A NATURAL INTERPRETATION OF CLASSICAL PROOFS

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Preface

My confusion began with the halting problem for Turing machines and Gödel's undecidability results, and then developed in a way reminiscent of the debate on the foundations of mathematics of the early 20th century, with its vain attempts to reconcile Brouwer's intuitionism with Cantor's set theory. Before that, I thought of logic as something of which there can be no disagreement. I was wrong. Philosophers do disagree on the semantics of existence and consequently on what there is. The situation extends to mathematics.

There are two major schools of logic of importance for this problem, classical and constructive logic. Constructive logic developed from Brouwer's intuitionism and is for historical reasons also known as intuitionistic logic, but should not to be confused with Brouwer's intuitionism. The most obvious difference between classical and constructive logic is that they accept respectively reject the *law of excluded middle*, the principle that A or not-A is true for any proposition A. This is related to their respective view of the nature of mathematical objects. The school of classical logic holds that there exists a mathematical reality for the mathematicians to discover, while the school of constructive logic holds that mathematical objects are constructions for the mathematicians to invent.

Consider the proposition *every bounded increasing sequence of rational numbers converges*. The two schools disagree on the truth of this proposition, but they agree on the truth of the proposition *no bounded increasing sequence of rational numbers diverges*. To understand why, consider what it means for a sequence of rational numbers to converge and what it means for the same sequence to diverge.

Let $x_1, x_2, x_3, ...$ be a sequence of rational numbers. We say that the sequence *converges* provided that for every positive rational number ϵ there exists a natural number n such that for every natural number m greater than or equal to n the absolute difference $|x_m - x_n|$ is less than ϵ. Furthermore, we say that the sequence *diverges* provided that there exists a positive rational number ϵ, called the *witness*, such that for every natural number n there exists a natural number m greater than or equal to n such that the absolute difference $|x_m - x_n|$ is greater than or equal to ϵ.

Using the law of excluded middle one can argue that every sequence of rational numbers that does not diverge must converge. However, the knowledge that a particular sequence of rational numbers does not diverge is not enough to enable one to approximate the corresponding real number to an arbitrarily given degree of accuracy. On the other hand, the constructivist claims that to know that a particular sequence of rational numbers converges is to know how to approximate the corresponding real number to an arbitrarily given degree of accuracy. Hence the constructivist rejects the law of excluded middle.

Using the law of excluded middle one can also argue that every sequence of rational numbers that does not converge must diverge. However, one can not from the knowledge that a particular sequence of rational numbers does not converge construct the corresponding witness.

The example just given illustrates the direct nature of constructive existence, as opposed to the indirect nature of classical existence, and shows why the law of excluded middle is not accepted as a law of constructive logic.

Finally, I would like to thank my supervisor, Per Martin-Löf, for posing the problem of investigating how the double-negation interpretation operates on derivations and not only on formulas as well as for his continued guidance of my work. Without him, this thesis would never have come into existence.

Jens Brage

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Chapter 1

Introduction

The topic of this thesis is to interpret classical logic in constructive type theory and show how classical logic fits within the semantics of constructive type theory. It is a topic on the borderline between mathematical logic and computer science. Today, most research in this area is pursued from the perspective of computer science and the theory of continuation-passing-style translations, or CPS translations for short. Furthermore most such CPS translations of classical logic into constructive logic only use the simply typed λ-calculus and some algebraic types. See Fischer (1972), Reynolds (1972), and Plotkin (1975) for the foundations of CPS translations and Reynolds (1993) for the early history of continuations. To my knowledge there is no interpretation of classical logic in constructive logic that makes full use of the syntactic-semantic method of constructive type theory. With this thesis I hope to fill this gap.

The subject of interpretations of classical logic in constructive logic began with the double negation interpretation of classical logic in minimal logic due to Kolmogorov (1925). The double negation interpretation was then followed by the interpretation of Peano arithmetic in Heyting arithmetic due to Gödel (1933) and the interpretation of classical logic in intuitionistic logic due to Kuroda (1951). Yet it was not until Griffin (1990) showed how to extend the *formulae-as-types* correspondence to classical logic that significant growth took place. His solution was to include operations on the flow of control, similar to `call/cc` of Scheme, into the notion of computation given by a simply typed call-by-value λ-calculus. After that Parigot (1992) introduced his $\lambda\mu$-calculus to realize classical proofs as programs. The $\lambda\mu$-calculus extended the simply typed λ-calculus with operators that can be used to model operations on the flow of control. The development then took the form of CPS translations of different $\lambda\mu$-calculi into different λ-calculi. See Ong (1996) and Ong and Stewart (1997) for call-by-value respectively call-by-name CPS translations of Parigot's $\lambda\mu$-calculus into the simply typed λ-calculus. See Selinger (2001, p. 24) for an informal description of the semantics of the $\lambda\mu$-calculus.

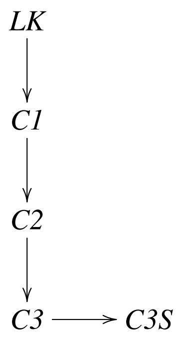

Figure 1.1: Rough picture of how the different calculi relate to each other.

This thesis grew out of the problem of how the double negation interpretation operates on derivations and not only on formulas. The solution can be understood as introducing a new set of logical constants, incorporating a CPS translation, in constructive type theory.

1.1 Summary

We shall introduce several calculi in this thesis. Figure 1.1 roughly pictures how they relate to each other.

The thesis may be considered as consisting of four parts. The first part is made up by Chapters 2 and 3, and constitutes the foundation on which the other parts rest. It contains the definition of the interpretation central to the thesis. The other three parts are Chapter 4, Chapter 5, and Chapter 6, respectively. In Chapter 4, we consider how the interpretation induces permutative rules for classical logic. In Chapter 5, we consider how the interpretation induces contraction rules for classical sequent calculi. In Chapters 6, we investigate how the interpretation can be used to make the Brouwer-Heyting-Kolmogorov semantics, due to Brouwer (1908, 1924), Heyting (1934), and Kolmogorov (1932), justify classical logic. We also consider how the interpretation relates to the theory of CPS translations. The plan of the thesis is as follows.

Chapter 2 is taken up by the introduction of the calculi *C1*, *C2*, and *C3* on which the thesis builds. The calculus *C1* is a formulation of Gentzen's *LK*. The idea behind *C1* is to interpret a classical sequent by a category in the sense of constructive type theory. We introduce signs in the sense of the classical tableau calculus of Beth (1959) and move the formulas of the succedent of a classical sequent to the antecedent, where the signs serve to distinguish the truth of not-A from the falsity of A. The signs are chosen to conform to the constructive concepts of truth and falsity in Martin-Löf (1995). The sign of falsity is interpreted in the sense of minimal logic.

The calculus *C2* is just *C1* in natural deduction style. The calculus *C3* is a natural deduction calculus chosen to induce certain plausible contraction rules for *C2* by means of an interpretation of *C2* in *C3*. The choice is made in a systematic way and yields a natural set of contraction rules for *C3*. The inversion principle of Gentzen (1935) and Prawitz (1965) holds for *C3*, which consequently shares the good normalization properties of intuitionistic natural deduction.

Chapter 3 is taken up by the interpretation of *C3* in constructive type theory. The idea is to let the introduction rules of *C3* determine the meanings of the logical constants of classical logic in the same way the introduction rules of intuitionistic logic determine the meanings of the logical constants of intuitionistic logic. The associated elimination rules are then used to interpret the elimination rules of *C3*. This offers a solution to the problem of what it means to be a classical proof of a proposition and what it means for two classical proofs of a proposition to be equal: the interpretation reduces the concept of a classical proof to the type theoretic concept of a proof, but for a classical proposition. In this way classical predicate logic becomes a fragment of constructive type theory on a par with intuitionistic predicate logic.

The interpretation obtained in this way is shown to respect convertibility and to be injective with respect to convertibility. If the concept of a classical proof is taken to mean a derivation modulo convertibility, then the interpretation can be said to map classical proofs to constructive proofs in an injective way.

We explore how the interpretation relates to constructive type theory. In particular, we merge the classical rules of implication and universal quantification into a dependent product and, moreover, prove the principles of mathematical and W-induction for classical logic and briefly consider the fate of accessibility induction.

Chapter 4 is taken up by the reduction of the interpretation to a translation of *C3* into the minimal fragment of Gentzen's *NJ* and the computation of the kernel of the translation. The translation is not injective with respect to convertibility, contrary to the interpretation of Chapter 3, but factorizes over an auxiliary calculus *C3S* in a way that makes the translation of *C3S* into *NJ* as good as injective. The auxiliary calculus is then used to compute the kernel, which constitutes a coarser equivalence relation of proofs than the identity of proofs in Chapter 3. The kernel is generated by a set of permutative rules for implication and negation that relate to *C3* in roughly the same way as those found by von Plato (2001) relate to his calculus.

Chapter 5 is devoted to the formalization of a contraction relation for *C2* with full precision, compatible with the interpretation of *C2* in constructive type theory with explicit substitution, where cuts are represented by explicit substitutions. The contraction relation is expressed as a term rewriting system using the a term notation closely related to the one introduced by Urban in his thesis (Urban 2000) and related papers (Urban and Bierman 1999, Urban 2001).

The contraction relation for *C2* has many of the good normalization properties of the contraction relation for *C3*. In particular, it is weakly normalizing and Church-Rosser. There are reasons to expect the contraction relation for *C2* to be strongly normalizing as well as it resembles a special case of the strongly normalizing contraction relations of Danos et al. (1997), Urban and Bierman (1999), and Urban (2000, 2001). However, we do not prove that the contraction relation for *C2* is strongly normalizing, but only discuss the reasons why this is to be expected.

Chapter 6 opens with a short discussion of the BHK semantics due to Brouwer (1908, 1924), Heyting (1934), and Kolmogorov (1932). We then demonstrate how the said semantics can be made to justify classical logic. This is done by means of slight shifts in meaning at certain points in the meaning explanations of the logical constants, differentiating between the notions of proof and *classical proof*, the latter incorporating a double negation.

The two notions of proof lend themselves to the introduction of another interpretation of classical logic. The new interpretation does not reinterpret the consequence relation but only the notion of truth, contrary to the previous interpretation.

The two interpretations are shown to be related to the well-known call-by-value and call-by-name CPS translation analyzed by Plotkin (1975) and latter extended to $\lambda\mu$-calculus by Ong (1996) and Ong and Stewart (1997), respectively. To determine the precise relationship, we compare how the the two interpretations act on certain derivations in classical sequent calculus with how the two CPS translations act on the corresponding derivations in $\lambda\mu$-calculus.

We think that these two interpretations have the potential of contributing to the theory of CPS translations as the meaning explanations can be used to give intentional meaning to the logical constants of $\lambda\mu$-calculus. In particular, this indicates that $\&$, $\vee$, $\forall$, $\exists$, $\top$, and $\perp$ should be interpreted in their respective ways independently of whether a call-by-value or call-by-name semantics is used.

Chapter 2

Calculi *C1*, *C2*, and *C3*

We shall in this chapter introduce the calculi *C1*, *C2*, and *C3*, on which the thesis builds, and, moreover, present interpretations of *C1* in *C2* and *C2* in *C3*.

We will use the words *interpretation* and *translation* in a technical sense, as inductively defined functions on syntactic objects (i.e. terms, formulas, derivations), but with a difference with respect to semantics. An interpretation of a source language in a target language serves to give the semantics of the source language in terms of the semantics of the target language, while a translation of a source language into a target language has nothing to do with semantics *a priori*, but only serves as a means to compare the two languages. The interpretation of Chapter 3, of *C3* in constructive type theory, serves to give the semantics of *C3*, *C2*, and *C1*, while the translation of Chapter 4, of *C3* into natural deduction, only serves to compare two languages with already given semantics.

2.1 Preliminary remarks

We presuppose a first order language and a sufficient supply of formal variables. The formal variables are syntactic objects whose precise nature will be of no concern. We shall use the variables of the metalanguage without ever having to display the formal variables themselves.

The logical constants are $\&$, $\vee$, $\supset$, $\neg$, $\forall$, and $\exists$ and follow the usual grammatical rules of formulas. The concept of a formula is defined in the usual way.

We shall use the symbol ψ to denote the type of proofs of the absurdity in the sense of minimal logic. The symbol derives from the Greek word $\psi\epsilon\tilde{\upsilon}\delta o\varsigma$, meaning falsehood. We shall furthermore use the symbols T and F to denote the signs of truth and falsity in the sense of the classical tableau calculus of Beth (1959).

Definition 2.1.1. A *signed formula* is a pair of a sign of truth or falsity and a formula. The sign is written in front of the formula.

Definition 2.1.2. A *signed formula marked by a variable* is a pair of a signed formula and a variable. The variable is written on top of the signed formula.

The variable is analogous to the mark placed on top of a discharged assumption in natural deduction.

Definition 2.1.3. A *context of signed formulas marked by variables* is a finite set of signed formulas marked by variables such that every variable marks no more than one signed formula.

Here the concept of a set is used in the informal sense. We regard contexts of signed formulas marked by variables as syntactic objects, i.e. they are treated as equal provided that they are equal as finite sets.

Definition 2.1.4. A *sequent* is here defined as a pair $(\mathit{\Gamma}, \alpha)$ where $\mathit{\Gamma}$ is a context of signed formulas marked by variables and α is a signed formula or ψ. A sequent is written

$$\mathit{\Gamma} \Rightarrow \alpha.$$

The symbol $\Rightarrow$ should not be confused with the turnstile $\vdash$ which belongs to the metalanguage. See Negri and von Plato (2001, Ch. 1) for a discussion of these and related concepts.

Definition 2.1.5. A *classical sequent* is here defined as a pair $(\mathit{\Gamma}, \mathit{\Delta})$ where $\mathit{\Gamma}$ and $\mathit{\Delta}$ are *contexts of formulas marked by variables*. A classical sequent is written

$$\mathit{\Gamma} \Rightarrow \mathit{\Delta}.$$

The idea to represent a context by a set of marked formulas appears already in Zucker (1974, § 1.4). The representation provides a way around much of the bureaucracy that goes with the concept of a sequent, because the structural rules of contraction and interchange are inherent in the representation.

We consider syntactic objects (e.g. formulas, sequents, derivations) only up to syntactic equality modulo changes of bound variables. The relation of syntactic equality, including changes of bound variables, is written $\equiv$.

We follow the convention introduced by Barendregt (1984, §2.1.3) that bound variables are chosen different from the free variables in any mathematical context. This prevents any free variable from becoming accidentally bound as the result of a substitution.

The result of the substitution of a syntactic object K for the free variable x in the syntactic object M is written $M[K/x]$.

2.2 Calculus *C1*

The calculus *C1* is a formulation of Gentzen's *LK* (Gentzen 1935). The idea behind *C1* is to interpret a classical sequent by a category in the sense of constructive type theory. The first step is to move the formulas of the succedent of a classical sequent in the sense of Definition 2.1.5 to the antecedent, where the signs of truth and falsity serve to distinguish the truth of not-A from the falsity of A. The result can then be represented as a sequent in the sense of Definition 2.1.4 with conclusion ψ, that is, a sequent of the form

$$\varGamma \Rightarrow \psi.$$

The meanings of the signs are chosen to conform to the constructive interpretation of the concepts of truth and falsity in Martin-Löf (1995): the intended interpretation of a signed formula is $\mathrm{T}\,A = \mathrm{proof}(A) : \mathrm{type}\ (A : \mathrm{prop})$ and $\mathrm{F}\,A = (\mathrm{T}\,A)\psi : \mathrm{type}\ (A : \mathrm{prop})$, where ψ is an indeterminate type in the sense of constructive type theory. The intended interpretation of a signed formula marked by a variable is the assumption that the variable is of the type denoted by the signed formula, that is, a clause of the form $x : \mathrm{T}\,A$ or $x' : \mathrm{F}\,A$. The intended interpretation of a sequent $\varGamma \Rightarrow \alpha$ is a category $\alpha\ (..., \varGamma)$ in the sense of constructive type theory, where the ellipsis stands for additional assumptions about the variables that range over the individuals. The additional assumptions are kept to a minimum unless otherwise stated.

Note that the intended interpretation of sequents makes (syntactic) equality of sequents coincide with the equality of categories in the sense type theory.

A sequent of the form $\varGamma \Rightarrow \psi$ is simply written as a context $\varGamma$ when part of a derivation. This convention makes for a compact presentation of the inference figures of *C3*.

Definition 2.2.1. A *C1-derivation* is a derivation built up by means of the inference rules in Table 2.1, p. 18. The quantifier rules of Table 2.1 are subject to the usual restrictions on variables.

The subscript to the right of the $\&\,\mathrm{T}$-inference figure in Table 2.1 means that there are two $\&\,\mathrm{T}$-rules of which the inference figure only shows the first, the second rule being completely analogous. Things are similar for the $\vee\,\mathrm{F}$-inference figure in Table 2.1.

The variables to the left of an inference indicate the assumptions involved. The variables to the left of a semicolon indicate discharged assumptions and are considered bound in the corresponding derivation. The variables to the right of a semicolon indicate open assumptions and are said to be *introduced* by the inference. They are considered free in the corresponding derivation. Free and bound variables are handled according to the variable convention of Section 2.1. Note that, this is a type-theoretic notion of free and bound variables, where the variables range over proof objects as well as individuals.

The structural rules of contraction and interchange are inherent in the concept of a sequent. In particular, contraction can always be avoided by a suitable choice of variables, e.g.

$$\frac{\mathrm{T}\overset{x}{A}, \mathrm{T}\,\overset{z}{A\&B}}{\mathrm{T}\,\overset{z}{A\&B}}\ x; z$$

is an instance of the $\&\,\mathrm{T}$-rule. On the other hand, the structural rule of weakening is inherent in the appearance of an arbitrary context in the axiom rule. We have

	True	**False**
$\&$	$\dfrac{\mathrm{T}\,\overset{x}{A},\Gamma}{\mathrm{T}\,\overset{z}{A\&B},\Gamma}\ {}_{1(2)}x;z$	$\dfrac{\mathrm{F}\,\overset{x'}{A},\Gamma \quad \mathrm{F}\,\overset{y'}{B},\Gamma}{\mathrm{F}\,\overset{z'}{A\&B},\Gamma}\ x',y';z'$
$\vee$	$\dfrac{\mathrm{T}\,\overset{x}{A},\Gamma \quad \mathrm{T}\,\overset{y}{B},\Gamma}{\mathrm{T}\,\overset{z}{A\vee B},\Gamma}\ x,y;z$	$\dfrac{\mathrm{F}\,\overset{x'}{A},\Gamma}{\mathrm{F}\,\overset{z'}{A\vee B},\Gamma}\ {}_{1(2)}x';z'$
$\supset$	$\dfrac{\mathrm{F}\,\overset{x'}{A},\Gamma \quad \mathrm{T}\,\overset{y}{B},\Gamma}{\mathrm{T}\,\overset{z}{A\supset B},\Gamma}\ x',y;z$	$\dfrac{\mathrm{T}\,\overset{x}{A},\mathrm{F}\,\overset{y'}{B},\Gamma}{\mathrm{F}\,\overset{z'}{A\supset B},\Gamma}\ x,y';z'$
$\neg$	$\dfrac{\mathrm{F}\,\overset{x'}{A},\Gamma}{\mathrm{T}\,\overset{z}{\neg A},\Gamma}\ x';z$	$\dfrac{\mathrm{T}\,\overset{x}{A},\Gamma}{\mathrm{F}\,\overset{z'}{\neg A},\Gamma}\ x;z'$
$\forall$	$\dfrac{\mathrm{T}\,\overset{x}{A[t/v]},\Gamma}{\mathrm{T}\,\overset{z}{\forall vA},\Gamma}\ x;z$	$\dfrac{\mathrm{F}\,\overset{x'}{A},\Gamma}{\mathrm{F}\,\overset{z'}{\forall vA},\Gamma}\ x';z'$
$\exists$	$\dfrac{\mathrm{T}\,\overset{x}{A},\Gamma}{\mathrm{T}\,\overset{z}{\exists vA},\Gamma}\ x;z$	$\dfrac{\mathrm{F}\,\overset{x'}{A[t/v]},\Gamma}{\mathrm{F}\,\overset{z'}{\exists vA},\Gamma}\ x';z'$

Axiom	**Cut**
$\dfrac{}{\mathrm{T}\,\overset{x}{A},\mathrm{F}\,\overset{x'}{A},\Gamma}\ ;x,x'$	$\dfrac{\mathrm{F}\,\overset{x'}{A},\Gamma \quad \mathrm{T}\,\overset{x}{A},\Delta}{\Gamma,\Delta}\ x',x;$

Table 2.1: Calculus *C1*. In $\forall$-false and $\exists$-true the variable v must not appear free in the context Γ. The structural rules of contraction and interchange are inherent in the concept of a sequent. The structural rule of weakening has been reduced to the appearance of a context in the axiom inference figure.

chosen to keep the number of contexts to a minimum, not to burden the formal treatment of *C1*. The notable exception of the two contexts in the cut rule is to facilitate local rules of cut elimination in Chapter 5.

2.3 Calculus *C2*

The calculus *C2* is just *C1* in natural deduction style.

Definition 2.3.1. A *C2-derivation* is a derivation built up from assumptions by means of the inference rules in Table 2.2, p. 20. The quantifier rules of Table 2.2 are subject to the usual restrictions on variables.

The subscript to the right of the & T-inference figure in Table 2.2 means that there are two & T-rules of which the inference figure only shows the first, the second rule being completely analogous. Things are similar for the $\vee$ F-inference figure in Table 2.2.

The variables to the left of an inference indicate discharged assumptions and are considered bound by the inference in the side-premise derivations. The variables associated with open assumptions are said to be *introduced* by the inference and are considered free, see the variable convention of Section 2.1.

2.4 Interpretation of *C1* in *C2*

We shall define an interpretation $M \mapsto [\![M]\!]$ that maps *C1*-derivations to *C2*-derivations of the same syntactic category. The interpretation forgets the contexts of the *C1*-derivation. The definition is by induction on the structure of M and proceeds by case analysis of the last step of M.

Case 1. If the last step of M is an axiom then $M \equiv$

$$\frac{}{\overset{x}{\mathrm{T}}A, \overset{x'}{\mathrm{F}}A, \Gamma}\; ; x, x' \quad \mapsto \quad \frac{\overset{x'}{\mathrm{F}}A \quad \overset{x}{\mathrm{T}}A}{\psi}$$

Case 2. If the last step of M is a cut then $M \equiv$

$$\frac{\begin{matrix} M \\ \overset{x'}{\mathrm{F}}A, \Gamma \end{matrix} \quad \begin{matrix} K \\ \overset{x}{\mathrm{T}}A, \Delta \end{matrix}}{\Gamma, \Delta}\; x', x; \quad \mapsto \quad \frac{\begin{matrix} [\overset{x'}{\mathrm{F}}A] \\ [\![M]\!] \\ \psi \end{matrix} \quad \begin{matrix} [\overset{x}{\mathrm{T}}A] \\ [\![K]\!] \\ \psi \end{matrix}}{\psi}\; x', x$$

	True	**False**
$\&$	$\dfrac{\overset{z}{\mathrm{T}\,A\&B} \quad \begin{matrix}[\overset{x}{\mathrm{T}\,A}]\\ \psi\end{matrix}}{\psi}\ {}_{1(2)}x$	$\dfrac{\overset{z'}{\mathrm{F}\,A\&B} \quad \begin{matrix}[\overset{x'}{\mathrm{F}\,A}]\\ \psi\end{matrix} \quad \begin{matrix}[\overset{y'}{\mathrm{F}\,B}]\\ \psi\end{matrix}}{\psi}\ x', y'$
$\vee$	$\dfrac{\overset{z}{\mathrm{T}\,A \vee B} \quad \begin{matrix}[\overset{x}{\mathrm{T}\,A}]\\ \psi\end{matrix} \quad \begin{matrix}[\overset{y}{\mathrm{T}\,B}]\\ \psi\end{matrix}}{\psi}\ x, y$	$\dfrac{\overset{z'}{\mathrm{F}\,A \vee B} \quad \begin{matrix}[\overset{x'}{\mathrm{F}\,A}]\\ \psi\end{matrix}}{\psi}\ {}_{1(2)}x'$
$\supset$	$\dfrac{\overset{z}{\mathrm{T}\,A \supset B} \quad \begin{matrix}[\overset{x'}{\mathrm{F}\,A}]\\ \psi\end{matrix} \quad \begin{matrix}[\overset{y}{\mathrm{T}\,B}]\\ \psi\end{matrix}}{\psi}\ x', y$	$\dfrac{\overset{z'}{\mathrm{F}\,A \supset B} \quad \begin{matrix}[\overset{x}{\mathrm{T}\,A}, \overset{y'}{\mathrm{F}\,B}]\\ \psi\end{matrix}}{\psi}\ x, y'$
$\neg$	$\dfrac{\overset{z}{\mathrm{T}\,\neg A} \quad \begin{matrix}[\overset{x'}{\mathrm{F}\,A}]\\ \psi\end{matrix}}{\psi}\ x'$	$\dfrac{\overset{z'}{\mathrm{F}\,\neg A} \quad \begin{matrix}[\overset{x'}{\mathrm{T}\,A}]\\ \psi\end{matrix}}{\psi}\ x$
$\forall$	$\dfrac{\overset{z}{\mathrm{T}\,\forall v A} \quad \begin{matrix}[\overset{x}{\mathrm{T}\,A[t/v]}]\\ \psi\end{matrix}}{\psi}\ x$	$\dfrac{\overset{z'}{\mathrm{F}\,\forall v A} \quad \begin{matrix}[\overset{x'}{\mathrm{F}\,A}]\\ \psi\end{matrix}}{\psi}\ x'$
$\exists$	$\dfrac{\overset{z}{\mathrm{T}\,\exists v A} \quad \begin{matrix}[\overset{x}{\mathrm{T}\,A}]\\ \psi\end{matrix}}{\psi}\ x$	$\dfrac{\overset{z'}{\mathrm{F}\,\exists v A} \quad \begin{matrix}[\overset{x'}{\mathrm{F}\,A[t/v]}]\\ \psi\end{matrix}}{\psi}\ x'$

Axiom	**Cut**
$\dfrac{\overset{x'}{\mathrm{F}\,A} \quad \overset{x}{\mathrm{T}\,A}}{\psi}$	$\dfrac{\begin{matrix}[\overset{x'}{\mathrm{F}\,A}]\\ \psi\end{matrix} \quad \begin{matrix}[\overset{x}{\mathrm{T}\,A}]\\ \psi\end{matrix}}{\psi}\ x', x$

Table 2.2: Calculus *C2*. In $\forall\,\mathrm{F}$- and $\exists\,\mathrm{T}$-inferences the variable v must not appear free in the context of discourse.

Case 3. If the last step of M is a & T-inference then $M \equiv$

$$\dfrac{\begin{array}{c}K\\ \mathrm{T}\,\overset{x}{A},\Gamma\end{array}}{\mathrm{T}\,\overset{z}{A\&B},\Gamma}\;{}_1x;z \qquad\mapsto\qquad \dfrac{\mathrm{T}\,\overset{z}{A\&B}\qquad \begin{array}{c}[\mathrm{T}\,\overset{x}{A}]\\ [\![K]\!]\\ \psi\end{array}}{\psi}\;{}_1x$$

The other T-inferences follow the same pattern.

Case 4. If the last step of M is a & F-inference then $M \equiv$

$$\dfrac{\begin{array}{c}M_1\\ \mathrm{F}\,\overset{x'}{A},\Gamma\end{array}\quad \begin{array}{c}M_2\\ \mathrm{F}\,\overset{y'}{B},\Gamma\end{array}}{\mathrm{F}\,\overset{z'}{A\&B},\Gamma}\;x',y';z' \qquad\mapsto\qquad \dfrac{\mathrm{F}\,\overset{z'}{A\&B}\qquad \begin{array}{c}[\mathrm{F}\,\overset{x'}{A}]\\ [\![M_1]\!]\\ \psi\end{array}\quad \begin{array}{c}[\mathrm{F}\,\overset{y'}{B}]\\ [\![M_2]\!]\\ \psi\end{array}}{\psi}\;x',y'$$

The other F-inferences follow the same pattern.

The interpretation is surjective onto the set of *C2*-derivations.

Lemma 2.4.1. *For every C2-derivation N there exists a C1-derivation M such that $[\![M]\!] \equiv N$.*

Proof. By induction on the structure of N. □

The interpretation fails to be injective on the set of *C1*-derivations, e.g. the two *C1*-derivations

$$\dfrac{\dfrac{}{\mathrm{T}\,\overset{x}{A},\mathrm{F}\,\overset{x'}{A}}\;;x,x'}{\mathrm{T}\,\overset{z}{A\&B},\mathrm{F}\,\overset{x'}{A}}\;x;z \qquad\text{and}\qquad \dfrac{\dfrac{}{\mathrm{T}\,\overset{z}{A\&B},\mathrm{T}\,\overset{x}{A},\mathrm{F}\,\overset{x'}{A}}\;;x,x'}{\mathrm{T}\,\overset{z}{A\&B},\mathrm{F}\,\overset{x'}{A}}\;x;z$$

translate to the same *C2*-derivation.

2.5 Calculus *C3*

The calculus *C3* is a natural deduction calculus chosen to induce certain plausible contraction rules for *C2*. The choice is made in a systematic way and yields a natural set of contraction rules for *C3*. We shall see in Section 2.9 that *C3* shares the good normalization properties of intuitionistic natural deduction.

In search of an evident interpretation of *C2* in constructive type theory, we shall break down the inference rules of *C2* into components. We shall limit the search so that the interpretation will respect some more or less universally accepted contraction rules for classical sequent calculi. This so that the interpretation will induce a convincing concept of classical proof.

Consider the contraction rule

$$\dfrac{\dfrac{[\mathrm{F}\,\overset{z'}{A\&B}] \quad \begin{matrix}[\mathrm{F}\,\overset{x'}{A}]\\ M_1\\ \psi\end{matrix} \quad \begin{matrix}[\mathrm{F}\,\overset{y'}{B}]\\ M_2\\ \psi\end{matrix}}{\psi}\,x',y' \qquad \dfrac{[\mathrm{T}\,\overset{z}{A\&B}] \quad \begin{matrix}[\mathrm{T}\,\overset{x}{A}]\\ K\\ \psi\end{matrix}}{\psi}\,{}_1x}{\psi}\,z',z \qquad \rightarrow \qquad \dfrac{\begin{matrix}[\mathrm{F}\,\overset{x'}{A}]\\ M_1\\ \psi\end{matrix} \quad \begin{matrix}[\mathrm{T}\,\overset{x}{A}]\\ K\\ \psi\end{matrix}}{\psi}\,x',x$$

where the derivations K, M_1, and M_2 are subject to the restrictions that the variable z does not appear free in K and the variable z' does not appear free in M_1 or M_2. Such a cut can be interpreted by a substitution in two ways depending on the interpretation of the signs of truth and falsity. The interpretation compatible with the intended interpretation of the signs is

$$\dfrac{\begin{matrix}[\mathrm{F}\,\overset{x'}{A}]\\ M\\ \psi\end{matrix} \quad \begin{matrix}[\mathrm{T}\,\overset{x}{A}]\\ K\\ \psi\end{matrix}}{\psi}\,x',x \quad \mapsto \quad \begin{matrix}\dfrac{\begin{matrix}[\mathrm{T}\,\overset{x}{A}]\\ [\![K]\!]\\ \psi\end{matrix}}{\mathrm{F}\,A}\,x\\ [\![M]\!]\\ \psi\end{matrix}$$

On this interpretation of cut, the above contraction rule translates to

$$\dfrac{\dfrac{\dfrac{[\mathrm{T}\,\overset{z}{A\&B}] \quad \begin{matrix}[\mathrm{T}\,\overset{x}{A}]\\ K\\ \psi\end{matrix}}{\psi}\,{}_1x}{\mathrm{F}\,A\&B}\,z \quad \begin{matrix}[\mathrm{F}\,\overset{x'}{A}]\\ M_1\\ \psi\end{matrix} \quad \begin{matrix}[\mathrm{F}\,\overset{y'}{B}]\\ M_2\\ \psi\end{matrix}}{\psi}\,x',y' \quad \rightarrow \quad \begin{matrix}\dfrac{\begin{matrix}[\mathrm{T}\,\overset{x}{A}]\\ K\\ \psi\end{matrix}}{\mathrm{F}\,A}\,x\\ M_1\\ \psi\end{matrix}$$

Next, we consider two ways to split the $\&\,\mathrm{T}$- and $\&\,\mathrm{F}$-rules into parts in an attempt to bring the premise derivations K and M_1 into contact with each other. The first approach, which yields the most natural solution, focuses on the $\&\,\mathrm{F}$-rule. The second approach focuses on the $\&\,\mathrm{T}$-rule.

Splitting of the $\&\,\mathrm{F}$-rule. If the $\&\,\mathrm{F}$-rule is divided into an application and another inference rule,

$$\dfrac{\mathrm{F}\,\overset{z'}{A\&B} \quad \begin{matrix}[\mathrm{F}\,\overset{x'}{A}]\\ M_1\\ \psi\end{matrix} \quad \begin{matrix}[\mathrm{F}\,\overset{y'}{B}]\\ M_2\\ \psi\end{matrix}}{\psi}\,x',y' \quad \mapsto \quad \dfrac{\mathrm{F}\,\overset{z'}{A\&B} \quad \dfrac{\begin{matrix}[\mathrm{F}\,\overset{x'}{A}]\\ M_1\\ \psi\end{matrix} \quad \begin{matrix}[\mathrm{F}\,\overset{y'}{B}]\\ M_2\\ \psi\end{matrix}}{\mathrm{T}\,A\&B}\,x',y'}{\psi}$$

then β-conversion can be applied,

$$\dfrac{\dfrac{\dfrac{\overset{z}{[\mathrm{T}\,A\&B]} \quad \begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{array}}{\psi}\,{}_1x}{\mathrm{F}\,A\&B}\,z \quad \begin{array}{c}\overset{x'}{[\mathrm{F}\,A]}\\ M_1\\ \psi\end{array} \quad \begin{array}{c}\overset{y'}{[\mathrm{F}\,B]}\\ M_2\\ \psi\end{array}}{\psi}\,x',y' \quad \to_\beta \quad \dfrac{\dfrac{\begin{array}{c}\overset{x'}{[\mathrm{F}\,A]}\\ M_1\\ \psi\end{array} \quad \begin{array}{c}\overset{y'}{[\mathrm{F}\,B]}\\ M_2\\ \psi\end{array}}{\mathrm{T}\,A\&B}\,x',y' \quad \begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{array}}{\psi}\,{}_1x$$

and so the reduction can be completed by a new reduction step

$$\dfrac{\dfrac{\begin{array}{c}\overset{x'}{[\mathrm{F}\,A]}\\ M_1\\ \psi\end{array} \quad \begin{array}{c}\overset{y'}{[\mathrm{F}\,B]}\\ M_2\\ \psi\end{array}}{\mathrm{T}\,A\&B}\,x',y' \quad \begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{array}}{\psi}\,{}_1x \quad \to \quad \begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \dfrac{\psi}{\mathrm{F}\,A}\,x\\ M_1\\ \psi\end{array}$$

The other F-rules can be treated in the same way.

We prefer this approach on the ground that the new rules have the form of introduction rules in constructive type theory and so can be taken to define the meanings of the logical constants. How these meanings of the logical constants fit together with the new reduction steps will be explained in Chapter 3. This method yields the introduction rules of *C3*. The elimination rules of *C3* are chosen identical to the T-rules of *C2*.

The second approach, to split the $\&$ T-rule into parts, is unsatisfying for several reasons. First, it requires η-conversion, something uncommon in natural deduction as well as in most type-theoretic situations; second, it does not provide any alternative to the attractive introduction rules gained by the first approach. The exotic solution is included for completeness.

Splitting of the $\&$ T-rule. If the $\&$ T-rule is divided into an application and another inference rule,

$$\dfrac{\overset{z}{\mathrm{T}\,A\&B} \quad \begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{array}}{\psi}\,{}_1x \quad \mapsto \quad \dfrac{\dfrac{\begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{array}}{\mathrm{F}\,A\&B}\,{}_1x \quad \overset{z}{\mathrm{T}\,A\&B}}{\psi}$$

then η-conversion can be applied,

$$\dfrac{\dfrac{\dfrac{\overset{z}{[\mathrm{T}\,A\&B]} \quad \begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{array}}{\psi}\,{}_1x}{\mathrm{F}\,A\&B}\,z \quad \begin{array}{c}\overset{x'}{[\mathrm{F}\,A]}\\ M_1\\ \psi\end{array} \quad \begin{array}{c}\overset{y'}{[\mathrm{F}\,B]}\\ M_2\\ \psi\end{array}}{\psi}\,x',y' \quad \to_\eta \quad \dfrac{\dfrac{\begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{array}}{\mathrm{F}\,A\&B}\,{}_1x \quad \begin{array}{c}\overset{x'}{[\mathrm{F}\,A]}\\ M_1\\ \psi\end{array} \quad \begin{array}{c}\overset{y'}{[\mathrm{F}\,B]}\\ M_2\\ \psi\end{array}}{\psi}\,x',y'$$

and so the reduction can be completed by another kind of reduction step

$$
\dfrac{\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}\,A]\\ K\\ \psi\end{array}}{\mathrm{F}\,A\&B}\,{}_1x \quad \begin{array}{c}[\overset{x'}{\mathrm{F}}\,A]\\ M_1\\ \psi\end{array} \quad \begin{array}{c}[\overset{y'}{\mathrm{F}}\,B]\\ M_2\\ \psi\end{array}}{\psi}\,x',y' \quad \to \quad \begin{array}{c}\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}\,A]\\ K\\ \psi\end{array}}{\mathrm{F}\,A}\,x\\ M_1\\ \psi\end{array}
$$

The other T-rules can be treated in the same way.

Definition 2.5.1. A *C3-derivation* is a derivation built up from assumptions by means of the inference rules of Table 2.3, p. 25. The quantifier rules of Table 2.3 are subject to the usual restrictions on variables.

The elimination rules of *C3*, except those for implication and negation, are all formal instances of the general elimination rules due to von Plato (2001). See Schroeder-Heister (1984) for related earlier development. Instantiation of von Plato's elimination rules with conclusion ψ and, in the case of &-elimination, further specialization into cases, yield the corresponding elimination rules of C3, except for implication and negation. How the elimination rules for implication and negation fit into this scheme of things will be revealed in Section 4.3.

The two $\vee$-introduction rules of *C3* makes disjunction decidable in a way similar to the case of disjunction in minimal logic. Yet the law of excluded middle holds because of the interpretation of the classical consequence relation, see Section 2.7.

The variables to the left of an inference indicate discharged assumptions and are considered bound in the side premise derivations by the inference. The variables associated with open assumptions are considered free, see the variable convention of Section 2.1.

2.6 Interpretation of *C2* in *C3*

We shall define an interpretation $M \mapsto [\![M]\!]$ that maps *C2*-derivations to *C3*-derivations of the same syntactic category. The definition is by induction on the structure of M and proceeds by case analysis of the last step of M.

Case 1. If the last step of M is an axiom then $M \equiv$

$$
\dfrac{\overset{x'}{\mathrm{F}}\,A \quad \overset{x}{\mathrm{T}}\,A}{\psi} \quad \mapsto \quad \dfrac{\overset{x'}{\mathrm{F}}\,A \quad \overset{x}{\mathrm{T}}\,A}{\psi}
$$

	Introduction	**Elimination**
$\&$	$\dfrac{\begin{matrix}[\overset{x'}{\mathrm{F}}\,A] & [\overset{y'}{\mathrm{F}}\,B]\\ \psi & \psi\end{matrix}}{\mathrm{T}\,A\&B}\ x',y'$	$\dfrac{\begin{matrix} & [\overset{x}{\mathrm{T}}\,A]\\ \mathrm{T}\,A\&B & \psi\end{matrix}}{\psi}\ {}_1x \quad \dfrac{\begin{matrix} & [\overset{y}{\mathrm{T}}\,B]\\ \mathrm{T}\,A\&B & \psi\end{matrix}}{\psi}\ {}_2y$
$\vee$	$\dfrac{\begin{matrix}[\overset{x'}{\mathrm{F}}\,A]\\ \psi\end{matrix}}{\mathrm{T}\,A\vee B}\ {}_1x' \quad \dfrac{\begin{matrix}[\overset{y'}{\mathrm{F}}\,B]\\ \psi\end{matrix}}{\mathrm{T}\,A\vee B}\ {}_2y'$	$\dfrac{\begin{matrix} & [\overset{x}{\mathrm{T}}\,A] & [\overset{y}{\mathrm{T}}\,B]\\ \mathrm{T}\,A\vee B & \psi & \psi\end{matrix}}{\psi}\ x$
$\supset$	$\dfrac{\begin{matrix}[\overset{x}{\mathrm{T}}\,A,\overset{y'}{\mathrm{F}}\,B]\\ \psi\end{matrix}}{\mathrm{T}\,A\supset B}\ x,y'$	$\dfrac{\begin{matrix} & [\overset{x'}{\mathrm{F}}\,A] & [\overset{y}{\mathrm{T}}\,B]\\ \mathrm{T}\,A\supset B & \psi & \psi\end{matrix}}{\psi}\ x',y$
$\neg$	$\dfrac{\begin{matrix}[\overset{x}{\mathrm{T}}\,A]\\ \psi\end{matrix}}{\mathrm{T}\,\neg A}\ x$	$\dfrac{\begin{matrix} & [\overset{x'}{\mathrm{F}}\,A]\\ \mathrm{T}\,\neg A & \psi\end{matrix}}{\psi}\ x'$
$\forall$	$\dfrac{\begin{matrix}[\overset{x'}{\mathrm{F}}\,A]\\ \psi\end{matrix}}{\mathrm{T}\,\forall vA}\ x'$	$\dfrac{\begin{matrix} & [\overset{x}{\mathrm{T}}\,A[t/v]]\\ \mathrm{T}\,\forall vA & \psi\end{matrix}}{\psi}\ x$
$\exists$	$\dfrac{\begin{matrix}[\overset{x'}{\mathrm{F}}\,A[t/v]]\\ \psi\end{matrix}}{\mathrm{T}\,\exists vA}\ x'$	$\dfrac{\begin{matrix} & [\overset{x}{\mathrm{T}}\,A]\\ \mathrm{T}\,\exists vA & \psi\end{matrix}}{\psi}\ x$

Abstraction	**Application**
$\dfrac{\begin{matrix}[\overset{x}{\mathrm{T}}\,A]\\ \psi\end{matrix}}{\mathrm{F}\,A}\ x$	$\dfrac{\mathrm{F}\,A \quad \mathrm{T}\,A}{\psi}$

Table 2.3: Calculus *C3*. In $\forall$-introduction and $\exists$-elimination the variable v must not appear free in the context of discourse.

Case 2. If the last step of M is a cut then $M \equiv$

$$\dfrac{\begin{array}{cc} \overset{x'}{[\mathrm{F}\,A]} & \overset{x}{[\mathrm{T}\,A]} \\ M & K \\ \psi & \psi \end{array}}{\psi}\, x', x \quad \mapsto \quad \begin{array}{c} \overset{x}{[\mathrm{T}\,A]} \\ [\![K]\!] \\ \dfrac{\psi}{\mathrm{F}\,A}\, x \\ [\![M]\!] \\ \psi \end{array}$$

Here the idea is to interpret the cut by a substitution of an abstraction in accordance with the preliminary discussion in Section 2.4.

Case 3. If the last step of M is a $\&\,\mathrm{T}$-inference then $M \equiv$

$$\dfrac{\begin{array}{cc} & \overset{x}{[\mathrm{T}\,A]} \\ & K \\ \mathrm{T}\,\overset{z}{A\&B} & \psi \end{array}}{\psi}\, {}_1x \quad \mapsto \quad \dfrac{\begin{array}{cc} & \overset{x}{[\mathrm{T}\,A]} \\ & [\![K]\!] \\ \mathrm{T}\,\overset{z}{A\&B} & \psi \end{array}}{\psi}\, {}_1x$$

The other T-inferences follow the same pattern.

Case 4. If the last step of M is a $\&\,\mathrm{F}$-inference then $M \equiv$

$$\dfrac{\begin{array}{ccc} & \overset{x'}{[\mathrm{F}\,A]} & \overset{y'}{[\mathrm{F}\,B]} \\ & M_1 & M_2 \\ \mathrm{F}\,\overset{z'}{A\&B} & \psi & \psi \end{array}}{\psi}\, x', y' \quad \mapsto \quad \dfrac{\mathrm{F}\,\overset{z'}{A\&B} \quad \dfrac{\begin{array}{cc} \overset{x'}{[\mathrm{F}\,A]} & \overset{y'}{[\mathrm{F}\,B]} \\ [\![M_1]\!] & [\![M_2]\!] \\ \psi & \psi \end{array}}{\mathrm{T}\,A\&B}\, x', y'}{\psi}$$

Here the idea is to split the inference into an application whose side premise is the conclusion of an introduction in accordance with the preliminary discussion in Section 2.5. The other F-inferences follow the same pattern.

We use von Plato's (2001) notion of normal derivation in order to discuss the properties of the interpretation.

Definition 2.6.1. We say that a *C3*-derivation is *normal* provided that all main premises of applications and eliminations are assumptions.

Later on, we shall introduce yet another notion of normal derivation, the notion of a derivation that cannot be further reduced. However, this will have to wait until reduction has been defined in Section 2.8. Although based on different concepts, the two notions are equivalent and will be used interchangeably in Section 2.9.

The interpretation correlates the cut-free *C2*-derivations with certain normal *C3*-derivations, those with conclusion ψ.

Proposition 2.6.2. *A C2-derivation M is cut-free if and only if $[\![M]\!]$ is normal.*

Proof. If M is cut-free then $[\![M]\!]$ is normal by induction on the structure of M. On the other hand, if M is not cut-free then $[\![M]\!]$ is not normal. Hence, if $[\![M]\!]$ is normal then M is cut-free, because M is either cut-free or not cut-free. □

The correspondence between cut-free *C2*-derivations and normal *C3*-derivations with conclusion ψ is 1-1 as evident from Propositions 2.6.3 and 2.6.4 below.

Proposition 2.6.3. *If a C3-derivation N is normal with conclusion ψ, then there exists a cut-free C2-derivation M such that $N \equiv [\![M]\!]$.*

Proof. By induction on the height of N and Proposition 2.6.2 to conclude that M is cut-free. □

Proposition 2.6.4. *If M and N are cut-free C2-derivations and $[\![M]\!] \equiv [\![N]\!]$, then $M \equiv N$.*

Proof. By induction on the height of $[\![M]\!]$. □

2.7 The law of excluded middle

Under the interpretation of the classical consequence relation the law of excluded middle becomes $\mathrm{F}\,A \vee \neg A \vdash \psi$. It can be derived in *C1* as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{T}\overset{x}{\,}A,\mathrm{F}\overset{x'}{\,}A}\;;x,x'}{\mathrm{T}\overset{x}{\,}A,\mathrm{F}\,A\overset{z'}{\vee}\neg A}\;x';z'}{\mathrm{F}\overset{y'}{\,}\neg A,\mathrm{F}\,A\overset{z'}{\vee}\neg A}\;x;y'}{\mathrm{F}\,A\overset{z'}{\vee}\neg A}\;y';z'$$

The above derivation corresponds to the derivation below via the interpretations of Section 2.4 and Section 2.7.

$$\dfrac{\mathrm{F}\,A\overset{z'}{\vee}\neg A \qquad \dfrac{\dfrac{[\mathrm{F}\overset{y'}{\,}\neg A] \qquad \dfrac{\dfrac{\mathrm{F}\,A\overset{z'}{\vee}\neg A \qquad \dfrac{\dfrac{[\mathrm{F}\overset{x'}{\,}A] \quad [\mathrm{T}\overset{x}{\,}A]}{\psi}}{\mathrm{T}\,A\vee\neg A}\;x'}{\psi}}{\mathrm{T}\,\neg A}\;x}{\psi}}{\mathrm{T}\,A\vee\neg A}\;y'}{\psi}$$

In this sense the law of excluded middle can be said to hold in *C3*.

2.8 The *C3*-contraction relation

For reduction we use a suitably modified version of the nomenclature and notation in Barendregt (1984, §§ 3.1.2, 3.1.5, 3.1.8).

Definition 2.8.1. A *contraction relation* on a set of derivations S is a binary relation R on S.

Definition 2.8.2. Let R be a contraction relation on a set of derivations S. Then R induces the inductively defined relations $\rightarrow_R$, $\twoheadrightarrow_R$, and $=_R$. For S equals the set of *C3*-derivations,

$\rightarrow_R$ is the binary relation defined by the rules

$$\begin{array}{rcl}
R(M,N) & \Rightarrow & M \rightarrow_R N \\
M \rightarrow_R N & \Rightarrow & \dfrac{\overset{Z}{\mathrm{F}A} \quad \overset{M}{\mathrm{T}A}}{\psi} \;\rightarrow_R\; \dfrac{\overset{Z}{\mathrm{F}A} \quad \overset{N}{\mathrm{T}A}}{\psi} \\
M \rightarrow_R N & \Rightarrow & \dfrac{\overset{M}{\mathrm{F}A} \quad \overset{Z}{\mathrm{T}A}}{\psi} \;\rightarrow_R\; \dfrac{\overset{N}{\mathrm{F}A} \quad \overset{Z}{\mathrm{T}A}}{\psi} \\
M \rightarrow_R N & \Rightarrow & \dfrac{\begin{matrix}[\overset{x}{\mathrm{T}A}] \\ M \\ \psi\end{matrix}}{\mathrm{F}A}\, x \;\rightarrow_R\; \dfrac{\begin{matrix}[\overset{x}{\mathrm{T}A}] \\ N \\ \psi\end{matrix}}{\mathrm{F}A}\, x
\end{array}$$

and so on for the other inference figures. The relation $\rightarrow_R$ is called *one-step R-reduction* and we say that M *R-reduces to* N *in one step* provided that $M \rightarrow_R N$.

The other one-step reduction relations found in this thesis can be defined similarly and will be taken for granted.

$\twoheadrightarrow_R$ is the binary relation on S defined as the reflexive and transitive closure of $\rightarrow_R$. The relation $\twoheadrightarrow_R$ is called *R-reduction* and we say that M *R-reduces to* N provided that $M \twoheadrightarrow_R N$.

$=_R$ is the binary relation on S defined as the reflexive, symmetric, and transitive closure of $\rightarrow_R$. The relation $=_R$ is called *R-equality* and we say that M is *R-convertible to* N provided that $M =_R N$.

Definition 2.8.3. Let R be a notion of reduction on some set of derivations. Let M and N be derivations such that $R(M, N)$. Then M is said to be an *R-redex* and, moreover, N is said to be an *R-contractum* of M. The process of stepping from a redex to a contractum is called *contraction*.

Definition 2.8.4. Let R be a notion of reduction on a set of derivations S. A derivation M of S that does not contain any R-redex is called *R-normal*. A derivation N is said to be an *R-normal form* of M provided that N is R-normal and $M =_R N$. Note that if M is an R-normal form and $M \twoheadrightarrow_R N$ then $M \equiv N$.

We shall often suppress the name of the contraction relation when it is clear from the context, and just write $M \to N$, $M \twoheadrightarrow N$, and $M = N$.

A *contraction rule* is a clause of the form $C : M \to N$, where M and N are schematic derivations subject to conditions, and defines a notion of reduction by $C(M, N)$. We define the notion of *C3*-reduction on the set of *C3*-derivations as the union of the contraction rules that follow. The contraction rules for *C3* are subject to Barendregt's variable convention recalled in Section 2.1 and so we do not have to worry about the capture of free variables. The contraction rules for *C3* are as follows.

F-contraction:

$$\dfrac{\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}A]\\ K\\ \psi\end{array}}{\mathrm{F}\,A}\,x \qquad \begin{array}{c}M\\ \mathrm{T}\,A\end{array}}{\psi} \quad\to\quad \begin{array}{c}M\\ \mathrm{T}\,A\\ K\\ \psi\end{array}$$

&-contraction 1(2):

$$\dfrac{\dfrac{\begin{array}{c}[\overset{x'}{\mathrm{F}}A]\\ M_1\\ \psi\end{array} \quad \begin{array}{c}[\overset{y'}{\mathrm{F}}B]\\ M_2\\ \psi\end{array}}{\mathrm{T}\,A\&B}\,x',y' \qquad \begin{array}{c}[\overset{x}{\mathrm{T}}A]\\ K\\ \psi\end{array}}{\psi}\,{}_1x \quad\to\quad \begin{array}{c}\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}A]\\ K\\ \psi\end{array}}{\mathrm{F}\,A}\,x\\ M_1\\ \psi\end{array}$$

The second rule of &-contraction is defined similarly.

$\vee$-contraction 1(2):

$$\dfrac{\dfrac{\begin{array}{c}[\overset{x'}{\mathrm{F}}A]\\ N\\ \psi\end{array}}{\mathrm{T}\,A \vee B}\,{}_1x' \qquad \begin{array}{c}[\overset{x}{\mathrm{T}}A]\\ L_1\\ \psi\end{array} \quad \begin{array}{c}[\overset{y}{\mathrm{T}}B]\\ L_2\\ \psi\end{array}}{\psi}\,x,y \quad\to\quad \begin{array}{c}\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}A]\\ L_1\\ \psi\end{array}}{\mathrm{F}\,A}\,x\\ N\\ \psi\end{array}$$

The second rule of $\vee$-contraction is defined similarly.

$\supset$-contraction:

$$\dfrac{\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}A, \overset{y'}{\mathrm{F}}B]\\ M\\ \psi\end{array}}{\mathrm{T}\,A \supset B}\,x,y' \qquad \begin{array}{c}[\overset{x'}{\mathrm{F}}A]\\ N\\ \psi\end{array} \quad \begin{array}{c}[\overset{y}{\mathrm{T}}B]\\ K\\ \psi\end{array}}{\psi}\,x',y \quad\to\quad \begin{array}{c}\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}A], \dfrac{\begin{array}{c}[\overset{y}{\mathrm{T}}B]\\ K\\ \psi\end{array}}{\mathrm{F}\,B}\,y\\ M\\ \psi\end{array}}{\mathrm{F}\,A}\,x\\ N\\ \psi\end{array}$$

¬-contraction:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A}]\\ M\\ \psi\end{array}}{\mathrm{T}\,\neg A}\,x \qquad \begin{array}{c}[\mathrm{F}\,\overset{x'}{A}]\\ N\\ \psi\end{array}}{\psi}\,x' \quad\rightarrow\quad \begin{array}{c} \dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A}]\\ M\\ \psi\end{array}}{\mathrm{F}\,A}\,x \\ N\\ \psi\end{array}$$

∀-contraction:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{F}\,\overset{x'}{A}]\\ M\\ \psi\end{array}}{\mathrm{T}\,\forall v A}\,x' \qquad \begin{array}{c}[\mathrm{T}\,\overset{x}{A}[t/v]]\\ K\\ \psi\end{array}}{\psi}\,x \quad\rightarrow\quad \begin{array}{c} \dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A}[t/v]]\\ K\\ \psi\end{array}}{\mathrm{F}\,A[t/v]}\,x \\ M[t/v]\\ \psi\end{array}$$

∃-contraction:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{F}\,\overset{x'}{A}[t/v]]\\ N\\ \psi\end{array}}{\mathrm{T}\,\exists v A}\,x' \qquad \begin{array}{c}[\mathrm{T}\,\overset{x}{A}]\\ L\\ \psi\end{array}}{\psi}\,x \quad\rightarrow\quad \begin{array}{c} \dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A}[t/v]]\\ L[t/v]\\ \psi\end{array}}{\mathrm{F}\,A[t/v]}\,x \\ N\\ \psi\end{array}$$

2.9 Normalization

The contraction rules of *C3* are similar to those of intuitionistic natural deduction. This is captured by the fact that the inversion principle of Gentzen (1935) and Prawitz (1965) holds for *C3*. The inversion principle states that the conclusion of an elimination should be implicit in any set that contains derivations sufficient to conclude the main premise of the elimination together with the side premise derivations of the elimination. It is clear from the contraction rules of *C3* that the inference rules of *C3* enjoy this property. Hence the inversion principle holds for *C3*.

The notion of *C3*-reduction shares the good normalization properties of intuitionistic natural deduction. This is so because normalization and the Church-Rosser property can be proved by the same methods as for intuitionistic natural deduction. The situation is in fact simpler than for intuitionistic natural deduction. Since the elimination rules and application rule of *C3* all have conclusion ψ, the conclusion of an elimination or application cannot appear as the main premise of another elimination or application, and so there is no need for permutative reductions.

Since only an assumption or the conclusion of an introduction or abstraction can appear as the main premise of an elimination or application, it readily follows that the notion of a *C3*-normal form coincides with the notion of a normal *C3*-derivation in the sense of Definition 2.6.1. These notions will be used interchangeably in what follows.

We argue that the *C3*-contraction relation is *weakly normalizing*, that is, that every *C3*-derivation has a *C3*-normal form. The idea is to use the method of *computable terms*[1] invented by Tait (1967) to prove normalization for Gödel's theory **T** (Gödel 1958) of functions of finite type. This method has been carried over to proofs by Martin-Löf (1971a) and can be used to prove Theorem 2.9.1 to the effect that the *C3*-contraction relation is weakly normalizing.

Theorem 2.9.1. *Every C3-derivation reduces to a C3-normal form.*

We write *C3*-$\mathrm{nf}(M)$ for the *C3*-normal form of a *C3*-derivation M.

The result of Theorem 2.9.1 can be pulled back to a Hauptsatz for *C2* by means of the interpretation of *C2* in *C3*.

Corollary 2.9.2. *For every C2-derivation M there exists a cut-free C2-derivation N such that $[\![M]\!] \twoheadrightarrow [\![N]\!]$.*

Proof. By Proposition 2.6.3 there then exists a cut-free *C2*-derivation N such that $[\![N]\!] \equiv$ *C3*-$\mathrm{nf}([\![M]\!])$. □

It is easy to see that a Hauptsatz similar to Corollary 2.9.2 holds for *C1*.

We shall furthermore argue that the notion of *C3*-reduction is *Church-Rosser*. The idea is to use the method developed by Church and Rosser (1936) to prove the Church-Rosser property for combinatory logic. The method was later perfected by Tait and Martin-Löf (1971b) for untyped combinatory logic and λ-calculus, respectively, see Hindley et al. (1972, App. 1), Hindley and Seldin (1986, App. 1), or Barendregt (1984, § 3.2). See also Barendregt (1984, § 11.2 and § 14.2) for other proofs of the Church-Rosser theorem. The method developed for combinatory logic can be used to prove Theorem 2.9.3 to the effect that the notion of *C3*-reduction is Church-Rosser.

Theorem 2.9.3. *For all C3-derivations K, L, and M such that $K \twoheadrightarrow L$ and $K \twoheadrightarrow M$ there exists a C3-derivation N such that $L \twoheadrightarrow N$ and $M \twoheadrightarrow N$.*

Corollary 2.9.4. *For all C3-derivations L and M such that $L = M$ there exists a C3-derivation N such that $L \twoheadrightarrow N$ and $M \twoheadrightarrow N$.*

Proof. Proof, following Barendregt (1984, § 3.1.12), by induction on the definition of *C3*-equality. If $L = M$ beacuse $L \equiv M$ or $L \rightarrow M$ then let $N \equiv M$. If $L = M$ because $M = L$ then N exists by the induction hypothesis. If $L = M$ because $L = K$ and $K = M$ then there exists *C3*-derivations P and Q such that $L, K \twoheadrightarrow P$ and $K, M \twoheadrightarrow Q$ by the induction hypothesis. Then there exists a *C3*-derivation N such that $P, Q \twoheadrightarrow N$ by Theorem 2.9.3 and so $L, M \twoheadrightarrow N$. □

[1]Tait used the word convertible rather than the word computable.

It follows from Coroallary 2.9.4 that *C3*-normal forms are unique and this is the content of Corollary 2.9.5. In Corollary 2.9.6, the result is pulled back to *C2*.

Corollary 2.9.5. *For all C3-normal forms L and M, if $L = M$ then $L \equiv M$.*

Proof. By Corollary 2.9.4 there exists a *C3*-derivation N such that $L \twoheadrightarrow N$ and $M \twoheadrightarrow N$. Now, $N \equiv L$ and $N \equiv M$, because a normal form only reduces to itself. Hence $L \equiv M$. □

Corollary 2.9.6. *For all cut-free C2-derivations L and M, if $[\![L]\!] = [\![M]\!]$ then $L \equiv M$.*

Proof. By Corollary 2.9.5 and Proposition 2.6.4. □

Proposition 2.9.6 does not generalize to *C1* because the interpretation of *C1* in *C2* fails to be injective on cut-free *C1*-derivations, see Section 2.4.

Chapter 3

Interpretation of *C3*

In this chapter, we interpret *C3* in constructive type theory. The idea behind the interpretation is to let the introduction rules of *C3* determine the meanings of the logical constants of classical logic in the same way the introduction rules of intuitionistic logic determine the meanings of the logical constants of intuitionistic logic in Gentzen (1935). The associated elimination rules are then used to interpret the elimination rules of *C3*. This offers a solution to the problem of what it means to be a classical proof of a proposition and what it means for two classical proofs of a proposition to be equal: the interpretation reduces the concept of a classical proof to the type theoretic concept of a proof, but for a classical proposition. In this way classical predicate logic becomes a fragment of constructive type theory on a par with intuitionistic predicate logic. It then becomes possible to combine classical and intuitionistic reasoning in the same derivation in the spirit of the search for unity initiated by Girard (1993).

We prove some basic properties of the interpretation, in particular, that the interpretation respects *C3*-equality and is injective with respect to type theoretic definitional equality. If the concept of a classical proof is taken to mean a *C3*-derivation modulo *C3*-equality, then the interpretation can be said to map classical proofs to constructive proofs in an injective way.

We also explore how the interpretation relates to constructive type theory. In Section 3.4 we merge the classical rules of implication and universal quantification into a dependent product. In Section 3.5 we prove the principles of mathematical and W-induction for classical logic and briefly consider the fate of accessibility induction. Unfortunately the technique used to prove the principles of mathematical and W-induction for classical logic does not generalize to accessibility induction.

3.1 Preliminary remarks

We use Martin-Löf's polymorphic type theory with the general scheme of inductive definitions for the interpretation of *C3*. The logical constants of classical logic are defined by general inductive definitions on a par with the definitions of the logical constants of intuitionistic logic. See Martin-Löf (1984, 1998) and Nordström et al. (1990) for the formal theory of Martin-Löf's type theory with the general scheme of inductive definitions.

Read from left to right the equality rules of constructive type theory can be understood as contraction rules, which together define a contraction relation for constructive type theory. The corresponding reduction relations are here written without any subscript. Similarly the normal form of a type theoretic object M is here written $\mathrm{nf}(M)$.

The notion of reduction for constructive type theory includes both β-contraction and η-expansion (both involving abstraction and application). The β-contraction rule and the subsequent relation of β-reduction are used on their own in Section 3.3. η-contraction is only used in Section 6.3, and then explicitely mentioned. η-expansion is not used at all.

Section 3.2 includes some lengthy computations in the proofs of several lemmas. Although trivial, the invloved substitutions are handled according to the rules of explicit substitution, so that the computations can be reused later in Chapter 5.

We introduce a type theoretic term notation for *C3*-derivations. The term notation is one-to-one between *C3*-derivations and *C3*-terms.

Definition 3.1.1. A *C3*-term is built up from variables and function constants (i.e. $\&I$, $\&E_1$, $\&E_2$, $\vee I_1$, $\vee I_2$, $\vee E$, $\supset I$, $\supset E$, $\neg I$, $\neg E$, $\forall I$, $\forall E$, $\exists I$, $\exists E$) by means of formal abstraction and formal application. The syntax will become evident in Section 3.2.

We use the same symbol (i.e. K, L, M, N, etc.) for a *C3*-derivation and the corresponding *C3*-term. For example, the *C3*-derivation

$$\dfrac{\dfrac{\dfrac{\overset{z}{[\mathrm{T}\,A\&B]} \quad \begin{matrix}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{matrix}}{\psi}\;{}_1x}{\mathrm{F}\,A\&B}\;z \qquad \dfrac{\begin{matrix}\overset{x'}{[\mathrm{F}\,A]}\\ M_1\\ \psi\end{matrix} \quad \begin{matrix}\overset{y'}{[\mathrm{F}\,B]}\\ M_2\\ \psi\end{matrix}}{\mathrm{T}\,A\&B}\;x',y'}{\psi}$$

corresponds to the term $([z]\&E_1(z,[x]K))(\&I([x']M_1,[y']M_2))$.

3.2 Type-theoretic interpretation of *C3*

We shall define an interpretation $M \mapsto [\![M]\!]$ that maps *C3*-derivations to type theoretic objects of the same syntactic category. The definition is by induction on the structure of M and proceeds by case analysis of the last step of M. The complexity is much higher than for the interpretations of Sections 2.4 and 2.6.

The signs of truth and falsity

The signs of truth and falsity are interpreted according to the intended interpretations of Chapter 2, that is, $\mathrm{T}\,A = \mathrm{proof}(A) : \mathrm{type}\ (A : \mathrm{prop})$ and $\mathrm{F}\,A = (\mathrm{T}\,A)\psi : \mathrm{type}\ (A : \mathrm{prop})$, where ψ is interpreted as an indeterminate type in the sense of constructive type theory.

The abstraction and application rules of *C3* become special cases of the abstraction and application rules of constructive type theory:

Abstraction

$$\frac{\begin{array}{c}[x : \mathrm{T}\,A] \\ K : \psi\end{array}}{[x]K : \mathrm{F}\,A}\ x$$

Application

$$\frac{K : \mathrm{F}\,A \quad M : \mathrm{T}\,A}{K(M) : \psi}$$

Phrased in the term notation for *C3*-derivations this becomes $[\![[x]K]\!] \equiv [x][\![K]\!]$ and $[\![K(M)]\!] \equiv [\![K]\!]([\![M]\!])$.

We make no difference between $\mathrm{F}\,A$ and the function type $(\mathrm{T}\,A)\psi$, but simply write $\mathrm{F}\,A$ as syntactic sugar for $(\mathrm{T}\,A)\psi$. We will make sparse use of function types other than $\mathrm{F}\,A$, with the notable exception of $\mathrm{T}''\,A \equiv (\mathrm{F}\,A)\psi$ in Chapter 6.

The type ψ is identified with the set of proofs of an indeterminate proposition Ψ, i.e. $\psi \equiv \mathrm{T}\,\Psi$. The proposition Ψ will be used to interpret the elimination rules of *C3* in constructive type theory. The involved type-theoretic elimination rules require a family of propositions which cannot be specialized to the type ψ but only to the proposition Ψ. The proposition Ψ will also be used to define minimal negation in Chapter 4.

Conjunction

The constant $\&$ expresses classical conjunction. The rule for forming conjunctions is:

&-formation

$$\frac{A : \mathrm{prop} \quad B : \mathrm{prop}}{A\&B : \mathrm{prop}}$$

The canonical constant $\&\text{-in}$ is used to express canonical proofs of conjunctions. The interpretation of the &-introduction rule of *C3* is given by the equation

$$[\![\&I([x']M_1, [y']M_2)]\!] \equiv \&\text{-in}([x'][\![M_1]\!], [y'][\![M_2]\!])$$

whose right-hand member is governed by the rule

&-introduction

$$\frac{\begin{matrix}[x' : \mathrm{F}\,A] & [y' : \mathrm{F}\,B] \\ M_1 : \psi & M_2 : \psi\end{matrix}}{\text{\&-in}([x']M_1, [y']M_2) : \mathrm{T}\,A\&B}\; x', y'$$

We can use the well-known pattern of inductive defitions to write down the associated rules for forming equal canonical proofs, see Backhouse (1988). These associated rules are henceforth taken for granted for conjunction as well as for the other logical constants.

The corresponding elimination and equality rules are:

&-elimination

$$\frac{M : \mathrm{T}\,A\&B \quad \begin{matrix}[z : \mathrm{T}\,A\&B] \\ C : \text{prop}\end{matrix} \quad \begin{matrix}[x : (\mathrm{F}\,A)\psi, y : (\mathrm{F}\,B)\psi] \\ K : \mathrm{T}\,C[\text{\&-in}(x,y)/z]\end{matrix}}{\text{\&-el}(M, [x,y]K) : \mathrm{T}\,C[M/z]}\; x, y, z$$

&-equality

$$\frac{M_1 : (\mathrm{F}\,A)\psi \quad M_2 : (\mathrm{F}\,B)\psi \quad \begin{matrix}[z : \mathrm{T}\,A\&B] \\ C : \text{prop}\end{matrix} \quad \begin{matrix}[x : (\mathrm{F}\,A)\psi, y : (\mathrm{F}\,B)\psi] \\ K : \mathrm{T}\,C[\text{\&-in}(x,y)/z]\end{matrix}}{\text{\&-el}(\text{\&-in}(M_1, M_2), [x,y]K) = K[M_1/x, M_2/y] : \mathrm{T}\,C[\text{\&-in}(M_1, M_2)/z]}\; x, y, z$$

We will only use instances of the above rules compatible with predicate logic, that is, instances where the variable z does not appear in C. The conclusion C will furthermore be specialized to Ψ. This combined with the identity $\psi \equiv \mathrm{T}\,\Psi$ produces the following rules:

&-elimination (specialized)

$$\frac{M : \mathrm{T}\,A\&B \quad \begin{matrix}[x : (\mathrm{F}\,A)\psi, y : (\mathrm{F}\,B)\psi] \\ K : \psi\end{matrix}}{\text{\&-el}(M, [x,y]K) : \psi}\; x, y$$

&-equality (specialized)

$$\frac{M_1 : (\mathrm{F}\,A)\psi \quad M_2 : (\mathrm{F}\,B)\psi \quad \begin{matrix}[x : (\mathrm{F}\,A)\psi, y : (\mathrm{F}\,B)\psi] \\ K : \psi\end{matrix}}{\text{\&-el}(\text{\&-in}(M_1, M_2), [x,y]K) = K[M_1/x, M_2/y] : \psi}\; x, y$$

The interpretation of the first &-elimination rule of *C3* is given by the equation

$$[\![\&E_1(M, [x]K)]\!] \equiv \text{\&-el}([\![M]\!], [x'', y'']x''([x][\![K]\!])),$$

that is,

$$\dfrac{\begin{matrix} M \\ \mathrm{T}\,A\&B \end{matrix} \quad \begin{matrix} \overset{x}{[\mathrm{T}\,A]} \\ K \\ \psi \end{matrix}}{\psi}\ {}_1x \quad \mapsto \quad \dfrac{[\![M]\!] : \mathrm{T}\,A\&B \qquad \dfrac{[x'' : (\mathrm{F}\,A)\psi] \quad \dfrac{\begin{matrix}[x : \mathrm{T}\,A] \\ [\![K]\!] : \psi\end{matrix}}{[x][\![K]\!] : \mathrm{F}\,A}\ x}{x''([x][\![K]\!]) : \psi}}{\&\text{-el}([\![M]\!], [x'', y'']x''([x][\![K]\!])) : \psi}\ x'', y''$$

where the variables x'' and y'' do not appear free in $[x][\![K]\!]$. The second &-elimination rule is interpreted similarly.

Lemma 3.2.1. *The interpretation of classical conjunction commutes with the &-contraction rules for C3.*

Proof. The left- and right-hand sides of the first &-contraction rule translate into $\&\text{-el}_1(\&\text{-in}([x']M_1, [y']M_2), [x]K)$ and $M_1[[x]K/x']$, respectively. Computation yields

$$\begin{aligned}
&\&\text{-el}(\&\text{-in}([x']M_1, [y']M_2), [x'', y'']x''([x]K)) \\
&\twoheadrightarrow x''([x]K)[[x']M_1/x'', [y']M_2/y''] \\
&\twoheadrightarrow x''[[x']M_1/x'', [y']M_2/y'']((([x]K)[[x']M_1/x'', [y']M_2/y'']) \\
&\twoheadrightarrow ([x']M_1)([x]K) \\
&\twoheadrightarrow M_1[[x]K/x'].
\end{aligned}$$

The case of the second &-contraction rule is similar. □

Disjunction

The constant $\vee$ expresses classical disjunction. The rule for forming disjunctions is:

$\vee$-formation

$$\frac{A : \text{prop} \quad B : \text{prop}}{A \vee B : \text{prop}}$$

The canonical constants $\vee\text{-in}_1$ and $\vee\text{-in}_2$ are used to express canonical proofs of disjunctions. The interpretation of the first $\vee$-introduction rule of *C3* is given by the equation

$$[\![\vee I_1([x']N)]\!] \equiv \vee\text{-in}_1([x'][\![N]\!])$$

whose right-hand member is governed by the rule

$\vee$-introduction 1(2)

$$\dfrac{\begin{matrix}[x' : \mathrm{F}\,A] \\ N : \psi\end{matrix}}{\vee\text{-in}_1([x']N) : \mathrm{T}\,A \vee B}\ x'$$

The second $\vee$-introduction rule is interpreted similarly.

The corresponding elimination and equality rules are:

$\vee$-elimination

$$\frac{N : \mathrm{T}\, A \vee B \qquad \begin{matrix}[z : \mathrm{T}\, A \vee B] \\ C : \text{prop}\end{matrix} \qquad \begin{matrix}[x : (\mathrm{F}\, A)\psi] \\ L_1 : \mathrm{T}\, C[\vee\text{-in}_1(x)/z]\end{matrix} \qquad \begin{matrix}[y : (\mathrm{F}\, B)\psi] \\ L_2 : \mathrm{T}\, C[\vee\text{-in}_2(y)/z]\end{matrix}}{\vee\text{-el}(N, [x]L_1, [y]L_2) : \mathrm{T}\, C[N/z]}\ x, y, z$$

$\vee$-equality 1(2)

$$\frac{N : (\mathrm{F}\, A)\psi \qquad \begin{matrix}[z : \mathrm{T}\, A \vee B] \\ C : \text{prop}\end{matrix} \qquad \begin{matrix}[x : (\mathrm{F}\, A)\psi] \\ L_1 : \mathrm{T}\, C[\vee\text{-in}_1(x)/z]\end{matrix} \qquad \begin{matrix}[y : (\mathrm{F}\, B)\psi] \\ L_2 : \mathrm{T}\, C[\vee\text{-in}_2(y)/z]\end{matrix}}{\vee\text{-el}(\vee\text{-in}_1(N), [x]L_1, [y]L_2) = L_1[N/x] : \mathrm{T}\, C[\vee\text{-in}(N)/z]}\ x, y, z$$

Specializing C to Ψ produces the following rules:

$\vee$-elimination (specialized)

$$\frac{N : \mathrm{T}\, A \vee B \qquad \begin{matrix}[x : (\mathrm{F}\, A)\psi] \\ L_1 : \psi\end{matrix} \qquad \begin{matrix}[y : (\mathrm{F}\, B)\psi] \\ L_2 : \psi\end{matrix}}{\vee\text{-el}(N, [x]L_1, [y]L_2) : \psi}\ x, y$$

$\vee$-equality 1(2) (specialized)

$$\frac{N : (\mathrm{F}\, A)\psi \qquad \begin{matrix}[x : (\mathrm{F}\, A)\psi] \\ L_1 : \psi\end{matrix} \qquad \begin{matrix}[y : (\mathrm{F}\, B)\psi] \\ L_2 : \psi\end{matrix}}{\vee\text{-el}(\vee\text{-in}_1(N), [x]L_1, [y]L_2) = L_1[N/x] : \psi}\ x, y$$

The interpretation of the $\vee$-elimination rule of *C3* is given by the equation

$$[\![\vee E([x]L_1, [y]L_2)]\!] \equiv \vee\text{-el}([\![N]\!], [x'']x''([x][\![L_1]\!]), [y'']y''([y][\![L_2]\!])),$$

that is,

$$\frac{\begin{matrix} N \\ \mathrm{T}\, A \vee B\end{matrix} \qquad \begin{matrix}[\mathrm{T}\overset{x}{A}] \\ L_1 \\ \psi\end{matrix} \qquad \begin{matrix}[\mathrm{T}\overset{y}{B}] \\ L_2 \\ \psi\end{matrix}}{\psi}\ x, y \quad \mapsto$$

$$\frac{N : \mathrm{T}\, A \vee B \qquad \dfrac{[x'' : (\mathrm{F}\, A)\psi] \qquad \dfrac{\begin{matrix}[x : \mathrm{T}\, A] \\ [\![L_1]\!] : \psi\end{matrix}}{[x][\![L_1]\!] : \mathrm{F}\, A}\ x}{x''([x][\![L_1]\!]) : \psi} \qquad \dfrac{[y'' : (\mathrm{F}\, A)\psi] \qquad \dfrac{\begin{matrix}[y : \mathrm{T}\, B] \\ [\![L_2]\!] : \psi\end{matrix}}{[y][\![L_2]\!] : \mathrm{F}\, B}\ y}{y''([y][\![L_2]\!]) : \psi}}{\vee\text{-el}([\![N]\!], [x'']x''([x][\![L_1]\!]), [y'']y''([y][\![L_2]\!])) : \psi}\ x'', y''$$

where the variables x'' and y'' must not appear free in $[x][\![L_1]\!]$ and $[y][\![L_2]\!]$, respectively.

Lemma 3.2.2. *The interpretation of classical disjunction commutes with the $\vee$-contraction rules for C3.*

Proof. The left- and right-hand sides of the first $\vee$-contraction rule translate into $\vee\text{-el}(\vee\text{-in}_1([x']N), [x]L_1, [y]L_2)$ and $N[[x]L_1/x']$, respectively. Computation yields

$$\begin{aligned}
&\vee\text{-el}(\vee\text{-in}_1([x']N), [x'']x''([x]L_1), [y'']y''([y]L_2)) \\
&\twoheadrightarrow x''([x]L_1)[[x']N/x''] \\
&\twoheadrightarrow x''[[x']N/x'']((([x]L_1)[[x']N/x'']) \\
&\twoheadrightarrow ([x']N)([x]L_1) \\
&\twoheadrightarrow N[[x]L_1/x'].
\end{aligned}$$

The case of the second $\vee$-contraction rule is similar. □

Implication

The constant $\supset$ expresses classical implication. The rule for forming implications is:

$\supset$-formation

$$\frac{A : \text{prop} \quad B : \text{prop}}{A \supset B : \text{prop}}$$

The canonical constant $\supset\text{-in}$ is used to express canonical proofs of implications. The interpretation of the $\supset$-introduction rule of *C3* is given by the equation

$$[\![\supset I([x, y']M)]\!] \equiv \supset\text{-in}([x, y'][\![M]\!])$$

whose right-hand member is governed by the rule

$\supset$-introduction

$$\frac{\begin{array}{c}[x : \mathrm{T}\, A, y' : \mathrm{F}\, B] \\ M : \psi\end{array}}{\supset\text{-in}([x, y']M) : \mathrm{T}\, A \supset B}\; x, y'$$

The corresponding elimination and equality rules are:

$\supset$-elimination

$$\frac{M : \mathrm{T}\, A \supset B \qquad \begin{array}{c}[z : \mathrm{T}\, A \supset B] \\ C : \text{prop}\end{array} \qquad \begin{array}{c}[y : (\mathrm{T}\, A, \mathrm{F}\, B)\psi] \\ K : \mathrm{T}\, C[\supset\text{-in}(y)/z]\end{array}}{\supset\text{-el}(M, [y]K) : \mathrm{T}\, C[M/z]}\; y, z$$

$\supset$-equality

$$\frac{M : (\mathrm{T}\, A, \mathrm{F}\, B)\psi \qquad \begin{array}{c}[z : \mathrm{T}\, A \supset B] \\ C : \text{prop}\end{array} \qquad \begin{array}{c}[y : (\mathrm{T}\, A, \mathrm{F}\, B)\psi] \\ K : \mathrm{T}\, C[\supset\text{-in}(y)/z]\end{array}}{\supset\text{-el}(\supset\text{-in}(M), [y]K) = K[M/y] : \mathrm{T}\, C[\supset\text{-in}(M)/z]}\; y, z$$

Specializing C to Ψ produces the following rules:

$\supset$-elimination (specialized)

$$\frac{M : \mathrm{T}\,A \supset B \qquad \begin{array}{c}[y : (\mathrm{T}\,A, \mathrm{F}\,B)\psi]\\ K : \psi\end{array}}{\supset\text{-el}(M, [y]K) : \psi}\; y$$

$\supset$-equality (specialized)

$$\frac{M : (\mathrm{T}\,A, \mathrm{F}\,B)\psi \qquad \begin{array}{c}[y : (\mathrm{T}\,A, \mathrm{F}\,B)\psi]\\ K : \psi\end{array}}{\supset\text{-el}(\supset\text{-in}(M), [y]K) = K[M/y] : \psi}\; y$$

The interpretation of the $\supset$-elimination rule of *C3* is given by the equation

$$[\![\supset E(M, [x']N, [y]K)]\!] \equiv \supset\text{-el}([\![M]\!], [y''][\![N]\!][[x]y''(x, [y][\![K]\!])/x']),$$

that is,

$$\frac{\begin{array}{c} M\\ \mathrm{T}\,A \supset B\end{array} \qquad \begin{array}{c}\overset{x'}{[\mathrm{F}\,A]}\\ N\\ \psi\end{array} \qquad \begin{array}{c}\overset{y}{[\mathrm{T}\,B]}\\ K\\ \psi\end{array}}{\psi}\; x', y \quad \mapsto$$

$$\frac{[\![M]\!] : \mathrm{T}\,A \supset B \qquad \begin{array}{c}\dfrac{\dfrac{\dfrac{[y'' : (\mathrm{T}\,A, \mathrm{F}\,B)\psi] \quad [x : \mathrm{T}\,A]}{y''(x) : (\mathrm{F}\,B)\psi} \qquad \dfrac{\begin{array}{c}[y : \mathrm{T}\,B]\\ [\![K]\!] : \psi\end{array}}{[y][\![K]\!] : \mathrm{F}\,B}\; y}{y''(x, [y][\![K]\!]) : \psi}}{[x]y''(x, [y][\![K]\!]) : \mathrm{F}\,A}\; x\\ [\![N]\!][[x]y''(x, [y][\![K]\!])/x'] : \psi\end{array}}{\supset\text{-el}([\![M]\!], [y''][\![N]\!][[x]y''(x, [y][\![K]\!])/x']) : \psi}\; y''$$

where the variable y'' does not appear free in N or $[y][\![K]\!]$.

Lemma 3.2.3. *The interpretation of classical implication commutes with the $\supset$-contraction rule for C3.*

Proof. The left- and right-hand sides of the $\supset$-contraction rule translate into $\supset$-el$(\supset$-in$([x, y']M), [x']N, [y]K)$ and $N[[x]M[[y]K/y']/x']$, respectively.

Computation yields

$$\begin{aligned}
&\supset\text{-el}(\supset\text{-in}([x,y']M),[y'']N[[x]y''(x,[y]K)/x'])\\
&\twoheadrightarrow N[[x]M(x,[y]K)/x'][[x,y']M/y'']\\
&\twoheadrightarrow N[[x,y']M/y'',([x]y''(x,[y]K))[[x,y']M/y'']/x']\\
&\twoheadrightarrow N[([x]y''(x,[y]K))[[x,y']M/y'']/x']\\
&\twoheadrightarrow N[[x]y''(x,[y]K)[[x,y']M/y'']/x']\\
&\twoheadrightarrow N[[x]y''[[x,y']M/y''](x[[x,y']M/y''],([y]K)[[x,y']M/y''])/x']\\
&\twoheadrightarrow N[[x]([x,y']M)(x,[y]K)/x']\\
&\twoheadrightarrow N[[x]M[x/x,[y]K/y']/x']\\
&\twoheadrightarrow N[[x]M[[y]K/y']/x'].
\end{aligned}$$

□

Negation

The constant ¬ expresses classical negation. The rule for forming negations is:

¬-formation

$$\frac{A : \text{prop}}{\neg A : \text{prop}}$$

The canonical constant ¬-in is used to express canonical proofs of negations. The interpretation of the ¬-introduction rule of *C3* is given by the equation

$$[\![\neg I([x]M)]\!] \equiv \neg\text{-in}([x][\![M]\!])$$

whose right-hand member is governed by the rule

¬-introduction

$$\frac{\begin{array}{c}[x : \mathrm{T}\,A]\\ M : \psi\end{array}}{\neg\text{-in}([x]M) : \mathrm{T}\,\neg A}\ x$$

The corresponding elimination and equality rules are:

¬-elimination

$$\frac{M : \mathrm{T}\,\neg A \quad \begin{array}{c}[z : \mathrm{T}\,\neg A]\\ C : \text{prop}\end{array} \quad \begin{array}{c}[y : (\mathrm{T}\,A)\psi]\\ K : \mathrm{T}\,C[\neg\text{-in}(y)/z]\end{array}}{\neg\text{-el}(M,[y]K) : \mathrm{T}\,C[M/z]}\ y,z$$

¬-equality

$$\frac{M : (\mathrm{T}\,A)\psi \quad \begin{array}{c}[z : \mathrm{T}\,\neg A]\\ C : \text{prop}\end{array} \quad \begin{array}{c}[y : (\mathrm{T}\,A)\psi]\\ K : \mathrm{T}\,C[\neg\text{-in}(y)/z]\end{array}}{\neg\text{-el}(\neg\text{-in}(M),[y]K) = K[M/y] : \mathrm{T}\,C[\neg\text{-in}(M)/z]}\ y,z$$

Specializing C to Ψ produces the following rules:

¬-elimination (specialized)

$$\frac{M : \mathrm{T}\,\neg A \qquad \begin{matrix}[y : (\mathrm{T}\,A)\psi] \\ K : \psi\end{matrix}}{\neg\text{-el}(M, [y]K) : \psi}\; y$$

¬-equality (specialized)

$$\frac{M : (\mathrm{T}\,A)\psi \qquad \begin{matrix}[y : (\mathrm{T}\,A)\psi] \\ K : \psi\end{matrix}}{\neg\text{-el}(\neg\text{-in}(M), [y]K) = K[M/y] : \psi}\; y$$

The interpretation of the ¬-elimination rule of *C3* is given by the equation

$$[\![\neg E(M, [x']N)]\!] \equiv \neg\text{-el}([\![M]\!], [x'][\![N]\!]),$$

that is,

$$\frac{\begin{matrix}M \\ \mathrm{T}\,\neg A\end{matrix} \qquad \begin{matrix}[x' : \mathrm{F}\,A] \\ N \\ \psi\end{matrix}}{\psi}\; x' \quad \mapsto \quad \frac{[\![M]\!] : \mathrm{T}\,\neg A \qquad \begin{matrix}[x' : \mathrm{F}\,A] \\ [\![N]\!] : \psi\end{matrix}}{\neg\text{-el}([\![M]\!], [x'][\![N]\!]) : \psi}\; y$$

Lemma 3.2.4. *The interpretation of classical negation commutes with the ¬-contraction rule for C3.*

Proof. The left- and right-hand sides of the ¬-contraction rule translate into $\neg\text{-el}(\neg\text{-in}([x]M), [x']N)$ and $N[[x]M/x']$, respectively. These are equal in one step. □

Universal quantification

We adhere to the Curry-Howard correspondence and make no formal difference between $(\mathrm{set}, \mathrm{element})$ and $(\mathrm{prop}, \mathrm{proof})$. We furthermore write $\mathrm{T}\,V$ instead of $\mathrm{element}(V)$ for the sake of compact notation.

The constant ∀ expresses classical universal quantification. The rule for forming universal quantifications is:

∀-formation

$$\frac{\begin{matrix}[v : \mathrm{T}\,V] \\ A : \mathrm{prop}\end{matrix}}{\forall v A : \mathrm{prop}}\; v$$

The canonical constant $\forall\text{-in}$ is used to express canonical proofs of universal quantifications. The interpretation of the ∀-introduction rule of *C3* is given by the equation

$$[\![\forall I([v, x']M)]\!] \equiv \forall\text{-in}([v, x'][\![M]\!])$$

whose right-hand member is governed by the rule

∀-introduction

$$\frac{\begin{matrix}[v : \mathrm{T}\,V, x' : \mathrm{F}\,A] \\ M : \psi\end{matrix}}{\forall\text{-in}([v, x']M) : \mathrm{T}\,\forall vA}\ v, x'$$

The corresponding elimination and equality rules are:

∀-elimination

$$\frac{M : \mathrm{T}\,\forall vA \qquad \begin{matrix}[z : \mathrm{T}\,\forall vA] \\ C : \text{prop}\end{matrix} \qquad \begin{matrix}[x : (\mathrm{T}\,V, \mathrm{F}\,A)\psi] \\ K : \mathrm{T}\,C[\forall\text{-in}(x)/z]\end{matrix}}{\forall\text{-el}(M, [x]K) : \mathrm{T}\,C[M/z]}\ x$$

∀-equality

$$\frac{M : (\mathrm{T}\,V, \mathrm{F}\,A)\psi \qquad \begin{matrix}[z : \mathrm{T}\,\forall vA] \\ C : \text{prop}\end{matrix} \qquad \begin{matrix}[x : (\mathrm{T}\,V, \mathrm{F}\,A)\psi] \\ K : \mathrm{T}\,C[\forall\text{-in}(x)/z]\end{matrix}}{\forall\text{-el}(\forall\text{-in}(M), [x]K) = K[M/x] : \mathrm{T}\,C[\forall\text{-in}(M)/z]}\ x$$

Specializing C to Ψ produces the following rules:

∀-elimination (specialized)

$$\frac{M : \mathrm{T}\,\forall vA \qquad \begin{matrix}[x : (\mathrm{T}\,V, \mathrm{F}\,A)\psi] \\ K : \psi\end{matrix}}{\forall\text{-el}(M, [x]K) : \psi}\ x$$

∀-equality (specialized)

$$\frac{M : (\mathrm{T}\,V, \mathrm{F}\,A)\psi \qquad \begin{matrix}[x : (\mathrm{T}\,V, \mathrm{F}\,A)\psi] \\ K : \psi\end{matrix}}{\forall\text{-el}(\forall\text{-in}(M), [x]K) = K[M/x] : \psi}\ x$$

The interpretation of the ∀-elimination rule of *C3* is given by the equation

$$[\![\forall E(M, t, [x]K)]\!] \equiv \forall\text{-el}([\![M]\!], [x'']x''(t, [x][\![K]\!])),$$

that is,

$$\frac{\begin{matrix}M \\ \mathrm{T}\,\forall vA\end{matrix} \qquad \begin{matrix}[x : \mathrm{T}\,A[t/v]] \\ K \\ \psi\end{matrix}}{\psi}\ x \qquad \mapsto$$

$$\frac{[\![M]\!] : \mathrm{T}\,\forall vA \qquad \dfrac{\dfrac{[x'' : (V, \mathrm{F}\,A)\psi] \quad t : \mathrm{T}\,V}{x''(t) : (\mathrm{F}\,A[t/v])\psi} \qquad \dfrac{\begin{matrix}[x : \mathrm{T}\,A[t/v]] \\ [\![K]\!] : \psi\end{matrix}}{[x][\![K]\!] : \mathrm{F}\,A[t/v]}\ x}{x''(t, [x][\![K]\!]) : \psi}}{\forall\text{-el}([\![M]\!], [x'']x''(t, [x][\![K]\!])) : \psi}\ x''$$

where the variable x'' does not appear free in $[x][\![K]\!]$.

Lemma 3.2.5. *The interpretation of classical universal quantification commutes with the* $\forall$*-contraction rule for C3.*

Proof. The left- and right-hand sides of the $\forall$-contraction rule translate into $\forall\text{-el}(\forall\text{-in}([v,x']M),t,[x]K)$ and $M[t/v][[x]K/x']$, respectively. Computation yields

$$\begin{aligned}
&\forall\text{-el}(\forall\text{-in}([v,x']M),[x'']x''(t,[x]K))\\
&\twoheadrightarrow x''(t,[x]L)[[v,x']M/x'']\\
&\twoheadrightarrow x''[[v,x']M/x''](t[[v,x']M/x''],([x]L)[[v,x']M/x''])\\
&\twoheadrightarrow ([v,x']M)(t,[x]K)\\
&\twoheadrightarrow M[t/v][[x]K/x'].
\end{aligned}$$

□

Existential quantification

The constant $\exists$ expresses classical existential quantification. The rule for forming existential quantifications is:

$\exists$-formation

$$\frac{\begin{matrix}[v:\mathrm{T}\,V]\\ A:\mathrm{prop}\end{matrix}}{\exists vA:\mathrm{prop}}$$

The canonical constant $\exists\text{-in}$ is used to express canonical proofs of existential quantifications. The interpretation of the $\exists$-introduction rule of *C3* is given by the equation

$$[\![\exists I(t,[x']N)]\!] \equiv \exists\text{-in}(t,[x'][\![N]\!])$$

whose right-hand member is governed by the rule

$\exists$-introduction

$$\frac{t:\mathrm{T}\,V \qquad \begin{matrix}[x':\mathrm{F}\,A[t/v]]\\ N:\psi\end{matrix}}{\exists\text{-in}(t,[x']N):\mathrm{T}\,\exists vA}$$

The corresponding elimination and equality rules are:

$\exists$-elimination

$$\frac{N:\mathrm{T}\,\exists vA \qquad \begin{matrix}[z:\mathrm{T}\,\exists vA]\\ C:\mathrm{prop}\end{matrix} \qquad \begin{matrix}[v:\mathrm{T}\,V, x:(\mathrm{F}\,A)\psi]\\ L:\mathrm{T}\,C[\exists\text{-in}(v,x)/z]\end{matrix}}{\exists\text{-el}(N,[v,x]L):\mathrm{T}\,C[N/z]}\ v,x$$

∃-equality

$$\frac{t : \mathrm{T}\,V \quad N : (\mathrm{F}\,A)\psi \quad \begin{matrix}[z : \mathrm{T}\,\exists vA]\\ C : \mathrm{prop}\end{matrix} \quad \begin{matrix}[v : \mathrm{T}\,V, x : (\mathrm{F}\,A)\psi]\\ L : \mathrm{T}\,C[\exists\text{-in}(v,x)/z]\end{matrix}}{\exists\text{-el}(\exists\text{-in}(t,N),[v,x]L) = L[t/v,N/x] : \mathrm{T}\,C[\exists\text{-in}(t,N)/z]}\ v,x$$

Specializing C to Ψ produces the following rules:

∃-elimination (specialized)

$$\frac{N : \mathrm{T}\,\exists vA \qquad \begin{matrix}[v : \mathrm{T}\,V, x : (\mathrm{F}\,A)\psi]\\ L : \psi\end{matrix}}{\exists\text{-el}(N,[v,x]L) : \psi}\ v,x$$

∃-equality (specialized)

$$\frac{t : \mathrm{T}\,V \quad N : (\mathrm{F}\,A)\psi \qquad \begin{matrix}[v : \mathrm{T}\,V, x : (\mathrm{F}\,A)\psi]\\ L : \psi\end{matrix}}{\exists\text{-el}(\exists\text{-in}(t,N),[v,x]L) = L[t/v,N/x] : \psi}\ v,x$$

The interpretation of the ∃-elimination rule of *C3* is given by the equation

$$[\![\exists E(N,[v,x]L)]\!] \equiv \exists\text{-el}([\![N]\!],[v,x'']x''([x][\![L]\!])),$$

that is,

$$\frac{\begin{matrix}N\\ \mathrm{T}\,\exists vA\end{matrix} \qquad \begin{matrix}[x : \mathrm{T}\,A]\\ L\\ \psi\end{matrix}}{\psi}\ x \quad\mapsto\quad \frac{[\![N]\!] : \mathrm{T}\,\exists vA \qquad \dfrac{[x'' : (\mathrm{F}\,A)\psi] \quad \dfrac{\begin{matrix}[x : \mathrm{T}\,A]\\ [\![L]\!] : \psi\end{matrix}}{[x][\![L]\!] : \mathrm{F}\,A}\ x}{x''([x][\![L]\!]) : \psi}}{\exists\text{-el}([\![N]\!],[v,x'']x''([x][\![L]\!])) : \psi}\ v,x''$$

where the variable x'' does not appear free in $[x][\![L]\!]$.

Lemma 3.2.6. *The interpretation of classical existential quantification commutes with the ∃-contraction rule for C3.*

Proof. The left- and right-hand sides of the ∃-contraction rule translate into $\exists\text{-el}(\exists\text{-in}(t,[x']N),[v,x]L)$ and $N[[x]L[t/v]/x']$, respectively. Computation yields

$$\begin{aligned}&\exists\text{-el}(\exists\text{-in}(t,[x']N),[v,x'']x''([x]L))\\ &\twoheadrightarrow x''([x]L)[t/v,[x']N/x'']\\ &\twoheadrightarrow x''[t/v,[x']N/x'']((([x]L)[t/v,[x']N/x''])\\ &\twoheadrightarrow ([x']N)((([x]L)[t/v])\\ &\twoheadrightarrow ([x']N)([x]L[t/v])\\ &\twoheadrightarrow N[[x]L[t/v]/x'].\end{aligned}$$

□

3.3 Properties of the interpretation

We prove that the interpretation respects *C3*-equality, respects normal forms up to β-equality, and is injective with respect to type-theoretic definitional equality.

Proposition 3.3.1. *For all C3-derivations M and N, if $M \twoheadrightarrow N$ then $[\![M]\!] \twoheadrightarrow [\![N]\!]$.*

Proof. By Lemma 3.2.1–Lemma 3.2.6 and induction on the number of reduction steps in the reduction $M \twoheadrightarrow N$. □

The interpretation respects normal forms up to β-equality. In fact, if a *C3*-normal form does not contain any $\supset$-elimination, then it translates to a normal form, as can be shown by induction.

Lemma 3.3.2. *Let $N : \mathrm{T}\,C\ (x' : \mathrm{F}\,A, \Gamma)$ and $K : \psi\ (y : \mathrm{T}\,B, \Gamma)$ be normal type-theoretic terms such that x' only appears in N as a main premise and x does not appear free in K. Then*

$$N[[x]y''(x,[y]K)/x'] : \mathrm{T}\,C\ (y'' : (\mathrm{F}\,A, \mathrm{F}\,B)\psi, \Gamma)$$

β-reduces to a normal type-theoretic term.

Proof. By induction on the height of N. The base case, when N is an assumption, is trivial. In the general case, because $x' : \mathrm{F}\,A$ only appears in N as a main premise, N can be written on the form

$$\begin{array}{c} \dfrac{\overset{x'}{\mathrm{F}\,A} \quad \overset{N_1}{\mathrm{T}\,A}}{\psi} \;,\ldots,\; \dfrac{\overset{x'}{\mathrm{F}\,A} \quad \overset{N_n}{\mathrm{T}\,A}}{\psi} \\ N_0 \\ \mathrm{T}\,C \end{array}$$

where x' does not appear free in N_0 but can appear free in N_1,...,N_n. Hence

$$N[[x]y''(x,[y]K)/x'] \equiv N_0(x'(N_1),\ldots,x'(N_n))[[x]y''(x,[y]K)/x']$$

β-reduces to

$$N_0(y''(N_1[[x]y''(x,[y]K)/x'],[y]K),\ldots,y''(N_n[[x]y''(x,[y]K)/x'],[y]K)).$$

Now $N_i[[x]y''(x,[y]K)/x']$ $(i = 1,\ldots,n)$ β-reduces to a normal term L_i by the induction hypothesis. Hence $N[[x]y''(x,[y]K)/x']$ β-reduces to the normal term

$$N_0(y''(L_1,[y]K),\ldots,y''(L_n,[y]K)).$$

□

Proposition 3.3.3. *For every normal C3-derivation M, $[\![M]\!] \twoheadrightarrow_\beta \mathrm{nf}([\![M]\!])$.*

Proof. By induction on the height of M. The base case, when M is an assumption, is trivial. The proof proceeds by case analysis of the last step of M.

Suppose that $M \equiv$

$$\dfrac{\overset{z}{\mathrm{T}\,A \supset B} \qquad \begin{matrix}[\overset{x'}{\mathrm{F}\,A}]\\ N\\ \psi\end{matrix} \qquad \begin{matrix}[\overset{y}{\mathrm{T}\,B}]\\ K\\ \psi\end{matrix}}{\psi}\; x', y$$

Then $[\![N]\!] \twoheadrightarrow_\beta \mathrm{nf}([\![N]\!])$ and $[\![K]\!] \twoheadrightarrow_\beta \mathrm{nf}([\![K]\!])$ by the induction hypothesis. Furthermore, because $x' : \mathrm{F}\,A$ can only enter N as a main premise and hence only appear in $\mathrm{nf}([\![N]\!])$ as a main premise,

$$\mathrm{nf}([\![N]\!])[[x]y''(x, [y]\mathrm{nf}([\![K]\!]))/x']$$

β-reduces to a normal type-theoretic term by Lemma 3.3.2, provided that $x : \mathrm{T}\,A$ does not appear free in $\mathrm{nf}([\![K]\!])$. Hence $[\![M]\!] \equiv$

$$\supset\text{-el}(z, [y''][\![N]\!][[x]y''(x, [y][\![K]\!])/x']))$$

β-reduces to a normal term.

The other cases are similar, though less complex. □

The β-redexes involved in the proof of Proposition 3.3.3 are, in a sense, only of an administrative nature, and it is natural to ask how these redexes compare with the so-called administrative redexes of the various continuation-passing-style translations found in the litterature, see Section 6.4 for further remarks and references.

Lemma 3.3.4. *For all normal C3-derivations M and M', if $\mathrm{nf}([\![M]\!]) \equiv \mathrm{nf}([\![M']\!])$ then $M \equiv M'$.*

Proof. By induction on the structure of M. The base case is trivial and the proof proceeds by case analysis of the last step of M.

Suppose that $M \equiv$

$$\dfrac{\overset{z}{\mathrm{T}\,A \supset B} \qquad \begin{matrix}[\overset{x'}{\mathrm{F}\,A}]\\ N\\ \psi\end{matrix} \qquad \begin{matrix}[\overset{y}{\mathrm{T}\,B}]\\ K\\ \psi\end{matrix}}{\psi}\; x', y$$

Then, by the same argument and notation as in the proofs of Lemma 3.3.2 and Proposition 3.3.3, $\mathrm{nf}([\![M]\!]) \equiv$

$$\supset\text{-el}(z, [y'']N_0(y''(N_1, [y]\mathrm{nf}([\![K]\!])), ..., y''(N_n, [y]\mathrm{nf}([\![K]\!])))).$$

Because M' is normal and $\mathrm{nf}([\![M]\!]) \equiv \mathrm{nf}([\![M']\!])$, also M' must end by an $\supset$-elimination, say with side premise derivations N' and K'. Again, by the same argument and notation as in the proofs of Lemma 3.3.2 and Proposition 3.3.3, $\mathrm{nf}([\![M']\!]) \equiv$

$$\supset\text{-el}(z, [y'']N_0'(y''(N_1', [y]\mathrm{nf}([\![K']\!])), ..., y''(N_n', [y]\mathrm{nf}([\![K']\!]))))$$

and identification of parts yields $[\![N]\!] \equiv [\![N']\!]$ and $[\![K]\!] \equiv [\![K']\!]$. Then $N \equiv N'$ and $K \equiv K'$ by the induction hypothesis, and so $M \equiv M'$.

The other cases are similar, though less complex. □

It follows from Lemma 3.3.4 that the interpretation is injective with respect to type-theoretic definitional equality. This property is important as it guarantees that nothing is lost in the interpretation.

Proposition 3.3.5. *For all C3-derivations M and N, if $[\![M]\!] = [\![N]\!]$ then $M = N$.*

Proof. By Lemma 3.3.4. □

3.4 On dependent products and sums

We merge the introduction rules of implication and universal quantification into a dependent product similar to that of constructive type theory. We also explain why we can not in the same way merge the introduction rules of conjunction and existential quantification into a dependent sum.

From ⊃ and ∀ to a dependent product

The constant Π denotes the classical dependent product. The rule for forming dependent products is:

Π-formation

$$\frac{A : \mathrm{T}\,A \quad \begin{array}{c}[x : \mathrm{T}\,A] \\ B : \mathrm{prop}\end{array}}{(\Pi x : A)B : \mathrm{prop}}\; x$$

The canonical constant ΠI is used to express canonical proofs of dependent products. The constant is governed by the following introduction rule:

Π-introduction

$$\frac{\begin{array}{c}[x : \mathrm{T}\,A, y' : \mathrm{F}\,B] \\ M : \psi\end{array}}{\Pi\text{-in}([x, y']M) : \mathrm{T}(\Pi x : A)B}\; x, y'$$

The ⊃-introduction and ∀-introduction rules can be understood as special cases of the Π-introduction rule. If the variable x does not appear free in B, then we arrive at the ⊃-introduction rule. If A is specialized to V, then we arrive at the ∀-introduction rule.

The corresponding elimination and equality rules are:

Π-elimination

$$\frac{M : \mathrm{T}(\Pi x : A)B \qquad \begin{array}{c}[z : \mathrm{T}(\Pi x : A)B]\\ C : \mathrm{prop}\end{array} \qquad \begin{array}{c}[y'' : (x : \mathrm{T}\,A, \mathrm{F}\,B)\psi]\\ K : \mathrm{T}\,C[\Pi I(y'')/z]\end{array}}{\Pi\text{-el}(M, [y'']K) : \mathrm{T}\,C[M/z]}\ y''$$

Π-equality

$$\frac{M : (\mathrm{T}\,A, \mathrm{F}\,B)\psi \qquad \begin{array}{c}[z : \mathrm{T}(\Pi x : A)B]\\ C : \mathrm{prop}\end{array} \qquad \begin{array}{c}[y'' : (x : \mathrm{T}\,A, \mathrm{F}\,B)\psi]\\ K : \mathrm{T}\,C[\Pi I(y'')/z]\end{array}}{\Pi\text{-el}(\Pi I(M), [y'']K) = K[M/y''] : \mathrm{T}\,C[\Pi I(M)/z]}\ y''$$

Also the elimination and equality rules for ⊃ and ∀ can be understood as special cases of the elimination and equality rules for Π.

Specializing C to Ψ produces the following rules:

Π-elimination (specialized)

$$\frac{M : \mathrm{T}(\Pi x : A)B \qquad \begin{array}{c}[y'' : (x : \mathrm{T}\,A, \mathrm{F}\,B)\psi]\\ K : \psi\end{array}}{\Pi\text{-el}(M, [y'']K) : \psi}\ y''$$

Π-equality (specialized)

$$\frac{M : (x : \mathrm{T}\,A, \mathrm{F}\,B)\psi \qquad \begin{array}{c}[y'' : (x : \mathrm{T}\,A, \mathrm{F}\,B)\psi]\\ K : \psi\end{array}}{\Pi\text{-el}(\Pi I(M), [y'']K) = K[M/y''] : \psi}\ y''$$

There exists an elimination rule for Π that follows the same pattern as the elimination rules of *C3*:

$$\frac{\mathrm{T}(\Pi x : A)B \qquad \begin{array}{c}[\overset{x'}{\mathrm{F}}A]\\ \psi\end{array} \qquad \begin{array}{c}[\overset{x}{\mathrm{T}}A, \overset{y}{\mathrm{T}}B]\\ \psi\end{array}}{\psi}\ x', x, y$$

It is interpreted according to

$$\frac{\begin{array}{c}M\\ \mathrm{T}(\Pi x : A)B\end{array} \qquad \begin{array}{c}[\overset{x'}{\mathrm{F}}A]\\ N\\ \psi\end{array} \qquad \begin{array}{c}[\overset{x}{\mathrm{T}}A, \overset{y}{\mathrm{T}}B]\\ K\\ \psi\end{array}}{\psi}\ x', x, y \quad \mapsto$$

$$\frac{\llbracket M\rrbracket : \mathrm{T}(\Pi x : A)B \qquad \begin{array}{c}\dfrac{\dfrac{\dfrac{[y'' : (x : \mathrm{T}\,A, \mathrm{F}\,B)\psi] \quad [x : \mathrm{T}\,A]}{y''(x) : (\mathrm{F}\,B)\psi} \qquad \dfrac{\begin{array}{c}[x : \mathrm{T}\,A], [y : \mathrm{T}\,B]\\ \llbracket K\rrbracket : \psi\end{array}}{[y]\llbracket K\rrbracket : \mathrm{F}\,B}\ y}{y''(x, [y]\llbracket K\rrbracket) : \psi}}{[x]y''(x, [y]\llbracket K\rrbracket) : \mathrm{F}\,A}\ x\\ \llbracket N\rrbracket[[x]y''(x, [y]\llbracket K\rrbracket)/x'] : \psi\end{array}}{\Pi\text{-el}(\llbracket M\rrbracket, [y'']\llbracket N\rrbracket[[x]y''(x, [y]\llbracket K\rrbracket)/x']) : \psi}\ y''$$

where y'' does not appear free in $[\![N]\!]$ or $[y][\![K]\!]$.

The above elimination rule for Π easily specializes to the $\supset$-elimination rule of *C3* in a way that commutes with the interpretation. The situation is different for the $\forall$-elimination rule of *C3*, for which the translation

$$\dfrac{\begin{array}{c}M\\ \mathrm{T}\,\forall vA\end{array} \qquad \begin{array}{c}[x:\mathrm{T}\,A[t/v]]\\ K\\ \psi\end{array}}{\psi}\,x \qquad\mapsto\qquad \dfrac{\begin{array}{c}M\\ \mathrm{T}(\Pi v:V)A\end{array}\qquad \dfrac{[\mathrm{F}\,\overset{v'}{V}]\quad \mathrm{T}\,\overset{t}{V}}{\psi} \qquad \begin{array}{c}[\mathrm{T}\,\overset{v}{V},\mathrm{T}\,\overset{x}{A}]\\ K\\ \psi\end{array}}{\psi}\,v',v,x$$

only commutes up to β-equality with the interpretation.

The interpretation of the elimination rule for Π that follows the same pattern as the elimination rules of *C3* commutes with the contraction rule

Π-contraction:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A},\mathrm{F}\,\overset{y'}{B}]\\ M\\ \psi\end{array}}{\mathrm{T}(\Pi x:A)B}\,x,y' \qquad \begin{array}{c}[\mathrm{F}\,\overset{x'}{A}]\\ N\\ \psi\end{array} \qquad \begin{array}{c}[\mathrm{T}\,\overset{x}{A},\mathrm{T}\,\overset{y}{B}]\\ K\\ \psi\end{array}}{\psi}\,x',x,y \qquad\mapsto\qquad \begin{array}{c}\dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A}], \quad \dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A}],[\mathrm{T}\,\overset{y}{B}]\\ K\\ \psi\end{array}}{\mathrm{F}\,B}\,y\\ M\\ \psi\end{array}}{\mathrm{F}\,A}\,x\\ N\\ \psi\end{array}$$

The proof involves a computation similar to that for classical implication, see the proof of Lemma 3.2.3.

On dependent sums

Any attempt similar to that for Π to merge the introduction rules of conjunction and existential quantification into a dependent sum will have to overcome the obstacle that $\&\text{-in}([x']M_1,[y']M_2):\mathrm{T}\,A\&B$ and $\exists\text{-el}(t,[y']N):\mathrm{T}\,\exists vB$ enclose objects of different form: $[x']M_1$ is a function while t is just an element. This prevents the merging of conjunction and existential quantification into a dependent sum.

Note that wraping $[x']M_1$ in a negation introduction to get around the difference between $[x']M_1$ and t would introduce one extra logical reduction step in the computation in the proof of Lemma 3.2.1.

3.5 On induction in classical logic

We prove the principles of mathematical and W-induction for classical logic. The idea is to pull back mathematical and W-induction from constructive type theory to *C3*.

Recall that the principle of mathematical induction is implicit in the N-elimination rule of Martin-Löf (1984). To make the N-elimination rule compatible with the inference rules of *C3*, we have to restrict it to predicate logic and conclusions of the form $\neg\neg C(n)$. We can then derive the fifth Peano axiom

$$C(0) \supset \big((\forall v : \mathrm{N})C(v) \supset C(\mathrm{s}(v))\big) \supset (\forall n : \mathrm{N})C(n),$$

see Figure 3.1, p. 53. The case of structural induction is similar to that of mathematical induction.

The proof of the fifth Peano axiom in Figure 3.1 holds for any property C of the natural numbers. It is hence more general than the proof due to Gödel (1933), which requires C to be stable.

The case of W-induction differs from that of mathematical induction in the way double-negations are incorporated into the elimination rule. Consider the W-elimination rule of Martin-Löf's set theory, that is, the version of the W-elimination rule that uses dependent products rather than dependent function types. Suppressing the proof objects it becomes

$$\dfrac{\overset{w}{\mathrm{T}\,\mathrm{W}(U,V)} \qquad \begin{array}{c}[\overset{u}{\mathrm{T}\,U}, \overset{f}{\mathrm{T}\prod_{v:V(u)}\mathrm{W}(U,V)}, \overset{x}{\mathrm{T}\prod_{v:V(u)} C(\mathrm{app}(f,v))}] \\ \mathrm{T}\,C(\sup(u,f))\end{array}}{\mathrm{T}\,C(w)}\; u,f,x$$

where U, V, $\mathrm{W}(U,V)$, and $\prod_{v:V(u)}\mathrm{W}(U,V)$ are sets of individuals. Taking $C(w)$ to be doubly negated, the above rule becomes

$$\dfrac{\overset{w}{\mathrm{T}\,\mathrm{W}(U,V)} \qquad \begin{array}{c}[\overset{u}{\mathrm{T}\,U}, \overset{f}{\mathrm{T}\prod_{v:V(u)}\mathrm{W}(U,V)}, \overset{x}{\mathrm{T}\prod_{v:V(u)} \neg\neg C(\mathrm{app}(f,v))}] \\ \mathrm{T}\,\neg\neg C(\sup(u,f))\end{array}}{\mathrm{T}\,\neg\neg C(w)}\; u,f,x$$

Now $\prod_{v:V(u)}\neg\neg C(\mathrm{app}(f,v))$ is equivalent to $(\forall v : V(u))C(\mathrm{app}(f,v))$, where the latter quantifier is understood in the sense of Section 3.2, and so the above rule is equivalent to the rule

$$\dfrac{\overset{w}{\mathrm{T}\,\mathrm{W}(U,V)} \qquad \begin{array}{c}[\overset{u}{\mathrm{T}\,U}, \overset{f}{\mathrm{T}\prod_{v:V(u)}\mathrm{W}(U,V)}, \overset{x}{\mathrm{T}\,(\forall v : V(u))C(\mathrm{app}(f,v))}] \\ \mathrm{T}\,\neg\neg C(\sup(u,f))\end{array}}{\mathrm{T}\,\neg\neg C(w)}\; u,f,x$$

by means of which we can derive a W-induction scheme similar to the fifth Peano axiom. See Figure 3.2, p. 54, for the derivation of the W-induction scheme.

Note that the concept of a function present in the set $\prod_{v:V(u)}\mathrm{W}(U,V)$ coincides with the constructive concept of a function. This is contrary to what happens when it comes to the concept of a function implicit in the proof objects of implication, see Sections 6.1 and 6.2. This supports the idea that any extension of the interpretation should only reinterpret propositions and not sets of individuals.

W-induction is intimately related to transfinite induction, which traditionally is formulated in terms of an accessibility predicate

$$\mathrm{Acc}(v) : \text{prop } (v : \mathrm{T}\, V),$$

where V is a set of individuals. The accessibility predicate is governed by the introduction and elimination rules

Acc-introduction

$$\dfrac{\begin{array}{c}[\mathrm{T}\, v \overset{x}{<} t] \\ \mathrm{T}\,\mathrm{Acc}(v)\end{array}}{\mathrm{T}\,\mathrm{Acc}(t)}\ x$$

and

Acc-elimination

$$\dfrac{\mathrm{T}\,\mathrm{Acc}(t) \qquad \begin{array}{c}[\mathrm{T}\,(\forall u \overset{z}{<} v)C(u)] \\ \mathrm{T}\, C(v)\end{array}}{\mathrm{T}\, C(t)}\ z$$

where $<$ is a binary relation on V.

Unfortunately, the rules for the accessibility predicate are not validated by the interpretation. Any classical counterpart ought to be governed by the introduction rule

Acc-introduction (classical counterpart)

$$\dfrac{\begin{array}{c}[\mathrm{T}\, v \overset{x}{<} t, \mathrm{F}\, \overset{y'}{\mathrm{Acc}}(v)] \\ \psi\end{array}}{\mathrm{T}\,\mathrm{Acc}(t)}\ x, y'$$

which however is not strictly positive and so falls outside the general scheme of inductive definitions.

$$
\left.
\dfrac{
[\mathrm{T}\,\overset{n}{\mathrm{N}}] \quad
\begin{matrix}\mathrm{T}\,\overset{x_0}{C}(0)\\ \vdots \\ \mathrm{T}\,\neg\neg C(0)\end{matrix}
\quad
\dfrac{
\dfrac{
\mathrm{T}\,\forall v\big(C(v)\overset{b}{\supset} C(\mathrm{s}(v))\big)
\quad
\dfrac{
[\mathrm{T}\,A(v)\overset{z}{\supset} C(\mathrm{s}(v))]
\quad
\begin{matrix}[\mathrm{T}\,\neg\neg \overset{x''}{C}(v)], [\mathrm{F}\,\overset{x'}{C}(v)]\\ \vdots \\ \psi\end{matrix}
\quad
\begin{matrix}[\mathrm{T}\,\neg \overset{y'}{C}(\mathrm{s}(v))], [\mathrm{T}\,\overset{y}{C}(\mathrm{s}(v))]\\ \vdots \\ \psi\end{matrix}
}{\psi}\; x', y
}{\psi}\; z
}{\mathrm{T}\,\neg\neg C(\mathrm{s}(v))}\; y'
}{
\begin{matrix}\mathrm{T}\,\neg\neg C(n), [\mathrm{F}\,\overset{c'}{C}(n)]\\ \vdots \\ \dfrac{\psi}{\mathrm{T}\,\forall n C(n)}\; n, c'\end{matrix}
}\; v, x''
\right\} *
$$

$$
\dfrac{
[\mathrm{F}\,\forall v\big(C(v)\supset \overset{e'}{C}(\mathrm{s}(v))\big)\supset \forall n C(n)]
\quad
\dfrac{
\dfrac{
[\mathrm{F}\,\forall n \overset{d'}{C}(n)]
\quad
\begin{matrix}[\mathrm{T}\,\overset{x_0}{C}(0)], [\mathrm{T}\,\forall v\big(C(v)\overset{b}{\supset} C(\mathrm{s}(v))\big)]\\ \vdots\, * \\ \mathrm{T}\,\forall n C(n)\end{matrix}
}{\psi}
}{\mathrm{T}\,\forall v\big(C(v)\supset C(\mathrm{s}(v))\big)\supset \forall n C(n)}\; b, d'
}{
\dfrac{\psi}{\mathrm{T}\,C(0)\supset \forall v\big(C(v)\supset C(\mathrm{s}(v))\big)\supset \forall n C(n)}\; x_0, e'
}
$$

Figure 3.1: Mathematical induction in *C3*. In the lower derivation the subderivation marked by an asterisk is understood to equate the whole upper derivation. Some trivial subderivations have been left out.

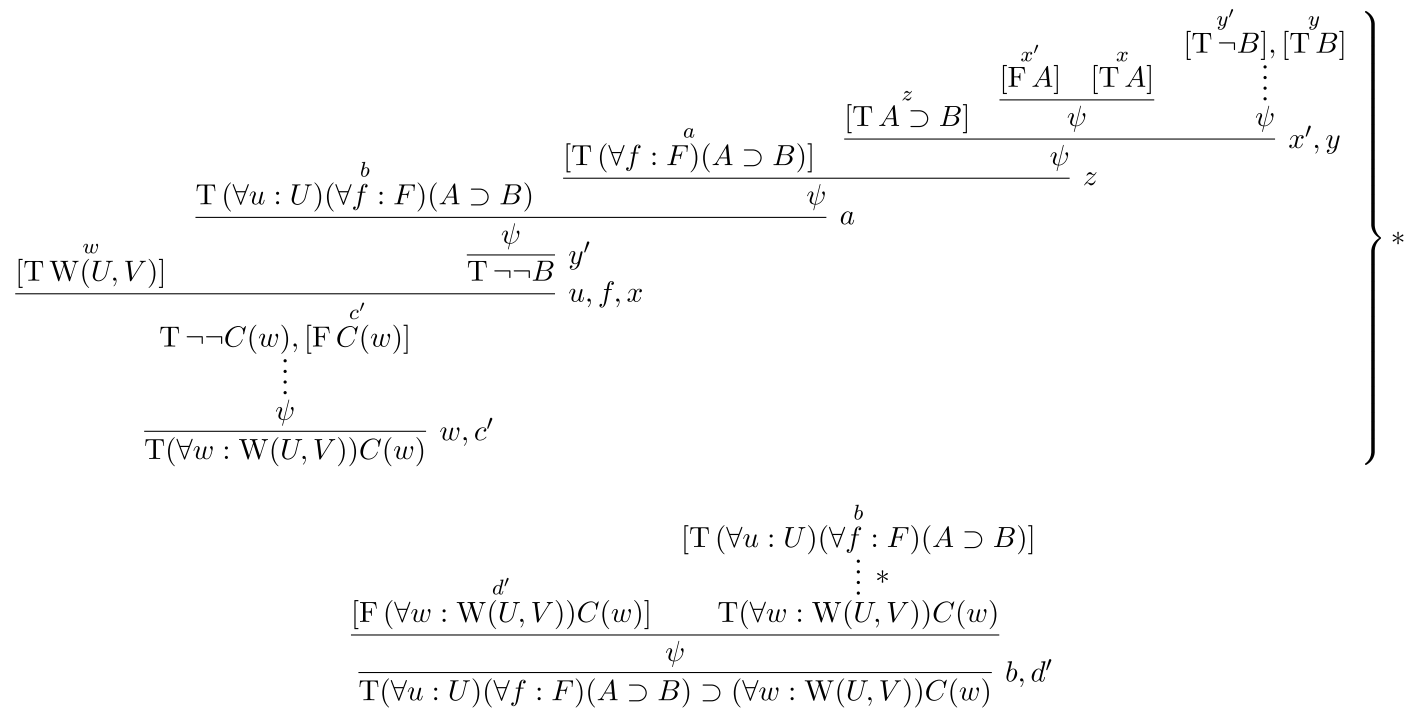

Figure 3.2: W-induction in *C3*. In the lower derivation the subderivation marked by an asterisk is understood to equate the whole upper derivation. Some trivial subderivations have been left out. Notation: $F \equiv \prod_{v:V(u)} \mathrm{W}(U,V)$, $A \equiv (\forall v : V(u))C(\mathrm{app}(f,v))$, and $B \equiv C(\sup(u,f))$.

Chapter 4

Translation into *NJ*

In this chapter, we reduce the interpretation to a translation of *C3* into the minimal fragment of Gentzen's *NJ* and then compute the kernel of the translation. The translation is not injective with respect to *C3*-equality, contrary to the interpretation of Chapter 3, but factorizes over an auxiliary calculus *C3S* in a way which makes the translation of *C3S* into *NJ* as good as injective. The auxiliary calculus is used to compute the kernel, which constitutes a coarser equivalence relation between proofs than the identity relation in Chapter 3. The kernel is generated by a set of permutative rules for implication and negation that relate to *C3* in roughly the same way as those found by von Plato (2001) relate to his calculus.

The inference figures of the minimal fragment of *NJ* can be found in Table 4.1, p. 56. We use the same set of symbols to denote the logical constants of classical and minimal logic. The minimal negation of *NJ* is defined by $\neg A \equiv A \supset \Psi$, where $\supset$ is the intuitionistic implication of *NJ*. We take the concepts of *NJ*-derivation and reduction of *NJ*-derivations for granted, see Prawitz (1965) for the contraction rules for *NJ*.

4.1 Translation of *C3* into *NJ*

We shall define a translation $M \mapsto [\![M]\!]$ that maps *C3*-derivations to *NJ*-derivations. The definition is similar to the definition of the interpretation in Chapter 3.

The translation can be said to exchange the classical introduction and elimination operators (&-in, &-el, etc.) for their intuitionistic counterparts and objects of the form $(\mathrm{F}\,A)\psi$ for objects of the form $\mathrm{T}\,\neg\neg A$. This induces the translation of formulas in Table 4.2, p. 57, which place double-negations in the positive positions of the logical symbol in question.

The translation of formulas is compositional in the sense that $[\![B[A/X]]\!] = [\![B]\!][[\![A]\!]/X]$, where X is a propositional variable.

The logical constants of truth ($\top$) and falsehood ($\bot$) are in classical and intuitionistic logic considered equivalent to the empty conjunction respectively dis-

	Introduction	**Elimination**
$\&$	$\dfrac{A \quad B}{A\&B}$	$\dfrac{A\&B}{A}\,1 \quad \dfrac{A\&B}{B}\,2$
$\lor$	$\dfrac{A}{A\lor B}\,1 \quad \dfrac{B}{A\lor B}\,2$	$\dfrac{A\lor B \quad \begin{matrix}[\overset{x}{A}]\\ C\end{matrix} \quad \begin{matrix}[\overset{y}{B}]\\ C\end{matrix}}{C}$
$\supset$	$\dfrac{\begin{matrix}[\overset{x}{A}]\\ B\end{matrix}}{A\supset B}\,x$	$\dfrac{A\supset B \quad A}{B}$
$\neg$	$\dfrac{\begin{matrix}[\overset{x}{A}]\\ \Psi\end{matrix}}{\neg A}\,x$	$\dfrac{\neg A \quad A}{\Psi}$
$\forall$	$\dfrac{A}{\forall v A}$	$\dfrac{\forall v A}{A[t/v]}$
$\exists$	$\dfrac{A[t/v]}{\exists v A}$	$\dfrac{\exists v A \quad \begin{matrix}[\overset{x}{A}]\\ C\end{matrix}}{C}\,x$

Table 4.1: Minimal fragment of the Calculus *NJ*. In $\forall$-introduction and $\exists$-elimination the variable v must not appear free in the context of discourse.

junction of formulas, that is, $\top \equiv A_1\&...\&A_n$ and $\bot \equiv A_1 \lor ... \lor A_n$ for $n = 0$. Hence $\llbracket\top\rrbracket = \neg\neg\llbracket A_1\rrbracket\&...\&\neg\neg\llbracket A_n\rrbracket$ and $\llbracket\bot\rrbracket = \neg\neg\llbracket A_1\rrbracket \lor ... \lor \neg\neg\llbracket A_n\rrbracket$ for $n = 0$. It then follows that $\llbracket\top\rrbracket = \top$ and $\llbracket\bot\rrbracket = \bot$, that is, classical truth should be translated by intuitionistic truth and classical falsehood should be translated by intuitionistic falsehood $\neq \Psi$.

The signs of truth and falsity

The signs of truth and falsity are interpreted according to $\llbracket \mathrm{T}\,A\rrbracket = \llbracket A\rrbracket$ and $\llbracket \mathrm{F}\,A\rrbracket = \neg\llbracket A\rrbracket$, where $\neg$ is the minimal negation of *NJ*. This effectively amounts to the abandonment of the signs of truth and falsity. Consequently, the translation will be many-to-one on sequents.

The translation of formulas coincides with Kolmogorov's original version of the double-negation interpretation (Kolmogorov 1925) for formulas not containing implication or negation, provided that the interpretation of classical sequents is

$$\begin{array}{rcl}
[\![A\&B]\!] & = & \neg\neg[\![A]\!]\&\neg\neg[\![B]\!] \\
[\![A\vee B]\!] & = & \neg\neg[\![A]\!]\vee\neg\neg[\![B]\!] \\
[\![A\supset B]\!] & = & [\![A]\!]\supset\neg\neg[\![B]\!] \\
[\![\neg A]\!] & = & \neg[\![A]\!] \\
[\![\forall v A]\!] & = & \forall v\neg\neg[\![A]\!] \\
[\![\exists v A]\!] & = & \exists v\neg\neg[\![A]\!] \\
[\![A]\!] & = & A,\ \text{provided that } A \text{ is atomic}
\end{array}$$

Table 4.2: Translation of classical formulas into formulas of minimal logic.

taken into account. Let A be such a first order formula and consider a classical derivation of A in the form of a *C3*-derivation of ψ from $\mathrm{F}\,A$. The derivation translates to a *NJ*-derivation of Ψ from $\neg[\![A]\!]$, equivalent to a *NJ*-derivation of $\neg\neg[\![A]\!]$. The latter is as it stands a *NJ*-derivation of Kolmogorov's translation of A.

Conjunction

We shall here and henceforth in the definition of the translation suppress the brackets $[\![\ldots]\!]$ in favor of readability. The case of conjunction then reads as follows.

$\&$-introduction

$$\dfrac{\begin{matrix}\overset{x'}{[\mathrm{F}\,A]} \\ M_1 \\ \Psi\end{matrix} \quad \begin{matrix}\overset{y'}{[\mathrm{F}\,B]} \\ M_2 \\ \Psi\end{matrix}}{\mathrm{T}\,A\&B}\, x',y' \quad \mapsto \quad \dfrac{\dfrac{\begin{matrix}\overset{x'}{[\neg A]}\\ M_1\\ \Psi\end{matrix}}{\neg\neg A}\,x' \quad \dfrac{\begin{matrix}\overset{y'}{[\neg B]}\\ M_2 \\ \Psi\end{matrix}}{\neg\neg B}\,y'}{\neg\neg A\&\neg\neg B}$$

$\&$-elimination 1(2)

$$\dfrac{\begin{matrix}M\\ \mathrm{T}\,A\&B\end{matrix} \quad \begin{matrix}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \Psi\end{matrix}}{\Psi}\,{}_1x \quad\mapsto\quad \dfrac{\dfrac{\begin{matrix}M\\ \neg\neg A\&\neg\neg B\end{matrix}}{\neg\neg A}\,1 \quad \dfrac{\begin{matrix}\overset{x}{[A]}\\ K\\ \Psi\end{matrix}}{\neg A}\,x}{\Psi}$$

The case of the second $\&$-elimination rule is similar.

Lemma 4.1.1. *The translation of classical conjunction commutes with the* $\&$-*contraction rules for C3.*

Proof. The left- and right-hand sides of the first &-contraction rule translate into the first and last *NJ*-derivation of the following reduction.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\overset{x'}{[\neg A]}\\ M_1\\ \Psi\end{matrix}}{\neg\neg A}\,x' \quad \dfrac{\begin{matrix}\overset{y'}{[\neg B]}\\ M_2\\ \Psi\end{matrix}}{\neg\neg B}\,y'}{\neg\neg A \& \neg\neg B}}{\neg\neg A}\,1 \qquad \dfrac{\begin{matrix}\overset{x}{[A]}\\ K\\ \Psi\end{matrix}}{\neg A}\,x}{\Psi} \quad\longrightarrow\quad \dfrac{\dfrac{\begin{matrix}\overset{x'}{[\neg A]}\\ M_1\\ \Psi\end{matrix}}{\neg\neg A}\,x' \quad \dfrac{\begin{matrix}\overset{x}{[A]}\\ K\\ \Psi\end{matrix}}{\neg A}\,x}{\Psi} \quad\longrightarrow\quad \begin{matrix}\dfrac{\begin{matrix}\overset{x}{[A]}\\ K\\ \Psi\end{matrix}}{\neg A}\,x\\ M_1\\ \Psi\end{matrix}$$

The case of the second &-contraction rule is similar. ☐

Disjunction

$\vee$-introduction 1(2)

$$\dfrac{\begin{matrix}\overset{x'}{[\mathrm{F}\,A]}\\ N\\ \Psi\end{matrix}}{\mathrm{T}\,A\vee B}\,{}_1x' \quad\mapsto\quad \dfrac{\dfrac{\begin{matrix}\overset{x'}{[\neg A]}\\ N\\ \Psi\end{matrix}}{\neg\neg A}\,x'}{\neg\neg A\vee\neg\neg B}\,1$$

The case of the second $\vee$-introduction rule is similar.

$\vee$-elimination

$$\dfrac{\begin{matrix}N\\ \mathrm{T}\,A\vee B\end{matrix} \quad \begin{matrix}\overset{x}{[\mathrm{T}\,A]}\\ L_1\\ \Psi\end{matrix} \quad \begin{matrix}\overset{y}{[\mathrm{T}\,B]}\\ L_2\\ \Psi\end{matrix}}{\Psi}\,x,y \quad\mapsto$$

$$\dfrac{\begin{matrix}N\\ \neg\neg A\vee\neg\neg B\end{matrix} \quad \dfrac{\overset{x''}{[\neg\neg A]} \quad \dfrac{\begin{matrix}\overset{x}{[A]}\\ L_1\\ \Psi\end{matrix}}{\neg A}\,x}{\Psi} \quad \dfrac{\overset{y''}{[\neg\neg B]} \quad \dfrac{\begin{matrix}\overset{y}{[B]}\\ L_2\\ \Psi\end{matrix}}{\neg B}\,y}{\Psi}}{\Psi}\,x'',y''$$

Lemma 4.1.2. *The translation of classical disjunction commutes with the* $\vee$*-contraction rules for C3.*

Proof. Similar to that of Lemma 4.1.1. ☐

Implication

$\supset$-introduction

$$\dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A},\mathrm{F}\,\overset{y'}{B}]\\ M\\ \Psi\end{array}}{\mathrm{T}\,A\supset B}\;x,y' \quad\mapsto\quad \dfrac{\begin{array}{c}[\overset{x}{A}],[\neg\overset{y'}{B}]\\ M\\ \dfrac{\Psi}{\neg\neg B}\;y'\end{array}}{A\supset\neg\neg B}\;x$$

$\supset$-elimination

$$\dfrac{\begin{array}{c}M\\ \mathrm{T}\,A\supset B\end{array}\quad \begin{array}{c}[\mathrm{F}\,\overset{x'}{A}]\\ N\\ \Psi\end{array}\quad \begin{array}{c}[\mathrm{T}\,\overset{y}{B}]\\ K\\ \Psi\end{array}}{\Psi}\;x',y \quad\mapsto\quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}M\\ A\supset\neg\neg B\end{array}\quad[\overset{x}{A}]}{\neg\neg B}\quad \dfrac{\begin{array}{c}[\overset{y}{B}]\\ K\\ \Psi\end{array}}{\neg B}\;y}{\Psi}}{\neg A}\;x\\ N\\ \Psi\end{array}$$

Lemma 4.1.3. *The translation of classical implication commutes with the $\supset$-contraction rules for C3.*

Proof. Similar to that of Lemma 4.1.1. □

Negation

$\neg$-introduction

$$\dfrac{\begin{array}{c}[\mathrm{T}\,\overset{x}{A}]\\ M\\ \Psi\end{array}}{\mathrm{T}\,\neg A}\;x \quad\mapsto\quad \dfrac{\begin{array}{c}[\overset{x}{A}]\\ M\\ \Psi\end{array}}{\neg A}\;x$$

$\neg$-elimination

$$\dfrac{\begin{array}{c}M\\ \mathrm{T}\,\neg A\end{array}\quad \begin{array}{c}[\mathrm{F}\,\overset{x'}{A}]\\ N\\ \Psi\end{array}}{\Psi}\;x' \quad\mapsto\quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}M\\ \neg A\end{array}\quad[\overset{x}{A}]}{\Psi}}{\neg A}\;x\\ N\\ \Psi\end{array}$$

Lemma 4.1.4. *The translation of classical negation commutes with the $\neg$-contraction rules for C3.*

Proof. Similar to that of Lemma 4.1.1. □

Universal quantification

∀-introduction

$$\dfrac{\begin{array}{c}\overset{x'}{[\mathrm{F}\,A]}\\ M\\ \Psi\end{array}}{\mathrm{T}\,\forall v A}\;x' \quad\mapsto\quad \dfrac{\dfrac{\begin{array}{c}\overset{x'}{[\neg A]}\\ M\\ \Psi\end{array}}{\neg\neg A}\;x'}{\forall v\neg\neg A}$$

∀-elimination

$$\dfrac{\begin{array}{c}M\\ \mathrm{T}\,\forall v A\end{array}\qquad \begin{array}{c}\overset{x}{[\mathrm{T}\,A[t/v]]}\\ K\\ \Psi\end{array}}{\Psi}\;x \quad\mapsto\quad \dfrac{\dfrac{\begin{array}{c}M\\ \forall v\neg\neg A\end{array}}{\neg\neg A[t/v]}\qquad \dfrac{\begin{array}{c}\overset{x}{[A[t/v]]}\\ K\\ \Psi\end{array}}{\neg A[t/v]}\;x}{\Psi}$$

Lemma 4.1.5. *The translation of classical universal quantification commutes with the* ∀*-contraction rules for C3.*

Proof. Similar to that of Lemma 4.1.1. □

Existential quantification

∃-introduction

$$\dfrac{\begin{array}{c}\overset{x'}{[\mathrm{F}\,A[t/v]]}\\ N\\ \Psi\end{array}}{\mathrm{T}\,\exists v A}\;x' \quad\mapsto\quad \dfrac{\dfrac{\begin{array}{c}\overset{x'}{[\neg A[t/v]]}\\ N\\ \Psi\end{array}}{\neg\neg A[t/v]}\;x'}{\exists v\neg\neg A}$$

∃-elimination

$$\dfrac{\begin{array}{c}N\\ \mathrm{T}\,\exists v A\end{array}\qquad \begin{array}{c}\overset{x}{[\mathrm{T}\,A]}\\ L\\ \Psi\end{array}}{\Psi}\;x \quad\mapsto\quad \dfrac{\begin{array}{c}N\\ \exists v\neg\neg A\end{array}\qquad \dfrac{\overset{x''}{[\neg\neg A]}\qquad \dfrac{\begin{array}{c}\overset{x}{[A]}\\ L\\ \Psi\end{array}}{\neg A}\;x}{\Psi}}{\Psi}\;x''$$

Lemma 4.1.6. *The translation of classical existential quantification commutes with the* ∃*-contraction rules for C3.*

Proof. Similar to that of Lemma 4.1.1. □

4.2 Properties of the translation

We prove that the translation respects *C3*-equality but does not respect normal forms. However the latter was to be expected from Lemma 3.3.2 and Proposition 3.3.3. More important is that the translation in an essential way fails to be injective with respect to *NJ*-equality. Hence something is lost.

Proposition 4.2.1. *For all C3-derivations M and N, if $M \twoheadrightarrow N$ then $[\![M]\!] \twoheadrightarrow [\![N]\!]$.*

Proof. By Lemmas 4.1.1–4.1.6 and induction on the number of reduction steps in the reduction $M \twoheadrightarrow N$. □

Proposition 4.2.2. *There exists a normal C3-derivation M whose translation $[\![M]\!]$ is not normal in NJ.*

Proof. Let $M \equiv$

$$\dfrac{\mathrm{T}\,A \overset{z}{\supset} B \qquad \dfrac{\overset{x'}{[\mathrm{F}\,A]} \quad \overset{x}{\mathrm{T}\,A}}{\psi} \qquad \dfrac{\overset{y'}{\mathrm{F}\,B} \quad \overset{y}{[\mathrm{T}\,B]}}{\psi}}{\psi}\; x', y$$

Then $[\![M]\!] \equiv$

$$\dfrac{\dfrac{\Psi}{\neg A}\,x \qquad \overset{x}{A}}{\Psi} \quad \text{with} \quad \Psi \equiv \dfrac{\dfrac{A \overset{z}{\supset} \neg\neg B \quad \overset{x}{[A]}}{\neg\neg B} \qquad \dfrac{\dfrac{\overset{y'}{\neg B} \quad \overset{y}{[B]}}{\Psi}}{\neg B}\,y}{\Psi} \quad \longrightarrow \quad \dfrac{\dfrac{A \overset{z}{\supset} \neg\neg B \quad \overset{x}{A}}{\neg\neg B} \qquad \dfrac{\dfrac{\overset{y'}{\neg B} \quad \overset{y}{[B]}}{\Psi}}{\neg B}\,y}{\Psi}$$

□

A similar situation occurs for negation elimination. However, if a normal *C3*-derivation does not contain any implication or negation elimination, then it translates to a normal *NJ*-derivation, as can be shown by induction on the height of the *C3*-derivation in question.

The translation fails to be injective with respect to *NJ*-equality for two reasons. First, the translation of the sign of falsity fails to distinguish between abstraction and ¬-introduction, e.g. any two *C3*-derivations

$$\dfrac{\begin{matrix}\overset{x}{[\mathrm{T}\,A]}\\ M\\ \psi\end{matrix}}{\mathrm{F}\,A}\,x \quad \text{and} \quad \dfrac{\begin{matrix}\overset{x}{[\mathrm{T}\,A]}\\ M\\ \psi\end{matrix}}{\mathrm{T}\,\neg A}\,x$$

translate to the same *NJ*-derivation. However this can be circumvented by restricting attention to derivations of the same conclusion from the same assumptions, see Proposition 4.3.4. The second reason for failure is of a completely different nature.

	Introduction	**Elimination**
$\supset$	$\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}A, \overset{y'}{\mathrm{F}}B]\\ \psi\end{array}}{\mathrm{T}\,A \supset B}\,x, y'$	$\dfrac{\mathrm{T}\,A \supset B \quad \mathrm{T}\,A \quad \begin{array}{c}[\overset{y}{\mathrm{T}}B]\\ \psi\end{array}}{\psi}\,y$
$\neg$	$\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}A]\\ \psi\end{array}}{\mathrm{T}\,\neg A}\,x$	$\dfrac{\mathrm{T}\,\neg A \quad \mathrm{T}\,A}{\psi}$

Table 4.3: Implication and negation rules of the Calculus *C3S*.

Proposition 4.2.3. *There exist different normal C3-derivations of the same conclusion from the same assumptions that translate to the same NJ-derivation.*

Proof. Any two normal *C3*-derivations

$$\dfrac{\overset{z}{\mathrm{T}}A \supset B \quad \begin{array}{c}\dfrac{[\overset{x'}{\mathrm{F}}A] \quad \begin{array}{c}N_1\\ \mathrm{T}\,A\end{array}}{\psi}\\ N_0\\ \psi\end{array} \quad \begin{array}{c}[\overset{y}{\mathrm{T}}B]\\ K\\ \psi\end{array}}{\psi}\,x', y \qquad \text{and} \qquad \begin{array}{c}\dfrac{\overset{z}{\mathrm{T}}A \supset B \quad \dfrac{[\overset{x'}{\mathrm{F}}A] \quad \begin{array}{c}N_1\\ \mathrm{T}\,A\end{array}}{\psi} \quad \begin{array}{c}[\overset{y}{\mathrm{T}}B]\\ K\\ \psi\end{array}}{\psi}\,x', y\\ N_0\\ \psi\end{array}$$

translate to the same *NJ*-derivation provided that the variable x' does not appear free in N_0. Furthermore, if N_0 ends by something else than an implication elimination, then the two *C3*-derivations are different as well. □

A similar situation occurs for negation elimination.

4.3 Calculus *C3S*

Consider von Plato's general elimination rules for intuitionistic natural deduction and the way they specialize to the elimination rules of *NJ*. We can in a similar manner specialize the implication and negation elimination rules of *C3* to a new kind of elimination rules. This yields the calculus *C3S*, which has the same inference rules as *C3* except for implication and negation. The latter rules can be found in Table 4.3, p. 62. We take the concept of a *C3S*-derivation for granted.

The implication and negation elimination rules of *C3S* are formal instances of the general elimination rules due to Schroeder-Heister (1984) and von Plato (2001) in the same way as the other elimination rules of *C3* and *C3S*.

The calculus *C3S* has the same good normalization properties as *C3*. In particular, the existence and uniqueness of *C3S*-normal forms. The corresponding contraction rules for implication and negation are as follows.

⊃-contraction:

$$\dfrac{\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}\,A, \overset{y'}{\mathrm{F}}\,B]\\ M\\ \psi\end{array}}{\mathrm{T}\,A \supset B}\;x, y' \qquad \begin{array}{c}N\\ \mathrm{T}\,A\end{array} \qquad \begin{array}{c}[\overset{y}{\mathrm{T}}\,B]\\ K\\ \psi\end{array}}{\psi}\;x', y \quad \rightarrow \quad \begin{array}{c} \\ \\ \overset{x}{\mathrm{T}}\,A,\end{array}\!\!\begin{array}{c}[\overset{y}{\mathrm{T}}\,B]\\ K\\ \dfrac{\psi}{\mathrm{F}\,B}\;y\end{array} \;\begin{array}{c}M\\ \psi\end{array}$$

¬-contraction:

$$\dfrac{\dfrac{\begin{array}{c}[\overset{x}{\mathrm{T}}\,A]\\ M\\ \psi\end{array}}{\mathrm{T}\,\neg A}\;x \qquad \overset{x}{\mathrm{T}}\,A}{\psi}\;x' \quad \rightarrow \quad \begin{array}{c}N\\ \mathrm{T}\,A\\ M\\ \psi\end{array}$$

The translation of *C3* into *NJ* factorizes over *C3S* into the following two translations in a way that serves to isolate the two reasons, mentioned in Section 4.2, as to why the original translation failed to be injective with respect to *NJ*-equality.

Translation of *C3* into *C3S*

The translation of *C3* into *C3S* only affects implication and negation eliminations, which translate in the following way.

⊃-elimination

$$\dfrac{\begin{array}{c}M\\ \mathrm{T}\,A \supset B\end{array} \qquad \begin{array}{c}[\overset{x'}{\mathrm{F}}\,A]\\ N\\ \psi\end{array} \qquad \begin{array}{c}[\overset{y}{\mathrm{T}}\,B]\\ K\\ \psi\end{array}}{\psi}\;x', y \quad \mapsto \quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}M\\ \mathrm{T}\,A \supset B\end{array} \qquad [\overset{x}{\mathrm{T}}\,A] \qquad \begin{array}{c}[\overset{y}{\mathrm{T}}\,B]\\ K\\ \psi\end{array}}{\psi}\;x', y}{\mathrm{F}\,A}\;x\\ N\\ \psi\end{array}$$

¬-elimination

$$\dfrac{\begin{array}{c}M\\ \mathrm{T}\,\neg A\end{array} \qquad \begin{array}{c}[\overset{x'}{\mathrm{F}}\,A]\\ N\\ \psi\end{array}}{\psi}\;x' \quad \mapsto \quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}M\\ \mathrm{T}\,\neg A\end{array} \qquad [\overset{x}{\mathrm{T}}\,A]}{\psi}}{\mathrm{F}\,A}\;x\\ N\\ \psi\end{array}$$

The translation respects *C3*-equality; the proof is similar to that of Proposition 4.2.1. The translation fails to be injective with respect to *C3S*-equality in the way of Proposition 4.2.3.

Translation of *C3S* into *NJ*

The translation of *C3S* into *NJ* coincides with the translation of *C3* into *NJ* except for implication and negation eliminations, which translate in the following way.

$\supset$-elimination

$$\dfrac{\begin{array}{c} M \\ \mathrm{T}\,A \supset B \end{array} \quad \begin{array}{c} N \\ \mathrm{T}\,A \end{array} \quad \begin{array}{c} \overset{y}{[\mathrm{T}\,B]} \\ K \\ \psi \end{array}}{\psi}\,y \quad \mapsto \quad \dfrac{\dfrac{\begin{array}{c} M \\ A \supset \neg\neg B \end{array} \quad \begin{array}{c} N \\ A \end{array}}{\neg\neg B} \quad \dfrac{\begin{array}{c} \overset{y}{[B]} \\ K \\ \Psi \end{array}}{\neg B}\,y}{\Psi}$$

$\neg$-elimination

$$\dfrac{\begin{array}{c} M \\ \mathrm{T}\,\neg A \end{array} \quad \begin{array}{c} N \\ \mathrm{T}\,A \end{array}}{\psi} \quad \mapsto \quad \dfrac{\begin{array}{c} M \\ \neg A \end{array} \quad \begin{array}{c} N \\ A \end{array}}{\Psi}$$

The translation respects *C3S*-equality; the proof is similar to that of Proposition 4.2.1.

Now, let the brackets $[\![...]\!]$ denote the translation of *C3S* into *NJ*. We can then prove that the translation also respects normal forms, contrary to the translation of *C3* into *NJ*.

Lemma 4.3.1. *For every C3S-derivation N, if N is normal then $[\![N]\!]$ is normal.*

Proof. Suppose that N cannot be further reduced. Then N is also normal in the sense of von Plato (2001), that is, all main premises of applications and eliminations are assumptions. Induction on the height of N then yields that $[\![N]\!]$ cannot be further reduced. □

Proposition 4.3.2. *For every C3S-derivation M,*

$$[\![\textit{C3S}\text{-nf}(M)]\!] \equiv \textit{NJ}\text{-nf}([\![M]\!]).$$

Proof. Given the existence of *C3S*-normal forms, $M \twoheadrightarrow \textit{C3S}\text{-nf}(M)$ and so $[\![M]\!] \twoheadrightarrow [\![\textit{C3S}\text{-nf}(M)]\!]$ by Proposition 4.2.1. Furthermore, the existence of *NJ*-normal forms gives $[\![M]\!] \twoheadrightarrow \textit{NJ}\text{-nf}([\![M]\!])$. Then the result follows by Lemma 4.3.1 and the uniqueness of *NJ*-normal forms. □

The translation fails to be injective with respect to *NJ*-equality in the first way mentioned in Section 4.2. This obstruction can be circumvented in the way also mentioned in Section 4.2 by means of the following lemma. The result is summed up in Proposition 4.3.4.

Lemma 4.3.3. *For all normal C3S-derivations M and M' of the same conclusion from the same assumptions, if $[\![M]\!] \equiv [\![M']\!]$ then $M \equiv M'$.*

Proof. By induction on the height of $\llbracket M \rrbracket$. The base case is trivial. The proof proceeds by case analysis of the last steps of M and M'.

Suppose that $M \equiv$

$$\frac{\overset{z}{\mathrm{T}\,A \supset B} \quad \begin{matrix} N \\ \mathrm{T}\,A \end{matrix} \quad \begin{matrix} [\overset{y}{\mathrm{T}\,B}] \\ K \\ \psi \end{matrix}}{\psi}\; x', y$$

Then $\llbracket M \rrbracket, \llbracket M' \rrbracket \equiv$

$$\frac{\dfrac{\overset{z}{A \supset \neg\neg B} \quad \begin{matrix} \llbracket N \rrbracket \\ A \end{matrix}}{\neg\neg B} \quad \dfrac{\begin{matrix} [\overset{y}{B}] \\ \llbracket K \rrbracket \\ \Psi \end{matrix}}{\neg B}\; y}{\Psi}$$

It then follows that M' must end by a $\supset$-elimination, say with side premise derivations N' and K'. Identification of the different parts of $\llbracket M \rrbracket$ and $\llbracket M' \rrbracket$ yields that $\llbracket N \rrbracket \equiv \llbracket N' \rrbracket$ and $\llbracket K \rrbracket \equiv \llbracket K' \rrbracket$. Then $N \equiv N'$ and $K \equiv K'$ by the induction hypothesis, and so $M \equiv M'$.

The other cases are similar. □

Proposition 4.3.4. *For all C3S-derivations M and N of the same conclusion from the same assumptions, if $\llbracket M \rrbracket = \llbracket N \rrbracket$ then $M = N$.*

Proof. By Proposition 4.3.2 and Lemma 4.3.3. □

4.4 The π-contraction relation

The proof of Proposition 4.2.3 contains a kind of permutation rule for *C3*. This permutation rule can be broken down into elementary parts, though of a different nature than the permutative rules of intuitionistic natural deduction. There is one set of such permutative rules for implication and another set for negation, and together they generate a permutative contraction relation for *C3*. We call this relation *C3*-π-contraction for short or just π-contraction when the calculus is clear from the context. The disambiguation will be used later in Section 5.3.

The π-equality relation coincides with the kernel of the translation of *C3* into *C3S*. This is captured by Propositions 4.4.6 and 4.4.7 and should be compared to the result, due to Dyckhoff and Pinto (1999), that two derivations in a cut-free sequent calculus for intuitionistic propositional logic are inter-permutable if and only if they determine the same natural deduction.

We only write down the rules for moving an $\supset$-elimination past an abstraction, an application, a &-introduction, or a $\&_1$-elimination. The rules for moving an $\supset$-elimination past any other inference can be defined similarly. For a compact notation, we use the shorthand

$$\underline{N} \equiv \supset E(M, [x']N, [y]K),$$

taking for granted that x' does not appear bound in N.

$\supset E$-permutations:

$$\underline{u'(u)} \to u'(u) \quad \text{provided } x' \not\equiv u', \tag{4.1}$$

$$\underline{([u]L)(u)} \to ([u]\underline{L})(u), \tag{4.2}$$

$$\underline{\&E_1(w, [u]L)} \to \&E_1(w, [u]\underline{L}), \tag{4.3}$$

$$\underline{\&E_1(\&I([u']N_1, [v']N_2), [u]L)} \to \&E_1(\&I([u']\underline{N_1}, [v']\underline{N_2}), [u]\underline{L}), \tag{4.4}$$

$$\underline{w'(\&I([u']N_1, [v']N_2))} \to w'(\&I([u']\underline{N_1}, [v']\underline{N_2})) \quad \text{provided } x' \not\equiv w', \tag{4.5}$$

$$\underline{([u]L)(\&I([u']N_1, [v']N_2))} \to ([u]\underline{L})(\&I([u']\underline{N_1}, [v']\underline{N_2})), \tag{4.6}$$

and

$$\underline{x'(\&I([u']N_1, [v']N_2))} \to \supset E(M, [x']x'(\&I([u']\underline{N_1}, [v']\underline{N_2})), [y]K) \tag{4.7}$$

where in (4.7) x' must appear free in $[u']N_1$ or $[v']N_2$. The latter condition helps to ensure the existence of π-normal forms. Without it (4.7) could be applied indefinitely.

Schwichtenberg (1999) proved that the notion of permutative reduction involved in the result due to Dyckhoff and Pinto (1999) is strongly normalizing. Unfortunately, our notion of π-reduction is not strongly normalizing. To see this, just consider how the rule for moving an $\supset$-elimination past another $\supset$-elimination, analogous to (4.3), acts on the *C3*-term

$$\supset E(z, [x_1']\supset E(z, [x_2']N, [y]K), [y]K).$$

The permutation rules for negation are defined similarly.

The following two lemmas will be used to prove Proposition 4.4.3 to the content that the π-contraction relation is weakly normalizing.

Lemma 4.4.1. *Every C3-derivation P, that ends by an $\supset$-elimination whose premise derivations are π-normal, π-reduces to a π-normal derivation.*

Proof. By induction on the height of the side premise derivation N of $P \equiv \underline{N}$. The proof proceeds by case analysis of the last steps of N. If no π-rule applies to P, then let $\pi\text{-nf}(P) \equiv P$. Otherwise, apply the π-rule in question and then apply the induction hypothesis to the new underlined terms. □

Lemma 4.4.2. *Every C3-derivation P, that ends by a $\neg$-elimination whose premise derivations are π-normal, π-reduces to a π-normal derivation.*

Proof. Similar to the proof of Lemma 4.4.2. □

Proposition 4.4.3. *Every C3-derivation π-reduces to a π-normal derivation.*

Proof. By Lemma 4.4.2, Lemma 4.4.3, and induction on the number of permutative redexes in M. □

A π-normal derivation has the following structure.

Lemma 4.4.4. *A C3-derivation is π-normal if and only if every $\supset$-elimination and every $\neg$-elimination has the form*

$$\frac{\begin{array}{c}M\\ \mathrm{T}\,A\supset B\end{array} \qquad \dfrac{\stackrel{x'}{[\mathrm{F}\,A]} \quad \stackrel{N}{\mathrm{T}\,A}}{\psi} \qquad \begin{array}{c}\stackrel{y}{[\mathrm{T}\,B]}\\ K\\ \psi\end{array}}{\psi}\; x', y \qquad \textit{resp.} \qquad \frac{\begin{array}{c}M\\ \mathrm{T}\,\neg A\end{array} \qquad \dfrac{\stackrel{x'}{[\mathrm{F}\,A]} \quad \stackrel{N}{\mathrm{T}\,A}}{\psi}}{\psi}\; x'$$

where x' does not appear free in N.

Proof. It is easy to see that if a *C3*-derivation has the above-mentioned form then it is π-normal. The other direction requires a case analysis to exclude any other form than the one above.

Consider any π-normal term $\supset E(M, [x']X, K)$. The last step of X can not be an elimination, because then (4.3), (4.4), or their like would apply, and so the last step of X must be an application, $X \equiv L(N)$. The derivation L can not be an abstraction, because then (4.2), (4.6), or their like would apply, and so L must be an assumption. Then $L \equiv x'$ because otherwise (4.1), (4.5), or their like would apply. Then the variable x' can not appear free in N because otherwise (4.7), or its like would apply.

The case of $\neg$-elimination is similar. □

The proof of Lemma 4.4.4 is in fact constructive despite all the negative statements.

It is now easy to see that there is a one-to-one correspondence between π-normal *C3*-derivations and *C3S*-derivations. In particular, we have the following corollary.

Corollary 4.4.5. *For all π-normal C3-derivations L and M, if $[\![L]\!] \equiv [\![M]\!]$ then $L \equiv M$.*

Proof. By Lemma 4.4.4 and induction on the height of $[\![L]\!]$. □

We are now ready to prove that the π-equality relation coincides with the kernel of the translation of *C3* into *C3S*.

Proposition 4.4.6. *For all C3-derivations L and M, if $L =_\pi M$ then $[\![L]\!] \equiv [\![M]\!]$.*

Proof. By inspection of the translation of the left- and right-hand sides of the permutation rules. □

Proposition 4.4.7. *For all C3-derivations L and M, if $[\![L]\!] \equiv [\![M]\!]$ then $L =_\pi M$.*

Proof. Suppose that $[\![L]\!] \equiv [\![M]\!]$. Then $[\![\pi\text{-nf}(L)]\!] \equiv [\![\pi\text{-nf}(M)]\!]$ by Proposition 4.4.3 and Proposition 4.4.6 and so $\pi\text{-nf}(L) \equiv \pi\text{-nf}(M)$ by Corollary 4.4.5. Hence $L =_\pi M$ by the definition of π-equality. □

Chapter 5

Contraction rules for *C2*

In this chapter, we formalize with full precision the contraction relation for *C2*, compatible with the interpretation of *C2* in constructive type theory with explicit substitution, where cuts are represented by explicit substitutions. The contraction relation is expressed as a term rewriting system using a term notation closely related to the one introduced by Urban in his thesis (Urban 2000) and related papers (Urban and Bierman 1999, Urban 2001).

The *C2*-contraction relation shares many of the good normalization properties of the *C3*-contraction relation. In particular, it is weakly normalizing and Church-Rosser. There are also reasons to expect it to be strongly normalizing as well as it resembles a special case of the strongly normalizing contraction relations of Danos et al. (1997), Urban and Bierman (1999), and Urban (2000, 2001). However, we do not prove that the *C2*-contraction relation is strongly normalizing, but only discuss the reasons why this is to be expected.

It is straightforward to pull back the *C2*-contraction relation to *C1*. However, the process is somewhat unwieldy and is left out.

5.1 Preliminary remarks

We use the formalization of explicit substitution in Tasistro (1997) but with some minor adjustments to the notation. In particular, the result of the explicit substitution of a syntactic objects K for the free variable x in the syntactic object M is written $M\,\langle K/x\rangle$. The notation for simultaneous explicit substitution is similar. However, we will only make use of unary explicit substitution.

As previously mentioned in Section 3.1, the substitutions involved in Lemmas 3.2.1–3.2.6 are equally valid under the rules of explicit substitution in Tasistro (1997). Hence these lemmas also hold for explicit substitutions. This will be used to decide the correct contraction rules for so-called *critical cuts*, cuts to be replaced by other cuts with simpler cut formulas, see Section 5.2.

In his thesis (Urban 2000) and related papers, Urban introduced a term notation sufficient to fully encode the structure of sequent proofs, which was then used to define a complete cut-elimination procedure as a term rewriting system and, among other things, to prove strong normalization. We shall introduce an almost identical term notation for *C2*-derivations. The term notation is one-to-one between *C2*-derivations and *C2*-terms similarly to Definition 3.1.1.

Definition 5.1.1. A *C2*-term is built up from variables and function constants (i.e. $\&\,\mathrm{T}_1$, $\&\,\mathrm{T}_2$, $\&\,\mathrm{F}$, $\vee\,\mathrm{T}$, $\vee\,\mathrm{F}_1$, $\vee\,\mathrm{F}_2$, $\supset\mathrm{T}$, $\supset\mathrm{F}$, $\neg\,\mathrm{T}$, $\neg\,\mathrm{F}$, $\forall\,\mathrm{T}$, $\forall\,\mathrm{F}$, $\exists\,\mathrm{T}$, $\exists\,\mathrm{F}$) by means of abstraction, application, and the two constants Ax and Cut. The syntax and semantics is implicit in Table 5.1, p. 71, taken together with the interpretation of *C3* in constructive type theory with explicit substitution.

The two constants Ax and Cut increase readability but are not strictly necessary as they could be replaced by expressions built up from abstraction, application, and explicit substitution.

We use the same symbol (i.e. K, L, M, N, etc.) for an indeterminate *C2*-derivation and the corresponding *C2*-term. For example, the *C2*-derivation

$$\dfrac{\dfrac{\overset{z'}{[\mathrm{F}\,A\&B]} \quad \begin{matrix}\overset{x'}{[\mathrm{F}\,A]}\\ M_1\\ \psi\end{matrix} \quad \begin{matrix}\overset{y'}{[\mathrm{F}\,B]}\\ M_2\\ \psi\end{matrix}}{\psi}\,x',y' \qquad \dfrac{\overset{z}{[\mathrm{T}\,A\&B]} \quad \begin{matrix}\overset{x}{[\mathrm{T}\,A]}\\ K\\ \psi\end{matrix}}{\psi}\,{}_1x}{\psi}\,z,z'$$

corresponds to the term $\mathrm{Cut}([z']\&\,\mathrm{F}(z', [x']M_1, [y']M_2), [z]\&\,\mathrm{T}_1(z, [x]K))$.

Definition 5.1.2. We say that a *C2*-derivation M *freshly introduces* a variable x provided the last step of M introduces x but no other step of M introduces x.

The same kind of terminology works for *C2*-terms. For example, $\mathrm{Ax}(x', x)$ freshly introduces both x' and x, $\&\,\mathrm{T}(z, [x]K)$ freshly introduces z provided that z does not appear free in K, and $\&\,\mathrm{F}(z', [x']M_1, [y]M_2)$ freshly introduces z' provided that z' does not appear free in M_1 or M_2.

5.2 The *C2*-contraction relation

The *C2*-contraction relation is made up of 53 contraction rules. These are roughly of two kinds: permutative rules (permuting cuts relative to logical inferences) and critical rules (replacing cuts by other cuts with simpler cut formulas). The contraction rules for axiom take up a position in between. We have structured the three axiom-contraction rules in a way similar to those of the contraction rules for the logical constants, suggesting that the first (5.1) and second (5.2) axiom-contraction rules be considered permutative rules while the third (5.3) axiom-contraction rule be considered a critical rule.

The permutative rules for *C2* are unrelated to the permutative rules for *C3* in Section 4.4. However, the latter can be translated to *C2* where they induce, what

$$
\begin{aligned}
[\![\& \,\mathrm{T}_1(z,[x]K)]\!] &\equiv \&E_1(z,[x][\![K]\!])\\
[\![\& \,\mathrm{T}_2(z,[y]K)]\!] &\equiv \&E_2(z,[y][\![K]\!])\\
[\![\& \,\mathrm{F}(z',[x']M_1,[y']M_2)]\!] &\equiv z'(\&I([x'][\![M_1]\!],[y'][\![M_2]\!]))\\
[\![\vee \mathrm{T}(z,[x]L_1,[y]L_2)]\!] &\equiv \vee E(z,[x][\![L_1]\!],[y][\![L_2]\!])\\
[\![\vee \mathrm{F}_1(z',[x']N)]\!] &\equiv z'(\vee I_1([x'][\![N]\!]))\\
[\![\vee \mathrm{F}_2(z',[y']N)]\!] &\equiv z'(\vee I_2([y'][\![N]\!]))\\
[\![\supset \mathrm{T}(z,[x']N,[y]K)]\!] &\equiv \supset E(z,[x'][\![N]\!],[y][\![K]\!])\\
[\![\supset \mathrm{F}(z',[x,y']M)]\!] &\equiv z'(\supset I([x,y'][\![M]\!]))\\
[\![\neg \mathrm{T}(z,[x']N)]\!] &\equiv \neg E(z,[x'][\![N]\!])\\
[\![\neg \mathrm{F}(z',[x]M)]\!] &\equiv z'(\neg I([x][\![M]\!]))\\
[\![\forall \mathrm{T}(z,t,[x]K)]\!] &\equiv \forall E(z,t,[x][\![K]\!])\\
[\![\forall \mathrm{F}(z',[x']M)]\!] &\equiv z'(\forall I([x'][\![M]\!]))\\
[\![\exists \mathrm{T}(z,[v,x]L)]\!] &\equiv \exists E(z,[v,x][\![L]\!])\\
[\![\exists \mathrm{F}(z',[x']N)]\!] &\equiv z'(\forall I([x'][\![N]\!]))\\
[\![\mathrm{Ax}(K,M)]\!] &\equiv [\![K]\!]([\![M]\!])\\
[\![\mathrm{Cut}([x']M,[x]K)]\!] &\equiv [\![M]\!]\left\langle [x][\![K]\!]/x' \right\rangle
\end{aligned}
$$

Table 5.1: Interpretation of *C2* in constructive type theory with explicit substitution. The terms for *C3*-derivations occurring to the right are here understood as standing for the corresponding type-theoretic terms with explicit substitutions.

is to the author's knowledge, a novel set of contraction rules for classical sequent calculi. We consider this topic in Section 5.3.

We focus on the contraction rules for axiom and conjunction. The line of argument leading up to these rules transfers to the other logical constants and makes it a routine matter to write down the other contraction rules.

We shall take for granted that the following condition is met: To prevent the variables x' and x bound in a cut redex $\mathrm{Cut}([x']M,[x]K)$ from becoming mixed up with other variables in the corresponding contractum, the variables bound by the cut must not appear bound in the respective premise derivations, that is, x' must not appear bound in M and x must not appear bound in K.

Axiom

The following contractions cover cuts interacting with axioms. In (5.2), the term P freshly introduces the variable x'.

Axiom-contractions:

$$\text{Cut}([x']\,\text{Ax}(u',u),[x]K) \quad \text{where } x' \not\equiv u' \\ \to \text{Ax}(u',u), \tag{5.1}$$

$$\text{Cut}([x']P,[x]\,\text{Ax}(u',u)) \quad \text{where } x \not\equiv u \\ \to \text{Ax}(u',u), \tag{5.2}$$

and

$$\text{Cut}([x']\,\text{Ax}(x',u),[x]\,\text{Ax}(u',x)) \\ \to \text{Ax}(u',u). \tag{5.3}$$

Lemma 5.2.1. *The axiom-contraction rules for C2 commute up to equality with the interpretation of C2 in constructive type theory with explicit substitution.*

Proof. The left-hand sides of (5.1), (5.2), and (5.3) translate to terms that reduce to $[\![\text{Ax}(u',u)]\!]$. Note that, in the first two cases, a term is substituted for a variable, in another term which does not depend on that variable. □

Conjunction

The following contractions cover cuts interacting with $\&\,\text{T}_1$-inferences and $\&\,\text{F}$-inferences. The contractions for cuts interacting with $\&\,\text{T}_2$-inferences are defined similarly. In (5.7–5.9), the term P freshly introduces the variable z'.

&-contractions:

$$\text{Cut}([z']\&\,\text{T}_1(w,[x]K),[z]L) \\ \to \&\,\text{T}_1(w,[x]\,\text{Cut}([z']K,[z]L)), \tag{5.4}$$

$$\text{Cut}([z']\&\,\text{F}(w',[x']M_1,[y']M_2),[z]L) \quad \text{where } z' \not\equiv w' \\ \to \&\,\text{F}(w',[x']\,\text{Cut}([z']M_1,[z]L),[y']\,\text{Cut}([z']M_2,[z]L)), \tag{5.5}$$

$$\text{Cut}([z']\&\,\text{F}(z',[x']M_1,[y']M_2),[z]L) \\ \to \text{Cut}([z']\&\,\text{F}(z',[x']\,\text{Cut}([z']M_1,[z]L), \\ [y']\,\text{Cut}([z']M_2,[z]L)),[z]L), \tag{5.6}$$

$$\text{Cut}([z']P,[z]\&\,\text{F}(w',[x']M_1,[y']M_2)) \\ \to \&\,\text{F}(w',[x']\,\text{Cut}([z']P,[z]M_1),[y']\,\text{Cut}([z']P,[z]M_2)), \tag{5.7}$$

$$\text{Cut}([z']P,[z]\&\,\text{T}_1(w,[x]K)) \quad \text{where } z \not\equiv w \\ \to \&\,\text{T}_1(w,[x]\,\text{Cut}([z']P,[z]K)), \tag{5.8}$$

$$\text{Cut}([z']P,[z]\&\,\text{T}_1(z,[x]K)) \\ \to \text{Cut}([z']P,[z]\&\,\text{T}(z,[x]\,\text{Cut}([z']P,[z]K))), \tag{5.9}$$

and

$$\text{(5.10)} \qquad \begin{aligned} &\mathrm{Cut}([z']\&\,\mathrm{F}(z',[x']M_1,[y']M_2),[z]\&\,\mathrm{T}_1(z,[x]K)) \\ &\to \mathrm{Cut}([x']M_1,[x]K) \end{aligned}$$

where z' does not appear free in $[x']M_1$ or $[y']M_2$ and z does not appear free in $[x]K$. We insist however on z' appearing free in $[x']M_1$ or $[y']M_2$ in (5.6) and z appearing free in $[x]K$ in (5.9). The latter conditions help to ensure the existence of *C2*-normal forms. Without them (5.6) and (5.9) could be applied indefinitely.

Lemma 5.2.2. *The* $\&$*-contraction rules for C2 commute up to equality with the interpretation of C2 in constructive type theory with explicit substitution.*

Proof. The translations of the left-hand sides of (5.4), (5.5), and (5.10) reduce to the translations of the corresponding right-hand sides. For (5.10), the reduction agrees with that in the proof of Lemma 3.2.1. Critical cuts for other logical constants behaves similarly.

The translations of the left- and right-hand sides of (5.6) reduce to the same term modulo the equalities $[\![M_i]\!]\,\langle[z]L/z'\rangle\,\langle[z]L/z'\rangle = [\![M_i]\!]\,\langle[z]L/z'\rangle\ (i=1,2)$. The translations of the left- and right-hand sides of (5.7) have a common reduct. The same applies to (5.8). The translations of the left- and right-hand sides of (5.9) reduce to the same term modulo the equality $[\![K]\!]\,\langle Q/z\rangle\,\langle Q/z\rangle = [\![K]\!]\,\langle Q/z\rangle$.

The cases for cuts interacting with $\&\,\mathrm{T}_2$-inferences are similar. □

Next, we compare the contraction rules just introduced with Gentzen's rules of cut elimination.

A comparison with Gentzen's cut-elimination procedure

We consider how our rules compare with Gentzen's rules of cut elimination. The latter are implicit in his Hauptsatz (Gentzen 1935). We make no attempt at completeness, but only consider how (5.6), (5.9), and (5.10) act on the derivation

$$\dfrac{\dfrac{\begin{matrix}M_1\\ \mathrm{F}\,\overset{x'}{A},\Gamma\end{matrix}\quad\begin{matrix}M_2\\ \mathrm{F}\,\overset{y'}{B},\Gamma\end{matrix}}{\mathrm{F}\,\overset{z'}{A\&B},\Gamma}\,x',y';z' \qquad \dfrac{\begin{matrix}K\\ \mathrm{T}\,\overset{x}{A},\Delta\end{matrix}}{\mathrm{T}\,\overset{z}{A\&B},\Delta}\,{}_1x;z}{\Gamma,\Delta}\,z',z;$$

and mention how Gentzen's rules act on a corresponding derivation in *LK*.

Focusing on the contraction rules just introduced, there are four cases to consider depending on whether $z' : \mathrm{F}\,A\&B$ appears in Γ and whether $z : \mathrm{T}\,A\&B$ appears in Δ or, what amounts to the same thing, whether z' appears free in M_1 or M_2 and whether z appears in K in the term

$$\mathrm{Cut}([z']\&\,\mathrm{F}(z',[x']M_1,[y']M_2]),[z]\&\,\mathrm{T}_1(z,[x]K)).$$

Case 1. Suppose that z' does not appear free in M_1 or M_2 and z does not appear free in K. Then (5.10) gives the term

$$\mathrm{Cut}([x']M_1, [x]K).$$

Case 2. Suppose that z' appears free in M_1 or M_2 but z does not appear free in K. Then (5.6) and (5.10) give the term

$$\mathrm{Cut}([x']\,\mathrm{Cut}([z']M_1, [z]\&\,\mathrm{T}_1(z, K)), [x]K).$$

Case 3. Suppose that z' does not appear free in M_1 or M_2 but z appears free in K. Then (5.9) and (5.10) give the term

$$\mathrm{Cut}([x']M_1, [x]\,\mathrm{Cut}([z']\&\,\mathrm{F}(z', M_1, M_2), [z]K)).$$

Case 4. Suppose that z' appears free in M_1 or M_2 and z appears free in K. Then (5.6), (5.9), and (5.10) give the term

$$\mathrm{Cut}([x']\,\mathrm{Cut}([z']M_1, [z]\&\,\mathrm{T}_1(z, [x]K)), [x]\,\mathrm{Cut}([z']N, [z]K)),$$

where $N \equiv$

$$\&\,\mathrm{F}(z', [x']\,\mathrm{Cut}([z']M_1, [z]\&\,\mathrm{T}_1(z, K)), [y']\,\mathrm{Cut}([z']M_2, [z]\&\,\mathrm{T}_1(z, K))).$$

The first three cases yield results comparable to the ones obtained from Gentzen's rules; only the fourth case yields a different result. In the fourth case, Gentzen's rules produce an *LK*-derivation comparable to the *C2*-term

$$\mathrm{Cut}([x']\,\mathrm{Cut}([z']M_1, [z]\&\,\mathrm{T}_1(z, [x]K)), [x]\,\mathrm{Cut}([z']L, [z]K)),$$

where $L \equiv \&\,\mathrm{F}(z', [x']M_1, [y']M_2])$. Hence, although Gentzen's rules are valid in the majority of cases, the semantics sometimes requires that the cuts are removed in a different way. The contraction rules for the other logical constants exhibit the same pattern.

Disjunction

The following contractions cover cuts interacting with $\vee\,\mathrm{T}$-inferences and $\vee\,\mathrm{F}_1$-inferences. The contractions for cuts interacting with $\vee\,\mathrm{F}_2$-inferences are defined similarly. As before, P freshly introduces z'.

$\vee$-contractions:

(5.11) $\mathrm{Cut}([z']\vee\mathrm{T}(w,[x]L_1,[y]L_2),[z]K)$
$\to \vee\mathrm{T}(w,[x]\,\mathrm{Cut}([z']L_1,[z]K),[y]\,\mathrm{Cut}([z']L_2,[z]K)),$

(5.12) $\mathrm{Cut}([z']\vee\mathrm{F}_1(w',[x']N),[z]K)$ where $z'\not\equiv w'$
$\to \vee\mathrm{F}_1(w',[x']\,\mathrm{Cut}([z']N,[z]K)),$

(5.13) $\mathrm{Cut}([z']\vee\mathrm{F}_1(z',[x']N),[z]K)$
$\to \mathrm{Cut}([z']\vee\mathrm{F}_1(z',[x']\,\mathrm{Cut}([z']N,[z]K)),[z]K),$

(5.14) $\mathrm{Cut}([z']P,[z]\vee\mathrm{F}_1(w',[x']N))$
$\to \vee\mathrm{F}_1(w',[x']\,\mathrm{Cut}([z']P,[z]N)),$

(5.15) $\mathrm{Cut}([z']P,[z]\vee\mathrm{T}(w,[x]L_1,[y]L_2))$ where $z\not\equiv w$
$\to \vee\mathrm{T}(w,[x]\,\mathrm{Cut}([z']P,[z]L_1),[y]\,\mathrm{Cut}([z']P,[z]L_2)),$

(5.16) $\mathrm{Cut}([z']P,[z]\vee\mathrm{T}(z,[x]L_1,[y]L_2))$
$\to \mathrm{Cut}([z']P,[z]\vee\mathrm{T}(z,[x]\,\mathrm{Cut}([z']P,[z]L_1),$
$[y]\,\mathrm{Cut}([z']P,[z]L_2))),$

and

(5.17) $\mathrm{Cut}([z']\vee\mathrm{F}_1(z',[x']N),[z]\vee\mathrm{T}(z',[x]L_1,[y]L_2))$
$\to \mathrm{Cut}([x']N,[x]L_1)$

where z' does not appear free in $[x']N$ and z does not appear free in $[x]L_1$ or $[y]L_2$. We insist on z' appearing free in $[x']N$ in (5.13) and z appearing free in $[x]L_1$ or $[y]L_2$ in (5.16).

Lemma 5.2.3. *The $\vee$-contraction rules for C2 commute up to equality with the interpretation of C2 in constructive type theory with explicit substitution.*

Proof. Similar to that of Lemma 5.2.2. □

Implication

The following contractions cover cuts interacting with $\supset$T-inferences and $\supset$F-inferences. As before, P freshly introduces z'.

$\supset$-contractions:

(5.18) $$\begin{aligned}&\mathrm{Cut}([z']{\supset}\,\mathrm{T}(w,[x']N,[y]K),[z]L)\\ &\to {\supset}\,\mathrm{T}(w,[x']\,\mathrm{Cut}([z']N,[z]L),[y]\,\mathrm{Cut}([z']K,[z]L))),\end{aligned}$$

(5.19) $$\begin{aligned}&\mathrm{Cut}([z']{\supset}\,\mathrm{F}(w',[x,y']M),[z]L)\quad\text{where } z'\not\equiv w'\\ &\to {\supset}\,\mathrm{F}(w',[x,y']\,\mathrm{Cut}([z']M,[z]L)),\end{aligned}$$

(5.20) $$\begin{aligned}&\mathrm{Cut}([z']{\supset}\,\mathrm{F}(z',[x,y']M),[z]L)\\ &\to \mathrm{Cut}([z']{\supset}\,\mathrm{F}(z',[x,y']\,\mathrm{Cut}([z']M,[z]L)),[z]L),\end{aligned}$$

(5.21) $$\begin{aligned}&\mathrm{Cut}([z']P,[z]{\supset}\,\mathrm{F}(w',[x,y']M))\\ &\to {\supset}\,\mathrm{F}(w',[x,y']\,\mathrm{Cut}([z']P,[z]M)),\end{aligned}$$

(5.22) $$\begin{aligned}&\mathrm{Cut}([z']P,[z]{\supset}\,\mathrm{T}(w,[x']N,[y]K))\quad\text{where } z\not\equiv w\\ &\to {\supset}\,\mathrm{T}(w,[x']\,\mathrm{Cut}([z']P,[z]N),[y]\,\mathrm{Cut}([z']P,[z]K)),\end{aligned}$$

(5.23) $$\begin{aligned}&\mathrm{Cut}([z']P,[z]{\supset}\,\mathrm{T}(w,[x']N,[y]K))\\ &\to \mathrm{Cut}([z']P,[z]{\supset}\,\mathrm{T}(z,[x']\,\mathrm{Cut}([z']{\supset}\,\mathrm{F}(z',[x,y']M),[z]N),\\ &\qquad [y]\,\mathrm{Cut}([z']{\supset}\,\mathrm{F}(z',[x,y']M),[z]K))),\end{aligned}$$

and

(5.24) $$\begin{aligned}&\mathrm{Cut}([z']{\supset}\,\mathrm{F}(z',[x,y']M),[z]{\supset}\,\mathrm{T}(z,[x']N,[y]K))\\ &\to \mathrm{Cut}([x']N,[x]\,\mathrm{Cut}([y']M,[y]K))\end{aligned}$$

where z' does not appear free in $[x,y']M$ and z does not appear free in $[x']N$ or $[y]K$. We insist on z' appearing free in $[x,y']M$ in (5.20) and z appearing free in $[x']N$ or $[y]K$ in (5.23).

Lemma 5.2.4. *The $\supset$-contraction rules for C2 commute up to equality with the interpretation of C2 in constructive type theory with explicit substitution.*

Proof. Similar to that of Lemma 5.2.2. □

There is an alternative to (5.24), namely the contraction rule

$$\begin{aligned}&\mathrm{Cut}([z']{\supset}\,\mathrm{F}(z',[x,y']M),[z]{\supset}\,\mathrm{T}(z,[x']N,[y]K))\\ &\to \mathrm{Cut}([y']\,\mathrm{Cut}([x']N,[x]M),[y]K).\end{aligned}$$

Ignoring the differences caused by explicit substitution makes the translation of the right-hand side of the above contraction rule syntactically equal to the translation of the right-hand sides of (5.24). However, only (5.24) agrees with the reduction in the proof of Lemma 3.2.3.

Negation

The following contractions cover cuts interacting with $\neg\,$T-inferences and $\neg\,$F-inferences. As before, P freshly introduces z'.

$\neg$-contractions:

(5.25) $\mathrm{Cut}([z']\neg\,\mathrm{T}(w,[x']N),[z]L)$
$\to \neg\,\mathrm{T}(w,[x']\,\mathrm{Cut}([z']N,[z]L),$

(5.26) $\mathrm{Cut}([z']\neg\,\mathrm{F}(w',[x]M),[z]L)$ where $z' \not\equiv w'$
$\to \neg\,\mathrm{F}(w',[x]\,\mathrm{Cut}([z']M,[z]L)),$

(5.27) $\mathrm{Cut}([z']\neg\,\mathrm{F}(z',[x]M),[z]L)$
$\to \mathrm{Cut}([z']\neg\,\mathrm{F}(z',[x]\,\mathrm{Cut}([z']M,[z]L)),[z]L),$

(5.28) $\mathrm{Cut}([z']P,[z]\neg\,\mathrm{F}(w',[x]M))$
$\to \neg\,\mathrm{F}(w',[x]\,\mathrm{Cut}([z']P,[z]M)),$

(5.29) $\mathrm{Cut}([z']P,[z]\neg\,\mathrm{T}(w,[x']N))$ where $z \not\equiv w$
$\to \neg\,\mathrm{T}(w,[x']\,\mathrm{Cut}([z']P,[z]N)),$

(5.30) $\mathrm{Cut}([z']P,[z]\neg\,\mathrm{T}(z,[x']N))$
$\to \mathrm{Cut}([z']z'(c),[z]\neg\,\mathrm{T}(z,[x']\,\mathrm{Cut}([z']P,[z]N))),$

and

(5.31) $\mathrm{Cut}([z']\neg\,\mathrm{F}(z',[x]M),[z]\neg\,\mathrm{T}(z,[x']N))$
$\to \mathrm{Cut}([x']N,[x]M)$

where z' does not appear free in $[x]M$ and z does not appear free in $[x']N$. We insist on z' appearing free in $[x]M$ in (5.27) and z appearing free in $[x']N$ in (5.30).

Lemma 5.2.5. *The $\neg$-contraction rules for C2 commute up to equality with the interpretation of C2 in constructive type theory with explicit substitution.*

Proof. Similar to that of Lemma 5.2.2. □

Universal quantification

The following contractions cover cuts interacting with $\forall\,$T-inferences and $\forall\,$F-inferences. As before, P freshly introduces z'.

$\forall$-contractions:

(5.32) $$\mathrm{Cut}([z']\forall\,\mathrm{T}(w,t,[x]K),[z]L)$$ $$\to \forall\,\mathrm{T}(w,t,[x]\,\mathrm{Cut}([z']K,[z]L)),$$

(5.33) $$\mathrm{Cut}([z']\forall\,\mathrm{F}(w'[v,x']M),[z]L) \quad \text{where } z' \not\equiv w'$$ $$\to \forall\,\mathrm{F}(w'[v,x']\,\mathrm{Cut}([z']M,[z]L)),$$

(5.34) $$\mathrm{Cut}([z']\forall\,\mathrm{F}(z'[v,x']M),[z]L)$$ $$\to \mathrm{Cut}([z']\forall\,\mathrm{F}(z',[v,x']\,\mathrm{Cut}([z']M,[z]L)),[z]L),$$

(5.35) $$\mathrm{Cut}([z']P,[z]\forall\,\mathrm{F}(w',[v,x']M))$$ $$\to \forall\,\mathrm{F}(w',[v,x']\,\mathrm{Cut}([z']P,[z]M)),$$

(5.36) $$\mathrm{Cut}([z']P,[z]\forall\,\mathrm{T}(w,t,[x]K)) \quad \text{where } z \not\equiv w$$ $$\to \forall\,\mathrm{T}(w,t,[x]\,\mathrm{Cut}([z']P,[z]K)),$$

(5.37) $$\mathrm{Cut}([z']P,[z]\forall\,\mathrm{T}(z,t,[x]K))$$ $$\to \mathrm{Cut}([z']P,[z]\forall\,\mathrm{T}(z,t,[x]\,\mathrm{Cut}([z']P,[z]K))),$$

and

(5.38) $$\mathrm{Cut}([z']\forall\,\mathrm{F}(z',[v,x']M),[z]\forall\,\mathrm{T}(z,t,[x]K))$$ $$\to \mathrm{Cut}([x']M[t/v],[x]K)$$

where z' does not appear free in $[v,x']M$ and z does not appear free in t or $[x]K$. Note that z can not appear free in t in first-order logic. We insist on z' appearing free in $[v,x']M$ in (5.34) and z appearing free in $[x]K$ in (5.37).

Lemma 5.2.6. *The $\forall$-contraction rules for C2 commute up to equality with the interpretation of C2 in constructive type theory with explicit substitution.*

Proof. Similar to that of Lemma 5.2.2. □

Existential quantification

The following contractions cover cuts interacting with $\exists\,\mathrm{T}$-inferences and $\exists\,\mathrm{F}$-inferences. As before, P freshly introduces z'.

$\exists$-contractions:

(5.39) $\mathrm{Cut}([z']\exists\,\mathrm{T}(w,[v,x]L),[z]K)$
$\to \exists\,\mathrm{T}(w,[v,x]\,\mathrm{Cut}([z']L,[z]K)),$

(5.40) $\mathrm{Cut}([z']\exists\,\mathrm{F}(w',t,[x']N),[z]K)$ where $z' \not\equiv w'$
$\to \exists\,\mathrm{F}(w',t,[x']\,\mathrm{Cut}([z']L,[z]K)),$

(5.41) $\mathrm{Cut}([z']\exists\,\mathrm{F}(z',t,[x']N),[z]K)$
$\to \mathrm{Cut}([z']\exists\,\mathrm{F}(z',t,[x']\,\mathrm{Cut}([z']N,[z]K)),[z]K),$

(5.42) $\mathrm{Cut}([z']P,[z]\exists\,\mathrm{F}(w',t,[x']N))$
$\to \exists\,\mathrm{F}(w',t,[x']\,\mathrm{Cut}([z']P,[z]N)),$

(5.43) $\mathrm{Cut}([z']P,[z]\exists\,\mathrm{T}(w,[v,x]L))$ where $z \not\equiv w$
$\to \exists\,\mathrm{T}(w,[v,x]\,\mathrm{Cut}([z']P,[z]L)),$

(5.44) $\mathrm{Cut}([z']P,[z]\exists\,\mathrm{T}(z,[v,x]L))$
$\to \mathrm{Cut}([z']P,[z]\exists\,\mathrm{T}(z,[v,x]\,\mathrm{Cut}([z']P,[z]L[t/v])))$,

and

(5.45) $\mathrm{Cut}([z']\exists\,\mathrm{F}(z',t,[x']N),[z]\exists\,\mathrm{T}(z,[v,x]L))$
$\to \mathrm{Cut}([x']N,[x]L)$

where z' does not appear free in $[x']N$ and z does not appear free in N or $[v,x]L$. We insist on z' appearing free in $[x']N$ in (5.41) and z appearing free in $[v,x]L$ in (5.44).

Lemma 5.2.7. *The $\exists$-contraction rules for C2 commute up to equality with the interpretation of C2 in constructive type theory with explicit substitution.*

Proof. Similar to that of Lemma 5.2.2. □

5.3 Additional contraction rules

We discuss some additional contraction rules for moving a cut past another cut as well as the permutative rules for *C3* (Section 4.4) translated to *C2*.

Cut-contraction

There are two ways that a cut can move past another cut. These are captured by the following permutative rules, which commute up to equality with the interpretation of *C2* in constructive type theory with explicit substitution.

Cut-permutations:

(5.46)
$$\mathrm{Cut}([z']\,\mathrm{Cut}([x']M,[x]K),[z]L) \\ \to \mathrm{Cut}([x']\,\mathrm{Cut}([z']M,[z]L),[x]\,\mathrm{Cut}([z']K,[z]L))$$

and

(5.47)
$$\mathrm{Cut}([z']P,[z]\,\mathrm{Cut}([x']M,[x]K)) \\ \to \mathrm{Cut}([x']\,\mathrm{Cut}([z']P,[z]M),[x]\,\mathrm{Cut}([z']P,[z]K))$$

where z' is freshly introduced in P.

The first rule (5.46) is easily seen to be incompatible with strong normalization. This was also to be expected from the theory of explicit substitutions as (5.46) corresponds to the improper rule

$$M\left\langle [x]K/x'\right\rangle \left\langle [z]L/z\right\rangle \to M\left\langle [z]L/z\right\rangle \left\langle [x]K\left\langle [z]L/z\right\rangle /x'\right\rangle,$$

which should be compared with the proper rule

$$M\left\langle [x]K/x'\right\rangle \left\langle [z]L/z\right\rangle \to M\left\langle [z]L/z,[x]K\left\langle [z]L/z\right\rangle /x'\right\rangle.$$

The latter use of simultaneous substitution raises the question of what kind of multicut could play this role in *C2*. We leave this question unanswered.

On the other hand, the *C2*-contraction relation extended with (5.47) resembles a special case of a known strongly normalizing contraction relation and so we expect the *C2*-contraction relation extended with (5.47) to be strongly normalizing, see the paragraph following the proof of Theorem 5.4.1.

π-contraction

We discuss how the permutative rules for *C3* (Section 4.4) translate to *C2*. The new permutative rules resemble the permutative rules for *C3*: there is one set of permutative rules for implication and another set for negation, and together they generate a permutative contraction relation for *C2*. We call this relation *C2*-π-contraction for short or just π-contraction when the calculus is clear from the context.

We only write down the rules for moving an $\supset\mathrm{T}$-inference past an axiom, a $\&\,\mathrm{T}_1$-inference, a $\&\,\mathrm{F}$-inference, or a cut. The rules for moving an $\supset\mathrm{T}$-inference past any other inference are defined similarly. We use the shorthand

$$\underline{N} \equiv\ \supset\mathrm{T}(z,[x']N,[y]K),$$

taking for granted that x' does not appear bound in N.

$\supset$ T-permutations:

(5.48) $$\underline{\mathrm{Ax}(u', u)} \rightarrow \mathrm{Ax}(u', u) \quad \text{provided } x' \not\equiv u',$$

(5.49) $$\underline{\&\,\mathrm{T}_1(w, [u]L)} \rightarrow \&\,\mathrm{T}_1(w, [u]\underline{L}),$$

(5.50) $$\underline{\&\,\mathrm{F}(w', [u']N_1, [v']N_2)} \rightarrow \&\,\mathrm{F}(w', [u']\underline{N_1}, [v']\underline{N_2}) \quad \text{provided } x' \not\equiv w',$$

(5.51) $$\underline{\mathrm{Cut}([u']N, [u]L)} \rightarrow \mathrm{Cut}([u']\underline{N}, [u]\underline{L}),$$

and

(5.52) $$\underline{\&\,\mathrm{F}(x', [u']N_1, [v']N_2)} \rightarrow \underline{\&\,\mathrm{F}(x', [u']\underline{N_1}, [v']\underline{N_2})}$$

where in (5.52) x' must appear free in $[u']N_1$ or $[v']N_2$. The latter condition helps to ensure the existence of π-normal forms. Without it (5.52) could be applied indefinitely.

The permutative rules for negation are defined similarly.

The *C2*-π-contraction relation is weakly normalizing. The proof is similar to that for the *C3*-π-contraction relation.

Proposition 5.3.1. *Every C2-derivation π-reduces to a π-normal derivation.*

Proof. Similar to the proof of Proposition 4.4.3. □

The *C2*-π-equality relation coincides with the kernel of the translation of *C2* into *C3S*. Hence the π-equality relations for *C2* and *C3* agree. Let the brackets $[\![...]\!]$ denote the translation of *C2* into *C3S* in the following two propositions.

Proposition 5.3.2. *For all C2-derivations L and M, if $L =_\pi M$ then $[\![L]\!] \equiv [\![M]\!]$.*

Proof. Similar to the proof of Proposition 4.4.6. □

Proposition 5.3.3. *For all C2-derivations L and M, if $[\![L]\!] \equiv [\![M]\!]$ then $L =_\pi M$.*

Proof. Similar to the proof of Proposition 4.4.7. □

5.4 Normalization

The *C2*-contraction relation shares many of the good normalization properties of the *C3*-contraction relation. In particular, it is weakly normalizing and Church-Rosser. There are reasons to expect the *C2*-contraction relation to be strongly normalizing as well. However, we do not prove that the *C2*-contraction relation is strongly normalizing, but only discuss the reasons why this is to be expected.

The *C2*-contraction relation induces a notion of *C2*-reduction and a notion of a *C2*-normal form. It is easy to see that the notion of a *C2*-normal form coincides with the notion of a cut-free *C2*-derivation. These notions will be used interchangeably in what follows.

We shall now prove that the *C2*-contraction relation is weakly normalizing. The proof resembles Gentzen's proof of his Hauptsatz, but differs from the latter in that it has to deal with contraction rules that sometimes increase the rank of a cut, i.e. (5.6), (5.9), and their like.

Theorem 5.4.1. *Every C2-derivation reduces to a C2-normal form.*

Proof. By induction on the number of cuts in the *C2*-derivation, the maximal degree of the cuts in the *C2*-derivation, and the structure of the premise derivations of the *topmost* cuts in the *C2*-derivation, that is, those cuts whose premise derivations are *C2*-normal forms.

If the maximal degree of the cuts in a *C2*-derivation P is 0, then P reduces to a *C2*-normal form *C2*-$\mathrm{nf}(P)$ by (5.1), (5.3), induction on the structure of the first premise derivation of a topmost cut in P, and induction on the number of cuts in P.

Next, assume that the theorem holds whenever the maximal degree of the cuts in the *C2*-derivation is less than the maximal degree of the cuts in a *C2*-derivation P. The proof proceeds by cases depending on the structure of the premise derivations of a topmost cut $Q \equiv \mathrm{Cut}([x']M, [x]K)$ in P. There are three cases to consider.

If M freshly introduces x' and K freshly introduces x, then Q reduces by (5.3), (5.10), (5.17), (5.24), (5.30), (5.38), or (5.45) to a *C2*-derivation Q' where every cut is a cut of degree less than the maximal degree of the cuts in P. Then Q' reduces to a *C2*-normal form *C2*-$\mathrm{nf}(Q)$ by the induction hypothesis.

If M freshly introduces x', then Q reduces by the preceding paragraph, (5.2), (5.7), (5.8), (5.9), and their like, as well as induction on the structure of K, to a *C2*-normal form *C2*-$\mathrm{nf}(Q)$.

If M does not freshly introduce x', then Q reduces by the preceding paragraph, (5.1), (5.4), (5.5), (5.6), and their like, as well as induction on the structure of M, to a *C2*-normal form *C2*-$\mathrm{nf}(Q)$.

Hence Q reduces to a *C2*-normal form *C2*-$\mathrm{nf}(Q)$ by the three preceding paragraphs. Then P reduces to a *C2*-normal form *C2*-$\mathrm{nf}(P)$ by induction on the number of cuts in P. □

The *C2*-contraction relation constitutes a special case of the strongly normalizing contraction relations of Danos et al. (1997), Urban and Bierman (1999), and Urban (2000, 2001). In particular, we expect that there exists a kind of bisimulation between the *C2*-contraction relation extended with (5.47) and a subset of the contraction relation of Urban (2000, 2001), provided the latter is extended to all of the logical constants. The idea is to let a cut $\mathrm{Cut}([z']M, [z]L)$, where z' is not freshly introduced in M, correspond to a cut $\overleftarrow{\mathrm{Cut}}([z']M, [z]L)$ labeled by a left arrow and to let a cut $\mathrm{Cut}([z']P, [z]L)$, where z' is freshly introduced in P, correspond to a cut $\overrightarrow{\mathrm{Cut}}([z']P, [z]L)$ labeled by a right arrow. The labeling is to be understood in the sense of Urban. Hence we expect the *C2*-contraction relation extended with (5.47) to be strongly normalizing.

There exists a connection between Urban's contraction relation and the contraction relation of Danos et al. (1997). The connection indicates a correspondence between the *C2*-contraction relation and the q-protocol of the latter article. However, the provocative style of Danos et al. (1997) and their use of linear logic makes further comparisons difficult.

We now prove that the two notions of *C2*-equality and equality in constructive type theory explicit substitution agree. This result is the content of Proposition 5.4.2 and Proposition 5.4.3.

Proposition 5.4.2. *For all C2-derivations L and M, if $L = M$ then $[\![L]\!] = [\![M]\!]$.*

Proof. By the Lemmas 5.2.1–5.2.7 and induction on the definition of *C2*-equality. □

Proposition 5.4.3. *For all C2-derivations L and M, if $[\![L]\!] = [\![M]\!]$ then $L = M$.*

Proof. Suppose that $[\![L]\!] = [\![M]\!]$. Then $[\![C2\text{-nf}(L)]\!] = [\![C2\text{-nf}(M)]\!]$ by Theorem 5.4.1 and Proposition 5.4.2 and so $C2\text{-nf}(M) \equiv C2\text{-nf}(N)$ by Proposition 3.3.5 and Corollary 2.9.6. Hence $M = N$ by the definition of *C2*-equality. □

Next we focus our attention on the uniqueness of *C2*-normal forms.

Corollary 5.4.4. *For all normal C2-derivations L and M, if $L = M$ then $L \equiv M$.*

Proof. The contraction rules for *C2* commutes up to equality with the interpretation of *C2* into *C3*, analogous to Lemmas 5.2.1-5.2.7. Hence $[\![L]\!] = [\![M]\!]$ in *C3* and so $L \equiv M$ by Corollary 2.9.6. □

Note that, when combined with Theorem 5.4.1, Corollary 5.4.4 implies that the notion of *C2*-reduction enjoys the Church-Rosser property.

Chapter 6

Applications

After a short discussion of the Brouwer-Heyting-Kolmogorov semantics, we demonstrate how the said semantics can be made to justify classical logic. This is done by means of slight shifts in meaning at certain points in the meaning explanations of the logical constants, differentiating between the notions of proof and *classical proof*, the latter incorporating a double negation.

The two notions of proof lend themselves to the introduction of a second interpretation of classical logic. The new interpretation does not reinterpret the consequence relation but only the notion of truth.

The two interpretations are shown to be related to the well-known call-by-value and call-by-name CPS translations analyzed by Plotkin (1975) and later extended to $\lambda\mu$-calculus by Ong (1996) and Ong and Stewart (1997), respectively. To determine the precise relationship, we compare how the two interpretations act on certain derivations in classical sequent calculus with how the two CPS translations act on the corresponding derivations in $\lambda\mu$-calculus.

We think that the two interpretations have the potential of contributing to the theory of CPS translations as the meaning explanations can be used to give meaning to the logical constants of $\lambda\mu$-calculus. In particular, this indicates that $\&$, $\vee$, $\forall$, $\exists$, $\top$, and $\bot$ should be interpreted in their respective ways independently of whether a call-by-value or call-by-name semantics is used.

6.1 A BHK semantics justifying classical logic

It is well-known that the Brouwer-Heyting-Kolmogorov semantics, due to Brouwer (1908, 1924), Heyting (1934), and Kolmogorov (1932), makes sense in both the intuitionistic and the classical settings. This was evident to Kolmogorov (1932). For the historical context, see Troelstra (1990, 1991). Although the BHK semantics makes sense in the classical setting, it does not justify classical logic. The law of excluded middle has to be taken for granted. We will show, among other things, how to transform the interpretation of Chapter 3 into an equivalent semantics similar to the BHK semantics. Since the interpretation of Chapter 3 justifies the law of excluded middle in the sense that it is a constructive interpretation of classical logic, the corresponding BHK-like semantics can be used to justify classical logic.

We will use the sign $\mathrm{T}'' A \equiv (\mathrm{F}\, A)\psi$ as well as some nonstandard terminology. We say that an object of $\mathrm{T}\, A$ constitutes a *proof* of A while an object of $\mathrm{T}''\, A$ constitutes a *classical proof* of A. Suppressing the proof objects, we have the judgments *A true* and *A classically true*, respectively, where the latter is akin to the notion of pseudotruth in Kolmogorov (1925). We may thus speak about both *truth* and *classical truth*. The way these two concepts enter the BHK-like semantics elucidates what we have to supply to make the BHK semantics justify classical logic.

Consider the transformation of the $\supset$-introduction rule of *C3* into the rule

$$\dfrac{\begin{matrix}[\mathrm{T}\overset{x}{\,} A] \\ \mathrm{T}''\, B\end{matrix}}{\mathrm{T}\, A \supset B}\; x$$

These two rules are the same in the sense that they define the same connective. According to the latter rule, *a canonical proof of $A \supset B$ is a construction that maps any proof of A into a classical proof of B*. The meanings of the other logical constants can be explained similarly. The meaning explanations for the logical constants of *C3* are:

- a canonical proof of $A\&B$ is a pair of a classical proof of A and a classical proof of B;
- a canonical proof of $A \vee B$ is a classical proof of A or a classical proof of B together with the means to tell which is the case;
- a canonical proof of $A \supset B$ is a construction that maps any proof of A into a classical proof of B;
- a canonical proof of $\neg A$ is a construction that maps any proof of A into an object of type ψ;
- a canonical proof of $\forall v A$ is a construction that maps any individual t into a classical proof of $A[t/v]$;
- a canonical proof of $\exists v A$ is a pair of an individual t and a classical proof of $A[t/v]$.

The above meaning explanations follow the same pattern as those of the BHK semantics. The only difference is that they distinguish between proofs and classical proofs. Let A be a proposition built up from the logical constants defined by these meaning explanations. Interpreting the classical judgment A true by the judgment A classically true then justifies classical logic. Hence we only have to distinguish between truth and classical truth to make the BHK semantics justify classical logic.

6.2 A second interpretation of classical proofs

We can get the meaning explanations of Section 6.1, except those of implication and negation, from the corresponding intuitionistic ones by replacing "proof" by "classical proof" while leaving "canonical proof" without change. Making the same replacement in the intuitionistic meaning explanations of implication and negation gives the following meaning explanations instead:

- a canonical proof of $A \supset B$ is a construction that maps any classical proof of A into a classical proof of B;
- a canonical proof of $\neg A$ is a construction that maps any classical proof of A into a classical proof of $\perp$ or, what amounts to the same thing, an object of type ψ.

Henceforth, we write $\supset'$ and $\supset''$ for the logical constants of implication in the first and second set of meaning explanations, respectively. The corresponding logical constants of negation are written $\neg'$ and $\neg''$.

The first set of meaning explanations was arrived at by interpreting a classical sequent $\Gamma \Rightarrow \Delta$ as $\mathrm{T}\,\Gamma, \mathrm{F}\,\Delta \Rightarrow \psi$. In the same way, we can arrive at the second set of meaning explanations by interpreting a classical sequent $\Gamma \Rightarrow \Delta$ as $\mathrm{T}''\,\Gamma, \mathrm{F}\,\Delta \Rightarrow \psi$. The second interpretation agrees completely with Kolmogorov's double-negation interpretation.

Applied to a hypothetical judgement

$$A_1\ \mathit{true}, ..., A_m\ \mathit{true} \vdash B\ \mathit{true},$$

the first interpretation gives

$$A_1\ \mathit{true}, ..., A_m\ \mathit{true} \vdash_{\mathit{classically}} B\ \mathit{true}$$

while the second interpretation gives

$$A_1\ \mathit{classically\ true}, ..., A_m\ \mathit{classically\ true} \vdash B\ \mathit{classically\ true}.$$

Here an essential difference appears. The first interpretation reinterprets the consequence relation while keeping the constructive interpretation of the notion of truth, whereas the second interpretation reinterprets the notion of truth while keeping the constructive interpretation of the consequence relation. Moreover, this extends to the meaning explanations of the logical constants.

6.3 Interplay with CPS translation theory

The two interpretations in Section 6.2 are related to the well-known call-by-value and call-by-name CPS translations analyzed by Plotkin (1975) and later extended to $\lambda\mu$-calculus by Ong (1996) and Ong and Stewart (1997), respectively. The relationship can be discerned already from how they interpret a classical sequent.

To determine the relationship, we compare how the two interpretations act on the derivations $L \equiv$

$$\dfrac{\begin{array}{c}M\\ \Gamma \Rightarrow \overset{z'}{A} \supset B, \Delta\end{array} \qquad \dfrac{\begin{array}{c}N\\ \Gamma \Rightarrow \overset{x'}{A}, \Delta\end{array} \quad \dfrac{}{\overset{y}{B}, \Gamma \Rightarrow \overset{y'}{B}, \Delta}\;; y, y'}{\overset{z}{A} \supset B, \Gamma \Rightarrow \overset{y'}{B}, \Delta}\; x', y; z}{\Gamma \Rightarrow \overset{y'}{B}, \Delta}\; z', z;$$

and $R \equiv$

$$\dfrac{\begin{array}{c}M\\ \overset{x}{A}, \Gamma \Rightarrow \overset{y'}{B}, \Delta\end{array}}{\Gamma \Rightarrow \overset{z'}{A} \supset B, \Delta}\; x, y'$$

with how the two CPS interpretations act on the derivations $L_{\lambda\mu} \equiv$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash M : \psi | z' : A \supset B, \Delta}{\Gamma \vdash \mu z'.M : A \supset B | \Delta}\; z'; \qquad \dfrac{\Gamma \vdash N : \psi | x' : A, \Delta}{\Gamma \vdash \mu x'.N : A | \Delta}\; x';}{\Gamma \vdash @_\lambda(\mu z'.M, \mu x'.N) : B | \Delta}}{\Gamma \vdash @_\mu(@_\lambda(\mu z'.M, \mu x'.N), y') : \psi | y' : B, \Delta}\; ; y'$$

and $R_{\lambda\mu} \equiv$

$$\dfrac{\dfrac{\dfrac{x : A, \Gamma \vdash M : \psi | y' : B, \Delta}{x : A, \Gamma \vdash \mu y'.M : B | \Delta}\; y';}{\Gamma \vdash \lambda x.\mu y'.M : A \supset B | \Delta}\; x;}{\Gamma \vdash @_\mu(\lambda x.\mu y'.M, z') : \psi | z' : A \supset B, \Delta}\; ; z'$$

The two CPS translations can be found in Table 6.3, p. 89. These translations take a $\lambda\mu$-term to a λ-term. The inference rules of the λ-calculus and the $\lambda\mu$-calculus can be found in Table 6.1 and Table 6.2, p. 88. Restricted to the λ-calculus, the translation coincides with the call-by-value respectively call-by-name translations due to Plotkin (1975). We have deviated from the standard syntax of λ-calculi and $\lambda\mu$-calculi to avoid confusion with the present syntax of constructive type theory.

The call-by-value CPS translations of the two $\lambda\mu$-terms

$$L_{\lambda\mu} \equiv @_\mu(@_\lambda(\mu z'.M, \mu x'.N), y')$$

and

$$R_{\lambda\mu} \equiv @_\mu(\lambda x.\mu y'.M, z')$$

are

$$\begin{aligned}
[\![L_{\lambda\mu}]\!]_v &\equiv @_\lambda(\lambda y'.@_\lambda(\lambda z'.[\![M]\!]_v, \lambda z.@_\lambda(\lambda x'.[\![N]\!]_v, \lambda x.@_\lambda(@_\lambda(z,x), y'))), y') \\
&\to @_\lambda(\lambda z'.[\![M]\!]_v, \lambda z.@_\lambda(\lambda x'.[\![N]\!]_v, \lambda x.@_\lambda(@_\lambda(z,x), y'))) \\
&\to [\![M]\!]_v[\lambda z.@_\lambda(\lambda x'.[\![N]\!]_v, \lambda x.@_\lambda(@_\lambda(z,x), y'))/z'] \\
&\to [\![M]\!]_v[\lambda z.[\![N]\!]_v[\lambda x.@_\lambda(@_\lambda(z,x), y')/x']/z']
\end{aligned}$$

Abstraction	**Application**
$\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$	$\dfrac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash @_\lambda(M, N) : B}$
Variable	
$\dfrac{}{\Gamma, x : A \vdash x : A}$	

Table 6.1: Minimalistic λ-calculus. The structural rules are understood to be inherent in the concept of sequent and the inference figures used.

	Abstraction	**Application**
λ	$\dfrac{\Gamma, x : A \vdash M : B \vert \Delta}{\Gamma \vdash \lambda x.M : A \to B \vert \Delta}$	$\dfrac{\Gamma \vdash M : A \to B \vert \Delta \quad \Gamma \vdash N : A \vert \Delta}{\Gamma \vdash @_\lambda(M, N) : B \vert \Delta}$
μ	$\dfrac{\Gamma \vdash M : \psi \vert \Delta, x' : A}{\Gamma \vdash \mu x'.M : A \vert \Delta}$	$\dfrac{\Gamma \vdash M : A \vert \Delta}{\Gamma \vdash @_\mu(M, x') : \psi \vert x' : A, \Delta}$
	Variable	
	$\dfrac{}{\Gamma, x : A \vdash x : A \vert \Delta}$	

Table 6.2: Minimalistic $\lambda\mu$-calculus. The structural rules are understood to be inherent in the concept of sequent and the inference figures used. In λ-abstraction, B must not equal ψ. The restriction is not a general feature of $\lambda\mu$-calculi, but makes the above calculus more similar to the other calculi of this thesis.

and

$$\begin{aligned}[\![R_{\lambda\mu}]\!]_v &\equiv @_\lambda(\lambda z'.@_\lambda(z', \lambda x.\lambda y'.[\![M]\!]_v), z') \\ &\to @_\lambda(z', \lambda x.\lambda y'.[\![M]\!]_v),\end{aligned}$$

respectively. Similarly, the corresponding call-by-name CPS translations are

$$\begin{aligned}[\![L_{\lambda\mu}]\!]_n &\equiv @_\lambda(\lambda y'.@_\lambda(\lambda z'.[\![M]\!]_n, \lambda z.@_\lambda(@_\lambda(z, \lambda x'.[\![N]\!]_n), y')), y') \\ &\to @_\lambda(\lambda z'.[\![M]\!]_n, \lambda z.@_\lambda(@_\lambda(z, \lambda x'.[\![N]\!]_n), y')) \\ &\to [\![M]\!]_n[\lambda z.@_\lambda(@_\lambda(z, \lambda x'.[\![N]\!]_n), y')/z']\end{aligned}$$

and

$$\begin{aligned}[\![R_{\lambda\mu}]\!]_n &\equiv @_\lambda(\lambda z'.@_\lambda(z', \lambda x.\lambda y'.[\![M]\!]_n), z') \\ &\to @_\lambda(z', \lambda x.\lambda y'.[\![M]\!]_n),\end{aligned}$$

$$\begin{aligned}
[\![\Gamma \vdash M : A|\Delta]\!]_v &= [\![\Gamma]\!]_v, [\![\Delta]\!]'_v \vdash [\![M]\!]_v : [\![A]\!]''_v \\
[\![A \to B]\!]_v &\equiv [\![A]\!]_v \to [\![B]\!]''_n \\
[\![\alpha]\!]_v &\equiv \alpha, \text{ provided that } \alpha \text{ is a type constant}
\end{aligned}$$

$$\begin{aligned}
[\![\lambda x.M]\!]_v &\equiv \lambda z'.@_\lambda(z', \lambda x.[\![M]\!]_v) \\
[\![@_\lambda(M,N)]\!]_v &\equiv \lambda y'.@_\lambda([\![M]\!]_v, \lambda z.@_\lambda([\![N]\!]_v, \lambda x.@_\lambda(@_\lambda(z,x), y'))) \\
[\![\mu x'.M]\!]_v &\equiv \lambda x'.[\![M]\!]_v \\
[\![@_\mu(M,x')]\!]_v &\equiv @_\lambda([\![M]\!]_v, x') \\
[\![x]\!]_v &\equiv \lambda x'.@(x', x)
\end{aligned}$$

$$\begin{aligned}
[\![\Gamma \vdash M : A|\Delta]\!]_n &\equiv [\![\Gamma]\!]''_n, [\![\Delta]\!]'_n \vdash [\![M]\!]_n : [\![A]\!]''_n \\
[\![A \to B]\!]_n &\equiv [\![A]\!]''_n \to [\![B]\!]''_n \\
[\![\alpha]\!]_n &\equiv \alpha, \text{ provided that } \alpha \text{ is a type constant}
\end{aligned}$$

$$\begin{aligned}
[\![\lambda x.M]\!]_n &\equiv \lambda z'.@_\lambda(z', \lambda x.[\![M]\!]_n) \\
[\![@_\lambda(M,N)]\!]_n &\equiv \lambda y'.@_\lambda([\![M]\!]_n, \lambda z.@_\lambda(@_\lambda(z, [\![N]\!]_n), y')) \\
[\![\mu x'.M]\!]_n &\equiv \lambda x'.[\![M]\!]_n \\
[\![@_\mu(M,x')]\!]_n &\equiv @_\lambda([\![M]\!]_n, x') \\
[\![x]\!]_n &\equiv x
\end{aligned}$$

Table 6.3: A call-by-value and a call-by-name translation of the $\lambda\mu$-calculus of Table 6.2 into the λ-calculus of Table 6.1. The translations are denoted by $[\![...]\!]_v$ and $[\![...]\!]_n$, respectively. We use the abbreviation $A' \equiv A \to \psi$ for types. Moreover, both the abbreviation and the translations are understood to be distributive over contexts.

respectively.

We now change to consider L and R.

The first interpretation, which has been defined in Chapters 2 and 3, gives

$$\begin{aligned}[\![L]\!] &\equiv [\![M]\!][[z]\supset'\text{-el}(z,[y''][\![N]\!][[x]y''(x,[y]y'(y))/x'])/z'] \\ &\to_\eta [\![M]\!][[z]\supset'\text{-el}(z,[y''][\![N]\!][[x]y''(x,y')/x'])/z']\end{aligned}$$

by η-contraction and

$$[\![R]\!] \equiv z'(\supset'\text{-in}([x,y'][\![M]\!])).$$

The terms $[\![L]\!]$ and $[\![R]\!]$ obviously have the same structure as $[\![L_{\lambda\mu}]\!]_v$ respectively $[\![R_{\lambda\mu}]\!]_v$, although they have a different meaning independently of how λ and $@_\lambda$ are interpreted.

The second interpretation, which can be defined in a similar way, gives

$$\begin{aligned}[\![L]\!] &\equiv z''([z]\supset''\text{-el}(z,[y'']y(y')[y''([x][\![N]\!])/y]))[[z'][\![M]\!]/z''] \\ &\to ([z'][\![M]\!])([z]\supset''\text{-el}(z,[y'']y(y')[y''([x][\![N]\!])/y])) \\ &\to [\![M]\!][[z]\supset''\text{-el}(z,[y'']y(y')[y''([x][\![N]\!])/y])/z'] \\ &\to [\![M]\!][[z]\supset''\text{-el}(z,[y'']y''([x][\![N]\!],y'))/z']\end{aligned}$$

and

$$[\![R]\!] \equiv z'(\supset''\text{-in}([x,y'][\![M]\!])).$$

Here $[\![L]\!]$ and $[\![R]\!]$ obviously have the same structure as $[\![L_{\lambda\mu}]\!]_n$ and $[\![R_{\lambda\mu}]\!]_n$, respectively.

This settles how the interpretations relate to CPS translations.

6.4 Implications for CPS translation theory

The two sets of meaning explanations are well suited for a calculus of natural deduction in which $\lambda\mu$-calculus can be interpreted. For the first set of meaning explanations, the introduction and elimination rules can be read off from the corresponding introduction rules and elimination rules in Chapter 3. The introduction and elimination rules of $\supset''$ and $\neg''$ can be arrived at in the same way as those of $\supset'$ and $\neg'$. All these inference rules can be found in Table 6.4, p. 91.

The elimination rules in Table 6.4 have the same form as the elimination rules due to von Plato (2001) and, consequently, similar permutative rules exist. However, specialization of the elimination rules from C to Ψ makes these permutative rules superfluous. Furthermore, specialization hides many details unnecessary for interpretation of $\lambda\mu$-calculus.

The two sets of meaning explanations have the potential of contributing to the theory of CPS translations. The meaning explanations can be used to give meaning

	Introduction	**Elimination**
$\&$	$\dfrac{\mathrm{T}''\,A \quad \mathrm{T}''\,B}{\mathrm{T}\,A\&B}$	$\dfrac{\mathrm{T}\,A\&B \quad \begin{matrix}[\overset{x}{\mathrm{T}''}\,A, \overset{y}{\mathrm{T}''}\,B]\\ \mathrm{T}\,C\end{matrix}}{\mathrm{T}\,C}\;x,y$
$\vee$	$\dfrac{\mathrm{T}''\,A}{\mathrm{T}\,A \vee B}\;1 \qquad \dfrac{\mathrm{T}''\,B}{\mathrm{T}\,A \vee B}\;2$	$\dfrac{\mathrm{T}\,A \vee B \quad \begin{matrix}[\overset{x}{\mathrm{T}''}\,A]\\ \mathrm{T}\,C\end{matrix} \quad \begin{matrix}[\overset{y}{\mathrm{T}''}\,B]\\ \mathrm{T}\,C\end{matrix}}{\mathrm{T}\,C}\;x$
$\supset'$	$\dfrac{\begin{matrix}[\overset{x}{\mathrm{T}}\,A]\\ \mathrm{T}''\,B\end{matrix}}{\mathrm{T}\,A \supset' B}\;x$	$\dfrac{\mathrm{T}\,A \supset' B \quad \begin{matrix}[(\mathrm{T}\,A)\overset{y}{}\,\mathrm{T}''\,B]\\ \mathrm{T}\,C\end{matrix}}{\mathrm{T}\,C}\;y$
$\supset''$	$\dfrac{\begin{matrix}[\overset{x}{\mathrm{T}''}\,A]\\ \mathrm{T}''\,B\end{matrix}}{\mathrm{T}\,A \supset'' B}\;x$	$\dfrac{\mathrm{T}\,A \supset'' B \quad \begin{matrix}[(\mathrm{T}''\,A)\overset{y}{}\,\mathrm{T}''\,B]\\ \mathrm{T}\,C\end{matrix}}{\mathrm{T}\,C}\;y$
$\neg'$	$\dfrac{\begin{matrix}[\overset{x}{\mathrm{T}}\,A]\\ \psi\end{matrix}}{\mathrm{T}\,\neg' A}\;x$	$\dfrac{\mathrm{T}\,\neg' A \quad \begin{matrix}[\overset{x'}{\mathrm{F}}\,A]\\ \mathrm{T}\,C\end{matrix}}{\mathrm{T}\,C}\;x'$
$\neg''$	$\dfrac{\begin{matrix}[\overset{x}{\mathrm{T}''}\,A]\\ \psi\end{matrix}}{\mathrm{T}\,\neg'' A}\;x$	$\dfrac{\mathrm{T}\,\neg'' A \quad \begin{matrix}[(\mathrm{T}''\,A)\overset{x'}{}\psi]\\ \mathrm{T}\,C\end{matrix}}{\mathrm{T}\,C}\;x'$
$\forall$	$\dfrac{\mathrm{T}''\,A}{\mathrm{T}\,\forall v A}$	$\dfrac{\mathrm{T}\,\forall v A \quad \begin{matrix}[\mathrm{T}''\,\overset{x}{A}[t/v]]\\ \mathrm{T}\,C\end{matrix}}{\mathrm{T}\,C}\;x$
$\exists$	$\dfrac{\mathrm{T}''\,A[t/v]}{\mathrm{T}\,\exists v A}$	$\dfrac{\mathrm{T}\,\exists v A \quad \begin{matrix}[\overset{x}{\mathrm{T}''}\,A]\\ \mathrm{T}\,C\end{matrix}}{\mathrm{T}\,C}\;x$

Table 6.4: Introduction and elimination rules derived from the BHK semantics, not including the rules for F and T$''$. In $\forall$-introduction and $\exists$-elimination the variable v must not appear free in the context of discourse.

to the logical constants of $\lambda\mu$-calculus. In particular, this indicates that $\&$, $\vee$, $\forall$, $\exists$, $\top$, and $\bot$ should be interpreted in their respective ways independently of whether a call-by-value or call-by-name semantics is used, see Section 4.1 for the cases of $\top$ and $\bot$. Furthermore, some of the works on CPS translations concern administrative redexes and how they can be avoided, see for example Danvy and Filinski (1992), Sabry and Felleisen (1992), Sabry (1994), Sabry and Wadler (1997), and Danvy and Nielsen (2003). In our interpretations, there exists a clear distinction between logical redexes and β-redexes, and it is tempting to identify β-redexes and administrative redexes. Whether this is will indeed work out, how it relates to the state of the art, and what it can contribute to the theory of CPS translations remains to be seen.

Concluding remarks

My objective in this thesis has been to explore how the syntactic-semantic method of constructive type theory can be used to give a semantics of classical logic. Although the objective originated from the question of how the double-negation translation operates on derivations and not only on formulas, the thesis has gone beyond answering that question. In particular, it has given a semantics of the logical constants of classical logic whose development parallels that of intuitionistic logic in constructive type theory. Among other things, this semantics clarifies the notion of proof in classical logic.

Knowledge has also been gained on induction in classical logic, on the identity of proofs induced by the interpretation, and on full precision contraction rules for classical sequent calculus. In Section 3.5, we proved the principles of mathematical and W-induction for classical logic. However, the technique that worked for W-induction could not be extended to accessibility induction in classical logic, as the accessibility predicate could not be formalized in a strictly positive way. Chapter 4 and Section 5.3, which were rather technical, made clear that the identity of proofs induced by the interpretation is sensitive to the choice of the target calculus of the interpretation, and so the target calculus has to be chosen with care. Finally, the full precision contraction relation for classical sequent calculus in Chapter 5 was derived from constructive type theory, unlike previous full precision contraction relations, which have been derived from other considerations.

With hindsight, I think the correct way to present classical logic is to start with the notion of classical proof and an interpretation of the consequence relation, and then state the corresponding Brouwer-Heyting-Kolmogorov semantics of the logical constants of classical logic as in Chapter 6. Except for the ambiguity in the choice of the interpretation of the consequence relation, this development parallels that of intuitionistic logic in constructive type theory. Whether a choice can be made between the two interpretations of the consequence relation on semantical grounds remains to be seen. However, I lean toward the interpretation that reinterprets the notion of truth while keeping the constructive interpretation of the consequence relation, which is the interpretation that matches Kolmogorov's (1925) double-negation translation, as shown in Section 6.2. The semantics of the logical constants of classical logic can then be formalized in constructive type theory as in Chapter 3.

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