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Chapter 7

TRACMASS—A Lagrangian Trajectory Model

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Abstract A detailed description of the Lagrangian trajectory model TRACMASS is presented. The theory behind the original scheme for steady state velocities is derived for rectangular and curvilinear grids with different vertical coordinates for the oceanic and atmospheric circulation models. Two different ways to integrate the trajectories in time in TRACMASS are presented. These different time schemes are compared by simulating inertial oscillations, which show that both schemes are sufficiently accurate not to deviate from the analytical solution.

The TRACMASS are exact solutions to differential equations and can hence be integrated both forward and backward with unique solutions. Two low-order trajectory subgrid parameterizations, which are available in TRACMASS, are explained. They both enable an increase of the Lagrangian dispersion, but are, however, too simple to simulate some of the Lagrangian properties that are desirable. The mass conservation properties of TRACMASS are shown to make it possible to follow the water or air masses both forward and backward in time, which also opens up for all sorts of calculations of water/air mass exchanges as well as Lagrangian stream functions.

7.1 Introduction

The specification of a flow field can be made in an Eulerian or a Lagrangian frame of reference. The Eulerian method is when the fluid flow is observed from a point fixed in space, while the Lagrangian method is instead working from the perspective of the

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flow. This can be illustrated by a cyclist, who passes an immobile traffic jam. In this case the static car driver sees the moving cyclist from an Eulerian perspective, while the moving cyclist observes the static traffic jam from a Lagrangian perspective. The zigzagging path of the cyclist between the cars constitutes a Lagrangian trajectory.

Most analytical and numerical models in fluid dynamics are made in the Eulerian framework, since it is then straightforward to describe the motion as a function of position and time. This is why in nearly all ocean general circulation models the equations of motion are discretized with finite differences on a fixed grid so that the motion of the water and its tracers such as salinity and temperature are described from the Eulerian perspective with different values in each grid box, even if the vertical discretization often has a time dependent component related to the motion of the fluid. Lagrangian trajectories are, however, still possible to calculate from the model simulated Eulerian velocity fields on the model grid.

The present chapter will present the TRACMASS Lagrangian trajectory model, which uses the Eulerian velocity fields, which have been simulated by ocean or atmosphere general circulation models (GCM). The trajectories are calculated off-line, i.e., after the GCM has been integrated and the velocity fields have been stored. This makes it possible to calculate many more trajectories than would be possible on-line, i.e., simultaneously with the GCM run. TRACMASS has been applied to many different general circulation models, both for the ocean and for the atmosphere.

The original feature of the method is that it solves the trajectory path through each grid cell with an analytical solution of a differential equation which depends on the velocities on the walls of the grid box. The scheme was originally developed in Döös (1995), Blanke and Raynaud (1997) for stationary velocity fields and hereafter further developed in de Vries and Döös (2001) for time-dependent fields by solving a linear interpolation of the velocity field both in time and in space over each grid box. This is in contrast to the time schemes such as simple Euler forward or more advanced fourth order Runge–Kutta methods (Butcher 2008; Fabbri 2009) where the trajectories are integrated forward in time with as short time steps as possible.

A consequence of solving the trajectory paths analytically over a certain time is that the solutions are unique and can be integrated forward in time and then backward in time and arriving exactly at the same position, which is not possible with the other trajectory methods. This makes it possible to trace origins of water or air masses as long as the subgrid parameterization is not activated.

The TRACMASS code has been further developed over the years and used in many studies of the global ocean (Döös and Coward 1997; Drijfhout et al. 2003; Döös et al. 2008) and regional ones for the Mediterranean and Baltic Seas (Döös et al. 2004; Jönsson et al. 2004; Engqvist et al. 2006; Soomere et al. 2011) as well as the large scale atmospheric circulation (Kjellsson and Döös 2012).

The code was originally written in Fortran 77 for the FRAM ocean model at the Institute of Oceanographic Sciences, Deacon Laboratory (IOSDL) in Wormley, UK in the early 1990s. The name TRACMASS comes from the EU project with the same name, where it served together with the similar trajectory code Ariane by

Blanke and Raynaud (1997). The present code is written in Fortran 95 and can be driven by velocity fields from most GCMs based on finite differences. The TRACMASS code is continuously upgraded and adapted. The code can be downloaded from <http://tracmass.org/>. The user must be familiar, in order to be able to use the TRACMASS code, with (1) the equations of motion for the ocean-atmosphere circulation (described in Chaps. 2, 4 and 6), (2) the finite differences of these equations (Chap. 3), (3) the TRACMASS theory (the introduction to which is presented in this chapter), (4) Unix and (5) Fortran.

The Lagrangian trajectory approach has many similarities with the Eulerian tracer approach but at the same time many differences. The two approaches are often confused due to their similarities. They are both advected passively by the velocity fields of the GCM, which makes it possible to trace water/air masses or substances such as pollutants as they are carried with the ocean currents or winds. The tracer equation generally needs to be integrated ‘on-line’ with the GCM while the Lagrangian trajectories can be both ‘on-line’ and ‘off-line’. The ‘off-line’ calculation of Lagrangian trajectories is by far the most rapid way since one only needs to read the already simulated velocity fields in order to calculate the trajectories.

The tracer equation includes explicitly a diffusion term, which represents a parameterization of the unresolved subgrid scales. There is also a numerical reason to include this since GCMs generally need some diffusion and viscosity to remain numerically stable in order to dissipate energy or to eliminate numerical noise due to the truncation errors in the numerical schemes. The passive tracers also have a numerical diffusion due to the finite difference approximation error, which by itself often would be enough as diffusion. The tracer approach is therefore often too diffusive but has been improved with better numerical advection schemes during the last decade. The Lagrangian trajectories are passively advected with the currents or winds and the subgrid parameterization is included in the sense that the GCM has been integrated with viscosity and diffusion. An extra diffusion can, however, if desired, be added to the trajectories. Another advantage of the trajectories is that it is possible to follow particles from their release points to the end both forward and backwards, which is impossible with passive tracers that cannot be integrated backward in time due to the numerical and parameterized diffusion.

The present chapter will describe the basic theory for the TRACMASS trajectory calculations and is organized as follows. In Sect. 7.2 we present the basic equations for a rectangular grid, which is then extended in Sect. 7.3 to the more general case with non-rectangular grids and for atmospheric GCMs in Sect. 7.4. The TRACMASS analytical time dependent scheme based on de Vries and Döös (2001) is presented in Sect. 7.5 followed by the presentation of two simple sub-grid parameterizations in Sect. 7.6 and how the mass conservation in TRACMASS enables analysis of the water/air mass transports in Sect. 7.7. In Sect. 7.8, we summarize and discuss the TRACMASS approach and its possible improvements in the future.

Fig. 7.1 Left: B-grid, Right: C-grid

Fig. 7.2 Illustration of a trajectory $[x(t), y(t)]$ through one grid box. The model velocities are defined at the walls of the grid box

7.2 Trajectory Solution for Rectangular Grids

This section is here only for pedagogical reasons, since it is only valid for rectangular Cartesian grids. The TRACMASS code is written in a more general way in order to enable TRACMASS to work with curvilinear grids, which are used by most GCMs, and will be presented in the next section.

Most finite difference GCMs use B- or C-grids (Mesinger and Arakawa 1976) as shown in Fig. 7.1, where i, j, k denote the discretized longitude, latitude and model level, respectively. The zonal velocity $u_{i,j,k}$ and meridional velocity $v_{i,j,k}$ are located differently in these two grids, while the vertical velocity $w_{i,j,k}$ is located in the middle at the top of the box in both grids (Figs. 7.2, 7.3a). Both these types of grids can be used in TRACMASS. The velocities in TRACMASS are set on a C-grid, which makes it straightforward when using a C-grid model. Although B-grid velocities just need to be projected on the C-grid by making a meridional average $u_{i,j,k}^C = 0.5(u_{i,j,k}^B + u_{i,j-1,k}^B)$ of two zonal velocities and a zonal average $v_{i,j,k}^C = 0.5(v_{i,j,k}^B + v_{i-1,j,k}^B)$ of two meridional velocities for each grid box.

In a finite difference model there is no information of scales below the grid size. The tracers are regarded as homogeneous within each grid box and the velocities

Fig. 7.3 Vertical trajectory discretization in model grids

are only defined on the grid side walls. It is, however, possible to define the velocity inside a grid box by interpolating linearly between the discretized velocity values of the opposite walls. For the zonal x -direction one obtains

$$u(x) = u_{i-1,j,k} + \frac{x - x_{i-1}}{\Delta x} (u_{i,j,k} - u_{i-1,j,k}). \quad (7.1)$$

Local zonal velocity and position are related by $u = dx/dt$. The approximation in Eq. (7.1) can now be written in terms of the following differential equation:

$$\frac{dx}{dt} + \beta x + \delta = 0, \quad (7.2)$$

with $\beta \equiv (u_{i-1,j,k} - u_{i,j,k})/\Delta x$ and $\delta \equiv -u_{i-1,j,k} - \beta x_{i-1}$. Using the initial condition $x(t_0) = x_0$, the zonal displacement of the trajectory inside the considered grid box can be solved analytically and is given by

$$x(t) = \left(x_0 + \frac{\delta}{\beta} \right) e^{-\beta(t-t_0)} - \frac{\delta}{\beta}. \quad (7.3)$$

The time t_1 when the trajectory reaches a zonal wall can be determined explicitly:

$$t_1 = t_0 - \frac{1}{\beta} \log \frac{x_1 + \delta/\beta}{x_0 + \delta/\beta}, \quad (7.4)$$

where $x_1 = x(t_1)$ is given by either x_{i-1} or x_i . For a trajectory reaching the wall $x = x_i$, for instance, the velocity u_i must necessarily be positive, so in order for Eq. (7.4) to have a solution, the velocity u_{i-1} must then be positive also. If this is not the case, then the trajectory either reaches the other wall at x_{i-1} or the signs of the transports are such that there is a zero zonal transport somewhere inside the grid box that is reached exponentially slow. For the meridional and vertical directions, similar calculations of t_1 are performed determining the meridional and vertical displacements of the trajectory, respectively, inside the considered grid box. The smallest transit time $t_1 - t_0$ and the corresponding x_1 denote at which wall of the grid box

Fig. 7.4 The Ocean Conveyor Belt with velocities simulated by the OCCAM model. The *red trajectories* are part of the shallow warmer part of the Conveyor Belt with transports toward the North Atlantic. The *blue trajectories* represent the flow of the dense and cold North Atlantic Deep Water from the North Atlantic into the Indo-Pacific

the trajectory will exit and move into the adjacent one. The exact displacements in the other two directions are then computed using the smallest t_1 in the corresponding Eq. (7.3). The resulting trajectory through the grid box is illustrated by Figs. 7.2 and 7.3a.

Note that a consequence of solving the trajectories analytically with Eq. (7.3) is that the solution is unique. The trajectory can hence be integrated forward in time and then backward in time and arriving back exactly in the same point where it started.

7.3 Scheme for Volume or Mass Transports and Non-rectangular Grids

The disadvantage with the scheme presented in the previous section is that it requires rectangular grid cells and GCMs generally use some sort of spherical or curvilinear grids as in the case of the Ocean Circulation and Climate Advanced Model (OCCAM) model presented in Fig. 7.4, where two spherical grids have been used for the world ocean. The longitudinal ($\Delta x_{i,j}$) and the latitudinal ($\Delta y_{i,j}$) grid lengths will hence be a function of their horizontal positions i, j on a curvilinear grid. The depth level thickness Δz_k will similarly vary but with layer level k .

Trajectories can, however, be calculated for the curvilinear grids by replacing the velocities by volume transports. The transport $U_{i,j,k}$ through the eastern wall of the i, j, k grid box is given by

$$U_{i,j,k} = u_{i,j,k} \Delta y_{i,j} \Delta z_k. \quad (7.5)$$

The distance is non-dimensionalized by using $r = x/\Delta x$, and the linear interpolation of the velocity (Eq. (7.1)) is replaced by

$$U(r) = U_{i-1,j,k} + (r - r_{i-1})(U_{i,j,k} - U_{i-1,j,k}). \quad (7.6)$$

The local transport and position are now related by $U = dr/ds$, where the scaled time variable is $s \equiv t/(\Delta x_{i,j} \Delta y_{i,j} \Delta z_k)$, the denominator being the volume of the particular grid box. The differential equation (7.2) is replaced by

$$\frac{dr}{ds} + \beta r + \delta = 0, \quad (7.7)$$

with $\beta \equiv U_{i-1,j,k} - U_{i,j,k}$ and $\delta \equiv -U_{i-1,j,k} - \beta r_{i-1}$. Using the initial condition $r(s_0) = r_0$, the zonal displacement of the trajectory is now given by

$$r(s) = \left(r_0 + \frac{\delta}{\beta}\right)e^{-\beta(s-s_0)} - \frac{\delta}{\beta}. \quad (7.8)$$

The scaled time s_1 becomes

$$s_1 = s_0 - \frac{1}{\beta} \log \frac{r_1 + \delta/\beta}{r_0 + \delta/\beta}, \quad (7.9)$$

where $r_1 = r(s_1)$ is given by either r_{i-1} or r_i . With the use of Eq. (7.5), the logarithmic factor can be expressed as $\log[U(r_1)/U(r_0)]$.

For a trajectory reaching the wall $r = r_i$, for instance, the transport $U(r_1)$ must necessarily be positive, so in order for Eq. (7.9) to have a solution, the transport $U(r_0)$ must then be positive also. If this is not the case, then the trajectory either reaches the other wall at r_{i-1} or the signs of the transports are such that there is a zero zonal transport somewhere inside the grid box that is reached exponentially slow. The calculations of s_1 are performed determining the zonal, meridional and vertical displacements of the trajectory, respectively, inside the considered grid box. The smallest transit time $s_1 - s_0$ and the corresponding r_1 denote at which wall of the grid box the trajectory will exit and move into the adjacent one. The exact displacements in the other two directions are then computed using the smallest s_1 in the corresponding Eq. (7.8).

The scheme is mass conserving since it deals with the transport across the grid walls just as in the GCM and the transport is only linearly interpolated between two opposite walls in a grid box.

The trajectories will hence never cross a grid wall.

The solutions for the meridional and vertical directions are calculated similarly as the zonal one but using the meridional and vertical transport, respectively, defined as

$$V_{i,j,k} = v_{i,j,k} \Delta x \Delta z_k, \quad (7.10)$$

$$W_{i,j,k} = w_{i,j,k} \Delta x \Delta y. \quad (7.11)$$

The scheme is also mass conserving in the sense that the vertical transport is directly calculated from the continuity equation in the same way as in the ocean GCM, which is due to the incompressibility in the ocean

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (7.12)$$

that is discretized with finite differences on a C-grid into

$$\frac{u_{i,j,k} - u_{i-1,j,k}}{\Delta x_{i,j}} + \frac{v_{i,j,k} - v_{i,j-1,k}}{\Delta y_{i,j}} + \frac{w_{i,j,k} - w_{i,j,k-1}}{\Delta z_k} = 0. \quad (7.13)$$

Equation (7.13) simply reflects the condition that the sum of all the volume fluxes in or out of the grid box is zero. The vertical volume transport through the top of the grid box is obtained from Eqs. (7.11) and (7.13),

$$W_{i,j,k} = W_{i,j,k-1} - (U_{i,j,k} - U_{i-1,j,k} + V_{i,j,k} - V_{i,j-1,k}), \quad (7.14)$$

which can be computed by integration from the bottom and upwards with the bottom boundary condition $W_{i,j,0} = 0$. Since the trajectory solutions are exact and the continuity equation is respected the TRACMASS trajectories will therefore never hit any solid boundary such as the coast or the sea floor. This feature should be taken into account when the TRACMASS model is used for calculations of the transport of tracers or pollution to the coast. As described in Chap. 9, the virtual coastline should be set to a certain distance from the model coastline.

The depth level thickness Δz in the above derivations depends only on the depth level k . TRACMASS can, however, be integrated, with other GCM vertical coordinates that may depend on something more than just the depth level. Options of vertical coordinates for TRACMASS hence exist for (1) depth level models, (2) sigma-coordinate models, where the thickness depends on the total depth, which varies in each horizontal grid point, (3) z -star coordinates, where the layer thickness depends on sea surface elevation, (4) isopycnal models, where Δz is the density layer thickness, which was implemented in TRACMASS by Marsh and Megann (2002) and (5) hybrid vertical coordinates for atmospheric GCMs, which will be presented in the next section. See Chap. 3 for a discussion of some properties of such models.

7.4 Scheme for Atmospheric Hybrid Vertical Coordinates

The atmospheric version of TRACMASS uses conservation of mass instead of volume. Most atmospheric GCMs today use terrain-following vertical coordinates. Following Simmons and Burridge (1981) the atmosphere is divided into N_{LEV} layers, which are defined by the pressures at the interfaces between them and these pressures are given by $p_{k+1/2} = A_{k+1/2} + B_{k+1/2} p_S$ for $k = 0, 1, \dots, N_{LEV}$, with $k = 0$ at the top of the atmosphere and $k = N_{LEV}$ at the Earth's surface. The $A_{k+1/2}$ and $B_{k+1/2}$ are constants, whose values effectively define the vertical coordinate and p_S

is the surface pressure. The dependent variables, which are the zonal wind u , the meridional wind v , the temperature T and the specific humidity q are defined in the middle of the layers, where the pressure is defined by $p_k = \frac{1}{2}(p_{k-1/2} + p_{k+1/2})$, for $k = 1, 2, \dots, N_{LEV}$. The vertical coordinate is $\eta = \eta(p, p_S)$ and has the boundary value $\eta(0, p_S) = 0$ at the top of the atmosphere and $\eta(p_S, p_S) = 1$ at the Earth's surface.

For the ocean, in the previous sections, we used volume transport because of the incompressibility approximation. In the atmosphere we need instead to use mass transport so Eq. (7.5) is now replaced by the zonal and meridional mass transports in the model layers:

$$U_{i,j,k} = u_{i,j,k} \frac{\Delta y \Delta p_k}{g}, \quad V_{i,j,k} = v_{i,j,k} \frac{\Delta x_j \Delta p_k}{g}, \quad (7.15)$$

where $\Delta p_k = \Delta A_k + \Delta B_k p_{Si,j,k}$, $\Delta A_k = A_{k+1/2} - A_{k-1/2}$ and $\Delta B_k = B_{k+1/2} - B_{k-1/2}$. Note that with hybrid coordinates, the pressure at model layer interfaces $p_{i,j,k}$ varies in both space and time as surface pressure varies.

The mass transport between model layer interfaces, here denoted W as a vertical flux, can be calculated using the continuity equation from Simmons and Burridge (1981) as done in Kjellsson and Döös (2012):

$$\left(\dot{\eta} \frac{\partial p}{\partial \eta} \right)_k = - \frac{\partial p_k}{\partial t} - \sum_{m=1}^k \nabla \cdot (u_m, v_m) \Delta p_m. \quad (7.16)$$

This gives the vertical velocity due to the variations in the pressure at the interface and the divergence above. This quantity can be translated into mass flux by multiplying by $\Delta x_j \Delta y$:

$$\begin{aligned} W_{i,j,k} &= \Delta x \Delta y \left(\dot{\eta} \frac{\partial p}{\partial \eta} \right)_k \\ &= - \sum_{m=1}^k \left(U_{i,j,m} - U_{i-1,j,m} + V_{i,j,m} - V_{i,j-1,m} \right. \\ &\quad \left. + \Delta x \Delta y \Delta B_m \frac{p_s^n - p_s^{n-1}}{\Delta t} \right), \end{aligned} \quad (7.17)$$

where we use $\partial p_k / \partial t = \partial (B_k p_s) / \partial t$.

The mass conservation of a grid box is illustrated in Fig. 7.3b. The integration over the model levels is done from the top down, with the assumption $W_{i,j,0} = 0$. This may lead to $W_{i,j,N_{LEV}} \neq 0$, if the fields are not perfectly mass-conserving, which is the case for reanalysis data (see Berrisford et al. 2011 for a study on ERA-Interim) as used by Kjellsson and Döös (2012). In such a case, the vertical flux at the surface must be explicitly set to zero.

The trajectory differential equation (7.7) and its solutions (7.8)–(7.9) remain the same but now fed with mass transports on atmospheric terrain-following vertical

Fig. 7.5 Example of mid-tropospheric atmospheric trajectories. The wind velocities are from the ERA-Interim reanalysis from the ECMWF

coordinates and the scaled time is now $s \equiv gt/(\Delta x_{i,j} \Delta y_{i,j} \Delta p_k)$. The atmospheric TRACMASS code has been used to study the atmospheric Hadley and Ferrel cells as well as the inter-hemispheric air mass exchange (Kjellsson and Döös 2012). Figure 7.5 shows an example of atmospheric TRACMASS trajectories calculated with winds from the ERA-Interim reanalysis from the European Centre for Medium-Range Weather Forecasts (ECMWF).

7.5 Time Integration

The trajectory schemes in the previous sections with the differential Eqs. (7.2) and (7.7) are only valid for stationary velocity fields. We will now present two possible ways to incorporate the temporal variability of the velocity and surface elevation fields in the TRACMASS trajectory calculations. One (called time-stepping) method is based on previous sections and one is more advanced, where the differential equation is extended in time and solved analytically in both space and time.

Note that nearly all GCMs today have some sort of free surface, which will make the level thickness Δz also dependent of time and will hence affect the mass transport across the grid walls. It is therefore necessary to have both the velocity and the surface elevation fields in order to compute the trajectories with TRACMASS.

7.5.1 Time-Stepping Method

The time-stepping method consists of assuming that the velocity and surface elevation fields are in steady state during a limited time interval. The fields are then

Fig. 7.6 Schematic illustration of how the velocity fields $u(t)$ can be updated in time, with new GCM data at regular intervals Δt_G in green and linearly interpolated velocity points in red with the time step Δt_i . The number of intermediate time steps between two GCM velocities is in this example $I_S = \Delta t_G / \Delta t_i = 4$

updated successively as new fields are available. If this is made ‘on-line’, i.e., in the same time as the GCM is integrated, then this time interval will simply be the same as the time step the GCM is integrated with, which is typically of the order of minutes in a global GCM. If instead the trajectories are calculated ‘off-line’ it will be at least as often as the fields have been stored by the GCM.

A linear time interpolation of the velocity fields between two GCM velocity fields enables a simple way to have shorter time steps by which the fields are updated in time. The time interval between two GCM velocity fields is Δt_G and the shorter time interval at which the fields are interpolated is Δt_i as illustrated by Fig. 7.6. The number of intermediate time steps is hence the ratio $I_S = \Delta t_G / \Delta t_i$.

7.5.2 Analytical Time Integration

In the present section, we will present a time dependent scheme, which was introduced in TRACMASS by de Vries and Döös (2001) that is solved analytically in time over Δt_G between two GCM time steps.

Given a set of velocities $\mathbf{V}_n$ for each model point, where n represents a discretized time, a bi-linear interpolation of transport in space as well as in time leads to

$$\begin{aligned}
 F(r, s) = & F_{i-1, n-1} + (r - r_{i-1})(F_{i, n-1} - F_{i-1, n-1}) \\
 & + \frac{s - s_{n-1}}{\Delta s} [F_{i-1, n} - F_{i-1, n-1} \\
 & + (r - r_{i-1})(F_{i, n} - F_{i-1, n} - F_{i, n-1} + F_{i-1, n-1})], \quad (7.18)
 \end{aligned}$$

which is the general expression for the three directions where i signifies either a longitudinal, meridional, or vertical direction. The transport is $F = (U, V, W)$ and as before $r = (x/\Delta x, y/\Delta y, z/\Delta z)$, $s \equiv t/(\Delta x \Delta y \Delta z)$, where the denominator is the volume of the particular grid box and Δs is the scaled time step between two data sets:

$$\Delta s = s_n - s_{n-1} = (t_n - t_{n-1})/(\Delta x \Delta y \Delta z) = \Delta t_G/(\Delta x \Delta y \Delta z), \quad (7.19)$$

where Δt_G is the time step between two data sets in true time dimension (seconds).

Connecting the local transport to the time derivative of the position with $F = dr/ds$, we get the differential equation

$$\frac{dr}{ds} + \alpha rs + \beta r + \gamma s + \delta = 0, \quad (7.20)$$

where the coefficients are defined by

$$\alpha \equiv -\frac{1}{\Delta s}(F_{i,n} - F_{i-1,n} - F_{i,n-1} + F_{i-1,n-1}), \quad (7.21)$$

$$\beta \equiv F_{i-1,n-1} - F_{i,n-1} - \alpha s_{n-1}, \quad (7.22)$$

$$\gamma \equiv -\frac{1}{\Delta s}(F_{i-1,n} - F_{i-1,n-1}) - \alpha r_{i-1}, \quad (7.23)$$

$$\delta \equiv F_{i-1,n-1} + r_{i-1}(F_{i,n-1} - F_{i-1,n-1}) - \gamma s_{n-1}. \quad (7.24)$$

Differently shaped analytical solutions exist for the three cases: $\alpha > 0$, $\alpha < 0$ and $\alpha = 0$, which together cover all possible values of α . Note that inside the grid box, the acceleration $d^2r/ds^2 = -\alpha r - \gamma$ consists of a constant and a linear r -dependent term proportional to α . For $\alpha > 0$, the latter term implies a varying deceleration across the grid box.

If $\alpha > 0$, we define the time-like variable $\xi = (\beta + \alpha s)/\sqrt{2\alpha}$ and get

$$r(s) = \left(r_0 + \frac{\gamma}{\alpha}\right)e^{\xi_0^2 - \xi^2} - \frac{\gamma}{\alpha} + \frac{\beta\gamma - \alpha\delta}{\alpha}\sqrt{\frac{2}{\alpha}}[D(\xi) - e^{\xi_0^2 - \xi^2}D(\xi_0)] \quad (7.25)$$

using Dawson's integral

$$D(\xi) \equiv e^{-\xi^2} \int_0^\xi e^{x^2} dx \quad (7.26)$$

and the initial condition $r(s_0) = r_0$. If $\alpha < 0$, ξ becomes imaginary. By defining $\zeta \equiv i\xi = (\beta + \alpha s)/\sqrt{-2\alpha}$, Eq. (7.25) can be re-expressed as

$$r(s) = \left(r_0 + \frac{\gamma}{\alpha}\right)e^{\zeta^2 - \zeta_0^2} - \frac{\gamma}{\alpha} - \frac{\beta\gamma - \alpha\delta}{\alpha}\sqrt{\frac{\pi}{-2\alpha}}e^{\zeta^2}[\text{erf}(\zeta) - \text{erf}(\zeta_0)], \quad (7.27)$$

where the error function is $\text{erf}(\zeta) = (2/\sqrt{\pi}) \int_0^\zeta e^{-x^2} dx$. The case $\alpha = 0$ will occur occasionally in practice. The corresponding solution of Eq. (7.20) is

$$r(s) = \left(r_0 + \frac{\delta}{\beta}\right)e^{-\beta(s-s_0)} - \frac{\delta}{\beta} + \frac{\gamma}{\beta^2}[1 - \beta s + (\beta s_0 - 1)e^{-\beta(s-s_0)}]. \quad (7.28)$$

A major difference compared with the time-stepping method (solution of Eq. (7.8)) is that now the transit times $s_1 - s_0$ cannot in general be obtained explicitly. Using the solutions (7.25)–(7.28), the relevant root s_1 of

$$r(s_1) - r_1 = 0 \quad (7.29)$$

has to be computed numerically for each direction. In the following subsection, we describe how the roots s_1 and the corresponding exiting wall r_1 can be determined. The displacement of the trajectory inside the considered grid box then proceeds as discussed previously for stationary velocity fields.

We will now determine the roots s_1 of Eq. (7.29) and the corresponding r_1 needed to compute trajectories inside a grid box. In the following, $s_{n-1} \leq s_0 < s_n$ and the relevant roots s_1 are to obey $s_0 < s_1 \leq s_n$. We also focus on the cases $\alpha > 0$ and $\alpha < 0$, since the considerations below can easily be adapted for $\alpha = 0$. For numerical purposes, we use

$$\frac{\beta\gamma - \alpha\delta}{\alpha} = \frac{F_{i,n}F_{i-1,n-1} - F_{i,n-1}F_{i-1,n}}{F_{i,n} - F_{i-1,n} - F_{i,n-1} + F_{i-1,n-1}}, \quad (7.30)$$

$$\frac{\gamma}{\alpha} = \frac{F_{i-1,n} - F_{i-1,n-1}}{F_{i,n} - F_{i-1,n} - F_{i,n-1} + F_{i-1,n-1}} - r_{i-1}, \quad (7.31)$$

$$\xi = \frac{F_{i-1,n-1} - F_{i,n-1} + \alpha(s - s_{n-1})}{\sqrt{2\alpha}}, \quad (7.32)$$

$$\zeta = \frac{F_{i-1,n-1} - F_{i,n-1} + \alpha(s - s_{n-1})}{\sqrt{-2\alpha}}. \quad (7.33)$$

The coefficient in (7.30) appearing in (7.25) and (7.27) is exactly zero when either the r_{i-1} or r_i wall represents land, the transport F_i or F_{i-1} being zero for all n , respectively. In these instances, the opposite wall fixes r_1 , and the root $s_1 > s_0$ can then be computed analytically. If there is no solution, we take $s_1 = s_n$. When all three transit times equal s_n , the trajectory will not move into an adjacent grid box but will remain inside the original one. Its new position is subsequently computed, and the next time interval is considered.

If $(\beta\gamma - \alpha\delta)/\alpha \neq 0$, the computation of the roots of Eq. (7.29) can only be done numerically. This is also true for locating the extrema of the solutions (7.25) and (7.27). Alternatively, one can consider $F(r, s) = 0$ using Eq. (7.18) to analyse where possible extrema are located. It follows that in the $(s-r)$ -plane, extrema lie on a hyperbola of the form $r = (as + b)/(c + ds)$. Of course, only the parts defined by $s_{n-1} \leq s \leq s_n$ and $r_{i-1} \leq r \leq r_i$ are relevant. Depending on which parts of the hyperbola, if any, lie in this ‘box’ and on the initial condition $r(s_0) = r_0$, the trajectory $r(s)$ exhibits none, one, or at most two extrema. In the latter case, the trajectory will not cross either the wall at r_{i-1} or the one at r_i (see Fig. 7.7 for an example). Hence, those trajectories $r(s)$ determining the transit time $s_1 - s_0$ will have at most one extremum, that is, there is at most one change of sign in the local transport F .

An efficient way to proceed then is as follows. First, consider the wall at r_i . For a trajectory to reach this wall, the local transport must be nonnegative, which depends on the signs of the transport $F_{i-1,n}$ and $F_{i,n}$. Four distinct configurations may arise:

1. $F(r_i, s) > 0$ for $s_{n-1} < s < s_n$.
2. Sign of $F(r_i, s)$ changes from positive to negative at $s = \tilde{s} < s_n$.

Fig. 7.7 Example of trajectory $r(s)$ exhibiting two extrema (zero-transport points) inside the relevant r - s 'box.' Regions with positive and negative transports are shown. Extrema for trajectories with differing initial conditions must lie on the hyperbola (dotted curves)

3. Sign of $F(r_i, s)$ changes from negative to positive at $s = \hat{s} < s_n$.

4. $F(r_i, s) < 0$ for $s_{n-1} < s < s_n$.

For case 1, we evaluate $r(s_n)$ using the appropriate analytical solution. If $r(s_n) \geq r_i$, the trajectory has crossed the grid-box wall for $s_1 \leq s_n$. If the initial transport $F(r_0, s_0) < 0$, the trajectory may have crossed the opposite wall at an earlier time. The latter is only possible if case 3 applies for the wall at r_{i-1} and $\hat{s} > 0$, in which case one checks whether $r(\hat{s}) \leq r_{i-1}$. If this is not so, then there is a solution to $r(s_1) - r_1 = 0$ for $r_1 = r_i$ and $s_0 < s_1 \leq s_n$. Subsequently, this root can be simply calculated numerically using a root-solving algorithm. But if $r(s_n) < r_i$ or, if applicable, $r(\hat{s}) \leq r_{i-1}$, we continue with considering the other wall. The arguments for the wall at r_{i-1} are similar to those relating to r . If case 2 applies and $s_0 < \hat{s}$, we follow the considerations given for case 1 using $\hat{s}$ instead of s_n . If there is a root for $r_1 = r_i$, then $s_0 < s_1 \leq \hat{s}$. For case 3, we follow the considerations given for case 1. If there is a root for $r_1 = r_i$, then $\hat{s} < s_1 \leq s_n$. For case 4, no solution of Eq. (7.29) is possible for $r_1 = r_i$. We must then turn attention to the other wall instead. All these considerations are applied to each direction.

7.5.3 Evaluation of the Two Time Integration Methods

The two possible time schemes by which TRACMASS can be integrated in time, which have been presented above, will here be evaluated by testing them on inertial oscillations. Exact analytical solutions of the trajectories for inertial oscillations can be found as well as the corresponding velocity fields. The experiment was originally set up by Fabbri (2009) to test four different trajectory algorithms. One of these algorithms was Ariane (Blanke and Raynaud 1997), which is based on the same equations as the version of TRACMASS that uses the time-stepping method. The three other trajectory algorithms were based on Euler forward and Runge–Kutta schemes. The trajectories, simulated by Ariane, deviated clearly from the analytical solution and the other trajectory schemes. It was thus concluded that Ariane was not as accurate as the other schemes.

In the present study we will repeat one of the Fabbroni (2009) tests for the two TRACMASS schemes and evaluate them by comparing them with the exact analytical inertial oscillation solution. The test consists of using the analytical solution of damped inertial oscillations, which are carried away with a mean geostrophic current so that the equations of motion are

$$\begin{aligned}\frac{\partial u}{\partial t} - fv &= -\gamma u, \\ \frac{\partial v}{\partial t} + fu &= -\gamma v + fu_g,\end{aligned}\tag{7.34}$$

which describe particle circles with a drift to the east due to a geostrophic velocity u_g and with a decreasing oscillation radius depending on the linear friction coefficient γ . The solutions for the velocities are

$$\begin{aligned}u &= u_g e^{-\gamma_g t} + (u_0 - u_g) e^{-\gamma t} \cos ft, \\ v &= -(u_0 - u_g) e^{-\gamma t} \sin ft\end{aligned}\tag{7.35}$$

and for the particle trajectories

$$\begin{aligned}x &= x_0 + \frac{u_g}{\gamma_g} (1 - e^{-\gamma_g t}) + \frac{(u_0 - u_g)f}{f^2 + \gamma^2} \left[\frac{\gamma}{f} + e^{-\gamma t} \left(\sin ft - \frac{\gamma}{f} \cos ft \right) \right], \\ y &= y_0 - \frac{(u_0 - u_g)f}{f^2 + \gamma^2} \left[1 - e^{-\gamma t} \left(\cos ft + \frac{\gamma}{f} \sin ft \right) \right].\end{aligned}\tag{7.36}$$

We used the same coefficients as Fabbroni (2009) with $u_0 = 0.3$ m/s, $u_g = 0.04$ m/s and a damping time of $t_d = 1/\gamma = 2.89$ days and $t_g = 1/\gamma_g = 28.9$ days. The latitude was set to be 45° N. The velocities are read into TRACMASS every hour ($\Delta t_G = 1$ hour) to mimic a GCM that stores the data once an hour. This in contrast to Fabbroni (2009) who read in the velocities as often as every 3 minutes, which is unrealistically high to be run off-line with.

TRACMASS was then integrated forward in time using the time-stepping method with intermediate time steps so the velocities were updated with linear interpolation between two such mimicked ‘GCM’ velocities. The results are shown in Fig. 7.8. Only the red curve that corresponds to the case with no intermediate time steps deviates clearly from the true analytical solution. The blue curve calculated with 10 intermediate time steps has a slight difference. The green (for which 1000 intermediate time steps have been used) lies almost exactly under the purple curve (that reflects the analytical time integration scheme). The small differences between the results from the truly analytical solution and the analytical time integration scheme and time-stepping scheme with 1000 intermediate time steps are likely due to that the velocities are only read into TRACMASS every hour on the model grid and not continuously in both time and space since it is suppose to mimic the reading of GCM fields, which are stored only every hour.

We do not know why we obtain clearly different and better results using TRACMASS here compared to what (Fabbroni 2009) got with Ariane, since both codes,

Fig. 7.8 Comparison of solutions for inertial oscillations. The *black curve* describe the pure analytical solution, the *red, blue and green curves* reflect the results from the time-stepping method using 0, 10 and 1000 intermediate steps between two ‘GCM’ velocities. The *purple curve* depicts the results obtained using the analytical time integration method

we believe, should be based on the same method. A model bug on some level in the Fabbri (2009) experiment is one possible explanation unless Ariane is not as similar to TRACMASS as we have supposed.

7.6 Subgrid Turbulence Parameterizations

The trajectory solutions in the previous sections only include the implicit large scale diffusion due to along-trajectory changes of temperature and salinity/humidity, and by the GCM’s parameterization of turbulent mixing in the momentum equations. These trajectories do not, however, explicitly represent subgrid scale turbulence.

There are two ways to incorporate a representation of subgrid-scale turbulence in TRACMASS. One where an additional random velocity is added called the ‘turbulence parameterization’ and one that adds a random displacement to the trajectory position, which is named ‘diffusion’. These two subgrid turbulence parameterizations will be presented here.

7.6.1 Turbulence Parameterization

This scheme, which was introduced by Döös and Engqvist (2007), adds a fluctuation u' , v' to the GCM-simulated velocity fields U , V . These fluctuations are expected to somehow model the deviations of the trajectories from the exact ones owing to the impact of subgrid turbulence, which is illustrated by Fig. 7.9. These are the instantaneous GCM velocities U , V , which are updated with the GCM output time step and from which the trajectories are calculated when no subgrid parameterization is added.

The turbulent velocities u' , v' are added to each horizontal grid-cell wall for each trajectory calculation and changed at every trajectory time step Δt . The trajectories

Fig. 7.9 Schematic illustration of the changed particle position by the subgrid turbulence parameterization due to the added random velocities u' , v'

are hence calculated with the TRACMASS code as it is, but with a velocity field, $u = U + u'$, that is somewhat shaken, resulting in a stirring of the trajectory particles.

The amplitude of the random turbulent velocity is proportional to the velocity of the circulation model velocity U so that $u' = RU$. Here R is a random number uniformly distributed between $-a$ and a , with standard deviation equal to $\sqrt{3}a$. This amplitude was set to the constant $a = 1$ in Döös and Engqvist (2007), but has here been tuned to obtain a relative dispersion similar to that of the surface drifters. The amplitude was furthermore adapted in Döös et al. (2011) so that the trajectory time step Δt in the TRACMASS code did not affect the results. This was obtained by setting $a = \kappa/(\Delta t)^{1/3}$. The best fit for an amplitude of the relative dispersion similar to that of the surface drifters was obtained for $\kappa = 160$. Using this scheme in practice we add a random noise with a standard deviation on the order of $\sqrt{3}a\sigma_u$, where σ_u is the Lagrangian standard deviation of the unperturbed velocity field.

The effect of this superimposed subgrid turbulence is clearly visible in Fig. 7.10, where a particle cluster is traced with and without this subgrid parameterization. The turbulence smoothes the trajectory positions and spreads them more evenly. The stirred particles in Fig. 7.10b fill visibly regions where no particles were present without subgrid turbulence in Fig. 7.10a.

7.6.2 Diffusion

This scheme adds a random displacement to the trajectory position in order to incorporate a subgrid parameterization of the non-resolved scales as illustrated by Fig. 7.11. The scheme was introduced in TRACMASS in Levine (2005) and tested in a relative dispersion study (Döös et al. 2011).

The horizontal advection-diffusion equation is

$$\frac{\partial P}{\partial t} + U \frac{\partial P}{\partial x} + V \frac{\partial P}{\partial y} = \nabla \cdot (A_H \nabla P), \quad (7.37)$$

where A_H is the horizontal eddy viscosity coefficient. Equation (7.37) is equivalent (see, e.g., Rupolo 2007) to the zeroth-order Markov process:

Fig. 7.10 A cluster of particles released and followed as trajectories until they exit the model domain. The *colour scale* indicates time in hours from the release, with trajectories' positions plotted every hour. The *black line* is the mean position of the trajectory cluster as it evolves in time. (a) without and (b) with subgrid turbulence parameterization. From Döös and Engqvist (2007)

Fig. 7.11 The added displacement due to diffusion. *Left panel* shows in *blue* the original trajectory and in *light blue* the changed one due to the added displacement. *Right panel* zooms in on the added random displacement, where R is defined by Eq. (7.42)

$$\frac{dx_i}{dt} = U_i + \sqrt{2A_H} \frac{dw}{dt}. \quad (7.38)$$

Here the stochastic impulse is represented by the increment $d\eta = (x_d^2 + y_d^2)^{1/2}$. It equals to $d\eta = \sqrt{2A_H}dw$, where w is a Wiener process with a zero mean and a second order moment $\langle dw \cdot dw \rangle = 2dt$. The corresponding Gaussian distribution is

$$P(x_d, y_d, \Delta t) = \frac{1}{\sqrt{2\pi A_H \Delta t}} \exp\left(-\frac{x_d^2 + y_d^2}{2A_H \Delta t}\right). \quad (7.39)$$

Figure 7.11 illustrates the displacements added to the original position of the particle after each time step of length Δt . The added random walk for the particles is given by

$$x_d = \sqrt{-A_H \Delta t \log(1 - q_1)} \cos 2\pi q_2, \quad (7.40)$$

$$y_d = \sqrt{-A_H \Delta t \log(1 - q_1)} \sin 2\pi q_2, \quad (7.41)$$

$$z_d = \sqrt{-A_v \Delta t \log(1 - q_3)} \cos 2\pi q_4. \quad (7.42)$$

Here A_H and A_v are the horizontal and vertical eddy viscosity coefficients and q_n are random numbers between 0 and 1. The added displacement in the horizontal and vertical planes will hence be respectively

$$r_H = \sqrt{x_d^2 + y_d^2} = \sqrt{-\Delta t A_H \log(1 - q_1)}, \quad (7.43)$$

$$r_V = \sqrt{-\Delta t A_v \log(1 - q_3)},$$

with horizontal and vertical standard deviations that are respectively

$$R_H = \sqrt{\Delta t A_H} \quad \text{and} \quad R_V = \sqrt{\Delta t A_v}. \quad (7.44)$$

This implies that about 70 % of the particles will be within this distance from their original positions and that the new velocity field will be characterized by an extra standard horizontal deviation on the order of $(A_H/dt)^{1/2}$, where dt is the Lagrangian integration time step.

It is important to distinguish between this subgrid parameterization of the horizontal and vertical mixing of the Lagrangian trajectories and that of the GCM itself. The velocity fields are generally simulated by the GCM with some sort of Laplacian diffusion. The mixing is hence included in a trajectory as it progresses and changes its tracer properties by contact with its surroundings (Koch-Larrouy et al. 2008). On the one hand one could therefore argue that adding a component to this velocity field would be redundant since the mixing has already been included in the GCM. These trajectories in themselves do not, however, explicitly represent subgrid-scale turbulent motion since they are passively advected by the model-simulated currents with no subgrid scales apart from the linear interpolations of the velocities between the grid points. On the other hand, Lagrangian trajectories are the equivalent of integrating Eq. (7.37) with no effects of velocity scales under the grid scale, which clearly must exist in the real ocean. Furthermore when Eq. (7.37) is discretized and integrated in an OGCM for the tracers it will also include the numerical diffusion, which is not the case for our trajectories since they are exact analytical solutions to the velocity fields in TRACMASS. It is however important to note that we can only evaluate or validate the OGCM itself when we do not add any subgrid parameterization to the model trajectories.

7.6.3 Subgrid Parameterization Questions

Döös et al. (2011) compared the relative dispersion of 5854 pairs of surface drifters with that of simulated TRACMASS trajectories. The coefficients were tuned in order to match the magnitude of the relative dispersion of the surface drifters after 32 days. The ‘diffusion’ parameterization, which adds a stochastic term to the trajectory in accordance with Eq. (7.38), attains realistic relative dispersion rates for $A_H = 2500 \text{ m}^2/\text{s}$. By calibrating the amplitude of the extra horizontal turbulent velocities u' , v' (cf. Appendix B of Döös et al. 2011), also the turbulence parameterization reaches realistic values. The absolute dispersion is not much affected when the diffusion parameterization is added, but gives far too high values for the ‘turbulence’ subgrid parameterization. The modelled trajectories with added diffusion/turbulence also manifest values of the residual velocities which are similar to real data, but with decidedly smaller values of the Lagrangian correlation time. In other words, realistic particle separation rates are obtained using a large diffusivity value, but at the cost of totally changing correlation properties and energy partitioning in the frequency domain.

A more realistic representation of the unresolved scales would require a higher order subgrid parameterization. Griffa (1996) showed that a random walk does not describe the turbulent dispersion behaviour of ocean tracers and that a better quantitative agreement can be reached using an Ornstein–Uhlenbeck process. This work has been refined by Pasquero et al. (2001) who observed that the Ornstein–Uhlenbeck model assumes Gaussian velocity distributions, while the ocean displays exponential-like tails associated with the mesoscale dynamics (Bracco et al. 2000a). Those tails are common to 2D turbulent flows (Bracco et al. 2000b) and to Lagrangian trajectories in an oceanic eddy-resolving model (Bracco et al. 2003). Based on these similarities Pasquero et al. (2001) built a family of two-process stochastic models that provided a better parameterization of turbulent dispersion in rotating barotropic flows.

Berloff and McWilliams (2002) and Berloff et al. (2002) also explored in detail the issue of (horizontal) stochastic parameterizations for oceanic flows, suggesting an alternative model to the one of Pasquero et al. (2001). It is therefore to be expected that the zeroth-order Markov process used in the present study will not provide a good representation of the surface drifters. The relative dispersion rates can hence only be tuned to match the total value at a particular moment. The shape of the power spectrum of the modelled velocity without parameterizations is therefore more realistic in its shape even if too weak.

7.7 Mass Transport and Lagrangian Stream Functions

The mass conservation of the TRACMASS schemes makes it possible to calculate mass transports between different sections in the model domain. A particular water or air mass can be isolated and followed as a set of trajectories between specific

Fig. 7.12 The Lagrangian stream function discretization on a grid box seen from above, with the grid lengths Δx and Δy . An example of one trajectory passing through so that the transport through the walls is

$$T_{i,j,k,n}^y = T_{i,j-1,k,n}^x = T_n \text{ and}$$

$$T_{i-1,j,k,n}^y = T_{i,j,k,n}^x = 0$$

Fig. 7.13 Schematic illustration of how the transport of two trajectories is counted on each grid cell wall. The *orange dots* correspond to meridional transport and the *red dots* to vertical transport, which are then summed in order to compute the Lagrangian stream functions

initial and final sections. Each trajectory, indexed by n , is associated with a volume transport T_n given by the velocity, initial area, and number of trajectories released (Fig. 7.12). During transit from the initial to the final section the volume transport remains unchanged; the transport/velocity field is thus non-divergent, permitting representation in terms of stream functions. The volume transport linked to each trajectory is inversely proportional to the number of trajectories released, *viz* the Lagrangian resolution (which should be sufficiently high to ensure that the stream function does not change when the number of trajectories is further increased).

A non-divergent 3-D volume-transport field is obtained by recording every instance of a trajectory passing a grid-box wall (Fig. 7.13). Every trajectory entering a grid box also exits, and hence this field exactly satisfies

$$T_{i,j,k,n}^x - T_{i-1,j,k,n}^x + T_{i,j,k,n}^y - T_{i,j-1,k,n}^y + T_{i,j,k,n}^z - T_{i,j,k-1,n}^z = 0, \quad (7.45)$$

where $T_{i,j,k,n}^x$, $T_{i,j,k,n}^y$ and $T_{i,j,k,n}^z$, are the trajectory-derived volume transports in the zonal (i), meridional (j), and vertical (k) directions, respectively.

A Lagrangian stream function can be calculated by summing over trajectories representing a desired path (Blanke et al. 1999). By integrating vertically over the transport and over the trajectories one obtains the Lagrangian barotropic stream

Fig. 7.14 Lagrangian zonal overturning stream function in the Gulf of Finland decomposed with water particle trajectories starting in the east at the exit of the River Neva (*left panel*) or at the longitude 23° E from the Northern Gotland Basin (*right panel*). The *green arrows* show where the particles have been released and the *black thin arrows* the direction of the flow. Contours of 500 m³/s

function $\Psi_{i,j}^{LB}$:

$$\Psi_{i,j}^{LB} - \Psi_{i-1,j}^{LB} = \sum_k \sum_n T_{i,j,k,n}^y \quad \text{or} \quad \Psi_{i,j}^{LB} - \Psi_{i,j-1}^{LB} = - \sum_k \sum_n T_{i,j,k,n}^x. \quad (7.46)$$

By instead integrating zonally one obtains the Lagrangian meridional overturning stream function $\Psi_{j,k}^{LM}$:

$$\Psi_{j,k}^{LM} - \Psi_{j,k-1}^{LM} = - \sum_i \sum_n T_{i,j,k,n}^y \quad \text{or} \quad \Psi_{j,k}^{LM} - \Psi_{j-1,k}^{LM} = \sum_i \sum_n T_{i,j,k,n}^z. \quad (7.47)$$

Finally by integrating meridionally one obtains the Lagrangian zonal overturning stream function $\Psi_{i,k}^{LZ}$:

$$\Psi_{i,k}^{LZ} - \Psi_{i,k-1}^{LZ} = - \sum_i \sum_n T_{i,j,k,n}^y \quad \text{or} \quad \Psi_{i,k}^{LZ} - \Psi_{i-1,k}^{LZ} = \sum_i \sum_n T_{i,i,k,n}^z. \quad (7.48)$$

An example of a zonal Lagrangian stream function is shown in Fig. 7.14.

The indices i, j, k do not have to be the horizontal or vertical discretization of the model grid. They can also be replaced by, e.g., temperature, salinity, density, specific humidity, geopotential height or pressure.

7.8 Conclusion and Discussion

In this chapter we have presented the theory behind the trajectory model TRACMASS by summarizing many articles, which have introduced new options and im-

proved TRACMASS. There are, however, still things that would be desirable to improve or add. The TRACMASS subgrid parameterizations, which were introduced in Levine (2005), Döös and Engqvist (2007), Döös et al. (2011) will need to be ameliorated with a higher order Markov model (see, e.g., Rupolo 2007).

It would also be desirable to evaluate the precision of the different TRACMASS schemes in more detail and compare them with other trajectory schemes such as the Runge–Kutta scheme. Fabbroni (2009) compared Ariane (Blanke and Raynaud 1997), which is based on the same equations as the time step version of TRACMASS with other trajectory schemes. The trajectories, simulated by Ariane, deviated clearly from the analytical solution and the other trajectory schemes in her study and Ariane was concluded not to be as accurate as the other schemes. In the present study we repeated the Fabbroni (2009) test of inertial oscillations, with exact analytical solutions. We found in contrast to her test that the TRACMASS scheme gave nearly exactly the same results as the analytical solution. The TRACMASS time-stepping method, which is comparable to the Ariane method, requires, however, that one uses sufficiently intermediate velocity time steps between the GCM time steps. The TRACMASS time-stepping method, when using 1000 intermediate time steps, gave almost exactly the same precision as the TRACMASS the method of analytical time integration. From these tests, we would like to argue that the TRACMASS schemes give at least as accurate trajectories as any other scheme and it is hard to argue that it would be of any use to have even more accurate schemes for geophysical fluid applications given all the missing physics and scales in a GCM. A more detailed and quantitative study would, however, be necessary to measure this.

One of the major advantages of TRACMASS is that it is mass conserving and now can calculate all sorts of mass transports between different sections in the ocean or the atmosphere as well as Lagrangian stream functions for chosen water/air masses. This can be particularly useful when performing analysis of, e.g., the inter-ocean exchange of water masses or the large scale atmospheric circulation. TRACMASS has also turned out to be very useful in completely different applications such as studies of genetic connectivity, dispersion of radionuclides or identification of transport patterns in the surface layer as in the present book. The number of possible TRACMASS applications will certainly continue to grow in the future as long as GCMs are based on finite differences.

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