A representation of extra-special 2-group, entanglement, and Berry phase of two qubits in Yang-Baxter system

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Abstract In this paper we show another representations of extra-special 2-groups. Based on this new representation, we infer a $\mathbb{M}$ matrix which obeys the extra-special 2-groups algebra relations. We also derive a unitary $\check{R}(\theta, \varphi)$ matrix from the $\mathbb{M}$ using the Yang-Baxterization process. A Hamiltonian for the two qubits is constructed from the unitary $\check{R}(\theta, \varphi)$ matrix. In this way, we study the Berry phase and entanglement of the two-qubit system. The results also establish relations between topological and holonomic quantum computation.

Keywords Qubits · Extra-special 2-group · Braiding group · Berry phase · Entanglement · Yang-Baxter system

1 Introduction

Recently, braiding quantum gates have played an important role in quantum computation [1–3]. It was shown that [2] a braiding quantum gate provide a universal two qubit quantum gate. More recently, three qubit braiding quantum gates have been constructed [4] and it can be shown that these gates produce maximally entangled GHZ states when they act on product states. In a recent paper [5] extra-special 2 groups has be used to produce higher dimensional unitary braiding gates. And it was realized that [6] the images of the unitary braid group representations are denoted as extensions of extra-special 2-groups. Therefore together with the braid group, extra-special 2-groups are seen to be important in quantum computation. Almost complex structures, which is fundamental in complex and symplectic geometries can be recognized as a representation of the extra-special 2-group [4].

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Solutions of the Yang-Baxter equation (YBE) namely R matrices play an increasing role both in quantum information theory and quantum field theories. In [3] unitary Yang-Baxter R matrices are also universal for quantum computation and therefore it enables us a novel way to study entanglement properties of braiding operators which have topological origin. To search physical properties of a Yang-Baxter system, such as evolution of entangled states, usually one constructs Hamiltonian from the unitary $\check{R}(\theta, \varphi)$ via Yang-Baxterization approach by which R matrix acquires θ dependence.

In this work our aim is three fold. One is to obtain a new class of representations of extra-special 2-groups based on the result of [5], the other is to construct 4×4 $\mathbb{M}$ where the matrix $\mathbb{M}$ satisfies the algebraic relations of extra-special 2-groups. And the third one, by using the $\mathbb{M}$ matrix, is to study the Berry phase and entanglement of the two-qubit Yang-Baxter system. The paper is organized as follows. In Sect. 2 we briefly overview extra-special 2-groups and almost complex structures based on [5, 4] and present another representation by using Hadamard gate and σ_y where σ_y is a pauli spin operator. In Sect. 3.2 we construct a new type of $\mathbb{M}$ matrix which satisfies extra-special 2-group relations. In Sect. 4, using the Yang-Baxterization approach, we obtain an $\check{R}(\theta, \varphi)$ matrix and then we investigate the entanglement properties of $\check{R}(\theta, \varphi)$ on two-qubit states by using concurrence as a measure of entanglement. From $\check{R}(\theta, \varphi)$ matrix, we also construct a Hamiltonian which enables us to study Berry phase. Finally, in Sect. 5 we discuss the procedure to obtain $\mathbb{M}$ matrix in terms of almost complex structures. Through out the paper, I denotes the identity matrix and I_m denotes the $m \times m$ identity matrix. Hadamard and Pauli matrices $H, \sigma_x, \sigma_y, \sigma_z$ have the usual forms

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.0.1)$$

2 Braiding group and extra-special 2-groups

In this section we review the construction of braiding group and extra-special 2-groups. Braid group B_n can be defined in terms of generators $b_1, \dots, b_{n-1}$ which satisfy the braid relations:

$$b_i b_j = b_j b_i, |i - j| \geq 2, \quad (2.0.2)$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, 1 \leq i \leq n - 2. \quad (2.0.3)$$

From these relations we obtain braided version of YBE, namely

$$(\check{R} \otimes I_d)(I_d \otimes \check{R})(\check{R} \otimes I_d) = (I_d \otimes \check{R})(\check{R} \otimes I_d)(I_d \otimes \check{R}), \quad (2.0.4)$$

with $d^2 \times d^2$ invertible matrix $\check{R} : V \otimes V \rightarrow V \otimes V$, where $V = \mathbb{C}^d$. Extra-special-2 groups are important for the studies of the images of the braid group B_n . Now, we will briefly review these groups based on [5]. Extra-special 2-group $\mathbf{E}_m$ is a group with generators $e_1, \dots, e_m$ satisfying the following relations:

- (i) $e_i^2 = -I$,
- (ii) $e_i e_j = e_j e_i$, $|i - j| \geq 2$, and
- (iii) $e_{i+1} e_i = -e_i e_{i+1}$, $1 \leq i, j \leq m - 1$,

where $-I$ is an order 2 central element. $\mathbf{E}_m$, of order 2^{m+1} , is an extra-special 2-group if the center $Z(G)$ and the commutator subgroup G' (which is generated by $[e_i, e_j] = e_i^{-1} e_j^{-1} e_i e_j$) are isomorphic to $\mathbb{Z}_2$ and $G/Z(G) \simeq (\mathbb{Z}_2)^m$. The commutator subgroup of $\mathbf{E}_m$ is $\{\pm I\}$ because of (anti-)commutation relations but when m is odd, we say $\mathbf{E}_m$ is nearly extra-special 2-group since the center of $\mathbf{E}_m$ does not coincide with the commutator subgroup but by adding or removing a generator we can obtain an extra-special 2-group. The center of $\mathbf{E}_m$ can be given as

$$Z(\mathbf{E}_m) = \begin{cases} \{\pm I\} & m \text{ even} \\ \{\pm I, e_1 e_3 \cdots e_m\} & m \text{ odd} \end{cases} \quad (2.0.5)$$

and when $m = 2k - 1$ is odd, the form of the center of $\mathbf{E}_{2k-1}$ depends on the parity of k

$$Z(\mathbf{E}_m) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & k \text{ even} \\ \mathbb{Z}_4 & k \text{ odd}. \end{cases} \quad (2.0.6)$$

In [5, 4] generators of nearly extra-special 2-group are defined in terms of almost complex structures which is a linear map from a real vector space to itself and satisfies the property $M^2 = -I$, where M denotes the almost complex structure. With the aid of the following proposition which is proved in [5], we can realize the representations of the braid group B_n from extra-special 2-group $\mathbf{E}_m$.

Proposition 2.0.1 *Let $\{T_1, \dots, T_{n-1}\}$ be the images of the generators $\{e_1, \dots, e_{n-1}\}$ of $\mathbf{E}_{n-1}$ under a representation ϕ_{n-1} of $\mathbf{E}_{n-1}$ such that:*

1. $T_i^2 = -I$,
2. $T_i T_j = T_j T_i$ if $|i - j| \geq 2$, and
3. $T_i T_{i+1} = -T_{i+1} T_i$ for all $1 \leq i \leq n - 2$.

Then

- (a) $\phi_{n-1}(e_i) = T_i$ defines a representation of $\mathbf{E}_{n-1}$ and ϕ_{n-1} has no 1-dimensional sub-representations.
- (b) If $\check{R}_i = \frac{1}{\sqrt{2}}(I + T_i)$, $1 \leq i \leq n - 1$, then $b_i \rightarrow \check{R}_i$ is an representation of braid group B_n .
- (c) And if the T_i are anti-Hermitian, in other words $T_i = -T_i^\dagger$, then B_n representation is unitary.

In the next section, we will present representations of extra-special 2-group $\mathbf{E}_m$ via almost complex structures.

3 Representations of $\mathbf{E}_m$ and almost complex structures

In the previous sections we have observed that extra-special 2-groups $\mathbf{E}_m$ can be represented in terms of almost complex structures. In this section we will elaborate this.

We will construct two classes of representations $\phi_m^i, i = 1, 2$ already appeared in [5, 4]. First class can be used to generate GHZ states of an even number of qubits.

Class1 Almost complex structures are $(2k)^2 \times (2k)^2$ matrix M^{JJ} with deformation parameters $q_{ij} \in \mathbb{C}$ can be given by

$$M^{JJ} = \sum_{i,j=-J}^J \epsilon(i)q_{ij} |ij\rangle \langle \bar{i}\bar{j}|, \quad \bar{i} = -i, \bar{j} = -j, \quad J = k - \frac{1}{2}, k \in \mathbb{N},$$

where q_{ij} is the deformation parameters and satisfy the constraints such that M^{JJ} becomes a representation of the extra-special 2-groups $q_{ij}q_{\bar{i}\bar{j}} = 1, q_{ij}q_{\bar{i}j} = q_{jl}q_{j\bar{i}}, q_{ij}^*q_{ij} = 1$, here q_{ij}^* is the complex conjugation of the corresponding deformation parameter and $\epsilon(i)$ is arbitrary but subject to the conditions $\epsilon(k)\epsilon(k) = 1, \epsilon(k)\epsilon(\bar{k}) = -1$ which have solutions $\epsilon(k) = \pm 1, \epsilon(\bar{k}) = \mp 1$ and $\bar{k} = -k$ or $\bar{k} = 2n - k + 1$ depending on the range of k . One can use separation of variables so that we can express $q_{ij} = q_i q_j$ and using this property, M^{JJ} recast to tensor product of two matrices M'_{2k} and P'_{2k}

$$M^{JJ} = M'_{2k} \otimes P'_{2k}, \quad M'_{2k} = \sum_{i=-J}^J \epsilon(i)q_i |i\rangle \langle \bar{i}|, \quad P'_{2k} = \sum_{j=-J}^J q_j |j\rangle \langle \bar{j}|. \quad (3.0.7)$$

M'_{2k} satisfy $(M'_{2k})^2 = -I_{2k}, (M'_{2k})^\dagger = -M'_{2k}$, and P'_{2k} satisfy

$$(P'_{2k})^2 = I_{2k}, (P'_{2k})^\dagger = P'_{2k} M'_{2k} P'_{2k} = -P'_{2k} M'_{2k}. \quad (3.0.8)$$

Class2 We can generalize the arguments above if we consider almost complex structure such that it takes the form $M_{2k_1} \otimes P_{2k_2}$ where $k_1 \neq k_2$. Note that in the above arguments we implicitly used the case $k_1 = k_2$ and if we consider the braided (YBE) that satisfy (2.0.2) and GHZ states we may define, for example $k_1 = 1, k_2 = 2^{N-2}, N \geq 2$ and hence we can obtain one class of representations of $\mathbf{E}_m$

$$M_{2N} = M_2 \otimes P_{2^{N-1}} = \sqrt{-1}\sigma_y \otimes \sigma_x^{\otimes N-1}.$$

Next, we will present another construction of M_{2N} using Hadamard gate and σ_y .

3.1 Another construction of M_{2N}

To present another possible construction of M_{2N} , we let M''_{2k_1} and P''_{2k_2} with $k_1 = 1, k_2 = 2^{N-2}, N \geq 2$ be defined as in the previous section. Then we obtain

$$M''_{2N} = M''_2 \otimes P''_{2^{N-1}} = \sqrt{-1}H \otimes \sigma_y^{\otimes N-1}, \quad (3.1.1)$$

where we used the superscript " to avoid confusion and H, σ_y are the Hadamard gate and pauli matrix respectively. For example, taking $N = 2$ yields a 4×4 matrix

$$M''_{4,2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}. \quad (3.1.2)$$

One can check that $M''_{4,2}$ can also give rise to unitary braid group representation under the proposition in 2.1 (b)

$$B''_{4,2} = \frac{1}{\sqrt{2}} (I_4 + M''_{4,2}). \quad (3.1.3)$$

Having armed with these ideas, we can present the following theorem in search of representations of extra-special 2-groups $\mathbf{E}_m$. The theorem and its proof are very similar to the theorem given in [5, 7].

Theorem 3.1.1 *Let ϕ_m^2 be a map into the set of unitary matrices $U(2^{N+k(m-1)})$ on generators of $\mathbf{E}_m$ by*

$$\phi_m^2(e_i) = I_{2^k}^{\otimes i-1} \otimes \sqrt{-1}H \otimes \sigma_y^{\otimes N-1} \otimes I_{2^k}^{\otimes m-i}, \quad i = 1, \dots, m, \quad (3.1.4)$$

where H is an Hadamard matrix. Then ϕ_m^2 is an anti-Hermitian representation of $\mathbf{E}_m$ for all $m \geq 2$ if and only if $\frac{N}{2} \leq k \leq N - 1$.

Proof Since H and σ_y is Hermitian $M_{2^N} = \sqrt{-1}H \otimes \sigma_y$ is anti-Hermitian because of the existence of $\sqrt{-1}$ in the expression. Similarly since $H^2 = I_2$, $(\sigma_y^{\otimes N-1})^2 = I_{2^{N-1}}$ the expression satisfies $M_{2^N}^2 = -I_{2^N}$ which implies $\phi_m^2(e_i)$ are also anti-Hermitian and satisfies the property $[\phi_m^2(e_i)]^2 = -I_{2^{N+k(m-1)}}$. Next, when $i \leq j$, $\phi_m^2(e_i)$ and $\phi_m^2(e_j)$ anti-commute if and only if $1 \leq k(j-i) \leq N-1$. For the third condition in Proposition 2.1 to be satisfied, we consider $j = i+1$ which yields $1 \leq k \leq N-1$. $\phi_m^2(e_i)$ and $\phi_m^2(e_j)$ commute if and only if $N-1 \leq k(j-i)$. For the second condition in Proposition 2.1 to be satisfied we take $j = i+2$, considering the case $m \geq 3$, we get $\frac{N}{2} \leq k \leq N-1$.

Next, we will give two examples that visualize the proof. The first one is very similar to the one given in [7]. To show the anti-commutation relations hold, we consider $\phi_2^2(e_1), \phi_2^2(e_2)$ such that

$$\begin{aligned} \phi_2^2(e_1)\phi_2^2(e_2) &= \left(\sqrt{-1}H \otimes \sigma_y^{\otimes N-1} \otimes I_{2^k}\right) \left(I_{2^k} \otimes \sqrt{-1}H \otimes \sigma_y^{\otimes N-1}\right) \\ &= \sqrt{-1}H \otimes \sigma_y^{\otimes k-1} \otimes \sqrt{-1}H \otimes I_{2^{N-k-1}} \otimes \sigma_y^{\otimes k}. \end{aligned} \quad (3.1.5)$$

$$\begin{aligned} \phi_2^2(e_2)\phi_2^2(e_1) &= \left(I_{2^k} \otimes \sqrt{-1}H \otimes \sigma_y^{\otimes N-1}\right) \left(\sqrt{-1}H \otimes \sigma_y^{\otimes N-1} \otimes I_{2^k}\right) \\ &= \sqrt{-1}H \otimes \sigma_y^{\otimes k-1} \otimes \sqrt{-1}H \otimes I_{2^{N-k-1}} \otimes \sigma_y^{\otimes k}. \end{aligned} \quad (3.1.6)$$

For $1 \leq k \leq N-1$, anti-commutation relations hold since Hadamard and σ_y anti-commutes. In order to show the commutation relations hold we consider the case $m \geq 3$ with $\phi_2^2(e_1), \phi_2^2(e_3)$ such that

$$\begin{aligned}\phi_2^2(e_1)\phi_2^2(e_3) &= \left(\sqrt{-1}H \otimes \sigma_y^{\otimes N-1} \otimes I_{2^k}\right) \left(I_{2^{2k}} \otimes \sqrt{-1}H \otimes \sigma_y^{\otimes N-1}\right) \\ &= \sqrt{-1}H \otimes \sigma_y^{\otimes N-1} \otimes I_{2^{2k-N}} \otimes \sqrt{-1}H \otimes \sigma_y^{\otimes N-1}\end{aligned}\quad (3.1.7)$$

$$\begin{aligned}\phi_2^2(e_3)\phi_2^2(e_1) &= \left(I_{2^{2k}} \otimes \sqrt{-1}H \otimes \sigma_y^{\otimes N-1}\right) \left(\sqrt{-1}H \otimes \sigma_y^{\otimes N-1} \otimes I_{2^{2k}}\right) \\ &= \sqrt{-1}H \otimes \sigma_y^{\otimes N-1} \otimes I_{2^{2k-N}} \otimes \sqrt{-1}H \otimes \sigma_y^{\otimes N-1}\end{aligned}\quad (3.1.8)$$

And this clearly commutes exactly when $\frac{N}{2} \leq k \leq N-1$. $\square$

3.2 Construction of $4 \times 4 \mathbb{M}$ matrix

In the following we present a $\mathbb{M}$ matrix which satisfies extra-special 2-group relations. For this purpose we will use Hadamard gate and pauli spin operator σ_y of the previous section. Define two 2×2 matrices such that

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & q \\ q^{-1} & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & -q \\ q^{-1} & 0 \end{pmatrix}, \quad (3.2.1)$$

where we choose the deformation parameters as $q = e^{i\varphi}$. Note that we have not deformed Hadamard gate and σ_y . We replace the deformation parameter q by hand. Now, $\mathbb{M}$ is given by

$$\mathbb{M} = A \otimes B. \quad (3.2.2)$$

In terms of spin operators $\mathbb{M}$ matrix reads

$$\begin{aligned}\mathbb{M} &= \frac{1}{\sqrt{2}} (e^{2i\varphi} (S_i^+ S_{i+1}^+) + e^{-2i\varphi} (S_i^- S_{i+1}^-) + S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+) \\ &\quad - 2e^{i\varphi} (S_i^z S_{i+1}^+) + 2e^{-i\varphi} (S_{i+1}^- S_i^z),\end{aligned}\quad (3.2.3)$$

where $S_i^+ = S_i^1 + iS_i^2$, $S_i^- = S_i^1 - iS_i^2$, and S_i^1, S_i^2 are the spin-1/2 angular momentum operators. In the computational basis of two qubits $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ $\mathbb{M}$ takes the following form

$$\mathbb{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -e^{i\varphi} & 0 & -e^{2i\varphi} \\ e^{-i\varphi} & 0 & 1 & 0 \\ 0 & -1 & 0 & e^{i\varphi} \\ e^{-2i\varphi} & 0 & -e^{-i\varphi} & 0 \end{pmatrix}. \quad (3.2.4)$$

Here $\mathbb{M}$ satisfies extra-special 2-group relations which can also be given as below [8]

1. $\mathbb{M}_{i,i+1}^2 = -I$,
2. $\mathbb{M}_{i,i+1}\mathbb{M}_{i+1,i+2}\mathbb{M}_{i,i+1} = \mathbb{M}_{i+1,i+2}$,
3. $\mathbb{M}_{i+1,i+2}\mathbb{M}_{i,i+1}\mathbb{M}_{i+1,i+2} = \mathbb{M}_{i,i+1}$.

In the following we Yang-Baxterize (3.2.4) (the procedure obtaining $\check{R}$ matrix for a given braiding operator) so that we can investigate the physical properties of the system.

4 Yang-Baxterization of the $\mathbb{M}$

It is known that [5] braiding operator can be obtained from the solutions of extra-special 2-group relations. In this section we present Yang-Baxterization of the $\mathbb{M}$ matrix. The Yang-Baxterization of the extra-special 2-group operator $\mathbb{M}$ [8]

$$\check{R}(x) = \rho(x)(\mathbb{I} + G(x)\mathbb{M}), \quad (4.0.5)$$

where $\rho(x)$, $G(x)$ are arbitrary functions to be determined and $\mathbb{I} = I_i \otimes I_{i+1}$ is the identity matrix. We can choose suitable $\rho(x)$ and $G(x)$ so that $\check{R}(x)$ is unitary. $\check{R}$ -matrix satisfies parameter dependent YBE which can be written as

$$\check{R}_{12}(x)\check{R}_{23}(xy)\check{R}_{12}(y) = \check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{12}(x), \quad (4.0.6)$$

where x and y are spectral parameters. Now, the condition on $\check{R}(x)$ becomes clear if we choose it to be unitary. Hence $\check{R}^\dagger(x) = \check{R}^{-1}(x)$ where $\check{R}^{-1}(x)$ is the inverse of corresponding matrix. Functions $\rho(x)$ and $G(x)$ are calculated in [9] in the case of 2 qubits. To obtain a unitary $\check{R}(x)$ matrix the functions can be given as $\check{R}(x) = \frac{1}{\sqrt{2(x^2+x^{-2})}}((x+x^{-1})I + (x-x^{-1})\mathbb{M})$. Following [9], if we define a new variable θ as $x = (-\cos 2\theta)^{-1/2}(\cos \theta + \sin \theta)$ and $x^{-1} = (-\cos 2\theta)^{-1/2}(\cos \theta - \sin \theta)$, then unitary $\check{R}(x)$ matrix takes the following form

$$\check{R}(\theta, \varphi) = \sin \theta \mathbb{I} + \cos \theta \mathbb{M}. \quad (4.0.7)$$

In the next section, we will use (4.0.7) to construct Hamiltonian and calculate the Berry phase afterwards.

4.1 R Matrix, Hamiltonian, Berry phase and entanglement of 2 qubits

We can also write unitary 4×4 $\check{R}(\theta, \varphi)$ in terms of spin operators as

$$\begin{aligned} \check{R}(\theta, \varphi) = & (\sin \theta (I_i \otimes I_{i+1}) + \frac{1}{\sqrt{2}} ((\cos \theta (e^{2i\varphi} (S_i^+ S_{i+1}^+) + e^{-2i\varphi} (S_i^- S_{i+1}^-) \\ & + S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+) - 2e^{i\varphi} (S_i^z S_{i+1}^+) + 2e^{-i\varphi} (S_{i+1}^z S_i^z))). \end{aligned} \quad (4.1.1)$$

There are two parameters θ, φ in the unitary matrix $\check{R}(\theta, \varphi)$. As in [10], if we let $\varphi = \omega t$ be time dependent while θ be time-independent, we can construct an

Hamiltonian by solving the Schrödinger equation as follows. If we define evolution of the wavefunction as

$$|\psi(\theta, \varphi(t))\rangle = \check{R}_i(\theta, \varphi(t)) |\psi(0)\rangle, \quad (4.1.2)$$

and recall the Schrödinger equation as

$$i\hbar \frac{\partial |\psi(\theta, \varphi(t))\rangle}{\partial t} = H(\theta, \varphi(t)) |\psi(\theta, \varphi(t))\rangle \quad (4.1.3)$$

and if we substitute the (4.1.2) into Schrödinger equation (4.1.3), few steps after we arrive at a Hamiltonian which is of the form

$$H(\theta, \varphi) = i\hbar \frac{\partial \check{R}_i(\theta, \varphi)}{\partial t} \check{R}_i^\dagger(\theta, \varphi), \quad (4.1.4)$$

where we drop the time dependence for convenience. Therefore, Hamiltonian can be written in matrix form as follows

$$H(\theta, \varphi) = \frac{\hbar\dot{\varphi} \cos \theta}{\sqrt{2}} \begin{pmatrix} -\frac{3}{\sqrt{2}} \cos \theta & e^{i\varphi} \sin \theta & \frac{1}{\sqrt{2}} e^{i\varphi} \cos \theta & 2e^{2i\varphi} \sin \theta \\ e^{-i\varphi} \sin \theta & \frac{1}{\sqrt{2}} \cos \theta & 0 & \frac{1}{\sqrt{2}} e^{i\varphi} \cos \theta \\ \frac{1}{\sqrt{2}} e^{-i\varphi} \cos \theta & 0 & -\frac{1}{\sqrt{2}} \cos \theta & -e^{i\varphi} \sin \theta \\ 2e^{-2i\varphi} \sin \theta & \frac{1}{\sqrt{2}} e^{-i\varphi} \cos \theta & -e^{-i\varphi} \sin \theta & \frac{3}{\sqrt{2}} \cos \theta \end{pmatrix}. \quad (4.1.5)$$

In the standard basis of two qubits $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ non degenerate eigenvalues of the Hamiltonian reads $E_{00} = \frac{1}{2}(-2 + \sqrt{2})\hbar\dot{\varphi} \cos \theta$, $E_{01} = \frac{1}{2}(2 - \sqrt{2})\hbar\dot{\varphi} \cos \theta$, $E_{10} = -\frac{1}{2}(2 + \sqrt{2})\hbar\dot{\varphi} \cos \theta$, $E_{11} = \frac{1}{2}(2 + \sqrt{2})\hbar\dot{\varphi} \cos \theta$. Corresponding eigenstates can be given as

$$|\chi_{00}(\theta, \varphi)\rangle = \frac{1}{N_+} \left(e^{2i\varphi} \cos \frac{\theta}{2} |00\rangle - (1 + \sqrt{2})e^{i\varphi} \sin \frac{\theta}{2} |01\rangle \right. \\ \left. + (1 + \sqrt{2})e^{i\varphi} \cos \frac{\theta}{2} |10\rangle + \sin \frac{\theta}{2} |11\rangle \right), \quad (4.1.6)$$

$$|\chi_{01}(\theta, \varphi)\rangle = \frac{1}{N_+} \left(-e^{2i\varphi} \sin \frac{\theta}{2} |00\rangle - (1 + \sqrt{2})e^{i\varphi} \cos \frac{\theta}{2} |01\rangle \right. \\ \left. - (1 + \sqrt{2})e^{i\varphi} \sin \frac{\theta}{2} |10\rangle + \cos \frac{\theta}{2} |11\rangle \right), \quad (4.1.7)$$

$$|\chi_{10}(\theta, \varphi)\rangle = \frac{1}{N_-} \left(-e^{2i\varphi} \cos \frac{\theta}{2} |00\rangle + (-1 + \sqrt{2})e^{i\varphi} \sin \frac{\theta}{2} |01\rangle \right. \\ \left. + (-1 + \sqrt{2})e^{i\varphi} \cos \frac{\theta}{2} |10\rangle + \sin \frac{\theta}{2} |11\rangle \right), \quad (4.1.8)$$

$$|\chi_{11}(\theta, \varphi)\rangle = \frac{1}{N_-} \left(e^{2i\varphi} \sin \frac{\theta}{2} |00\rangle + (-1 + \sqrt{2}) e^{i\varphi} \cos \frac{\theta}{2} |01\rangle \right. \\ \left. - (-1 + \sqrt{2}) e^{i\varphi} \sin \frac{\theta}{2} |10\rangle + \cos \frac{\theta}{2} |11\rangle \right), \quad (4.1.9)$$

where $N_{\pm} = \sqrt{4 \pm 2\sqrt{2}}$. Berry's theory [11] tells us that under the adiabatic approximation, $\varphi(t)$ can not be removed from the system and it will give rise to non-trivial phase called Berry phase. Therefore Berry phases for the above eigenstates read

$$\gamma_{00,11} = i \int_0^T dt \langle \chi_{\pm}(\theta, \varphi) | \frac{\partial}{\partial t} | \chi_{\pm}(\theta, \varphi) \rangle = \pi \left(-2 + \left(\pm 1 + \frac{1}{\sqrt{2}} \right) \cos \theta \right) \quad (4.1.10)$$

$$\gamma_{01,10} = \pi \left(-2 + \frac{\cos \theta}{\pm 2 + \sqrt{2}} \right), \quad (4.1.11)$$

where subscripts in the geometric phase γ denotes corresponding eigenstates. For example $\gamma_{00} = i \int_0^T dt \langle \chi_{00}(\theta, \varphi) | \frac{\partial}{\partial t} | \chi_{00}(\theta, \varphi) \rangle$. As one can observe, these Berry phases are deformed. And we could not find any transformation that transfer it to a standard form.

4.1.1 Entanglement of two qubit

The concurrence as a measure of entanglement can easily be calculated. When $\check{R}(\theta, \varphi)$ matrix acts on direct product states $|kl\rangle = |k_i\rangle \otimes |l_{i+1}\rangle$, it produces entangled states. In this way, one can find the concurrences as

$$C(\check{R}(\theta, \varphi) |kl\rangle) = \frac{1}{\sqrt{2}} |\sin 2\theta|, \quad \text{for } k, l = 0, 1, \quad (4.1.12)$$

where C is calculated for all states and remarkably, C possess the same entanglement degree. But it can easily be observed that they do not give rise to Bell states when especially $\theta = \pi/2$ because of the factor $1/\sqrt{2}$. These non-maximally entangled states can be used in quantum teleportation and could be realized in optical techniques [12]. However, concurrences of eigenstates read

$$C(|\chi_{kl}(\theta, \varphi)\rangle) = |\sin \theta|, \quad \text{for } k, l = 0, 1, \quad (4.1.13)$$

where $C(|\chi_{kl}(\theta, \varphi)\rangle)$ is evaluated for all eigenstates. It is worth noting that $C(|\chi_{kl}(\theta, \varphi)\rangle)$ for all k and l has the same degree of entanglement.

5 Conclusions and discussions

In summary we have shown that Hadamard gate together with Pauli matrix(σ_y) give rise to a new class of representations of extra-special 2-groups. Based on this representation

we have inferred $\mathbb{M}$ matrix which satisfies extra-special 2-group relations and derived a unitary $\check{R}(\theta, \varphi)$ matrix via Yang-Baxterization of the $\mathbb{M}$ matrix. Then, the evolution of the Yang-Baxter system is evaluated by constructing a Hamiltonian from the $\check{R}(\theta, \varphi)$ matrix. Berry phase and entanglement properties of the systems are investigated and calculation of the concurrence shows that constructed unitary $\check{R}(\theta, \varphi)$ matrix is not perfect entangler, e.g., it can not generate Bell states when it acts on direct product states. However, the results are interesting in construction of topological and holonomic quantum gates. Let us make a discussion to end the paper. Although the matrix $\mathbb{M}$ satisfies the algebraic relations of extra-special 2-groups, the construction of $\mathbb{M}$, especially when the deformation parameters in $\mathbb{M}$ is considered, may not be expressed as in terms of almost complex structures.

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