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AROUND MULTIVARIATE SCHMIDT-SPITZER THEOREM

PER ALEXANDERSSON AND BORIS SHAPIRO

Abstract. Given an arbitrary complex-valued infinite matrix $A = (a_{ij})$, $i = 1, \ldots, \infty; j = 1, \ldots, \infty$ and a positive integer $n$ we introduce a naturally associated polynomial basis $B_A$ of $\mathbb{C}[x_0, \ldots, x_n]$. We discuss some properties of the locus of common zeros of all polynomials in $B_A$ having a given degree $m$; the latter locus can be interpreted as the spectrum of the $m \times (m+n)$-submatrix of $A$ formed by its $m$ first rows and $(m+n)$ first columns. We initiate the study of the asymptotics of these spectra when $m \to \infty$ in the case when $A$ is a banded Toeplitz matrix. In particular, we present and partially prove a conjectural multivariate analog of the well-known Schmidt-Spitzer theorem which describes the spectral asymptotics for the sequence of principal minors of an arbitrary banded Toeplitz matrix. Finally, we discuss relations between polynomial bases $B_A$ and multivariate orthogonal polynomials.

1. Introduction

The approach of this paper is motivated by the modern interpretation of the Heine-Stieltjes multiparameter spectral problem as presented in [9] and [10]. Let us recall some relevant results in the matrix set-up.

Given integers $m > 0$ and $n \geq 0$ consider the space $\text{Mat}(m, m+n)$ of complex-valued $m \times (m+n)$-matrices. For $s = 0, \ldots, n$ define the $s$-th unit matrix $I_s := (\delta_{s+i-j}) \in \text{Mat}(m, m+n)$.

(In what follows the sizes of matrices can be infinite.)

Definition 1 (see [10]). Given a matrix $A \in \text{Mat}(m, m+n)$ define its eigenvalue locus $E_A$ as

$$E_A := \left\{(x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} : \text{rank} \left( A - \sum_{s=0}^{n} x_s I_s \right) < m \right\}.$$ 

For $n = 0$, $E_A$ coincides with the usual set of eigenvalues of a square matrix $A$.

Proposition 2 (Lemma 1 of [10]). For arbitrary $A \in \text{Mat}(m, m+n)$ the eigenvalue locus $E_A$ consists of $\binom{m+n}{n+1}$ points counting multiplicities. In other words, counting multiplicities there exist $\binom{m+n}{n+1}$ eigenvalue tuples $(x_0, x_1, \ldots, x_n)$ such that $A - \sum_{s=0}^{n} x_s I_s$ has rank smaller than $m$.

Remark 3. Notice that for $n > 0$, the locus $E_A$ is not a complete intersection since it is given by the vanishing of all maximal minors of $A$. (A similar phenomenon can be observed for common zeros of multivariate Schur polynomials, since Schur polynomials are given by determinant formulas.)

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Notation 4. Given an infinite matrix \( A = (a_{ij}), \ i = 1, \ldots, \infty; \ j = 1, \ldots, \infty \), an integer \( n \geq 0 \), and an \( m \)-tuple of positive integers \( I = (i_1, i_2, \ldots, i_m) \) satisfying \( 1 \leq i_1 < i_2 < \ldots < i_m \leq m + n \), consider the submatrix \( A_I \) of \( A - \sum_{s=0}^{n} x_s I_s \) formed by the first \( m \) rows and \( m \) columns indexed by \( I \). Define

\[ P_A^I(x_0, x_1, \ldots, x_n) := \det A_I. \]

Evidently, \( P_A^I(x_0, \ldots, x_n) \) is a maximal minor of the principal \( m \times (m + n) \) submatrix of \( A - \sum_{s=0}^{n} x_s I_s \) formed by its \( m \) first rows and \( m + n \) first columns. Therefore \( P_A^I(x_0, \ldots, x_n) \) is a polynomial in \( x_0, \ldots, x_n \).

Proposition 5. In the above notation the following holds:

(i) for any multiindex \( I \) with \( |I| = m \), \( \deg P_A^I(x_0, \ldots, x_n) = m \);

(ii) all \( \binom{m+n}{m} \) polynomials \( P_A^I(x_0, \ldots, x_n) \in \mathbb{C}[x_0, \ldots, x_n] \) with \( |I| = m \) are linearly independent which implies that the totality of all polynomials \( P_A^I(x_0, \ldots, x_n) \) is a linear basis of \( \mathbb{C}[x_0, \ldots, x_n] \);

(iii) the set \( E_A^{(m)} \) of common zeros of all \( P_A^I(x_0, \ldots, x_n) \) with \( |I| = m \) is a finite subset of \( \mathbb{C}^{n+1} \) of cardinality \( \binom{m+n}{n+1} \) counting multiplicities.

Remark 6. Notice that for \( \binom{m+n}{m} \) randomly chosen polynomials in \( \mathbb{C}[x_0, x_1, \ldots, x_n] \) of degree \( m \) the set of their common zeros is typically empty.

Proposition 5 together with our numerical experiments motivate the following question.

Given an arbitrary infinite matrix \( A \) as above, associate to each \( E_A^{(m)} \) its "root-counting" measure \( \mu_A^{(m)} \) supported on \( E_A^{(m)} \subset \mathbb{C}^{n+1} \) by assigning to every point \( p \in E_A^{(m)} \) the point mass \( \kappa(p)/\binom{m+n}{n+1} \) where \( \kappa(p) \) is the multiplicity of \( p \). (Obviously, \( \mu_A^{(m)} \) is a discrete probability measure.)

Main Problem. Under which assumptions on \( A \) does the weak limit \( \mu_A = \lim_{m \to \infty} \mu_A^{(m)} \) exist? In case when \( \mu_A \) exists, is it possible to describe the support and density of the measure?

In the classical case \( n = 0 \), the above problem was intensively studied by many authors. The main focus has been when \( A \) is either a Jacobi or a Toeplitz matrix (or their generalizations such as block-Toeplitz matrices etc.), see e.g. [3, 4, 11, 12].

The main goal of this note is to present a multivariate analogue of the well-known theorem by P. Schmid and F. Spitzer [8], where they describe \( \mu_A \) for an arbitrary banded Toeplitz matrix \( A \) in the case \( n = 0 \).

Namely, let \( A = (c_{i-j}) \), with \( i, j = 1, 2, \ldots \) be an infinite, banded Toeplitz matrix, where \( c_i = 0 \) if \( i < -k \) or \( i > h \). Fixing \( n \geq 0 \) as above, we obtain for each positive integer \( m \) the eigenvalue locus \( E_A^{(m)} \) of the principal \( m \times (m + n) \) submatrix \( A^{(m)} \) of \( A \).

Define the limit set \( B_A \) of eigenvalue loci as

\[ B_A = \left\{ x \in \mathbb{C}^{n+1} : x = \lim_{m \to \infty} x_m, x_m \in E_A^{(m)} \right\}, \ x = (x_0, \ldots, x_n). \]

In other words, \( B_A \) is the set of limit points of the sequence \( \{E_A^{(m)}\} \). Thus \( B_A \) is the support of the limiting measure \( \mu_A \) if it exists. (For a general infinite matrix \( A \) as above, its limit set \( B_A \) might be empty.)
Set
\[ Q(t, x) = t^k \left( \sum_{j=-k}^{h} c_j t^j - \sum_{j=0}^{n} x_j t^j \right), \]  
and let \( \alpha_1(x), \alpha_2(x), \ldots, \alpha_{k+h}(x) \) be the roots of \( Q(t, x) = 0 \), ordered according to their absolute values, i.e. \( |\alpha_i(x)| \leq |\alpha_{i+1}(x)| \) for all \( 0 < i < k + h \). Let \( C_A \) be the real semi-algebraic set given by the condition:
\[ C_A = \{ x \in \mathbb{C}^{n+1} : |\alpha_k(x)| = |\alpha_{k+1}(x)| = \cdots = |\alpha_{k+n+1}(x)| \}. \]

Our main conjecture is as follows.

**Conjecture 7.** For any banded Toeplitz matrix \( A \), if \( B_A \) is defined by (2) and \( C_A \) defined by (4) then \( B_A = C_A \).

By Conjecture 7 the set \( B_A \) is a real semi-algebraic \((n+1)\)-dimensional subset of \( \mathbb{C}^{n+1} \). In the classical case \( n = 0 \), Conjecture 7 is settled by P. Schmidt and F. Spitzer in [8]. Another important case when Conjecture 7 has been proved follows from some known results on multivariate Chebyshev polynomials, which is is presented in Example 8 below. Namely, when \( k = 1 \) and \( h = n + 1 \) with \( c_{-1} \) and \( c_{n+1} \) non-zero, we may do an affine change of the variables and a scaling of \( A \). This reduces to the latter case to \( c_{-1} = c_{n+1} = 1 \) and all other \( c_i = 0 \).

For these particular values, the family \( \{ P_A^2(x) \} \) becomes the multivariate Chebyshev polynomials of the second kind, see e.g. [5, 7, 2, 13]. These polynomials also have a close connection to another well-known family of polynomials, namely the Schur polynomials.

**Example 8.** For the above matrices corresponding to the multivariate Chebyshev polynomials the eigenlocus \( E_A^{(m)} \) can be described explicitly, see for example [6].

More precisely, the points in \( E_A^{(m)} \) lie on a real, \( n \)-dimensional surface \( C_A \subset \mathbb{C}^{n+1} \) which is naturally parametrized by an \((n+1)\)-dimensional torus \( T^{n+1} \). This parametrization is given by
\[ C_A = \{ x \in \mathbb{C}^{n+1} | x_j = -e_{j+1} (\exp(i\theta_1), \ldots, \exp(i\theta_{n+1}), \exp(i\theta_{n+2})) \} \]
where \((\theta_1, \ldots, \theta_{n+1})\) lie on \( T^{n+1} \), \( \sum_{j=0}^{n+2} \theta_j = 0 \), and \( e_j \) is the \( j \)-th elementary symmetric function in \( n + 2 \) variables.

Notice that for \( x \in C_A \),
\[ Q(t, x) = 1 + x_0 t + x_1 t^2 + \ldots + x_{n} t^{n+1} + t^{n+2} = \prod_j (t + e^{i\theta_j}) \]
by the Vieta formulas. Thus, for \( x \in C_A \), all roots of \( Q \), (as a polynomial in \( t \)) have absolute value equal to 1 when the \( x_j \) are parametrized as in (5).

Furthermore, the points in \( E_A^{(m)} \) are also expressed by (5), with the parameters \((\theta_1, \ldots, \theta_{n+2})\) being certain rational multiples of \( \pi \), distributed in a regular lattice. The mapping from the 2-torus to the eigenlocus is illustrated in Fig. 1.

Another interesting aspect of Example 8 is that all the points \( x = (x_0, \ldots, x_n) \) in the sets \( E_A^{(m)} \) satisfy the conditions \( x_j = \frac{\pi}{n-j}, j = 0, 1, \ldots, n \), which explains why we can draw \( C_A \subset \mathbb{C}^2 \) in Fig. 1a as a 2-dimensional set. For larger \( n \), \( C_A \) is a \((n+1)\)-dimensional analogue of the two-dimensional deltoid, shown in Fig. 1a.
For general \( \mathcal{A} \), we do not have the inclusion \( \mathcal{E}^{(m)}_{\mathcal{A}} \subseteq C_{\mathcal{A}} \) for arbitrary finite \( m \). However, if \( \mathcal{A} \) has an additional extra symmetry, this seems to be the case.

**Definition 9.** A banded Toeplitz matrix such that its \( Q(t, \mathbf{x}) \) in (3) satisfies
\[
Q(t, x_0, x_1, \ldots, x_n) = t^{h+k-1}Q(1/t, x_{n-1}, \ldots, x_0)
\]
is called multihermitian of order \( n \).

**Conjecture 10.** If \( \mathcal{A} \) is multihermitian of order \( n \), then each point \( \mathbf{x} = (x_0, x_1, \ldots, x_n) \in \mathcal{E}^{(m)}_{\mathcal{A}} \) satisfies \( x_j = \overline{x}_{n-j} \) for \( j = 0, 1, \ldots, n \).

Conjecture 10 obviously holds for the case \( n = 0 \), as it reduces to the fact that hermitian matrices have real eigenvalues. It is also straightforward to check that Conjecture 10 is true for the Chebyshev case of Example 8 above.

We have extensive numerical evidence for this conjecture. Another strong indication supporting Conjecture 10 is that if \( \mathcal{A} \) is multi-hermitian, then every point \( \mathbf{x} \in C_{\mathcal{A}} \) (which by Conjecture 7 is in the limit eigenlocus) satisfies the required symmetry \( x_j = \overline{x}_{n-j} \) for \( j = 0, 1, \ldots, n \).

**Example 11.** The bivariate case when \( Q(t, \mathbf{x}) = 1 + t^d x_0 + t^{d+1} x_1 + t^{2d+1}, d \geq 1 \) gives sets in \( C^2 \) where \( x_0 = \overline{x_1} \), by Conjecture 10. They correspond to Toeplitz

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**Figure 1.** The eigenvalue locus \( \mathcal{E}^{(20)}_2 \) and its pull-back to \( T^2 \). The torus \( T^2 \) is covered with a hexagon, where each triangle is mapped to the eigenlocus. The 6-fold symmetry is due to the \( S_3 \)-action by permutation of the arguments \( \theta_1, \theta_2, \theta_3 \) in (5). (Notice \( \theta_1 + \theta_2 + \theta_3 = 0 \) and this is the subspace which is illustrated in the figure to the right.)

The next group of examples are bivariate analogues of special univariate cases originally studied in [8], and later in [4], where they are referred to as “star-shaped curves”:

**Example 11.** The bivariate case when \( Q(t, \mathbf{x}) = 1 + t^d x_0 + t^{d+1} x_1 + t^{2d+1}, d \geq 1 \) gives sets in \( C^2 \) where \( x_0 = \overline{x_1} \), by Conjecture 10. They correspond to Toeplitz
matrices of the form
\[
\begin{pmatrix}
  x_0 & x_1 & 1 & 0 & 0 & \cdots \\
  1 & x_0 & x_1 & 1 & 0 & \cdots \\
  0 & 1 & x_0 & x_1 & 1 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\begin{pmatrix}
  x_0 & x_1 & 0 & 1 & 0 & 0 & \cdots \\
  0 & x_0 & x_1 & 0 & 1 & 0 & \cdots \\
  1 & 0 & x_0 & x_1 & 0 & 1 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}, \ldots
\]

The above two matrices represent \( d = 1 \) and \( d = 2 \).

Figures 2 and 3 present the distributions of \( x_0 \in \mathbb{C} \), for \( d = 2, 3, 4 \). (Recall that \( x_1 = \bar{x}_0 \).) The points shown on these figures belong to \( E^{(m)}_A \) for \( m = 13, 14, 15 \), and the curves are certain hypocycloids, parametrizing the boundary of \( C_A \). More explicitly, for a given integer \( d \geq 1 \), the hypocycloid boundary for \( x_0 \in \mathbb{C} \) is given by
\[
x_0 = (-1)^d e^{-i(d+2)\theta} \left( (d+2) e^{i(2d+3)\theta} + d + 1 \right) \text{ where } \theta \in [0, 2\pi],
\]
which is one of the implications of Conjecture 7.

Finally, the main result of this note is as follows;

**Theorem 12.** For any banded Toeplitz matrix \( A \), where \( B_A \) is defined by (2) and \( C_A \) is defined by (4), one has \( B_A \subseteq C_A \).

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2. Proofs

**Proof of Proposition 5.** We shall prove items (i) and (ii) simultaneously by calculating the leading homogeneous part of \( P^I_A(x_0, \ldots, x_n) \). Let us order the set of all admissible indices \( I = (1 \leq i_1 < \ldots < i_m \leq m + n) \) lexicographically. We can
also order lexicographically all monomials of degree $m$ in $x_0, \ldots, x_n$. By equation (1) $P_I^A(x_0, \ldots, x_n) = \det A_I$ where the columns of $A_I$ are indexed by $I$. Let $\tilde{P}_I^A(x_0, \ldots, x_n)$ be the homogeneous part of $P_I^A(x_0, \ldots, x_n)$ of degree $m$. One can easily see that the product of all entries on the main diagonal of $A_I$ contains the monomial $m_I$ of degree $m$ given by $m_I = \prod_{j=1}^m x_i^{j-1}$. Moreover it is straightforward that $\tilde{P}_I^A(x_0, \ldots, x_n) = m_I + \ldots$ where $\ldots$ stands for the linear combination of monomials $m_{I'}$ of degree $m$ coming other $I'$ which are lexicographically smaller than $I$. In other words, the matrix formed by $\tilde{P}_I^A(x_0, \ldots, x_n)$ versus monomials is triangular in the lexicographic ordering with unitary main diagonal which proves items (i) and (ii).

Item (iii) is just a reformulation of Proposition 2 above. □

Throughout the rest of the paper, we use the convention $\alpha = (\alpha_1, \ldots, \alpha_{h+k})$. We will also assume that $c_h = 1$, which corresponds to a rescaling of the original matrix $A$. This is equivalent to the assumption that $Q(t, x)$ is monic. By shifting the variables in $x$, we may also assume, without loss of generality, that $c_0 = c_1 = \ldots = c_n = 0$ in $A$.

In the above notation, it is convenient to work with the roots of $Q(t, x)$. This motivates the following definitions. Let $\Gamma_j \subset \mathbb{C}^{h+k}$, $j = k, \ldots, k+n$ denote the real semi-algebraic hypersurface consisting of all $\alpha = (\alpha_1, \ldots, \alpha_{h+k})$ such that when the $\alpha_j$ are ordered with increasing modulus, $|\alpha_j| = |\alpha_{j+1}|$. Similarly, let $G_j$ be defined as the real semi-algebraic set

$$\{ x \in \mathbb{C}^{n+1} : Q(t, x) = (t - \alpha_1) \ldots (t - \alpha_{h+k}) \text{ where } \alpha \in \Gamma_j \}.$$  

Then, by definition, $C_A = \bigcap_{j=k}^{k+n} G_j$.

**Proposition 13.** For any banded Toeplitz matrix $A$ and any non-negative $n < h$, the set $C_A$ defined by (3)-(4) is compact.

**Proof.** As discussed above, we may without loss of generality assume that $c_h = 1$ and $c_0 = c_1 = \ldots = c_n = 0$. Since $Q$ may be assumed to be monic, we have $c_j = e_{h-j}(\alpha)$ for $-k \leq j < 0$ and $n < j \leq h$, and $x_j = -e_{h-j}(\alpha)$ when
0 \leq j \leq n. Thus, it suffices to show that the set of \( \alpha \in \mathbb{C}^{h+k} \) that satisfies the conditions (3)-(4), is compact. It is also evident that the set \( C_A \) is closed, so we only need to show that it is bounded. We show this fact by contradiction.

Assume we have a sequence of roots \( \{\alpha^m\}_{m=1}^{\infty} \) of (3) such that \( \|\alpha^m\| \rightarrow \infty \) where (4) is satisfied for each \( \alpha^m \). We assume that the modulus of the roots are always ordered increasingly. There are two cases to consider.

**Case 1:** Assume that for some \( 0 \leq b < k \), a sequence of individual roots satisfies the condition \( |\alpha^m_b| \rightarrow \infty \) but \( |\alpha^m_j| \) are bounded for all \( m \) and \( j \leq b \). Then consider \( e_{h+k-b}(\alpha) \). Since \( b < k \), in our notation \( e_{h+k-b}(\alpha) \) equals the coefficient \( c_{b-k} \).

Notice that \( e_{h+k-b} \) contains the term \( \alpha_{b+1}\alpha_{b+2}\cdots\alpha_{h+k} \) which grows quicker than all other terms in the expansion of \( e_{h+k-b}(\alpha) \). This contradicts the assumption \( e_{h+k-b}(\alpha) = c_{b-k} \).

**Case 2:** Assume that for some \( b \) with \( k+n \leq b < h+k \), we have a sequence of individual roots \( |\alpha^m_{b+1}| \rightarrow \infty \) but \( |\alpha^m_j| \) are bounded for all \( m \) and \( j \leq b \). Consider

\[
e_b(\alpha) = e_b(\alpha_1, \ldots, \alpha_{h+k}) = \sum_{\sigma \in \{1^{b+h-k}\}} \frac{e_0}{\alpha_{\sigma_1}\alpha_{\sigma_2}\cdots\alpha_{\sigma_b}}.
\]

This contains an expression with the denominator \( \alpha_1\alpha_2\cdots\alpha_b \), i.e. the product of all bounded roots. Now, since \( h+k-b \) roots among all \( h+k \) roots grow in absolute value, and the product \( \alpha_1\cdots\alpha_{h+k} \) equals \( c_h \), it follows that \( |\alpha_1\alpha_2\cdots\alpha_b| \rightarrow 0 \) as \( m \rightarrow \infty \), and this term converges to 0 quicker than any other product \( \alpha_{\sigma_1}\alpha_{\sigma_2}\cdots\alpha_{\sigma_b} \). Thus, \( e_b \) should grow. This contradicts the assumption \( e_b(\alpha) = c_{h-b} \).

Notice that under our assumptions, the above cases cover all possible ways for a sequence of roots to diverge. Since both cases yield a contradiction, it follows that any sequence of roots of (3) satisfying (4) must be bounded. Thus, \( C_A \) is compact.

The following result is multivariate analog of a known fact in the case \( n = 0 \), see [3, Prop. 11.18 and 11.19].

**Proposition 14.** In the notation of (3)-(4), for any \( x \) belonging to the boundary \( \partial C_A \) of \( C_A \), at least one of the following three conditions is satisfied:

1. The discriminant of \( Q(t, x) \) with respect to \( t \) vanishes, i.e. \( Q(t, x) \) has (at least) a double root in \( t \).

2. \( |\alpha_{k-1}(x)| = |\alpha_k(x)| = |\alpha_{k+1}(x)| = \cdots = |\alpha_{k+n+1}(x)| \).

3. \( |\alpha_k(x)| = |\alpha_{k+1}(x)| = \cdots = |\alpha_{k+n+1}(x)| = |\alpha_{k+n+2}(x)| \).

**Proof.** We need the following two simple statements.

**Lemma 15.** Let \( \text{Pol}_d \) be the set of all monic polynomials of degree \( d \) with complex coefficients. Let \( \Sigma_{p,q} \subset \text{Pol}_d \) be the subset of polynomials satisfying

\[
|\alpha_p| = |\alpha_{p+1}| = \cdots = |\alpha_q|,
\]

where \( 1 \leq p < q \leq d \) and \( \alpha_1, \alpha_2, \ldots, \alpha_d \) being the roots of polynomials ordered according to their increasing absolute values. Then \( \Sigma_{p,q} \) is a real semi-algebraic set of codimension \( q - p \) whose boundary is the union of three pieces: \( \Sigma_{p-1,q} \), \( \Sigma_{p,q+1} \) and the intersection of \( \Sigma_{p,q} \) with the standard discriminant in \( \text{Pol}_d \), i.e. the set of polynomials having multiple roots. (Notice that if \( p = 1 \) then \( \Sigma_{p-1,q} \) is empty, and if \( q = d \) then \( \Sigma_{p,q+1} \) is empty by definition.)
\textbf{Proof.} $\Sigma_{p,q}$ is obtained as the image under the Vieta map of an obvious semi-algebraic set $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_p| = |\alpha_{p+1}| = \cdots = |\alpha_q| \leq |\alpha_{q+1}| \leq \cdots \leq |\alpha_d|$. Notice that the Vieta map is a local diffeomorphism outside the preimage of the standard discriminant. Therefore the boundary of $\Sigma_{p,q}$ must either belong to the standard discriminant or to one of $\Sigma_{p-1,q}$ or $\Sigma_{p,q+1}$. The former is the common boundary between $\Sigma_{p,q}$ and $\Sigma_{p-1,q-1}$ and the latter is the common boundary between $\Sigma_{p,q}$ and $\Sigma_{p+1,q+1}$.

Given a closed Whitney stratified set $X$ (for example, semi-analytic) we say that $X$ is a $k$-dimensional stratified set without boundary if

(i) the top-dimensional strata of $X$ have dimension $k$;

(ii) for any point $x$ lying in any stratum of dimension $k-1$, one can choose orientation of the (germs of) $k$-dimensional strata of a sufficiently small neighborhood of $x$ in $X$ so that $\partial X = 0$.

\textbf{Lemma 16.} The boundary of the intersection of any closed semi-algebraic set $\Gamma$ with any closed algebraic set $\Theta$ is included in the intersection of the boundary $\partial \Gamma$ with $\Theta$.

\textbf{Proof.} Observe that any real algebraic variety $X$ of dimension $k$ is a stratifiable set without boundary. Indeed, the fact we are proving is local, and it suffices to prove it for generic $x$ on $(k-1)$-dimensional strata.

Consider an embedding of $X$ in some high-dimensional linear space, take the Whitney stratification with $x$ on its stratum $Y \subset B$ of dimension $k-1$, and a transversal to $Y$ of codimension $k-1$ at $x$.

Therefore, we may now assume that the germ of $X$ near $x$ is topologically a product of a germ of algebraic curve and a germ of a smooth manifold of dimension $k-1$. Furthermore, a germ of any real algebraic curve $\Gamma$ can be always oriented so that $\partial \Gamma = 0$ which follows from the existence of Puiseux series for an arbitrary branch of algebraic curve. This argument shows that any point in the intersection $\Gamma \cap \Theta$ which does not belong to the boundary of $\Gamma$ can not lie on the boundary of this intersection which settles Lemma 16.

\textbf{Proof of Theorem 12.} In our notation, let $D_j^m(x)$ be the determinant of the $m \times m$-matrix $A_I$ with $I = \{j+1, j+2, \ldots, j+m\}$ for $0 \leq j \leq n$. It is evident that $E^{(m)}_A$ is a subset of the set $\tilde{E}^{(m)}_A$ of solutions to the system of polynomial equations

\begin{equation}
D_0^m(x) = D_1^m(x) = \cdots = D_n^m(x) = 0.
\end{equation}

We will show a stronger statement that, in notation of Theorem 12,

$$\lim_{m \to \infty} \tilde{E}^{(m)}_A \subseteq C_A.$$  

Although each individual $\tilde{E}^{(m)}_A$ (considered as a points set with multiplicities) is strictly bigger than $E^{(m)}_A$ the limits $B_A = \lim_{m \to \infty} E^{(m)}_A$ and $\lim_{m \to \infty} \tilde{E}^{(m)}_A$ seem to coincide as infinite sets.

The next proposition accomplishes the proof of Theorem 12.
In Theorem 4 of [1] it was shown that each sequence of determinants \( \{D^m_j(x)\}_{m=1}^{\infty} \) as above satisfies a linear recurrence relation with coefficients depending on \( x \). The characteristic polynomial \( \chi_j(t) \) of the \( j \)-th recurrence can be factorized as

\[
\chi_j(t, x) = \prod_{\sigma}(t - r_{j\sigma}), \quad \text{where } r_{j\sigma} = (-1)^{k+j}(\alpha_{\sigma_1} \cdots \alpha_{\sigma_{k+j}})^{-1}, \sigma \in \binom{[k+h]}{k+j}.
\]

Proposition 17. Suppose that \( \{x_m\}_{m=1}^{\infty} \), is a sequence of points in \( \mathbb{C}^{n+1} \) satisfying the system of equations:

\[
D_j^m(x_m) = 0 \quad \text{for } j = 0, 1, \ldots, n \text{ and } m = 1, 2, \ldots
\]

and such that the limit \( \lim_{m \to \infty} x_m =: x^* \) exists. Then for all \( j = 0, \ldots, n \)

\[
|\alpha_{k+j}(x^*)| = |\alpha_{k+j+1}(x^*)| \quad \text{when the } \alpha_i \text{ are indexed with increasing order of their modulus.}
\]

Proof. Provided that all the roots of \( \chi_j(t, x) \) are distinct, by using a version of Widom’s formula, (see [1, 3]) we have

\[
D_j^m(x) = \sum_{\sigma} \prod_{l \in \sigma,i \notin \sigma} \left(1 - \frac{\alpha_l(x)}{\alpha_i(x)}\right)^{-1} \cdot r_{j\sigma}(x)^m.
\]

We may assume that for \( x^* \) and fixed \( j \), the \( r_{j\sigma}(x^*) \) are ordered decreasingly with respect to their modulus (for some ordering \( \sigma = 1, 2, \ldots \)). The goal is to prove that \( |r_{j1}(x^*)| > |r_{j2}(x^*)| \geq \ldots \geq |r_{jb}(x^*)| \), i.e. that the largest root is simple and has modulus strictly larger than any other root of the characteristic equation (10). By examining (12), it is evident that \( r_{j1}(x_m)^m \) is the dominating term for sufficiently large \( m \), that is, \( D_j^m(x_m)/r_{j1}(x_m)^m \to L \neq 0 \) as \( m \to \infty \).

By standard properties of linear recurrences, this holds even when there are multiple zeros among the smaller roots; remember that our assumption was that \( r_{j1}(x_m) \) is a simple zero of (10) when \( m \) is large enough.

Hence, for sufficiently large \( m \), \( D_j^m(x_m) \approx Lr_{j1}(x_m)^m \), which is non-zero for sufficiently large \( m \). This contradicts the condition that \( x_m \) satisfies (11). Consequently, \( |r_{j1}(x^*)| = |r_{j2}(x^*)| \) for \( j = 0, 1, \ldots, n \) and this implies Proposition 17. \( \square \)

Proposition 17 implies that \( x \) lies in \( B_\mathcal{A} \) only if \( x \) is a limit of solutions to (11), but such limit \( x \) must satisfy that \( |\alpha_k(x)| = |\alpha_{k+1}(x)| = \ldots = |\alpha_{k+n+1}(x)| \). Therefore, \( B_\mathcal{A} \subseteq C_\mathcal{A} \).

3. Further directions

1. It seems relatively easy to describe the stratified structure of \( C_\mathcal{A} \) at least in case of generic \( \mathcal{A} \). In particular, in the Chebyshev case of Example 8 the set \( C_\mathcal{A} \) has the same stratification as a simplex of corresponding dimension. One can also understand the stratified structure of the sets \( \Sigma_{p,q} \) introduced in Lemma 15. Since each \( C_\mathcal{A} \) is obtained from a corresponding \( \Sigma_{p,q} \) by intersecting it with an affine subspace the stratified structure of the former for generic \( \mathcal{A} \) is also describable. On the other hand, our Example 11 seems to show more complicated stratified structure due to the presence of additional symmetry.
2. We say that an (infinite) complex-valued matrix $A$ has a \textit{weak univariate orthogonality property} if the sequence of characteristic polynomials of its principal minors obeys the standard 3-term recurrence relation with complex coefficients. There is a straightforward version of this notion for finite square matrices. Obviously, any Jacobi matrix has this property. However, it seems that for any $m \geq 3$ the set $WO_m \subset \text{Mat}(m,m)$ of all $m \times m$-matrices with the latter property has a bigger dimension than the set $Jac_m \subset \text{Mat}(m,m)$ of all Jacobi $m \times m$-matrices.

**Problem 18.** Find the dimension of $WO_m$?

3. Analogously, given a non-negative integer $n$, we say that an (infinite) complex-valued matrix $A$ has a \textit{weak $n$-variate orthogonality property} if the above family $\{P_A(x_0, x_1, \ldots, x_n)\}$ (see Definition 4) satisfies the 3-term recurrence relation (2.2) of Theorem 2.1 of [13] with complex coefficients.

There are many similarities between families $\{P_A(x_0, x_1, \ldots, x_n)\}$ and families of multivariate orthogonal polynomials which by one of the standard definitions of such polynomials also satisfy (2.2) of Theorem 2.1 of [13] with real coefficients.

Our computer experiments show that in this aspect the case $n > 0$ is quite different from the classical case $n = 0$. In particular, we believe that the following conjecture holds.

**Conjecture 19.** Given $n > 0$, a banded matrix $A$ has a weak $n$-variate orthogonality property if it is of the form

$$A = \begin{pmatrix}
  a_0 & a_1 & a_2 & \ldots & a_{n+1} & 0 & 0 & 0 & \ldots \\
  d_{-1} & d_0 & d_1 & \ldots & d_n & d_{n+1} & 0 & 0 & \ldots \\
  0 & d_{-1} & d_0 & \ldots & d_{n-1} & d_n & d_{n+1} & 0 & \ldots \\
  0 & 0 & d_{-1} & \ldots & d_{n-2} & d_{n-1} & d_n & d_{n+1} & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix},$$

where $a_0, \ldots, a_{n+1}, d_{-1}, \ldots, d_{n+1} \in \mathbb{C}$.

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