A PROOF THAT COMMUTATIVE ARTINIAN RINGS ARE NOETHERIAN

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The origin of this note is a desire, on behalf of the author, to gain a better understanding of the well-known fact that each commutative ring which satisfies the descending chain condition on ideals also satisfies the ascending chain condition, or in other words that all Artinian rings are Noetherian. This is proved in most introductory texts in commutative algebra, such as e.g. [2]. An early proof, still very readable, may be found in Cohen's classical paper [3] (Theorem 1). Cohen refers to a paper by Akizuki ([1]) as a first source. It should be pointed out that this theorem is valid also for noncommutative rings, which was shown by Hopkins ([4]).

In this paper, however, we only treat the commutative case and it should always be tacitly understood that *R is a commutative ring and M is a module over R.*

That the descending chain condition on a ring implies the ascending chain condition could be interpreted in the following informal way. It seems to be easier to take small (and hence many) steps downwards than upwards in a ring.

A very small step would of course be a step of length one. The following lemma (which of course is essentially well known) gives the exact condition under which it is possible to take a length-one step downwards in a module.

Lemma 1. *The following two conditions are equivalent for M.*

1) $M = \mathfrak{m}M$ *for each maximal ideal* $\mathfrak{m}$ *in* R*.*

2) $\ell(M/N)$ *is infinite for each proper submodule* N *of* M*.*

Proof. Suppose first that $\ell(M/N)$ is finite for some submodule N. We may then choose N such that $\ell(M/N) = 1$. If x is nonzero in M/N we have a surjective homomorphism $R \to M/N$ defined by $r \mapsto rx$. Since M/N is simple the kernel of this homomorphism is a maximal ideal $\mathfrak{m}$ in R. Thus $\mathfrak{m}(M/N) = (0)$ and hence $\mathfrak{m}M \neq M$. This proves 1) $\Rightarrow$ 2). Suppose next that $M \neq \mathfrak{m}M$ for some $\mathfrak{m}$. Put $V = M/\mathfrak{m}M$. Then V is a nontrivial vectorspace over the field $R/\mathfrak{m}$. Clearly there is a subspace W such that $\dim_{R/\mathfrak{m}} V/W = 1$ (simply omit one element from a basis for V). Now just lift W back to get a submodule N of M such that $\ell(M/N) = 1$.

We shall now concentrate on rings satisfying the *descending chain condition on products of maximal ideals*. Thus we shall assume that there is an ideal in R of the form $\mathfrak{a} = \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r$, where $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_r$ are maximal ideals (not necessarily distinct) and where $\mathfrak{a}\mathfrak{m} = \mathfrak{a}$ for every maximal ideal $\mathfrak{m}$. For example R could be an Artinian ring. Clearly $\mathfrak{a}$ is unique. In this situation condition 1) of Lemma 1 can be reformulated as $M = \mathfrak{a}M$.

Proposition 2. *Suppose that* R *satisfies the descending chain condition on products of maximal ideals and let* $\mathfrak{a}$ *be the unique minimal product of maximal ideals. Then we have the following two possibilities.*

1) $\mathfrak{a}M \neq (0)$ *and* M *is neither Noetherian nor Artinian.*

2) $\mathfrak{a}M = (0)$ *and in this case* M *is Artinian if and only if* M *is Noetherian.*

Proof. 1) Suppose $\mathfrak{a}M \neq (0)$ and put $N = \mathfrak{a}M$. Note that $\mathfrak{a}N = N \neq (0)$ and hence $(0 :_N \mathfrak{a})$ is a proper submodule of N. Thus, by Lemma 1, $\ell(N/(0 :_N \mathfrak{a})) = \infty$ so there is certainly a submodule L of M which lies strictly between $(0 :_N \mathfrak{a})$ and N. We then have $\mathfrak{a}L \neq (0)$ and $L \neq \mathfrak{a}M$ i.e. $\mathfrak{a}(M/L) \neq (0)$. We may thus repeat the argument twice with M replaced by L and M/L respectively. This will yield submodules L_{-1} and L_1 of M such that $L_{-1} \subset L \subset L_1$ and $\mathfrak{a}L_{-1} \neq (0)$, $\mathfrak{a}(M/L_1) \neq (0)$. Proceeding in this way we obtain an infinite (ascending and descending) chain $\cdots \subset L_{-2} \subset L_{-1} \subset L \subset L_1 \subset L_2 \subset \cdots$.
2) If $\mathfrak{a}M = (0)$, say $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r M = (0)$, then indeed it is well known and easy to see that M is Artinian if and only if M is Noetherian (cf for example [2] Corollary 6.11).

An immediate corollary is the following.

Corollary 3. *Suppose that R satisfies the descending chain condition on products of maximal ideals. Then M is Noetherian if and only if M is Artinian. In particular every Artinian ring is Noetherian.*

Remark 1. Summarizing we see that there are three alternative "reasons" why a non-Noetherian ring is non-Artinian:

(i) there is an infinite descending chain of products of maximal ideals

(ii) the zero ideal is a product of maximal ideals and hence there is a step of the form $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{r-1}/\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_r$ which is infinite-dimensional as a vector space

(iii) there is a minimal product of maximal ideals, but this ideal is different from zero. Then an infinite descending chain is provided by the proof of Proposition 2. Such a ring is the one in our example below.

Remark 2. There is a general dimension theory for Artinian modules due to Roberts ([7]) and further developed by Kirby ([5]). We adopt Kirby's terminology and notation and refer to Noetherian dimension of a module, denoted N-dim and defined recursively in the following way. Let, for $n = -1, 0, 1, 2, \ldots$, $\mathcal{C}_n = \{M \mid \text{N-dim}\, M \leq n\}$. Then $\mathcal{C}_{-1} := \{(0)\}$ and $\mathcal{C}_n := \{M \notin \mathcal{C}_{n-1} \mid$ every ascending chain of submodules $M_0 \subset M_1 \subset M_2 \subset \cdots$, such that $M_i/M_{i-1} \notin \mathcal{C}_{n-1}$ for all i, is finite$\}$. Thus, for example, $\mathcal{C}_0$ is the class of Noetherian modules. The concept of Noetherian dimension is dual to the concept of Krull dimension (K-dim) due to Rentschler and Gabriel ([6]), which uses descending chains instead of ascending chains. Returning to the first part of the proof of Proposition 2 and the chain $\cdots \subset L_{-1} \subset L \subset L_1 \subset \cdots$ we observe that the condition $\mathfrak{a}M \neq (0)$ is inherited by each step in this chain, i.e. $\mathfrak{a}(L_{n+1}/L_n) \neq (0)$. Thus we may prove by induction that $\text{N-dim}\, M > r$ and $\text{K-dim}\, M > r$ for all integers r. Hence, if $\mathfrak{a}M \neq (0)$, the Noetherian dimension and the Krull dimension of M are both infinite.

Remark 3. The fact that every Artinian module is Noetherian if R satisfies the descending chain condition on products of maximal ideals also follows from the general dimension theory, using an analogue of the usual Hilbert polynomial ([5] theorem 2.6).

Remark 4. Recall that if M is both Noetherian and Artinian, i.e. of finite length, then M is essentially a module over an Artinian ring because, as an easy argument shows, $R/\text{Ann}\, M$ is Artinian.

Next follows a nontrivial example of a ring with descending chain condition on products of maximal ideals.

Example. Let Q_+ be the additive monoid of nonnegative rational numbers and let $A = k[Q_+]$ be the monoidalgebra over a field k. Let $y_1, y_2, \ldots$ be an infinite set of variables and put $T = A[y_1, y_2, \ldots]/(y_1, y_2, \ldots)^n$. The elements of T can be uniquely represented in the form $f = f_0(x) + f_1(x)p_1(\overline{y}) + \cdots + f_{n-1}(x)p_{n-1}(\overline{y})$, where $p_i(\overline{y})$, $i = 1, 2, \ldots, n-1$ are monomials of degree i in the variables $y_1, y_2, \ldots$ and where $f_i(x) = \sum_k a_{ik} x^{r_{ik}}$ for rational numbers $0 \leq r_{i0} < r_{i1} < \cdots$ and $a_{ik} \in k$. A maximal ideal of T is $M = \{f; f_0(0) = 0\}$ i.e. the set of all $f \in T$ such that the "constant term" of $f_0(x)$ is zero. M is the kernel of the homomorphism $T \to k$ given by $f \mapsto f_0(0)$. It is easy to show that $M^i = \{f; f_0(0) = f_1(0) = \cdots = f_{i-1}(0) = 0\}$ for $i = 1, 2, \ldots, n$. It follows that $M \supset M^2 \supset M^3 \supset \cdots \supset M^n = M^{n+1} = \cdots$. Localize at M and put $R = T_M$ and $\mathfrak{m} = M_M$. Then R is a local ring and $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots \supset \mathfrak{m}^n = \mathfrak{m}^{n+1} = \cdots$ so R certainly satisfies the descending chain condition on products of maximal ideals. Thus an R-module M is Noetherian if and only if it is Artinian. Note that $R/\mathfrak{m}^n$ is neither Artinian nor Noetherian.

The lemma which follows is dual to Lemma 1 and hence gives an answer to the question when it is possible to take a length-one step upwards in a module. Although this lemma probably is more well known than Lemma 1 we include its short proof for sake of completeness.

Lemma 4. *The following two conditions are equivalent for M*

1) $(0 :_M \mathfrak{m}) = (0)$ for each maximal ideal $\mathfrak{m}$ in R.

2) $\ell(N)$ is infinite for each submodule $N \neq (0)$.

Proof. Suppose first that $\ell(N)$ is finite for some submodule $N \neq (0)$ and choose, as we may, N such that $\ell(N) = 1$. Then there is a maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}N = (0)$ and hence $(0 :_M \mathfrak{m}) \neq (0)$. Suppose next that $(0 :_M \mathfrak{m}) \neq (0)$ for a certain $\mathfrak{m}$. As $(0 :_M \mathfrak{m})$ is an $R/\mathfrak{m}$-vector space it is clear that $(0 :_M \mathfrak{m})$ has a finite dimensional subspace N.

We finally use Lemma 4 to prove a slight sharpening of part 1) of Proposition 2.

Proposition 5. *Suppose there is an ideal $\mathfrak{b}$ in R such that $\mathfrak{b}M \neq (0)$ and such that $(L :_M \mathfrak{b}) = (L :_M \mathfrak{b}\mathfrak{m}) = (L :_M \mathfrak{b}^2)$ holds for all submodules L of M and all maximal ideals $\mathfrak{m}$ in R. Then M is neither Noetherian nor Artinian.*

Proof. Like in the proof of Proposition 2 we show that there is a proper non-zero submodule L of M such that the hypotheses are inherited by L and M/L. Put $N = (0 :_M \mathfrak{b})$. Then $N \neq M$, $\mathfrak{b}M \neq N$ and $(N :_M \mathfrak{m}) = N$ for all maximal ideals $\mathfrak{m}$. Thus, by Lemma 4, there is a submodule L such that

$N \subset L \subset N + \mathfrak{b}M$. It is simple routine to verify that L and M/L satisfy the hypotheses of the proposition.

Note that we may remark here, as we did in Remark 2 after Proposition 2, that it actually follows from the proof that N-dim M and K-dim M are both infinite.

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