \mathbf{e}, \mathbf{h} \) and \( \mathbf{m} \). \( \square \)

After using the same kind of correction for the problem at the grazing angle as in the acoustic case and by replacing \( \frac{1}{\mathcal{H}} \) with the mean curvature \( \mathcal{H} \), we get a second order approximation

Theorem 3 (GPO2 in electromagnetics). A second order generalized physical optics approximation is
\[ J_s^{tot} = \begin{cases} 2\mathbf{n} \times \mathbf{H}^{inc} - \frac{\mathcal{H}}{k} \frac{|\mathbf{n} \times \mathbf{m}|^2}{|\mathbf{n} \cdot \mathbf{m}|^2 + 1 - |\mathbf{n} \cdot \mathbf{m}|^2} \left( H_\theta^{inc} \hat{\phi} + H_\phi^{inc} \hat{\theta} \right), & \text{on } \Gamma_{lit} \\ 0, & \text{on } \Gamma_{shad} \end{cases} \]

The third order GPO approximation is
\[ J_s^{tot} \big|_{\Gamma_{lit}} = 2\mathbf{n} \times \mathbf{H}^{inc} + \left( -\frac{\mathcal{H}}{ik} |\mathbf{n} \times \mathbf{m}|^2 + \delta_1 (1 - |\mathbf{n} \cdot \mathbf{m}|^2) \right) + \frac{\mathcal{H}^2}{k^2} |\mathbf{n} \times \mathbf{m}|^2 + \frac{9|\mathbf{n} \times \mathbf{m}|^2 + |\mathbf{n} \times \mathbf{m}|^4}{1 - |\mathbf{n} \cdot \mathbf{m}|^2} \left( H_\phi^{inc} \hat{\theta} + H_\theta^{inc} \hat{\phi} \right) \]
\[ + \left( -\frac{\mathcal{H}}{ik} |\mathbf{n} \times \mathbf{m}|^2 + \delta_1 (1 - |\mathbf{n} \cdot \mathbf{m}|^2) \right) + \frac{\mathcal{H}^2}{k^2} |\mathbf{n} \times \mathbf{m}|^2 - \frac{5|\mathbf{n} \times \mathbf{m}|^2 + 1 - |\mathbf{n} \cdot \mathbf{m}|^2}{1 - |\mathbf{n} \cdot \mathbf{m}|^2} \left( H_\phi^{inc} \hat{\theta} + H_\theta^{inc} \hat{\phi} \right), \]
where the parameters \( \delta_1 \) and \( \delta_2 \) need to be specified. The choice \( \delta_1 = \frac{C_1 \mathcal{H}^2}{k^2} \), \( \delta_2 = \frac{C_2 \mathcal{H}}{k} \) yields a third order approximation for small angles, but where the error at the grazing angle becomes very large. Another choice is \( \delta_1 = \frac{C_1 \mathcal{H}}{k}, \delta_2 = \frac{C_2 \mathcal{H}}{k} \). The two choices are presented in Figure 4.13 and it looks like the second choice \( \delta_1 = \frac{C_1 \mathcal{H}}{k} \) has a smaller error.
Figure 4.13: Relative error in current $\mathbf{J}$ using different PO approximations for a sphere, for wave number $k = 20$ and 80.
Chapter 5

Wave Scattering

The techniques of chapter 4 generate the potentials or in the electromagnetic case the currents on the scattering surfaces. Based on these quantities the frequency domain scattered wave field can be computed from its integral representations (1.17), (1.27), (1.28). The oscillatory potential is multiplied with the oscillatory Green’s function and integrated over the boundary to produce the scattered far-field.

5.1 Restriction of the Integration Domain

When we compute the radar cross section for large wave numbers $k$, both the potential $\partial_n u$ and the Green’s function are highly oscillatory and the integration over the scatterer $\Gamma$ essentially cancels out except for a region around the reflection point, $\phi_0$. This phenomenon is investigated by Bruno et.al in [16]. In the monostatic case, $\phi_0 = 0$. The integrand $G_\infty \partial_n u^{tot}$ is given in Figure 5.1, for a cylinder and four different observation angles.

In Figure 5.1(a)-5.1(c), the integrand is highly oscillatory except at two angles, $\phi_0$ and $\pi + \phi_0$. The angle $\pi + \phi_0$ is on the shadow side of the scatterer, where the integrand is negligible, except when $\phi_0$ approaches $90^\circ$, in figure 5.1(d). Ignoring the problems in Figure 5.1(d), we will truncate the integrand with a filter function around $\phi_0$ eliminating the contribution from the shadow side. Consider a $C^{2P+1}$ filter function $F(\phi, \phi_0, \theta, \psi)$, with the property

$$F_{2P+1}(\phi, \phi_0, \theta, \psi) = f_P(\phi - \phi_0, \theta, \theta + \psi)(1 - f_P(\phi - \phi_0, -\theta, -\psi, -\theta)) \quad (5.1)$$
Figure 5.1: The real part of the integrand in RCS calculations for various observation angles on a circular cylinder, with $k = 40$.

where, with $c(x) = \cos \left( \frac{\pi x - x_0}{x_1 - x_0} \right)$

$$f_P(x, x_0, x_1) = \begin{cases} 1, & x < x_0, \\ g_P(x), & x \in [x_0, x_1], \\ 0, & x > x_1, \end{cases}$$

$$g_0(x) = \frac{1}{2} + \frac{1}{2} c(x),$$
$$g_1(x) = \frac{1}{2} + \frac{3}{4} c(x) - \frac{1}{4} c^3(x),$$
$$g_2(x) = \frac{1}{2} + \frac{15}{16} c(x) - \frac{10}{16} c^3(x) + \frac{3}{16} c^5(x),$$
$$g_3(x) = \frac{1}{2} + \frac{35}{32} c(x) - \frac{35}{32} c^3(x) + \frac{21}{32} c^5(x) - \frac{5}{32} c^7(x).$$
The coefficients in $g_P$ are found by matching derivatives at $x_0$ and $x_1$.

![Figure 5.2: Integration filter $F_1$ with center $\phi_0 = 60^\circ$, $\theta = 90^\circ$ and transition region $\psi = 30^\circ$.](image)

The scattered field of a Dirichlet cylinder computed from (1.18) can be rewritten

$$u^{sc} = - \int F_{2P+1} G \partial_n u^{tot} d\Gamma_y - \int (1 - F_{2P+1}) G \partial_n u^{tot} d\Gamma_y. \quad (5.2)$$

For a given wave number $k$, we want to determine $\phi_0$, $\theta$ and $\psi$ such that the integral

$$\int (1 - F_{2P+1}) G \partial_n u^{tot} d\Gamma_y \quad (5.3)$$

is negligible.

We want to examine how the integration region $\theta$ and transition region $\psi$ depend on the wave number $k$, in order to get a certain relative error in an observation point located at different angles. In [16] it is shown that $\theta \sim \frac{1}{\sqrt{k}}$ is the right order and that $\psi$ depends on the regularity of the transition-function $F_{2P+1}$ when the integrand is slightly simplified. In Figure 5.3 the importance of the smoothness of the transition-function $F_{2P+1}$ is presented. As $F_{2P+1}$ gets more regular, we need a larger transition region, but the relative error decreases more rapidly with increasing $k$. We have used 10 integration points per wavelength, which means that the regularity is not completely resolved for the smallest wave-numbers. In Figure 5.4, we fix $\psi = \frac{\pi}{4} \theta$ and vary $\alpha$, such that $\theta = \alpha \sqrt{\frac{2\pi}{k}}$. A reasonable choice of $\alpha$ is 2 or 3. We have so far only considered the monostatic RCS. When the bistatic
RCS is considered, then the wave-number increases with the angle and therefore \( \theta = \alpha \frac{2\pi}{\sqrt{k \cos \phi_0}} \) is a natural choice of \( \theta \), where \( \phi_0 \) is the stationary point, which is the center of the integration domain. In practise, we take a slightly larger \( \alpha \).

![Figure 5.3: Relative error of monostatic RCS \( \sigma(0, 0) \), with \( k = 5, 20, 80 \) and 320 as a function of \( \frac{\psi}{\theta} \) when \( \theta = 2\sqrt{\frac{2\pi}{k}} \).]

### 5.2 Determination of the Shadow Boundary

In the case where we for instance use physical optics approximation for the potential or current of a general scatterer, we need to find the boundary of the shadow region. This could be done using ray tracing but also in a levelset framework, [69]. The scatterers are then represented by a continuous levelset function \( \Phi \), such that \( \Phi > 0 \).
5.2. DETERMINATION OF THE SHADOW BOUNDARY

![Graphs showing relative error of RCS for different values of k.]

Figure 5.4: Relative error of monostatic RCS $\sigma(0,0)$, with $k = 5, 20, 80$ and $320$ as a function of $\alpha$ when $\theta = \alpha \sqrt{\frac{2\pi}{k}}$.

in the exterior $\Omega'$ and $\Phi < 0$ in the interior $\Omega$. The scatterer boundary is defined by the curve $\Phi(\mathbf{r}) = 0$. A new levelset function $\Psi$ describes the shadow line and is defined by

$$\Psi(\mathbf{r}) = \min_{\mathbf{r}' \in \mathcal{L}(\mathbf{r}, \mathbf{r}_0)} \Phi(\mathbf{r}')$$

where $\mathcal{L}$ is the line segment between the source point $\mathbf{r}_0$ and the current point $\mathbf{r}$. The function $\Psi$ will be zero at the boundary of the shadow and the shadow line is defined by $\Psi$ and $\Phi$ both being zero.
5.3 Radar Cross Section Calculations

An important property when constructing military aircraft is to remain undetected on radar. How well such an airplane is detected is measured by the radar cross section, RCS, which measures the scattered far-field for different angles relative to the incoming field. A useful characterization of the scattering properties of an object is given by the monostatic and bistatic radar cross section. The monostatic RCS is proportional to the apparent size of an object in the direction of the incoming field. The bistatic RCS describes the response in all directions, given the direction of the incoming field. The bistatic RCS

\[ \sigma(\phi, \phi^{\text{inc}}) = \lim_{r \to \infty} 2\pi r \frac{|u^{\text{sc}}(r, \phi)|^2}{|u^{\text{inc}}(0, 0)|^2}, \quad \text{in} \ 2D, \]  

(5.4)

\[ \sigma(\theta, \phi, \theta^{\text{inc}}, \phi^{\text{inc}}) = \lim_{r \to \infty} 4\pi r^2 \frac{|u^{\text{sc}}(r, \theta, \phi)|^2}{|u^{\text{inc}}(0, 0, 0)|^2}, \quad \text{in} \ 3D. \]  

(5.5)

There are simplifications to the Green’s function for Helmholtz equation, when the RCS is to be computed. Defining the far-field pattern

\[ u^{\infty}(\hat{r}) = \begin{cases} e^{ik|\hat{r}|} \sqrt{|\hat{r}|} \left( u^{\infty}(\hat{r}) + O\left(\frac{1}{|\hat{r}|}\right) \right), & \text{in} \ 2D, \\ e^{ik|\hat{r}|} \left( u^{\infty}(\hat{r}) + O\left(\frac{1}{|\hat{r}|}\right) \right), & \text{in} \ 3D, \end{cases} \]  

(5.6)

where \( \hat{r} = \frac{r}{|r|} \).

The radar cross section can now be reformulated,

\[ \sigma(\phi, \phi^{\text{inc}}) = 2\pi \frac{|u^{\infty}(\phi)|^2}{|u^{\text{inc}}(0, 0)|^2}, \quad \text{in} \ 2D, \]  

(5.7)

\[ \sigma(\theta, \phi, \theta^{\text{inc}}, \phi^{\text{inc}}) = 4\pi \frac{|u^{\infty}(\theta, \phi)|^2}{|u^{\text{inc}}(0, 0, 0)|^2}, \quad \text{in} \ 3D. \]  

(5.8)

The limiting Green’s function is given by, [21],

\[ G^{\infty}(\hat{r}, \hat{r}') = \begin{cases} \frac{e^{ik|\hat{r}|}}{\sqrt{|\hat{r}|}} e^{-ik\hat{r}'}, & \text{in} \ 2D, \\ \frac{1}{4\pi} e^{-ik\hat{r}'}, & \text{in} \ 3D. \end{cases} \]  

(5.9)

The far-field pattern in acoustics can now be computed

\[ u^{\infty}(\hat{r}) = \iint (\partial_n G^{\infty}(\hat{r}, \hat{r}')) u^{\text{tot}}(\hat{r}') - G^{\infty}(\hat{r}, \hat{r}') (\partial_n u^{\text{tot}}(\hat{r}')) d\Gamma'. \]  

(5.10)

5.4 Error in Far-field Computations With OSRC

We want to get an estimate of the error when OSRC conditions are used in far-field computations. The coordinate system is chosen such that the incoming field is a
plane wave along the negative x-axis, $u^{inc} = e^{-ikr \cos \phi}$. A far-field computation in the $\hat{r} = (\cos \phi, \sin \phi)$ direction using an OSRC condition $\partial_n u^{sc} \approx OSRC(u^{sc}) = A_0 u^{sc} + A_2 \partial^2_s u^{sc}$ on a cylinder scatterer in 2D with a Dirichlet boundary has an absolute error

$$E(\hat{r}) = \int G_\infty(\hat{r}, r') (\partial_n u^{sc}(r') + OSRC(u^{inc}(r'))) \, d\Gamma'.$$

The scatterer is parametrized by

$$x' = r(\phi) \cos \phi, \quad y' = r(\phi) \sin \phi.$$

It is assumed that the scatterer is sufficiently smooth, such that:

- It is convex,
- $\partial_n u^{sc} = \mu u^{inc}$, where $\mu$ is a slowly varying function,
- $q_2 = \frac{\partial^2 u^{inc}}{u^{inc}}$ is a slowly varying function,
- $\rho_{sv} = r(\phi) (\mu + A_0 + A_2q_2)$ is a slowly varying function.

**Remark 7.** A slowly varying function does not oscillate with a frequency which depends on the wave number $k$.

In order to make life simple, we need the following identities,

$$\hat{r} \cdot r' = r(\phi) \cos (\phi - \hat{\phi}), \quad (5.11)$$

$$\cos (\phi - \hat{\phi}) + \cos \phi = 2 \cos \left( \phi - \frac{\hat{\phi}}{2} \right) \cos \left( \frac{\hat{\phi}}{2} \right). \quad (5.12)$$

The error in the RCS calculation using the OSRC approximations for a Dirichlet problem in 2D is then

$$E(\hat{r}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{-\pi}^{\pi} e^{-ikr(\phi) \cos \hat{\phi} \cos \left( \phi - \frac{\hat{\phi}}{2} \right)} \rho_{sv}(\phi) d\phi. \quad (5.13)$$

By using the method of stationary phase [13], the most significant term in the expansion over $k$ can be computed. Looking for zeroes of the derivative of the exponent w.r.t. $\phi$ in (5.13) yields an equation for the stationary phase $\phi_0$,

$$\phi_0 = \frac{\hat{\phi}}{2} + \arctan \frac{\partial_\phi r(\phi_0)}{r(\phi_0)}.$$
Taylor expansion around the stationary point \( \phi_0 \) yields with \( r_p = \partial^p_{\phi} r(\phi_0) \),

\[
r(\phi) \cos \left( \phi - \frac{\phi_0}{2} \right) = f_0 - \frac{1}{2} f_2 (\phi - \phi_0)^2 + \frac{1}{6} f_3 (\phi - \phi_0)^3 + O \left( (\phi - \phi_0)^4 \right),
\]

\[
f_0 = \frac{r_0}{\sqrt{1 + \frac{r_0^2}{r_0^2}}},
\]

\[
f_2 = \frac{r_0}{\sqrt{1 + \frac{r_0^2}{r_0^2}}} \left( 1 + \frac{2r_1^2}{r_0^2} - \frac{r_2}{r_0} \right),
\]

\[
f_3 = \frac{r_0}{\sqrt{1 + \frac{r_0^2}{r_0^2}}} \left( \frac{r_3}{r_0} - \frac{2r_1}{r_0} - \frac{3r_1r_2}{r_0} \right).
\]

Since we have assumed that the scatterer is convex, it holds that \( r_0 > 0 \). In the case where the object has locally constant curvature (that means \( r_j = 0 \), \( j \geq 1 \)), it is clear that the cubic coefficient \( f_3 = 0 \). For a convex object, there exist two stationary phases. One point of stationary phase is in the lit region and one is in the shadow region (except when the outgoing angle is \( \hat{\phi} = 180^\circ \), where the stationary phase is at \( \pm 90^\circ \)). We will use partition of unity,

\[
F_{\delta_0}(\phi - \phi_0) + F_{\delta_1}(\phi - \phi_1) + G(\phi) = 1,
\]

where

\[
F(\phi) = \begin{cases} 1, & \phi \leq \alpha < 1, \in \mathcal{C}^\infty, \\ 0, & \phi \geq 1, \end{cases} \quad F_\delta(\phi) = F \left( \frac{\phi}{\delta} \right)
\]

with \( F_{\delta_0}(\phi - \phi_0) = 1 \) in a sufficiently large neighborhood of the point \( \phi_0 \) of stationary phase in the lit region and \( F_{\delta_1}(\phi - \phi_1) = 1 \) in the corresponding neighborhood of \( \phi_1 \) in the shadow region. This yields

\[
E(\hat{r}) = E_0(\hat{r}) + E_1(\hat{r}) + E_G(\hat{r}),
\]

where

\[
E_j(\hat{r}) = \frac{e^{i\pi/4}}{\sqrt{2\pi k}} \int_{-\pi}^{\pi} F_{\delta_j}(\phi - \phi_j) e^{-2ikr(\phi) \cos \frac{\hat{\phi}}{2} \cos \left( \phi - \hat{\phi} \right)} \rho_{sv}(\phi) d\phi.
\]

and \( E_G \) is defined by replacing \( F_{\delta_j} \rightarrow G \). In [16] Bruno et.al. prove that when \( p \geq 1, 0 < \varepsilon < A \) and \( 0 < \alpha < 1 \), then

\[
\int_{-\pi}^{\pi} F_A(\phi) e^{ik\phi_0} d\phi = \int_{-\varepsilon}^{\varepsilon} F_\varepsilon(\phi) e^{ik\phi_0} d\phi + O \left( (k\varepsilon)^{-n} \right), \quad \forall n \geq 1. \quad (5.14)
\]

If the regularity of the cutoff function \( F \) is decreased to \( \mathcal{C}^N \), then the result holds for \( n \leq N \). For sufficiently large \( k \), only the first two terms in the expansion
around the stationary phase are important. This indicates that we should take
\( \delta_j = b \sqrt{\frac{2\pi}{k}} = b \sqrt{\lambda} \), \( \alpha = \frac{\pi}{2} < 1 \). Numerical computations indicate that \( a \geq 2 \) is
sufficient in order to get a reasonable small error. As a result of the theorem proved
by Bruno et al., we can deduce that \( E_G \) is negligible, and the domain of integration
can be reduced to around the stationary points, when \( \delta_j \) is chosen as indicated
above.

Define
\[
\begin{align*}
k_0 &= \frac{-2kr_0 \cos \frac{\phi}{2}}{\sqrt{1 + \frac{r_1^2}{r_0^2}}}, \\
k_2 &= \frac{-1}{2} k_0 \left( 1 + 2 \frac{r_1^2}{r_0^2} - \frac{r_2^2}{r_0^2} \right) = kr_0 C_2 (r_0, r_1, r_2) \cos \frac{\phi}{2}, \\
-2ikr(\phi) \cos \frac{\phi}{2} \cos \left( \phi - \frac{\phi_0}{2} \right) &= ik_0 + ik_2 (\phi - \phi_j)^2 + O((\phi - \phi_j)^3).
\end{align*}
\]

The function \( C_2 \) is equal to 1 when the scatterer is a circle.

We can now compute an approximation for \( E_j(\hat{r}) \),
\[
E_j(\hat{r}) = \frac{e^{i\pi/4}+ik_0}{\sqrt{8\pi k}} \int_{-\pi}^{\pi} F_{\delta_j}(\phi - \phi_j) e^{ik_2(\phi - \phi_j)^2 + O((\phi - \phi_j)^3)} \rho_{sv}(\phi) d\phi.
\]

If the third order term in \( \phi - \phi_0 \) is neglected, then we can directly use Bruno’s
theorem and get
\[
E_j(\hat{r}) \leq \frac{1}{\sqrt{8\pi k}} 2\delta_j \cdot \max_{\phi \in (\phi_0 - \delta_j, \phi_0 + \delta_j)} \rho_{sv} + O((k\delta_j)^{-N}).
\]

If we choose \( \delta_j = b \sqrt{\frac{2\pi}{k}} \), then the error from one stationary point is
\[
E_j(\hat{r}) \leq \frac{b}{k} \frac{1}{r_0 C_2 \cos \frac{\phi_j}{2}} \cdot \max_{\phi \in (\phi_0 - \delta_j, \phi_0 + \delta_j)} \rho_{sv} + O\left( \left( \frac{kr_0 C_2 \cos \frac{\phi_j}{2}}{4\pi^2 b^2} \right)^N \right).
\]

For sufficiently large wave number \( k \) and filter parameter \( b \), it is sufficient to
show that the absolute error in the On Surface Radiation Condition is first order
in \( k \), i.e. \( \rho_{sv} = O(k^{-1}) \). (This corresponds to a second order relative error.)

5.5 Numerical computations of the far-field

We have developed several different schemes to approximate the potential of a scat-
terer in the incoming field for scalar Helmholtz equation and Maxwell’s equations.
In this chapter we apply these schemes on different geometries. We will present far-
field computations for different geometries for wave numbers \( k = 20 \) and \( k = 80 \).
For $k = 80$, we also introduce a filter $F_5(\phi, 0, 60^\circ, 30^\circ)$ defined in (5.1). The absolute error of the PO approximations are decreased when the filter is used. The error decreases also the implicit OSRC approximation. In the computation of the far-field, we will use the far-field Green’s functions (5.4).

Circular Cylinder With a Dirichlet Boundary

In previous chapters, several different schemes are derived, which approximates the analytical solution of a circular cylinder with either Dirichlet or Neumann boundaries. These schemes should naturally work well for circular objects, as in Figure 5.5(a).

In Figure 5.5, the bistatic RCS $\sigma(\phi_{obs}, 0)$ defined in (5.4) is computed. In Figure 5.5(a) and 5.5(b), the absolute error in the far-field are computed without any filter function when $k = 20$, with different approximations without any filter-function. The approximations when $k = 80$ are presented in Figure 5.5(c) and 5.5(d). In Figure 5.5(e) and 5.5(f) we introduce a filter $F_5(\phi, 0, 60^\circ, 30^\circ)$ defined in (5.1). The OSRC gives a substantially better approximation for small angles than the original PO approximation.

Elliptic Cylinder With a Dirichlet Boundary

When the scatterer is no longer of a circular shape, then we approximate the scatterer locally with the osculating circle and apply the conditions for the circle locally. The elliptical cylinder has major axis $a = 1$ and minor axis $b = \frac{1}{2}$, see Figure 5.5(b).

In Figure 5.6, we present the bistatic far-field, when the incoming field propagates in the negative $x$-direction. The absolute error is smaller with a filter for the PO approximations and the implicit OSRC approximation.

2 radii Cylinder With a Dirichlet Boundary

In Figure 5.7, we present the bistatic far-field, when the incoming field propagates in the negative $x$-direction for two wave numbers $k = 20$ and $80$, where the scatterer is the cylinder in Figure 5.5(c), with large radius 1 and small radius 0.1.
Figure 5.5: Absolute error of bistatic RCS for a circular cylinder with Dirichlet boundary and with wavenumber $k = 20$ and 80. The RCS solution is denoted by reference (ref).
Figure 5.6: Absolute error of bistatic RCS for an elliptic cylinder with Dirichlet boundary and with wavenumber $k = 20$ and 80. The RCS solution is denoted by reference (ref).
Figure 5.7: Absolute error of bistatic RCS for an 2 radii cylinder with Dirichlet boundary and with wavenumber $k = 20$ and 80. The RCS solution is denoted by reference (ref).
Appendix A

The GEMS project

The center of excellence: Parallel and Scientific Computing Institute (PSCI) was established to facilitate research collaboration between academic institutions and industry. It was supported by the Swedish government agency Vinnova and hosted by the Royal Institute of Technology (KTH) in close cooperation with Uppsala University during the period 1995-2005. One of the programs in PSCI was computational electromagnetics and the central project in this program as well as overall in PSCI was the project: General Electromagnetic Solver (GEMS). The other programs in PSCI were computational fluid dynamics, bio computing, and high performance computing.

The GEMS project involved KTH, Uppsala University and Chalmers University of Technology on the academic side, the national defense research agency FOI, and Ericsson and Saab on the industrial side. The overarching goal of GEMS was the development of a simulation system that could be used for electromagnetic simulations in industry as well as functioning as a platform for academic research. This development was done by engineers from industry, by programmers directly working for PSCI and by graduate students. A research program supported the development and the contribution of this thesis was part of that program.

The GEMS system consists of two parts: a time domain simulation system and a frequency domain simulation system. Both of these systems are of hybrid type, which means that they include a combination of different numerical algorithms that are applied in different parts of the computational domain.

The time domain hybrid solver contains a finite difference time domain (FDTD) algorithm based on the Yee scheme [70] for structured grids. The Yee scheme is highly efficient but lacks in accuracy when modeling general geometries. Therefore this method is hybridized with unstructured grids close to boundaries that do not fit the structured grid. The algorithms for the unstructured parts are either an explicit finite volume time domain method (FVTD) or an implicit finite element time domain (FETD) method. Since the unstructured domains are only near the boundary the FETD solver is the standard unstructured choice. The extra cost for
the implicit structure is then minimal and the flexibility of no time step restriction even for small elements is desirable. There are two options available for treating the far-field. One is radiation or absorbing boundary conditions and the other is perfectly matched layers. Different material models and sub-grid models for wires are available.

The study on time domain integral equation techniques that is described in chapter 2 was partially done in order to evaluate its potential as another option in the time domain system. Time domain integral equation methods have also been successfully used as far-field boundary conditions.

The frequency domain system is built around a method of moments (MoM) solver with a fast multipole method (FMM) to reduce the computational complexity. The MoM method allows for different types of boundaries as, for example, perfect conductors (PEC), dielectric materials and ports. Wire models are also available in this case. For high frequencies the MoM integral equation technique is still too computationally costly and a physical optics (PO) approximation can be applied for parts of the boundary. The OSRC research described in this thesis is aimed at improving the accuracy of the PO technique.

For even higher frequencies relative to the geometry the frequency domain system has a geometrical theory of diffraction (GTD) ray tracing module. This module can be used stand alone or coupled to the other frequency domain components.

Both the time domain and the frequency domain systems are implemented for distributed computing using MPI message passing. They also are integrated with a user interface that can handle geometric input and visualize solutions. An important component of this IO system is the generation of a computational grid. There also are a few other routines that are used in both the time domain and the frequency domain systems. One such routine simulates waveguides and is used to generate data for ports.
Figure A.1: The GEMS hybrid system
Appendix B

Differential Geometry

The idea of On Surface Radiation Conditions is to use surface derivatives in order to approximate certain quantities, such as the potential in acoustics and the current in electromagnetics. In this appendix, we will define different kinds of surface derivatives, which are used in our OSRC conditions.

Let the function $u$ and the vector field $v$ be defined on a surface $\Gamma$. In order to take surface derivatives, $u$ and $v$ have to be extended to a tubular neighborhood of $\Gamma$. We define $\tilde{u}$ and $\tilde{v}$ in the tubular neighborhood by

$$
\tilde{u}(r) = u(\mathcal{P}_\Gamma r), \quad \tilde{v}(r) = v(\mathcal{P}_\Gamma r),
$$

where $\mathcal{P}_\Gamma r$ is the orthogonal projection of $r$ onto the continuously differentiable surface $\Gamma$. For a sufficiently small neighborhood, this projection is unique. One can now define

$$
\text{grad}_\Gamma u = \left(\text{grad}\tilde{u}\right)|_\Gamma, \quad \text{div}_\Gamma v = \left(\text{div}\tilde{v}\right)|_\Gamma, \quad \text{curl}_\Gamma u = \left(\hat{n} \times \text{grad}\tilde{u}\right)|_\Gamma, \quad \text{curl}_\Gamma v = \left(\hat{n} \cdot \text{curl}\tilde{v}\right)|_\Gamma
$$

In spherical coordinates the explicit expressions are

$$
\text{grad}_\Gamma u = \frac{1}{R} \frac{\partial u}{\partial \theta} \hat{\theta} + \frac{1}{R \sin \theta} \frac{\partial u}{\partial \phi} \hat{\phi}, \quad \text{div}_\Gamma v = \frac{1}{R \sin \theta} \left( \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{\partial v_\phi}{\partial \phi} \right), \quad \text{curl}_\Gamma u = \frac{1}{R} \frac{\partial u}{\partial \phi} \hat{\theta} - \frac{1}{R \sin \theta} \frac{\partial u}{\partial \theta} \hat{\phi}, \quad \text{curl}_\Gamma v = \frac{1}{R \sin \theta} \left( \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right).
$$

145
In order to obtain variational formulations containing surface derivatives, we need some Stoke identities, Nedelec [55],

\[
\int (\nabla \Gamma \cdot \div \Gamma u) \cdot v \, d\Gamma = -\int (\div \Gamma u)(\div \Gamma v) \, d\Gamma, \quad (B.9)
\]
\[
\int (\nabla \times \Gamma \times \nabla \times \Gamma u) \cdot v \, d\Gamma = \int (\nabla \times \Gamma u)(\nabla \times \Gamma v) \, d\Gamma. \quad (B.10)
\]

The laplacian of the scalar function \( u \) can in 3D be expressed as in Nedelec [55],

\[
\Delta u = \Delta \Gamma u + 2\mathcal{H}\partial_n u + \partial^2_n u, \quad (B.11)
\]

where \( \mathcal{H} \) is the mean curvature.
Appendix C

Sobolev Spaces

The integral formulations of the Helmholtz equation contain the incoming field and the potential, which both are defined on the surface $\Gamma$. We need to define the Sobolev spaces to understand the required regularity in order to get a well-defined integral formulation of the different problems considered. The Sobolev space $H^m(\Omega)$ is the space of functions in $L^2(\Omega)$, with derivatives up to order $m$ are in $L^2(\Omega)$. This can be extended to non-integer $m$ by using the Fourier transform,

$$H^s(\mathbb{R}^2) = \left\{ u \in S'(\mathbb{R}^2) : \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

where $S'(\mathbb{R}^2)$ is the dual space of the infinitely differentiable, rapidly decreasing functions in $\mathbb{R}^2$. The associated norm of the Sobolev space is

$$\|u\|_{H^s(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

The Sobolev space is valid also for negative $s$. For variational formulations in timedomain, we need to define a Sobolev space $H^s_\omega(\mathbb{R}_+, E)$,

$$f \in H^s_\omega(\mathbb{R}_+, E) \iff \int_{-\infty}^{+\infty} |k|^{2s} \| \hat{f}(k) \|^2_E dk < \infty.$$  \hfill (C.1)

The trace theorem [55] states that if $u \in H^m(\Omega)$, then the restriction of the function onto $\Gamma$ belongs to $H^{m-\frac{1}{2}}(\Gamma)$. We define [15]

$$H^s_{loc}(\Omega) = \{ f : f \in H^s(K), \forall \text{ compact } K \subset \text{ interior } \Omega \}.$$

Denote by $TH^s(\Gamma)$ the space of vectors tangent to $\Gamma$ such that the components in a local basis in the tangent plane are in the space $H^s(\Gamma)$, where $\Gamma$ is a smooth
surface. We introduce the Hilbert spaces

\[
H^s_{\text{curl}}(\Gamma) = \{ g \in TH^s(\Gamma) : \text{curl}_\Gamma g \in H^s(\Gamma) \},
\]

\[
H^s_{\text{div}}(\Gamma) = \{ g \in TH^s(\Gamma) : \text{div}_\Gamma g \in H^s(\Gamma) \}.
\]
Appendix D

Conforming Basis Functions

When On Surface Radiation Conditions are to be used on electromagnetic scattering problems, we need to project our incoming field and currents onto a divergence-conforming and a curl-conforming part, such that

\[ P_{\text{div}} E_{\text{tan}} = E_{\text{div}}, \]
\[ P_{\text{curl}} E_{\text{tan}} = E_{\text{curl}}, \]
\[ E_{\text{tan}} = E_{\text{div}} + E_{\text{curl}}, \]
\[ \text{curl}_T (E_{\text{div}}) = 0, \]
\[ \text{div}_T (E_{\text{curl}}) = 0. \]

The projection is done by using divergence-conforming RWG-elements. These elements are also called Raviart-Thomas-elements. The curl-conforming elements are called Whitney-elements.

The RWG-elements can be defined as

\[ \Phi_{\text{div}}^m (r) = \begin{cases} \frac{l_m}{2A^+_m} \mathbf{v}^+_m, & \text{for } r \in K^+_m; \\ 0, & \text{for } r \notin K^+_m, \end{cases} \]

where \( l_m = |r_3 - r_2| \) is the length of the edge, \( A^+_m \) is the area of triangle \( K^+_m \) and \( K^-_m \), respectively and \( \mathbf{v}^+_m = r - r_1, \mathbf{v}^-_m = r_4 - r \). The involved quantities are described in figure D.1. The Whitney elements are orthogonal to the RWG-elements, i.e.

\[ \Phi_{\text{curl}}^m (r) = \mathbf{n} \times \Phi_{\text{div}}^m (r). \]
We list some identities for the basis functions
\[
\begin{align*}
\text{div}\Phi_m^{\text{div}} &= \pm \frac{l_m}{A^\pm}, \\
\text{curl}\Phi_m^{\text{div}} &= 0, \\
\text{div}\Phi_m^{\text{curl}} &= 0, \\
\text{curl}\Phi_m^{\text{curl}} &= \pm \frac{l_m}{A^\pm}, \\
\Phi_m^{\text{div}} \cdot \Phi_n^{\text{div}} &= \frac{l_m l_n}{4A_m^\pm A_n^\pm} \mathbf{v}_m^\pm \cdot \mathbf{v}_n^\pm, \\
\Phi_m^{\text{curl}} \cdot \Phi_n^{\text{curl}} &= \frac{l_m l_n}{4A_m^\pm A_n^\pm} \mathbf{v}_m^\pm \cdot \mathbf{v}_n^\pm, \\
\Phi_m^{\text{div}} \cdot \Phi_n^{\text{curl}} &= -\frac{l_m l_n}{4A_m^\pm A_n^\pm} \hat{n} \cdot (\mathbf{v}_m^\pm \times \mathbf{v}_n^\pm), \\
\Phi_m^{\text{curl}} \cdot \Phi_n^{\text{div}} &= \frac{l_m l_n}{4A_m^\pm A_n^\pm} \hat{n} \cdot (\mathbf{v}_m^\pm \times \mathbf{v}_n^\pm).
\end{align*}
\]

In order to get the tangential part of e.g. the electric field, one can use these basis functions. The weight of \(\Phi_m^{\text{div}}\) is the normal flow over the edge and the weight of \(\Phi_m^{\text{curl}}\) is the tangential flow on the edge. We can write
\[
\begin{align*}
\mathbf{r}_m &= \frac{\mathbf{r}_m^m + \mathbf{r}_m^n}{2}, \\
\hat{\mathbf{t}}_m &= \frac{\mathbf{r}_m^3 - \mathbf{r}_m^2}{|\mathbf{r}_m^3 - \mathbf{r}_m^2|}, \\
\hat{\mathbf{n}}_m &= \frac{\hat{\mathbf{n}}_m^+ + \hat{\mathbf{n}}_m^-}{2}, \quad \text{(or analytical, if available)} \\
\mathbf{E}_{\text{tan}}(\mathbf{r}) &= \sum_m (\mathbf{E}(\mathbf{r}_m) \cdot \hat{\mathbf{t}}_m)\Phi_m^{\text{curl}}(\mathbf{r}) + (\mathbf{E}(\mathbf{r}_m) \cdot (\hat{\mathbf{t}}_m \times \hat{\mathbf{n}}_m))\Phi_m^{\text{div}}(\mathbf{r}).
\end{align*}
\]
Appendix E

List of Symbols and Abbreviations

\( \nabla u \) Gradient w.r.t. spatial variable.
\( \nabla \cdot \mathbf{v} \) Divergence of vector \( \mathbf{v} \).
\( \nabla \times \mathbf{v} \) Curl of vector \( \mathbf{v} \).
\( \partial_n \) Normal derivative, \( \partial_n u = \hat{n} \cdot \nabla u \).
\( \nabla_{\Gamma} \) Surface gradient.
\( J_m(x) \) Bessel function of order \( m \).
\( Y_m(x) \) Neumann function of order \( m \).
\( H^{(1)}_m(x) \) Hankel function of first kind of order \( m \).
\( \mathcal{H}^{(1)}_m(x) \) Spherical Hankel function of first kind of order \( m \).
\( P_m \) Legendre function of order \( m \).
\( P^\mu_m \) Associated Legendre function of order \( m \).
\( G \) Green’s function.
\( (2p-1)!! \) Odd semi-factorial \( 1 \cdot 3 \cdot \ldots \cdot (2p-1) \).
\( (2p)!! \) Even semi-factorial \( 2 \cdot 4 \cdot \ldots \cdot 2p \).
\( \varepsilon_m \) Neumann symbol, \( \varepsilon_0 = 1 \) and \( \varepsilon_m = 2 \) for \( m = 1, 2, 3, \ldots \)
\( k \) Wave number in Acoustics.
\( \kappa \) Curvature.
\( \mathcal{H} \) Mean curvature \( \frac{\kappa_1 + \kappa_2}{2} \), where \( \kappa_j \) are the principal curvatures.
\( \delta_{\text{lit}} \) 1 on illuminated (lit) region and 0 on shadow region.
## APPENDIX E. LIST OF SYMBOLS AND ABBREVIATIONS

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>GO</td>
<td>Geometrical optics.</td>
</tr>
<tr>
<td>GTD</td>
<td>Geometrical Theory of Diffraction.</td>
</tr>
<tr>
<td>PO</td>
<td>Physical Optics.</td>
</tr>
<tr>
<td>PTD</td>
<td>Physical Theory of Diffraction.</td>
</tr>
<tr>
<td>MoM</td>
<td>Method of Moments.</td>
</tr>
<tr>
<td>TDIE</td>
<td>Time Domain Integral Equations.</td>
</tr>
<tr>
<td>MOT</td>
<td>Marching On in Time.</td>
</tr>
<tr>
<td>DtN-operator</td>
<td>Dirichlet-to-Neumann operator.</td>
</tr>
<tr>
<td>OSRC</td>
<td>On Surface Radiation Condition.</td>
</tr>
<tr>
<td>RCS</td>
<td>Radar Cross Section.</td>
</tr>
<tr>
<td>Monostatic RCS</td>
<td>RCS as a function of the wave number $k$ for backscattering.</td>
</tr>
<tr>
<td>Bistatic RCS</td>
<td>RCS as a function of the angle for a fixed wave number $k$.</td>
</tr>
<tr>
<td>PEC</td>
<td>Perfectly Electric Conductor.</td>
</tr>
<tr>
<td>GEMS</td>
<td>General ElectroMagnetic Solver.</td>
</tr>
<tr>
<td>PSCI</td>
<td>Parallel and Scientific Computing Institute.</td>
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Bibliography


