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# Topics in Computational Algebraic Geometry and Deformation Quantization

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# Abstract

This thesis consists of two parts, a first part on computations in algebraic geometry, and a second part on deformation quantization. More specifically, it is a collection of four papers. In the papers I, II and III, we present algorithms and an implementation for the computation of degrees of characteristic classes in algebraic geometry. Paper IV is a contribution to the field of deformation quantization and actions of the Grothendieck-Teichmüller group.

In Paper I, we present an algorithm for the computation of degrees of Segre classes of closed subschemes of complex projective space. The algorithm is based on the residual intersection theorem and can be implemented both symbolically and numerically.

In Paper II, we describe an algorithm for the computation of the degrees of Chern-Schwartz-MacPherson classes and the topological Euler characteristic of closed subschemes of complex projective space, provided an algorithm for the computation of degrees of Segre classes. We also explain in detail how the algorithm in Paper I can be implemented numerically. Together this yields a symbolical and a numerical version of the algorithm.

Paper III describes the Macaulay2 package CharacteristicClasses. It implements the algorithms from papers I and II, as well as an algorithm for the computation of degrees of Chern classes.

In Paper IV, we show that $L_\infty$-automorphisms of the Schouten algebra $T_{\text{poly}}(\mathbb{R}^d)$ of polyvector fields on affine space $\mathbb{R}^d$ which satisfy certain conditions can be globalized. This means that from a given $L_\infty$-automorphism of $T_{\text{poly}}(\mathbb{R}^d)$ an $L_\infty$-automorphism of $T_{\text{poly}}(M)$ can be constructed, for a general smooth manifold $M$. It follows that Willwacher's action of the Grothendieck-Teichmüller group on $T_{\text{poly}}(\mathbb{R}^d)$ can be globalized, i.e., the Grothendieck-Teichmüller group acts on the Schouten algebra $T_{\text{poly}}(M)$ of polyvector fields on a general manifold $M$.

# List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

PAPER I: **A method to compute Segre classes of subschemes of projective space**
D. Eklund, C. Jost, C. Peterson,
*Journal of Algebra and its Applications* **12**(2), 2013

PAPER II: **An algorithm for computing the topological Euler characteristic of complex projective varieties**
C. Jost, manuscript

PAPER III: **A Macaulay2 package for characteristic classes and the topological Euler characteristic of complex projective schemes**
C. Jost, manuscript

PAPER IV: **Globalizing $L_\infty$-automorphisms of the Schouten algebra of polyvector fields**
C. Jost, to appear in *Differential Geometry and its Applications*

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# Contents

# 1. Introduction

This thesis consists of two parts. In the first part, the papers I, II and III, we present algorithms computing invariants in algebraic geometry, namely the degrees of certain chararacteristic classes of complex projective schemes. The motivation for computing characteristic classes comes from intersection theory, enumerative geometry and algebraic statistics. The second part, Paper IV, is a contribution to the the field of deformation quantization. We show that the action of the Grothendieck-Teichmüller group on polyvector fields on affine space $\mathbb{R}^d$ can be globalized, i.e., we construct an action of the Grothendieck-Teichmüller group on polyvector fields on a general smooth manifold, using Willwacher's action on polyvector fields on affine space $\mathbb{R}^d$.

We start by giving a short introduction to the characteristic classes relevant for this thesis, followed by a motivating example. We then summarize the content of the first three papers. Briefly, in Paper I we present an algorithm for the computation of the degrees of Segre classes of complex projective schemes. In Paper II we describe an algorithm for the computation of the degrees of Chern-Schwartz-MacPherson classes and the topological Euler characteristic. Paper III is a presentation of the Macaulay2 package CharacteristicClasses, which contains implementations of the algorithms from Paper I and II as well as computations of the degrees of Chern classes.

We continue with an introduction to the topics relevant for Paper IV. At first, we present the Schouten algebra of polyvector fields, $L_\infty$-algebras and $L_\infty$-automorphisms. Furthermore, we shortly introduce the problem of deformation quantization, from which the study of polyvector fields arises. We also give a short introduction to the Grothendieck-Teichmüller group. At last, we summarize Paper IV. In this article, we show that Willwacher's action of the Grothendieck-Teichmüller group on the Schouten algebra can be globalized, i.e., that one can construct an action of the Grothendieck-Teichmüller group on the Schouten algebra $T_{\text{poly}}(M)$ on a general smooth manifold $M$.

# 2. Computing characteristic classes in algebraic geometry

## 2.1 Chern classes, generalized Chern classes and Segre classes of projective schemes

In this chapter, all schemes are defined over the field of complex numbers. We introduce the characteristic classes relevant for this thesis: Chern classes, Segre classes and Chern-Schwartz-MacPherson classes. In the following, let $X$ be a closed subscheme of $\mathbb{P}^n$. A subvariety of $X$ is a closed, reduced and irreducible subscheme. The standard reference for intersection theory, including Chow groups and Segre classes is [17]. Chern-Schwartz-MacPherson classes are described in several articles by Aluffi, e.g., [1].

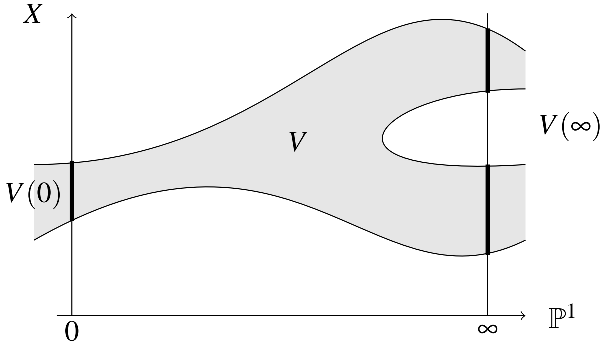
The habitat of the characteristic classes of $X$ is the Chow group $A_*(X)$ of cycles of $X$. A $k$-cycle on $X$ is a finite formal sum $\sum_{i=1}^m a_i V_i$ of $k$-dimensional subvarieties of $X$. The abelian group of $k$-cycles on $X$ is denoted by $Z_k(X)$. We define rational equivalence of cycles, an equivalence relation on $Z_k(X)$, following Section 1.6 in [17]. Let $V$ be a $(k+1)$-dimensional subvariety of the product $X \times \mathbb{P}^1$ such that the projection to $\mathbb{P}^1$ is dominant. For a point $P$ in $\mathbb{P}^1$, denote by $V(P)$ the projection to $X$ of the fiber of $P$ under the projection $V \to \mathbb{P}^1$, see also Figure 2.1. By definition, a $k$-cycle $\alpha$ is rationally equivalent to zero if there are $(k+1)$-dimensional subvarieties $V_1, \ldots, V_t$ of $X \times \mathbb{P}^1$ such that the projections to $X$ are dominant, and such that

$$\alpha = \sum_{i=1}^{t} V_i(0) - V_i(\infty).$$

Quoting out rational equivalence yields the abelian group $A_k(X)$ of $k$-cycle classes $\sum_{i=1}^m a_i [V_i]$. The $k$-cycle classes for all dimensions $k$ together form the (graded) Chow group $A_*(X) = \bigoplus_{k=1}^d A_k(X)$, where $d$ is the dimension of $X$. For example, the Chow group of projective space $\mathbb{P}^n$ is $\mathbb{Z}[H]/(H^{n+1})$, where $H$ denotes the class of hyperplanes and $H^i$ the class of codimension $i$ linear spaces. Its generators as a free abelian group are $[\mathbb{P}^n], H, \ldots, H^n$, where $[\mathbb{P}^n]$ is the class of the projective space $\mathbb{P}^n$ itself.

In general, however, Chow groups are difficult to compute. Hence instead

**Figure 2.1:** Rational equivalence




of computing the elements of the Chow group themselves, it is convenient to compute a coarser invariant, their degrees. For a $k$-cycle class $\sum_{i=1}^m a_i[V_i]$, define its degree $\deg(\sum_{i=1}^m a_i[V_i])$ to be the weighted sum $\sum_{i=1}^m a_i \deg(V_i)$ of the degrees of its representatives. The degree of a subvariety $V_i$ of $X$ is its degree seen as a subvariety of the ambient space $\mathbb{P}^n$. Observe that this definition of degree disagrees with Definition 1.4 in [17], where the degree $\int \alpha$ denotes what we call the degree of the zero-dimensional part of a cycle $\alpha$. A different point of view, applied in [1], is to compute the push-forward of characteristic classes to the Chow group of $\mathbb{P}^n$. This profits from the fact that the Chow group of $\mathbb{P}^n$ is well-known and small. Denote by $i$ the imbedding $X \hookrightarrow \mathbb{P}^n$. Then the push-forward $i_* \colon A_*(X) \to A_*(\mathbb{P}^n)$ will map $\alpha = \sum_{i=1}^m a_i[V_i] \in A_k(X)$ to $\sum_{i=1}^m a_i \deg(V_i) H^{n-k} = \deg(\alpha) H^{n-k}$. Hence computing the push-forward of a $k$-cycle class to $\mathbb{P}^n$ is equivalent to computing its degree.

If $E$ is globally generated, then for general sections $s_1, \ldots, s_d$ of $E$ it holds that the degeneracy locus

$$c_i' = \{x \in X \mid s_1(x), \ldots, s_{d-i+1}(x) \text{ are dependent}\}$$

has pure dimension $i$ in $X$ or is empty. Then define $c_i(E) = [c_i']$, where $[c_i'] \in A_{d-i}(X)$ denotes the class of $c_i'$. We omit the proof of well-definedness. If $E$ is not globally generated, then one can twist $E$ with a line bundle $L$ on $X$ such that $E \otimes L$ has enough global sections. The Chern classes of $E$ can then be defined in terms of the Chern classes of $E \otimes L$ and the first Chern class of $L$, by the formula

$$c_p(E) = \sum_{i=0}^{p} (-1)^{p-i} \binom{d-i}{p-i} c_1(L)^{p-i} c_i(E \otimes L).$$

The first Chern class of a line bundle is just the cycle class of the corresponding divisor. The total Chern class $c(E)$ of $E$ is defined as the sum $c(E) = 1 + c_1(E) + \ldots + c_d(E)$.

By now, we have defined Chern classes of vector bundles on schemes. The Chern classes of a smooth scheme are by definition the Chern classes of the tangent bundle of that scheme.

One can define the total Segre class of a vector bundle as the formal inverse of the total Chern class of that bundle, which we have already defined. However, for the definition of the Segre classes of a scheme $X$, we need the more involved definition of the Segre classes of cones, applied to the normal cone $C_X\mathbb{P}^n$. Fortunately, there exists a shortcut, allowing us to define the Segre classes $s_i(X,\mathbb{P}^n)$ for $1 \leq i \leq d$ directly. Let $E_X$ be the exceptional divisor of the blow-up $\mathrm{Bl}_X\mathbb{P}^n$ of $\mathbb{P}^n$ along $X$, and let $\eta : E_X \to X$ be the projection. Then $s_i(X,\mathbb{P}^n) := (-1)^p \eta_*(E_X^p)$ with $p := n+i-d$, where $E_X^p$ denotes the $p$-th self intersection of the exceptional divisor $E_X$. Intersection of Cartier divisors is defined in [17], Definition 2.4.2. The total Segre class $s(X,\mathbb{P}^n)$ of $X$ in $\mathbb{P}^n$ is then defined as $s(X,\mathbb{P}^n) = 1 + s_1(X,\mathbb{P}^n) + \ldots + s_d(X,\mathbb{P}^n)$.

There are several generalizations of Chern classes to possibly singular schemes, including Chern-Schwartz-MacPherson classes. They were defined independently by MacPherson [29], proving a conjecture of Grothendieck, and Schwartz [31], and were shown to agree in [6]. Chern-Schwartz-MacPherson classes enjoy nice functorial properties, which we describe by summarizing section 2.2-2.3 of [1]. Let $S$ be a proper scheme. The Chern-Schwartz-MacPherson class of a closed subscheme $X$ of $S$ is an element $c_{\mathrm{SM}}(X)$ in $A_*(X)$ such that $c_{\mathrm{SM}}(X) = c(X)$, the total Chern class of $X$, for nonsingular schemes $X$. The $i$-th Chern-Schwartz-MacPherson class $(c_{\mathrm{SM}})_i(X) \in A_{d-i}(X)$ is then the codimension $i$ part of the total Chern-Schwartz-MacPherson class $c_{\mathrm{SM}}(X)$. Chern-Schwartz-MacPherson classes extend to constructible functions by

$$c_{\mathrm{SM}}(\sum_{V \subset S} m_V \mathbf{1}_V) = \sum_{V \subset S} m_V c_{\mathrm{SM}}(V),$$

where the sum runs over the closed subvarieties of $S$. This actually gives a natural transformation $\mathcal{C} \rightsquigarrow \mathcal{A}$ from the functor of constructible functions $\mathcal{C}$ to the Chow group functor $\mathcal{A}$. The functor $\mathcal{C}$ of constructible functions maps a scheme to the abelian group of constructible functions on it. A morphism $f : S \to T$ of schemes is mapped to the morphism $\mathcal{C}(f)$ of abelian groups by $\mathcal{C}(f)(\mathbf{1}_V)(y) = \chi(f^{-1}(y) \cap V)$, for $V \subseteq S$ a subscheme and $y \in T$ a closed point. Here $\chi$ is the topological Euler characteristic and hence $\chi(f^{-1}(y) \cap V)$ is the topological Euler characteristic of the fiber of the point $y$. The topological Euler characteristic of a complex projective scheme is defined as the alternating sum of the Betti numbers of the scheme in the usual Euclidean topology.

As a special case of the functorial properties, we get that Chern-Schwartz-MacPherson classes can be used to compute the Euler characteristic of the support of schemes. Let $\kappa : X \to$ point. Then

$$\kappa_* c_{\mathrm{SM}}(X) = c_{\mathrm{SM}}(\mathcal{C}(\kappa)\mathbf{1}_X) = c_{\mathrm{SM}}(\chi(X_{\mathrm{red}})\mathbf{1}_{\mathrm{point}}) = \chi(X_{\mathrm{red}})[\mathrm{point}],$$

hence $\int c_{\mathrm{SM}}(X) = \chi(X_{\mathrm{red}})$, where $\int c_{\mathrm{SM}}(X)$ denotes the degree of the top class $(c_{\mathrm{SM}})_d(X)$. Observe that constructible functions follow laws of exclusion-inclusion, e.g.,

$$\mathbf{1}_{X_1 \cap X_2} = \mathbf{1}_{X_1} + \mathbf{1}_{X_2} - \mathbf{1}_{X_1 \cup X_2}.$$

Hence Chern-Schwartz-MacPherson classes follow similar laws of exclusion-inclusion, e.g.,

$$c_{\mathrm{SM}}(X_1 \cap X_2) = c_{\mathrm{SM}}(X_1) + c_{\mathrm{SM}}(X_2) - c_{\mathrm{SM}}(X_1 \cup X_2).$$

We continue with an example for the use of Segre classes of vector bundles, a formula for the degree of secant varieties.

*Example.* The secant variety $\sec(X)$ of a variety $X$ embedded in $\mathbb{P}^n$ is defined to be the Zariski closure of the union of all the secants of $X$, i.e., all the lines going through two points in $X$. By Corollary 8.2.9 in [16], for a generic variety X (i.e., almost all varieties) of dimension $d$ it holds that

$$\deg \sec(X) = \sum_{k \geq 0} \binom{2d+1}{k} \deg s_k(T_X).$$

Besides being interesting in their own right, secant varieties have recently gained much interest in algebraic statistics, because they correspond to mixture models. Secant varieties are also closely connected to the problem of tensor decomposition.

More generally, characteristic classes are important concepts in algebraic geometry, especially enumerative geometry and intersection theory. In Paper II and III, we also mention a recent application in algebraic statistics.

## 2.2 Summary of Paper I

We start by shortly summarizing the results of [11], of which Paper I is a generalization. In [11], the authors present a numerical algorithm for the computation of the degrees of the Chern classes of smooth projective schemes. The idea is to relate the degrees of the Chern classes of a variety $X \hookrightarrow \mathbb{P}^n$ to the degrees of so-called residuals. Choose $n$ general hypersurfaces containing $X$ by choosing $n$ random elements of the ideal of $X$. By Bertini's theorem, the hypersurfaces intersect in $X$ and a possibly empty zero scheme, the residual. Using Proposition 9.1.1 from [17], a theorem on the equivalence of a connected component in the intersection product, the authors find a linear relation for the degrees of the Chern classes, depending on the number of points in the residual. The number of points in the residual can be computed using Bertini [3], a software for the numerical solution of polynomial equation systems. Repeating the procedure several times yields enough linear relations between the degrees of the Chern classes to compute them.

In Paper I, we use ideas similar to those in [11] to compute the degrees of the Segre classes of a scheme. One main difference is that the scheme may be singular. We present an algorithm that computes the degrees of the Segre classes of a possibly singular scheme $X \hookrightarrow \mathbb{P}^n$. The idea is again to relate the degrees of the Segre classes to the degrees of residuals. Choose $d$ hypersurfaces containing $X$, where $d$ goes from the codimension of $X$ to $n$. The components complementary to $X$ in the intersection of the hypersurfaces are called the residuals. It does not follow directly from Bertini's theorem that the residual has the expected dimension. However, by blowing up $\mathbb{P}^n$ along $X$, using Bertini's theorem on the blowup and using the residual intersection formula from [17], we prove that the residuals do have the expected dimension. Furthermore, we find a linear relation between the degrees of the Segre classes, depending on the degrees of the residual. Letting $d$ go from the codimension of $X$ to $n$, this yields a triangular linear equation system with ones on the diagonal for the degrees of the Segre classes.

The degrees of the residual can be computed both symbolically and numerically. Symbolically, the residual can be computed as the saturation of the ideal of the intersection of the hypersurfaces with the ideal of $X$. Moreover, we claim that the degree of the residual can be computed numerically as a by-product of the regenerative cascade, an algorithm for numerical irreducible decomposition developed in [20]. This yields a numerical method for the computation of the residuals and hence the degrees of the Segre classes.

Another method for computing degrees of Segre classes of complex projective schemes was presented by Aluffi in [1]. We compare the symbolic implementation of our algorithm to Aluffi's implementation. According to our experiments, the implementations complement each other.

The results in Paper I have been generalized in [30], by replacing the ambient space $\mathbb{P}^n$ with general toric varieties. The authors present an algorithm for the computation of the pushforward to the ambient space of the Segre classes of subvarieties of general toric varieties.

## 2.3 Summary of Paper II

In Paper II, we present an algorithm for the computation of the degrees of Chern-Schwartz-MacPherson classes of complex projective schemes, provided an algorithm for the computation of Segre classes. Together with the Segre class algorithm from Paper I, this yields an algorithm for the computation of the degrees of Chern-Schwartz-MacPherson classes. As the degree of the top Chern-Schwartz-MacPherson class is the topological Euler characteristic, this also yields a method to compute the topological Euler characteristic of complex projective schemes. The algorithm described in this paper relies heavily on the algorithms presented by Aluffi in [1]. One of the algorithms computes the degrees of Chern-Schwartz-MacPherson classes from so-called shadows of graphs, and the other computes the degrees of Segre classes from the shadows of graphs. We prove that the second algorithm can be reversed, i.e., the shadows of the graph can be computed from the degrees of the Segre classes. Together with the first algorithm this yields a method to compute degrees of Chern-Schwartz-MacPherson classes.

We also describe in detail how to use the regenerative cascade from [20] to compute the degrees of Segre classes numerically, which was already claimed in Paper I. This yields numerical algorithms for the computation of the degrees of both Segre classes and Chern-Schwartz-MacPherson classes.

We compare the running times of the implementation of our algorithm to those of Aluffi's implementation described in [1] and the Macaulay2 command euler. The latter computes the topological Euler characteristic of smooth schemes by computing the Hodge numbers. The symbolic implementations complement each other well. The numerical implementation is actually slower than the symbolic implementations for the examples considered in Paper II. However, the numerical version has other advantages, e.g., it is parallelizable.

The motivation for computing Chern-Schwartz-MacPherson classes and the topological Euler characteristic stems from algebraic statistics. As Huh shows in [22], there is a close connection between the maximum likelihood degree of certain statistical models and Chern-Schwartz-MacPherson classes. When estimating parameters of polynomial statistical models from given data by maximizing the likelihood function, the likelihood function may have several stationary points. Their number is called the maximum likelihood degree [7]. As showed in [22], the maximum likelihood degree of a large class of models defined by polynomials is their signed topological Euler characteristic. Using these results, we compute the maximum likelihood degree of an example model by computing its topological Euler characteristic.

## 2.4 Summary of Paper III

In Paper III, we describe the Macaulay2 package CharacteristicClasses. The package is an implementation of the algorithms for the computation of degrees of Chern classes [11], Segre classes (Paper I), Chern-Schwartz-MacPherson classes and the topological Euler characteristic (Paper II). Both the symbolic and the numerical version of the algorithms are implemented, the latter via an interface to Bertini [3]. The input data can be either an ideal in a polynomial ring or a projective variety. The output can be given either as a list of degrees, or as the push-forward to the Chow ring of $\mathbb{P}^n$.

The target group of the article are potential users of the package. We start by reviewing Chow groups, degrees of Chern and Segre classes, and the topological Euler characteristic. Then we describe how to use the package with the help of examples. We compute the degrees of the Chern classes of the twisted cubic, the degrees of the Segre classes of the singular locus of the Whitney umbrella, and the topological Euler characteristic of the statistical model described in Paper II.

# 3. Globalizing the action of the Grothendieck-Teichmüller group on the Schouten algebra

## 3.1 Polyvector fields, $L_\infty$-algebras and the Grothendieck-Teichmüller group

We introduce the notions used in Paper IV and shortly sketch the background. At first, we fix notation and review graded vector spaces. Then we introduce the Schouten algebra of polyvector fields on a manifold and $L_\infty$-algebras and their morphisms. We finish by giving a short overview about the problem of deformation quantization and the Grothendieck-Teichmüller group.

A graded vector space is a direct sum $V = \bigoplus_{n=-\infty}^{\infty} V_n$ of vector spaces $V_n$. An element $f \in V_n$ is called homogeneous of degree $n$, one writes that $|f| = n$. We denote by $V[i]$ the graded vector space $\bigoplus_{n=-\infty}^{\infty} V[i]_n$ where $V[i]_n = V_{i+n}$. So $V[i]$ is isomorphic to $V$ as a vector space, but has a different grading. It is called the vector space $V$ shifted by $i$. The image of $v \in V_{i+n}$ under the isomorphism $V[i]_n = V_{i+n}$ is denoted by $v[i]$. A linear map $f$ such that $f(V_n) \subset V_{n+i}$ is said to have degree $i$. The tensor product $V \otimes W$ of graded vector spaces $V$ and $W$ is defined as $V \otimes W = \bigoplus_{n=-\infty}^{\infty}(V \otimes W)_n$ with $(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$. The exterior and symmetric powers of a graded vector space are defined analogously, with $v \wedge w = -(-1)^{|v||w|} w \wedge v$ for $v \wedge w \in V \wedge V$ and $v \cdot w = (-1)^{|v||w|} w \cdot v$ for $v \cdot w \in V \cdot V$. The décalage isomorphism identifies $\Lambda^n(V[1]) \cong (S^n V)[n]$ via $v_1[1] \wedge \ldots \wedge v_n[1] \mapsto (-1)^{\sum_{i=1}^n i \cdot |v_i|}(v_1 \cdot \ldots \cdot v_n)[n]$.

A Lie bracket of degree $n$ on a graded vector space is a graded degree 0 Lie bracket on $V[n]$, i.e., a bilinear map $[-,-] \colon V[n] \otimes V[n] \to V[n]$ satisfying

- Antisymmetry: $[f,g] = -(-1)^{|f||g|}[g,f]$, and
- Jacobi identity: $[f,[g,h]] = [[f,g],h] + (-1)^{|f||g|}[g,[f,h]]$.

A degree 1 Lie bracket $[-,-]$ is equivalent to a bilinear symmetric bracket $\langle -,- \rangle \colon V \otimes V \to V[1]$ via

$$\langle x,y \rangle = (-1)^{|x|}\,[\,x[1], y[1]\,]\,[-1].$$

The bracket $\langle -,- \rangle$ is symmetric and satisfies the Jacobi identity. See section 13.3.10 in [28].

### 3.1.1 The Schouten algebra of polyvector fields

Let $M$ be a smooth manifold of dimension $d$. Denote by $C^\infty(M)$ the $\mathbb{R}$-algebra of smooth functions on $M$, and denote the tangent bundle by $TM$. The Schouten algebra $T_{\mathrm{poly}}(M)$ of polyvector fields on $M$ is a degree 1 Lie algebra whose underlying set is the exterior algebra over the $C^\infty(M)$-module of vector fields, i.e.,

$$T_{\mathrm{poly}}(M) = \bigoplus_{n=0}^{d} \Lambda^n(\Gamma(TM)).$$

The Schouten algebra obtains a grading by $|f| = n$ for $f \in \Lambda^n(\Gamma(TM))$.

The Lie bracket on vector fields extends to the Schouten algebra via the graded Leibniz rule,

$$[f, g \wedge h] = [f,g] \wedge h + (-1)^{|f||g|} g \wedge [f,h]$$

for homogeneous polyvector fields $f, g$ and $h$. The extended bracket is called the Schouten bracket. It is a symmetric bracket corresponding to a degree 1 Lie bracket, i.e., it satisfies

- Symmetry: $[f,g] = (-1)^{|f||g|}[g,f]$, and
- Jacobi identity: $[f,[g,h]] = [[f,g],h] + (-1)^{|f||g|}[g,[f,h]]$.

The Schouten bracket can be defined explicitely in an open set with local coordinates $x^1, \ldots, x^n$. We use Einstein notation, hence using upper indices for local coordinates indicates that they transform contravariantly. A polyvector field can be written as

$$f^{i_0 \cdots i_k}(x) \frac{\partial}{\partial x^{i_0}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_k}}.$$

for smooth $f^{i_0 \cdots i_k}(x)$. Observe that the Einstein summation convention is used. We write $\varphi_i$ instead of $\frac{\partial}{\partial x^i}$. Then a polyvector field, written in local coordinates, is an element of the graded commutative polynomial ring $C^\infty(U)[\varphi_1, \ldots, \varphi_d]$, where the $\varphi_i$ are of degree 1. This allows us to write the Schouten bracket in local coordinates simply as

$$[f,g] = (-1)^{|f|} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \varphi_i} + (-1)^{|f||g|+|g|} \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial \varphi_i}.$$

Here $f$ and $g$ are elements of $C^\infty(U)[\varphi_1, \ldots, \varphi_d]$, i.e., polyvector fields in local coordinates. Observe that $C^\infty(U)[\varphi_1, \ldots, \varphi_d]$ is graded commutative, i.e.,

$$\varphi_i \varphi_j = (-1)^{|\varphi_i||\varphi_j|} \varphi_j \varphi_i = -\varphi_j \varphi_i.$$

### 3.1.2 $L_\infty$-algebras and Maurer-Cartan elements

$L_\infty$-algebras are generalizations of Lie algebras, consisting not only of a bilinear bracket, but an $n$-linear operation for any positive integer $n$. The idea is that the bilinear operation is not exactly a Lie bracket, but only a Lie bracket *up to homotopy*, i.e., the failure to be a Lie bracket is expressed in terms of the trilinear bracket. Similar conditions have to hold for the higher brackets.

In more detail, an $L_\infty$-structure on a graded vector space $g$ is a collection of skew-symmetric maps

$$Q_n \colon \bigotimes^n g \to g$$

of degree $2-n$ such that

$$\sum_{i+j=n+1} \sum_{\sigma} \pm Q_i(Q_j(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \ldots \otimes v_{\sigma(n)}) = 0 \qquad (3.1)$$

for all $n \geq 1$. The inner sum runs over all permutations $\sigma$ such that $\sigma(1) < \ldots < \sigma(j)$ and $\sigma(j+1) < \ldots < \sigma(n)$. The sign $\pm$ depends on $i, j$ and the permutation $\sigma$. If all the maps except $Q_2$ are zero, then Equation (3.1) for $n = 3$ is the usual graded Jacobi identity. In general, $n = 3$ yields a Jacobi identity up to higher terms, or up to homotopy.

Another approach to defining $L_\infty$-algebras will prove to be convenient. Consider the reduced graded symmetric coalgebra

$$\bar{S}(g[1]) = \bigoplus_{i=1}^{\infty} S^n g[1],$$

where $S^n$ denotes the $n$-th symmetric tensor power. It is endowed with a coalgebra structure $\Delta$ which is uniquely determined by $\Delta(v) = v \otimes 1 + 1 \otimes v$ for $v \in g$ and by being an algebra homomorphism.

An $L_\infty$-structure on $g$ is a coalgebra differential and coderivation $Q$ on $\bar{S}(g[1])$, that is, a degree 1 map $Q \colon \bar{S}(g[1]) \to \bar{S}(g[1])$ such that $Q^2 = 0$ and $Q$ is a coderivation, i.e.,

$$\Delta \circ Q = (Q \otimes Q) \circ \Delta.$$

The differential $Q$ is determined by maps $S^n g[1] \to \bar{S}(g[1])$ for every $n$. Because $Q$ is a coderivation, these maps are in turn determined by their composition with the projection $S^n(g[1]) \to g[1]$. Hence a differential $Q$ on the symmetric coalgebra $\bar{S}(g[1])$ can be given by degree 1 maps

$$Q_n \colon S^n g[1] \to g[1].$$

They correspond to degree $2-n$ maps

$$Q_n \colon \Lambda^n g \to g.$$

The condition $Q^2 = 0$ translates to the conditions (3.1) on the $Q_n$ and vice versa. We say that the pair $(g,Q)$ is an $L_\infty$-algebra.

The second definition makes it easier to define the notion of an $L_\infty$-morphism. Let $(g,Q)$ and $(h,R)$ be $L_\infty$-algebras. Then an $L_\infty$-morphism from $(g,Q)$ to $(h,R)$ is given by a coalgebra morphism

$$\Phi\colon \bar{S}(g[1]) \to \bar{S}(h[1])$$

that commutes with $Q$ and $R$, i.e., $\Phi \circ Q = R \circ \Phi$.

In a way similar to $L_\infty$-algebra structures, an $L_\infty$-morphism $\Phi\colon \bar{S}(g[1]) \to \bar{S}(h[1])$ is uniquely determined by its composition with the projection $\bar{S}(h[1]) \to h[1]$. Hence an $L_\infty$-morphism $\Phi$ can also be given by linear maps $\Phi_n\colon \Lambda^n g \to h$ satisfying compatibility conditions coming from the fact that $\Phi$ respects the $L_\infty$-structures $Q$ and $R$.

We turn to the topic of twisting $L_\infty$-algebras and $L_\infty$-morphisms with a so-called Maurer-Cartan element. For a more detailed introduction see, e.g., [12; 34]. In a graded Lie algebra, a Maurer-Cartan element is a degree 1 element $\lambda$ such that $[\lambda,\lambda]$ is zero. In an $L_\infty$-algebra $(g,Q)$, a Maurer-Cartan element is a degree 1 element $\pi$ of $g$ satisfying

$$\sum_{i=1}^{\infty} \frac{1}{i!} Q_i(\pi,\ldots,\pi) = 0.$$

Given a Maurer-Cartan element in an $L_\infty$-algebra, one can construct a new $L_\infty$-algebra structure on the same underlying set by twisting with a Maurer-Cartan element as follows.

Let $(g,Q)$ and $(h,R)$ be $L_\infty$-algebras as before, $\pi$ an element of $g[1]$ and $\Phi\colon (g,Q) \to (h,R)$ an $L_\infty$-morphism. Define $\exp(\pi)\colon \bar{S}(g[1]) \to \bar{S}(g[1])$ by

$$\exp(\pi)(X) := \sum_{i=0}^{\infty} \frac{1}{i!} \pi^i X$$

for $X$ in $\bar{S}(g[1])$. One checks that $\exp(-\pi) \circ \exp(\pi) = \mathrm{id}$.

Suppose that $\pi$ is a Maurer-Cartan element. Then the map $Q_\pi$ defined by

$$Q_\pi = \exp(-\pi) \circ Q \circ \exp(\pi)$$

makes $(g,Q_\pi)$ into an $L_\infty$-algebra. This is the twisting of $(g,Q)$ with $\pi$. It may also be given explicitly by the formula $Q_\pi(X) = \sum_{i=0}^{\infty} \frac{1}{i!} Q(\pi^i X)$.

We can also twist the $L_\infty$-morphism $\Phi\colon (g,Q) \to (h,R)$ with the Maurer-Cartan element $\pi$ of $(g,Q)$. The first step is to find a corresponding Maurer-Cartan element in $(h,R)$. It is given by $\omega = \sum_{i=1}^{\infty} \frac{1}{i!} \Phi_i(\pi^i)$. We can twist $(g,Q)$ with $\pi$ and $(h,R)$ with $\omega$ and get the $L_\infty$-algebras $(g,Q_\pi)$ and $(h,R_\omega)$. The twisted $L_\infty$-morphism $\Phi_\pi$ between them is given by $\Phi_\pi = \exp(-\omega) \circ Q \circ \exp(\pi)$. An explicit formula for $\Phi_\pi$ is given by $\Phi_\pi(X) = \sum_{i=0}^{\infty} \frac{1}{i!} \Phi(\pi^i X)$.

### 3.1.3 Deformation Quantization

In order to give some background and motivation for the study of polyvector fields and $L_\infty$-algebras, we sketch the problem of deformation quantization. We introduce fundamental objects in deformation quantization, polydifferential operators and the Hochschild cochain complex, as well as the important Hochschild-Kostant-Rosenberg theorem and the Formality Theorem.

Let $M$ be a smooth $d$-dimensional manifold as before. We start by introducing the $C^\infty(M)$-module of polydifferential operators $D_{\text{poly}}(M)$. On an open set $U$ of the manifold $M$ with local coordinates $x^1,\dots,x^d$, a polydifferential operator is a map

$$C^\infty(U)\otimes\ldots\otimes C^\infty(U)\to C^\infty(U)$$

of the form

$$\Delta_U = \Delta_U^{I_1\ldots I_n}(x)\frac{\partial^{|I_1|}}{\partial x^{i_1^{(1)}}\ldots\partial x^{i_{k_1}^{(1)}}}\otimes\ldots\otimes\frac{\partial^{|I_n|}}{\partial x^{i_1^{(n)}}\ldots\partial x^{i_{k_n}^{(n)}}}, \tag{3.2}$$

where the $\Delta_U^{I_1\ldots I_n}(x)$ are smooth functions, the $I_j$ are multi-indices

$$I_j = (i_1^{(j)},\dots,i_{k_j}^{(j)})$$

and $|I_j| = k_j$. On smooth functions $a_1,\dots,a_n$, they act as

$$\Delta(a_1,\dots,a_n) = \Delta^{I_1\ldots I_n}\frac{\partial^{|I_1|}a_1}{\partial x^{i_1^{(1)}}\ldots\partial x^{i_{k_1}^{(1)}}}\cdot\ldots\cdot\frac{\partial^{|I_n|}a_n}{\partial x^{i_1^{(n)}}\ldots\partial x^{i_{k_n}^{(n)}}}.$$

Having described polydifferential operators locally, we define polydifferential operators on the manifold $M$ as maps

$$\Delta\colon C^\infty(M)\otimes\ldots\otimes C^\infty(M)\to C^\infty(M)$$

that are locally of the form (3.2). More precisely, there is a covering of $M$ with open subsets $U_i$ such that $\Delta$ restricted to $C^\infty(U_i)^{\otimes n}$ is of the form (3.2).

The polydifferential operators form a complex with the grading

$$D_{\text{poly}}(M) = \bigoplus_{n=0}^{\infty} D^n_{\text{poly}}(M),$$

where $D^n_{\text{poly}}(M)$ consists of the polydifferential operators from $C^\infty(M)^{\otimes n}$ to $C^\infty(M)$. It is a subcomplex of the Hochschild cochain complex $C^\bullet(A,A)$ for the associative algebra $A := C^\infty(M)$, where

$$C^n(A,A) = \mathrm{Hom}_{\mathbb{R}}(A^{\otimes n+1},A).$$

The differential in this complex is given by $d\colon C^{n-1}(A,A) \to C^n(A,A)$ with

$$(d\Delta)(a_0,\dots a_n) =$$
$$a_0\Delta(a_1,\dots,a_n) \;+\; \sum_{i=1}^{n}(-1)^i\Delta(a_0,\dots,a_{i-1}a_i,\dots,a_n) \;+\; \Delta(a_0,\dots,a_{n-1})a_n.$$

The cohomology of the Hochschild cochain complex is denoted by $\mathrm{HH}^*(A,A)$, the cohomology of the subcomplex $D_{\mathrm{poly}}(M)$ is denoted by $\mathrm{HH}^*_{\mathrm{diff}}(M)$. We mention also that the polydifferential operators form a Lie algebra with respect to the so-called Gerstenhaber bracket, which induces a bracket on cohomology. So both the space of polydifferential operators and its cohomology are Lie algebras. They are connected to the Schouten algebra by the following famous theorem.

*Hochschild-Kostant-Rosenberg-Theorem* ([21]). The cohomology of the differential Hochschild complex of polydifferential operators is isomorphic as a Lie algebra to the Lie algebra of polyvector fields with the Schouten bracket:

$$(\mathrm{HH}^*_{\mathrm{diff}}, [-,-]_G) = (T_{\mathrm{poly}}(M), [-,-]_S).$$

One motivation for studying the Hochschild cochain complex of $C^\infty(M)$ comes from the work of Gerstenhaber [18]. The Hochschild cohomology groups of an associative algebra control the deformations of this algebra. Here, we consider a special deformation problem, deformation quantization, posed by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in [4] and [5].

Let $A$ be a commutative algebra over a field $\mathbb{K}$. We introduce a formal parameter $v$. A deformation of $A$ is an associative algebra $A[[v]]$ over $\mathbb{K}[[v]]$ with a product

$$\star\colon A[[v]] \otimes_{\mathbb{K}[[v]]} A[[v]] \to A[[v]]$$

such that the original product of $A$ can be obtained by setting $v = 0$. Any such product can be written in the form

$$a \star b = ab + vB_1(a,b) + v^2B_2(a,b) + \dots\ ,$$

where $B_1(a,b), B_2(a,b),\dots$ are elements in $A$. Consider the important special case when $A$ is the algebra $C^\infty(M)$ of smooth functions on a manifold $M$. From the associativity of the star product, it follows that $\{-,-\}$ defined by

$$\{a,b\} = \frac{B_1(a,b) - B_1(b,a)}{2}$$

makes $A[[v]]$ into a Poisson algebra, i.e., a special Lie algebra. If there exists a Poisson algebra structure on $A = C^\infty(M)$, then $M$ is called a Poisson manifold.

The problem of deformation quantization is to prove existence and uniqueness of an equivalence class of star products for any given Poisson manifold.

An attempt to give some physical intuition behind this problem is to interpret the algebra $A$ as the algebra of classical observables and the deformed associative algebra as the algebra of quantum observables. The formal parameter $v$ is often set to $\frac{i\hbar}{2}$ where $\hbar$ is the Planck constant. Finding a deformation of the algebra of classical observables to the algebra of quantum observables means explaining why quantum world appears to follow classical laws.

The solution to the deformation quantization problem for symplectic manifolds, i.e., nondegenerate Poisson manifolds, was given by Fedosov [15]. Its main idea is the Fedosov construction, which is used heavily in Paper IV. The solution for general Poisson structures is due to Kontsevich [25; 26]. It consists of the proof of the Formality Theorem (stated as Formality Conjecture in [25]), which is a stronger version of the Hochschild-Kostant-Rosenberg Theorem.

*Formality Theorem* (Kontsevich [26]). There exists an $L_\infty$-quasi-isomorphism between the Lie algebras $T_{\text{poly}}(M)$ and $D_{\text{poly}}(M)$.

The connection to deformation quantization is roughly the following: If the Formality Conjecture is true, one can relate the Maurer-Cartan elements of $D_{\text{poly}}(M)$ and $T_{\text{poly}}(M)$. The Maurer-Cartan elements of $T_{\text{poly}}(M)$ are the Poisson structures of $M$, and the Maurer-Cartan elements of $D_{\text{poly}}(M)$ can be identified with deformations of the usual multiplication of smooth functions on $M$. Hence, the existence of a star product as above follows from the Formality Conjecture. For more details we refer to [25], [32] and the introduction [10].

Kontsevich actually proved the formality theorem only for affine space $\mathbb{R}^d$, but sketched a proof of a globalization to general manifolds. This allowed Cattaneo, Felder and Tomassini [8; 9] as well as Dolgushev [13] to establish the globalization of Kontsevich's result.

### 3.1.4 The Grothendieck-Teichmüller group and Kontsevich's graph complex

The Grothendieck-Teichmüller group was defined by Drinfel'd in [14] in his study of associators, but its history goes back to Grothendieck's Esquisse d'un Programme [19]. In this research proposal for the Centre National de la Recherche Scientifique, Grothendieck proposes to study the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$ by studying its action on the so-called Teichmüller tower. The Teichmüller tower is the collection of the Teichmüller groupoids, i.e., the fundamental groupoids of the moduli spaces $M_{g,n}$ of compact Riemann surfaces $X$ of genus $g$ with $n$ distinguished points, connected by a certain set of homomorphisms. As stated by Grothendieck, there is an injection of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

into the automorphism group of the profinite version $\{\widehat{T_{g,n}}\}_{g,n}$ of the full Teichmüller tower. Drinfel'd proved that the Grothendieck-Teichmüller group GT is the automorphism group of the subtower $\{\widehat{T_{0,n}}\}_n$. Hence the Grothendieck-Teichmüller group is indeed closely connected to the Galois group of $\mathbb{Q}$, as stated by Drinfel'd in the title of [14].

In [14], Drinfel'd actually constructs two related groups called *GT* and *GRT*, meaning Grothendieck-Teichmüller group and its graded version. Both groups are defined in terms of rather complicated generators and relations. *GT* acts on the set *Ass* of Drinfel'd associators on the right, and *GRT* on the left. A more combinatoric definition is given by Bar-Nathan in [2] following Lochak and Schneps [27]. Bar-Nathan defines *GT* as the group of structure preserving automorphisms of the category **PaB** of parenthesized braids, and *GRT* as the automorphism group of **PaCD**, the category of parentesized chord diagrams. As every associator induces an isomorphism between **PaB** and **PaCD**, one obtains Drinfel'd's right action of *GT* and left action of *GRT*.

In the following, we focus on the graded version *GRT* of the Grothendieck-Teichmüller group and describe Willwacher's recent result on the role of GRT in deformation quantization. At first, we give an informal presentation of the Kontsevich graph complex $\mathrm{GC}_2$, introduced in [23] and [24], whose cohomology had long been a riddle. The underlying space of the Kontsevich graph complex $\mathrm{GC}_2$ is the space of formal sums of graphs, each of whose vertices are at least trivalent. It is a Lie algebra, where the Lie bracket of two graphs is defined as the signed sum of all possible insertions of one graph into the other. One checks that the graph $\bullet\!\!-\!\!\bullet$ satisfies the Maurer-Cartan equation $[\bullet\!\!-\!\!\bullet,\bullet\!\!-\!\!\bullet]=0$, hence it equips $\mathrm{GC}_2$ with the differential $[\bullet\!\!-\!\!\bullet,-]$. We have thus constructed a differential graded Lie algebra $(\mathrm{GC}_2,[-,-],[\bullet\!\!-\!\!\bullet,-])$. Willwacher's main result in [33] is that the zeroth cohomology of this complex is isomorphic as a Lie algebra to the Lie algebra of yet another version of the Grothendieck-Teichmüller group: $H^0(\mathrm{GC}_2)\cong\mathfrak{grt}_1$. In Drinfel'd's notation, $\mathfrak{grt}_1$ is the Lie algebra of the nilpotent part $GRT_1$ of the Grothendieck-Teichmüller group *GRT*. Following Willwacher, we abuse notation and call these versions $\mathfrak{grt}$ and *GRT* as well.

## 3.2 Summary of Paper IV

As said above, Willwacher showed in [33] that the zero cohomology of the Kontsevich graph complex is the Lie algebra $\mathfrak{grt}$ of the Grothendieck-Teichmüller group *GRT*. In the same article, he shows that the Grothendieck-Teichmüller algebra also acts by $L_\infty$-automorphisms on the Schouten algebra of polyvector fields $T_{\mathrm{poly}}(\mathbb{R}^d)$. In Paper IV, we show that this action can be globalized, i.e., that *GRT* also acts on the Schouten algebra of polyvector fields $T_{\mathrm{poly}}(M)$ on a general smooth manifold $M$. The action depends on the choice of a connection on $M$. More generally, we prove that a class of $L_\infty$-automorphisms of $T_{\text{poly}}(\mathbb{R}^d)$, including Willwacher's, can be globalized.

The method we use is the Fedosov globalization. In [15], Fedosov describes how to globalize the Moyal star product using an ingenious construction, and hence proving deformation quantization for arbitrary symplectic manifolds. The same method was used by Dolgushev in [13] to globalize Kontsevich's $L_\infty$-quasiisomoprhism between $T_{\text{poly}}(\mathbb{R})$ and $D_{\text{poly}}(\mathbb{R})$. We in turn use the same method to globalize $L_\infty$-automorphisms of $T_{\text{poly}}(\mathbb{R})$. Many steps of the construction correspond directly to results in [13] which in turn correspond to results in [15].

We start with the main construction, vertical polyvector fields $T_{\text{poly}}^{\text{vert}}(M)$ and differential forms $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ with values in vertical polyvector fields. Intuitively, vertical polyvector fields are sections of a vector bundle of $M$ each of whose fibers is isomorphic to $T_{\text{poly}}(\mathbb{R}^d)$. Therefore, $L_\infty$-automorphisms of $T_{\text{poly}}(\mathbb{R}^d)$ are easily extended to so-called vertical $L_\infty$-automorphisms of $T_{\text{poly}}^{\text{vert}}(M)$ and $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$. The following steps consist in constructing a differential on $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ such that $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ is a resolution of $T_{\text{poly}}(M)$. In a first step, one constructs a differential $\delta$ yielding a resolution as vector space. This differential is then twisted by a Maurer-Cartan element depending on the choice of a torsion-free connection on $M$. With the resulting differential $D$, one obtains that $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ is a resolution of $T_{\text{poly}}(M)$ as a Lie algebra. The last step is to show that, from the vertical $L_\infty$-automorphism, another automorphism of $\Omega(M, T_{\text{poly}}^{\text{vert}}(M))$ that respects the differential $D$ can be constructed. This induces an $L_\infty$-automorphism of $T_{\text{poly}}(M)$, yielding a globalization of the $L_\infty$-automorphism of $T_{\text{poly}}$ that we started with.

In the last part of Paper IV, we show that Willwacher's $L_\infty$-automorphisms belong to the class of automorphisms that can be globalized with the method presented before. This shows that Willwacher's $L_\infty$-action of $GRT$ on the Schouten algebra $T_{\text{poly}}(\mathbb{R})$ can be globalized.

# Sammanfattning

Avhandlingen består av två delar, en del om beräkningar inom algebraisk geometri och en del inom deformationskvantisering. Närmare bestämt består den av fyra artiklar. I artiklarna I, II och III presenteras algoritmer och en implementation för beräkningen av invarianter inom algebraisk geometri, nämligen grader av karakteristiska klasser av komplexa projektiva scheman. Den andra delen, artikel IV, är ett bidrag till deformationskvantisering och Grothendieck-Teichmüllergruppens verkan på objekt som studeras inom deformationskvantisering.

Karakteristiska klasser, såsom Chernklasser och Segreklasser, är viktiga invarianter inom algebraisk geometri, särskilt inom snitteorin och enumerativ geometri. Chernklasser av vektorknippen kan på ett förhållandevis konkret sätt definieras som degenereringslokusen av generiska sektioner av vektorknippet, d.v.s. delvarietéer där ett antal generiska sektioner är linjärt beroende. Chernklasserna av ett vektorknippe på ett schema $X$ är element av Chowgruppen $A_*(X)$ av $X$, d.v.s. formella summor av delvarietéer definierade upp till rationell ekvivalens. Chowgruppen är i allmänhet svår att beräkna. Istället för att beräkna Chernklasserna själva kan man därför nöja sig med att beräkna en grövre invariant, Chernklassernas grader. Liknande gäller för Segreklasser och generaliserade Chernklasser, som vi definierar i avhandlingens kappa.

I artikel I presenteras en algoritm för beräkningen av grader av Segreklasser av slutna delscheman av komplexa projektiva rummet $\mathbb{P}^n$. Segreklasserna av ett schema inbäddat i projektiva rummet är definierade som Segreklasserna av inbäddningens normalkon. Algoritmen använder sig av snitteori och teorin av residualer, och kan implementeras både symboliskt och numeriskt.

I artikel II beskrivs en algoritm för beräkningen av Chern-Schwartz-MacPhersonklassernas grader samt den topologiska Eulerkarakteristiken av slutna delscheman av komplexa projektiva rummet $\mathbb{P}^n$. Förutsättningen är att en algoritm för beräkningen av grader av Segreklasser är given. Vi förklarar även noggrant hur algoritmen i artikel I kan implementeras numeriskt. Tillsammans får man en symbolisk och en numerisk version av algoritmen.

I artikel III beskrivs Macaulay2-paketet CharacteristicClasses. I detta paket är algoritmerna från artiklarna I och II implementerade. Dessutom ingår implementationen av en algoritm för beräkningen av grader av Chernklasser.

Avhandlingens andra del handlar om Grothendieck-Teichmüllergruppens roll inom deformationskvantisering. Grothendieck-Teichmüllergruppen är en komplicerad grupp som knyter ihop flera olika delar inom matematiken. Den går tillbaka till Grothendieck som försökte studera den absoluta Galoisgruppen av kroppen $\mathbb{Q}$ genom att studera dess verkan på det så kallade Teichmüllertornet. Drinfel'd definierade senare Grothendieck-Teichmüllergruppen i en artikel om generaliserade Hopfalgebror. De senaste åren har det visat sig att det finns ett samband även mellan Grothendieck-Teichmüllergruppen och viktiga objekt inom området deformationskvantisering, som Kontsevichs grafkomplex och Schoutenalgebran av polyvektorfält. Nyligen visade Willwacher att första kohomologin av Kontsevichs grafkomplex är Liealgebran av en variant av Grothendieck-Teichmüllergruppen. Han visar även att denna Liealgebra verkar på Schoutenalgebran av polyvektorfält på affina rummet $\mathbb{R}^d$ genom $L_\infty$-automorfier. Dessa begrepp introduceras avhandlings kappa.

I artikel IV visas att $L_\infty$-automorfier av Schoutenalgebran $T_{\text{poly}}(\mathbb{R}^d)$ av polyvektorfält på affina rummet $\mathbb{R}^d$ kan globaliseras om dessa uppfyller vissa villkor. Det betyder att man kan konstruera en $L_\infty$-automorfi av $T_{\text{poly}}(M)$ från en given $L_\infty$-automorfi av $T_{\text{poly}}(\mathbb{R}^d)$, där $M$ är en slät mångfald. Det följer att Willwachers verkan av Grothendieck-Teichmüllergruppen på $T_{\text{poly}}(\mathbb{R}^d)$ kan globaliseras, d.v.s. Grothendieck-Teichmüllergruppen verkar på Schoutenalgebran $T_{\text{poly}}(M)$ av polyvektorfält på en allmän mångfald $M$.

# Zusammenfassung

Die hier vorgelegte Arbeit besteht aus zwei Teilen. Der erste Teil befasst sich mit Berechnungen im Bereich der algebraischen Geometrie, der zweite mit dem Gebiet der Deformationsquantisierung. Genauer gesagt besteht die Arbeit aus vier Artikeln. In den Artikeln I, II und III werden mehrere Algorithmen und deren Implementierung vorgestellt, die Invarianten im Gebiet der algebraischen Geometrie berechnen, nämlich die Grade charakteristischer Klassen von komplexen projektiven Schemata. Artikel IV ist ein Beitrag zur Deformationsquantisierung und der Wirkung der Grothendieck-Teichmüllergruppe auf Objekte, die in der Deformationsquantisierung studiert werden.

Charakteristische Klassen, wie beispielsweise Chern- und Segreklassen, sind wichtige Invarianten in der algebraischen Geometrie, speziell in der enumerativen Geometrie und Schnittheorie. Die Chernklassen eines Vektorbündels können verhältnismäßig konkret als Degenerationsloci von generischen Schnitten des Vektorbündels definiert werden, das heißt als Untervarietäten auf denen die Schnitte linear abhängig sind. Die Chernklassen eines Vektorbündels auf einem Schema $X$ sind Elemente der Chowgruppe $A_*(X)$ von $X$, das heißt formelle Summen von Untervarietäten modulo rationelle Äquivalenz. Die Chowgruppe ist im Allgemeinen schwer zu berechnen. Deshalb kann man sich, statt die Chernklassen selbst zu berechnen, auf die Berechnung gröberer Invarianten beschränken, nämlich der Grade der Chernklassen. Ähnliches gilt für Segreklassen und verallgemeinerte Chernklassen, die in der Einleitung definiert werden.

Im Artikel I wird ein Algorithmus für die Berechnung der Grade der Segreklassen von geschlossenen Unterschemata des komplexen projektiven Raumes $\mathbb{P}^n$ vorgestellt. Die Segreklassen eines in ein gegebenes Schema eingebetteten Schemas sind definiert als die Segreklassen des Normalenkegels der Einbettung. Der Beweis der Korrektheit des Algorithmus verwendet Schnittheorie und die Theorie der Residuale. Der Algorithmus kann sowohl symbolisch als auch numerisch implementiert werden.

Im Artikel II wir ein Algorithmus für die Berechnung der Grade der Chern-Schwartz-MacPhersonklassen sowie der topologischen Eulercharakteristik von geschlossenen Unterschemata des komplexen projektiven Raumes $\mathbb{P}^n$ beschrieben. Dabei wird angenommen, dass ein Algorithmus für die Berechnung von Graden von Segreklassen gegeben ist. Zudem wird im Detail erklärt, wie der

Algorithmus in Artikel I numerisch implementiert werden kann. Zusammen erhält man sowohl eine symbolische als auch eine numerische Version des Algorithmus.

Im Artikel III wird das Macaulay2-Paket CharacteristicClasses beschrieben, in welchem die Algorithmen aus den Artikeln I und II implementiert sind. Außerdem ist eine Implementierung eines Algorithmus für die Berechnung von Graden von Chernklassen enthalten.

Der zweite Teil der Arbeit behandelt die Rolle der Grothendieck-Teichmüllergruppe in der Deformationsquantisierung. Die Grothendieck-Teichmüllergruppe ist eine komplizierte Gruppe, die mehrere verschiedene Gebiete der Mathematik verknüpft. Sie geht zurück auf Grothendiecks Versuch, die absolute Galoisgruppe des Körpers $\mathbb{Q}$ über deren Wirkung auf den sogenannten Teichmüllerturm zu studieren. Später definierte Drinfel'd die Grothendieck-Teichmüllergruppe in einer Arbeit zu verallgemeinerten Hopfalgebren. In den letzten Jahren wurde außerdem ein Zusammenhang mit Objekten aus dem Gebiet der Deformationsquantisierung entdeckt, wie Kontsevichs Graphenkomplex und die Schoutenalgebra der Polyvektorfelder. Vor kurzem zeigte Willwacher, dass die erste Kohomologie des Graphenkomplexes isomorph ist zur Liealgebra einer Variante der Grothendieck-Teichmüllergruppe. Ebenfalls zeigte er, dass diese Liealgebra auf die Schoutenalgebra der Polyvektorfelder über dem affinen Raum $\mathbb{R}^d$ durch $L_\infty$- Automorphismen wirkt. Alle diese Begriffe werden in der Einleitung dieser Arbeit erklärt.

Im Artikel IV zeigen wir, dass $L_\infty$-Automorphismen der Schoutenalgebra $T_{\text{poly}}(\mathbb{R}^d)$ von Polyvektorfeldern auf dem affinen Raum $\mathbb{R}^d$ globalisiert werden können, wenn diese gewisse Bedingungen erfüllen. Dies bedeutet, dass ein $L_\infty$-Automorphismus von $T_{\text{poly}}(M)$ konstruiert werden kann, wenn ein $L_\infty$-Automorphismus von $T_{\text{poly}}(\mathbb{R}^d)$ vorliegt. Hierbei ist $M$ eine beliebige glatte Mannigfaltigkeit. Es folgt, dass Willwachers Wirkung der Grothendieck-Teichmüllergruppe auf $T_{\text{poly}}(\mathbb{R}^d)$ globalisiert werden kann, d.h. die Grothendieck-Teichmüllergruppe wirkt auf die Schoutenalgebra $T_{\text{poly}}(M)$ von Polyvektorfeldern auf einer beliebigen glatten Mannigfaltigkeit $M$.

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