A short proof of Glivenko theorems for intermediate predicate logics

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Abstract

We give a simple proof-theoretic argument showing that Glivenko's theorem for propositional logic and its version for predicate logic follow as an easy consequence of the deduction theorem, which also proves some Glivenko type theorems relating intermediate predicate logics between intuitionistic and classical logic. We consider two schemata, the double negation shift (DNS) and the one consisting of instances of the principle of excluded middle for sentences (REM). We prove that both schemata combined derive classical logic, while each one of them provides a strictly weaker intermediate logic, and neither of them is derivable from the other. We show that over every intermediate logic there exists a maximal intermediate logic for which Glivenko's theorem holds. We deduce as well a characterization of DNS, as the weakest (with respect to derivability) scheme that added to REM derives classical logic.

We say that a set $\mathcal{L}$ of first order formulas is an intermediate predicate logic between intuitionistic and classical predicate logic if:

1. every formula provable in intuitionistic predicate logic IL belongs to $\mathcal{L}$, and every formula in $\mathcal{L}$ is provable in the classical predicate logic CL.

2. $\mathcal{L}$ is closed under modus ponens, substitution and generalization.

(we refer the reader to [1] for a complete definition of the notions in 2).

Let $\mathcal{L}$ and $\mathcal{M}$ be two intermediate predicate logics, where the signature has been previously fixed. We say that Glivenko's theorem for $\mathcal{L}$ over $\mathcal{M}$ holds if and only if, whenever $\vdash_{\mathcal{L}} \phi$, then we also have $\vdash_{\mathcal{M}} \neg\neg\phi$. The case where $\mathcal{L}$ (resp. $\mathcal{M}$) are precisely the classical (resp. intuitionistic) propositional logic had been first considered by Glivenko in [4]. As observed by Kleene in [5], this situation cannot be generalized to the predicate case; instead, one gets a Glivenko theorem setting $\mathcal{M}$ to be the logic obtained from the intuitionistic logic by adding the so called *double negation shift* (DNS), which is the axiom scheme $\forall x\neg\neg\phi(x) \to \neg\neg\forall x\phi(x)$. Several proofs of this fact are known, both syntactic and semantic, involving generally some kind of induction at the meta-level (see [6], [7], [9]) or descriptions of Kripke models which are enough to characterize DNS ([3]). Other generalizations to substructural logics are also known ([2]). The main purpose of this note is to prove that Glivenko's theorem, in the propositional and the predicate version, are immediate consequences of the corresponding deduction theorems.

Consider the *restricted excluded middle* scheme REM consisting of those instances of the principle of excluded middle that involve only closed formulas. In what follows, we write $\mathcal{L}+\mathcal{M}$ for the logic axiomatized by formulas in both $\mathcal{L}$ and $\mathcal{M}$. We use the Hilbert style axiomatization of intuitionistic logic, as exposed, for example, in [8]. Results obtained by using Gentzen systems, as those of [9] and [10], can be transferred to this context in view of the equivalence of such systems, which is proved in [8]. Our main result is:

*This research was financed by the project "Constructive and category-theoretic foundations of mathematics (dnr 2008-5076)" from the Swedish Research Council (VR).

Proposition 1. *Given any set S of classical tautologies, Glivenko's theorem holds for $IL + REM + S$ over $IL + S$.*

Proof. Suppose $\vdash_{IL+REM+S} \phi$. By the deduction theorem, there is a finite number of sentences $\{\phi_i\}_{1\leq i\leq n}$ such that $\vdash_{IL+S} (\bigwedge_{i=1}^n \phi_i \vee \neg\phi_i) \to \phi$, and hence such that $\vdash_{IL+S} \neg\neg(\bigwedge_{i=1}^n \phi_i \vee \neg\phi_i) \to \neg\neg\phi$. Since $\vdash_{IL} \neg\neg(\bigwedge_{i=1}^n \phi_i \vee \neg\phi_i)$, we must also have $\vdash_{IL+S} \neg\neg\phi$. □

Remark 2. Glivenko's theorem for propositional logic can be proved in essentially the same way as above. If we remove S, replace IL by intuitionistic propositional logic IL_p and replace REM by the principle of excluded middle for propositional logic PEM_p, the argument above goes through, since the deduction theorem for propositional intuitionistic logic (as well as the other elementary properties used) hold.

Remark 3. Glivenko's theorem for predicate logic can be seen as a special case of Proposition 1. Take S to be the DNS scheme, then one can show that $IL + REM + DNS$ is precisely CL. Indeed, given any instance of excluded middle, say, $\phi(\overline{x}) \vee \neg\phi(\overline{x})$ (were $\overline{x}$ represents an n-tuple of variables), we note that $\vdash_{IL} \forall\overline{x}\neg\neg(\phi(\overline{x}) \vee \neg\phi(\overline{x}))$, and hence we can prove that $\vdash_{IL+DNS} \neg\neg\forall\overline{x}(\phi(\overline{x}) \vee \neg\phi(\overline{x}))$. Now, since REM allows to derive the scheme $\neg\neg\phi \to \phi$ for closed formulas ϕ, we deduce also that $\vdash_{IL+REM+DNS} \forall\overline{x}(\phi(\overline{x})\vee\neg\phi(\overline{x}))$, from which we conclude using $\forall$-elimination that every instance of excluded middle is provable in $IL + REM + DNS$.

Remark 4. The same proof applies if one considers minimal logic instead of intuitionistic logic, since again, the deduction theorem as well as the other elementary properties used in the proof hold for minimal logic.

Proposition 1 gives, hence, a variety of Glivenko type theorems that, at least for the special case in which S is a $\neg\neg$-stable scheme, happen to be non trivial and optimal, in a sense that we will specify below. For this purpose we need first some characterizations of DNS:

Proposition 5. *The following holds:*

1. *DNS is the weakest (with respect to derivability) scheme that in addition to REM derives classical logic.*

2. *DNS is the strongest (with respect to derivability) scheme amongst the $\neg\neg$-stable classical tautologies.*

3. *REM does not derive DNS nor any $\neg\neg$-stable classical tautology that is not intuitionistically valid.*

4. *DNS does not derive REM.*

Proof. 1. Suppose that $IL + REM + S = CL$ for some set S of classical tautologies. Then, by the deduction theorem, since every instance ϕ of DNS is classically true, there is a finite number of sentences $\{\phi_i\}_{1\leq i\leq n}$ such that $\vdash_{IL+S} (\bigwedge_{i=1}^n \phi_i \vee \neg\phi_i) \to \phi$, and hence, as before, $\vdash_{IL+S} \neg\neg\phi$. Since DNS is a $\neg\neg$-stable scheme, we deduce that $S \vdash_{IL} \phi$.

2. This follows from Glivenko's theorem for predicate logic. Alternatively, suppose that S is a set of $\neg\neg$-stable classical tautologies. Since $IL + REM + DNS = CL$, given any instance ϕ of S, we see by the deduction theorem that there is a finite number of sentences $\{\phi_i\}_{1\leq i\leq n}$ such that $\vdash_{IL+DNS} (\bigwedge_{i=1}^n \phi_i \vee \neg\phi_i) \to \phi$, and hence $DNS \vdash_{IL} \neg\neg\phi$. Since ϕ is $\neg\neg$-stable, this finishes the proof.

3. If REM derives some $\neg\neg$-stable classical tautology ϕ, then by the deduction theorem there is a finite number of sentences $\{\phi_i\}_{1\leq i\leq n}$ such that $\vdash_{IL} (\bigwedge_{i=1}^n \phi_i \vee \neg\phi_i) \to \phi$, and hence $\vdash_{IL} \phi$.

4. If DNS derived REM, we would have $IL + DNS = IL + REM + DNS = CL$, while it is already proven in [10] that $IL + DNS$ is strictly weaker than classical logic.

□

Proposition 6. *The following holds:*

1. *Given any set S of classical tautologies, $IL + REM + S$ is the maximum amongst intermediate predicate logics $\mathcal{L}$ for which Glivenko's theorem over $IL + S$ holds.*

2. *Given any set S of $\neg\neg$-stable classical tautologies, $IL + S$ is the minimum amongst intermediate predicate logics $\mathcal{M}$ over which Glivenko's theorem for $IL + REM + S$ holds.*

Proof. 1. Assume that $\mathcal{L}$ is an intermediate predicate logic for which Glivenko's theorem over $IL + S$ holds, and suppose that $\vdash_{\mathcal{L}} \phi$. Then $\vdash_{\mathcal{L}} \overline{\phi}$, where $\overline{\phi}$ is the universal closure of ϕ. By Glivenko's theorem, $\vdash_{IL+S} \neg\neg\overline{\phi}$, which allows us to deduce that $\vdash_{IL+S} (\overline{\phi} \vee \neg\overline{\phi}) \rightarrow \overline{\phi}$. Since $\overline{\phi} \vee \neg\overline{\phi}$ is an instance of REM, it follows that $\overline{\phi}$ (and hence ϕ) is derivable in the logic $IL + REM + S$. This proves that $\mathcal{L} \subseteq IL + REM + S$.

2. Suppose that Glivenko's theorem holds for $IL + REM + S$ over $\mathcal{M}$. Then every formula of S, being $\neg\neg$-stable, would be provable in $\mathcal{M}$. Therefore $IL + S \subseteq \mathcal{M}$.

□

Proposition 1 can be seen to provide an infinite number of different Glivenko type theorems. Indeed, Umezawa considers in [10], among others, the following $\neg\neg$-stable schemata:

$$M^o : \neg\neg\forall x(\neg\phi(x) \vee \neg\neg\phi(x))$$

$$D^\dagger : \neg\neg\forall y(\forall x(\phi(x) \vee \psi(y)) \rightarrow (\forall x\phi(x) \vee \psi(y)))$$

$$P_n^o : \neg\neg\forall x \left(\bigvee_{i \neq j}^{n} \phi_i(x) \rightarrow \phi_j(x) \right), \quad n \geq 2$$

and shows that each of them is strictly weaker than DNS, though none is intuitionistically provable. In particular, he proves that there is a strict inclusion of logics $IL \subsetneq ... \subsetneq IL + P_n^o \subsetneq ... \subsetneq IL + P_2^o \subsetneq IL + DNS$. It follows immediately that Proposition 1 provides a maximal logic $IL + REM + P_n^o$ for which Glivenko's theorem over $IL + P_n^o$ holds. These results are non trivial in the sense that $IL + REM + P_n^o$ is strictly stronger than $IL + P_n^o$. Indeed, if that was not the case, REM would be derivable from P_n^o, which, being $\neg\neg$-stable, is in turn derivable from DNS by the second part of Proposition 5, and this contradicts the fourth part of that proposition. Moreover, all the logics $IL + REM + P_n^o$ are also strictly weaker than CL due to the first part of the cited proposition (in fact, the proof of that part can be adapted to show that there are strict inclusions $IL + REM \subsetneq ... \subsetneq IL + REM + P_n^o \subsetneq ... \subsetneq IL + REM + P_2^o \subsetneq IL + REM + DNS$). This provides, hence, an infinite number of different instances of Proposition 1 besides the known case of predicate logic.

As a final remark, we note that the proof of Proposition 1 has some computable content in the sense that it allows to reconstruct the proof for $\neg\neg\phi$ in $IL + S$ from the available proof of ϕ in $IL + REM + S$, since this approach relies on the deduction theorem, for which it is possible to give a constructive proof.

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