Multiplier sequences for Laguerre bases
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MULTIPLIER SEQUENCES FOR LAGUERRE BASES

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In memory of Julius Borcea

Abstract
Let the real sequence \( \{\lambda_k\}_{k=0}^\infty \) define a linear operator \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) by the action \( T(x^k) = \lambda_k x^k \). Pólya and Schur characterized all such operators in their seminal work from 1914. The corresponding sequence \( \{\lambda_k\}_{k=0}^\infty \) is called a multiplier sequence. In 2009 Borcea and Brändén gave a complete characterization of general linear operators preserving real-rootedness (and stability) via the symbol. Relying on these results we completely characterize multiplier sequences for generalized Laguerre bases – that is, we characterize all real-rootedness preserving linear operators, \( T \), such that \( T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x) \), where \( L_n^{(\alpha)}(x) \) is the \( n \)th generalized Laguerre polynomial. We also apply our methods to reprove the characterization of Hermite multiplier sequences achieved by Piotrowski in 2007.

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1. Introduction

The distribution of zeros of entire functions and the theory of equations is a vast and versatile area that has interested many mathematicians, including Laguerre, Pólya, Cauchy, Sturm, Jensen, Marden and Turan. The dynamics of root loci under linear transformations is of fundamental importance in this area. Recently a complete characterization of operators preserving real-rootedness (and stability) was achieved by Borcea and Brändén in [3] and thereby settled the following problem in some of the most fundamental cases.

**Problem 1.** Characterize all linear transformations (operators) such that

\[ T : \pi(\Omega) \to \pi(\Omega) \cup 0 \]

where \( \Omega \in \mathbb{C} \) and \( \pi(\Omega) \) is the class of all (complex or real) univariate polynomials whose zeros lie in \( \Omega \).

The work in this thesis relies heavily on the characterization proved by Borcea and Brändén. We will study linear operators acting on real univariate polynomials, which are diagonal with respect to a sequence of orthogonal polynomials. The Hermite and Laguerre polynomial bases are investigated in detail, and we characterize the linear operators that preserve real-rootedness with respect to these bases.

One motivation to solve problems of this type stems from the study of the analytic continuation of the function

\[ \zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}, \]

known as the Riemann \( \zeta \)-function. Riemann conjectured that \( \zeta(x) \) has all its non-trivial zeros on the line \( \{ x : \text{Re}(x) = 1/2 \} \). Consider the function

\[ \xi(x) = \Gamma\left( \frac{x}{2} + 1 \right)(x - 1)^{-x/2} \zeta(x). \]

This is the \( \xi \)-function, which is entire. The Riemann hypothesis can be equivalently reformulated as the claim that the zeros of \( \xi(1/2 + ix) \), are real. The characteristics of the class of entire functions with only real zeros is thus of great interest.

As pointed out by Bleecker and Csordas in [2], the work by Turán ([19], 1950) regarding the Riemann function shed some light on how to investigate entire functions with zeros in a strip. Let the strip of half-width \( A \) around the real axis be

\[ S(A) = \{ z \in \mathbb{C} : |\text{Im}(z)| \leq A \} \]

for some \( A \geq 0 \). Turán suggested that it may be beneficial to expand such functions in terms of Hermite polynomials,

\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left( e^{-x^2} \right), \quad n = 0, 1, 2, \ldots. \quad (1.1) \]

Intuitively the reason for this was that the level sets for the standard monomial basis \( \{ z^n \}_{n=0}^{\infty} \) are given by concentric circles,

\[ \{ z \in \mathbb{C} : |z|^k = c, \ c > 0 \}, \]

whereas the level sets of Hermite polynomials, \( H_n \), when \( n \) is large are approximately horizontal lines. It is therefore natural to study linear operators
that are defined by their action on Hermite polynomials, when it comes to preserving real-rootedness or the property of having all zeros in a strip.

Very important contributions to the problem of classifying linear operators preserving real-rootedness were made by Pólya and Schur in the beginning of the 20th century. They introduced and characterized multipler sequences. Starting in the late seventies, investigations of Craven and Csordas led them to extensions of classical theorems of Polya and others, initiating a renaissance in the field. A survey of this field as of 2004 is found in [9]. The field has continued to emerge with many contributors, in particular by additions from Borcea and Brändén, who proved classification theorems for linear operators and extended them to the multivariate setting.

**Problem 2.** Let \( \mathcal{P} = \{ P_n(x) \}_{n=0}^{\infty} \) be a sequence of real polynomials. For a sequence \( \{ \lambda_n \}_{n=0}^{\infty} \) of real numbers, define a linear operator \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) by
\[
T(P_n(x)) = \lambda_n P_n(x), \quad \text{for all } n \in \mathbb{N} := \{0, 1, 2, \ldots\}.
\]
Characterize the sequences \( \{ \lambda_n \}_{n=0}^{\infty} \) for which \( T \) preserves real-rootedness.

We call such a sequence a \( \mathcal{P} \)-multiplier sequence, while the term multiplier sequence is reserved for the classical case \( \mathcal{P} = \{ x^n \}_{n=0}^{\infty} \). The case of Problem 2 when \( \mathcal{P} = \{ x^n \}_{n=0}^{\infty} \) goes back to Laguerre and Jensen and was completely solved by Pólya and Schur in [16], see also [9, 14]. Turán [19] and subsequently Bleecker and Csordas [2] provided classes of multiplier sequences for the Hermite polynomials \( \mathcal{H} = \{ H_n(x) \}_{n=0}^{\infty} \), while Piotrowski completely characterized \( \mathcal{H} \)-multiplier sequences in [15]. Recently partial results regarding multiplier sequences for the generalized Laguerre bases [11], and for the Legendre bases [1], were achieved. A more general approach can be found in [12]. In [1] also the following set inclusions were established:

![Diagram of multiplier sequences](image)

The characterizations of multiplier sequences for Legendre polynomials and other important bases are still open. In this report we completely characterize the multiplier sequences for Laguerre polynomial using the theory on stability preservers established in [3]. The results in this thesis are joint work with Brändén [8].

As we shall see in Section 4.1 the (generalized) Laguerre polynomials, \( \mathcal{L}_\alpha = \{ L_n^{(\alpha)}(x) \}_{n=0}^{\infty} \), may be defined by
\[
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \frac{(-x)^k}{k!}, \quad \alpha > -1,
\]
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see [18]. Let the linear operator \( T \) be identified by

\[
T(L^{(\alpha)}_n(x)) = \lambda_n L^{(\alpha)}_n(x)
\]

where \( L^{(\alpha)}_n \) is the \( n \)th Laguerre polynomial. If \( T \) is real-rootedness preserving then \( \{\lambda_n\} \) is called a \( L^{(\alpha)} \)-multiplier sequence. Later on we shall see that any \( P \)-multiplier sequence where \( P \) is a simple set of polynomials is also a multiplier sequence in the classical sense. In particular it was proved in [1] that any \( L^{(\alpha)} \)-multiplier sequence is a multiplier sequence in the classical sense. Our main result is the following

**Theorem 1.** Let \( p(y) = \sum_{k=0}^{\infty} \binom{k+\alpha}{k} a_k y^k \) be a formal power series where \( \alpha > -1 \), and let \( \{\lambda_n\}_{n=0}^{\infty} \) be defined by

\[
\lambda_n := \sum_{k=0}^{n} a_k \binom{n}{k}.
\]

Then \( \{\lambda_n\}_{n=0}^{\infty} \) is a \( L^{(\alpha)} \)-multiplier sequence if and only if \( p(y) \) is a real-rooted polynomial with all its zeros contained in the interval \([-1,0]\).

**Remark 1.** Let \( \{\lambda_n\}_{n=0}^{\infty} \) be defined as in Theorem 1. Then since \( \binom{n}{k}^{-1} = [(n-k)\binom{n}{k}] \) we may also solve for \( a_k \)

\[
a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \lambda_k.
\]

It follows that we can express \( p(y) \), defined in Theorem 1 in terms of the sequence \( \lambda_n \) as

\[
p(y) = \sum_{k=0}^{\infty} \lambda_k \binom{\alpha+k}{k} y^k (1+y)^{-k-\alpha-1}.
\]

As in Theorem 1 the zeros of \( p(y) \) are real and contained in \([-1,0]\) if and only if \( \lambda_n \) is a \( L^{(\alpha)} \)-multiplier sequence.

The idea behind the proof of this turns out to be useful in other cases and we are able give an alternative proof of the theorem regarding Hermite polynomials due to Piotrowski [15]. Consider the case when \( \mathcal{P} \) is the set of Hermite Polynomials, \( H_n(x) \) as in (1.1). If the diagonal operator with respect to this basis preserves real-rootedness we say that the corresponding sequence of eigenvalues \( \{\lambda_n\}_{n=0}^{\infty} \) is a \( \mathcal{H} \)-multiplier sequence, where \( \mathcal{H} = \{H_n\}_{n=0}^{\infty} \). The following theorem characterizes \( \mathcal{H} \)-multiplier sequences.

**Theorem 2** (Piotrowski [15]). The sequence of real numbers \( \{\lambda_k\}_{k=0}^{\infty} \) is a \( \mathcal{H} \)-multiplier sequence if and only if it is a multiplier sequence with \( 0 \leq \lambda_k \leq \lambda_{k+1} \) for all \( k \geq 0 \).

In Theorem 2 it is sufficient to consider only positive \( \lambda_k \), since \( \{\lambda_k\}_{k=0}^{\infty} \) is a \( \mathcal{H} \)-multiplier sequence if and only if \( \{(-1)^k \lambda_k\}_{k=0}^{\infty} \) is a \( \mathcal{H} \)-multiplier sequence.
2. Preliminaries

There are many classical results in complex analysis that concern the classification of entire functions with zeros in a certain prescribed domain or the counting of zeros. Consider for example the Principle of Argument that counts the difference between the number of zeros and the number of poles via integration of the logarithmic derivative on a Jordan curve. Actually the Fundamental Theorem of Algebra is yet another exercise in counting zeros of a polynomial. Although results like these are of relevance to our subject it is not in our scope to present them here. Instead we refer the reader to [17].

However we shall need some fundamental theory from complex analysis. Recall that an entire function \( f(z) \) is a holomorphic function of a complex variable \( z \). Thus \( f(z) \) can be represented as

\[
 f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots
\]

where this power series is everywhere convergent. One can therefore think of entire functions as a natural generalization of a polynomial.

For the readers convenience we now state the classical result due to Hurwitz.

**Theorem 3 (Hurwitz, [17]).** Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of analytic functions defined in a region \( \Omega \subset \mathbb{C} \). Suppose that this sequence converges to a function \( f \neq 0 \), uniformly on every compact subset of \( \Omega \). Then \( \zeta \in \Omega \) is a zero of \( f \) of multiplicity \( m \) if and only if there exists a neighbourhood \( V \subset \Omega \) of \( \zeta \) such that, in every disc \( D(\zeta, \varepsilon) \subset V \), each polynomial \( f_n \) whose index exceeds some bound \( n(\varepsilon) \) has exactly \( m \) zeros, counted according to their multiplicities.

2.1. The order and type of an entire function. Weierstrass showed that an entire function can be represented as a product in terms of its zeros and then the connection between polynomials and entire functions was furthered investigated by Laguerre. For example, Laguerre introduced the notion of the genus of an entire function. The growth rate of a polynomial is closely related to the number of zeros via the degree and Borel, Hadamard and Lindelöf discovered that a similar connection between growth and the distribution of zeros of an entire function exists. In this section we introduce the order and type of an entire function. These notions are entities that describe the growth of an entire function. The order and type are thoroughly discussed in [14].

Let \( f(z) \) be an entire function and \( M(r) := \max |f(z)| \) where \( |z| = r \). Assume there exists a positive constant \( k \) such that \( M(r) < e^{r^k} \). Then the greatest lower bound of \( k \) is called the order of \( f(z) \) and this may be defined as:

\[
 \rho := \lim_{r \to \infty} \frac{\ln \ln M(r)}{\ln r}.
\]

The type of an entire function \( f(z) \) of finite order \( \rho \) is

\[
 \sigma := \lim_{r \to \infty} \frac{\ln M(r)}{r^\rho},
\]
and this is the greatest lower bound of a positive number $A$ for which, asymptotically, $M(r) < e^{Ar^p}$. Theorem 2 in [14] now gives the following relation between the order and the type in terms of its Taylor coefficients:

**Theorem 4.** The order, $\rho$, and the type $\sigma$ of an entire function are expressed in terms of its Taylor coefficients by the following equations:

$$\rho = \lim_{n \to \infty} \frac{n \ln n}{\ln |c_n|}, \quad (\sigma e^\rho)^{\frac{1}{\rho}} = \lim_{n \to \infty} \frac{1}{n} c_n^{\frac{1}{\rho}}$$

where $c_n$ is the $n$th Taylor coefficient.

**Remark 2.** In the above $\lim_{r \to \infty} \phi(r) = \lim_{r \to \infty} (\sup \phi(t))_{t \geq r}$.

### 2.2. The Laguerre–Pólya class and multiplier sequences.

We now introduce the Laguerre–Pólya class which is a necessary definition for the Pólya–Schur characterization of multiplier sequences.

**Definition 1** (The Laguerre–Pólya class). A real, entire function is said to be in the Laguerre–Pólya class, $L^P$, if it can be expressed as

$$\sum_{k=0}^{\infty} \frac{\lambda_k x^k}{k!} = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}}$$

where $c, \beta, x_k \in \mathbb{R}$, $c \neq 0$, $\alpha \geq 0$, $n$ is a non-negative integer and $\sum_{k=1}^{\infty} \frac{1}{x_k^2} < \infty$. If $\lambda_k \geq 0$ for all $k$ we denote this by $\varphi \in L^P^+$. The functions in this class, and only these, are the uniform limits of polynomials with only real zeros on compact subsets of $\mathbb{C}$. Hence $L^P$ is closed under differentiation. The following theorem now provides the necessary and sufficient conditions for $\lambda_k$ to be a multiplier sequence.

**Theorem 5** (Pólya-Schur, [16], [14]). Let $\lambda_n : \mathbb{N} \to \mathbb{R}$ be a sequence of real numbers and let $T$ be the corresponding diagonal linear operator given by $T(x^n) = \lambda_n x^n$ and define $\Phi$ by by the formal power series

$$\Phi(x) := \sum_{n=0}^{\infty} \frac{\lambda_n x^n}{n!}$$

The following are equivalent:

(i) $\{\lambda_n\}_{n=0}^{\infty}$ is a multiplier sequence.

(ii) $\Phi(x)$ defines an entire function which is the limit, uniformly on compact sets, of polynomials with only real zeros of the same sign.

(iii) Either $\Phi(x)$ or $\Phi(-x)$ is an entire function that can be written as

$$cx^n e^{sx} \prod_{k=1}^{\infty} (1 + \alpha_k x)$$

where $n \in \mathbb{N}$, $c \in \mathbb{R}$, $s, \alpha_k \geq 0$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \alpha_k < \infty$.

(iv) For all non-negative integers $n$, the polynomial $T((x + 1)^n)$ is real-rooted with all zeros and of the same sign.

We note that (ii)-(iii) above gives a transcendental description of multiplier sequences while (iv) is the algebraic version. We shall provide a proof of this by use of the symbol and Theorem 7 presented in the next section.
algebraic version of this statement says that for a real-rootedness preserving operator \(T\) and all non-negative integers \(n\) the polynomial \(T((x+1)^n)\) has only real zeros of the same sign. In the next section we shall see that this polynomial is the corresponding symbol of this operator. The following properties of multiplier sequences are consequences of Theorem 5.

**Corollary 1.** If \(\{\lambda_n\}_{n=0}^{\infty}\) is a multiplier sequence and \(\lambda_k \lambda_l \neq 0\) for some \(k < l\), then \(\lambda_i \neq 0\), for all \(k \leq i \leq l\).

This follows from (iii) above. We say that a sequence \(\{\lambda_n\}_{n=0}^{\infty}\) is trivial if there is a number \(k\) such that \(\lambda_n = 0\) for all \(n \notin \{k, k+1\}\).

**Corollary 2.** Let \(\{\lambda_n\}_{n=0}^{\infty}\) be a sequence of real numbers and let \(T\) be the corresponding diagonal operator. Then \(T\) has rank at most two and \(\{\lambda_n\}_{n=0}^{\infty}\) is a multiplier sequence if and only if \(\{\lambda_n\}_{n=0}^{\infty}\) is a trivial sequence (as defined in the introduction).

This follows directly from Corollary 1.

**Theorem 6** ([10]). Let \(\Phi(x) = \sum_{k=0}^{\infty} \frac{\lambda_k x^k}{k!}\) be a transcendental entire function in the Laguerre–Pólya class. Suppose that the product representation of \(\Phi(x)\) has the form

\[
\Phi(x) = c x^n e^{sx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{\alpha_k}\right) \quad 0 \leq \omega \leq \infty
\]

where \(s \geq 0, \alpha_n > 0, c > 0, \sum \alpha_n^{-1} < \infty\) and \(n\) is a non-negative integer. Then \(s \geq 1\) if and only if \(0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots\).

### 2.3. The symbol and stability.

In this section we state the Borcea-Brändén theorem which characterizes the linear operators preserving real-rootedness and stability.

#### 2.3.1. The symbol.

To each linear operator we may associate the symbol as follows:

**Definition 2.** To any linear operator \(T : \mathbb{C}[x] \rightarrow \mathbb{C}[x]\) we define the symbol \(G_T(x, y)\) to be the formal power series given by

\[
G_T(x, y) = \sum_{n=0}^{\infty} \frac{(-1)^n T(x^n)}{n!} y^n
\]

From this definition we see that we may take the symbol to be

\[G_T(x, y) = T(e^{-xy}).\]

#### 2.3.2. Stability and the multivariate Laguerre–Pólya class.

A multivariate polynomial \(P(x) \in \mathbb{C}[x_1, \ldots, x_n]\) is \(\Omega\)-stable if \(P(x) \neq 0\) for all \(x \in \Omega\), where \(\Omega\) is some subset of \(\mathbb{C}^n\). When

\[
\Omega = \{z \in \mathbb{C} : \text{Im}(z) > 0\}^n
\]
we simply say that $P(x)$ is stable. If all the coefficients of a stable polynomial $P(x)$ are real we say that $P(x)$ is real-stable. In the univariate case real-stability coincides with the property of having only real zeros, since the non-real zeros come in conjugate pairs. An operator preserving this stability is called a stability preserving operator. The notion of stability allows us to extend the Laguerre–Pólya class to the multivariate case as the class of entire functions in $n$ variables that are uniform limits on compact subset of $\mathbb{C}$ of real-stable polynomials in $n$ variables. We denote this by $\mathcal{L}-\mathcal{P}_n$.

**Definition 3.** The Laguerre–Pólya class of entire functions in $n$ variables, $\mathcal{L}-\mathcal{P}_n$, consists of all entire functions that are uniform limits on compact subset of $\mathbb{C}$ of real-stable polynomials in $n$ variables.

The symbol and the next two theorems provides a way to decide if an operator preserves stability or not.

**Theorem 7** (Borcea-Brändén, [3]). A linear operator $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ preserves real-rootedness if and only if either

1. The range of $T$ is at most two-dimensional and $T$ is of the form
   $$T(P(x)) = \alpha(P)Q + \beta(P)R$$
   where $\alpha, \beta : \mathbb{R}[x] \rightarrow \mathbb{R}$ are linear functional and $Q + iR$ is a stable polynomial, or

2. $G_T(x, y) \in \mathcal{L}-\mathcal{P}_2$ or

3. $G_T(-x, y) \in \mathcal{L}-\mathcal{P}_2$.

Let $\mathbb{R}_N[x] = \{f \in \mathbb{R}[x] : \deg(f) \leq N\}$. The algebraic version of the above result is now the following.

**Theorem 8** (Borcea-Brändén, [3]). A linear operator $T : \mathbb{R}_N[x] \rightarrow \mathbb{R}[x]$ preserves real-rootedness if and only if either

1. The range of $T$ is at most two-dimensional and $T$ is of the form
   $$T(P(x)) = \alpha(P)Q + \beta(P)R$$
   where $\alpha, \beta : \mathbb{R}_N[x] \rightarrow \mathbb{R}$ are linear functional and $Q + iR$ is a stable polynomial, or

2. $T((x + y)^n)$ is real-stable, or

3. $T((x - y)^n)$ is real-stable.

In (2) and (3) $T$ acts on $x$ and treats $y$ as a "dummy-variable". The proofs of these results can be found in [3]. In view of this we are now able to provide the proof of the Pólya–Schur Theorem (Theorem 5 as in [3]).

**Proof.** If the range of $T$ is no larger than 2 it follows from Corollary 2 that $\{\lambda_n\}_{n=0}^\infty$ is a trivial multiplier sequence and by definition statements (i)-(iv) holds. Assume that the range of $T > 2$. By Theorem 7 we must thus show that either $G_T(x, y)$ or $G_T(-x, y)$ is stable.

Now consider $f(x) \in \mathbb{R}[x]$. We then claim that $f(xy)$ is a real stable, bivariate polynomial if and only if the zeros of $f$ are real and nonnegative.

First, assume that $f$ has only real nonnegative zeros. Then $f(xy)$ factors as

$$f(xy) = c \prod_{j=0}^n(xy - a_j)$$
where \( a_j \in \mathbb{R} \) for all \( 0 \leq j \leq n \). Moreover \( xy - a_j \) is real stable if and only if \( a_j \geq 0 \).

On the other hand if \( f(xy) \) is real stable, and let \( x = y = t \) it follows that \( f(t^2) \) is hyperbolic. This shows that the zeros of \( f \) are all nonnegative.

Now Theorem 8 implies that either \( T((1-xy)^n) \) or \( T((1+xy)^n) \) is real stable if and only if \( T \) preserves real-rootedness. From this we conclude that \( (i) \Leftrightarrow (iv) \). Combining Theorem 7 and Theorem 8 we get that \( (ii) \Leftrightarrow (iv) \).

Definition 1 yields \( (iii) \Leftrightarrow (ii) \). □

2.3.3. The Laguerre–Pólya class for bivariate functions. The notion of stability allowed us to define \( \mathcal{L} \mathcal{P}_n \) and in particular \( \mathcal{L} \mathcal{P}_2 \) which is of importance for our purposes. Similar to functions in \( \mathcal{L} \mathcal{P} \) we may represent functions in \( \mathcal{L} \mathcal{P}_2 \) by a expression of restricted order in two variables.

**Theorem 9** ([14]). Let \( f(x, y) \) be an entire function of two variables. Then \( f \) is in \( \mathcal{L} \mathcal{P}_2 \) if and only if \( f \) has the following representation
\[
f(x, y) = e^{-ax^2-by^2}f_1(x, y),
\]
where \( a \) and \( b \) are non-negative numbers and \( f_1 \) is in \( \mathcal{L} \mathcal{P}_2 \) of order at most one in each of its variables under the condition that the other variable is fixed in the lower half-plane.

This result will be of use when proving Theorem 2.

2.4. Interlacing zeros and the Hermite-Biehler theorem. We shall need a version of the Hermite-Biehler theorem and the notions of interlacing zeros and proper position. Let \( x_1 \leq x_2 \leq \cdots \leq x_n \) and \( y_1 \leq y_2 \leq \cdots \leq y_m \) be the (real) zeros of two polynomials \( p \) and \( q \) where \( \deg p = n \), \( \deg q = m \) and \( |n-m| \leq 1 \). We say that the zeros of \( p(x) \) and \( q(y) \) interlace if they can be ordered so that \( x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \) or \( y_1 \leq x_1 \leq y_2 \leq x_2 \leq \cdots \) If the zeros of two polynomials \( p \) and \( q \) interlace then the Wronskian
\[
W[p, q] := p'q - pq'
\]
is either non-negative or non-positive on the whole of \( \mathbb{R} \). In the case when \( W[p, q] \leq 0 \) we say that \( p \) and \( q \) are in proper position and we denote this by \( p \ll q \). By convention we say that \( 0 \ll p \) and \( p \ll 0 \) for any real-rooted polynomial \( p \). Moreover if \( p, q \) are any polynomials of degree less than 2 we write \( p \ll q \) if and only if \( W[p, q] \leq 0 \) on the whole of \( \mathbb{R} \).

**Theorem 10** (Hermite-Biehler, [17]). Let \( p, q \in \mathbb{R}[x] \) be two polynomials. Then \( q \ll p \) if and only if the polynomial
\[
f(x) = p(x) + iq(x)
\]
is stable.

We may extend the notion of proper position to \( \mathcal{L} \mathcal{P} \) by \( q \ll p \) if and only if \( p + iq \in \mathcal{L} \mathcal{P}(\mathbb{C}) \), where \( \mathcal{L} \mathcal{P}(\mathbb{C}) \) is the complex Laguerre–Pólya class which is defined to be the set of entire functions that are limits, uniformly on compact subsets of \( \mathbb{C} \) of stable polynomials.
3. Multiplier sequences for simple set of polynomials

A simple set of polynomials $\mathcal{P} = \{P_n\}_{n=0}^{\infty}$ is a set of polynomial such that $\deg P_n = n$. By this definition we see that $\mathcal{P}$ is a basis of $\mathbb{R}[x]$.

Consider the simple set of polynomials, $\mathcal{P} = \{P_n\}_{n=0}^{\infty}$. Let the linear operator $T$ as usual be such that

$$T(P_n(x)) = \lambda_n P_n(x)$$

and $T$ is real-zero preserving, so that $\{\lambda_n\}_{n=0}^{\infty}$ is a $\mathcal{P}$-multiplier sequence. We shall make some observations for the simple sets of polynomials and the $\mathcal{P}$-multiplier sequences in the finite case. The first one is the following.

**Lemma 1.** Any $\mathcal{P}$-multiplier sequence is a multiplier sequence in the classical sense.

**Proof.** We shall prove this by use of Theorem 8. Assume that $T$ is a real-rootedness preserving operator acting on $\mathcal{P}$ as in (3.1) so that $\{\lambda_k\}_{k=0}^{\infty}$ is a $\mathcal{P}$-multiplier sequence. We will now see that $\{\lambda_k\}_{k=0}^{\infty}$ is also a multiplier sequence in the classical sense. Restrict $T$ to $\mathbb{R}_N[x]$ and consider the symbol $F_T$ given by

$$F_T(x, y) = T((x+y)^N).$$

one might as well consider $F_T(-x, y)$. From (3.2) we see that

$$F_T(x, y) = \sum_{k=0}^{N} \binom{N}{k} T(x^k)y^{N-k}.$$ 

Let $0 \leq k \leq n$. We may w.l.o.g assume that $P_n(x)$ is monic and thus write

$$x^n = P_n(x) + \sum_{k=0}^{n-1} a_k P_k(x)$$

for some coefficients $a_k$. Thus, since $\deg P_k(x) = k$

$$T(x^n) = \lambda_n P_n(x) + \sum_{k=0}^{n-1} \lambda_k a_k P_k(x) = \lambda_n x^n + O(x^{n-1})$$

where $O(x^{n-1})$ is a polynomial in $x$ of degree $n - 1$. Let now $t > 0$ and let $x \to tx$ and $y \to ty$ and consider

$$t^{-n}T((xt)^n) = \lambda_n x^n + \frac{1}{t} O(x^{n-1})$$

and let $t \to \infty$ so that $t^{-n}T((xt)^n) \to \lambda_n x^n$. It then follows that

$$t^{-N} F_T(tx, ty) \to \sum_{k=0}^{N} \binom{N}{k} \lambda_k x^k y^{N-k}.$$ 

This is the symbol for an operator acting on $\{x^n\}_{n=0}^{\infty}$ and since $T$ was assumed to be real-zero preserving when acting on $\mathcal{P}$ it follows by Theorem 8 that $F_T(x, y)$ is stable and thus

$$\sum_{k=0}^{N} \binom{N}{k} \lambda_k x^k y^{N-k} \in \mathcal{L}(\mathcal{P}).$$
by Theorem 3. Consequently \( \{ \lambda_k \}_{k=0}^n \) is a multiplier sequence. By the change of variables \( x \to xt \) and \( y \to y/t \) a similar argument holds in the transcendental case.

Actually for any real-rootedness and degree preserving operator \( T : \mathbb{R}[x] \to \mathbb{R}[x] \) such that
\[
T(x^n) = \lambda_n x^n + O(x^{n-1}),
\]
then \( \{ \lambda_n \}_{n=0}^{\infty} \) is a multiplier sequence. To see this, consider the matrix
\[
[T[1], T[x], \ldots, T[x^n]] = \begin{pmatrix}
\lambda_0 & A_{01} & A_{02} & \cdots & A_{0n} \\
0 & \lambda_1 & A_{12} & \cdots & A_{1n} \\
0 & 0 & \lambda_2 & \cdots & A_{2n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \lambda_n
\end{pmatrix}
\]
where thus each \( T[x^j] \) is a column vector. Since this matrix is diagonalizable it follows that \( \{ \lambda_n \}_{n=0}^{\infty} \) is a \( P \)-multiplier sequence w.r.t the eigenpolynomials \( P_n \) and hence, by Lemma 1 above, \( \{ \lambda_n \}_{n=0}^{\infty} \) is a multiplier sequence in the classical sense.

Now we show that for any increasing multiplier sequence the corresponding eigenpolynomials are real rooted. Assume that \( \{ \lambda_n \}_{n=0}^{\infty} \) is increasing and as in the proof of Lemma 1, express the standard monomial basis as
\[
x^n = P_n(x) + \sum_{k=0}^{n-1} a_k P_k(x).
\]
Then
\[
T(x^n) = \lambda_n P_n(x) + \sum_{k=0}^{n-1} \lambda_k a_k P_k(x).
\]
is real-rooted. We then claim that all the \( P_j(x) \)s are real-rooted polynomials for all \( j = 1, \ldots, n \). Indeed, since \( T \) preserves real-rootedness, then so does \( T^m \) and we get for this iterated operator
\[
T^m(x^n) = \lambda_n^m P_n(x) + \sum_{k=0}^{n-1} \lambda_k^m a_k P_k(x)
\]
and thus
\[
\frac{T^m(x^n)}{\lambda_n^m} = P_n(x) + \sum_{k=0}^{n-1} \left( \frac{\lambda_k}{\lambda_n} \right)^m a_k P_k(x)
\]
is real-rooted for all \( m \). Now since \( \{ \lambda_n \}_{n=0}^{\infty} \) is increasing it follows that \( \frac{\lambda_k}{\lambda_n} \to 0 \) as \( m \to \infty \) and thus, since
\[
\frac{T^m(x^n)}{\lambda_n^m}
\]
is real-rooted it follows by Theorem 3 that \( P_n \) is real-rooted.

One can also show that for increasing \( \lambda_n \) also the zeros of \( P_n(x) \) interlace the zeros of \( P_{n-1}(x) \).
4. Orthogonal Polynomials

An orthogonal polynomial sequence is a simple set of polynomials, where any two polynomials in this set, not equal, are orthogonal to each other with respect to some inner product. Orthogonal polynomials have applications in many areas, including physics and probability theory. Just as with Fourier series the orthogonal polynomials provide useful methods for solving and interpreting solutions to differential equations. In Chihara's book, [7], one can find the following straightforward introduction to orthogonal polynomials via the Chebyshev polynomials:

Consider the trigonometric identity

\[ 2 \cos m\phi \cos n\phi = \cos(m + n)\phi + \cos(m - n)\phi \]  

 Integrating both sides then yields

\[ \int_0^\pi \cos m\phi \cos n\phi d\phi = 0, \quad m \neq n \]  

and by the change of variables, \( x = \cos \phi \), we get

\[ \int_{-1}^1 T_m(x)T_n(x) \frac{dx}{\sqrt{1 - x^2}} = 0, \quad x \in [-1, 1], \quad m \neq n \]

where \( T_n(x) = \cos n\phi \). By the above and trigonometric identities we may now identify

\[ T_0(x) = 1 \]
\[ T_1(x) = x \]
\[ T_2(x) = 2x^2 - 1 \]
\[ T_3(x) = 4x^3 - 3x \]
\[ T_4(x) = 8x^4 - 8x^2 + 1 \]

We shall now see that any \( T_k(x) \) is a polynomial in \( x \) of degree \( k \). Consider (4.1) and let \( m = 1 \) to get

\[ 2 \cos \phi \cos n\phi = \cos(1 + n)\phi + \cos(1 - n)\phi. \]

Thus by change of variables we get the recursion formula.

\[ T_{n+1} = 2xT_n - T_{n-1} \]

and indeed \( T_{n+1} \) is a polynomial of degree \( n+1 \) assuming that \( T_n \) and \( T_{n-1} \) are polynomials of degree \( n \) and \( n - 1 \) respectively. Because of (4.2) we say that \( \{T_k\}_{k=0}^\infty \) is an orthogonal sequence with respect to the weight function \( (1 - x^2)^{-1/2} \) on \((-1, 1)\). Now more generally a simple set of polynomials \( \{P_n(x)\}_{n=0}^\infty \) is said to be orthogonal w.r.t \( w(x) \) if

\[ \int_a^b P_m(x)P_n(x)w(x)dx = 0, \quad m \neq n. \]
Here we have assumed \( w(x) \) to be positive on a subset of \((a, b)\) with positive Lebesgue measure. That is
\[
\int_a^b w(x) \, dx > 0.
\]
For an unbounded interval we shall also require that the moments,
\[
\mu_n = \int_a^b x^n w(x) \, dx, \quad n = 0, 1, 2 \ldots
\]
are finite. The zeros of a real polynomial \( P_n(x) \) orthogonal w.r.t \( w(x) \) over \((a, b)\) are all distinct and lie in this interval. Furthermore orthogonal polynomials \( P_n(x) \) satisfy the following three-term recurrence relation
\[
x P_n(x) = A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x)
\]
for a sequence of numbers \( A_n, B_n, C_n \) where \( A_n, C_n \neq 0 \) and \( n \geq 1 \).

The classical orthogonal polynomials are the Hermite, Laguerre and Jacobi polynomials. They have several properties in common. We will utilize the fact that the generating function for classical orthogonal polynomials may be explicitly expressed in terms of the weight function. We shall also use the recurrence relation and the differential equation they satisfy. We will introduce the Hermite and Laguerre polynomials, but the Legendre polynomials, which are a special case of the Jacobi polynomials, are also of interest for this subject, see [1]. The Chebychev polynomials seen above are a special case of the Jacobi polynomials.

4.1. Laguerre polynomials. The generalized Laguerre polynomials \( \{L_n^{(\alpha)}(x)\}_{n=1}^{\infty} \) form an orthogonal set over \((0, \infty)\) with weight function \( w(x) = x^\alpha e^{-x} \). Thus
\[
\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) \, dx = 0, \quad m \neq n, \quad \alpha > -1.
\]

We may define \( L_n^{(\alpha)}(x) \) by the following generating function:
\[
\frac{e^{-tx}}{(1 - t)^{1+\alpha}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n.
\]

The first few Laguerre polynomials are:
\[
L_0^{(\alpha)}(x) = 1 \\
L_1^{(\alpha)}(x) = x + \alpha + 1 \\
L_2^{(\alpha)}(x) = 1/2(x^2 - 2(\alpha + 2)x + (\alpha + 1)(\alpha + 2)) \\
L_3^{(\alpha)}(x) = 1/6(-x^3 + 3(\alpha + 3)x^2 - 3(\alpha + 2)(\alpha + 3)x + (\alpha + 1)(\alpha + 2)(\alpha + 3)).
\]

The term "Laguerre polynomials" commonly refers to the special case when \( \alpha = 0 \). However, we shall mean the generalized Laguerre polynomials whenever we speak of Laguerre polynomials (if not otherwise is stated). The differential equation for Laguerre polynomials is
\[
nL_n^{(\alpha)}(x) = (x - \alpha - 1) \frac{d}{dx} L_n^{(\alpha)}(x) - x \frac{d^2}{dx^2} L_n^{(\alpha)}(x).
\]
and in the special case when $\alpha = 0$ this becomes:

$$nL_n(x) = (x - 1) \frac{d}{dx} (x - 1)L_n - x \frac{d^2}{dx^2} L_n.$$  

As already mentioned in the introduction we have the following explicit formula for Laguerre polynomials.

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(n + \alpha)_n}{(n - k)_k} \frac{(-x)^k}{k!}, \quad \alpha > -1$$

and when $\alpha = 0$ one gets:

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!}.$$  

These may be deduced via the differential equation and the generating function.

4.2. Hermite polynomials. Hermite polynomials are orthogonal with respect to the weight function $e^{-x^2}$ over $(-\infty, \infty)$. Thus

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = 0, \quad m \neq n$$

The first few Hermite polynomials are:

- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$

There are several interesting relations for the Hermite polynomials.

Apell formula:

$$H'_n(x) = 2nH_{n-1}(x)$$

The pure recurrence relation:

$$H_n(x) = 2H_{n-1}(x) - 2(n - 1)H_{n-2}(x)$$

Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$  

Explicit formula:

$$H_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!}.$$  

Hermite’s differential equation:

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0.$$
5. Multiplier sequences for Hermite polynomials

By using the methods described earlier we will now give a short proof of the characterization of Hermite multiplier sequences achieved by Piotrowski [15]. The Hermite polynomials, \( \mathcal{H} = \{ H_n(x) \}_{n=0}^{\infty} \), may be defined by the generating function

\[
\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n,
\]

(5.1)

see [18]. Recall from the introduction that if the operator \( T \) acts on \( H_n \) as

\[
T[H_n(x)] = \lambda_n H_n(x),
\]

and \( T \) is a real-rootedness preserving operator then \( \{ \lambda_n \} \) is called a \( \mathcal{H} \)-multiplier sequence. Since Hermite polynomials are even or odd it is easy to see that \( \{ \lambda_n \}_{n=0}^{\infty} \) is an \( \mathcal{H} \)-multiplier sequence if and only if \( \{ (-1)^n \lambda_n \}_{n=0}^{\infty} \) is an \( \mathcal{H} \)-multiplier sequence. Also, any trivial sequence is an \( \mathcal{H} \)-multiplier sequence. By Lemma 1 all \( \mathcal{H} \)-multiplier sequences are multiplier sequences.

Since the entries of multiplier sequences either have the same sign or alternate in sign (by Theorem 5) it remains to characterize non-negative and non-trivial Hermite multiplier sequences. In [15] a generalization of Pólya’s curve theorem led to the following characterization, which we will now reprove:

**Theorem 11** (Piotrowski, [15]). Let \( \{ \lambda_n \}_{n=0}^{\infty} \) be a non-trivial sequence of non-negative numbers. Then \( \{ \lambda_n \}_{n=0}^{\infty} \) is a \( \mathcal{H} \)-multiplier sequence if and only if it is a (classical) multiplier sequence with \( \lambda_n \leq \lambda_{n+1} \) for all \( n \geq 0 \).

We prove this by means of the symbol, \( T(e^{-xy}) \) and the generating function for Hermite polynomials.

Let \( \{ \lambda_n \}_{n=0}^{\infty} \) be a non-trivial and non-negative classical multiplier sequence and let \( T \) be the corresponding operator. Note that (5.1) implies

\[
e^{-xy} = e^{y^2/4} \sum_{k=0}^{\infty} \frac{H_k(x)}{k!} \left( \frac{-y}{2} \right)^k,
\]

where \( t = -\frac{y}{2} \). Hence

\[
G_T(x, y) = T(e^{-xy}) = e^{y^2/4} \sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!}
\]

is the symbol of \( T \). By Theorem 7 we want to determine when \( G_T(x, y) \in \mathcal{L} - \mathcal{P}_2(\mathbb{R}) \) or \( G_T(-x, y) \in \mathcal{L} - \mathcal{P}_2(\mathbb{R}) \). First let us prove that \( G_T(-x, y) \) is never in \( \mathcal{L} - \mathcal{P}_2(\mathbb{R}) \). Suppose that \( G_T(-x, y) \in \mathcal{L} - \mathcal{P}_2(\mathbb{R}) \) and let \( m \) be the first index for which \( \lambda_m \neq 0 \). Then, since \( e^{-y^2/4} \in \mathcal{L} - \mathcal{P}_2(\mathbb{R}) \),

\[
y^{-m} e^{-y^2/4} G_T(-x, y) = \frac{\lambda_m H_m(x)}{2^m m!} + \frac{\lambda_{m+1} H_{m+1}(x)}{2^{m+1}(m+1)!} y + \cdots \in \mathcal{L} - \mathcal{P}_2(\mathbb{R}).
\]

(5.2)

Since \( \{ \lambda_n \}_{n=0}^{\infty} \) is nonnegative, Corollary 1 implies \( \lambda_m, \lambda_{m+1} > 0 \), and since multiplier sequences preserve \( \mathcal{L} - \mathcal{P}_2 \) when applied to one variable at a time
(see [6] and Lemma 3.7 of [4]) we may truncate this expression above by the multiplier sequence \(\{1,1,0,\ldots\}\) (acting on \(y\)) to (5.2) and conclude

\[
\frac{\lambda_m H_m(x)}{2^m m!} + \frac{\lambda_{m+1} H_{m+1}(x)}{2^{m+1} (m+1)!} y \in \mathcal{L} - \mathcal{P}_2(\mathbb{R}).
\]

Letting \(y = i\) Theorem 10 implies that \(H_{m+1}(x) \ll H_m(x)\) which does not hold (although \(H_m(x) \ll H_{m+1}(x)\) is a standard fact about orthogonal polynomials). Hence we have arrived at a contradiction.

It remains to find necessary and sufficient conditions for \(G_T(x,y)\) to be in the Laguerre–Polya class. Considering the generating function again yields

\[
\sum_{k=0}^{\infty} \frac{H_k(x)(-y)^k}{2^k k!} = e^{-xy} e^{-\frac{x^2}{4}}
\]

which is in \(\mathcal{L} - \mathcal{P}_2\). Now for each non-negative multiplier sequence \(\lambda_k\) also

\[
\sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!}
\]

is in \(\mathcal{L} - \mathcal{P}_2\). By Theorem 9 we may write

\[
\sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!} = e^{-ax^2-by^2} g(x,y)
\]

for some entire function \(g(x,y) \in \mathcal{L} - \mathcal{P}_2\) of order at most 1 in each variable under the condition that the other variable is fixed in the open upper half plane. So the symbol of \(T\) is

\[
G_T(x,y) = e^{\frac{x^2}{4}} \sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!}
\]

\[
= e^{\frac{x^2}{4}} e^{-ax^2-by^2} g(x,y)
\]

\[
= e^{-ax^2-(b-\frac{1}{4})y^2} g(x,y)
\]

By Theorem 9 we must thus decide when \(b \geq \frac{1}{4}\).

As seen in Theorem 5 we may write

\[
\sum_{n} \frac{\lambda_n x^n}{n!} = c x^n e^{sx} \prod_{k=1}^{\infty} (1 + \alpha_k x)
\]

in other words for such series \(\rho \leq 1\) and \(\sigma = s\). From this and Theorem 4 we may deduce the following lemma:

**Lemma 2.** Let \(\{\lambda_n\}_{n=0}^{\infty}\) be a non-negative multiplier sequence with exponential generating function given by (5.3), and let \(\sum_{n=0}^{\infty} c_n x^n\) be an entire function in \(\mathcal{L} - \mathcal{P}_2(\mathbb{C})\) of order 2 and type c. Then

\[
\sum_{n=0}^{\infty} \lambda_n c_n z^n = \exp(-cs^2 x^2) f(x)
\]

where \(f(x)\) has order at most one.
Proof. By continuity we may assume that $s > 0$. The relation between the order and type in Theorem 4 asserts that the order of (5.4) is 2. Let $\sigma$ be the type of the left hand side of (5.4). By Theorem 4 again,

$$
(\sigma e^2)^{1/2} = \lim_{n \to \infty} n^{1/2} |\lambda_n|^{1/n} = \lim_{n \to \infty} n^{1/2} \left( \frac{\lambda_n}{n!} \right)^{1/n} (n!)^{1/n} |c_n|^{1/n}.
$$

By Stirlings formula we have that $(n!)^{1/n} \sim ne^{-1}$,

$$
(\sigma e^2)^{1/2} = e^{-1} \lim_{n \to \infty} n^{1/2} |c_n|^{1/n} \left( \frac{\lambda_n}{n!} \right)^{1/n} = e^{-1}(ce^2)^{1/2}se,
$$

that is, $\sigma = cs^2$. □

Since the order and type of

$$
\sum_{k=0}^{\infty} \frac{H_k(x)(-y)^k}{2^k k!}
$$

is 2 respectively $\frac{1}{4}$ w.r.t. $x$ it follows by Lemma 2 that the order and type of

$$
\sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!}
$$

is 2 and $\frac{s^2}{4}$ respectively where $s$ is the type of

$$
cx^n e^{ax} \prod_{k=1}^{\infty} (1 + \alpha_k x)
$$

in Theorem 5. But since we may also write

$$
\sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!} = e^{-ax^2-by^2}g(x, y)
$$

its clear that $b = \frac{s^2}{4}$. Then by Theorem 6 we get the condition that $s \geq 1$. This happens if and only if $\lambda_n$ is increasing by Theorem 6 and thus the statement follows.
6. Multiplier sequences for Laguerre polynomials

Let us return to the Laguerre polynomials. It would seem natural to speak of Laguerre multiplier sequences, but since that refers to other objects - as in [9] - we shall adopt the terminology of [11]. In this section we completely characterize the $L_\alpha$-multiplier sequences. Our main result is the following:

**Theorem 12.** Let $p(y) = \sum_{k=0}^{\infty} (-1)^{k+1} \binom{k+\alpha}{k} a_k y^k$ be a formal power series where $\alpha > -1$ and let $\{\lambda_n\}_{n=0}^\infty$ be defined by

$$\lambda_n := \sum_{k=0}^{n} a_k \binom{n}{k}.$$  

Then $\{\lambda_n\}_{n=0}^\infty$ is a $L_\alpha$-multiplier sequence if and only if $p(y)$ is a real-rooted polynomial with all its zeros contained in the interval $[-1,0]$.

Let the linear operator $T$ be identified by $T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x)$ where $L_n^{(\alpha)}(x)$ is the $n$th Laguerre polynomial. If $T$ is real-rootedness preserving then $\{\lambda_n\}$ is called a $L_\alpha$-multiplier sequence. Recall now that for each linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ the associated symbol, $G_T(x,y)$, is the formal power series given by:

$$G_T(x,y) := \sum_{n=0}^{\infty} \frac{(-1)^n T(x^n)}{n!} y^n.$$  

The symbol, the notion of stability and the next theorem are the keys to characterize the $L_\alpha$-multiplier sequences. Theorem 7 suggest that we should find necessary and sufficient conditions for the symbol, $G_T(x,y)$, of the operator given by $T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x)$ to be in $L_\alpha \mathcal{P}_2$. We shall need an expression for $G_T(x,y)$.

**Lemma 3.** Let $\{\lambda_n\}_{n=0}^\infty$ be a sequence of real numbers. The symbol of the operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x)$, for all $n \in \mathbb{N}$, is given by

$$G_T(x,y) = e^{-xy} \sum_{n=0}^{\infty} a_n x^n L_n^{(\alpha)}(xy + x).$$

where

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \lambda_k, \quad n \in \mathbb{N}.$$

**Proof.** Recall the differential equation satisfied by the Laguerre polynomials:

$$nL_n^{(\alpha)}(x) = (x - \alpha - 1) \frac{d}{dx} L_n^{(\alpha)}(x) - x \frac{d^2}{dx^2} L_n^{(\alpha)}(x),$$  

see [18]. Consider the operator $\delta := (x - \alpha - 1)d/dx - xd^2/dx^2$ and let

$$\binom{\delta}{k} := \delta(\delta - 1) \cdots (\delta - k + 1) \binom{k}{k}.$$  

Then $\binom{\delta}{k} L_n^{(\alpha)}(x) = \binom{n}{k} L_n^{(\alpha)}(x)$, and letting $T$ be the operator corresponding to $\{\lambda_n\}_{n=0}^\infty$, we have $T = \sum_{k=0}^{\infty} a_k \binom{\delta}{k}$. Let $S_k$ denote the operator $\binom{\delta}{k}$. Then,
by the change of variables $y = t/(t-1)$, in the generating function for the Laguerre polynomials:

$$\frac{e^{-xt/(1-t)}}{(1-t)^{1+\alpha}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n,$$

yields

$$G_{S_k}(x,y) = S_k(e^{-xy}) = \sum_{n=0}^{\infty} \binom{n}{k} L_n^{(\alpha)}(x)y^n(1+y)^{-n-\alpha-1}.$$

On the other hand, with the same change of variables as above, identity (9) on page 211 in [18] states that

$$\sum_{n=0}^{\infty} \binom{n}{k} L_n^{(\alpha)}(x)y^n(1+y)^{-n-\alpha-1} = e^{-xy} \sum_{n=0}^{\infty} a_n y^n L_n^{(\alpha)}(xy + x),$$

and thus we get

$$G_{S_k}(x,y) = y^k L_k^{(\alpha)}(xy + x).$$

We apply this to the symbol of $T$ and get

$$G_T(x,y) = \sum_{n=0}^{\infty} a_n y^n L_n^{(\alpha)}(xy + x).$$

The explicit expression (1.2) of the Laguerre polynomials now yields:

$$G_T(x,y) = e^{-xy} \sum_{n=0}^{\infty} a_n y^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x(y+1))^{k}}{k!}, \quad (6.1)$$

The $n$th order derivative of

$$p(y) = \sum_{k=0}^{\infty} \binom{k+\alpha}{k} a_k y^k$$

from Theorem 12 is equal to

$$p^{(n)}(y) = \sum_{k=n}^{\infty} \binom{k+\alpha}{k-n} (\alpha+1) \cdots (\alpha+n) a_k y^{k-n}.$$

Inserting $p^{(n)}(y)$ in expression (6.1) and changing the order of summation in (6.1) yields following consequence of Lemma 3

**Corollary 3.** The symbol of the operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x)$, for all $n \in \mathbb{N}$, is given by

$$G_T(x,y) = e^{-xy} \sum_{k=0}^{\infty} p^{(k)}(y) \frac{(-x(y+1))^{k}}{(\alpha+1) \cdots (\alpha+k)k!},$$

where $p(y)$ is defined as in Theorem 12.
for any

Now the Laguerre inequality (see e.g. [9, Corollary 3.7]) states that

To see that there is no cancellation we write

We now compute the Wronskian

other words

which belongs to \( \mathcal{L}_2(\mathbb{R}) \).

In (6.2) let \( q(y) = yp(y) + \frac{y(1+y)}{(1+\alpha)}p'(y) \) and \( x = i \). Theorem 10 extended to entire functions now asserts that the Wronskian satisfies \( W[p,q] \leq 0 \). In other words

We now compute the Wronskian

for all \( y \in \mathbb{R} \). Evaluating \( W[p,q] \) at a simple real zero \( y_0 \) of \( p(y) \) yields

(6.3)

To see that there is no cancellation we write

and obtain

Now the Laguerre inequality (see e.g. [9, Corollary 3.7]) states that

for any \( f(x) \in \mathcal{L}_2(\mathbb{R}) \). Thus \( R(y_0) \geq Ms(y_0)^2 > 0 \) which proves that (6.3) is the dominating term near \( y_0 \) and from which it follows that \( y_0 \in [-1,0] \).

We shall now see that \( p(y) \) is in fact a polynomial. We know that \( p(y) \) is an entire function in \( \mathcal{L}_2(\mathbb{R}) \) so it has the form (1). Since the zeros lie
in the interval \([-1, 0]\) it follows that \(p(y)\) can only have a finite number of zeros, that is, \(p(y) = e^{xy-by^2}K(y)\), where \(K(y)\) is a real-rooted polynomial with zeros only in \([-1, 0]\), and \(a, b \in \mathbb{R}\) with \(b \geq 0\). It remains to prove that \(a = b = 0\). Now \(Q(x, y) = e^{xy-by^2}(K(y) - xF(y))\) where

\[
F(y) = yK(y) + \frac{y(y + 1)}{1 + \alpha}((a - 2by)K(y) + K'(y)).
\]

The zeros of \(F(y)\) and \(K(y)\) interlace by Theorem 10 (set \(x = i\)). Notice that \(\deg F \geq \deg K + 2\), unless \(a = b = 0\). Hence \(a = b = 0\) and there is no exponential factor.

This altogether proves that \(p(y)\) is indeed a real-rooted polynomial with all its zeros contained in \([-1, 0]\), and finishes the proof of necessity in the case when \(G_T(x, y) \in \mathcal{L}\mathcal{P}_2(\mathbb{R})\). It remains to prove that we cannot have \(G_T(-x, y) \in \mathcal{L}\mathcal{P}_2(\mathbb{R})\).

Assume \(G_T(-x, y) \in \mathcal{L}\mathcal{P}_2(\mathbb{R})\). Then proceeding as for the case when \(G_T(x, y) \in \mathcal{L}\mathcal{P}_2(\mathbb{R})\), we get \(q(y) \ll p(y)\) where \(q(y)\) is defined as above. Thus

\[
W[p, q](y) = -p^2(y) + \frac{y(y + 1)}{1 + \alpha}((p'(y))^2 - p(y)p(y)') - \frac{2y + 1}{1 + \alpha}p(y)p'(y) \geq 0,
\]

for all \(y \in \mathbb{R}\). If \(p(-1/2) \neq 0\), then Laguerre’s inequality implies that the middle term in (6.4) is non-positive and thus \(W[p, q](-1/2) < 0\). Suppose \(y = -1/2\) is a zero of \(p(y)\) of multiplicity \(M \geq 1\). Then, since

\[
(y + 1/2)\frac{p'(y)}{p(y)} \approx M,
\]

near \(y = -1/2\) we see that also the last term in (6.4) is negative near \(y = -1/2\). Hence we cannot have \(G_T(-x, y) \in \mathcal{L}\mathcal{P}_2(\mathbb{R})\).

6.0.2. Proof of sufficiency. We now prove that the conditions on \(p(y)\) in Theorem 12 implies \(G_T(x, y) \in \mathcal{L}\mathcal{P}_2(\mathbb{R})\), which will then prove sufficiency by Theorem 7. Assume that the zeros of

\[
p(y) = \sum_{k=0}^{n} \binom{k + \alpha}{k} a_k y^k = \prod_{j=0}^{n} (y - \theta_j)
\]

are real and lie in \([-1, 0]\) and consider again the symbol in view of Corollary 3.

\[
G_T(x, y) = e^{-yx} \sum_{k=0}^{\infty} p^{(k)}(y) \frac{(-yx(y + 1))^k}{(\alpha + 1) \cdots (\alpha + k)k!}.
\]

Since \(((\alpha + 1) \cdots (\alpha + k))^{-1})k=0^{\infty}\) is a non-negative multiplier sequence as proved already by Laguerre [13], and as such preserves \(\mathcal{L}\mathcal{P}_2(\mathbb{R})\) when acting on \(x\) (see [6] and Lemma 3.7 of [4]), it is enough to prove that

\[
\sum_{k=0}^{n} \binom{k + \alpha}{k} \frac{(-yx(y + 1))^k}{k!}
\]
is a stable polynomial in two variables. Now
\[
\sum_{k=0}^{n} p^{(k)}(y) \frac{(-xy(y+1))^k}{k!} = p(y - xy(y+1))
\]
\[
= \prod_{j=0}^{n} (y - xy(y+1) - \theta_j)
\]
where \( \theta_j \in [-1, 0] \). Observe that \( y - \theta_j \ll y(y+1) \) for all \( \theta_j \in [-1, 0] \).
and thus, by e.g. [5, Lemma 2.8], it follows that each factor is stable. This finishes the proof of Theorem 12.
References