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PORTFOLIO CHOICE WITH NON-EXPECTED UTILITY
IN CONTINUOUS TIME

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PORTFOLIO CHOICE WITH NON-EXPECTED UTILITY IN CONTINUOUS TIME

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Abstract

Non-expected utility preferences, which can distinguish intertemporal substitution from attitudes towards risk, are extended to continuous time. The consumption/savings and portfolio problem is solved for constant intertemporal elasticity of substitution and constant relative risk aversion.

⁰ I have benefitted from a discussion with Miles Kimball and from comments by Ingrid Werner. I thank Maria Gil for secretarial and editorial assistance.

I. Non-expected Utility in Discrete Time

It is well known that the standard intertemporal additively separable utility function cannot distinguish separate intertemporal substitution from attitudes towards risk. Selden (1978, 1979) has developed a formalization of two-period preferences that allows for such a distinction. The formalization specifies that preferences in the first period can be expressed as an intertemporal utility function $u_1 = U(c_1, \hat{c}_2)$ over consumption in period 1, c_1 , and *certainty-equivalent* consumption in period 2, $\hat{c}_2$. The certainty equivalent consumption in period 2 is defined via a separate risk utility function according to $\hat{c}_2 = V^{-1}(EV(\tilde{c}_2))$, where E is the expectations operator and $\tilde{c}_2$ the stochastic period 2 consumption. Hence the overall preferences can be written

$$(1) \quad u_1 = U(c_1, V^{-1}(E_1 V(\tilde{c}_2))).$$

Intertemporal substitution is captured by the U -function, and attitudes towards risk by the V -function.

A problem with Selden's formulation is that the preferences are not time consistent if there are more than two periods. Several authors (Kreps-Porteus (1978), Chew and Epstein (1987), Epstein and Zin (1988), Farmer (1987), and Weil (1987a, 1987b)) have constructed a generalization to many periods that involve a recursive definition of the utility function. More precisely, utility in period t , u_t , can be written as an intertemporal utility function ("aggregator function")

$$(2a) \quad u_t = U_t(c_t, \hat{u}_{t+1})$$

of consumption in period t , c_t , and certainty-equivalent utility $\hat{u}_{t+1}$, in period $t+1$.

Certainty equivalent utility is in turn defined by a risk utility function according to

$$(2b) \quad V_t(\hat{u}_{t+1}) = E_t V_t(\tilde{u}_{t+1}),$$

where E_t is the expectations operator given information of period t and $\tilde{u}_{t+1}$ is the stochastic utility in period $t+1$. Then the overall preferences in period t can be written

$$(3) \quad u_t = U_t(c_t, V^{-1}(E_t V(\tilde{u}_{t+1}))).$$

The case when both the intertemporal and risk utility functions are CES corresponds to constant intertemporal elasticity of substitution (CIES) and constant relative risk aversion (CRRA). This case has been thoroughly examined in the literature.

II. Non-expected Utility in Continuous Time

Let us consider the most straightforward extension of non-expected utility to continuous time. Let us consider preferences for a small time interval Δt and then take the limit when Δt approaches zero. We simply specify that preferences in period t can be written as the intertemporal utility function

$$(4a) \quad u(t) = \lim_{\Delta t \rightarrow 0+} U[c(t), \hat{u}(t; t+\Delta t); t, t+\Delta t]$$

of consumption $c(t)$ and certainty equivalent utility $\hat{u}(t; t+\Delta t)$. The latter is defined with the risk utility function,

$$(4b) \quad \hat{u}(t; t+\Delta t) \equiv V_t^{-1}[E_t V_t(\tilde{u}(t+\Delta t))],$$

where $\tilde{u}(t+\Delta t)$ is the stochastic utility in time $t+\Delta t$.

We shall assume CIES and CRRA and use the functional forms

$$(5a) \quad U(c, \hat{u}; t, t+\Delta t) = e^{-\delta t} [c^\rho \Delta t + e^{-\delta \Delta t} \hat{u}^\rho]^{1/\rho}, \quad \delta > 0, \quad \rho < 1, \quad \rho \neq 0,$$

where the intertemporal elasticity of substitution is $1/(1-\rho) > 0$, and

$$(5b) \quad V(u) = u^{1-\gamma},$$

where the relative risk aversion is $\gamma > 0$, $\gamma \neq 1$.¹

III. Portfolio Choice

Next, we consider an agent who faces a portfolio and consumption problem as in Merton (1971), but has non-expected utility preferences as above. Let there be a riskless asset with instantaneous return r , and let there be n risky assets $i = 1, \dots, n$, whose rates of return $dq_i(t)/q_i(t)$ follow the Brownian motions

$$(6) \quad dq_i(t)/q_i(t) = \nu_i dt + \sum_j S_{ij} d\omega_j(t), \quad i = 1, \dots, n,$$

where $\omega(t) = (\omega(t))_{j=1}^m$ is an m -vector of independent Wiener processes ($m \geq n$). Let $W(t)$ denote wealth. For given consumption $c(t) = c$ and portfolio shares of risky assets $w(t) =$

¹ I thank Miles Kimball for giving me hints on how to formulate the recursive definition of the indirect utility function. After this was written, I have received a draft from Miles Kimball and research notes from Darrel Duffie and Larry Epstein with very related material. Kimball shows how the recursive definition can be written more compactly by taking the limit without specifying the functional forms to be CES. Duffie and Epstein specify preferences as an integral over instantaneous utility of consumption, indirect utility and instantaneous variance of indirect utility, and derive equilibrium restrictions on asset returns.

$(w_i(t))_{i=1}^n$ we can write the change in wealth during a short interval Δt as

$$(7) \quad W(t + \Delta t) - W(t) = [W(t)(w' \nu + r) - c] \Delta t + W(t) w' S \Delta \omega,$$

where $\nu = (\nu_i)$ is the n -vector of expected rates of return and $S = [S_{ij}]$ is the $n \times m$ "standard deviation" matrix of rates of return.

The decision problem of the agent is then to choose, at each instant of time, for given $W = W(t)$ a portfolio and a consumption level to maximize (4) subject to (7). The corresponding value function is denoted by $J(W, t)$.

We restrict the discussion to the special case CIES and CRRA in (5). We can exploit the special time dependence of the intertemporal utility function and guess that the indirect utility function fulfills

$$(8) \quad J(W, t) = e^{-\delta t} I(W).$$

The indirect utility function is then defined recursively as

$$(9) \quad I(W(t)) = \lim_{\Delta t \rightarrow 0+} \max_{c, w} \{c^\rho \Delta t + e^{-\delta \Delta t} [E_t I(W(t + \Delta t))]^{1-\gamma} \rho / (1-\gamma)\}^{1/\rho}$$

subject to (7).

Given the homogeneity of the budget constraint and the chosen functional forms, we may guess that the indirect utility and consumption functions are linear in wealth,

$$(10) \quad I(W) = A W, \quad A > 0, \quad \text{and} \quad c(W) = B W, \quad B > 0.$$

Substituting (10) in (9), applying Ito's Lemma, and taking the limit gives the Bellman equation (see appendix for details)

$$(11) \quad 0 = \max_{B, w} \{(B/A)^\rho - \delta + \rho[(w' \nu + r - B) - \gamma w' \sigma w / 2]\},$$

where $\sigma = SS'$ is the instantaneous covariance matrix for the rates of return (assumed nonsingular).

The first-order condition for the optimal portfolio gives

$$(12) \quad w^* = \sigma^{-1} \nu / \gamma.$$

Hence *the optimal portfolio depends on the risk aversion parameter but not on the intertemporal elasticity of substitution.*

The first-order condition for B gives

$$(13) \quad B^* = A^\rho / (\rho - 1).$$

Substitution of (12) and (13) in (11) finally gives

$$(14a) \quad A = \left[\frac{\delta - \rho[r + \nu' \sigma^{-1} \nu / 2 \gamma]}{1 - \rho} \right]^{(1 - 1/\rho)} \quad \text{and}$$

$$(14b) \quad B^* = \frac{\delta - \rho[r + \nu' \sigma^{-1} \nu / 2 \gamma]}{1 - \rho}.$$

We see that *the optimal consumption and saving decision via B depends on both the intertemporal elasticity of substitution and the risk aversion.*

When the intertemporal elasticity of substitution and relative risk aversion fulfill

$$(15) \quad \rho = 1 - \gamma,$$

the non-expected utility function above collapses to the standard additively separable intertemporal CES function. It is easy to see that the optimum portfolio and consumption/saving decision above then coincides with the one in Merton (1971).

Appendix

Substitution of (10) in (9) and division by $A W$ gives

$$(A.1) \quad 1 = \lim_{\Delta t \rightarrow 0+} \max_{B, w} \left\{ (B/A)^\rho \Delta t + e^{-\delta \Delta t} \left[E_t [1 + (w' \nu + r - B) \Delta t + w' S \Delta \omega]^a \right]^{\rho/a} \right\}^{1/\rho},$$

where $a = 1 - \gamma$. The term on the right hand side of (A.1) multiplying $e^{-\delta \Delta t}$ can be written

$$(A.2) \quad 1 + \rho[(w' \nu + r - B) + (a - 1)w' \sigma w / 2] \Delta t + o(\Delta t^2),$$

where we have applied Ito's Lemma to the expression under the expectations operator,

$$(A.3) \quad \begin{aligned} E_t [1 + (w' \nu + r - B) \Delta t + w' S \Delta \omega]^a \\ = 1 + a(w' \nu + r - B) \Delta t + a(a - 1)w' \sigma w \Delta t / 2 + o(\Delta t^2). \end{aligned}$$

(We compute $E_t df$, where $f(t, \omega) = [1 + (w' \nu + r - B)t + w' S \omega]^a$.) The expression in curly brackets on the right hand side of (A.1) can then be written

$$(A.4) \quad 1 + (1/\rho) \{ (B/A)^\rho - \delta + \rho[(w' \nu + r - B) + (a - 1)w' \sigma w / 2] \} \Delta t + o(\Delta t^2).$$

Subtracting unity, dividing by Δt , and taking the limit gives (11).

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