Filosofie licentiatavhandling

The Ekedahl Invariants for finite groups

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Abstract
In 2009 T. Ekedahl introduced some cohomological invariants for affine finite groups of finite type over the algebraically closed field \(\mathbb{k}\) of characteristic zero. These relate, naturally, to invariant theory for groups and also to the Noether Problem (one wonders about the rationality of the extension \(\mathbb{F}(x_g : g \in G)^G\) over \(\mathbb{F}\), for a field \(\mathbb{F}\) and a finite group \(G\)). In this work, we introduce these invariants, we state the literature results and we show that these invariants are trivial for every finite group in \(\text{GL}_3 (\mathbb{C})\) and for the fifth discrete Heisenberg group \(H_5\).
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1 Introduction

In this introduction, we give two definitions of the Ekedahl invariants: the first one does not involve the concept of stack. We also present some related topics like the Noether problem and the Bogomolov multiplier $B_0(G)$ of a group $G$. We are interested in $B_0(G)$, because the second Ekedahl invariants (the first relevant) is equal to the class of its dual group, $e_2(G) = \{B_0(G)^\vee\}$, in a Grothendieck type structure $L_0(\text{Ab})$ (see the main Theorem). After sketching the relevant facts in literature, we present our results.

Notation. In all this manuscript, $k$ is an algebraically close field of characteristic zero and $G$ is an affine finite group of finite type over $k$ (we will just write finite group). Moreover every cohomology group (if not explicitly expressed differently) is the singular cohomology group with integer coefficient, that is $H^k(\mathcal{X}) = H^k(\mathcal{X}; \mathbb{Z})$. Similarly for the cohomology of a finite group $G$, $H^k(G) = H^k(G; \mathbb{Z})$.

The Noether problem

Let $\mathbb{F}$ be a field and let $G$ be a finite group. We denote by $\mathbb{F}(x_g : g \in G)$ the field of rational functions with variables indexed by the elements of the group $G$. The group acts on it via $h \cdot x_g = x_{hg}$. We consider the field extension $\mathbb{F} \subset \mathbb{F}(x_g : g \in G)^G = \mathbb{F}(G)$, where the latter denotes the field of invariants\(^1\). In 1914, Emmy Noether\(^2\) wondered if the field extension $\mathbb{F} \subset \mathbb{F}(x_g : g \in G)^G$ is rational (i.e. purely transcendental).

To fix the idea let us consider an easy example. Let $\mathbb{F} = \mathbb{Q}$ and $G = \mathbb{Z}/2\mathbb{Z}$. The zero element in $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Q}(x_0, x_1)$ by fixing both variables and 1 switches them. The extension $\mathbb{Q}(x_0, x_1)^{\mathbb{Z}/2\mathbb{Z}} \subset \mathbb{Q}(x_0, x_1)$ is proper because $x_0 \notin \mathbb{Q}(x_0, x_1)^{\mathbb{Z}/2\mathbb{Z}}$. One observes that $x_0 + x_1$ and $x_0x_1$ are invariants generating $\mathbb{Q}(x_0, x_1)^{\mathbb{Z}/2\mathbb{Z}}$. Therefore, the extension $\mathbb{Q} \subset \mathbb{Q}(x_0, x_1)^{\mathbb{Z}/2\mathbb{Z}}$ is rational.

Mathematicians conjectured a positive answer to the Noether problem, until the breakthrough result of of Swan [23] in 1969: ”... the conjecture has proved to be extremely intractable. I will show here that there is a good reason for this. The conjecture is false even in the simplest case of a cyclic permutation group ...”. Indeed, he proved that the extension $\mathbb{Q} \subset \mathbb{Q}(x_g : g \in \mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}/p\mathbb{Z}}$ is not rational for $p = 47, 113$ and 233.

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\(^1\) $\mathbb{F}(G)$ is a field because $1 \in \mathbb{F}$ is fixed and if $ab = 1$ in $\mathbb{F}$ and $G$ fixes $a$, then it fixes also $b$.

\(^2\) Amalie Emmy Noether (Germany, 23 March 1882 - 14 April 1935).
After this, a lot of effort was spent on the Noether problem, but one needs to wait for Saltman’s result [21] in 1984 to get a more complete picture. He proved that for any field $F$ and for any prime $p$ with $(\text{char } F, p) = 1$, there exists a group $G$ of order $p^9$ such that the Noether problem has negative answer.

Saltman used a cohomological invariant introduced by Artin and Mumford [2]. Bogomolov [4], in 1988, showed a concrete way to compute this invariant that is now called Bogomolov multiplier (or Bogomolov group): it is a cohomological obstruction to the rationality of $F(G)/F$, i.e. the rationality of $F(G)/F$ implies $B_0(G) = 0$. In [4], is proved that the Bogomolov group, $B_0(G)$, is a subgroup of the Schur Multiplier $M(G) = H^2(G; \mathbb{C}^*)^4$, defined as the cohomology classes in $H^2(G; \mathbb{C}^*)$ such that their restrictions to any abelian subgroup of $G$ is zero. This means that

$$B_0(G) = \bigcap_A \text{Ker } (H^2(G; \mathbb{C}^*)) \to H^2(A; \mathbb{C}^*)) .$$

where the intersection runs over the abelian subgroups $A \subseteq G$.

The Grothendieck group of varieties and Bittner’s presentation

The Grothendieck group of varieties, $K_0(\textbf{Var}_k)$, is the group generated by the isomorphism classes, $\{X\}$, of algebraic $k$-varieties $X$, subject to the relation

$$\{X\} = \{Z\} + \{X \setminus Z\} .$$

for any closed subvarieties $Z$ of $X$. It is possible to see that $K_0(\textbf{Var}_k)$ has a ring structure give by $\{X\} \cdot \{Y\} = \{X \times Y\}$.

For instance, in $K_0(\textbf{Var}_k)$ there are the following classes: the class of the empty set $\{\emptyset\}$, the class of the point $\{\ast\}$, the class of the affine line $\mathbb{L} = \{A^1_k\}$ (also called Lefschetz class). Using the multiplication operation, $\{A^n_k\} = \mathbb{L}^n$; moreover, one shows that $\mathbb{P}^n = \mathbb{L}^0 + \mathbb{L}^1 + \cdots + \mathbb{L}^n$.

In [3], Bittner proves that, $K_0(\textbf{Var}_k)$ is generated by the class of smooth and proper varieties modulo the relations $\{X\} + \{E\} = \{Bl_Y(X)\} + \{Y\}$

---

3In [4], Bogomolov also improved Saltman’s statement from $p^9$ to $p^6$. Other results in this direction are [6, 7, 20].

4One can also show that $H^2(G; \mathbb{C}^*) \simeq H^3(G; \mathbb{Z})$.

5If $A$ is a subgroup of $G$, the restriction map is a natural surjective cohomological map $H^*(G, -) \to H^*(A, -)$. 

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with $Bl_Y(X)$ being the blow up of $X$ along $Y$ with exceptional divisor $E$:

$$\xymatrix{Bl_Y(X) \ar[r] & X \\
E \ar[u] & Y \ar[u]}
$$

Therefore, with the help of compactification and resolution of singularities one writes the class of a scheme $\{X\} \in K_0(\text{Var}_k)$ as a sum of classes of smooth and proper varieties $\{X_j\}$: $\{X\} = \sum_j n_j \{X_j\}$, with $n_j \in \mathbb{Z}$.

A non-stacky definition of the Ekedahl invariants

In mathematics, to study manifolds, varieties and more in general topological space one constructs abstract structure like the cohomology groups, the homotopy groups et. Those can be seen as functors from the category of our preferred objects, $C$, to the category of abelian groups, $\text{Ab}$. In other words, we associate to any object (manifold, variety,...) an abelian group.

The invariants we are going to define need a more refined target: Let $L_0(\text{Ab})$ be the group generated by the isomorphism classes, $\{G\}$, of finitely generated abelian groups $G$ under the relation $\{A \oplus B\} = \{A\} + \{B\}$. For clarification, $\{\mathbb{Z}\}$ and $\{\mathbb{Z}/p^n\}$ belong to $L_0(\text{Ab})$ and there are elements in $L_0(\text{Ab})$ that do not correspond to any group: while $\{\mathbb{Z}\} + \{\mathbb{Z}/5\}$ is the class of $\{\mathbb{Z} \oplus \mathbb{Z}/5\}$, the element $\{\mathbb{Z}\} - \{\mathbb{Z}/5\}$ is not the class of any group.

Let $V$ be a faithful representation of a finite group $G$. The group $G$ acts naturally on $V$ and one considers the quotient $V/G$. Let $m$ be a positive integer and let $G$ act componentwise on $V^m$: one obtains $V^m/G$. The latter is usually a non smooth scheme.

**Definition 1.** Let $V$ be a faithful $k$-representation of a finite group $G$. Let $X$ be a smooth and proper resolution of $V^m/G$:

$$X \xrightarrow{\pi} V^m/G.$$  

For $m$ large enough, the $i$-th Ekedahl invariant, $e_i(G)$, is given as following:

$$e_i(G) = \{H^{2m-i}(X; \mathbb{Z})\} + \sum_j n_j \{H^{2m-i}(X_j; \mathbb{Z})\} \in L_0(\text{Ab})$$

where $\{V^m/G\} \in K_0(\text{Var}_k)$ is written as the sum of classes of smooth and proper varieties $\{X\}$ and $\{X_j\}$, $\{V^m/G\} = \{X\} + \sum_j n_j \{X_j\}$. 

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Of course, the previous definition is not well given. Writing ‘$m$ large enough’ leaves a picky reader very unsatisfied. There is a deep reason for this. Indeed, this $m$ comes from the stacky world and (the author thinks) there is no way to avoid it.

A stacky definition of the Ekedahl invariants
The Motivic ring of algebraic $k$-varieties is $K_0(\text{Var}_k)[\mathbb{L}^{-1}]$. We naturally define a dimension filtration

$$\text{Fil}^n (K_0(\text{Var}_k)[\mathbb{L}^{-1}]) = \{\{X\}/\mathbb{L}^i : \text{dim } X - i \leq n\}.$$  

We denote by $\widehat{K}_0(\text{Var}_k)$ the completion of the Motivic ring with respect to this filtration. This ring is called Kontsevich’s value ring.

We now introduce the Ekedahl invariants as done in [10]. For any integer $k$, Ekedahl defined a cohomological map

$$\text{H}^k : \widehat{K}_0(\text{Var}_k) \to L_0(\text{Ab}).$$

The Ekedahl invariants are the negative cohomologies of the class of the classifying stack $BG$ seen inside $\widehat{K}_0(\text{Var}_k)$.

To every smooth and proper $k$-variety, $X$, one can naturally associate its integral cohomology group $\text{H}^k (X)$ and we consider its class, $\{\text{H}^k (X)\} \in L_0(\text{Ab})$. This construction produces a well defined map

$$\text{H}^k : K_0(\text{Var}_k) \to L_0(\text{Ab}).$$

Now, we define for every integer $k$ the map (with an abuse of notation)

$$\text{H}^k : \widehat{K}_0(\text{Var}_k) \to L_0(\text{Ab}),$$

sending $\{X\}/\mathbb{L}^m$ to $\{\text{H}^{k+2m} (X; \mathbb{Z})\}$, for any $X$ smooth and proper. This map is just an extension of the namesake map defined over $K_0(\text{Var}_k)$. We observe that

1. Since $\widehat{K}_0(\text{Var}_k)$ is the completion of $K_0(\text{Var}_k)[\mathbb{L}^{-1}]$ under the filtration in dimension $\text{Fil}^n = \{\{X\}/\mathbb{L}^i : \text{dim } X - i \leq n\}$, one only needs to define the map $\text{H}^k$ on the elements of the type $\{X\}/\mathbb{L}^m$.

2. It is sufficient to define $\text{H}^k$ for the elements $\{X\}/\mathbb{L}^m$, with $X$ smooth and proper.
For instance $H^k(\{\ast\}) = \{\mathbb{Z}\}$ for $k = 0$ and zero otherwise. More about this cohomological map can be found Section 2.

**Definition.** A $G$-torsor $P$ over a scheme $X$ over $k$, $P \to X$, is a scheme with a regular $G$-action.

**Definition.** The classifying stack of a group $G$, $BG$, is a pseudo-functor from the category of schemes over $k$, $\text{Sch}_k$, to the category of groupoids over $k$, $\text{Gpd}_k$, sending any open scheme $U$ to the groupoid of $G$-torsors over $U$:

$$BG : \text{Sch}_k \to \text{Gpd}_k$$

$$U \mapsto \{G\text{-torsors over } U\}.$$

Equivalently, the classifying stack of the group $G$ is usually defined as the stack quotient $BG = [\ast/G]$.

Similarly to algebraic varieties, the algebraic stacks deserve their own Grothendieck ring, $K_0(\text{Stack}_k)$. One proves that

1. $K_0(\text{Stack}_k) = K_0(\text{Var}_k)[L^{-1}, (L^n - 1)^{-1}, \text{for any } n]$.
2. The completion map from the Motivic to the Kontsevich’s value ring factors through $K_0(\text{Stack}_k)$.
3. $\{BG\}$ can be seen inside of $\hat{K}_0(\text{Var}_k)$.

Finally, we get the goal.

**Definition 2.** The $i$-th Ekedahl invariant is $e_i(G) = H^{-i}(\{BG\}) \in L_0(\text{Ab})$. We say that $e_i(G)$ are trivial if $e_i(G) = 0$ for $i \neq 0$.

We prove in Section 2 that the definitions we gave are equivalent.

**The main facts about the Ekedahl invariants**

One way to study $e_i(G)$ is focusing on the class $\{BG\}$. Indeed if $\{BG\} = 1$ then the $e_i(G)$ are trivial. It has been proved that:

**Proposition 1.1** (Proposition 3.2, Proposition 3.9 and Theorem 4.3 [10]). In the following cases $\{BG\} = 1 \in \hat{K}_0(\text{Var}_k)$:

1) $G = \mu_n$ for any $n$;
2) \( G \) is the symmetric group \( S_n \) for any \( n \);  
3) \( G \) is a finite unipotent group;  
4) \( G \) is a finite subgroup of \( GL_1 \);  
5) \( G \) is a finite subgroup of the affine transformation of \( \mathbb{A}^1_k \).

Therefore one has the following triviality result:

**Corollary 1.2.** *For the cases 1), ..., 5) the Ekedahl invariants are trivial.*

In Section 3 we are going to extend this list by proving that if \( G \) is a subgroup of \( GL_3(C) \), then \( \{BG\} = 1 \).

Vice versa, to study if the Ekedahl invariants are not trivial, one uses that \( e_2(G) = \{B_0(G)^\vee\} \). The next theorem summarizes this and other properties.

**Theorem** (Theorem 5.1 [10]). Let \( G \) be a finite group. Let \( k \) be a field of characteristic zero. The following holds:

a) \( e_i(G) = 0 \), for \( i < 0 \);  
b) \( e_0(G) = \{\mathbb{Z}\} \);  
c) \( e_1(G) = 0 \);  
d) \( e_2(G) = \{B_0(G)^\vee\} \), where \( B_0(G)^\vee \) is the dual of the Bogomolov multiplier of the group \( G \);  
e) The class \( \{\mathbb{Z}\} \) does not appear in \( e_i(G) \), for \( i > 0 \).

We prove this result in Section 2. We observe that the point a) gives sense to the switching of sign in the Definition 2.

We remark that \( B_0(G) \) is a cohomological obstruction for the Noether problem (that is, \( B_0(G) \neq 0 \Rightarrow \mathbb{F}(G)/\mathbb{F} \) is not rational) and we have seen example when \( B_0(G) \neq 0 \). Therefore, the given counterexamples to the Noether problem (with \( B_0(G) \neq 0 \)) are also the first easy examples of finite groups \( G \) with non trivial Ekedahl invariants. In these examples, the class of the classifying stack, \( \{BG\} \), is different from the class of the point \( \{\ast\} \).

**Proposition 1.3** (Non triviality). *The second Ekedahl invariant is non trivial for every field \( k \) (with \( k = k \) and \( \text{char}(k) = 0 \)) and for the groups of order \( p^9 \) in Saltman’s paper [21] and of order \( p^6 \) in Bogomolov’s paper [4]. Moreover in these cases, \( \{BG\} \neq 1 \) in \( \widetilde{\mathbb{K}}_0(\text{Var}_k) \).*
Another connection between the Noether problem and the non-triviality of \( \{B G\} \) is also the next proposition.

**Proposition 1.4** (Corollary 5.8 [10]). \( \{B^{\mathbb{Z}/47\mathbb{Z}}\} \neq 1 \) in \( \widehat{K}_0(\text{Var}_\mathbb{Q}) \).

**The goal of the work**

In this work we first deal with some properties of the class of \( B G \) in the Kontsevich value ring. Then we focus on the finite subgroups of \( \text{GL}_n(\mathbb{C}) \) and we prove that:

**Theorem.** Let \( G \) be a finite subgroup of \( \text{GL}_3(\mathbb{C}) \). Then \( \{B G\} = 1 \) and the Ekedahl invariants of \( G \) are trivial.

A general approach for \( \text{GL}_4 \) has stopped for the hardness (and the lack of literature) of the study of the affine varieties \( k^3 / G \) even in the case \( k = \mathbb{C} \).

Therefore, we focus on the discrete Heisenberg group \( H_p \), the subgroup of upper unitriangular matrices of \( \text{GL}_3(\mathbb{F}_p) \). This is an interesting candidate for the study of \( e_i(H_p) \), because \( B_0(H_p) = 0 \) and so the first unknown Ekedahl invariants is \( e_3(H_p) \). We show that

\[
e_3(H_p) = e_4(H_p) = \{\text{tor}(H^{2p-5}(X_p;\mathbb{Z}))\}.
\]

where \( X_p \) is smooth and projective resolution of \( \mathbb{P}(V)/A_p \), \( V \) is a faithful linear \( p \)-dimensional complex representation of \( H_p \) and \( A_p \) is \( H_p \), modulo its center.

The use of toric varieties tools constricts our computation to \( p = 5 \): we show that \( \text{tor}(H^5(X_p;\mathbb{Z})) \) is zero.

**Theorem.** The Ekedahl invariants of the fifth discrete Heisenberg group, \( e_i(H_5) \), are trivial.
Background

In Section 2 and 3, we require a good knowledge of the theory of algebraic stacks [18]. In Section 3, we require a modest knowledge of representation theory of group and we suggest for this [9] or [14]. Moreover in Section 4 we use the representations of the discrete Heisenberg group $H_p$; the author suggests for this Section 12 of [9].

In Sections 6 and 7 the reader should know the theory of toric varieties and how to compute their cohomologies: the text abounds of observations and remarks that always refer to [12], but we also suggest [11]. In particular, in Section 7, we use intersection theory tools over toric varieties [14] (or more in general [13]).

In Section 4, we use the Leray - Cartan spectral sequence (see Theorem 8$_{bis}$.9 in [19]) and, in Section 7, we use the spectral sequences induced by an open covering (see Chapter 3.4 [17]). These techniques are standard and in both cases, for more technical facts, we refer to [19, 24].

Finally, in the first five sections we use the theory of cohomology of groups. For this the author follows [1, 5].
2 The Ekedahl invariants for finite groups

We denote by $K_0(\text{Var}_k)$ and $K_0(\text{Var}_k)[L^{-1}]$ the Grothendieck ring and the Motivic ring of algebraic $k$-varieties. In the latter, one defines a dimension filtration

$$\text{Fil}^n(K_0(\text{Var}_k)[L^{-1}]) = \{(X)/\mathbb{L}^i : \dim X - i \leq n\}.$$ 

and its completion under this filtration is the Kontsevich value ring, $\widehat{K}_0(\text{Var}_k)$.

**Definition 2.1.** We denote by $K_0(\text{Stack}_k)$ the Grothendieck group of algebraic $k$-stacks. This is the group generated by the isomorphism classes $\{X\}$ of algebraic $k$-stacks, $X$, of finite type all of whose automorphism group scheme are affine\(^6\). The elements of this group fulfill the following relations:

1. for each closed substack $Y$ of $X$, $\{X\} = \{Y\} + \{Z\}$ where $Z$ is the complement of $Y$ in $X$;
2. for each vector bundle $E$ of constant rank $n$ over $X$, $\{E\} = \{X \times \mathbb{A}^n\}$.

Similarly to $K_0(\text{Var}_k)$, $K_0(\text{Stack}_k)$ has a ring structure.

**Lemma 2.2.** One has that $K_0(\text{Stack}_k) = K_0(\text{Var}_k)[L^{-1}, (L^n - 1)^{-1}, \forall n \in \mathbb{N}]$ Moreover, the completion map $K_0(\text{Var}_k)[L^{-1}] \rightarrow \widehat{K}_0(\text{Var}_k)$ factors through $K_0(\text{Var}_k)[L^{-1}] \rightarrow K_0(\text{Stack}_k) \rightarrow \widehat{K}_0(\text{Var}_k)$.

**Proof.** The first part is proved in Theorem 1.2 of [10]. Regarding the second one, we observe that $L^n - 1 = L^n(1 - L^{-n})$ is invertible in $\widehat{K}_0(\text{Var}_k)$. Indeed, $(1 - L^{-n})^{-1} = 1 + L^{-n} + L^{-2n} + \ldots$ and each truncation $x_k = \sum_{j=0}^{k} L^{-jn}$ belongs to $\text{Fil}^{-kn}$. So, the serie converges in $\widehat{K}_0(\text{Var}_k)$. \qed

**Definition 2.3.** A special group $G$ is a connected algebraic group scheme of finite type all of whose torsors over any extension field $k \subseteq K$ are trivial.

**Lemma 2.4.** Let $G$ be a special group and let $H$ be a closed subgroup scheme of $G$. Then

a) $\{BG\} \cap \{B/h\} = \{G\} \{BG\}$;

b) $\{G\} \{BG\} = 1$

\(^6\)Shortly, algebraic $k$-stack of finite type with affine stabilizer.
Proof. See Proposition 1.1 of [8] items v) and ix).

We narrow down our investigation to finite groups.

Lemma 2.5. Let $V$ be an $n$-dimensional linear representation of $G$. Then

\begin{align*}
\{[V^m/G]\} &= \mathbb{L}^m \{BG\} \quad (1) \\
\{[\mathbb{P}(V)/G]\} &= (1 + \mathbb{L}^1 + \cdots + \mathbb{L}^{n-1}) \{BG\} \quad (2)
\end{align*}

Proof. From the vector bundle $[V/G] \to BG$, using the second property in Definition 2.1, one has that $\{[V/G]\} = \mathbb{L}^n \{BG\}$. Similarly, one proves the first equation.

Let $O$ be the origin of $V$. The natural map $[V^{\{O\}}/G] \to \mathbb{P}(V)/G$ is a $\mathbb{G}_m$-torsor and this implies $\{[V^{\{O\}}/G]\} = (\mathbb{L} - 1)\{[\mathbb{P}(V)/G]\}$. Moreover, $\{[V^{\{O\}}/G]\} = (\mathbb{L}^n - 1) \{BG\} = (\mathbb{L} - 1) \{[\mathbb{P}(V)/G]\}$. \qed

Formula (1) expresses how $\{BG\}$ is connected with $\{[V^m/G]\}$. The next result links $\{BG\}$ to $\{V^m/G\}$. Behind this proposition there is the study of the difference between $\{[V^m/G]\}$ and $\{V^m/G\}$ in $K_0(Var_k)[\mathbb{L}^{-1}]$.

We write an element of $V^m$ as

$$v = (v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, \ldots, v_{n(k-1)+1}, \ldots, v_{kn}, v_{kn+1}, \ldots, v_m)$$

with $k = \lfloor m/n \rfloor$. In other words, we consider $v \in V^m$ as a sequence of $n$ vectors. Let $U$ be the subset of $V^m$ such that at least one of $v_{nj+1}, \ldots, v_{jn+n}$ is a basis for $V$.

We denote by $M$, the complement of $U$ in $V^m$. This is a closed subset of $V$, because it is defined by $k$ equations $\det(v_{nj+1}, \ldots, v_{jn+n}) = 0$, for $j = 0, \ldots, k$. Therefore, $\text{codim}(M) = \text{codim}(M/G) = k$. We also observe that $U$ is $\text{GL}_n(k)$-invariant, because any linear transformation in $\text{GL}_n(k)$ moves a basis of $V$ into another one. Moreover, $\text{GL}_n(k)$ (and so $G$) acts freely on it\textsuperscript{7}: $[U/G] = U/G$.

The difference $\{[V^m/G]\} - \{V^m/G\}$ becomes

$$\begin{align*}
\{[V^m/G]\} - \{V^m/G\} &= (\{Z\} + \{U/G\}) - (\{M/G\} + \{U/G\}) \\
&= \{Z\} - \{M/G\}, \quad (3)
\end{align*}$$

\textsuperscript{7}If $\text{GL}_n(k)$ fixes an element $u = (u_1, \ldots, u_m) \in U$, then it fixes a basis of $V$ (by definition of $U$). If $M \in \text{GL}_n(k)$ fixes a basis of $V$, then $M$ is the identity map.
where \( Z \) is a stack, complement of \( U/G \) in \([V^m/G]\). Similarly to \( M/G \), \( Z \) has codimension \( k \) because both are the complement of the same object \( U/G \), but in two different environments \( V^m/G \) and \([V^m/G]\) with the same dimension. The class of the difference \( \{[V^m/G]\} - \{V^m/G\} \) is so determined by the class of these complements. We see, in the next proposition, how this implies that

\[
\{BG\} = \lim_{m \to \infty} \frac{\{V^m/G\}}{L^{mn}}.
\]

**Proposition 2.6** (Proposition 3.1 [8]). Let \( V \) be an \( n \)-dimensional faithful linear representation of \( G \). Then

a) \( \{BG\} = \frac{\{GL(V)/G\}}{\{GL(V)\}} \);

b) The image of \( \{BG\} \) in \( \hat{K}_0(Var_k) \) is equal to \( \lim_{m \to \infty} \{V^m/G\} L^{-mn} \);

**Proof.** The general linear group is a special group and we apply Lemma 2.4.a for \( G \subseteq GL(V) \): \( \{BG\} = \{GL(V)/G\}\{BGV\} \). Using Lemma 2.4.b, one gets \( \{BGV\} = \frac{1}{\{GL(V)\}} \) and so, we prove the first point.

Using formula (1) and formula (3) one has

\[
\{BG\} - \{V^m/G\} L^{-mn} = \left(\{[V^m/G]\} - \{V^m/G\}\right) L^{-mn} = \left(\{Z\} - \{M/G\}\right) L^{-mn}
\]

where \( Z \) and \( M/G \) are respectively the complement of \( U/G \) firstly seen inside of \([V^m/G]\) and then inside of \( V^m/G \). The open set \( U \) was defined just before this lemma.

We remark that \( Fil^j(K_0(Var)_k)[L^{-1}] \) is equal to \( \{X/L^i : \dim X - i \leq j\} \). Then, \( \{M/G\} L^{-mn} \) belongs to \( Fil^j(K_0(Var)_k)[L^{-1}] \) if and only if \( \dim M/G - mn \leq j \). One knows that \( \dim M/G - mn \) is equal to \( \text{codim}(M/G) = -k \) and so we prove that \( \{M/G\} L^{-mn} \) is in \( Fil^j \) for any \( j \geq -k \). Therefore, \( \lim_{m \to \infty} \{M/G\} L^{-mn} = 0 \) and, with similar arguments, \( \lim_{m \to \infty} \{Z\} L^{-mn} = 0 \). Thus, \( \{BG\} - \{V^m/G\} L^{-mn} \) converges to zero in \( \hat{K}_0(Var_k) \). \( \square \)

We want to define a cohomological maps, \( H^k \), for \( \hat{K}_0(Var_k) \). The first thing we need is a target: this is \( L_0(\text{Ab}) \). We recall its definition.

**Definition 2.7.** We denote by \( L_0(\text{Ab}) \) the group generated by isomorphism classes of finitely generated abelian groups under the relation that \( \{A \oplus B\} = \{A\} + \{B\} \). We equip \( L_0(\text{Ab}) \) with the discrete topology.

Moreover, let \( L_0^f(\text{Ab}) \) be the subgroup of \( L_0(\text{Ab}) \) generated by the classes of finite groups.
The elements of $L_0(\text{Ab})$ are essentially $\{\mathbb{Z}\}$ and $\{\mathbb{Z}/p^n\}$. In the introduction we saw that this structure has the interesting property to have elements that do not correspond to any group: for example $\{\mathbb{Z}\} - \{\mathbb{Z}/5\}$.

It is natural to define a cohomological map $H^k : K_0(\text{Var}_k) \rightarrow L_0(\text{Ab})$, by assigning to any smooth and proper $k$-variety $X$ the class of its integral cohomology group $H^k(X; \mathbb{Z})$. This map is well defined and we extend it to $\widehat{K}_0(\text{Var}_k)$ sending $(X)/L^m$ to $\{H^{k+2m}(X; \mathbb{Z})\}$, for any $X$ smooth and proper,

$$H^k : \widehat{K}_0(\text{Var}_k) \rightarrow L_0(\text{Ab}).$$

**Theorem 2.8** (Proposition 3.2.i) and ii), Proposition 3.3.ii) [8]). The following cohomological map

$$H^* : \widehat{K}_0(\text{Var}_k) \rightarrow L_0(\text{Ab})((t))$$

$$\{Y\} \mapsto \sum_{k \in \mathbb{Z}} H^k(\{Y\}) t^k.$$

is well defined. For each $k \in \mathbb{Z}$, $H^k : \widehat{K}_0(\text{Var}_k) \rightarrow L_0(\text{Ab})$ is also a continuous group homomorphism.

**Notation.** Without confusion we denote by 1, the class of a point, $L^0 = \{\ast\}$, in $\widehat{K}_0(\text{Var}_k)$ and also $1 = \{\mathbb{Z}\} \in L_0(\text{Ab})$. With this notation, $H^* (1) = 1$.

We finally introduce the object of our work.

**Definition.** The $i$-th Ekedahl invariant is $e_i(G) = H^{-i}(\{BG\}) \in L_0(\text{Ab})$. We say that $e_i(G)$ are trivial if $e_i(G) = 0$ for $i \neq 0$.

The following theorem gives the main tools (this is the main theorem in the introduction).

**Theorem 2.9** (Theorem 5.1 [10]). Let $G$ be a finite group. Let $k$ be a field of characteristic zero. Definition 1 and 2 of the Ekedahl invariants are equivalent and

a) $e_i(G) = 0$, for $i < 0$;

b) $e_0(G) = \{\mathbb{Z}\}$;

c) $e_1(G) = 0$;
d) $e_2 (G) = \{B_0 (G)^{\vee}\},$ where $B_0 (G)^{\vee}$ is the dual of the Bogomolov multiplier of the group $G$;

e) $e_i (G) \in L^i_0 (\text{Ab}),$ for $i > 0.$

Proof. The proof is technical but very interesting. We refer to Theorem 5.1 in [10] for the point e).

We consider an $n$-dimensional faithful representation $V$ of $G.$ We have seen in Proposition 2.6.b) that $\{BG\} = \lim_{m \to \infty} \{V^m / G\}L^{-mn} \in K_0 (\text{Var}_k).$ From Theorem 2.8, the map $H^k$ is continuous and so, for $m$ large enough,

$$H^{-i} (\{BG\}) = \{H^{2mn-i} (\{V^m / G\}; \mathbb{Z})\},$$

where the shifting $2mn-i$ comes from the multiplication for $L^{-mn}$ of $\{V^m / G\}L^{-mn}$.

We, now, consider a compactification and resolution of the singularities of $V^m / G.$ This allows to write $\{V^m / G\}$ as a suitable sum $\{X\} + \sum_j n_j \{X_j\}$ where $\{X\}$ is smooth, projective and birational to $V^m / G,$ the $X_j$’s are smooth and proper with dimension strictly less then $\dim (V^m / G) = mn$ and $n_j \in \mathbb{Z}.$ Therefore,

$$e_i (G) = H^{-i} (\{BG\}) = \{H^{2mn-i} (X; \mathbb{Z})\} + \sum_j n_j \{H^{2mn-i} (X_j; \mathbb{Z})\}.$$ 

This proves that the two definitions are equivalent.

Let $i = 0.$ The only surviving cohomology is $H^{2mn} (X; \mathbb{Z}) = \mathbb{Z},$ because $\dim (X_j) < \dim (V^m / G) = mn.$ Thus, $e_0 (G) = H^0 (\{BG\}) = 1.$

If $i = 1,$ for similar reasons, $e_1 (G) = H^{-1} (\{BG\}) = \{H^{2mn-1} (X; \mathbb{Z})\}.$ Since $X$ is birational to $V^m / G,$ one has the inclusion $k(X) \simeq k(V^m / G) \subseteq k(V).$ This implies that $X$ is unirational$^8$ and, therefore, simply connected$^9.$ Hence, $H_1 (X; \mathbb{Z}) \simeq H^{2mn-1} (X; \mathbb{Z}) = 0$ and thus $e_1 (G) = 0.$

Regarding $e_2 (G),$ one firstly observe, by Poincaré duality, that

$$\text{tor}(H^{2mn-2} (X; \mathbb{Z})) \cong \text{tor}(H_3 (X; \mathbb{Z})).$$

Artin and Mumford [2] have proved that $\text{tor}(H^3 (X; \mathbb{Z}))$ is a birational invariant and Bogomolov [4] proved that this is exactly $B_0 (G)^{10}.$

Up to now, we have proved that $e_2 (G) = \{B_0 (G)^{\vee}\} + \alpha \{\mathbb{Z}\}$ for $\alpha \in \mathbb{Z}.$ The point e) of the statement also implies that $\alpha = 0.$

---

$^8$A scheme $X$ is unirational if there exists an embedding $k(X) \subseteq k(x_1, \ldots, x_n).$

$^9$In [22], Serre proved that if $V$ is a projective and nonsingular unirational variety then the fundamental group $\pi_1 (V)$ is zero.

$^{10}$He proved, in Theorem 1.1, that if $X$ is smooth, projective and unirational the Brauer
Therefore, one gets Proposition 1.3. We recall it.

**Proposition 2.10 (Non triviality).** There exist groups of order $p^9$ [21] and groups of order $p^6$ [4], such that the second Ekedahl invariant is non trivial, $e_2(G) \neq 0$. Moreover in these cases, $\{BG\} \neq 1$ in $\hat{K}_0(\text{Var}_k)$.

**Observation 2.11.** Saltman and Bogomolov counterexamples to the Noether problem need a field $k$ such that its characteristic is coprime to $p$. In this sections $\text{char}(k) = 0$ so we lose this condition.

There are no examples in literature, to the authors knowledge, of finite group $G$ such that $B_0(G) = 0$ and $e_3(G) \neq 0$.

---

*group $Br_v(\mathbb{K})$ is isomorphic to $\text{tor}(H^3(X;\mathbb{Z}))$, with $\mathbb{K} = k(X)$. Moreover he defined $Br_v(G) = Br_v(k(X))$, where $X$ is a smooth, proper and birational to $V/\mathcal{G}$ with $V$ being any generically free representation of $G$. Thus, in Theorem 3.1, he has proved that $Br_v(G) = B_0(G)$.*
3 The finite subgroups of $\text{GL}_3$

Given a $G$-representation $V$, Proposition 2.5 links $\{BG\}$ to $\{[V]/G\}$ and $\{[P(V)/G]\}$. To get information about $\{BG\}$ one can think to study directly $\{[V]/G\}$ and the results in Section 3 of [10] are in this direction\(^{11}\).

Here, instead, we study $\{[P(V)/G]\}$. We prove that $\{[P(V)/G]\} = \{[P(V)/H]\}$, where $H$ is the reduction of $G$ in $\text{PGL}_n$. Similarly as $\{[V]/G\} - \{V/G\}$, we compare the classes $\{[P(V)/H]\}$ and $\{[P(V)/G]\}$ and we prove that the Ekedahl invariants for every finite group $G$ in $\text{GL}_3(\mathbb{C})$ are trivial. In the end we set up some facts we are going to use in the next sections.

**Notation.** From now on, we set $k = \mathbb{C}$.

**Notation.** For all this section, $G$ is a subgroup of $\text{GL}_n$ and $H$ is its reduction in $\text{PGL}_n$:

$$
0 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 0
$$

$$
0 \longrightarrow \mathbb{G}_m \longrightarrow \text{GL}_n \longrightarrow \text{PGL}_n \longrightarrow 0.
$$

**Notation.** Let $Y$ be a smooth quasi-projective algebraic variety and let $A$ be a finite group of automorphisms of $Y$. Let $T_y Y$ be the tangent space of $Y$ at the point $y$. We denote by $Y \longrightarrow Y/A$ the canonical quotient map and by $\bar{y} = \rho(y)$.

**Proposition 3.1.** $\{[P(V)/G]\} = \{[P(V)/H]\} \in \widehat{K}_0(\text{Var}_k)$.

**Proof.** Let $V$ be an $n$-dimensional faithful representation of $G$ and we denote by $V_0 = V \setminus \{O\}$, where $O$ is the origin of $V$. Since $[V_0/G] \rightarrow [P(V)/G]$ is a $\mathbb{G}_m$-torsor, $\{[V_0/G]\} = (\mathbb{L} - 1)\{[P(V)/G]\}$. Similarly from $([V_0/K]/H) \rightarrow [P(V)/H]$, one gets $\{[V_0/K]/H\} = (\mathbb{L} - 1)\{[P(V)/H]\}$. The statement follows from $[V_0/K]/H = [V_0/G]$.

\(^{11}\)Theorem 3.4 in [10] shows that

$$\{BG\} \mathbb{L}^n = \{U/G\} + \sum_{\text{flag } f} (-1)^{n_f} \{\mathcal{B}N_G(f)\} \mathbb{L}^{d_f},$$

where $U$ is the trivial stabilizer open set of $V$, the sum runs over the $G$-conjugacy class representative of strict non trivial stabilizer flag of $V$ and $N_G(f)$ is the normalizer of the flag $f$. We refer to Section 3 of [10] for the missing details. Anyway, $\{BG\}$ always appears on the right hand side and this makes this approach difficult.
Before going on we give some basic definitions regarding the quotient of algebraic varieties by finite groups.

**Definition 3.2.** A non-zero element $M$ in $GL(V)$ is a pseudo-reflection if it fixes a codimension one subspace of $V$. The pseudo reflection subgroup of $G \subset GL(V)$, $\mathcal{H}(G)$, is its subgroup generated by pseudo-reflection. The group $G$ is called a pseudo-reflection group, if $G = \mathcal{H}(G)$.

**Theorem 3.3** (Chevalley-Shephard-Todd). Let $V$ be a finite dimensional vector space over a field $\mathbb{F}$. Let $G$ be a finite subgroup of $GL(V)$, such that the characteristic of the field $\mathbb{F}$ does not divide the order of the group, $|G|$.

Then, $G$ is a pseudo-reflection group if and only if $\mathbb{F}[V^\vee]^G$ is a polynomial algebra.

One way to rephrase the previous theorem is to say that the quotient $V/G$ is smooth if and only if $G = \mathcal{H}(G)$. We call $\text{Sing}(V/G)$ the singular points of $V/G$.

We denote by $\text{Stab}_y(A)$ the stabilizer of $y \in Y$, that is the group of elements in $A$ that fix $y$.

**Lemma 3.4** (Cartan). For all the points $y$ of $Y$, the action of $\text{Stab}_y(A)$ on $Y$ induces an action of $\text{Stab}_y(A)$ on $T_yY$. Moreover the analytic germe $(Y/A, \bar{y})$ is isomorphic to $(T_yY/A, \bar{O})$, where $\bar{O}$ is the image of the origin $O \in T_yY$ under the quotient map $T_yY \to T_yY/A$.

Cartan's lemma links the study of the singularities of $Y/A$ to the study of $T_yY/A$, that is $C^{\dim(Y))/A}$. Moreover it gives the following results.

**Corollary 3.5.** For all the points $y$ of $Y$, $\text{Stab}_y(A) \subseteq GL_{\dim(Y)}$.

It is possible to prove that $p \in \text{Sing}(V/G)$ if and only if $\mathcal{H}(\text{Stab}_p(G)) \neq \text{Stab}_p(G)$. In addiction, we give another useful well known fact.

**Definition 3.6.** We say that $y$ in $Y$ is a simplicial toroidal singularity if it is locally isomorphic (in the analytic topology) to the origin in a simplicial toric affine variety.

**Lemma 3.7.** Let $\bar{y} \in Y/A$. It is a simplicial toroidal singularity if and only if the quotient $\text{Stab}_y(A)/\mathcal{H}(\text{Stab}_y(A))$ in $T_yY$ is abelian.

Point 4) in Proposition 1.1 shows that if $G$ is a finite subgroup of $GL_1$ then $\{BG\} = 1$ and so its Ekedahl invariants are trivial. We extend this to any finite group in $GL_3(\mathbb{C})$. 19
Proposition 3.8. Let $G$ be a finite subgroup of $\text{GL}_2(\mathbb{C})$. Then $\{BG\} = 1$ and the Ekedahl invariants of $G$ are trivial.

Proof. Let $U$ be the open subset of $\mathbb{P}^1$ where $H$ acts freely. Then

$$\{[\mathbb{P}^1/H]\} = \{U/H\} + \sum_p \{[p/\text{Stab}_p(H)]\}$$

$$= \{U/H\} + \sum_p \{\mathcal{B}\text{Stab}_p(H)\}.$$ 

where the sum runs over the points with non trivial stabilizer and we have used that $[p/\text{Stab}_p(H)] = \mathcal{B}\text{Stab}_p(H)$. Similarly $\{\mathbb{P}^1/H\} = \{U/H\} + \sum_p \{*\}$ and so

$$\{[\mathbb{P}^1/H]\} = \{\mathbb{P}^1/H\} + \sum_p (\{\mathcal{B}\text{Stab}_p(H)\} - \{*\}).$$

Using (in order) the equation (2), Proposition 3.1, the previous formula and $\mathbb{P}^1/H \cong \mathbb{P}^1$, one has

$$\{BG\}(1 + \mathbb{L}) = \{[\mathbb{P}^1/G]\} = \{[\mathbb{P}^1/H]\} = \{\mathbb{P}^1/H\} + \sum_p (\{\mathcal{B}\text{Stab}_p(H)\} - \{*\})$$

Using the Corollary 3.5, $\text{Stab}_p(H)$ is a subset of $\text{GL}_1$ and, hence, for Proposition 1.1.4), $\{\mathcal{B}\text{Stab}_p(H)\} = 1$ for every non trivial stabilizer point $p$. Hence, $\{BG\}(1 + \mathbb{L}) = \{\mathbb{P}^1\}$ and this implies $\{BG\} = 1$, because $\mathbb{L}^n - 1$ is invertible in $\widetilde{K}_0(\text{Var}_k)$, $\mathbb{L}^2 - 1 = (\mathbb{L} - 1)(\mathbb{L} + 1)$ and so $1 + \mathbb{L}$ is invertible too.

Proposition 3.9. Let $G$ be a finite subgroup of $\text{GL}_3(\mathbb{C})$. Then $\{BG\} = 1$ and the Ekedahl invariants of $G$ are trivial.

Proof. Using equation (2) and Proposition 3.1, we know that

$$\{BG\}\{\mathbb{P}^2\} = \{[\mathbb{P}^2/H]\}.$$ 

Since $\{\mathbb{P}^2\}$ is invertible in $\widetilde{K}_0(\text{Var}_k)$, it is sufficient to prove that $\{[\mathbb{P}^2/H]\} = \{\mathbb{P}^2\}$. 

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Let $U$ be the open subset of $\mathbb{P}^2$ where $H$ acts freely and let $C$ be the complement of $U$ in $\mathbb{P}^2$. We denote by $C_0$ and $C_1$ respectively the dimension zero and the dimension one closed subsets of $C$ so that $C = C_0 \sqcup C_1$.

We first focus on $C_0$. One observes that $[C_0/H]$ is the disjoint union of a finite number of quotient stacks $[O_i/H]$ where $O_i$ is the orbits of $P_i \in C_0$ under the action of $H$. We note that $[O_i/H] = [P_i/\text{Stab}_{P_i}(H)] = \mathcal{B}\text{Stab}_{P_i}(H)$ where $\text{Stab}_{P_i}(H)$ is the stabilizer of $P_i$ in $H$. For Corollary 3.5, $\text{Stab}_{P_i}(H)$ is a subgroup of $\text{GL}_2(\mathbb{C})$ and then, using the previous proposition, $\{[O_i/H]\} = \{\mathcal{B}\text{Stab}_{P_i}(H)\} = \{\ast\} = 1$. Therefore $\{[C_0/H]\} = \{\ast\} = 1$.

Let us study $[C_1/H]$. The set $C_1$ is the union of a finite number of lines $L_i$. We denote by $I$, the subset of $C_1$, made by the intersection points of those lines $L_i$. We call $C^*_1$ be the complement of $I$ in $C_1$.

Let $L$ be a line in $C_1$ and $S_L = \text{Stab}_L(H) = \{g \in H : gL \subseteq L\}$. Since $H \subset \text{PGL}_3$, then one sees that $S_L \subset \text{Stab}_L(\text{PGL}_3) \cap H$. A class in $\text{Stab}_L(\text{PGL}_3)$ is $\begin{pmatrix} 1 & 0 & 0 \\ \vdots & \text{GL}_2 \end{pmatrix}$ and therefore $\text{Stab}_L(\text{PGL}_3) \cong \text{GL}_2 \ltimes \mathbb{C}^2$. The group $S_L$ stabilizes $L$ and so there exists a morphism $S_L \xrightarrow{\alpha} \text{PGL}_2$.

Since $S_L \subset \text{Stab}_L(\text{PGL}_3)$, $\alpha$ factors through the following:

$$
\begin{align*}
\text{GL}_2 \ltimes \mathbb{C}^2 & \to \text{GL}_2 \to \text{PGL}_2 \\
(g, x) & \mapsto g \mapsto [g]
\end{align*}
$$

We note that the kernel of the map $\text{GL}_2 \ltimes \mathbb{C}^2 \to \text{GL}_2$ restricted to $S_L$ is trivial, because the $\ker(\text{GL}_2 \ltimes \mathbb{C}^2 \to \text{GL}_2) = \mathbb{C}^2$ and then $\ker(S_L \to \text{GL}_2) = S_L \cap \mathbb{C}^2 = 0$. Thus, $S_L \subset \text{GL}_2$ and, using the previous proposition, one gets $\{[L/s_L]\} = \{L/s_L\}$.

We set $L' = L \cap C^*_1$. Then $[L'/s_L] = [L'/s_L] \cup [L\setminus L'/s_L]$. For what we said for the zero dimensional case $\{[L\setminus L'/s_L]\} = \{L\setminus L'/s_L\}$ and so $\{[L'/s_L]\} = \{L'/s_L\}$. We call $O_j'$ the orbit of $L_j'$ under $H$. Since $C^*_1$ is the disjoint union of a finite
number of orbits $O'_j$, then

$$
\{[C_i/H]\} = \{[C_i/H]\} + \{[I/H]\}
$$

$$
= \sum_j \{[O'_j/H]\} + \{[I/H]\}
$$

$$
= \sum_j \{[L'_j/H]\} + \{[I/H]\}
$$

$$
= \sum_j \{L'_j/H\} + \{I/H\}
$$

$$
= \{C_1/H\}.
$$

Summarizing the proven facts, one has

$$
\{[\mathbb{P}^2/H]\} = \{[U/H]\} + \{[C_0/H]\} + \{[C_1/H]\}
$$

$$
= \{U/H\} + \{C_0/H\} + \{C_1/H\}
$$

$$
= \{\mathbb{P}^2/H\}.
$$

Therefore there remains to prove that $\{\mathbb{P}^2/H\} = \{\mathbb{P}^2\}$. For this reason let $X$ be a resolution of the singularities of $\mathbb{P}^2/H$ that is also birational to $\mathbb{P}^2$:

$$
X \xrightarrow{\pi} \mathbb{P}^2/H
$$

$$
\downarrow \pi'
$$

$$
\mathbb{P}^2
$$

It is well know that the quotient singularities of $\mathbb{P}^2/H$ are rational singularities and the exceptional divisor $D_y$ of $y \in \text{Sing}(\mathbb{P}^2/H)$ is a tree of $\mathbb{P}^1$: we mean that each irreducible component of $D_y$ is $\mathbb{P}^1$ and the graph associated to the resolution of $y$ is a tree. This implies that $D_y = \bigcup_{j=1}^{n_y} \mathbb{P}^1$, where $n_y$ is the number of irreducible components of $D_y$. Then $\{D_y\} = n_y \{\mathbb{P}^1\} - \sum_{i,j} \{\ast\}$. Since the graph of the resolution is a tree, then there are exactly $n_y - 1$ intersection points in $\sum_{i,j} \{\ast\}$. Hence $\{D_y\} = n_y \{\mathbb{P}^1\} - (n_y - 1) = n_y L + 1$.  

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Then,
\[
\{\mathbb{P}^2/H\} = \{X\} - \sum_y (\{D_y\} - \{y\})
\]
\[
= \{X\} - \sum_y (n_yL + 1 - 1)
\]
\[
= \{X\} - L\sum_y n_y
\]
\[
= \{X\} - Ln,
\]
where \(n = \sum_y n_y\) is the number of irreducible components in the full exceptional divisor \(D = \cup_y D_y\). Similarly, one gets
\[
\{\mathbb{P}^2\} = \{X\} - \mathbb{L}m,
\]
where \(m\) is the number of irreducible components in the full exceptional divisor \(E\) of the resolution \(X \xrightarrow{\pi'} \mathbb{P}^2\).

We shall prove that \(m = n\), that is the numbers of irreducible components is equal in both the resolutions \(\pi\) and \(\pi'\). Indeed we prove that the dimension of \(H^2(X; \mathbb{Q})\) is exactly such number plus one, hence independent by the resolutions. We study, for instance, the resolution \(X \xrightarrow{\pi'} \mathbb{P}^2\) and one has the long exact sequence:
\[
\cdots \to H^1(E(X; \mathbb{Q})) \to H^1(X; \mathbb{Q}) \to H^1(V; \mathbb{Q}) \to \cdots
\]
where \(V\) is the open set of \(X\) such that \(\pi'|_V\) is an isomorphism. One gets,
\[
\cdots \to H^1(V; \mathbb{Q}) \to H^2(E(X; \mathbb{Q})) \to H^2(X; \mathbb{Q}) \to H^2(V; \mathbb{Q}) \to H^3(X; \mathbb{Q}) \to \cdots
\]
Using Lemma 5.1, one proves that \(H^1(V; \mathbb{Q}) = 0\) and \(H^2(V; \mathbb{Q}) = \mathbb{Q}\). Moreover \(H^2(E(X; \mathbb{Q})) = \mathbb{Q}^m\) and \(H^3(X; \mathbb{Q}) = 0\). Indeed one defines, for any \(i\) and for \(k > 0\) the second quadrant spectral sequence
\[
E^{k,i}_{1} = \bigoplus_{i_1 < \cdots < i_k} H^i_{E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}}(X; \mathbb{Q})
\]
converging to \(H^*(X; \mathbb{Q})\). In the \(E_1\)-terms, the diagonal converging to \(H^2_E(X; \mathbb{Q})\) is everywhere zero a part \(E_1^{-1,2} = \oplus_j H^2_{E_j}(X; \mathbb{Q}) = \mathbb{Q}^m\) (each \(E_j \cong \mathbb{P}^1\)). No
differentials will change $E_1^{-1,2}$ and so $H^2_E (X; \mathbb{Q}) = E_\infty^{-1,2} = \mathbb{Q}^m$. Instead the differential
\[ d_1 : \oplus_{i,j} H^4_{E_i \cap E_j} (X; \mathbb{Q}) \to \oplus_j H^4_{E_j} (X; \mathbb{Q}) \]
is injective and $E_\infty^{-2,4} = E_2^{-2,4} = 0$. In the $E_2$-terms, the diagonal converging to $H^3_D (X; \mathbb{Q})$ is everywhere zero and so $H^3_D (X; \mathbb{Q}) = 0$.

Thus we obtain $0 \to \mathbb{Q}^m \to H^2 (X; \mathbb{Q}) \to \mathbb{Q} \to 0$ and $H^2 (X; \mathbb{Q}) = \mathbb{Q}^{m+1}$. \(\square\)

The difficulties for $G \subset GL_4$ arise from the study of the resolution of $\mathbb{P}^3/H$. In general for bigger $n$, Corollary 3.5 moves the question to the study of the quotients of the type $\mathbb{C}^{n-1}/G$ and those are not well known.

However, this is not the case if $H$ is abelian and if the singularities of $\mathbb{P}^{n-1}/H$ are zero dimensional. We show a technical lemma we need.

**Definition 3.10.** Let $X$ be a smooth and proper complex algebraic scheme. We denote by
\[ p_X(t) = \sum_{i \geq 0} \dim(H^i (X; \mathbb{Q})) t^i \]
the Poincaré polynomial of $X$. For a complex algebraic scheme $X$ (not necessarily smooth and proper) we define the virtual Poincaré polynomial as the polynomial $p_X(t)$ under the constraints $p_X(t) = p_Y(t) + P_{X\setminus Y}(t)$ for every closed subvariety $Y$ in $X$.

We observe that $\beta^i(X) = \dim(H^i (X; \mathbb{Q}))$ is the $i$-th Betti number of $X$. For instance, $p_{\mathbb{P}^n}(t) = 1 + t^2 + \cdots + t^{2n}$. The next two lemmas offer other examples.

**Lemma 3.11.** Let $Y$ be a smooth projective toric variety, then the odd degree coefficients of $p_Y(t)$ are zero.

*Proof.* See Section 5.2 of [12]. \(\square\)

**Lemma 3.12.** Let $G$ be a finite subgroup of $GL_n$ such that $H$ is abelian. Then $p_{\mathbb{P}^{n-1}/H}(t) = p_{\mathbb{P}^{n-1}}(t)$.

*Proof.* $H^* (\mathbb{P}^{n-1}/H; \mathbb{Q}) = H^* (\mathbb{P}^{n-1}; \mathbb{Q})^H = H^* (\mathbb{P}^{n-1}; \mathbb{Q})$. \(\square\)

**Lemma 3.13.** Let $G$ be a finite subgroup of $GL_n$. Let $\mathbb{P}^{n-1}/H$ have only zero dimensional singularities. Let $X$ be a smooth and proper resolution of $\mathbb{P}^{n-1}/H$. If $H$ is abelian, then:
i) \( \{BG\}(1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1}) = \{\mathbb{P}^{n-1}/H\} \) and, in particular,
\[
e_k (G) + e_{k+2} (G) + \cdots + e_{k+2(n-1)} (G) = H^{-k} (\{\mathbb{P}^{n-1}/H\}).
\]

ii) Every singularity of \( \mathbb{P}^{n-1}/H \) is a toroidal singularity and
\[
\{\mathbb{P}^{n-1}/H\} = \{X\} - \sum_y (\{D_y\} - \{y\}),
\]
where the sum runs over \( y \in \text{Sing} (\mathbb{P}^{n-1}/H) \); \( \{D_y\} \) is the exceptional toric divisor of \( y \) with irreducible components decomposition \( D_y = D^1_y \cup \cdots \cup D^r_y \); \( \{D_y\} \) = \( \sum_{q \geq 1} (-1)^{q+1} \sum_{i_1, \ldots, i_q} \{D^i_{y_1} \cap \cdots \cap D^i_{y_q}\} \).

iii) If \( k \neq 0 \), one has
\[
1 = \beta^k (X) - \sum_y \sum_{q \geq 1} (-1)^{q+1} \sum_{i_1, \ldots, i_q} \beta^k (D^i_{y_1} \cap \cdots \cap D^i_{y_q})
\]
and, for \( k = 0 \),
\[
1 = \beta^0 (X) - \sum_y \sum_{q \geq 1} (-1)^{q+1} \sum_{i_1, \ldots, i_q} (\beta^0 (D^i_{y_1} \cap \cdots \cap D^i_{y_q}) - 1).
\]

iv) \( \beta^{\text{odd}} (X) = 0 \).

**Proof.** By assumptions \( \mathbb{P}^{n-1}/H \) has only zero dimensional singularities. Regarding item i) we observe that
\[
\{\mathbb{P}^{n-1}/H\} = \{\mathbb{P}^{n-1}/H\} + \sum_j (\{B\text{Stab}_{P_j} (H)\} - \{\ast\})
\]
where the sum runs over the orbits of points with nontrivial stabilizer in \( \mathbb{P}^{n-1} \) and \( P_j \) is a point in such an orbit. Every stabilizer group of \( H \) is abelian and so using Proposition 1.1, we know that \( \{B\text{Stab}_{P_j} (H)\} = 1 \) and so \( \{\mathbb{P}^{n-1}/H\} = \{\mathbb{P}^{n-1}/H\} \). Using also Proposition 3.1, we obtain the first part of i). For the second one, we note that applying the cohomological map \( H^{-k} (\cdot) \) on the left hand side, one has:
\[
H^{-k} (\{BG\}(1 + \cdots + \mathbb{L}^{n-1})) = H^{-k} (\{BG\}) + \cdots + H^{-k} (\{BG\}\mathbb{L}^{n-1})
\]
\[
= H^{-k} (\{BG\}) + \cdots + H^{-k-2(n-1)} (\{BG\})
\]
\[
= e_k (G) + \cdots + e_{k+2(n-1)} (G).
\]
Every stabilizer group of $H$ is abelian and so it is for the quotient of $\text{Stab}_x(H)$ modulo $\mathcal{H}(\text{Stab}_x(H))$ in $T_xX$. Then, for Lemma 3.7, each singularity of $\{\mathbb{P}^{n-1}/H\}$ is an isolated simplicial toroidal singularities. One produces a toric resolution with normal crossing toric exceptional divisors [12]. We mean that calling $D_y$ the exceptional divisor of the resolution of the toroidal singularity $y$ in $\mathbb{P}^{n-1}/H$, $D_y = D^1_y \cup \cdots \cup D^r_y$ and each intersection $D^1_y \cap \cdots \cap D^q_y$ is an irreducible smooth toric varieties. Hence, one has equation (4) and $\{D_y\} = \sum_{q \geq 1} (-1)^{q+1} \sum_{i_1, \ldots, i_q} \{D^1_y \cap \cdots \cap D^q_y\}$. Thus, $p_{D_y}(t) = \sum_{q \geq 1} (-1)^{q+1} \sum_{i_1, \ldots, i_q} p_{D^1_y \cap \cdots \cap D^q_y}(t)$ and, using Lemma 3.11, the odd degree coefficients of $p_{D_y}(t)$ are zero.

We want to compute the virtual Poincaré polynomial of $X$. Via formula (4) and thus, using Lemma 3.12, we have

$$p_{\mathbb{P}^{n-1}}(t) = p_X(t) - \sum_y (p_{D_y}(t) - 1).$$

Comparing, degree by degree, the polynomial in the left hand side and in the right hand side, one gets the Betti numbers equalities and item iv).

We are so ready for the results we wanted:

**Theorem 3.14.** Let $G$ be a finite subgroup of $\text{GL}_n$. Let $\mathbb{P}^{n-1}/H$ have only zero dimensional singularities. Let $X$ be a smooth and proper resolution of $\mathbb{P}^{n-1}/H$. If $H$ is abelian, then

$$e_k(G) + e_{k+2}(G) + \cdots + e_{k+2(n-1)}(G) = \{H^{-k}(X; \mathbb{Z})\}.$$

**Proof.** From the first item of the previous lemma, we know that

$$e_k(G) + e_{k+2}(G) + \cdots + e_{k+2(n-1)}(G) = H^{-k}(\{\mathbb{P}^{n-1}/H\}).$$

To compute $H^{-k}(\{\mathbb{P}^{n-1}/H\})$, we study a resolution of the singularities of $\{\mathbb{P}^{n-1}/H\}$. Using the previous technical lemma we express $\{\mathbb{P}^{n-1}/H\}$ in (4) as a sum of smooth and proper varieties and $\{D_y\} = \sum_{q \geq 0} (-1)^{q+1} \sum_{i_1, \ldots, i_q} \{D^1_y \cap \cdots \cap D^q_y\}$ where $D_y$ is the exceptional divisor of the resolution of the singularity $y$ in $\{\mathbb{P}^{n-1}/H\}$. Moreover $D_y = D^1_y \cup \cdots \cup D^r_y$, where $D^i_y$ and each intersection $D^1_y \cap \cdots \cap D^q_y$ are irreducible smooth toric varieties.

If $k > 0$ or $k < -2(n-2)$ (for dimensional reason) and if $k$ is odd and between $0 \leq k \leq -2(n-2)$ (for Lemma 3.11), $H^{-k}(\{D_y\} - \{y\}) = 0$ and so the theorem is trivially true.
It remains the case $0 \leq k = 2j \leq -2(n-2)$. For these values, in the left hand side there are some negative Ekedahl invariants (so zero), $e_0(G)$ and some positive even Ekedahl invariants $e_2(G) + \cdots + e_{2(n-1)+j}(G) \in L_0^j(\mathbb{A}^n)$ (see Theorem 2.9.e).

On the right hand side the only possible torsion part is $\{\text{tor } H^{-k}(X; \mathbb{Z})\}$, because the cohomologies of a smooth toric variety is torsion free (see Section 5.2 in [12]). Hence, what remains to prove is that the free parts cancel each others: for $k \neq 0$,

$$\{Z\} = \beta^{-k}(X)\{Z\} - \sum_y \sum_{q \geq 1} (-1)^{q+1} \sum_{i_1, \ldots, i_q} \beta^{-k}(D_{i_1} \cap \cdots \cap D_{i_q}) \{Z\}$$

and

$$\{Z\} = \beta^0(X)\{Z\} - \sum_y \sum_{q \geq 1} (-1)^{q+1} \sum_{i_1, \ldots, i_q} (\beta^0(D_{i_1} \cap \cdots \cap D_{i_q}) - 1) \{Z\}.$$ 

These follow from the previous Lemma.  \qed
4 The discrete Heisenberg group $H_p$

We define the discrete Heisenberg Group $H_p$ as the following subgroup of $GL_3(\mathbb{F}_p)$:

$$H_p = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\}.$$ 

In this section we draw a method to calculate the Ekedahl invariants for $H_p$. From a $p$-dimensional faithful complex representation of $H_p$, we construct the scheme $\mathbb{P}^{p-1}/A_p$, where $A_p$ is the quotient group of $H_p$ modulo its center, $Z_{H_p}$. Since $A_p$ is abelian, using Theorem 3.14, we get

$$e_k(G) + e_{k+2}(G) + \cdots + e_{k+2(p-1)}(G) = \{H^{-k}(X_p; \mathbb{Z})\},$$

where $X_p$ is a smooth and projective resolution of $\mathbb{P}^{p-1}/A_p$. Because of Theorem 2.9, $e_0(H_p) = \{\mathbb{Z}\}$ and $e_1(H_p) = e_2(H_p) = 0$. Thus, the first possible unknown Ekedahl invariants for $H_p$ is $e_3(H_p)$. We show that

$$e_3(H_5) = e_4(H_5) = \{\text{tor}(H^5(X_5; \mathbb{Z}))\}.$$ 

We narrow down our investigation to $p = 5$ because of some difficulties in some cohomological computations. Therefore,

**Theorem.** The Ekedahl invariants for the group $H_5$ are trivial.

In Section 4.1, we define $\mathbb{P}^{p-1}/A_p$ and we show that it has $p + 1$ zero dimensional isolated toroidal singularities. In Section 4.2 we prove the main result of the work, Theorem 4.11, modulo two computational facts we deal with in the next sections: In Section 5, we show that $H^{\text{odd}}(U_p; \mathbb{Z}) = 0$ where $U_p$ is the smooth open set of $\mathbb{P}^{p-1}/A_p$; in Section 6 we construct the resolution $X_p \xrightarrow{f} \mathbb{P}^{p-1}/A_p$. Let $E$ be the union of exceptional divisors $D_y$ for $y \in \text{Sing}(\mathbb{P}^{p-1}/A_p)$. In Section 7, we calculate that the cohomologies, $H^5_E(X_p; \mathbb{Z})$ are torsion free.

4.1 The scheme $\mathbb{P}^{p-1}/A_p$

It is useful to note that $H_p$ is generated by

$$\mathcal{X} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{Z} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

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modulo the relations \( Z \mathcal{Y} \mathcal{X} = \mathcal{X} \mathcal{Y} \), \( Z^p = \mathcal{X}^p = \mathcal{Y}^p = \text{Id}, \) \( Z \mathcal{X} = \mathcal{X} Z \) and \( Z \mathcal{Y} = \mathcal{Y} Z \). The center of \( H_p \), \( Z H_p \), is generated by \( Z \) and we denote by \( A_p \) the group quotient \( H_p/Z H_p \). From these, one gets that \( A_p \cong (\mathbb{Z}/p\mathbb{Z})^2 \).

**Lemma 4.1.** The discrete Heisenberg group \( H_p \) is the central extension of \( \mathbb{Z}/p\mathbb{Z} \) by \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \):

\[
1 \to \mathbb{Z}/p\mathbb{Z} \to H_p \overset{\phi}{\to} \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to 1.
\]

**Lemma 4.2** ([4], Lemma 4.9). Let \( G \) be an extension of a cyclic group \( H \) by an abelian group \( Q \). Then \( B_0(G) = 0 \).

**Corollary 4.3.** The Bogomolov multiplier \( B_0(H_p) = 0 \), for every prime \( p \).

The discrete Heisenberg group has \( p^2 + p - 1 \) irreducible complex representations: \( p^2 \) of them are one dimensional and the remaining \( p - 1 \) are faithful and \( p \)-dimensional (see Section 12 of [9] or [16]).

Let \( V \) be a faithful irreducible \( p \)-dimensional complex representation of \( H_p \), \( \rho : H_p \to \text{GL}_p(\mathbb{C}) \).

**Notation.** For simplicity we use \( \mathbb{P}^{p-1} \) instead of \( \mathbb{P}(V) \). We also denote by \( \text{Id}_n \) the identity element of \( \text{GL}_n(\mathbb{C}) \).

There is a natural action of \( H_p \) on \( V \) and it induces an action on \( \mathbb{P}^{p-1} \). One so defines the quotient \( \mathbb{P}^{p-1}/H_p \).

Since \( Z \) belongs to the center, \( \rho(Z) = e^{2\pi i \frac{p}{p}} \text{Id}_p \), for some \( 0 < i < p \). Hence, the center acts trivially on \( \mathbb{P}^{p-1} \) and \( \mathbb{P}^{p-1}/H_p \cong \mathbb{P}^{p-1}/A_p \). We remark some notations that we have introduced in Section 3.

**Remark 4.4.** \( \mathcal{H}(G) \) is the pseudo-reflection subgroup of \( G \). The set of the singular points of \( \mathbb{P}^{p-1}/A_p \) is \( \text{Sing}(\mathbb{P}^{p-1}/A_p) \). The stabilizer of \( x \in \mathbb{P}^{p-1} \), \( \text{Stab}_x(A_p) \) is the group of elements in \( A_p \) that fix \( x \). We denote by \( \mathbb{P}^{p-1} \overset{\pi}{\rightarrow} \mathbb{P}^{p-1}/A_p \) the canonical quotient map and by \( \bar{x} = \pi(x) \).

Finally, we have seen that \( \bar{x} \in \text{Sing}(\mathbb{P}^{p-1}/A_p) \) if and only if \( \mathcal{H}(\text{Stab}_x(A_p)) \neq \text{Stab}_x(A_p) \) in the tangent space \( T_{\bar{x}}\mathbb{P}^{p-1} \) of \( \mathbb{P}^{p-1} \) at \( x \), where \( \text{Stab}_x(A_p) \) acts on \( T_{\bar{x}}\mathbb{P}^{p-1} \) in a natural way.

From Lemma 3.7, we know that if \( \mathbb{P}^{p-1}/A_p \) has singularities, then they are toroidal. We study these singularities and so we focus on \( \text{Stab}_x(A_p) \).

**Proposition 4.5.** Let \( x \in \mathbb{P}^{p-1} \). If the action of \( A_p \) at \( x \) is not free, then \( |\text{Stab}_x(A_p)| = p \).
Proof. Let \( W_x \) be the one dimensional subvector-space of \( V \) corresponding to \( x \). The stabilizer of \( x \) is a subgroup of \( A_p \) and, by Lagrange’s Theorem (and using the assumptions), it could have order \( p \) or \( p^2 \). If \( \text{Stab}_x(A_p) = A_p \), then for every \( g \in A_p \), \( gW_x = W_x \) and \( A_pW_x = W_x \). Then \( H_pW_x = W_x \). This implies that \( W_x \) is an one dimensional irreducible \( H_p \)-subrepresentation of \( V \) against the fact that \( H_p \) acts irreducibly.

There are exactly \( p + 1 \) subgroups of order \( p \) in \( A_p \). Let \( B \) be one of them. We define \( \hat{B} \) as a subgroup of \( H_p \) such that \( \phi|_{\hat{B}} \) is a group isomorphism:

\[
H_p \xrightarrow{\phi} A_p \rightarrow 1
\]

\[
\hat{B} \cong B.
\]

Observation 4.6. \( \hat{B} \cong \mathbb{Z}/p\mathbb{Z} \subset \phi^{-1}(B) \).

We restrict the representation \( H_p \xrightarrow{\rho} \text{GL}_p(\mathbb{C}) \) to the subgroup \( \hat{B} = \mathbb{Z}/p\mathbb{Z} \). Using elementary results of representation theory one writes the vector space \( V \) as a direct sum of one dimensional irreducible representations: \( V = \mathbb{C}^p = \bigoplus_{\chi \in \hat{B}^\vee} V_\chi \), where

\[
V_\chi = \{ v \in V : g \cdot v = \chi(g) \cdot v, \forall g \in G \}
\]

and \( \hat{B}^\vee \) is the dual group of \( \hat{B} \), that is \( \{ f : \hat{B} \rightarrow S^1, \text{group homomorphism} \} \) with \( S^1 \) unit circle in \( \mathbb{C} \). Since \( \hat{B} \) is commutative, each irreducible representation \( V_\chi \) is one dimensional. Moreover \( \hat{B}^\vee = (\mathbb{Z}/p\mathbb{Z})^\vee = \mathbb{Z}/p\mathbb{Z} \). Therefore, one gets the following eigenspaces decomposition

\[
V = \bigoplus_{\chi \in \mathbb{Z}/p\mathbb{Z}} V_\chi. \tag{5}
\]

In other words, \( \hat{B} \) fixes \( p \) one dimensional linear subspaces \( V_\chi \) and so \( B \) fixes \( p \) points \( P_\chi \in \mathbb{P}^{p-1} \), with \( \text{Stab}_{P_\chi}(A_p) = B \), that is \( (\mathbb{P}^{p-1})^B = \{ P_{\chi_0}, \ldots, P_{\chi_{p-1}} \} \).

Proposition 4.7. Let \( B \) and \( B' \) be two distinct \( p \)-subgroups of \( A_p \), then

\[
(\mathbb{P}^{p-1})^B \cap (\mathbb{P}^{p-1})^{B'} = \emptyset.
\]

Proof. Trivially, \( A_p = B \oplus B' \) and if \( P \in (\mathbb{P}^{p-1})^B \cap (\mathbb{P}^{p-1})^{B'} \), then \( \text{Stab}_P(A_p) = A_p \), against Proposition 4.5. \( \square \)
We observe that \( A_p/B \) acts regularly\(^{12} \) on \( (\mathbb{P}^{p-1})^B \). Thus, these points are a unique orbit under the action of \( A_p/B \) and this means that they correspond to a unique point \( y_B \) in \( \mathbb{P}^{p-1}/A_p \).

**Theorem 4.8.** \( \mathbb{P}^{p-1}/A_p \) has \( p + 1 \) simplicial toroidal singular points.

**Proof.** There are exactly \( p + 1 \) subgroups, \( B \), of order \( p \) in \( A_p \). Each of them corresponds to a point \( y_B \) in \( \mathbb{P}^{p-1}/A_p \). By Proposition 4.7, these points are distinct.

Let \( y \in \mathbb{P}^{p-1} \) such that \( \overline{y} = y_B \). We consider the action of \( \text{Stab}_y(A_p) \) on the tangent space \( T_y \mathbb{P}^{p-1} \). The pseudo-reflection group \( \mathcal{H}(\text{Stab}_y(A_p)) \) is zero, because it is a subgroup of \( \text{Stab}_y(A_p) \cong \mathbb{Z}/p\mathbb{Z} \) and, so, it is either the trivial group or \( \text{Stab}_y(A_p) \). The latter is not possible because \( \text{Stab}_y(A_p) \) stabilizes only the origin of the vector space \( T_y \mathbb{P}^{p-1} \). Thus, \( \mathcal{H}(\text{Stab}_y(A_p)) \neq \text{Stab}_y(A_p) \) in \( T_y \mathbb{P}^{p-1} \) and for Lemma 3.7 these singularities are also toroidal and simplicial.

The action of \( A_p/B \) on \( (\mathbb{P}^{p-1})^B \) is regular and using Lemma 3.4, we know that, for every \( \chi \), \( (\mathbb{P}^{p-1}/A_p, y_B) \) is locally isomorphic to \( (T_{\mathcal{P}(\chi)} \mathbb{P}^{p-1}/\text{Stab}_{\mathcal{P}(\chi)}(A_p), \overline{O}) \), where \( \overline{O} \) is the image of the origin of \( T_{\mathcal{P}(\chi)} \mathbb{P}^{p-1} \) under the quotient map \( T_{\mathcal{P}(\chi)} \mathbb{P}^{p-1} \to T_{\mathcal{P}(\chi)} \mathbb{P}^{p-1}/\text{Stab}_{\mathcal{P}(\chi)}(A_p) \).

In Section 6, we find a resolution for such toroidal points. To do it we need the explicit expression of the actions of \( B \) on the tangent space \( T_{\mathcal{P}(\chi)} \mathbb{P}^{p-1} \).

Let \( y \in \mathbb{P}^{p-1} \) and let \( W_y \) be the one dimensional subvector space of \( V \) corresponding to \( y \). Then \( T_y \mathbb{P}^{p-1} \cong \text{Hom}_\mathbb{C}(W_y, V/W_y) \). Using the vector space decomposition (5), we know that \( P_{\chi'} \) corresponds to \( V_{\chi'} \) for some character \( \chi' \in \mathbb{Z}/p\mathbb{Z} \). One has

\[ V/V_{\chi} = \bigoplus_{\chi \in \mathbb{Z}/p\mathbb{Z}, \chi \neq \chi'} V_{\chi} \]

and this implies that

\[
T_{\mathcal{P}(\chi)} \mathbb{P}^{p-1} = \text{Hom}_\mathbb{C}(V_{\chi'}, V/V_{\chi'}) = \text{Hom}_\mathbb{C}(V_{\chi'}, \oplus_{\chi \neq \chi'} V_\chi) = \bigoplus_{\chi \neq \chi'} \text{Hom}_\mathbb{C}(V_{\chi'}, V_\chi)
\]

\(^{12}\) A free action of \( G \) on a topological space \( X \) is regular if for every \( x \) and \( y \) in \( X \) there exists precisely one element \( g \) in \( G \) such that \( g \cdot x = y \).
We denote by $e_{\chi'} \cdot \chi^{-1}$ the generator of the one dimensional vector space $\text{Hom}_\mathbb{C}(V_{\chi'}, V_{\chi})$, so that $\text{Hom}_\mathbb{C}(V_{\chi'}, V_{\chi}) = \mathbb{C} e_{\chi'} \cdot \chi^{-1}$. One gets

$$TP_{\chi'_{P\mathbb{P}}-1} = \bigoplus_{1 \neq \chi \in \mathbb{Z}/p\mathbb{Z}} \mathbb{C} e_{\chi},$$

Without loss of generality let $\chi = 1^{13}$. We remark that by definition $v \in V_{\chi}$, $g \cdot v = \chi(g)v$ and then, $g \cdot e_{\chi} = \chi(g)e_{\chi}$ and this gives the action of $B$ on $TP_{\chi'_{P\mathbb{P}}-1}$.

**Theorem 4.9.** Let $V$ be a faithful irreducible $p$-dimensional complex representation of the discrete Heisenberg group $H_p$. Let $A_p$ be the quotient group $H_p/\mathbb{Z}H_p$. One has:

1. $\mathbb{P}^{p-1}/H_p \cong \mathbb{P}^{p-1}/A_p$;

2. The stabilizer group of $x \in \mathbb{P}^{p-1}$ such that $\bar{x} \in \text{Sing}(\mathbb{P}^{p-1}/A_p)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$;

3. The action of the stabilizer group, $B \cong \mathbb{Z}/p\mathbb{Z}$, on the tangent space $T_x\mathbb{P}^{p-1}$ induces the following inclusion:

$$\mathbb{Z}/p\mathbb{Z} \hookrightarrow (\mathbb{C}^*)^{p-1}$$

$$j \mapsto (\zeta^j, \zeta^{2j}, \ldots, \zeta^{(p-1)j});$$

(6)

where $\zeta$ is a $p$-root of unity.

4. $\mathbb{P}^{p-1}/A_p$ has $p+1$ zero dimensional simplicial toroidal singularities locally isomorphic to the origin of the toric affine variety $\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}$.

**Proof.** The action of $\mathbb{Z}/p\mathbb{Z}$ on the tangent space $T_x\mathbb{P}^{p-1}$ induces the following inclusion:

$$B \hookrightarrow (\mathbb{C}^*)^{p-1}$$

$$g \mapsto (\chi(g))_{\chi \in (\mathbb{Z}/p\mathbb{Z})^*};$$

Concretely, $(\mathbb{Z}/p\mathbb{Z})^* = \{\chi_j, j \in (\mathbb{Z}/p\mathbb{Z})^*\}$ and $\chi_j(i) = \zeta^j$, where $\zeta$ is a $p$-root of unity.

There remains to prove that each singularity is locally isomorphic to $\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}$. The latter is the compactification of $(\mathbb{C}^*)^{p-1}/\mathbb{Z}/p\mathbb{Z}$.

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13Then $e_{\chi', \chi^{-1}} = e_{\chi^{-1}}$ and $\mathbb{C} e_{\chi} = \text{Hom}_\mathbb{C}(V_1, V_{\chi})$
4.2 The third Ekedahl invariant of $H_p$

As we said in the beginning of this section, the first unknown Ekedahl invariant for $H_p$ is $e_3(H_p)$. The strategy to use Theorem 3.14 is to write $\{\mathbb{P}^{p-1}/A_p\}$ as a sum of classes of smooth and proper varieties.

Let $X_p \to \mathbb{P}^{p-1}/A_p$ be the resolution of the $p+1$ toroidal singularities of $\mathbb{P}^{p-1}/A_p$. We show a toric resolution in Section 6. This always exists, but we perform the computation only for $p=5$ and $p=7$.

Using Theorem 3.14 for $G = H_p$, $n = p$, $X = X_p$ and $k = -2p + 5$ and also applying Theorem 2.9.e), we have

$$e_3(G) = \{ \text{tor} (H^2_{p-5}(X_p; \mathbb{Z})) \} = \{ \text{tor} (H^4(X_p; \mathbb{Z})) \}.$$  

Similarly, for $p = 5$ and $k = -2p + 6$

$$e_4(G) = \{ \text{tor} (H^2_{p-6}(X_p; \mathbb{Z})) \} = \{ \text{tor} (H^4(X_5; \mathbb{Z})) \}.$$  

If we set $p = 5$ then $e_3(G)$ and $e_4(G)$ are the only invariants that could not be zero. Indeed,

$$e_5(G) = \{ \text{tor} (H^3(X_5; \mathbb{Z})) \} = e_2(G) = 0$$

$$e_6(G) = \{ \text{tor} (H^2(X_5; \mathbb{Z})) \} = e_1(G) = 0$$

and $e_i(G) = 0$ for $i > 6$ for dimensional reason.

**Theorem 4.10.** $\text{tor}(H^5(X_5; \mathbb{Z})) = 0$.

*Proof.* We recall the geometrical picture constructed during this section:

\[
\begin{array}{c}
\mathbb{P}(V) \leftarrow U \\
X_p \downarrow f \quad \quad \pi \quad \quad \pi \\
\quad \downarrow \phi \\
U_p \sim \quad \quad \downarrow f \\
\end{array}
\]

where

- $U$ is the open subset of $\mathbb{P}(V)$ where $A_p$ acts freely;
\[ S_p = \mathbb{P}(V) \setminus U; \]
\[ U_p = U/A_p; \]
\[ X_p \] is a smooth and projective resolution of \( \mathbb{P}^{p-1}/A_p; \)
\[ E \] is the union of exceptional divisors of the resolution \( X_p \xrightarrow{f} \mathbb{P}^{p-1}/A_p. \)

We now consider the long exact sequence
\[ \cdots \to H^*_{E}(X_5; \mathbb{Z}) \to H^*(X_5; \mathbb{Z}) \to H^*(U_5; \mathbb{Z}) \to \cdots \]
From Lemma 3.13.iii), we know that \( \beta^{odd}(X_5) = 0 \) and hence
\[ H^{odd}(X_5; \mathbb{Z}) = \text{tor}\left( H^{odd}(X_5; \mathbb{Z}) \right). \]
In Section 5, Theorem 5.12 shows that the cohomology \( H^{odd}(U_5; \mathbb{Z}) = 0. \)
Therefore, the sequence becomes
\[ \cdots \to H^5_{E}(X_5; \mathbb{Z}) \to \text{tor}(H^5(X_5; \mathbb{Z})) \to 0. \]
Theorem 7.6 says that \( H^5_{E}(X_5; \mathbb{Z}) = 0 \) and one has \( \text{tor}(H^5(X_5; \mathbb{Z})) = 0. \)

Therefore we have proved the main result.

**Theorem 4.11.** The Ekedahl invariants of the fifth discrete Heisenberg group are trivial.
5 The cohomology of the smooth open set $U_p$

In this section we compute the cohomology of the smooth open set $U_p \subseteq \mathbb{P}^{p-1}/A_p$. Since $A_p$ acts freely on $U$, we study $H^* (U_p; \mathbb{Z})$ via the Cartan-Leray spectral sequence (see Section 5 or Theorem 8 bis.9 in [19]) relative to the quotient map $\pi : U \to U_p$:

$$E_2^{i,j} = H^i (A_p; H^j (U; \mathbb{Z})) \Rightarrow H^{i+j} (U_p; \mathbb{Z}).$$

This is a first quadrant spectral sequence and to write the $E_2$-terms one need to focus on $H^* (U; \mathbb{Z})$.

**Lemma 5.1.** Let $U$ be an open space in $\mathbb{P}^{p-1}$ such that $S = \mathbb{P}^{p-1} \setminus U$ is a finite set of points. One has

$$H^i (U; \mathbb{Z}) = \begin{cases} H^i (\mathbb{P}(V); \mathbb{Z}) & \text{if } i < 2p - 3; \\ \mathbb{Z} [S]^0 & \text{if } i = 2p - 3; \\ 0 & \text{if } i > 2p - 3. \end{cases}$$

where $\mathbb{Z} [S]$ is the group freely generated over the $p(p + 1)$ points in $S$ and

$$\mathbb{Z} [S]^0 = \left\{ \sum_{x \in S} a_x x : \sum_{x \in S} a_x = 0 \right\}.$$

**Proof.** Using the Gysin exact sequence,

$$\cdots \to H^*_S (\mathbb{P}(V); \mathbb{Z}) \to H^* (\mathbb{P}(V); \mathbb{Z}) \to H^* (U; \mathbb{Z}) \to \cdots$$

and $H^i_S (\mathbb{P}(V); \mathbb{Z}) = 0$ for $i \neq 2p - 2$, we easily get that $H^i (\mathbb{P}(V); \mathbb{Z}) \cong H^i (U; \mathbb{Z})$ for $i < 2p - 3$. Moreover $H^{2p-2} (U; \mathbb{Z}) = 0$, because it is the top cohomology of a not-compact space.

Hence $H^{2p-3} (U; \mathbb{Z})$ lies in the exact sequence

$$0 \to H^{2p-3} (U; \mathbb{Z}) \to H^0 (S; \mathbb{Z}) \xrightarrow{r} H^{2p-2} (\mathbb{P}^{p-1}; \mathbb{Z}) \to 0,$$

that is

$$0 \to H^{2p-3} (U; \mathbb{Z}) \to \mathbb{Z}^{p(p+1)} \to \mathbb{Z} \to 0.$$

We know that the inclusion of $S$ into $\mathbb{P}^{p-1}$ gives a cohomological map

$$H^2_S (\mathbb{P}^{p-1}; \mathbb{Z}) \xrightarrow{r} H^{2(p-1)} (\mathbb{P}^{p-1}; \mathbb{Z}) .$$
Each point $Q \in S$ contributes with one copy of $\mathbb{Z}$ to $H^2_S(\mathbb{P}^{p-1}; \mathbb{Z})$. Let $1_Q$ be the generator of $H^2_S(\mathbb{P}^{p-1}; \mathbb{Z})$ inside of $H^2_S(\mathbb{P}^{p-1}; \mathbb{Z})$. This is mapped in the class $[Q] = 1 \in H^2(\mathbb{P}^{p-1}; \mathbb{Z})$.

Since $H^2(\mathbb{P}^{p-1}; \mathbb{Z}) = \ker(r)$ and from $r \left( \sum_{x \in S} a_x x \right) = \sum_{x \in S} a_x$, we obtain

$$H^2(\mathbb{P}^{p-1}; \mathbb{Z}) = \mathbb{Z}[S]^0 = \left\{ \sum_{x \in S} a_x x : \sum_{x \in S} a_x = 0 \right\}.$$ 

\[ \square \]

From now on we assume $p > 3$.

**Corollary 5.2.** Let $U$ be the open set of $\mathbb{P}^{p-1}$ where $A_p$ acts freely. Let $S_p = \mathbb{P}^{p-1} \setminus U$. One has

$$H^i(U; \mathbb{Z}) = \begin{cases} H^i(\mathbb{P}(V); \mathbb{Z}) & \text{if } i < 2p - 3; \\ \mathbb{Z}[S_p]^0 & \text{if } i = 2p - 3; \\ 0 & \text{if } i > 2p - 3. \end{cases}$$

**Proof.** We set $S = S_p$. \[ \square \]

Using the previous lemma we write the elements $E_{2}^{i,j}$ in Table 1.

**Lemma 5.3.** The cohomology of $A_p$ has a $\mathbb{Z}$-algebra structure:

$$H^*(A_p; \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, x_2, y]}{(y^2, px_1, px_2, py)},$$

where $\deg(x_1) = \deg(x_2) = 2$ and $\deg(y) = 3$.

\[ 14 \text{Let } Q \text{ be a point in } \mathbb{P}^{p-1}. \text{ The inclusion } Q \hookrightarrow \mathbb{P}^{p-1} \text{ induces the cohomological map } \\
H^2_Q(\mathbb{P}^{p-1}; \mathbb{Z}) \to H^2(\mathbb{P}^{p-1}; \mathbb{Z}). \]

Both cohomology groups are isomorphic to $\mathbb{Z}$ and we denote by $[Q]$ the image of 1 under this cohomological map.

$$\begin{align*}
\mathbb{Z} & \to \mathbb{Z} \\
1 & \mapsto [Q].
\end{align*}$$

This is the class of the subvariety $Q$ in $H^2(\mathbb{P}^{p-1}; \mathbb{Z})$. 

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\[
\begin{array}{cccc|cc}
2p - 2 & 0 & 0 & \cdots & 0 \\
2p - 3 & H^0(A_p; \mathbb{Z}[S_p]^0) & H^1(A_p; \mathbb{Z}[S_p]^0) & \cdots & H^i(A_p; \mathbb{Z}[S_p]^0) \\
2p - 4 & H^0(A_p; \mathbb{Z}) & H^1(A_p; \mathbb{Z}) & \cdots & H^i(A_p; \mathbb{Z}) \\
2p - 5 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & H^0(A_p; \mathbb{Z}) & H^1(A_p; \mathbb{Z}) & \cdots & H^i(A_p; \mathbb{Z}) \\
0 & H^0(A_p; \mathbb{Z}) & H^1(A_p; \mathbb{Z}) & \cdots & H^i(A_p; \mathbb{Z}) \\
\hline
j/i & 0 & 1 & \cdots & i \\
\end{array}
\]

Table 1: \( E_2^{i,j} = H^i(A_p; H^j(U; \mathbb{Z})) \)

**Proof.** We refer to Appendix A. \( \square \)

**Lemma 5.4.** One has:

\[
\begin{align*}
H^0(A_p; \mathbb{Z}[S_p]^0) &= \mathbb{Z}[S_p/A_p]^0 \cong \mathbb{Z}^p \\
H^1(A_p; \mathbb{Z}[S_p]^0) &= 0
\end{align*}
\]

**Proof.** From the exact sequence

\[
0 \to \mathbb{Z}[S_p]^0 \to \mathbb{Z}[S_p] \to \mathbb{Z} \to 0,
\]

one gets the long exact sequence in cohomology

\[
0 \to H^0(A_p; \mathbb{Z}[S_p]^0) \to H^0(A_p; \mathbb{Z}[S_p]) \to H^0(A_p; \mathbb{Z}) \to \\
\to H^1(A_p; \mathbb{Z}[S_p]^0) \to H^1(A_p; \mathbb{Z}[S_p]) \to H^1(A_p; \mathbb{Z}) \to \ldots
\]

the first row is

\[
0 \to (\mathbb{Z}[S_p]^0)_{A_p} \to (\mathbb{Z}[S_p])_{A_p} \to \mathbb{Z} \to 0,
\]

so \( H^0(A_p; \mathbb{Z}[S_p]^0) = (\mathbb{Z}[S_p]^0)_{A_p} \). This is also exact and gives \( H^1(A_p; \mathbb{Z}[S_p]^0) = 0 \).

We observe that the action of \( A_p \) on \( S_p \) gives an \( A_p \)-invariant components decomposition \( S_p = \bigcup_{a \in \mathbb{Z}/p\mathbb{Z}} S_a \). In details \( S_a \cong A_p/A_a \), where \( A_a \) is the stabilizer of the points in \( S_a \) and \( A_a \cong \mathbb{Z}/p\mathbb{Z} \). Thus \( (\mathbb{Z}[S_p]^0)_{A_p} = (\mathbb{Z}[S_p]_{A_p})^0 = (\mathbb{Z}[^{\mathbb{Z}/p\mathbb{Z}}]_{A_p})^0 \) and, thus, \( H^0(A_p; \mathbb{Z}[S_p]^0) \cong \mathbb{Z}^p \). \( \square \)
Notation. We denote by $h$ the first Chern class of $O(1)$ in $\mathbb{P}^{p-1}$. Hence $h^k$ generates $H^{2k}(\mathbb{P}^{p-1}; \mathbb{Z})$ and one has $H^*(\mathbb{P}^{p-1}; \mathbb{Z}) \cong \mathbb{Z}[h]/(h^p)$.

Using this notation one writes an element of $E_2^{i,2k}$ (for $2k < 2p - 3$) as $h^k \cdot \alpha$ where $\alpha \in H^i(A_\nu; H^{2k}(U; \mathbb{Z}))$. This is useful because it keep track of the differential $H^*(\mathbb{P}^{p-1}; \mathbb{Z})$-algebra structure (see Chapter 2 in [19]). Indeed, for $2k < 2p - 3$, $d_3(h^k) = kh^{k-1}d_3(h)$.

**Proposition 5.5.** In the Cartan-Leray spectral sequence the differential $d_2^{i,j}$ is zero if $j < 2p - 3$.

**Proof.** The differential $d_2$ moves one line down and two columns forward. As we see in Figure 1, from every line with $j < 2p - 3$, $d_2^{i,j}$ lands in a zero row.

![Figure 1: The differential $d_2(h)$](image)

**Proposition 5.6.** In the Cartan-Leray spectral sequence the differential $d_2^{i,j}$ does not modify $E_2^{0,2p-4}$ and $E_2^{1,2p-4}$.

**Proof.** Figure 2 shows that the differentials landing on $E_2^{0,2p-4}$ and $E_2^{1,2p-4}$ come from zero terms.

Table 2 shows that in the $E_3$ level nothing change a part the lines $j = 2p - 4, 2p - 3$.

**Proposition 5.7.** In the Cartan-Leray spectral sequence the differential $d_3$ is non trivial.
Proof. For the differential structure, we know that $d_3$ is given by the value of $d_3(h): h \in E_{3}^{0,2}$ and $d_3(h) \in E_{3}^{3,0}$ (see Figure 3).

We prove the statement by contradiction and we assume that $d_3(h) = 0$. Firstly, we observe that for $k > 3$, the differentials $d_k$ sets the values of $d_k(h) = 0$, because $d_k(h)$ always lands in a zero row (see Figure 4). $d_k(h)$ sets the values of $d_k(h^m)$ (then $d_k(h^m) = 0$) and for $k > 3$ the only non trivial differentials could be from the top line $j = 2p - 3$ and from the line $j = 2p - 4$ for $i > 2$.

In details, from $E_{-2p}^{i}$ the odd differentials are zero and the even ones move as follows

$$d_{2m}^{i,2p-3} : E_{2m}^{i,2p-3} \to E_{2m}^{i+2m,2p-2-2m}.$$  

Instead, from $E_{-2p}^{i,2p-4}$ the relevant differentials are the odd one:

$$d_{2m+1}^{i,2p-4} : E_{2m+1}^{i,2p-4} \to E_{2m+1}^{i+2m+1,2p-4-2m}.$$  

Let us focus on the diagonal $i + j = 2p - 1$. These terms are reached by the even differentials from the line $2p - 3$, $d_{2m}^{i,2p-3} : E_{2m}^{i,2p-3} \to E_{2m}^{i+2m,2p-2-2m}$, but $E_{2m}^{1,2p-3} = E_{2m}^{1,2p-3} = H^1(A_p; \mathbb{Z})[S_p]^0 = 0$ for Lemma 5.4. Thus, the only way to modify the terms in the diagonal $i + j = 2p - 1$ are the odd differentials $d_{2m+1}^{2,2p-4} : E_{2m+1}^{2,2p-4} \to E_{2m+1}^{2+2m+1,2p-4-2m}$. We observe that $E_{3}^{2,2p-4} = H^2(A_p; \mathbb{Z})/\text{im}(d_2^{2,2p-3}) \cong \mathbb{Z}/p^2/\text{im}(d_2^{2,2p-3})$.

We consider then $d_{2p-3}^{2,2p-4} : E_{2p-3}^{2,2p-4} \to E_{2p-3}^{2p-1,0}$, that is

$$d_{2p-3}^{2,2p-4} : \text{Ker}(d_{2p-5}^{2,2p-4}) \to H^{2p-1}(A_p; \mathbb{Z}).$$
Table 2: The level $E_{3}^{i,j}$: the gray cells could be modified by $d_{2}$.

<table>
<thead>
<tr>
<th></th>
<th>$2p-2$</th>
<th>$2p-3$</th>
<th>$2p-4$</th>
<th>$2p-5$</th>
<th>\vdots</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j/i$</td>
<td>0</td>
<td>0</td>
<td>$H^{0}(A_{p};Z)$</td>
<td>$H^{1}(A_{p};Z)$</td>
<td>\cdots</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 3: $d_{3}(h)$ belongs to $H^{3}(A_{p};Z)$

Trivially $\text{Ker}(d_{2}^{2,2p-4}) \subseteq \text{Ker}(d_{2}^{2,2p-4}) \subseteq \cdots \subseteq \text{Ker}(d_{3}^{2,2p-4}) \subseteq \frac{(Z/pZ)^{2}}{\text{Im}(d_{2}^{2,2p-3})}$ and, if $p > 3$, there is no way that $d_{2}^{2,2p-4}$ is surjective in $H^{2p-1}(A_{p};Z) \cong (Z/pZ)^{p-1}$.

Therefore $0 \neq E_{\infty}^{2p-1,0} = \frac{(Z/pZ)^{p-1}}{\text{Im}(d_{2}^{2,2p-4})} \subseteq H^{2p-1}(U_{p};Z) = 0$, for dimensional reasons. This contradicts the fact that $d_{3}(h) = 0$. \hfill \Box

We have proved that $d_{3}(h)$ is non-zero and this is an element of $H^{3}(A_{p};Z)$, generated by $y$ in $H^{*}(A_{p};Z)$ (see Lemma 5.3). Then, $d_{3}(h) = \alpha y$, where $\alpha \in Z/pZ^{*}$. The $d_{3}$ differential is a degree 3 homomorphism from $H^{*}(A_{p};Z)$
Figure 4: The differential $d_3$ in the $E_3$-terms to itself:

$$d_3^{i,2k} : H^i(A_p; \mathbb{Z}) \to H^{i+3}(A_p; \mathbb{Z})$$

$$1 \mapsto \alpha y.$$ for $i, k > 0$ (see Figure 5).

**Lemma 5.8.** Let $m > 0$ and $0 < k < p - 2$. One has

- $\ker(d_3^{2m,2k}) = 0$;
- $\ker(d_3^{2m+1,2k}) = H^{2m+1}(A_p; \mathbb{Z})$;
- $\text{Im}(d_3^{2m,2k}) = H^{2m+3}(A_p; \mathbb{Z})$;
- $\text{Im}(d_3^{2m+1,2k}) = 0$.

Moreover for $k = 0$,

- $\ker(d_3^{2m,0}) = H^{2m}(A_p; \mathbb{Z})$;
- $\ker(d_3^{2m+1,0}) = H^{2m+1}(A_p; \mathbb{Z})$;
- $\text{Im}(d_3^{2m,0}) = 0$;
- $\text{Im}(d_3^{2m+1,0}) = 0$. 
Finally for $k > 0$,
\[
\begin{align*}
\ker(d_3^{0,2k}) &= p\mathbb{Z} \cong \mathbb{Z}; \\
\ker(d_3^{1,2k}) &= 0; \\
\text{Im}(d_3^{0,2k}) &= H^3(A_p; \mathbb{Z}); \\
\text{Im}(d_3^{1,2k}) &= 0.
\end{align*}
\]

**Proof.** Using the $\mathbb{Z}$-algebra structure of $H^*(A_p; \mathbb{Z})$ given in Lemma 5.3, we know that in $H^{2m}(A_p; \mathbb{Z})$, there are the monomials $x_1^ax_2^b$ with $a+b = m$ and in $H^{2m+1}(A_p; \mathbb{Z})$ there are the monomials $yx_1^cx_2^d$ with $c+d = m-1$.

Let $m > 0$ and $k > 0$. The differential $d_3^{2m,2k}$ sends injectively $x_1^ax_2^b$ in $yx_1^cx_2^d$ and so one has the first and the third equations in the statement.

Since $d_3^{2m+1,2k}$ maps $yx_1^cx_2^d$ to $y^2x_1^cx_2^d = 0$ (because $y^2 = 0$ in $H^*(A_p; \mathbb{Z})$) we get the others.

The results for $(0,2k)$ comes from the kernel of the multiplication by $\alpha y$ being $p\mathbb{Z}$ and for and $(1,2k)$ we use that $E_3^{1,2k} = 0$.

For $k = 0$ the statements follows from similar argument and from Figure 6. \hfill $\Box$

**Proposition 5.9.** In the Cartan-Leray spectral sequence for $0 \leq j < 2p-4$, one has
\[
E_4^{i,j} \cong \begin{cases} 
H^{i}(A_p; \mathbb{Z}) & \text{for } i \text{ even and } j = 0; \\
\mathbb{Z} & \text{for } j \leq 2p-4 \text{ even and } i = 0; \\
0 & \text{otherwise}
\end{cases}
\]
Figure 6: The differential $d_3$ from the bottom line lands in a zero row and $E_4^{0,2p-4} = \mathbb{Z}$ and $E_4^{1,2p-4} = 0$.

**Proof.** Let $m, k > 0$. In each term $E_3^{2m,2k}$, we have

$$E_3^{2m-3,2k+2} \xrightarrow{d_3} E_3^{2m,2k} \xrightarrow{d_3} E_3^{2m+3,2k-2}$$

and similarly

$$E_3^{2m-2,2k+2} \xrightarrow{d_3} E_3^{2m+1,2k} \xrightarrow{d_3} E_3^{2m+4,2k-2}.$$

Thus, using the previous lemma,

$$E_4^{2m,2k} = \frac{\ker(d_3^{2m,2k})}{\text{Im}(d_3^{2m-3,2k+2})} = 0;$$

$$E_4^{2m+1,2k} = \frac{\ker(d_3^{2m+1,2k})}{\text{Im}(d_3^{2m-2,2k+2})} = \frac{H^{2m+1}(A_p; \mathbb{Z})}{H^{2m+1}(A_p; \mathbb{Z})} = 0.$$

For $i \leq 2$, we observe that the differentials coming from the left are zero because they start from a zero term (see Figure 7). For $i = 1$ the column is zero. For $i = 2 \ker(d_3^{2,2k}) = 0$. So $E_4^{1/2,j} = 0$ for $0 < j < 2p - 4$. These give the statement for $0 < j < 2p - 4$ and $i \neq 0$.

The rest is proved similarly. □

The $E_4$-terms are showed in Table 3. One proves that what is not in gray is $E_4^{i,j}$.
Figure 7: The differential $d_3$ landing on the first columns

<table>
<thead>
<tr>
<th>$2p-2$</th>
<th>$2p-3$</th>
<th>$2p-4$</th>
<th>$2p-5$</th>
<th>...</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$H^0 (A_p; \mathbb{Z})$</td>
<td>0</td>
<td>...</td>
<td>$H^{2k} (A_p; \mathbb{Z})$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: The $E_4$-terms: the gray cell are not useful for our computation.

**Proposition 5.10.** In the Cartan-Leray spectral sequence $E_{\infty}^{i,j} = E_4^{i,j}$ for

- $j = 0$ and $i < 2p - 2$;
- $0 < j < 2p - 4$;
- $j = 2p - 4$ and $i = 0, 1$.

**Proof.** For $j < 2p - 4$ the differentials $d_k$ for $k > 3$ are zero: the differential starting from $E_k^{0,2k}$ lands in a zero cell (see, for instance, Figure 8). Hence the only change could come from the lines $j = 2p - 4$ (for $i > 1$) and $j = 2p - 3$. In particular, for $i > 1$,

$$d_{2p-3}^{i,2p-4} : E_{2p-3}^{i,2p-4} \to E_{2p-3}^{i+2p-3,0},$$

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Figure 8: For $k > 3$, $d_k(h)$ lands in a zero line

and for every $i$ from

$$d_{2p-2}^{i,2p-3} : E_{2p-2}^{i,2p-3} \rightarrow E_{2p-2}^{i+2p-2,0}.$$ 

Therefore, these modifications could involve the lines $j = 2p - 4$ (for $i > 1$), $j = 2p - 3$ and the bottom line for $i \geq 2p - 2$.

\[ \square \]

**Corollary 5.11.** In the Cartan-Leray spectral sequence, one has

- $E_{\infty}^{i,j} = 0$ if $i + j < 2p - 2$ and $i, j \neq 0$;
- $E_{\infty}^{0,j} = \mathbb{Z}$ if $j < 2p - 3$ is even;
- $E_{\infty}^{i,j} = 0$ if $j < 2p - 3$ is odd;
- $E_{\infty}^{i,0} = H^i(A_p; \mathbb{Z})$ if $i < 2p - 2$ is even;
- $E_{\infty}^{i,0} = 0$ if $i < 2p - 2$ is odd.

From the $E_{\infty}$-level, one reads the information about $H^* (U_p; \mathbb{Z})$ via

$$E_{\infty}^{i,j} = \text{gr} \left( H^{i+j} (U_p; \mathbb{Z}) \right).$$

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The previous corollary implies that $E_{i,0}^\infty \subseteq H^i (U_p; \mathbb{Z})$ and, for $k$ even, $\mathbb{Z} = E_0^k \cong H^k (U_p; \mathbb{Z}) / E_0^k$.

**Theorem 5.12.** The cohomology of the smooth open subset $U_p$ of $\mathbb{P}(V)/A_p$ for $k < 2p - 2$ is

$$H^k (U_p; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0; \\
0 & \text{if } k \text{ is odd}; \\
\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})^{k+1} & \text{if } k \neq 0 \text{ and even.}
\end{cases}$$

**Proof.** For $k = 0$, we only have $H^0 (U_p; \mathbb{Z}) = \mathbb{Z} = E_0^0$. The $E_{i,j}^\infty$-terms in the odd diagonal ($i + j = k$ odd) is zero, then $H^k (U_p; \mathbb{Z}) = 0$. For $0 < k = 2m < 2p - 2$,

$$(\mathbb{Z}/p\mathbb{Z})^{m+1} \cong H^{2m} (A_p; \mathbb{Z}) \cong E_\infty^{2m,0} \subseteq H^{2m} (U_p; \mathbb{Z})$$

and $H^{2m} (U_p; \mathbb{Z}) / (\mathbb{Z}/p\mathbb{Z})^{m+1} \cong \mathbb{Z}$. \qed
6 The toroidal singularities of \(\mathbb{P}^{p-1}/\mathbb{A}_p\)

In this section we study the model \(X_p \overset{f}{\rightarrow} \mathbb{P}^{p-1}/\mathbb{A}_p\). We saw in Theorem 4.9 that \(\mathbb{P}^{p-1}/\mathbb{A}_p\) has \(p+1\) zero dimensional simplicial toroidal singularities locally isomorphic to the origin of the toric affine variety \(\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}\). Thus to resolve each of those singularities, it is sufficient to resolve the origin in \(\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}\) via a suitable (we do not require minimal) number of blowups.

In Theorem 6.3 we show the fan of \(\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}\). Adding a proper number of new rays, in Section 6.2 (using a Magma-algorithm), we construct the wanted resolution. The given algorithm works for any prime \(p\), but, actually, for \(p > 7\) the computation crashes for lack of memory space: I have used an AMD Athlon(tm) 64 Processor 3200+ stepping 00, 2010.528 MHz with op. system Linux and memory 3.0 GB.

We show a sketch of the resolution for the integer primes 2, 3, 5 and 7. Since in the rest of the work we are going to use only the case \(p = 5\), we show the details of the resolution of \(\mathbb{A}^4/\mathbb{Z}/5\mathbb{Z}\) in Appendix B.

For the reader that meets toric varieties for the first times we suggest to have a look in [12] for the undefined notions.

**Notation.** From now on on \(T = (\mathbb{C}^*)^{p-1}\), the \(p-1\) dimension torus. Given a fan \(\Delta\) in the toric lattice \(N = \mathbb{Z}^{p-1}\) we denote by \(X(\Delta)\) the toric variety associated to \(\Delta\). Let \(M\) be the dual lattice of \(N\). For a cone \(\sigma\) in \(N\), \(\sigma^*\) is a cone in \(M\) and it denotes its dual. When we tensor by \(\mathbb{Q}\) we use the subscript index, like \(M_{\mathbb{Q}} = M \otimes \mathbb{Q}\), or \(\sigma_{\mathbb{Q}} = \sigma \otimes \mathbb{Q}\). We set \(S_{\sigma} = \sigma_{\mathbb{Q}} \cap M\). This is a semigroup in \(M\) with respect to the lattice points sum. We denote by \(A_{\sigma} = \mathbb{C}[S_{\sigma}]\), the semigroup algebra of \(S_{\sigma}\), and \(U_{\sigma} = \text{Spec}(A_{\sigma})\), the affine algebraic variety associated to \(\sigma\).

### 6.1 The fan of \(\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}\)

To work with the toric variety \(\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}\), we need to find its fan, \(\Delta_p\).

In Theorem 4.9 we showed the inclusion (6). This gives an action of \(\mathbb{Z}/p\mathbb{Z}\) on \(T\) (from its product). Indeed, for every \(\chi_j \in \mathbb{Z}/p\mathbb{Z}\) and for every \((c_1, \ldots, c_{p-1}) \in T\), one has

\[
\chi_j \cdot (c_1, c_2, \ldots, c_{p-1}) = (c_1\zeta^j, c_2\zeta^{2j}, \ldots, c_{p-1}\zeta^{(p-1)j}).
\]

**Remark 6.1.** Let \(\Delta_1 \hookrightarrow \Delta_2\) be an inclusion of fans in \(N\). Then we have an inclusion of toric varieties \(X(\Delta_1) \subseteq X(\Delta_2)\). Figure 9 show a basic example:
it is the inclusion of the zero fan \{0\} into \(\Delta = \mathbb{N}^2\). It corresponds to \((\mathbb{C}^*)^2 \subseteq \)

\[
\begin{align*}
\text{Zero dimensional fan of } T \\
\text{Semigroup cone of } \mathbb{A}^2 \\
\text{Fan of } \mathbb{A}^2 \\
\text{Semigroup lattice } N \text{ of } T
\end{align*}
\]

Figure 9: Inclusion of fans and its dual

\(\mathbb{A}^2\) and similarly for higher dimensions. Another way to understand this is to translate it in term of semigroup algebra inclusion, that is we take the dual of the previous pictures.

The inclusion \(T \hookrightarrow \mathbb{A}^{p-1}\) gives

\[
\mathbb{C}[x_1, x_2, \ldots, x_{p-1}] \hookrightarrow \mathbb{C}[x_1^{\pm 1}, \ldots, x_{p-1}^{\pm 1}].
\]

It is also \(\mathbb{Z}/\mathbb{pZ}\)-equivariant, with respect to the \(\mathbb{Z}/\mathbb{pZ}\)-action given in (6) and so we obtain the inclusion of the invariant rings

\[
\mathbb{C}[x_1, \ldots, x_{p-1}]^{\mathbb{Z}/\mathbb{pZ}} \hookrightarrow \mathbb{C}[x_1^{\pm 1}, \ldots, x_{p-1}^{\pm 1}]^{\mathbb{Z}/\mathbb{pZ}}.
\]

**Lemma 6.2.** \(\mathbb{C}[x_1, \ldots, x_{p-1}]^{\mathbb{Z}/\mathbb{pZ}} = \mathbb{C}[x_1^{\pm}, \ldots, x_{p-1}^{\pm}]^{\mathbb{Z}/\mathbb{pZ}} \cap \mathbb{C}[x_1, \ldots, x_{p-1}]\).

**Proof.** From the previous inclusion of invariants ring and

\[
\mathbb{C}[x_1, \ldots, x_{p-1}]^{\mathbb{Z}/\mathbb{pZ}} \hookrightarrow \mathbb{C}[x_1, \ldots, x_{p-1}]
\]

we get what we claim. \(\square\)
The following proposition is in an exercises at page 35 in [12].

**Proposition 6.3.** Let $\mathbb{Z}/p\mathbb{Z}$ be embedded in the torus $T$ via (6). The toric lattice corresponding to $T/\mathbb{Z}/p\mathbb{Z}$ is

$$N(T/\mathbb{Z}/p\mathbb{Z}) = \left\{ v \in \frac{1}{p}\mathbb{Z}^{p-1} : \sum_{i=1}^{p-1} iv_i \in \mathbb{Z} \right\}. $$

The fan of the toric variety $\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}$ is made of one maximal cone $\sigma_p$ defined as

$$\sigma_p = \left\{ v \in \frac{1}{p}\mathbb{N}^{p-1} : \sum_{i=1}^{p-1} iv_i \in \mathbb{Z} \right\}. $$

In particular, the variety $\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}$ is not smooth.

**6.2 The resolution of the singularity of $\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}$**

Let $\Delta$ be a fan in $\mathbb{N}$ and let $X(\Delta)$ be its associated toric variety. To each maximal cone of $\Delta$ corresponds a point of $X(\Delta)$. Moreover, the index of this cone (see Section 2.6 in [12]) is one if and only if the corresponding point is smooth. To resolve such a singularity one needs to find a refinement of the fan $\Delta$, such that all the maximal cones (appeared in the new subdivision) have index one.

In our case, we have seen, in Proposition 6.3, that the fan of $\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}$ is composite of just one maximal cone $\sigma_p$: this is the positive orthant cone of $N(T/\mathbb{Z}/p\mathbb{Z})$. Moreover, it has non trivial index, that is the associated point (the origin of $\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}$) is singular.

Let us have a look at $\sigma_2$ and $\sigma_3$. In Figure 10 and 11 the new lattice points are in red color.

**Example 6.4.** Using Proposition 6.3,

$$\sigma_2 = \left\{ v \in \frac{1}{2}\mathbb{N} : v \in \mathbb{Z} \right\} = 2\mathbb{N}. $$

**Example 6.5.** We consider

$$\sigma_3 = \{(v_1, v_2) \in \frac{1}{3}\mathbb{N}^2 : v_1 + 2v_2 \in \mathbb{Z}\}. $$
Figure 10: The cone of $\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}$

Figure 11: The cone of $\mathbb{A}^2/\mathbb{Z}/3\mathbb{Z}$

We denote $v_i = m_i/3$ (with $m_i \geq 0$) and so we required $m_1 + 2m_2 \in 3\mathbb{Z}$. The cone is generated by the following lattice points: $(1, 0), (2/3, 1/3), (1/3, 2/3)$ and $(0, 1)$.

To refine $\sigma_p$, we need to add some rays, one dimensional cone, passing through the new lattice points of $T/\mathbb{Z}$ with respect to the lattice of $T$ (the red point). Each blowup corresponds to add a new ray. With a suitable number of blowup we will obtain a smooth toric variety, that is each maximal cones of the new fan $\Delta'_p$ have multiplicity one. We refer to Section 2.6 of [12] for the details of this construction. Even if $\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{A}^2/\mathbb{Z}/3\mathbb{Z}$ are not interesting for our work, they are nice examples.

**Example 6.6.** Let us consider the cone $\sigma_2$ given in Example 6.4. We observe that the origin is a double point, because the index of $\sigma_2$ is 2. We obtain the resolution adding the ray through the lattice point $1/2$ (see Figure 12).

Figure 12: The resolution of the origin in $\mathbb{A}^1/\mathbb{Z}/2\mathbb{Z}$

In the previous example the number of maximal cones, before and after the resolution, did not change. This does not happen for lattice’s dimension bigger that one. Let us see this phenomena in the following easy case.
Example 6.7. Let us refine the cone of the Example 6.5. The index of $\sigma_3$ is 3 and so the origin is a singular point. We add the rays $r_1$ and $r_2$ respectively generated by the lattice points $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$ (see Figure 13). Hence $\Delta'_3$

![Figure 13: The resolution of the origin in $\mathbb{A}^2/\mathbb{Z}/3\mathbb{Z}$](image)

has 3 maximal cone $\tau_1$, $\tau_2$ and $\tau_3$. They have multiplicity one and, in fact, corresponds to smooth points.

Let us focus on the resolution of the origin in $\mathbb{A}^{p-1}/\mathbb{Z}/p\mathbb{Z}$. We tackle this problem for a positive number $p$ (even not prime). We have used the computer algebra program Magma. I define a function, called **risoluzione**\(^{15}\), that, given a generic number $p$, returns the refined fan $\Delta'_p$. Here there is the code:

```plaintext
risoluzione:= function(p);
    L:=ToricLattice(p-1);
    v:=L![1..p-1];
    w:=L!(v/p);
    LL,f:=AddVectorToLattice(w);
    C:=PositiveQuadrant(L);
    CC:=Image(f,C);
    $\Delta_p$:=Fan(CC);
    $\Delta'_p$:=Resolution($\Delta_p$);
    return $\Delta'_p$;
end function;
```

\(^{15}\)The Italian for resolution.
Example 6.8. In this example we show how the algorithm works:

\[ \Delta_2' := \text{risoluzione}(2); \]
Fan \( \Delta_2' \) with one ray:
\[ (1) \]
and 1 maximal cones.

\[ \Delta_3' := \text{risoluzione}(3); \]
Fan \( \Delta_3' \) with 4 rays:
\[ (1, 0), \]
\[ (1, 3), \]
\[ (1, 2), \]
\[ (1, 1) \]
and 3 maximal cones.

\[ \Delta_5' := \text{risoluzione}(5); \]
Fan \( \Delta_5' \) with 10 rays:
\[ (0, 0, 1, 0), \]
\[ (0, 1, 0, 0), \]
\[ (1, 0, 0, 0), \]
\[ (1, 2, 3, 5), \]
\[ (1, 2, 3, 4), \]
\[ (1, 1, 2, 2), \]
\[ (1, 1, 1, 1), \]
\[ (1, 2, 3, 3), \]
\[ (1, 2, 2, 3), \]
\[ (2, 3, 4, 6) \]
and 21 maximal cones.

\[ \Delta_7' := \text{risoluzione}(7); \]
Fan \( \Delta_7' \) with 21 rays and 126 maximal cones.

Observation 6.9. The reader could ask why the minimal lattice point defining the rays have only integral component: Magma always rescales the lattice. To convince yourself, have a look of Examples 6.6, 6.7 and compare numbers appearing with the numbers in the previous example.

This rescaling is made using a proper linear transformation. It is crucial to stress that the computations in Section 7 do not change under this transformation.

For \( p = 11 \) the computation crashes for lack of local memory. I use a AMD Athlon(tm) 64 Processor 3200+ stepping 00, 2010.528 MHz with op. system
Linux and memory 3.0 GB. However this work is based on the resolution for $p = 5$. We list all the details of the resolution of the origin of $\mathbb{A}^4/\mathbb{A}^5$ in Appendix B.
7 The computation of $H^5_E(X_5; \mathbb{Z})$

In Section 4 we have defined the scheme $\mathbb{P}^p - 1/A_p$ and we have proved in Theorem 4.8 that it has $p + 1$ zero dimensional toric singular points locally isomorphic to the origin of $\mathbb{A}^p - 1/\mathbb{Z}$. We denoted by $\text{Sing}(\mathbb{P}^p - 1/A_p)$ the set of those singular points. We have also locally constructed a resolution, $X_p \xrightarrow{f} \mathbb{P}^p - 1/A_p$, blowing up a sufficient number of time the toric singularities $\mathbb{A}^p - 1/\mathbb{Z}$.

Let $p = 5$ and let $E = f^{-1}(\text{Sing}(\mathbb{P}^4/A_5))$, the union of exceptional divisors from the resolution of the six toroidal singularities locally isomorphic to $\mathbb{A}^4/\mathbb{Z}$. Then $E = \bigcup_{y \in \text{Sing}(\mathbb{P}^4/A_5)} E^{(y)}$, with $E^{(y)} = f^{-1}(y)$, and each $E^{(y)}$ is isomorphic to $D$, the exceptional normal crossing divisors from the resolution of the toric singularity $\mathbb{A}^4/\mathbb{Z}$. Thus

$$H^*_E(X_5; \mathbb{Z}) = \bigoplus_{y \in \text{Sing}(\mathbb{P}^4/A_5)} H^*_D(X_5; \mathbb{Z}) = H^*_D(X_5; \mathbb{Z})^{\oplus 6}$$

Then, from now on we focus on $H^*_D(X_5; \mathbb{Z})$ and we suggest to the reader to have at least a look to the Appendix B: using those computations we know the exceptional divisor $D$. In fact, keeping the notation of Appendix B, $\Delta_5$ is the fan of $\mathbb{A}^4/\mathbb{Z}$ and it is composite by four rays $r_1 = (0, 0, 1, 0)$, $r_2 = (0, 1, 0, 0)$, $r_3 = (1, 0, 0, 0)$ and $r_4 = (1, 2, 3, 5)$ and one maximal cone. In addition, $\Delta'_5$ is the fan of the resolution of $\mathbb{A}^4/\mathbb{Z}$: it is the refining of $\Delta_5$ and it has 10 rays:

$$
\begin{align*}
    r_1 &= (0, 0, 1, 0), \\
    r_2 &= (0, 1, 0, 0), \\
    r_3 &= (1, 0, 0, 0), \\
    r_4 &= (1, 2, 3, 5), \\
    r_5 &= (1, 2, 3, 4), \\
    r_6 &= (1, 1, 2, 2), \\
    r_7 &= (1, 1, 1, 1), \\
    r_8 &= (1, 2, 3, 3), \\
    r_9 &= (1, 2, 2, 3), \\
    r_{10} &= (2, 3, 4, 6). 
\end{align*}
$$

These rays made 21 maximal cones. Moreover,

$$D = D_5 \cup D_6 \cup D_7 \cup D_8 \cup D_9 \cup D_{10},$$

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where \( D_i = V(r_i) \), that is the toric variety defined by the fan \( \text{Star}(r_i) \) (see next remark).

**Remark 7.1.** Given a cone \( \tau \) of a generic fan \( \Delta \) in \( N \) (torus lattice of dimension \( n \)), we denote by \( N_\tau \) the sublattice of \( N \) generated, as a group, by \( \tau \cup N \). Moreover, let \( N(\tau) = N/N_\tau \) be the quotient lattice and, its dual, \( M(\tau) = \tau^\perp \cup M \). \( T_{N(\tau)} \) is the torus corresponding to this lattice, so it has dimension \( n - k \), where \( k = \dim(\tau) \).

We define the Star of the cone \( \tau \) as the set of cones in \( \Delta \) contain \( \tau \) as a face. These cones form a fan in \( N(\tau) \) and we denote this fan \( \text{Star}(\tau) \). \( V(\tau) \) is the toric variety defined by the fan \( \text{Star}(\tau) \) in \( N(\tau) \). For more details, have a look at Section 3.1 in [12].

**Lemma 7.2.** For \( i \geq 5 \), \( D_i = V(r_i) \) is a projective toric variety.

*Proof.* For \( i \geq 5 \) each exceptional rays \( r_i \) passes through lattice points inside of the main cone of \( k^4/\mathbb{Z}/\mathbb{Z} \). Hence \( \text{Star}(r_i) \) is a complete fan. \( \square \)

Since \( D_i \) are toric varieties then we keep track of the intersections \( D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k} \) using the following remark.

**Remark 7.3.** Let \( \sigma \) and \( \tau \) be two cones of \( \Delta \) then

\[
V(\sigma) \cap V(\tau) = \begin{cases} 
V(\langle \sigma, \tau \rangle) & \text{if } \langle \sigma, \tau \rangle \in \Delta \\
\emptyset & \text{otherwise}
\end{cases}
\]

where \( \langle \sigma, \tau \rangle \) is the cone generated by \( \sigma \) and \( \tau \).

Then, one has

\[
D_{i_1} \cap \cdots \cap D_{i_k} = \begin{cases} 
V(\langle r_{i_1}, \ldots, r_{i_k} \rangle) & \text{if } \langle r_{i_1}, \ldots, r_{i_k} \rangle \in \Delta_5' \\
\emptyset & \text{otherwise}
\end{cases}
\]

To compute \( H^*_D(X_5; \mathbb{Z}) \), we use the second quadrant spectral sequence with \( E_1 \) term given by

\[
E_1^{k,i} = \bigoplus_{i_1 < \cdots < i_k} H^i_{D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}}(X_5; \mathbb{Z})
\]

for any \( i \) and for \( k > 0 \). This spectral sequence converges to \( H^*_D(X_5; \mathbb{Z}) \), that is

\[
E_\infty^{k,i} = \text{gr} \left( H^i_D(x_5; \mathbb{Z}) \right)
\]
Since the columns are counted from \( k = -1 \), one produces a shift by one seen in the previous equation. Since \( D_{i_1} \cap \cdots \cap D_{i_k} \) are smooth, one knows that
\[
H^*_{D_{i_1} \cap \cdots \cap D_{i_k}} (X_5; \mathbb{Z}) = H^{*-2 \dim X_5} (D_{i_1} \cap \cdots \cap D_{i_k}; D_{i_1} \cap \cdots \cap D_{i_k}; \mathbb{Z}).
\]

**Remark 7.4.** The dimension of the a cone \( \sigma \) in \( \Delta \) is equal to the codimension of \( V(\sigma) \) in \( X(\Delta) \).

Thus, one has
\[
H^*_{D_{i_1} \cap \cdots \cap D_{i_k}} (X_5; \mathbb{Z}) = H^{*-2 \dim (r_{i_1} \cap \cdots \cap r_{i_k})} (D_{i_1} \cap \cdots \cap D_{i_k}; \mathbb{Z}). \tag{8}
\]

First of all, we want to find the \( E_1 \)-terms of the spectral sequence. From the resolution given in Appendix B, there are 6 exceptional divisors, but if \( k > 3 \), then \( D_{i_1} \cap \cdots \cap D_{i_k} = \emptyset \) (that is there exist no maximal cones of \( \Delta_5 \), generated only by exceptional divisors).

**Proposition 7.5.** \( E_1^{-k,i} = 0 \) if \( k > 3 \).

Using formula (8), we write in Table 4 the \( E_1 \)-terms of the spectral sequence. All the indexes of the direct sums run over \( 5 \leq i_j \leq 10 \), the labels of

\[
\begin{array}{cccc|c}
0 & 0 & 0 & 0 & 9 \\
\oplus H^2 (D_{i_1} \cap D_{i_2} \cap D_{i_3}) & d_1^{3,8} & \oplus H^4 (D_{i_1} \cap D_{i_2}) & d_1^{2,8} & \oplus H^6 (D_{i_1}) & 8 \\
0 & 0 & 0 & 0 & 7 \\
\oplus H^0 (D_{i_1} \cap D_{i_2} \cap D_{i_3}) & d_1^{3,6} & \oplus H^3 (D_{i_1} \cap D_{i_2}) & d_1^{2,6} & \oplus H^4 (D_{i_1}) & 6 \\
0 & 0 & 0 & 0 & 5 \\
0 & \oplus H^0 (D_{i_1} \cap D_{i_2}) & d_2^{2,6} & \oplus H^2 (D_{i_1}) & 4 \\
0 & 0 & 0 & 0 & 3 \\
0 & \oplus H^0 (D_{i_1}) & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\begin{tabular}{cccc}
-3 & -2 & -1 & \( -k \setminus i \) \\
\end{tabular}

Table 4: The \( E_1 \)-terms and the differentials \( d_1 \)

the exceptional divisors \( D_i \). All the cohomologies are integral cohomologies.
Now, we focus on the homomorphism
\[ \oplus H^2(D_{i_1} \cap D_{i_2}) \xrightarrow{d_{2,6}} \oplus H^4(D_{i_1}). \]

We claim the following fact:

Claim. \( \text{Ker}(d_{2,6}) = 0 \).

Then \( E_2 = 0 \) and so \( E_\infty = E_2 = 0 \). Moreover, form equation (7), we know that the terms of the spectral sequence involved in \( \text{gr} (H_D^5(X; \mathbb{Z})) \) are \( E_{-1,5}, E_{-2,6} \) and \( E_{-3,7} \) and all of them are zero. Thus, \( H_D^5(X; \mathbb{Z}) = 0 \).

Theorem 7.6. \( H_E^5(X; \mathbb{Z}) = 0 \).

Proof. \( H_E^5(X; \mathbb{Z}) = \bigoplus_{y \in \text{Sing}(\nu/\Delta)} H_D^5(X; \mathbb{Z}) = 0 \)

The rest of this section is devoted to prove the claim. To find \( \oplus H^2(D_{i_1} \cap D_{i_2}) \) and \( \oplus H^4(D_{i_1}) \) we first study the fan of the projective toric varieties \( D_{i_1} \cap D_{i_2} \) and \( D_{i_1} \), and then we compute their cohomologies. The following remark gives a tool to compute the cohomology of such varieties.

Remark 7.7. Given a nonsingular toric variety \( X \) with \( n \)-dimensional fan \( \Delta \), we know a lot of information about the cohomology: \( H^{2k}(X; \mathbb{Z}) \) is isomorphic to \( C^k(X) \) the Chow ring of \( X \).

For any ordering of the maximal cones \( \{\sigma_i\}_{i=1,\ldots,m} \) of \( \Delta \), let, for \( i = 1,\ldots,m \),
\[ \tau_i = \bigcap_{j \geq i} \sigma_i \]
\[ \dim(\sigma_j \cap \sigma_j) = n - 1. \]

Thus \( \tau_1 = \{0\} \), the zero dimensional cone and \( \tau_m = \sigma_m \).

If the condition
\[ \tau_i \subseteq \sigma_j \implies i \leq j, \]

is fulfilled, then \( \{[V(\tau_i)], [V(\tau_i)] \in C^*(X)\}_{i=1,\ldots,m} \) is a basis for \( H^*(X; \mathbb{Z}) \). In particular, if we consider all the elements \( [V(\tau_i)] \in C^{\dim(\tau_i)}(X) \), those \( [V(\tau_i)] \) form a basis for \( H^{2,\dim(\tau_i)}(X; \mathbb{Z}) \).

Let \( k = -2 \). To study the double intersections \( D_{i_1} \cap D_{i_2} \), we look for the two dimensional cones in \( \Delta'_5 \) generated by \( \langle r_{i_1}, r_{i_2} \rangle \).
Proposition 7.8. The double intersection $D_{i_1} \cap D_{i_2} \neq \emptyset$ if and only if $(i_1, i_2)$ is in the following list: $(5, 6)$, $(5, 7)$, $(5, 8)$, $(5, 9)$, $(5, 10)$, $(6, 7)$, $(6, 8)$ and $(9, 10)$.

Proof. Those are the pair such that the two dimensional cone $\langle r_{i_1}, r_{i_2} \rangle$ is contained in some maximal cone of $\Delta'_5$.

To study $E^{2,6}_1$, we need to find the fan of $D_{i_1} \cap D_{i_2}$.

Observation 7.9. A toric variety $X$ is usually given by its fan $\Delta$. Given a rational convex polytope $K$ in $\mathbb{N}_\mathbb{R}$ (containing the origin) and a subdivision of its boundary, $\partial K$, we construct a complete fan $\Delta_K$, whose cones are the cones over the proper faces of $K$. Thus $X(\Delta_K)$ is proper. Not all the fans come from a convex polytope in $\mathbb{N}_\mathbb{R}$.

Another important construction is the fan coming from a polytope $P$ (even not containing the origin) in $\mathbb{M}_\mathbb{R}$, the dual of $\mathbb{N}_\mathbb{R}$. $\Delta_P = \Delta_{P^o}$, where $P^o$ is a polytope in $\mathbb{N}_\mathbb{R}$, dual to $P$.

When we talk about polytope in this work we always mean the polytope $K$ in $\mathbb{N}_\mathbb{R}$. Actually, since we use only the boundary of $K$ and its subdivision, we only care about the simplicial complex that we construct on $\partial K$. Thus, in what follows the pictures show the simplicial complexes over their boundary.

Proposition 7.10. Figure 14 shows the simplicial complex, induced by the fan, in the boundary of the polytope of the non-trivial double intersections.

Proof. We focus on $D_5 \cap D_7$, but what we say holds also for the other double intersections. There are 3 maximal cones containing the two dimensional one $\langle r_5, r_7 \rangle$: $\langle r_6, r_3, r_5, r_7 \rangle$, $\langle r_6, r_2, r_5, r_7 \rangle$ and $\langle r_2, r_3, r_5, r_7 \rangle$. Going modulo $\langle r_5, r_7 \rangle$, we obtain a two dimensional fan. Let $r_6 = c_2 r_2 + c_3 r_3 + c_4 r_5 + c_5 r_7$ and so $r_6 = c_2 r_2 + c_3 r_3$ mod $(r_5, r_7)$. Let $r_3$ and $r_2$ be the axis $e_1$ and $e_2$ of the lattice $N(\langle r_5, r_7 \rangle)$ and let $v = (c_2, c_3)$; $c_2 = c_3 = -1$. Figure 15 shows the fan obtained and stresses the simplicial complex obtained. The study of another double intersection with three maximal cones changes only the vector $v$, but not the picture.

Now, we use Remark 7.7 to obtain $\text{H}^* (D_{i_1} \cap D_{i_2}; \mathbb{Z})$. 58
Proposition 7.11. The cohomologies in second degree of the non empty double intersections $D_i \cap D_j$ are the following

\[
\begin{align*}
\text{H}^2 (D_5 \cap D_6; \mathbb{Z}) &= \mathbb{Z}^4 \\
\text{H}^2 (D_5 \cap D_7; \mathbb{Z}) &= \mathbb{Z} \\
\text{H}^2 (D_5 \cap D_8; \mathbb{Z}) &= \mathbb{Z} \\
\text{H}^2 (D_5 \cap D_9; \mathbb{Z}) &= \mathbb{Z}^2 \\
\text{H}^2 (D_5 \cap D_{10}; \mathbb{Z}) &= \mathbb{Z} \\
\text{H}^2 (D_6 \cap D_7; \mathbb{Z}) &= \mathbb{Z}^2 \\
\text{H}^2 (D_6 \cap D_8; \mathbb{Z}) &= \mathbb{Z} \\
\text{H}^2 (D_9 \cap D_{10}; \mathbb{Z}) &= \mathbb{Z}
\end{align*}
\]

Proof. The proof is elementary and we show only the case of $D_5 \cap D_6$ as an example. We have seen that the polytopes of $D_5 \cap D_6$ looks like an hexagon. Following Remark 7.7, we order the maximal cones in its fan, hence, we order
the edge of the hexagon: for instance have a look at Figure 16. The ordering

respects condition (9) and hence the \([V(\tau_i)]\) (see Figure 17) generate the
cohomology of \(D_5 \cap D_6\). \(H^2(D_5 \cap D_6; \mathbb{Z})\) is generated by the class of the toric
variety \(V(\tau_i)\) coming from the one dimensional cones \(\tau_2, \tau_3, \tau_4\) and \(\tau_5\). Thus
the claim. \(\square\)

Corollary 7.12. \(E_1^{2.6} = \mathbb{Z}^{13}\).

Finally, let us consider \(E_1^{1.6}\). As the reader could image, the polytopes
and the pictures get one dimension more.

Proposition 7.13. Figures 18, 19 and 20 show the simplicial complexes, in-
duced by the fans, in the boundary of the polytopes of the exceptional divisors
\(D_5, ..., D_{10}\). The main simplex is hidden behind the picture so that these
complexes are always triangulation of a two dimensional sphere.
Figure 17: The basis of the cohomology of $D_5 \cap D_6$

Figure 18: The simplicial complex over the boundary of the polytope of the exceptional divisors $D_5$ and $D_6$

Proof. We prove only the $D_7$-case, because for the other divisors we proceed similarly. We select (via Magma) in $\Delta'_5$ the maximal cones containing the ray $r_7 = (1, 1, 1, 1)$ (have a look in Appendix B). We want to construct $\text{Star}(r_7)$, hence we go modulo the ray $(1, 1, 1, 1)$. $\text{Star}(r_7)$ is a 3 dimensional fan. We set

\[ e_1 = (1, 0, 0, 0) \mod(1, 1, 1, 1), \]
\[ e_2 = (0, 1, 0, 0) \mod(1, 1, 1, 1), \]
\[ e_3 = (0, 0, 1, 0) \mod(1, 1, 1, 1), \]

being the lattice generators of $N(\langle r_7 \rangle)$. Moreover we write

\[ (0, 0, 0, 1) = (1, 1, 1, 1) - (1, 0, 0, 0) - (0, 1, 0, 0) - (0, 0, 1, 0) \]
Figure 19: The simplicial complexes over the boundary of the polytope of the exceptional divisors $D_7$ and $D_8$

and so $(0,0,0,1) \mod (1,1,1,1) = -e_1 - e_2 - e_3$. In Appendix B we show the details of this new fan. Figure 21 shows the polytope of $D_7$ in $N_R$. The picture looks complicate and since we are interested in the boundary structure (as said in Observation 7.9) we only consider the two dimensional simplicial complex in the boundary. After an opportune deformation of the picture we move the face having vertices $r_5$, $r_6$ and $r_2$ behind and we obtain the claim.

We compute the cohomologies $H^4(D_i; \mathbb{Z})$ following Remark 7.7: we order the maximal cone of Star($r_i$), that is we order the 2-dimensional simplex of the simplicial complexes shown in the previous proposition.

**Proposition 7.14.** The integral cohomologies in second degree of the exceptional divisors $D_i$ are:

$$H^4(D_i; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}^6 & \text{for } i = 5 \\
\mathbb{Z}^4 & \text{for } i = 6 \\
\mathbb{Z}^2 & \text{for } i = 7 \text{ and } 9 \\
\mathbb{Z} & \text{for } i = 8 \text{ and } 10
\end{cases}$$

**Proof.** The orderings in Figures 22, 23, 24, 25, 26 and 27 satisfy the condition (9). Using Remark 7.7, one has the bases $\{\tau^{(i)}_j\}$ for the cohomology rings $H^*(D_i; \mathbb{Z})$ (see Figures 28, 29, 30 and 31).
Figure 20: The simplicial complexes over the boundary of the polytope of the exceptional divisors $D_9$ and $D_{10}$

Counting the cones of the basis of dimension 2, that is counting the simplices $\{\tau_j^{(i)}\}$ in the pictures of dimension 1, we obtain the rank of $H^4(D_i;\mathbb{Z})$.

\[\square\]

**Corollary 7.15.** $E_1^{1,6} = \mathbb{Z}^{16}$.

We go back to our aim: the study of kernel of the homomorphism

$$\mathbb{Z}^{13} \xrightarrow{d_1^{2,6}} \mathbb{Z}^{16}.$$  

The differential $d_1$ send an element of the basis, $[V(\tau)]$, of $H^2(D_i \cap D_j;\mathbb{Z})$ in its self but seen in $\oplus_i H^4(D_i;\mathbb{Z})$. Therefore, to know the map $d_1^{*,*}$, we need to know the image of the elements of the basis of $H^* (D_{i_1} \cap D_{i_2};\mathbb{Z})$.

**Notation.** The class of $V(\sigma)$ in $H^*(D_{i_1} \cap D_{i_2};\mathbb{Z})$ is $[V(\sigma)]_{(i_1,i_2)}$; similarly for $[V(\sigma)]_{(i_1)}$. If $[V(\tau_k)]$ is an element of the basis of $H^*(D_{i_1} \cap D_{i_2};\mathbb{Z})$, then we write $[V(\tau_k(i_1,i_2))]$; similarly for $[V(\tau_k^{(i_1)})]$.

**Proposition 7.16.** One has:

$$d_1^{2,6}([V(\tau_k^{(i,j)})]) = [V(\tau_k^{(i,j)})]_{(j)} - [V(\tau_k^{(i,j)})]_{(i)}$$
Figure 21: The polytope of $D_7$

Proof. The $d_{1,i}^2$ map arises from the injections $D_{i_1} \cap D_{i_2} \hookrightarrow D_{i_1}$ and $D_{i_1} \cap D_{i_2} \hookrightarrow D_{i_2}$. $V(\tau_{k}^{(i,j)})$ is a closed subvariety of $D_{i_1} \cap D_{i_2}$ and so it is also a closed subvariety of $D_{i_1}$ and $D_{i_2}$. Hence we assign to the class $[V(\tau_{k})]_{(i_1,i_2)}$ itself, but in the different varieties Chow groups of $D_{i_1}$ and of $D_{i_2}$. So the statement, up to sign. \hfill \qed

Therefore before going inside of the analysis of $d_1$, we need to discuss the methodology to solve the following problem.

Problem. Let $\sigma$ be a cone of the nonsingular, complete (and simplicial) fan $\Delta$ with $m$ maximal cones; let $\{[V(\tau_k)]\}_{k=1,...,m}$ be a basis for $H^*(X(\Delta); \mathbb{Z})$. We want to write

$$[V(\sigma)] = \sum_{\dim(\tau_k)=\dim(\sigma)} a_k [V(\tau_k)]. \quad (10)$$

To do that, we use some toric variety trick.

Remark 7.17. We want to find the coefficient $a_k$ in (10). We firstly observe that if $[V(\sigma)] \in H^i(X(\Delta); \mathbb{Z})$ and $[V(\rho)] \in H^j(X(\Delta); \mathbb{Z})$, then $[V(\sigma)] \cdot [V(\rho)] \in H^{i+j}(X(\Delta); \mathbb{Z})$. Secondly, for any maximal cone $\tau$ of $\Delta$, $[V(\tau)] \in H^{2 \dim(X)}(X(\Delta); \mathbb{Z})$. Using Remark 7.7 and the nonsingularity of the fan $\Delta$, $H^{2 \dim(X)}(X(\Delta); \mathbb{Z})$ is generated by $[V(\tau_m)]$: then $[V(\tau)] = [V(\tau_m)]$ for any maximal cone.
Moreover, since any element of $H^{2 \dim(X)}(X(\Delta); \mathbb{Z})$ can be written as $n[V(\tau_m)]$ with $n$ integer number, then, in what follow, we will write any element of $H^{2 \dim(X)}(X(\Delta); \mathbb{Z})$ as an integer number.

Now let us multiplying the (10) for a generic cone $[V(\tau')]$ in $\Delta$ such that $\dim(\sigma) + \dim(\tau') = \dim(X(\Delta))$. One has

$$[V(\sigma)] \cdot [V(\tau)] = \sum_{\dim(\tau_k) = \dim(\sigma)} a_k [V(\tau_k)] \cdot [V(\tau)].$$

$[V(\tau_k)] \cdot [V(\tau)]$ and $[V(\sigma)] \cdot [V(\tau)]$ lie in $H^{2 \dim(X)}(X(\Delta); \mathbb{Z})$ and we see that as integer numbers. Let us define the intersection matrix,

$$\mathcal{I}_{k,l} = [V(\tau_k)] \cdot [V(\tau_l)], \quad (11)$$

and

$$\mathbf{u}_e = [V(\sigma)] \cdot [V(\tau)], \quad (12)$$
Figure 23: The ordering of the maximal cones in the fan of $D_6$: the 2-simplex having vertices $r_1$, $r_5$ and $r_8$ is $\sigma_5$.

for all the $\tau_l$ such that $\dim(\tau_k) + \dim(\tau_l) = \dim(X(\Delta))$. It is well known that $I_{k,l} = I_{l,k}$. Thus,

$$u_l = [V(\sigma)] \cdot [V(\tau_l)] = \sum a_k [V(\tau_k)] \cdot [V(\tau_l)],$$

$$= \sum a_k I_{k,l}$$

$$= \sum I_{l,k}a_k.$$

If we call $u = (u_l)$, $a = (a_k)$ and $I = (I_{k,l})$, then to obtain $a$ we need to solve the linear system

$$I \cdot a = u.$$

Therefore, apart a solution of a linear system, we rewrite the previous problem into the following one:

**Problem.** Let $\sigma$ be a cone of the nonsingular, complete (and simplicial) fan $\Delta$; let $\{[V(\tau_k)]\}_{k=1,\ldots,m}$ be a basis for $H^* (X(\Delta); \mathbb{Z})$. We want to know the matrices $I$, and the vectors $u$ defined in the previous remark.

To find $I$ and $u$, we study the generic case of the intersection product

$$[V(\sigma)] \cdot [V(\tau)],$$

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where $\sigma$ and $\tau$ are cone in the fan $\Delta$ and $\dim(\sigma) + \dim(\tau) = \dim(X(\Delta))$.

There are two possibilities. The first one is that $\dim(\sigma + \tau) = \dim(X(\Delta))$: if $\sigma + \tau \in \Delta$, then $V(\sigma)$ and $V(\tau)$ intersect transversally in the point $V(\sigma + \tau)$.

**Example 7.18.** Let $X(\Delta)$ be given as in Example 6.7. From Figure 13, we know that $\Delta'$ has three maximal cones $\{\sigma_1, \sigma_2, \sigma_3\}$. The exceptional rays are $r_1 = \langle \frac{1}{3}, \frac{2}{3} \rangle$ and $r_2 = \langle \frac{2}{3}, \frac{1}{3} \rangle$. Moreover

$$V(e_1) \cap V(r_2) = V(\sigma_3)$$

and

$$V(e_1) \cap V(e_2) = \emptyset.$$

In the case of the transversal intersection the intersection product is

$$[V(\sigma)] \cdot [V(\tau)] = \begin{cases} [V(\sigma + \tau)] & \text{if } \sigma + \tau \in \Delta; \\
0 & \text{otherwise.} \end{cases}$$

The other case is $\dim(\sigma + \tau) < \dim(X(\Delta))$: the cones $\sigma$ and $\tau$ have some rays in common and the intersection is not more transversal.

For what concerns this work, we set $\dim(X(\Delta)) = 3$, with $\dim(\sigma) = 2$ and $\dim(\tau) = 1$; of course, $\tau \subset \sigma$. Let $\tau = \langle r_1 \rangle$ and $\sigma = \langle r_1, r_2 \rangle$. In addition, let $D_1$ be the first lattice point of $\tau$ and $D_2$ the first lattice point of the ray $\langle r_2 \rangle$.

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Figure 25: The ordering of the maximal cones in the fan of $D_8$: the 2-simplex having vertices $r_1$, $r_5$ and $r_6$ is $\sigma_2$

Since $\Delta$ is simplicial, complete and smooth, then there exist two three-dimensional cones $\rho'$ and $\rho''$ such that $\sigma$ is a face of them. Let $v'$ and $v''$ be the two first lattice points of the other rays, respectively, of $\rho'$ and $\rho''$. Since $\Delta$ is nonsingular, this implies that every cone is nonsingular, then also $\rho'$ and $\rho''$. Thus we can write

$$v' + v'' = a_1 D_1 + a_2 D_2.$$  

Moreover $[V(r_j)] \cdot [V(\sigma)] = c'[V(\rho')] + c''[V(\rho'')]$, where $c' + c'' = -a_j$ and, finally,

$$[V(r_j)] \cdot [V(\sigma)] = -a_j. \quad (13)$$

The general case, for greater dimension of $X$, is a simple generalization of this case (see [12] at page 99).

**Example 7.19.** Let $\Delta = \text{Star}(r_8)$. Then we have the polytope give in Proposition 7.13 with the basis shown in the proof of Proposition 7.14. Let $\sigma$ be the 2-dimensional cone generated by $r_2$ and $r_5$ (in the Figure 32 a 1-dimensional simplex). We want to write $[V(\sigma)]$ as in formula (10). There is only one basis cone of dimension 2 and so $[V(\sigma)] = n[V(\tau_3^{(8)})]$ with $n \in \mathbb{Z}$.

We apply the previous method and so we need to construct the numbers
of the fan Star(r_8) going from $\Delta'_5$ modulo $r_8$:

\[
D_1 \equiv e_3 \mod r_8, \\
D_2 \equiv e_2 \mod r_8, \\
D_5 \equiv e_4 \mod r_8, \\
D_6 \equiv -e_2 - e_3 - e_4 \mod r_8.
\]

Now we write

\[
D_1 + D_5 = e_3 + e_4 = -1(e_2) - 1(-e_2 - e_3 - e_4) = \alpha D_2 + \beta D_6.
\]

Thus, $[V(\tau_3^{(8)})] \cdot [V(\tau_2^{(8)})] = 1$. Multiplying $[V(\sigma)] = n[V(\tau_3^8)]$ on both sides for $[V(\tau_2^8)]$ we get:

\[
1 = [V(\tau_4^{(8)})] = [V(\sigma)] \cdot [V(\tau_2^{(8)})] = n[V(\tau_3^{(8)})] \cdot [V(\tau_2^8)] = n.
\]

Thus, $[V(\sigma)] = [V(\tau_3^{(8)})]$.

In this section we use intensively computations similar to the ones in the previous example. For this reason we decide to express those numbers in a table like the following:

\[
\begin{array}{cccc|ccc}
 i & j & k & z & D_i + D_j = \alpha D_k + \beta D_z & \alpha & \beta \\
 1 & 5 & 2 & 6 & e_3 + e_4 = -1(e_2) - 1(-e_2 - e_3 - e_4) & -1 & -1 \\
\end{array}
\]
Figure 27: The ordering of the maximal cones in the fan of $D_{10}$: the 2-simplex having vertices $r_3, r_4$ and $r_5$ is $\sigma_2$

The boldface text means that we are going to use this coefficient (better, the opposite of it) for the intersection product.

In terms of cones, if $[V(\tau_k^{(i,j)}))] \in H^2(D_i \cap D_j; \mathbb{Z})$ then $\tau_k^{(i,j)}$ is a one-dimensional cone of $\text{Star}(D_i \cap D_j)$ in the toric lattice $\mathbb{N} \langle \langle r_i, r_j \rangle \rangle$. We observe that $D_i \cap D_j$ is a subvariety of $D_i$ and so $D_i \cap D_j$ corresponds to the cones in $\text{Star}(D_i)$ containing $r_j$. For instance, Figure 33 shows the fan of $D_5 \cap D_6$ (see Figure 17) inside of the fan of $D_6$. The figure also shows the image of the basis elements of $H^2(D_5 \cap D_6; \mathbb{Z})$.

Doing the same procedure for all the element of the bases of $H^2(D_i \cap D_j; \mathbb{Z})$ one gets Figures 28, 29, 30 and 31. Comparing those with Figures 22, 23, 24, 25, 26 and 27, we prove the following propositions.

**Proposition 7.20.** The $D_8$-component and the $D_{10}$-component of the image of $d_1^{2,6}$ are:

\[
\begin{align*}
\sigma_1^{2,6}([V(\tau_2^{(5,8)})])_{(8)} &= [V(\tau_3^{(8)})] \\
\sigma_1^{2,6}([V(\tau_2^{(6,8)})])_{(8)} &= [V(\tau_3^{(8)})] \\
\sigma_1^{2,6}([V(\tau_2^{(9,10)})])_{(10)} &= \pm [V(\tau_3^{(10)})] \\
\sigma_1^{2,6}([V(\tau_2^{(5,10)})])_{(10)} &= [V(\tau_3^{(10)})]
\end{align*}
\]

**Proof.** We start proving the statement for $D_8$. We can always choose the basis for the cohomology such that $\tau_3^{(8)}$ coincides with one of the edge (and
so we did in Proposition 7.14). Hence \( d_1^{2,6}([V(\tau_2^{6,8})]) = [V(\tau_3^8)] \).

Regarding \( d_1^{2,6}([V(\tau_2^{5,8})]) \), one known that \( d_1^{2,6}([V(\tau_2^{5,8})]) = n[V(\tau_3^8)] \) with \( n \in \mathbb{Z} \). Using the computation done in Example 7.19, one has \( n = 1 \).

The proof follows similarly for \( D_{10} \): \( d_1^{2,6}([V(\tau_2^{9,10})]) = n[V(\tau_3^{10})] \). Multiplying for \([V(\tau_2^{10})]\) we get,

\[
1 = d_1^{2,6}([V(\tau_2^{9,10})]) \cdot [V(\tau_2^{10})] = n[V(\tau_3^{10})] \cdot [V(\tau_2^{10})].
\]

\( n \) has to be an integer and similarly for \([V(\tau_3^{10})] \cdot [V(\tau_2^{10})]\), then \( n = \pm 1 \). \( \square \)

**Proposition 7.21.** The \( D_7 \)-component of the image of \( d_1^{2,6} \) are:

\[
\begin{align*}
d_1^{2,6}([V(\tau_2^{5,7})])_{(7)} &= [V(\tau_5^{7})]
d_1^{2,6}([V(\tau_2^{6,7})])_{(7)} &= [V(\tau_5^{7})]
d_1^{2,6}([V(\tau_3^{6,7})])_{(7)} &= [V(\tau_4^{7})]
\end{align*}
\]
Figure 29: The basis of the cohomology of $D_6$

**Proof.** The only problem is the first equality, because the others coincide with the basis elements.

We compute the intersection matrix $I$ in (11) and the vector $u$ defined in (12).

Let $D_j$ be the first lattice points of the rays $r_j$ in $\text{Star}(D_7)$. We are going to use the following:

\[
D_1 \equiv e_3 \mod r_7 \\
D_2 \equiv e_2 \mod r_7 \\
D_3 \equiv e_1 \mod r_7 \\
D_5 \equiv -3e_1 - 2e_2 - e_3 \mod r_7 \\
D_6 \equiv -e_1 - e_2 \mod r_7
\]

\[
u = ([V(\tau_2^{(5,7)})] \cdot [V(\tau_3^{(7)})], [V(\tau_2^{(5,7)})] \cdot [V(\tau_3^{(7)})]) = (0, 1) \text{ and } I, \text{ is}
\]

\[
\begin{pmatrix}
    V(\tau_4^{(7)}) & V(\tau_5^{(7)}) \\
    V(\tau_2^{(7)}) & 1 & 0 \\
    V(\tau_3^{(7)}) & -2 & 1
\end{pmatrix}
\]

In the following table we show the computations needed:
Figure 30: The basis of the cohomology of $D_{10}$ and $D_7$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$k$</th>
<th>$z$</th>
<th>$D_i + D_j = \alpha D_k + \beta D_z$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>3</td>
<td>5</td>
<td>$-e_1 = -1(e_1) + 0(-3e_1 - 2e_2 - e_3)$</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>$-3e_1 - 2e_2 = -1(e_1) + 2(-e_1 - e_2)$</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>$e_1 + e_2 = 0(e_3) + -1(-e_1 - e_2)$</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Therefore, solving the linear system we get $\mathbf{a} = (0, 1)$.

**Proposition 7.22.** The $D_9$-component of the image of $d_{1^26}^2$ are:

\[
\begin{align*}
    d_{1^26}^2([V(\tau_{2(5,9)})])_{(9)} &= [V(\tau_5^{(9)})] \\
    d_{1^26}^2([V(\tau_{3(5,9)})])_{(9)} &= [V(\tau_4^{(9)})] \\
    d_{1^26}^2([V(\tau_{2(9,10)})])_{(9)} &= [V(\tau_5^{(9)})]
\end{align*}
\]

**Proof.** Similarly to the previous proposition the first equation needs some computation $d_{1^26}^2([V(\tau_{2(5,9)})])$, and the others are proved by comparing pictures.

So we compute the intersection matrix $\mathcal{I}$ and the vector $\mathbf{u}$. The first lattice points $D_j$ of the rays $r_j$ of Star($D_9$) are:

\[
\begin{align*}
    D_2 & \equiv e_2 \mod r_9 \\
    D_3 & \equiv -2e_2 - 2e_3 - 3e_4 \mod r_9 \\
    D_4 & \equiv e_3 + 2e_4 \mod r_9 \\
    D_5 & \equiv e_3 + e_4 \mod r_9 \\
    D_{10} & \equiv -e_2 \mod r_9
\end{align*}
\]

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Easily, \( u = (0, 1) \): both are transversal intersections. The intersection matrix \( \mathcal{I} \) is

\[
\begin{pmatrix}
V(\tau_4^{(9)}) & V(\tau_5^{(9)}) \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]

with the following computation table:

\[
\begin{array}{cccc|c}
 i & j & k & z & D_i + D_j = \alpha D_k + \beta D_z \\
2 & 10 & 5 & 4 & 0 = 0(e_3 + e_4) + 0(e_3 + 2e_4) \\
3 & 5 & 10 & 4 & -2e_2 - e_3 - 2e_4 = 2(-e_2) - 1(e_3 + 2e_4)
\end{array}
\]

with \( \alpha = 0 \) and \( \beta = 0 \).

Solving the system \( \mathcal{I} \cdot a = u \), we get the statement. \( \square \)

**Proposition 7.23.** The \( D_6 \)-component of the image of \( d_1^{2,6} \) are:

\[
\begin{align*}
d_1^{2,6}([V(\tau_2^{(5,6)})])_{(6)} &= [V(\tau_9^{(6)})] \\
d_1^{2,6}([V(\tau_3^{(5,6)})])_{(6)} &= [V(\tau_8^{(6)})] \\
d_1^{2,6}([V(\tau_4^{(5,6)})])_{(6)} &= [V(\tau_7^{(6)})] \\
d_1^{2,6}([V(\tau_5^{(5,6)})])_{(6)} &= [V(\tau_6^{(6)})] \\
d_1^{2,6}([V(\tau_2^{(6,7)})])_{(6)} &= [V(\tau_7^{(6)})] \\
d_1^{2,6}([V(\tau_3^{(6,7)})])_{(6)} &= [V(\tau_6^{(6)})] \\
d_1^{2,6}([V(\tau_2^{(6,8)})])_{(6)} &= -2[V(\tau_6^{(6)})] - [V(\tau_7^{(6)})] + [V(\tau_9^{(6)})]
\end{align*}
\]
Figure 32: The fan of $D_8$; $\sigma$ is generated by $r_2$ and $r_5$

\textbf{Proof.} The first four are true for coincidence with basis elements we see. For the last ones we need to work a bit more. The first lattice points $D_j$ of the rays $r_j$ of Star($D_6$) are:

\begin{align*}
D_1 & \equiv e_3 \mod r_6 \\
D_2 & \equiv e_2 \mod r_6 \\
D_3 & \equiv -e_2 - 2e_3 - 2e_4 \mod r_6 \\
D_4 & \equiv e_2 + e_3 + 3e_4 \mod r_6 \\
D_5 & \equiv e_2 + e_3 + 2e_4 \mod r_6 \\
D_7 & \equiv -e_3 - e_4 \mod r_6 \\
D_8 & \equiv e_2 + e_3 + e_4 \mod r_6
\end{align*}

The intersection matrix $I$ is

\[
\begin{array}{cccc}
\cdot & V(\tau_9^{(6)}) & V(\tau_8^{(6)}) & V(\tau_7^{(6)}) & V(\tau_6^{(6)}) \\
V(\tau_5^{(6)}) & 1 & -2 & 0 & 0 \\
V(\tau_2^{(6)}) & 0 & 1 & -2 & 1 \\
V(\tau_3^{(6)}) & 0 & 0 & 1 & -1 \\
V(\tau_4^{(6)}) & 0 & 0 & 0 & 1 \\
\end{array}
\]

where we use the following computations (starting from the right):
Figure 33: The basis of the cohomology of $D_5 \cap D_6$ inside $D_6$

Figure 34: The basis elements of the cohomology group $H^2 (D_i \cap D_j; \mathbb{Z})$ inside the fan of $D_8$ and $D_9$.
Figure 35: The basis elements of the cohomology group $H^2(D_i \cap D_j; \mathbb{Z})$ inside the fan of $D_5$ and $D_6$.

Figure 36: The basis elements of the cohomology group $H^2(D_i \cap D_j; \mathbb{Z})$ inside the fan of $D_7$ and $D_8$. 
The first five are mapped to a basis element. The first lattice points

Proof. The first five are mapped to a basis element. The first lattice points

The vectors are $u_{\Delta} = (0, -2, 1, 0)$, $u_{\Delta} = (1, 0, 0, 0)$ and $u_{\Delta} = (1, 0, 1, -2)$. These can be obtained from the following table:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$k$</th>
<th>$z$</th>
<th>$D_i + D_j = \alpha D_k + \beta D_z$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>8</td>
<td>2</td>
<td>5</td>
<td>$e_2 = 1(e_2) + 0(e_3 + e_4 + 2e_4)$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>7</td>
<td>5</td>
<td>$-2e_3 - 2e_4 = 2(-e_3 - e_4) + 0(e_2 + e_3 + 2e_4)$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>$e_2 + 2e_4 = 1(-e_2 - 2e_3 - 2e_4) + 2(e_2 + e_3 + 2e_4)$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>$-e_2 - e_3 - 2e_4 = 0(\ldots) - 1(e_2 + e_3 + 2e_4)$</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Solving the linear system we obtain the statement. 

**Proposition 7.24.** The $D_5$-component of the image of $d^{2,6}_1$ are:

$d^{2,6}_1([\Delta]) = [\Delta]$ 

Proof. The first five are mapped to a basis element. The first lattice points

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$k$</th>
<th>$z$</th>
<th>$D_i + D_j = \alpha D_k + \beta D_z$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>$-2e_3 - 2e_4 = 0(e_3) + 2(-e_3 - e_4)$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>5</td>
<td>7</td>
<td>$e_2 + 2e_3 + 2e_4 = -1(-e_2 - 2e_3 - 2e_4) + 0(\ldots)$</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>2</td>
<td>8</td>
<td>$e_2 + 2e_3 + 2e_4 = -1(e_2) + 2(e_2 + e_3 + 2e_4)$</td>
<td>-1</td>
<td>+2</td>
</tr>
</tbody>
</table>
$D_j$ of the rays $r_j$ of Star($D_5$) are:

\[
\begin{align*}
D_1 & \equiv e_3 \mod r_5 \\
D_2 & \equiv e_2 \mod r_5 \\
D_3 & \equiv -2e_2 - 3e_3 - 4e_4 \mod r_5 \\
D_4 & \equiv e_4 \mod r_5 \\
D_6 & \equiv -e_2 - e_3 - 2e_4 \mod r_5 \\
D_7 & \equiv -e_2 - 2e_3 - 3e_4 \mod r_5 \\
D_8 & \equiv -e_4 \mod r_5 \\
D_9 & \equiv -e_3 - e_4 \mod r_5 \\
D_{10} & \equiv -e_2 - 2e_3 - 2e_4 \mod r_5
\end{align*}
\]

The rest of the images is obtained via the intersection matrix

\[
\begin{pmatrix}
V(\tau_5^{(5)}) & V(\tau_6^{(5)}) & V(\tau_{10}^{(5)}) & V(\tau_{11}^{(5)}) & V(\tau_{12}^{(5)}) & V(\tau_{13}^{(5)}) \\
-2 & 1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 & 1 & -2
\end{pmatrix}
\]

and the vectors $u_{[V(\tau_2^{(5,9)})]} = (0, 0, 0, 1, -2, 1)$, $u_{[V(\tau_2^{(5,7)})]} = (0, 0, 0, 0, 1)$ and $u_{[V(\tau_2^{(5,10)})]}$ equals to $u_{[V(\tau_2^{(5,9)})]} = (1, -2, 1, 0, 0, 0)$.

We have got these via the following table:

\[
\begin{pmatrix}
i & j & k & z & D_i + D_j = \alpha D_k + \beta D_z & \alpha & \beta \\
2 & 10 & 9 & 4 & -2e_3 - 2e_4 = 2(-e_3 - e_4) + 0(e_4) & 2 & 0 \\
1 & 9 & 2 & 4 & -e_4 = 0(e_2) - 1(e_4) & 0 & -1 \\
1 & 3 & 6 & 4 & -2e_2 - 2e_3 - 4e_4 = 2(-e_2 - e_3 - 2e_4) + 0(e_4) & 2 & 0 \\
7 & 8 & 2 & 6 & -e_2 - 2e_3 - 4e_4 = 1(e_2) + 2(-e_2 - e_3 - 2e_4) & 1 & 2 \\
2 & 3 & 7 & 6 & -e_2 - 3e_3 - 4e_4 = 2(-e_2 - 2e_3 - 3e_4) - (\ldots) & 2 & -1 \\
7 & 4 & 3 & 6 & -e_2 - 2e_3 - 2e_4 = (-2e_2 - 3e_3 - 4e_4) - (\ldots) & 1 & -1 \\
1 & 6 & 2 & 8 & -e_2 - 2e_4 = -1(e_2) + 2(-e_4) & -1 & 2 \\
3 & 4 & 9 & 10 & -2e_2 - 3e_3 - 3e_4 = -1(-e_3 - e_4) + 2(\ldots) & -1 & +2
\end{pmatrix}
\]

Solving the linear system we get the results.
Table 5: The matrix \( M_{2,6} \). We decorate column and row with the basis elements.

The matrix in Table 5 show the matrix of the homomorphism \( d_{1,2}^{2,6} \). Using the computer algebra program \texttt{Magma} we compute its kernel over \( \mathbb{Z} \):

\[
X := \text{Matrix}(\text{IntegerRing}(), 16, 13, [0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0, \ldots])};
\]
\[
X1:=\text{Transpose}(X);
\]
\[
\text{Kernel}(X1);
\]
\[
\text{RSpace of degree 13, dimension 0 over Integer Ring}
\]
A The integral cohomology of $A_p$

In this appendix we compute the integral cohomology of $A_p = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, $H^* (A_p; \mathbb{Z})$.

We define $H^n (G; \mathbb{Z}) = \text{Ext}^n_{k[G]} (k, \mathbb{Z})$ where $k$ is the base field and $k[G]$ is the free abelian $k$-module with basis $G$ and with multiplication induced by the multiplication in the group $G$. For instance, $k[\mathbb{Z}/p\mathbb{Z}] = k(z)/(z^n - 1)$. We remark the Künneth formula for the trivial action of $G$ on the coefficients $\mathbb{Z}$:

$$H^n (G \times H; \mathbb{Z}) = \bigoplus_{i=0}^{n} H^i (G; \mathbb{Z}) \otimes \mathbb{Z} H^{n-i} (H; \mathbb{Z})$$

$$\bigoplus_{j=0}^{n+1} \bigoplus_{j=0}^{n+1} \text{Tor}_1^\mathbb{Z} (H^j (G; \mathbb{Z}), H^{n+1-j} (H; \mathbb{Z}))$$

We also remark that $H^* (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = \mathbb{Z} \bigoplus_{i=1}^\infty \mathbb{Z}/p\mathbb{Z}[-2i]$.

Since the group $G$ acts trivially on the coefficients $\mathbb{Z}$, we want to compute $H^* (A_p; \mathbb{Z})$ via the Künneth formula

$$H^n (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = \bigoplus_{i=0}^{n} H^i (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \otimes \mathbb{Z} H^{n-i} (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z})$$

$$\bigoplus_{j=0}^{n+1} \bigoplus_{j=0}^{n+1} \text{Tor}_1^\mathbb{Z} (H^j (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}), H^{n+1-j} (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}))$$

We denote the first part with $T_1^n$ and the latter with $T_2^n$. To help the reader for $0 \leq i \leq n$ we also denote by $T_1^{i,n} = H^i (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \otimes \mathbb{Z} H^{n-i} (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z})$ and for $0 \leq j \leq n + 1$ $T_2^{j,n} = \text{Tor}_1^\mathbb{Z} (H^j (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}), H^{n+1-j} (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}))$. Using these notations $T_1^n = \bigoplus_{i=0}^{n} T_1^{i,n}$ and $T_2^n = \bigoplus_{j=0}^{n+1} T_2^{j,n}$.

**Lemma A.1.** One has that

$$T_1^{i,n} = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ \mathbb{Z}/p\mathbb{Z} & \text{if } n > 0 \text{ and } i \text{ are even}, \\ \mathbb{Z} & \text{if } n = 0 \text{ and } i = 0 \end{cases}$$

Therefore,

$$T_1^n = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n \text{ is odd}, \\ (\mathbb{Z}/p\mathbb{Z})^{k+1} & \text{if } n = 2k > 0. \end{cases}$$
Proof. Let \( n \) be odd. \( T^{i,n}_1 \) involves a tensor product between an even and an odd cohomologies degree of \( \mathbb{Z}/p\mathbb{Z} \) and \( H^{odd} (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = 0 \). Similarly if \( n \) is even and \( i \) is odd.

Instead if \( n \) is even one obtains the following cases: if \( i = 0 \) the tensor product becomes \( H^0 (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \otimes \mathbb{Z} H^n (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = \mathbb{Z} \otimes \mathbb{Z} \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \); if \( i > 0 \) we have \( H^n (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \otimes \mathbb{Z} H^{n-1} (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z} \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z}. \)

\[ \square \]

**Lemma A.2.** One has that \( T^{j,n}_2 = \mathbb{Z}/p\mathbb{Z} \otimes (\mathbb{Z}/p\mathbb{Z})^2 \) if and only if \( j \) is non-zero and even, \( n \) is non-zero and odd and \( n + 1 > j \). Otherwise \( T^{j,n}_2 = 0 \).

Therefore, \( T^{n}_2 = \begin{cases} (\mathbb{Z}/p\mathbb{Z})^{k+1} & \text{if } n = 2k + 1, \\ 0 & \text{otherwise.} \end{cases} \)

**Proof.** We remark that if \( I \) and \( J \) are two ideals in \( \mathbb{Z} \), then \( \text{Tor}^\mathbb{Z}_I (\mathbb{Z}/I, \mathbb{Z}/J) = I \cap J/I \cdot J \). If we set \( I = (0) \) and \( J = \mathbb{Z}/p\mathbb{Z} \), \( \text{Tor}^\mathbb{Z}_I (\mathbb{Z}/I, \mathbb{Z}/J) = 0 \). If instead \( I = J = \mathbb{Z}/p\mathbb{Z} \), then \( \text{Tor}^\mathbb{Z}_I (\mathbb{Z}/I, \mathbb{Z}/J) = \mathbb{Z}/I^2 \).

Therefore, \( T^{j,n}_2 = \text{Tor}^\mathbb{Z}_I (\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0. \) If \( j \) is odd, then \( \text{H}^j (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = 0 \) and, similarly, if both \( n \) and \( j \) are even. Therefore so \( T^{j,n}_2 = 0 \). If \( j \) is non-zero and even, one gets \( T^{j,n}_2 = I/I^2 \) with \( I = \mathbb{Z}/p\mathbb{Z} \).

Gluing together all these fact one gets the following results.

\[ \text{H}^n (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} ; \mathbb{Z}) = \begin{cases} (\mathbb{Z}/p\mathbb{Z})^{k+1} & \text{if } n = 2k, \\ (\mathbb{Z}/p\mathbb{Z}^2)^{k} & \text{if } n = 2k + 1. \end{cases} \]

In Lemma 5.3 we claim also the \( \mathbb{Z} \)-algebra structure of \( \text{H}^* (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} ; \mathbb{Z}) \).

One knows that (see Corollary 4.3 in [1]) if \( p \neq 2 \),

\[ \text{H}^* (\mathbb{Z}/p\mathbb{Z}^n; \mathbb{Z}/p\mathbb{Z}) = \wedge_{\mathbb{F}_p} [a_1, \ldots, a_n] \otimes \mathbb{F}_p [b_1, \ldots, b_n]. \]

where \( \deg(a_i) = 1 \) and \( \deg(b_i) = 2 \). Hence \( \text{H}^* (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) = \wedge_{\mathbb{F}_p} [a] \otimes \mathbb{F}_p [b] \) and \( \text{H}^* (A_p; \mathbb{Z}/p\mathbb{Z}) \) is equal to \( \wedge_{\mathbb{F}_p} [a_1, a_2] \otimes \mathbb{F}_p [b_1, b_2] \).

From the short exact sequence \( 0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}^2 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0 \) one gets the Bockstein operator

\[ \beta : \text{H}^i (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) \rightarrow \text{H}^{i+1} (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) \]

that is also a derivation (see II.2 in [1]). One knows that \( \text{H}^1 (\mathbb{Z}/p\mathbb{Z}; (\mathbb{Z}/p\mathbb{Z})^2) \) is the group of homomorphism from \( \mathbb{Z}/p\mathbb{Z} \) to \( (\mathbb{Z}/p\mathbb{Z})^2 \) and it is zero, then \( \beta a \neq 0 \).

We set \( b = \beta a \). Similarly, one sets in \( \text{H}^* (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) \) that \( \beta a_i = b_i \).
For $i > 0$, $H^i(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z})$ has only $p$-torsion. Then, from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$, we get that the injection $H^i(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \hookrightarrow H^i(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z})$. Trivially, $H^1(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = 0$.

From the diagram

$$
\begin{array}{cccccc}
0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z}/p\mathbb{Z} & \to & 0 \\
& & \downarrow & & \downarrow & & Id & & \\
0 & \to & \mathbb{Z}/p\mathbb{Z} & \to & (\mathbb{Z}/p\mathbb{Z})^2 & \to & \mathbb{Z}/p\mathbb{Z} & \to & 0,
\end{array}
$$

one gets the diagram in cohomology:

$$
\begin{array}{cccccc}
0 & \to & H^1(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) & \overset{\delta}{\to} & H^2(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) & \\
& & Id & & \sim & \\
0 & \to & H^1(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) & \overset{\beta}{\to} & H^2(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}).
\end{array}
$$

Therefore $\delta a = \beta b = b$ and so $H^*(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}[\beta]/(\beta b)$.

We have seen that $H^* (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \otimes H^* (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \hookrightarrow H^{even} (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z})$ and for dimensional reason this is an isomorphism.

We have also seen that $H^3 (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}/(\mathbb{Z}/p\mathbb{Z})^2$. We call $y$ the generator of $H^3 (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z})$. Trivially, $y^2 = 0$. We want to show that $H^{odd} (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) = y \cdot H^{odd-3} (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z})$.

Again we look at the diagram

$$
\begin{array}{cccccc}
H^2 (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) & \to & H^2 (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) & \overset{\delta}{\to} & H^3 (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) & \\
& & Id & & \sim & \\
H^2 (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) & \to & H^2 (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) & \overset{\beta}{\to} & H^3 (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}).
\end{array}
$$

Since the Bockstein operator is a derivations $\beta b_i = \beta \beta b_i = 0$ and $\beta (a_1 a_2) = b_1 a_2 - b_2 a_1$.

The diagram commutes so we set $y$ (up to scalar multiplication) to be $\delta (a_1 a_2)$ and it corresponds to $b_1 a_2 - b_2 a_1$ in $H^3 (\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z})$. We now consider

$$
\begin{array}{cccccc}
H^{odd-3} (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) & \overset{-y}{\to} & H^{odd} (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) & \\
& & \downarrow & & \\
H^{odd-3} (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) & \overset{(b_1 a_2 - b_2 a_1)}{\to} & H^{odd} (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}).
\end{array}
$$
For dimensional reason, it is sufficient to prove that the multiplication by \( y \) is injective.

Let \( x \in H^{n-3}_{\text{odd}}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \) such that \( x \cdot y = 0 \). Since the diagram commutes there there is an element \( X \in H^{n-3}_{\text{odd}}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \), image of \( x \) such that \( X \cdot (b_1a_2 - b_2a_1) = 0 \). This is not possible because \( H^{n-3}_{\text{odd}}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \) goes injectively in \( H^{n-5}_{\text{odd}}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \) in the subgroup of elements of the type \( b_1^{j_1}b_2^{n-3/2-j} \) that multiplied by \( b_1a_2 - b_2a_1 \) are not zero.

**B  The resolution of \( \mathbb{A}^4/\mathbb{Z}/5\mathbb{Z} \)**

In this appendix we show all the details of the resolution of the origin of \( \mathbb{A}^4/\mathbb{Z}/5\mathbb{Z} \). Thus, let \( p = 5 \).

Using the Magma-algorithm give in Section 6.2, we refine the fan \( \Delta_5 \) having 4 rays with the following minimal (magma) lattice points \((0,0,1,0)\), \((0,1,0,0)\), \((1,0,0,0)\), and \((1,2,3,5)\).

> \( \Delta'_5 := \text{risoluzione}(5) \);

Fan \( \Delta'_5 \) with 10 rays:

\[
\begin{align*}
& (0, 0, 1, 0), \\
& (0, 1, 0, 0), \\
& (1, 0, 0, 0), \\
& (1, 2, 3, 5), \\
& (1, 2, 3, 4), \\
& (1, 1, 2, 2), \\
& (1, 1, 1, 1), \\
& (1, 2, 3, 3), \\
& (1, 2, 2, 3), \\
& (2, 3, 4, 6)
\end{align*}
\]

and 21 maximal cones:

\[
\text{4-dimensional simplicial cone with 4 minimal generators:}
\begin{align*}
& (1, 0, 0, 0), \\
& (0, 1, 0, 0), \\
& (0, 0, 1, 0), \\
& (1, 1, 1, 1),
\end{align*}
\]

4-dimensional simplicial cone with 4 minimal generators:

\[
\begin{align*}
& (1, 0, 0, 0),
\end{align*}
\]
(0, 0, 1, 0),
(1, 1, 2, 2),
(1, 1, 1, 1),
4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(0, 0, 1, 0),
(1, 1, 2, 2),
(1, 1, 1, 1),
4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(0, 1, 0, 0),
(1, 2, 3, 4),
(1, 1, 1, 1),
4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(1, 2, 3, 4),
(1, 1, 2, 2),
(1, 1, 1, 1),
4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(1, 2, 3, 4),
(1, 1, 2, 2),
(1, 1, 1, 1),
4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(0, 0, 1, 0),
(1, 2, 3, 4),
(1, 2, 3, 3),
4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(0, 0, 1, 0),
(1, 1, 2, 2),
(1, 2, 3, 3),
4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(1, 2, 3, 4),
(1, 1, 2, 2),
(1, 2, 3, 3),
4-dimensional simplicial cone with 4 minimal generators:
(0, 0, 1, 0),
(1, 2, 3, 4),
(1, 1, 2, 2),
(1, 2, 3, 3),
4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(0, 1, 0, 0),
(1, 2, 3, 5),
(1, 2, 2, 3),
4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(0, 1, 0, 0),
(1, 2, 3, 4),
(1, 2, 2, 3),
4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(1, 2, 3, 5),
(1, 2, 3, 4),
(2, 3, 4, 6),
4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(1, 2, 3, 5),
(1, 2, 2, 3),
(2, 3, 4, 6),
4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(1, 2, 3, 4),
(1, 2, 2, 3),
(2, 3, 4, 6),
4-dimensional simplicial cone with 4 minimal generators:
(1, 2, 3, 5),
(1, 2, 3, 4),
(1, 2, 2, 3),
(2, 3, 4, 6),
4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(1, 2, 3, 5),
4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(0, 0, 1, 0),
(1, 2, 3, 5),
(1, 1, 2, 2),

4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(1, 2, 3, 5),
(1, 2, 3, 4),
(1, 1, 2, 2),

4-dimensional simplicial cone with 4 minimal generators:
(0, 0, 1, 0),
(1, 2, 3, 5),
(1, 2, 3, 4),
(1, 1, 2, 2),

4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(0, 0, 1, 0),
(1, 2, 3, 5),
(1, 2, 3, 4)

With Magma we can also select the maximal cones containing a particular ray.

**Magma code:**
Maximal cones of $\Delta'_5$ containing the ray (1, 1, 1, 1)

4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(0, 1, 0, 0),
(0, 0, 1, 0),
(1, 1, 1, 1),

4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(0, 0, 1, 0),
4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(0, 0, 1, 0),
(1, 1, 2, 2),
(1, 1, 1, 1),

4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(0, 1, 0, 0),
(1, 2, 3, 4),
(1, 1, 1, 1),

4-dimensional simplicial cone with 4 minimal generators:
(1, 0, 0, 0),
(1, 2, 3, 4),
(1, 1, 2, 2),
(1, 1, 1, 1),

4-dimensional simplicial cone with 4 minimal generators:
(0, 1, 0, 0),
(1, 2, 3, 4),
(1, 1, 2, 2),
(1, 1, 1, 1),

].

If we go modulo the rays (1, 1, 1, 1) we obtain the following fan.

Star(r_7)

3-dimensional simplicial cone with 3 minimal generators:
(1, 0, 0),
(0, 1, 0),
(0, 0, 1),

3-dimensional simplicial cone with 3 minimal generators:
(1, 0, 0),
(0, 0, 1),
(-1, -1, 0),

3-dimensional simplicial cone with 3 minimal generators:
(0, 1, 0),
3-dimensional simplicial cone with 3 minimal generators:
  (1, 0, 0),
  (0, 1, 0),
  (-3, -2, -1),

3-dimensional simplicial cone with 3 minimal generators:
  (1, 0, 0),
  (-3, -2, -1),
  (-1, -1, 0),

3-dimensional simplicial cone with 3 minimal generators:
  (0, 1, 0),
  (-3, -2, -1),
  (-1, -1, 0),

].
References


