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IMPLEMENTATION IN UNDOMINATED NASH
EQUILIBRIA WITHOUT INTEGER GAMES

by

Tomas Sjöström

INSTITUTE FOR INTERNATIONAL ECONOMIC STUDIES
Stockholm University

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S-106 91 Stockholm
Sweden

Implementation in Undominated Nash Equilibria Without Integer Games

Tomas Sjöström¹
Department of Economics, Littauer Center
Harvard University
Cambridge, MA 02138

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Abstract: A mechanism which uses neither integer games nor modulo games is shown to implement any social choice function in a production economy with private and (excludable) public goods. More precisely, the socially optimal outcome is the unique undominated Nash equilibrium outcome of a game where each player reports his own preferences and those of two neighbors. This equilibrium also results from the iterated elimination of weakly dominated strategies. There are no equilibria in mixed strategies. The mechanism has a natural interpretation in terms of optimal taxation theory.

1. Introduction

The implementation problem is the problem of designing a game form, or mechanism, with the property that the equilibrium outcomes of the mechanism coincide with the recommendations of an a priori given social choice rule. If this can be done, the social choice rule is said to be "implementable".

The concept of Nash equilibrium implies non-trivial restrictions on the class of implementable social choice rules. For example, if the social choice rule is single valued, continuous, Pareto optimal and

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individually rational, then it can be Nash-implemented only if it is Walrasian [see Hurwicz (1979) and Maskin (1977)]. However, researchers using other equilibrium concepts have reached totally different conclusions [for example Moore and Repullo (1988), Palfrey and Srivastava (1986) and Abreu and Sen (1987)]. These recent results seem to imply that problems of incentives and incomplete information disappear if only the planner is clever enough. With enough ingenuity, he can design a mechanism that makes the agents act precisely the way he wants them to. In short, "anything" can be implemented. In particular, in an important paper Palfrey and Srivastava (1986) showed that in economic environments any social choice rule can be implemented in undominated Nash equilibria.

Unfortunately, the results pertaining to Nash refinements do not yield insights into the structure of optimal mechanisms. They are basically existence theorems. For implementation in Nash equilibria, on the other hand, considerable insight into the structure of mechanisms was provided by the literature on the size of strategy spaces [for example Reichelstein and Reiter (1988)], and the papers on Nash implementation of the Walrasian and the Lindahl correspondences [for example Schmeidler (1980) and Walker (1981)].

A basic problem with the literature on implementation using Nash refinements is the frequent use of "integer games". The players are asked to announce an integer, and if certain strategy combinations occur, the player who announced the highest integer "wins". It is not clear that the equilibrium concepts that have been used actually describe how rational players (acting independently) would solve such "unusual" decision problems. The closely related "modulo games" are no more appealing.

In an important contribution, Jackson (1989) showed that in fact integer games may be necessary in the sense that some social choice rules can only be implemented if integer games are used. Thus, integer games seem to be an important part of implementation. And yet this is unfortunate because of their highly unintuitive nature.

However, Jackson's result does not apply to the economic environments we shall be considering. In fact, a mechanism where each player only reports his own preferences and those of two other individuals (his "neighbors") will be shown to implement any social choice rule in undominated Nash equilibria. "Integer games" or "modulo games" are not used. The undominated Nash equilibrium can be obtained by two rounds of elimination of weakly dominated strategies. Since each player can eliminate dominated strategies without knowing anything about the preferences of players who are not his "neighbors", the informational assumptions needed for this mechanism are rather weak. In addition, the mechanism has a natural interpretation in terms of optimal taxation theory.

In independent research, Jackson, Palfrey and Srivastava (1990) have also found that integer games are unnecessary in economic environments. Their mechanism is not a "revelation mechanism", since the players announce outcomes in addition to preferences. It is interesting that in fact no such auxiliary messages, only the players' private information, need to be reported.

In an interesting paper, Abreu and Matsushima (1991) have further developed the concept of implementation with iterated elimination of weakly dominated strategies. Their mechanism has the advantage of not requiring large "punishments". However, many more than two iterations may be necessary to arrive at an equilibrium. Thus, there is a trade-off between the complexity of solving the game and the severity of the "punishments".

A simple example will serve to introduce my mechanism. Let there be a single divisible private good in fixed supply of one unit, and three players. Players 1 and 2 always prefer more of the good to less, but player 3 has single-peaked preferences. The preferences of players 1 and 2 are known to the planner, but player 3's type is either $t_{\frac{1}{2}}$ or $t_{\frac{1}{3}}$. Type $t_{\frac{1}{2}}$ preferences peak at consumption $\frac{1}{2}$, i.e. $\frac{1}{2}$ is the best possible consumption for type $t_{\frac{1}{2}}$. Type $t_{\frac{1}{3}}$ preferences peak at consumption $\frac{1}{3}$. If player 3 is of type $t_{\frac{1}{2}}$, the social choice rule recommends the outcome $f(t_{\frac{1}{2}}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, but if he is of type $t_{\frac{1}{3}}$, the rule recommends $f(t_{\frac{1}{3}}) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Thus, player 3 prefers the $t_{\frac{1}{3}}$ allocation when he is of type $t_{\frac{1}{2}}$ and the $t_{\frac{1}{2}}$ allocation when he is of type $t_{\frac{1}{3}}$. Faced with such a malevolent social standard, he would like to conceal his true preferences. By a result due to Maskin (1977), the social choice function cannot be implemented in Nash equilibria. However, it can be implemented in undominated Nash equilibria by a mechanism with the following two features: (i) player 3's consumption is either independent of his own report, or else his report is used to select his preferred consumption bundle, and (ii) players 1 and 2 have an incentive to match player 3's announcement.

Let the three players simultaneously make a statement about player 3's preferences. Let t^i be player i 's statement. The outcome function is represented in figure 1. Player 1 picks a row, player 2 picks a column, and player 3 picks a matrix. For example, if $t^1=t^2=t_{\frac{1}{2}}$ and $t^3=t_{\frac{1}{3}}$, players 1 and 2 get nothing while player 3 gets $\frac{1}{3}$.

With these rules, the only time player 3's consumption depends on t^3 is when $t^1 \neq t^2$, "off the diagonal". And if in fact $t^1 \neq t^2$, it is clearly best for player 3 to tell the truth. Delete player 3's weakly dominated untruthful strategy. Since player 1 and player 2's only hope of getting a non-zero consumption is to match t^3 , the iterated elimination of weakly dominated strategies results in everyone reporting player 3's true type. Moreover, truth telling is the unique undominated Nash, perfect, proper and stable equilibrium. When the truth is told, the optimal outcome results.

The plan of the paper is as follows. Definitions are given in section 2. In order to motivate my results, section 3 contains an informal discussion of the "taxation principle". Section 4 contains the main result. Section 5 contains an impossibility result for the two person case.

2. The model

The set of players is $I=\{1, \dots, n\}$. Assume $n \geq 3$.

There are m divisible goods. Each player's consumption set is $\mathbb{R}_+^m$. Let x_j^i be player i 's consumption of good j , let $x^i=(x_1^i, \dots, x_m^i)$ and $x =$

$(x^1, \dots, x^n)$. The first r goods ($0 \leq r \leq m$) are pure private goods. The next $m-r$ goods are excludable pure public goods. (A pure exchange economy is a special case of this model).

A production plan $y = (y_1, \dots, y_m) \in \mathbb{R}_+^m$, where y_j denotes the net output of the j th good, is feasible if $y \in Y$, the technology set. Assume $0 \in Y$. For the private goods, aggregate feasibility requires that the sum of the consumptions is no greater than the net output. For the public goods, aggregate feasibility requires that no single agent consumes more than the net output. However, it is feasible for any agent to consume less (and this can be enforced by the planner). The output of the j th good is

$$y_j(x) = \begin{cases} \sum_{i \in I} x_j^i & \text{if } 1 \leq j \leq r \\ \max_{i \in I} x_j^i & \text{if } r < j \leq m \end{cases}$$

Let $y(x) = (y_1(x), \dots, y_m(x))$. A consumption plan $x \in \mathbb{R}_+^{mn}$ is feasible if $y(x) \in Y$. Let X be the set of all feasible consumption plans. Make the following assumptions.

Assumption 1. Free disposal: if $y \in Y$ and $0 \leq y' \leq y$, then $y' \in Y$.

Assumption 2. Convexity of the set of feasible consumption plans: if $x, x' \in X$, then $tx + (1-t)x' \in X$ for all $t \in [0, 1]$.

Assumption 2 can be substantially weakened: see the remark following the proof of Lemma 2 below.

Each player i has binary relation R_i on $\mathbb{R}_+^m$ indicating weak preference.

Assumption 3. R_i is complete and transitive.

For strong preference, $x P_i y$ if and only if it is not true that $y R_i x$.

Let $\mathcal{R}_i$ be player i 's set of possible weak preference relations. Assume $\#\mathcal{R}_i \geq 3$ for all $i \in I$ (player i has at least 3 "types"). Let $\mathcal{R} \equiv \times_{i \in I} \mathcal{R}_i$. For all $i \in I$, make the assumptions:

Assumption 4. Strictly convex preferences: for any $R_i \in \mathcal{R}_i$ and any $x, z \in \mathbb{R}_+^m$, if $x \neq z$ and $x R_i z$, then $(tx + (1-t)z) P_i z$ for all $t \in (0,1)$.

Assumption 5. Unique worst consumption bundle: for any $R_i \in \mathcal{R}_i$, if $x \geq 0$ and $x \neq (0,0,\dots,0)$, then $x P_i (0,0,\dots,0)$.

Assumption 6. Preference reversal: for any $R_i, R'_i \in \mathcal{R}_i$, if $R_i \neq R'_i$, then there exist $x, \tilde{x} \in X$ such that $x^i P_i \tilde{x}^i$ and $\tilde{x}^i P'_i x^i$.

These assumptions do not rule out satiation. Assumption 6 was introduced by Palfrey and Srivastava (1986).

For any set $B \subseteq \mathbb{R}_+^m$ and any $R_i \in \mathcal{R}_i$, a choice correspondence is defined as follows: $c(B, R_i) = \{x \in B : \text{for all } z \in B, x R_i z\}$. A choice set is any nonempty, compact and convex subset of $\mathbb{R}_+^m$. Note that $c(B, R_i)$ is a singleton if B is a choice set.

A social choice function is a function $f: \mathcal{R} \rightarrow X$, where $f(R) = (f_1(R), \dots, f_n(R))$. Agent i 's consumption is $f_i(R)$. Multi-valued functions are not considered in this paper. Make the assumption:

Assumption 7. Individual rationality: for all i and R , $f_i(R) \neq (0,0,\dots,0)$.

A game form $\Gamma = (S, h)$ consists of pure strategy spaces $S = \times S_i$, where S_i is player i 's strategy space, and an outcome function $h: S \rightarrow X$. A generic strategy for player i is denoted s^i . Let $s \equiv (s^1, s^2, \dots, s^n)$. Player i 's consumption bundle is $h_i(s)$. The pair (Γ, R) defines a game.

For any vector x , define $x_{-i} \equiv (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Similarly, for a set of vectors $x = (x^1, x^2, \dots, x^n)$, define $x^{-i} \equiv (x^1, x^2, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$. For any set of strategies $F = \times F^i \subseteq S$, a strategy $\hat{s}^i \in F^i$ is weakly dominated in F in the game (Γ, R) if there exists $s^{*i} \in F^i$ such that $h_i(s^{-i}, s^{*i}) R_i h_i(s^{-i}, \hat{s}^i)$ for all $s^{-i} \in F^{-i}$, and $h_i(s^{-i}, s^{*i}) P_i h_i(s^{-i}, \hat{s}^i)$ for some $s^{-i} \in F^{-i}$. Define $U_i(F; (\Gamma, R)) \equiv \{s^i \in F^i : s^i \text{ is not weakly dominated in } F \text{ in the game } (\Gamma, R)\}$.

A strategy profile $s \in S$ is a pure strategy Nash equilibrium for (Γ, R) if $h(s) R_i h(s^{-i}, \hat{s}^i)$ for all i and all $\hat{s}^i \in S^i$. A strategy profile $s \in S$ is an undominated Nash equilibrium for (Γ, R) if it is a Nash equilibrium and $s^i \in U_i(S; (\Gamma, R))$ for all $i \in I$. Let $E^{UN}(\Gamma, R)$ be the set of undominated Nash equilibria for (Γ, R) .

For the iterated elimination of weakly dominated strategies, let $U_i^1(\Gamma, R) \equiv U_i(S; (\Gamma, R))$. If $U_i^k(\Gamma, R)$ has been defined for all $i \in I$, define $U_i^{k+1}(\Gamma, R) \equiv U_i[\times_{j \in I} U_j^k(\Gamma, R) : (\Gamma, R)]$. Let $U_i^\infty(\Gamma, R) \equiv \bigcap_{k=1}^\infty U_i^k(\Gamma, R)$ and $U^\infty(\Gamma, R) = \times_{i \in I} U_i^\infty(\Gamma, R)$.

The game form Γ implements f in undominated Nash equilibria if, for all $R \in \mathcal{R}$, $E^{UN}(\Gamma, R)$ is a singleton, and $h(E^{UN}(\Gamma, R)) = f(R)$. The game form Γ implements f with iterated elimination of dominated strategies if, for all $R \in \mathcal{R}$, $U_i^\infty(\Gamma, R)$ is a singleton for all $i \in I$, and $h(U^\infty(\Gamma, R)) = f(R)$.²

²Early studies of implementation via elimination of dominated strategies are Farquharson (1969) and Moulin (1979). These authors considered environments with finite sets of alternatives. They did not give general characterization results for implementable social choice rules.

3. The taxation principle

This section, which draws on Guesnerie (1990), contains an informal discussion in order to motivate the "taxation mechanism" developed in section 4.

Suppose there is a continuum of agents, whose characteristics are drawn independently from a fixed distribution. The planner knows the distribution, but not the characteristics of individual agents. Furthermore, each agent knows his own characteristics (and the fixed distribution) but not the characteristics of other individual agents.

Assume that, when agent i has characteristics $R_i \in \mathcal{R}_i$, the social choice function recommends the consumption bundle $f_i(R_i)$. Suppose the planner asks agent i to reveal his characteristics. Clearly, if agent i has characteristics R_i^* but prefers $f_i(R_i)$ to $f_i(R_i^*)$, he will not tell the truth. Then the social choice function will not be implemented.

Thus, a necessary condition for implementation seems to be that, for any $R_i^* \in \mathcal{R}_i$, $f_i(R_i^*) \in c(\cup_{R_i \in \mathcal{R}_i} f_i(R_i), R_i^*)$. Moreover, the condition seems to be sufficient for implementation. For suppose the condition holds, and consider the following simple taxation mechanism. For each $i \in I$, let $S^i = \mathcal{R}_i$ and define the set $B_i \equiv \cup_{R_i \in \mathcal{R}_i} f_i(R_i)$. Give player i the consumption bundle ³

$$h_i(s) = c(B_i, s^i) \quad (1)$$

³If $c(B_i, s^i)$ is not a singleton, pick $f_i(s^i)$.

Since B_i is exogenously given to agent i , it is intuitive that he has an incentive to reveal his true preferences (i.e. if his true preferences are R_i^* , he picks $s^i = R_i^*$). This informal argument suggests a taxation principle: a social choice function can be implemented if and only if it can be implemented by a simple taxation mechanism. The terminology was suggested by Rochet (1984). In fact, the informal argument can be made rigorous [Guesnerie (1990)]. When the space of characteristics is the same for each agent and the social choice rule is anonymous, the taxation principle is closely related to no-envy in the sense of Foley (1967).

Unfortunately, if the assumption of a continuum of agents who know nothing about each other is dropped, the argument breaks down [see Guesnerie (1990)]. First of all, agent i 's optimal consumption will in general depend on the particular characteristics of all other agents (i.e. player i 's optimal consumption is $f_i(R)$, not $f_i(R_i)$). Second, it may be possible to get agent i to reveal information about agent $j \neq i$.

In the next section, I present a generalization of the simple taxation mechanism which works when the set of agents is finite, and the agents have knowledge of each others preferences (in fact it will suffice if each agent knows his two neighbors' preferences). Any social choice function can be implemented by this mechanism.

4. A taxation mechanism

I will restrict my attention to pure strategies. It will be clear from the proofs that this is without loss of generality. If mixed strategies and von Neumann-Morgenstern utilities are defined in the usual way, the unique undominated Nash equilibrium in pure strategies remains the unique undominated Nash equilibrium in the game where mixed strategies are allowed.

A taxation mechanism $\Gamma=(S,h)$ is constructed as follows. For each $i \in I$, $S^i \equiv \mathcal{R}_{i-1} \times \mathcal{R}_i \times \mathcal{R}_{i+1}$. Thus, each agent i reports the characteristics of himself and his two neighbors.⁴ A generic strategy is $s^i = (R_{i-1}^i, R_i^i, R_{i+1}^i)$. For each $i \in I$ and each $s^{-i} \in S^{-i}$, a choice set $B_i(s^{-i})$ is defined (see below - of course, the precise specification will be crucial). Player i 's consumption is computed as follows (cf (1) above):

$$h_i(s) = \begin{cases} c(B_i(s^{-i}), R_i^i) & \text{if } R_{i-1}^i = R_{i-1}^{i-1} \text{ and } R_{i+1}^i = R_{i+1}^{i+1} \\ (0, 0, \dots, 0) & \text{otherwise} \end{cases} \quad (2)$$

Some terminology: A K -tuple $\{s^{j_1}, s^{j_2}, \dots, s^{j_K}\}$ of strategies is totally consistent if whenever players $i, k \in \{j_1, \dots, j_K\}$ both announce the preferences of player $j \in I$ (including the case $j \in \{i, k\}$), $R_j^i = R_j^k$ holds. Conversely, a K -tuple $\{s^{j_1}, s^{j_2}, \dots, s^{j_K}\}$ is totally inconsistent if, whenever $i, k \in \{j_1, \dots, j_K\}$ both announce the preferences of player $j \in I$, then $R_j^i \neq R_j^k$. Again, the case $j \in \{i, k\}$ is included.

The main result of the paper is:

⁴References to player $i-1$, $i+1$ etc are always taken modulo $n+1$.

Theorem. Any social choice function can be implemented by a taxation mechanism.

The Theorem refers to double implementation in undominated Nash equilibria and with iterated elimination of dominated strategies: $h(E^{UN}(\Gamma, R)) = h(U^{\infty}(\Gamma, R)) = f(R)$ for all $R \in \mathcal{R}$. It will be proved by means of two Lemmas.

The intuition behind Lemma 1 is the following. Since R_i^i will only affect player i 's consumption through $c(B_i(s^{-i}), R_i^i)$, and $B_i(s^{-i})$ is independent of s^i , player i has an incentive to reveal his true preferences. I will make sure that the family of choice sets $\{B_i(s^{-i})\}$ is rich enough to make any untruthful statement concerning agent i 's own preferences dominated: for any two preference orderings $\tilde{R}_i, \hat{R}_i \in \mathcal{R}_i$ with $\tilde{R}_i \neq \hat{R}_i$, there will be a strategy profile $s^{-i} \in S^{-i}$ such that $c(B_i(s^{-i}), \tilde{R}_i) \neq c(B_i(s^{-i}), \hat{R}_i)$. This is a separation property of the choice sets. A similar idea was used by Palfrey and Srivastava (1986).

The outcome function punishes player i whenever his statements about his neighbors differ from his neighbors' reports about themselves. In equilibrium player i will always manage to avoid the punishment. But, in equilibrium his neighbors will report their own characteristics truthfully. Therefore, player i will tell the truth, not only about himself, but also about his neighbors.

To formalize the intuition, suppose the taxation mechanism satisfies the following four properties.

CONS. If s is totally consistent, then $h(s) = f(R_1^1, R_2^2, \dots, R_n^n)$.

FEAS. For all $s \in S$, $h(s) \in X$.

SEP. For all $i \in I$, all $\tilde{R}_i, \hat{R}_i \in \mathcal{R}_i$ such that $\tilde{R}_i \neq \hat{R}_i$, and all $(\hat{R}_{i-1}, \hat{R}_{i+1}) \in \mathcal{R}_{i-1} \times \mathcal{R}_{i+1}$, there exists $s^{-i} \in S^{-i}$ such that $R_{i-1}^{i-1} = \hat{R}_{i-1}$, $R_{i+1}^{i+1} = \hat{R}_{i+1}$ and $c(B_i(s^{-i}), \tilde{R}_i) \neq c(B_i(s^{-i}), \hat{R}_i)$.

NON. For all $i \in I$ and all $s^{-i} \in S^{-i}$, $B_i(s^{-i}) \cap \mathbb{R}_+^m \setminus \{0\} \neq \emptyset$.

Lemma 1. Suppose the family $\{B_i(s^{-i})\}$ of choice sets is such that CONS, FEAS, SEP and NON are satisfied. Then the taxation mechanism (i) implements f with iterated elimination of dominated strategies, and (ii) implements f in undominated Nash equilibria.

Proof. Let $R^* = (R_1^*, \dots, R_n^*)$ denote the true preferences.

For any $i \in I$, any $\hat{s}^i = (\hat{R}_{i-1}^i, \hat{R}_i^i, \hat{R}_{i+1}^i) \in S^i$ and any $s^{-i} \in S^{-i}$, $h_i(s^{-i}, (\hat{R}_{i-1}^i, R_i^*, \hat{R}_{i+1}^i)) R_i^* h_i(s^{-i}, \hat{s}^i)$. If $\hat{R}_i^i \neq R_i^*$, then by SEP there is s^{-i} (with $R_{i-1}^{i-1} = \hat{R}_{i-1}^i$ and $R_{i+1}^{i+1} = \hat{R}_{i+1}^i$) such that $h_i(s^{-i}, (\hat{R}_{i-1}^i, R_i^*, \hat{R}_{i+1}^i)) = c(B_i(s^{-i}), R_i^*) P_i^* c(B_i(s^{-i}), \hat{R}_i^i) = h_i(s^{-i}, \hat{s}^i)$. Therefore, $\hat{s}^i \notin U_i(S(\Gamma, R^*))$. On the other hand, if $\hat{R}_i^i = R_i^*$, then clearly $\hat{s}^i \in U_i(S(\Gamma, R^*)) = \{s^i \in S^i : R_i^i = R_i^*\}$.

If $s^{-i} \in \times_{j \neq i} U_j(S(\Gamma, R^*))$ and $R_j^i \neq R_j^*$, then $h_i(s) = 0$ by (2). Since NON implies there is always non-zero consumption possibilities, $U_i^2(\Gamma, R^*) = U_i^\infty(\Gamma, R^*) = \{(R_{i-1}^*, R_i^*, R_{i+1}^*)\}$. Similarly, $E^{UN}(\Gamma, R^*) = \{s^*\}$, where $s^{*i} = (R_{i-1}^*, R_i^*, R_{i+1}^*)$ for all $i \in I$. By CONS, $h(E^{UN}(\Gamma, R^*)) = h(U^\infty(\Gamma, R^*)) = f(R^*)$. $\square$

The Theorem is proved if it is shown that there always exists a family of choice sets such that the four properties hold.

Lemma 2. For any social choice function f , there exists a family of choice sets such that CONS, FEAS, SEP, and NON hold.

Proof. Consider $R_i, R'_i \in \mathcal{R}_i$, where $R_i \neq R'_i$. By assumption 6, there are feasible bundles x, y such that $x P_i y$ and $y P'_i x$. I will choose a pair (x, y) with the additional property that $x P_i z$ for any z on the line segment between x and y . See figure 2. Then, for this pair (x, y) , I define $\beta_i(R_i, R'_i) \equiv \{ z \in \mathbb{R}_+^m : z = tx + (1-t)y \text{ for } t \in [0, 1] \}$. I repeat the procedure for all ordered pairs of preference orderings in $\mathcal{R}_i$. Obviously, for all $R_i, R'_i \in \mathcal{R}_i$, $c(\beta_i(R_i, R'_i), R_i) \neq c(\beta_i(R_i, R'_i), R'_i)$. Let $\phi(i, s) \equiv (R_1^n, R_2^1, \dots, R_i^{i-1}, R_{i+1}^{i+2}, R_{i+2}^{i+1}, \dots, R_n^{n-1})$, and define $B_i(s^{-i})$ as follows.

$$B_i(s^{-i}) = \begin{cases} \{ f_i(\phi(i, s)) \} & \text{if } s^{-i} \text{ is totally consistent} \\ \beta_i(R_i^{i-1}, R_i^{i+1}) & \text{if } s^{-i} \text{ is totally inconsistent} \\ \{ (\frac{1}{n})f_i(\phi(i, s)) \} & \text{otherwise} \end{cases}$$

CONS and NON are clearly satisfied. Consider SEP. Fix $\tilde{R}_i, \hat{R}_i \in \mathcal{R}_i$ such that $\tilde{R}_i \neq \hat{R}_i$, and let $(\hat{R}_{i-1}, \hat{R}_{i+1}) \in \mathcal{R}_{i-1} \times \mathcal{R}_{i+1}$ be arbitrary. Choose some totally inconsistent $s^{-i} \in S^{-i}$ such that $R_{i-1}^{i-1} = \hat{R}_{i-1}$, $R_{i+1}^{i+1} = \hat{R}_{i+1}$, $R_i^{i-1} = \tilde{R}_i$ and $R_i^{i+1} = \hat{R}_i$. Then $B_i(s^{-i}) = \beta_i(\tilde{R}_i, \hat{R}_i)$. Since $c(\beta_i(\tilde{R}_i, \hat{R}_i), \hat{R}_i) \neq c(\beta_i(\tilde{R}_i, \hat{R}_i), \tilde{R}_i)$, SEP holds.

Consider FEAS. There are four cases.

Case 1. For some i , s^{-i} is totally inconsistent. Then $h_j(s) = 0$ for all $j \neq i$, so $h(s) \in X$.

Case 2. There is a unique i such that s^{-i} is totally consistent, but no j such that s^{-j} is totally inconsistent. Then $B_i(s^{-i}) = \{f_i(\phi(i, s))\}$, and $B_j(s^{-j}) = \{f_j(\phi(j, s))/n\}$ for all $j \neq i$.

First, suppose $R_{i+1}^i = R_{i+1}^{i+1}$ and $R_{i-1}^i = R_{i-1}^{i-1}$. In this case, since s^{-i} is totally consistent, $R_j^{j-1} = R_j^{j+1}$ for all $j \in I$. Hence there is $R \in \mathcal{R}$ such that

$\phi(j,s) = R$ for all $j \in I$. Since $f(R) \in X$, $h(s) \in X$ by free disposal.

Second, suppose either $R_{i+1}^i \neq R_{i+1}^{i+1}$ or $R_{i-1}^i \neq R_{i-1}^{i-1}$. Either way, $h_i(s)=0$. For all $j \in I$, $[f_1(\phi(j,s)), f_2(\phi(j,s)), \dots, f_n(\phi(j,s))] \in X$. By convexity of X , $[(\frac{1}{n})\sum_{j \in I} f_1(\phi(j,s)), (\frac{1}{n})\sum_{j \in I} f_2(\phi(j,s)), \dots, (\frac{1}{n})\sum_{j \in I} f_n(\phi(j,s))] \in X$. By free disposal, $[(\frac{1}{n})f_1(\phi(1,s)), \dots, (\frac{1}{n})f_n(\phi(n,s))] \in X$. Replacing $(\frac{1}{n})f_i(\phi(i,s))$ by 0 and invoking free disposal again, $h(s) \in X$.

Case 3. There are at least two players i and j , $i \neq j$, such that s^{-i} and s^{-j} are both totally consistent. Then s is totally consistent, so for some $R \in \mathcal{R}$, $h(s)=f(R) \in X$.

Case 4. For all $i \in I$, s^{-i} is neither totally consistent nor totally inconsistent. Then $B_i(s^{-i}) = \{(\frac{1}{n})f_i(\phi(i,s))\}$ for all $i \in I$. By the argument of the second part of case 2, $h(s) \in X$. $\square$

Remark: It is clear from the proof of Lemma 2 that assumption 2 may be replaced by: there is $\varepsilon > 0$ such that whenever $x, x' \in X$, $\varepsilon x + \varepsilon x' \in X$. However, the convexity of the choice sets would then be lost. That would complicate the mechanism, but would not change the basic results.

5. Two person implementation

The following example shows that results like those in section 4 cannot be obtained for the case $n=2$.

Suppose there are no public goods ($m=r$) and $Y = \{y \in \mathbb{R}_+^m : \sum y^i \leq \omega\}$, where ω

$\gg 0$ is the aggregate endowment. Preferences are strictly monotonic. Suppose f is a social choice rule such that $f_1(\mathcal{R}) \subseteq \{x \in X: x^1 \leq \omega/2\}$.

Let $\Gamma = (S, h)$ be a mechanism with strategy spaces $S_1 = S_2 = \mathcal{R}$, and suppose $E^{UN}(\Gamma, R) = \{(s_1, s_2)\} = \{(R, R)\}$ for all $R \in \mathcal{R}$. Consider $R_1, R_1^* \in \mathcal{R}_1$ such that for all $x, y \in \mathbb{R}_+^m \setminus \{z \in \mathbb{R}_+^m : z \gg \omega/2\}$, $x R_1 y$ if and only if $x R_1^* y$. That is, R_1^* and R_1 differ only in the way they rank bundles in $\{z \in \mathbb{R}_+^m : z \gg \omega/2\}$.

It follows from the definition of Nash equilibrium and strict monotonicity of preferences that, for all $s \in S$, $h(s) \subseteq X \setminus \{x \in X : x^1 \gg \omega/2\}$. Therefore, for any strategy profiles $s', s'' \in S$, $h_1(s') R_1 h_1(s'')$ iff $h_1(s') R_1^* h_1(s'')$. But then $E^{UN}(\Gamma, (R_1, R_2)) = E^{UN}(\Gamma, (R_1^*, R_2))$, a contradiction.

6. Conclusions

The taxation mechanism has several desirable properties.

- (i) Any social choice function can be implemented in our environment.
- (ii) The taxation mechanism makes no use of artificial "games within the game" for knocking out unwanted equilibria. The agents are not asked to announce integers, or to "raise flags" [cf Palfrey and Srivastava (1986)], or anything of that sort.
- (iii) There are no mixed strategy (undominated Nash) equilibria.
- (iv) A player needs to know only his own and his two neighbors' preferences in order to eliminate dominated strategies. Therefore, the informational requirements are rather modest. This is in contrast to game forms with integer games, where the assumption of common knowledge of strategies is crucial.

(v) The taxation mechanism is immune to the criticism raised by Mookherjee and Reichelstein (1987) against implementation in undominated strategies: that an equilibrium may be vulnerable to further iterations in the process of removing strategies. This follows from the double implementation in undominated Nash equilibria and with iterated elimination of weakly dominated strategies.

(vi) If the number of possible types for each agent is finite, then strategy sets in the taxation mechanism will be finite. In this case the unique undominated Nash equilibrium will also be the unique perfect equilibrium, the unique proper equilibrium, and the unique stable equilibrium.⁵ This follows from well-known existence results, and the fact that perfect, proper and stable equilibria are undominated Nash equilibria.

(vii) The agents are "telling the truth" at equilibrium. Therefore the equilibrium strategies are "focal points" which the players should be able to arrive at quite easily. In any case, a player should understand the following speech, if made by the planner: "Tell me what you are like. I will use that information to give you your best bundle in a certain set. This set does not depend on what you say you are like. Therefore you should tell the truth, or else you might get the wrong bundle! Your two neighbors will hear me make the same speech, so of course they will report their true characteristics to me. Now what do you think those reports will be ? Think carefully before you answer, for a liar will be severely punished !"

⁵These three equilibrium concepts are not defined for the case of infinite strategy sets.

As formulated here, the taxation mechanism has one basic drawback. One player's choice set might not depend continuously on his neighbors' reports, and the severe punishments for lying introduce a another discontinuity. I have not explored the issue of continuous implementation in this setting.

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	$t^3=t_{\frac{1}{2}}$		$t^3=t_{\frac{1}{3}}$	
	$t^2=t_{\frac{1}{2}}$	$t^2=t_{\frac{1}{3}}$	$t^2=t_{\frac{1}{2}}$	$t^2=t_{\frac{1}{3}}$
$t^1=t_{\frac{1}{2}}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	$\frac{1}{3}, 0, \frac{1}{2}$	$0, 0, \frac{1}{3}$	$0, \frac{1}{3}, \frac{1}{3}$
$t^1=t_{\frac{1}{3}}$	$0, \frac{1}{3}, \frac{1}{2}$	$0, 0, \frac{1}{2}$	$\frac{1}{3}, 0, \frac{1}{3}$	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$

Figure 1

Figure 2