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ON THE CONTINUITY OF OPTIMAL INCOME-TAX SCHEDULES

by

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Abstract
This note develops a simple geometric argument suggesting that there to every budget function in Mirrlees\textsuperscript{*} model of optimal income taxation exists a continuous (and non-decreasing) budget function which achieves the same welfare distribution and individual allocations. Hence, without loss of generality, one may already at the outset assume optimal tax schedules, if such exist, to be continuous and non-decreasing. The result is derived under somewhat more general conditions than those in Mirrlees (1971).

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1. Introduction

It is well-known that even "smooth" optimal control problems may lack continuous solutions. This seems to be the motivation why Mirrlees (1971) at the outset permits discontinuous budget functions ("consumption functions" in op.cit.) The only requirement in his model is that they be upper semi-continuous (u.s.c.) - otherwise individuals may face decision problems which have no solution. However, a simple geometric argument suggests that there to every u.s.c. budget function exists a continuous (and non-decreasing) budget function which produces exactly the same solution set for each individual in the economy. This argument does not bear on convexity of preferences, nor does it require, as does Mirrlees (1971), individuals to have identical preferences. Hence, the argument will be developed in a somewhat more general setting.

2. The model

Consider a population of individuals, each of which selects consumption \( x \) and working time \( y \) in \( T = (0, +\infty) \times [0,1] \) so as to maximize his utility \( u_n(x,y) \) subject to \( x \leq c(ny) \). Here \( c : \mathbb{R}_+ \to \mathbb{R}_+ \) is an exogenously imposed budget function, of which we require \( c(0) > 0 \) and upper semi-continuity. Each \( u_n \) is assumed to be continuous, strictly increasing in \( x \), strictly decreasing in \( y \), and such that the closure of every indifference curve is contained in \( T \). Hence, the solution set is non-empty for every individual \( n \). The product \( z = ny \) is interpreted as the labour supplied when individual \( n \) works during \( y \) time units. Across the population, the productivity coefficient \( n \) is assumed to have a distribution with support \( I \), where \( I \) is an open interval in \( \mathbb{R}_+ \).
For any budget function $c$ and $n \in I$, let $v_n(c) = \max\{u(x, z/n) \mid x \leq c(z)\}$ and $A_n(c) = \{(x, z) \mid x \leq c(z) \text{ and } u(x, z/n) = v_n(c)\}$, where $A_n(c) \neq \emptyset$ as noted above. Two budget functions $c$ and $e$ are said to be welfare equivalent if $v_n(c) = v_n(e)$ $\forall n \in I$. If $A_n(c) = A_n(e)$ $\forall n \in I$, then they are said to be allocation equivalent. Clearly allocation equivalence implies welfare equivalence, but not vice versa.

3. The geometric argument

Let a budget function $c$ be given. For every individual $n \in I$, let $e_n$ be the function the graph of which is $n$'s optimal indifference curve, i.e. $e_n: [0, \overline{z}_n] \to \mathbb{R}_+$ is defined implicitly by $u_n(e_n(z), z/n) = v_n(c)$, where $\overline{z}_n = \sup\{z < n; u_n(x, z/n) > v_n(c)\}$ for some $x > 0 > 0$. Every $e_n$ is continuous, strictly increasing and nowhere exceeded by $c$. Let $e$ be the (lower) envelop of the family $\{e_n\}$, i.e. $e: [0, \overline{z}] \to \mathbb{R}_+$ is defined by $e(z) = \inf\{e_n(z); n \in I\}$ where $\overline{z} = \sup\{\overline{z}_n; n \in I\}$. Then $c \leq e$ and $e$ is u.s.c. and non-decreasing by construction. However, $e$ is not necessarily lower semi-continuous (l.s.c.). For example, suppose $I = (0, +\infty)$, $c(z) = 1$ on $[0, 1]$ and $= 2$ on $[1, +\infty)$. If, for every $n > 1$, $e_n(z) = 1 + z^n$ on $[0, 1)$, then $e(z) = 1$ on $[0, 1)$ while $e(1) = 2$. However, if individual preferences depend in a monotonic and continuous way on the productivity coefficient $n$ of the individual then $e$ can in fact be shown to be continuous (cf. assumptions (M) and (C), Lemma 1).

Since $c \leq e \leq e_n$ for every individual $n \in I$, it is clear that we would obtain the same welfare distribution if we were to replace $c$ by $e$. However, for certain budget functions $c$, we would also have
caused some individual solution sets to increase, as some indifference curves $e_n$ may coincide with $e$ on intervals where $c < e$.

In order to achieve allocation equivalence, we are hence led to seek a continuous and non-decreasing function, the graph of which should run between the graphs of $c$ and $e$, coinciding with $e$ only where $e$ coincides with $c$. The existence and allocation-equivalence of such a function is established by means of a general lemma (Lemma 2 and Theorem), cf also Figure 1.

4. Formal analysis

As for the differences in preferences among individuals, it is assumed that an individual's indifference curve through any point $(x, z)$ is flatter the higher is his productivity coefficient. More precisely, for every $n \in I$, $x > 0$ and $z \in [0, n)$, let

$$P_n(x, z) = \{ (x', z') ; u_n(x', z'/n) \geq u_n(x, z/n) \},$$

and assume monotonicity with respect to $n$ is the sense

(M) $\forall m, n \in I$ with $m < n$, $x > 0$ and $z \in [0, m)$:

$$P_n(x, z) \cap \{ (x', z') ; z' < z \} \subset P_m(x, z) \cap \{ (x', z') ; z' < z \}$$

$$P_m(x, z) \cap \{ (x', z') ; z' > z \} \subset P_n(x, z) \cap \{ (x', z') ; z' > z \}$$

This condition is implied by assumption (B) in Mirrlees (1971) (and also by the essentially equivalent assumption (AM) in Seade (1982)). Moreover, we assume the preference structure to be continuous with respect to $n$ in the sense

(C) $\forall x > 0, z \in I$ : $\inf_{x' > 0} \{ (x', 0) \in \bigcup_{n \geq z} P_n(x, z) \} = 0$
That this condition expresses continuity is seen from the "dual"
condition

\[ \forall x > 0, n \in I : \inf_{x' > 0; (x', 0) \in \bigcup_{z < n} P_n(x, z)} = 0 \]

which follows from the hypotheses made in section 2 concerning
preferences. Condition (C) is trivially satisfied in Mirrlees' model where all individuals have the same preferences.

A preference structure satisfying (M) but not (C) can be constructed as follows. Let I = (0, 2), and let u be a utility function satisfying the requirements in Section 2. Let \( u_n = \bar{u} \)
for all \( \bar{n} < 1 \), while for \( n \geq 1 \) let \( u_n \) depend on \( n \) in such a way that \( P_n(x, z) \cap \{(x, z); z < 1\} = P_2(x, z) \cap \{(x, z); z < 1\} \) and (M) still fulfilled. Then (C) is not satisfied, since for every \( x > 0; n > 1 \n \)
\[ \inf_{x' > 0; (x', 0) \in \bigcup_{n \geq 1} P_n(x, 1)} = \inf_{x' > 0; (x', 0) \in P_2(x, 1)} > 0. \]

Lemma 1: \( e: [0, \bar{z}] \to \mathbb{R}_+ \) is non-decreasing and continuous.

Proof: Each \( e_n \) being non-decreasing, \( e = \inf e_n \) is non-decreasing.
Each \( e_n \) being u.s.c., also \( e \) is u.s.c. since \( \{z \in (0, \bar{z}_n); e_n(z) < \alpha\} \) open \( \forall \alpha \in \mathbb{R} \) and hence \( \{z \in (0, \bar{z}); e(z) < \alpha\} = \bigcup_{n \in I} \{z \in (0, \bar{z}_n); e_n(z) < \alpha\} \) open. Thus \( e \) is also right-continuous. For left-continuity, suppose \( z_0 \in (0, \bar{z}) \) and let \( x_o = e(z_0) \). By (C) and the hypothesis \( c(0) > 0 \), there \( \exists m \in I \) such that \( (c(0), 0) \in P_m(x_o, z_o) \). Let \( f: [0, z_o] \to \mathbb{R} \) be defined by \( x = f(z) \) iff \( u_m(x, z/m) = u_m(x_o, z_o/m) \). By monotonicity and continuity of \( u \), \( f \) is continuous and strictly increasing with \( f(0) < c(0) \) and \( f(z_o) = x_o \). Moreover, \( f \leq e_m \) since \( z = f(x) \to u_m(x, z/m) \leq u_m(c(0), 0) < v_m(c) \). Applying (M) to \( (c(0), 0) \) we obtain
\( f_{\leq e_n} \forall n \leq m \), and applying (M) to \((x_0, z_0)\) we likewise obtain 

\( f_{< e_n} \forall n > m \). Hence \( f \leq e \) with \( f(z_0) = e(z_0) \). By continuity and monotonicity of \( f \), \( e \) is left-continuous at \( z_0 \). End of proof.

**Lemma 2:** Suppose \( D \subseteq \mathbb{R} \) open and \( f : D \to \mathbb{R}_+ \) l.s.c.. Then there exists a continuous \( g : D \to \mathbb{R}_+ \) such that \( g \leq f \) and \( \{ x ; g(x) > 0 \} = \{ x ; f(x) > 0 \} \).

**Proof:** \( D_+ = \{ x ; f(x) > 0 \} \) open since \( f \) l.s.c.. Hence there is an increasing sequence of compact sets \( K_1, K_2, \ldots \) such that \( \cup K_i = D_+ \). (Every open set in \( \mathbb{R} \) is a countable union of open intervals, each of which is the union of an increasing sequence of compact sub-intervals.) Let \( \alpha_i = \inf \{ f(x); x \in K_i \} \). Since \( f \) l.s.c., 

\[ f(x) = \alpha_i \text{ for some } x \in K_i. \] 

Hence, each \( \alpha_i > 0 \). For every \( i \), there is a continuous \( h_i : D \to [0, \alpha_{i+1}] \) such that \( h_i = \alpha_{i+1} \) on \( K_i \) and = 0 outside \( K_{i+1} \). Consequently, every \( h_i \leq f \). Let \( g = \max \{ h_i \} \). Then \( g \) is continuous, \( 0 \leq g \leq f \), and \( g > 0 \) on \( D_+ \). End of proof.

**Theorem:** For every (u.s.c.) budget function there exists an allocation-equivalent budget function that is continuous and non-decreasing.

**Proof:** First, we establish a candidate function \( e^* \) on \((0, \bar{z})\).

Let \( f = e - c \). Then \( f \geq 0 \) l.s.c. and hence, by Lemma 2, there is a continuous function \( g : (0, \bar{z}) \to \mathbb{R}_+ \) such that \( 0 < g \leq f \) and \( g(z) > 0 \) wherever \( f(z) > 0 \). Let \( e^* : (0, \bar{z}) \to \mathbb{R} \) be defined by \( e^*(z) = \sup \{ e(z') - g(z'); z' < z \} \). Then \( e^* \) is continuous, non-decreasing and \( c = e - f \leq e - g \leq e^* \leq e \). Secondly, we show \( A_n(e^*) = A_n(c) \) \( \forall n \in I \). Clearly
\( v_n(c) \leq v_n(e^*) \) since \( c \leq e^* \). Moreover, \( e^* \leq e_n \ \forall n \in I \), so \( v_n(e^*) \leq v_n(e_n) = v_n(c) \). Hence \( v_n(e^*) = v_n(c) \ \forall n \in I \). Now suppose \((x,z) \in A_n(c)\). Then

\[ x = e_n(z) = c(z). \]

However \( c \leq e^* \leq e_n \), so \( x = e^*(z) \), i.e. \( (x,z) \in A_n(e^*) \).

Reversely, if \((x,z) \in A_n(e^*)\), then \( x = e_n(z) = e^*(z) \), so \( e^*(z) = e(z) \).

Hence, \( g(z) = 0 \) and consequently \( f(z) = 0 \), i.e. \( x = e^*(z) = e(z) = c(z) \), so \((x,z) \in A_n(c)\). It remains to extend \( e^* \) from \((0, \tilde{z})\) to \( R_+ \). Let

\[ \tilde{z} = \sup\{z \in (0, \tilde{z}) ; (x,z) \in A_n(c) \text{ some } n \in I \text{ and } x > 0 \}. \]

Then \( e^* \) is bounded on \((0, \tilde{z})\) since \( c \) is bounded on \([0, \tilde{z}]\) by upper semi-continuity and \((x,z) \in A_n(c) \text{ some } n \in I \Rightarrow e^*(z) = c(z) \). Define \( \tilde{e} : R_+ \to R_+ \) by \( \tilde{e}(0) = \lim_{\tilde{z} \to 0} e^*(z) \), \( \tilde{e}(z) = e^*(z) \) for \( z \in (0, \tilde{z}) \) and \( \tilde{e}(z) = e^*(\tilde{z}) \) for \( z \geq \tilde{z} \). Clearly \( \tilde{e} \) is allocation-equivalent to \( c \). End of proof.

**References**


Footnotes

1 The condition \( c(0) > 0 \) is imposed for convenience and without loss of generality since if \( z_0 = \inf \{ z; c(z) > 0 \} > 0 \) and there were individuals with \( n < z_0 \), then their decision problems would have no solutions, while if there were no such individuals, then we could substitute \( z - z_0 \) for \( z \) everywhere in the model.

2 To see this, let \( x' > 0 \) and \( n \in I \) be fixed. Then, for any \( x > 0 \), \( u_n(x', 0) = u_n(x, z/n) \) for some \( z < n \), since the closure of the indifference curve through \( (x', 0) \) is by hypothesis contained in \( T \). Hence \( (x', 0) \in uP_n(x, z) \) for all \( z < n \).

3 In Mirrlees' model, \( (x', 0) \in uP_n(x, z) \) iff \( u(x', 0) > \lim_{n > z} u(x, y) \) as \( y \to 1 \).
Figure 1: Illustration of the geometric argument. The shaded area is the budget set generated by a budget function $c$. With $n_1 < n_2 < n_3 < n_4$, the graphs of the functions $e_{n_i}$ are ordered as in the diagram, by (M). The dotted line indicates the graph of an allocation-equivalent budget function $\tilde{e}$. 