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RATIONAL BEHAVIOR

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ABSTRACT

A set of strategy profiles is here said to be closed under rational behavior (*curb*) if it contains all its best replies. Each *curb* set contains the support of at least one Nash equilibrium in mixed strategies, but there are perfect Nash equilibria that are not contained in any minimal *curb* set. It is shown that every game with compact strategy sets and continuous payoff functions possesses at least one minimal *curb* set, that every minimal *curb* set is identical with its best replies and that it is contained in the set of rationalizable strategy profiles.

1. Introduction

In game theory and its applications, the use of the Nash equilibrium concept is usually justified in terms of some implicit pre-play communication. While a Nash equilibrium is a point in the Cartesian product of the players' strategy spaces such that no player can increase his payoff by a unilateral deviation, a strict Nash equilibrium requires that any unilateral deviation actually incurs a loss. Any strict Nash equilibrium has all the strategic stability properties that the refinement literature asks for, but many games lack such equilibria. It is usually argued that Nash equilibrium is a necessary prerequisite for a strategy profile to be a "self-enforcing" pre-play agreement and that strict Nash equilibrium is a sufficient condition for such self-enforcement. In any non-strict Nash equilibrium, at least one player is indifferent between some of his pure strategies even under his Nash equilibrium beliefs. As is well known, such indifference can make the equilibrium highly "unstable".

The point is illustrated in Figure 1 below. The game in diagram (a) has a unique Nash equilibrium, and in this equilibrium both players randomize between their first two strategies, player 1 choosing T with probability $2/3$ and M with probability $1/3$, and player 2 choosing L with probability $1/4$. However, under these Nash equilibrium beliefs, player 1 is indifferent between T and M, and player 2 is indifferent between L and R. If player 1 would assign probability $p > 1/3$ to the uncertain event that 2 will choose L, then 1's unique optimal strategy is B. And if player 2 would assign probability $q > 1/3$ to the event that 1 will choose M, then 2's unique strategy is L etc. Strategy T, which is assigned probability $2/3$ in Nash equilibrium, is optimal for player 1 only if he assesses $1/4 < p < 1/3$. It thus appears that the unique Nash equilibrium of this game (hence its only perfect equilibrium) is "unstable". In particular, it does not seem justified to exclude, as does the Nash

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	L	R
T	4,1	1,2
M	1,3	2,1
B	6,2	0,3

(a)

	L	R
T	2,2	0,0
M	0,1	1,1
B	0,0	0,2

(b)

Figure 1

equilibrium criterion, the possibility that player 1 will play B , even if pre-play communication is presumed.

While the example in Figure 1 (a) suggests a coarsening of the Nash equilibrium concept, the same type of argument applied to the game in Figure 1 (b) suggests a refinement of the Nash equilibrium requirement. For although the Nash equilibrium (M,R) is undominated and hence perfect, a pre-play agreement to play (M,R) does not appear to be "self-enforcing", since also L is a best reply to M , and in view of this indifference on behalf of player 2, player 1 may contemplate playing T . If player 2 assigns positive probability to the possibility that 1 will play T , and a smaller probability to the event that 1 will play B , then 2's unique optimal strategy is L , in which case player 1 should certainly play T instead of M . It hence seems that we are led to a rejection of the perfect but non-strict Nash equilibrium (M,R) in favor of the strict equilibrium (T,L).

Below we will argue that even if one agrees to treat pre-play communication as possible but implicit (unmodeled), it is not clear why one should presume that "agreements" take the form of a single strategy (pure or mixed) profile, rather than, say, a set of strategy profiles. In Figure 1 (a), a "set-wise" Nash equilibrium agreement would be for player 1 to play either T or M , and for player 2 to play either L or R . However, for certain subjective probability assessments compatible

with such an agreement, player 1 would optimally play B. Hence, once such a shift in approach has been made, one may be lead to the acceptance of certain strategies that are not in the support of any Nash equilibrium. Or put differently: while it is true that the unique Nash equilibrium in the game in Figure 1 (a) is the only self-enforcing strategy profile, in the sense that no player has an incentive to deviate if he believes that the other player will randomize according to the Nash equilibrium probabilities, it is not clear why he should hold such beliefs even under an implicit pre-play agreement. And once one questions such specific beliefs, one is lead to consider whole sets of pure strategies, and beliefs over such sets. Indeed, a set-valued solution concept has been developed by Kohlberg and Mertens (1986). However, while they consider sets of Nash equilibria, i.e. sets *contained* in their own best replies, we here develop a set-valued solution concept which generalizes the notion of strict equilibrium by instead considering sets *containing* all their own best replies, a "dual" approach.

One set-valued solution concept developed in this paper is the '*tight curb*' notion, i.e. sets which are *identical* with their own best replies, and the associated notion of a minimal tight curb set. In the light of the work of Bernheim (1984) and Pearce (1984) one sees that a maximal tight curb set has to coincide with the set of rationalizable strategy profiles in a game with continuous payoff functions and compact strategy sets. Hence, while their work may be seen as an investigation into the properties of a maximal tight curb set, the present paper may be viewed as a study of the opposite end of the spectrum of tight curb sets. Another reason for highlighting minimal tight curb sets is that these are the nearest *set*-valued generalization of strict equilibria. In addition to studying *curb* sets, defined in terms of best replies, we also investigate a weaker notion, *curb** sets, based on undominated best replies.

This study is only a first search in a new direction, and in some games minimality may indeed be too restrictive a requirement. Minimality has the

advantage of reducing strategic ambiguity. But of course such an advantage may be too costly in terms of other game theoretic desiderata. Hence, instead of espousing minimality as a necessary requirement, we view this paper only as a step *towards* a selection theory of sets closed under rational behavior, for which purpose minimal curb (and curb*) sets constitute useful benchmarks.

2. Notation and preliminaries

Attention in this paper is focused on normal-form games $G=(N,S,U)$, where $N = \{1,2,\dots,n\}$ is the set of players, S is the Cartesian product of the players' strategy sets S_i , and U is a mapping of S into $\mathbb{R}^n$, such that $U_i(s) \in \mathbb{R}$ is the i 'th player's vonNeumann–Morgenstern utility level when strategy profile s is played. We will let $\mathcal{G}$ denote the class of all games G in which each strategy set S_i is a compact set in some Euclidean space and each payoff function $U_i:S \rightarrow \mathbb{R}$ is continuous. For any game G , let P be the collection of all products of nonempty and compact subsets of the players strategy sets, i.e. $X \in P$ if and only if X is the Cartesian product of nonempty compact sets $X_i \subset S_i$ [$i=1,2,\dots,n$]. (In particular, $S \in P$ if $G \in \mathcal{G}$.) As is usual, we will write $s \setminus s_i^?$ when the i 'th component of a profile s is replaced by $s_i^?$. A strategy profile $s \in S$ is a *Nash equilibrium* if $U_i(s) \geq U_i(s \setminus s_i^?) \forall i \in N, \forall s_i^? \in S_i$, and it is a *strict (Nash) equilibrium* if $U_i(s) > U_i(s \setminus s_i^?) \forall i \in N, \forall s_i^? \in S_i - \{s_i\}$.

In the present paper, a player's *belief* about others' strategies takes the form of a product probability measure on all players' strategy sets (we include his own strategy set in the domain only for notational convenience). Hence, a player's belief is formally identical with a mixed strategy profile $m \in M(S)$, where $M(S)$ is the Cartesian product of the sets $M(S_i)$ of Borel probability measures over each strategy set S_i . For any Borel subset $X_i \subset S_i$, let $M(X_i)$ be the (Borel) probability measures with

support in X_i , i.e. $M(X_i) = \{m \in M(S_i) : m(X_i) = 1\}$, and for any $X \in P$ let $M(X)$ be the Cartesian product of the sets $M(X_i)$.

For each player $i \in N$, strategy $s_i \in S_i$ and belief $m \in M(S)$, let $u_i(s_i | m)$ be the player's *expected utility* under belief m when he plays s_i .¹ Let $\beta_i(m)$ be the i 'th player's set of *optimal* strategies in S_i under belief $m \in M(S)$, i.e., strategies $s_i \in S_i$ such that $u_i(s_i | m) \geq u_i(s'_i | m) \forall s'_i \in S_i$. If $G \in \mathcal{G}$ (i.e., payoff functions are continuous and strategy sets compact), then each set $\beta_i(m) \subset S_i$ is nonempty and compact. For any set $X \in P$, let $\beta_i(X)$ denote the i 'th player's set of optimal strategies under beliefs in $M(X)$:

$$(1) \quad \beta_i(X) = \bigcup_{m \in M(X)} \beta_i(m),$$

and write $\beta(X)$ for the Cartesian product of the sets $\beta_i(X)$. This study is restricted to games G in $\mathcal{G}$, a class of games for which $X \in P$ implies $\beta(X) \in P$.

3. Sets closed under rational behavior

A set X of strategy profiles will be said to be *closed under rational behavior* (*curb*) if $X \in P$ and $\beta(X) \subset X$. In words: a set X of (pure) strategies is curb if it is a nonempty and compact product set, and if the belief that strategies outside X will not be played implies that such strategies will indeed not be played by any player who is rational in the sense of never playing strategies that are suboptimal under all beliefs.

In spirit, the present criterion is related to the notion of strict equilibrium, and, indeed, every such equilibrium, viewed as a singleton set, meets the curb condition:

¹ More exactly, we define $u_i : S_i \times M(X) \rightarrow \mathbb{R}$ by $u_i(s_i | m) = \int U_i(z \setminus s_i) dm(z)$. Note that $u_i(s_i | m)$ is functionally independent of the component $m_i \in M(S_i)$.

$s^* \in S$ is a strict equilibrium if and only if $\{s^*\} = \beta(\{s^*\})$. However, the curb criterion is also met by the set $X=S$ of all strategy profiles in the game, the set $X=S$ thus being the maximal curb set. Conversely, one may ask whether there exist *minimal* curb sets, i.e., curb sets which do not contain any proper subset which is a curb set:

Proposition 1: Every game $G \in \mathcal{G}$ possesses at least one minimal curb set.

Proof: Let Q be the (nonempty) collection of curb sets in S , partially ordered by (weak) set inclusion. By Hausdorff's Maximality Principle, Q contains a maximal nested sub-collection. Let $Q' \subset Q$ be such a sub-collection, and, for each $i \in \mathbb{N}$, let $\tilde{X}_i$ be the intersection of all sets X_i' for which $X' \in Q'$. Since each set X_i' is nonempty and compact, so is $\tilde{X}_i$, by the Cantor Intersection Theorem. Hence, $\tilde{X} \in P$. Suppose $s_i \in \beta_i(\tilde{X})$. Since $M(\tilde{X}) \subset M(X') \forall X' \in Q'$, we have $s_i \in \beta_i(X') \forall X' \in Q'$, and thus $s_i \in X_i' \forall X' \in Q'$ (since all $X' \in Q'$ are curb). Hence, $s_i \in \tilde{X}_i$, so $\beta_i(\tilde{X}) \subset \tilde{X}_i \forall i \in \mathbb{N}$, i.e., $\tilde{X}$ is curb.

Extending the definition of strict equilibrium from singleton sets to arbitrary product sets, we call a curb set X *tight* if $\beta(X) = X \forall i \in \mathbb{N}$. In particular, a profile $s \in S$ is a strict equilibrium if and only if $\{s\}$ is a tight curb set. Note that a tight curb set is "immune" to iterated elimination of suboptimal strategies under beliefs in $M(X)$. For if it is common knowledge that no player will use a strategy outside X , then each player knows that other (rational) players will play in $\beta(X)$ and hence each player should play in $\beta(\beta(X))$ etc. If X is curb, then all iterates $\beta^n(X)$ are subsets of X , so iterated elimination of suboptimal strategies does not lead players out of X . If X is tight curb, then such iteration has no effect at all: $\beta^n(X) = X$, for all n .

While many games lack strict equilibria, we proceed to show that every game G in $\mathcal{G}$ possesses at least one tight curb set. In fact, in such games, every minimal

curb set is tight, indeed a *minimal* tight curb set, i.e., it contains no proper subset which is a tight curb set. Conversely, every minimal tight curb set is a minimal curb set.

Proposition 2: A set in a game $G \in \mathcal{G}$ is a minimal curb set iff it is a minimal tight curb set.

Proof: First, suppose X is minimal curb but not tight. Then there exists some player $j \in N$ for which $\beta_j(X) \subset X_j$ and $\beta_j(X) \neq X_j$. Let $X'_j = \beta_j(X)$ and $X'_i = X_i \ \forall i \neq j$. Then $X' \subset X$, $X' \neq X$, $\beta_j(X') = X'_j$ and $M(X') \subset M(X)$, so $\beta_i(X') \subset \beta_i(X) \subset X_i = X'_i \ \forall i \neq j$. Hence, $\beta(X') \subset X'$. The payoff function u_j being continuous, the correspondence β_j from $M(X)$ to S_j is nonempty— and compact—valued, and, by Berge's Maximum Theorem upper hemi—continuous.² Moreover, $M(X)$ is compact, so its image $\beta_j(X) = \beta_j(X') = X'_j$ is nonempty and compact, i.e., $X' \in P$. In sum: X' is curb, contradicting the hypothesis that X is minimal. Thus, any minimal curb set is tight, and, being minimal among curb sets, it is minimal among tight curb sets. Secondly, suppose X is a minimal tight curb set. Applying the proof of Proposition 1 to the curb set X in the role of S , one establishes the existence of a minimal curb set $X' \subset X$. By the first part of the present proof, such a set X' is tight, and, since by hypothesis X is a minimal tight curb set, $X' = X$. Hence, X is a minimal curb set.

In order to relate the concept of a minimal curb set to established solution concepts, two further observations are useful, both being valid for all games G in $\mathcal{G}$. The first observation is that every curb set in such a game contains the support of at

² We endow $M(S)$ with the *topology of weak convergence*. A sequence (m^t) from $M(S)$ is said to *converge weakly* to $m \in M(S)$ if $\int f dm^t \rightarrow \int f dm$ for all continuous functions $f: S \rightarrow \mathbb{R}$. Since S is a compact metric space, so is $M(S)$, see e.g. Theorem 6.4 in Parthasarathy (1967).

least one Nash equilibrium in mixed strategies. To see this, suppose XCS is a curb set in $G=(N,S,U)$ and consider the game $G'=(N,X,U)$ obtained when players are restricted to the (nonempty and compact) strategy subsets X_i . Like G , the "subgame" G' meets the conditions of the Glicksberg Theorem concerning the existence of Nash equilibrium in mixed strategies. Now, if $m \in M(X)$ is such an equilibrium of G' , then it is also a Nash equilibrium of G , since by hypothesis each restricted strategy set X_i contains all best replies in S_i to strategies in X . Secondly, as noted in Bernheim (1984), the set RCS of rationalizable strategy profiles is nonempty and is the largest product set XCS satisfying the equality $X=\beta(X)$. Hence, if the set R is compact, then $R \in P$ and R is the maximal tight curb set. In an appendix at the end of the paper, we prove that the set R is indeed compact (and nonempty) in games $G \in \mathcal{G}$. In the light of this observation and Proposition 2, it should be clear why a minimal curb set and the set of rationalizable strategy profiles can be thought of as the two ends of a spectrum.

In order to illustrate these general findings, let us briefly consider some examples. Reconsidering Figure 1, one notes that the only curb set in diagram (a) is the full strategy space $\{T,M,B\} \times \{L,R\}$ itself, and it contains as a proper subset the support $(\{T,M\} \times \{L,R\})$ of the unique (mixed strategy) Nash equilibrium, which also happens to be quasi-strict. The only minimal curb set in diagram (b) in the same Figure is $\{T\} \times \{L\}$, the support of the unique strict equilibrium. Note, in particular, that the support of the undominated and hence perfect Nash equilibrium (M,R) is not contained in any minimal curb set.

To highlight the difference between minimal curb and rationalizability, consider a two-player game on the square $[0,3]^2$ in which both players have the same continuous payoff function f defined as follows: for $s \in [1,2]^2$ f is constant, say $f(s)=1$, while outside this smaller square, f declines linearly to zero along any straight

line connecting a point on a side of the smaller square with a point on the corresponding side of the larger square. The graph of f is thus a pyramid on a square with the top sliced off. In such a game of pure coordination, every strategy is rationalizable and no strategy profile is a strict equilibrium, while the unique minimal curb set is precisely the smaller central square, as intuition would suggest.

Figure 2 (a) shows a version of the "battle of the sexes" game with three curb sets, $\{T\} \times \{L\}$, $\{B\} \times \{R\}$ and $\{T,B\} \times \{L,R\}$, all of which are tight, but only two of which are minimal. The non-minimal curb set appears to be a plausible pre-play "agreement", viz. if the two players cannot agree on any of the two strict equilibria. The game in Figure 2 (b) has two minimal curb sets, $\{T\} \times \{L\}$ and $\{M,B\} \times \{M,R\}$, the first containing a strict equilibrium and the second a non-strict equilibrium. Intuitively, the second set seems the more likely pre-play agreement. The game in Figure 2 (c), finally, suggests that the curb requirement may in some games be too restrictive. Consider the set $\{T\} \times \{L\}$ in that game. This set is evidently not curb, since player 1 can costlessly deviate from T . Yet one could argue that it is "closed" if not under "rational" play, at least under "rational and cautious" play, since the only other best reply for player 1 to 2's strategy L is his weakly dominated strategy B . In fact, the only curb set (hence tight) is the full set $S = \{T,B\} \times \{L,R\}$, suggesting that the curb criterion may indeed be unduly restrictive in some games.

	L	R
T	3,1	0,0
B	0,0	1,3

(a)

	L	M	R
T	1,1	0,0	0,0
M	0,0	2,3	3,2
B	0,0	3,2	2,3

(b)

	L	R
T	1,1	1,0
B	1,0	0,0

(c)

Figure 2

4. Sets closed under rational and cautious behavior

The above observation about optimal but weakly dominated strategies suggests the following weakening of the curb criterion. For each player $i \in N$, let $S_i^* \subset S_i$ be his subset of strategies that are *weakly undominated*, i.e., for which there exists no mixed strategy $m_i \in M(S_i)$ which weakly dominates s_i .³ For any belief $m \in M(S)$, let $\beta_i^*(m) = \beta_i(m) \cap S_i^*$, the i 'th player's undominated optimal strategies under m , and for any $X \in P$, let $\beta_i^*(X) = \beta_i(X) \cap S_i^*$. A set $X \subset S$ will be called *closed under rational and cautious behavior* (*curb**) if $X \in P$ and $\beta^*(X) \subset X$. In words: a set X of (pure) strategies is *curb** if it is a nonempty and compact product set, and if the belief that strategies outside X will not be played implies that such strategies will indeed not be played by any player who is (a) rational in the sense of never playing strategies that are suboptimal under all beliefs and (b) cautious in the sense of never playing a weakly dominated strategy.⁴ This weaker criterion is evidently met by the set $X = \{T\} \times \{L\}$ in Figure 2 (c).

It is not difficult to verify that minimal *curb** sets exist in all games with continuous payoff functions and compact strategy sets — just apply the proof of Proposition 1 to the collection Q^* of *curb** sets. The proof of Proposition 2 does not apply, though. For while (in games $G \in \mathcal{G}$) the set $\beta(X)$ is nonempty and compact if X is, this is not generally true of the corresponding subset $\beta^*(X)$. For example, consider the two-player game on the unit square $S = [0,1]^2$ in which both players have

³ More exactly, but with a slight abuse of notation: there exists no $m_i \in M(S_i)$ such that $U_i(s' \setminus m_i) \geq U_i(s' \setminus s_i)$ for all $s' \in S$, with strict inequality for at least one $s' \in S$.

⁴ As argued in Weibull (1990), play of weakly dominated strategies can actually be both "rational" and "cautious" in games in which some player moves more than once in some play of the game. Hence, we take the present criterion to be generally valid only for games in which each player moves at most once in every play of the game.

the same payoff function g defined as follows. Along the diagonal $s_1=s_2$ let g be an inverted parabola with maximum in the interior and vanishing at the end points, say $g(s_1, s_1) = 1 - (2s_1 - 1)^2$. For each s_2 , let $g(s_1, s_2)$ be linearly increasing from zero when $s_1=0$ to $g(s_2, s_2)$ when $s_1=s_2$, and let g decrease linearly to zero as s_1 increases from s_2 to 1. Each payoff function thus being positive on the interior of S and zero on its boundary, we clearly have $S^* \subset (0,1)^2$. In fact, $\beta^*(S) = (0,1)^2$, since every $s_1 \in (0,1)$ is the unique best reply to $s_2=s_1$ (while evidently $\beta(S) = [0,1]^2$).

However, by a slight extension of Lemma 1 in Chapter I of Moulin (1982), one can show that the set $\beta^*(X)$ is nonempty for every $X \in P$ (for games in $\mathcal{G}$, see Proposition B in the appendix). Hence, if we use the slightly weaker tightness condition that X be contained in the *closure* of $\beta^*(X)$, which we call *almost tight*, then a minor elaboration of the proof of Proposition 2 applies to curb^* , mutatis mutandis, leading to the following parallel conclusion: a set is a minimal curb^* set iff it is a minimal almost tight curb^* set.⁵

⁵ More precisely, we define a set $X \subset S$ to be an *almost tight curb** set if $X \in P$ and $X = \overline{\beta^*(X)}$. Note that if $X \in P$ and $\beta^*(X) \subset X$, then $\overline{\beta^*(X)} \subset X$.

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APPENDIX

Bernheim (1984) and Pearce (1984) define the set $R_i \subset S_i$ of rationalizable strategies for player i as the intersection of all sets $R_i^t \subset S_i$, for $t=0,1,2,\dots$, defined recursively by $R_i^0 = S_i$ and $R_i^{t+1} = \{s_i \in R_i^t : \exists m \in M(R^t) \text{ s.t. } u_i(s_i|m) \geq u_i(s'_i|m) \forall s'_i \in R_i^t\}$. The following theorem is a variation on Proposition 3.2 in Bernheim (1984):

Proposition A: If $G \in \mathcal{G}$, then each set R_i is nonempty and compact.

Proof: Every set R_i^0 belongs to P . Suppose that for some nonnegative integer t every set $R_i^t \in P$. Then each set R_i^{t+1} is nonempty by continuity of U_i and compactness of R_i^t . We claim that R_i^{t+1} is also compact. To see this, consider the correspondence γ_i from $M(R^t)$ to R_i^t , defined by $\gamma_i(m) = \{s_i \in R_i^t : u_i(s_i|m) \geq u_i(s'_i|m) \forall s'_i \in R_i^t\}$. By Berge's Maximum Theorem, γ_i is compact-valued and u.h.c. (in the weak topology on $M(S)$), and $M(R^t)$ is compact (in the same topology). Hence, the image $R_i^{t+1} = \gamma_i(M(R^t))$ is compact. In sum: $(R_i^t)_{t=0}^\infty$ is a nested sequence of nonempty and compact sets. By Cantor's Intersection Theorem, their intersection R_i is nonempty and compact.

The following observation is a slight extension of Lemma 1 in Chapter I of Moulin (1982):

Proposition B: If $G \in \mathcal{G}$, then $S^* \neq \emptyset$.

Proof: Let $i \in N$. In order to show $S_i^* \neq \emptyset$, let $\mu \in M(S)$ be such that $\mu_j(A) > 0$ for every $j \neq i$ and every open set $A \subset S_j$. Let $s_i^* \in \beta_i(\mu)$. Then $u_i(s_i^*|\mu) \geq \int u_i(s_i|\mu) d\mu_i(s_i)$

$\forall m_i \in M(S_i)$. We claim $s_i^* \in S_i^*$. For suppose, on the contrary, that $m_i \in M(S_i)$ weakly dominates s_i^* . Then $\int U_i(s' \setminus s_i) dm_i(s_i) \geq U_i(s' \setminus s_i^*)$ for all $s' \in S$, with strict inequality for some $s'' \in S$. By continuity of U_i , there exist open sets $A_j \subset S_j \ \forall j \neq i$, such that strict inequality holds for all $s' \in S$ with $s'_j \in A_j \ \forall j \neq i$. By hypothesis, $\mu_j(A_j) > 0 \ \forall j \neq i$, so integration of the inequality over S with respect to μ gives $\int u_i(s_i | \mu) dm_i(s_i) = \iint U_i(s' \setminus s_i) dm_i(s_i) d\mu(s') > \int U_i(s' \setminus s_i^*) d\mu(s') = u_i(s_i^* | \mu)$, a contradiction to $s_i^* \in \beta_i(\mu)$.