Cohomology of the moduli space of curves of genus three with level two structure

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Abstract

In this thesis we investigate the moduli space $\mathcal{M}_3[2]$ of curves of genus 3 equipped with a symplectic level 2 structure. In particular, we are interested in the cohomology of this space. We obtain cohomological information by decomposing $\mathcal{M}_3[2]$ into a disjoint union of two natural subspaces, $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$, and then making S_7- resp. S_8-equivariant point counts of each of these spaces separately.

Sammanfattning

Målet med denna uppsats är att undersöka modulirummet $\mathcal{M}_3[2]$ av kurvor av genus 3 med symplektisk nivå 2 struktur. Mer specifikt vill vi hitta information om kohomologin av detta rum. För att uppnå detta delar vi först upp $\mathcal{M}_3[2]$ i en disjunkt union av två naturliga delrum, $\mathcal{Q}[2]$ och $\mathcal{H}_3[2]$, och räknar därefter punkterna av dessa rum S_7- respektive S_8-ekvivariant.

Contents

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1. Introduction

The main object of interest of this thesis is the space $\mathcal{M}_3[2]$ of curves of genus three equipped with a symplectic level two structure and our goal will be to obtain cohomological information about this space. There are several groups which act as automorphisms on $\mathcal{M}_3[2]$ and the cohomology groups of $\mathcal{M}_3[2]$ therefore become representations of these groups. We shall therefore also be interested in the cohomology of $\mathcal{M}_3[2]$ as representations of these groups. Most importantly we shall investigate the cohomology of $\mathcal{M}_3[2]$ as representations of the symmetric group on seven elements, S_7.

The first step in our investigation will be to decompose $\mathcal{M}_3[2]$ into a disjoint union of two natural subspaces: the space $\mathcal{Q}[2]$ consisting of genus three curves with level two structure whose canonical curve is a smooth plane quartic and the space $\mathcal{H}_3[2]$ consisting of hyperelliptic curves of genus three equipped with a symplectic level two structure. We shall then investigate each of these spaces separately by counting points over finite fields. We finally obtain cohomological and representation theoretic information by applying the Lefschetz trace formula to these point counts.

The structure of the thesis is as follows. In Chapter 2 we introduce some theory which might not be standard to all algebraic geometers but is needed later in the thesis. We also introduce the spaces $\mathcal{M}_3[2]$, $\mathcal{Q}[2]$ and $\mathcal{H}_3[2]$ more thoroughly and, perhaps most importantly, we explain an isomorphism between the space $\mathcal{Q}[2]$ and a moduli space $\mathcal{P}^{2,7}_{\mathrm{gp}}$ of certain seventuples of ordered points in $\mathbb{P}^2$. This isomorphism goes via the moduli space of geometrically marked Del Pezzo surfaces of degree 2 so we also discuss these surfaces.

Chapter 2 contains little new information although some effort has been made to collect these facts and present them in an, hopefully, coherent manner. The isomorphism $\mathcal{Q}[2] \cong \mathcal{P}^{2,7}_{\mathrm{gp}}$ is described both in [DO88] and [GH04] and information about Del Pezzo surfaces can be found in [Dem76], [Kol96] and [Man74].

In Chapter 3 we discuss equivariant point counts and how they are connected to the cohomology of a space and to representation theory. The presentation of these results is quite brief and the theory is only given in the generality needed later in the thesis.

In Chapter 4 we apply the results of the two preceding chapters to the space $\mathcal{Q}[2]$ of plane quartics with level structure. More precisely, we make a S_7-equivariant point count of this space and use this to obtain cohomological information. The main results of this chapter are given in Table 4.1, Equation 4.17.1 and Equation 4.17.2.

With the results of Chapter 4 at hand, the only missing piece of information in order to make a S_7-equivariant point count of $\mathcal{M}_3[2]$ is a S_7-equivariant point count of $\mathcal{H}_3[2]$. Therefore, Chapter 5 is devoted to this problem. In the first part of the chapter we give a description of $\mathcal{H}_3[2]$ which is well suited to make a S_8-equivariant point count. This point count can then be used to obtain the desired S_7-equivariant point count. The main results of this chapter are given in Table 5.3.

Finally, in Chapter 6 we put all the pieces together and thus obtain a S_7-equivariant point count of $\mathcal{M}_3[2]$. The results are given in Table 6.2, Equation 6.0.1 and Equation 6.0.2.

The equivariant computations have all been done for a general, odd prime power q. However, the use of several programs written in Maple have been an indispensable tool in verifying the results and, even more importantly, searching for errors in the calculations. Some programs written in Maple has also been of use in obtaining some of the results of Section 5.3.

2. Background

The main object of this thesis is the moduli space of curves of genus three curves with symplectic level two structure, $\mathcal{M}_3[2]$. Two other objects which will be important to us are the moduli space of geometrically marked Del Pezzo surfaces of degree two, $\widetilde{\mathcal{DP}}_2$, and the moduli space of seventuples of points in general position, $\mathcal{P}^{2,7}_{\mathrm{gp}}$. The purpose of this chapter is to introduce these objects and explain a little of what is known about them. Since both level structures and markings are lattices, and since a large part of the discussion will involve these lattices, we will begin by reviewing some of the theory around this subject.

We shall work over an algebraically closed field K whose characteristic is different from 2.

2.1 Lattices

Let A be an integral domain. A *lattice* over A is a pair (L, b) where L is a free A-module and b is a nondegenerate, bilinear form $b : L \times L \to A$. Here, nondegenerate means that if $b(x, y) = 0$ for all $y \in L$, then $x = 0$. We will sometimes leave the bilinear form implicit and denote the lattice (L, b) simply by L.

Let (L, b) and (L', b') be A-lattices and let $\phi : L \to L'$ be an isomorphism of A-modules. If

$$b(x, y) = b'(\phi(x), \phi(y)),$$

for all $x, y \in L$, then ϕ is called an *isometry* and the lattices L and L' are said to be *isometric*.

Rather than developing the general theory of lattices further, we shall directly turn to the examples that will be of most importantance to us.

2.1.1. Hyperbolic lattices. Let $A = \mathbb{Z}$ and let $L = H_r$ be a free $\mathbb{Z}$-module of rank $r + 1$ with generators $e_0, \ldots, e_r$. We define a bilinear form b on H_r by setting

$$b(e_0, e_0) = 1, \quad b(e_i, e_i) = -1,\ i = 1, \ldots, r, \quad b(e_i, e_j) = 0,\ i \neq j.$$

The lattice (H_r, b) is called the *standard hyperbolic lattice* of rank $r+1$. The group of isometries $H_r \to H_r$ is called *the orthogonal group* of H_r and will be denoted $O(H_r)$.

2.1.2. Symplectic vector spaces. In this section we shall give a short introduction to the theory of symplectic vector spaces over the field of two elements. For a more complete treatment, we recommend [Art57].

Let $\mathbb{F}_2$ be the field of two elements and let V be a vector space over $\mathbb{F}_2$ of dimension $2g$. Fix a nondegenerate, alternating bilinear form

$$b : V \times V \to \mathbb{F}_2,$$

i.e. $b(v, v) = 0$ for all v and $v \mapsto b(v, -)$ gives an isomorphism from V to $\mathrm{Hom}(V, \mathbb{F}_2)$. The lattice (V, b) is called a *symplectic space* over $\mathbb{F}_2$. We will sometimes leave the bilinear form implicit and denote the symplectic space (V, b) simply by V.

A subspace $X \subset V$ such that $b(x, x') = 0$ for all x and x' in X is called *isotropic*. Any maximal isotropic subspace has dimension g. To see this, define

$$X^\perp := \{v \in V \mid b(v, x) = 0 \text{ for all } x \in X\},$$

and note that X is a subspace of $X^\perp$ if X is isotropic. Since $\dim X + \dim X^\perp = \dim V = 2g$ it now follows that $\dim X \leq g$. If the inequality is strict we can find a vector v in $X^\perp \setminus X$ and the space $X \oplus \mathbb{F}_2 v$ is then isotropic of dimension $\dim(X) + 1$.

Given a maximal isotropic subspace X, we may complete X into an *isotropic decomposition*

$$V = X \oplus Y,$$

i.e. a decomposition of V such that Y is also isotropic and such that X and Y are in mutual duality via b. Thus, given a basis $x_1, \ldots, x_g$ of X this duality provides a dual basis $y_1, \ldots y_g$ of Y such that $b(x_i, y_j) = \delta_{i,j}$. The basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ of V is then called a *symplectic basis*. The vector space $\mathbb{F}_2^{2g}$ with symplectic basis

$$\begin{aligned} x_i &= \text{ the } i\text{'th coordinate vector}, \quad i = 1, \ldots, g, \\ y_i &= \text{ the } i\text{'th coordinate vector}, \quad i = g+1, \ldots, 2g, \end{aligned}$$

will be called the *standard symplectic space* of dimension $2g$ over $\mathbb{F}_2$.

The *symplectic group* of V is defined as

$$\mathrm{Sp}(V) := \{T \in \mathrm{GL}(V) \mid b(Tu, Tv) = b(u, v), \text{ for all } u, v \in V\}.$$

It is clear that $\mathrm{Sp}(V)$ acts simply transitively on the set of symplectic bases of V, so by counting the symplectic bases one may find that

$$|\mathrm{Sp}(V)| = 2^{g^2}(2^{2g}-1)(2^{2g-2}-1)\cdots(2^2-1).$$

Further, one may show that $\mathrm{Sp}(V)$ is generated by the *transvections*

$$T_u(v) := v + b(u,v)u.$$

The idea of the proof is to first show that the subgroup generated by the tranvections acts transitively on V. Then one shows that it also also acts transitively on pairs of vectors (u,v) such that $b(u,v)=1$ and then, finally, one shows that it acts transitively on the set of symplectic bases.

Consider the transvection T_u. We have

$$\begin{aligned} T_u(T_u(v)) &= T_u(v+b(u,v)u) = \\ &= v + b(u,v)u + b(u, v+b(u,v)u)u = \\ &= v + 2b(u,v)u + b(u,v)b(u,u)u = \\ &= v, \end{aligned}$$

so the transvections are involutions.

2.2 Quadratic forms on symplectic vector spaces

We shall now give some definitions and key properties of quadratic forms on quadratic vector spaces over $\mathbb{F}_2$ and some related topics. See also [GH04].

Let (V,b) be a symplectic vector space over $\mathbb{F}_2$. A function $q: V \to \mathbb{F}_2$ is called a *quadratic form* (relative to b) if

$$q(u+v)+q(u)+q(v) = b(u,v), \tag{2.2.1}$$

for all u and v in V. We shall denote the set of quadratic forms on V by $Q(V)$. Note that we can recover the bilinear form from a fixed quadratic form via the relation 2.2.1. We remark that the action of the symplectic group on V induces an action on $Q(V)$ by

$$T.q(v) = q(T^{-1}v),$$

for $T \in \mathrm{Sp}(V)$.

Given an isotropic decomposition $V = X \oplus Y$ we define a standard quadratic form $q_{X\oplus Y}$ in the following way. Let v be any vector in V and write $v = x + y$ with $x \in X$ and $y \in Y$. Now define

$$q_{X\oplus Y}(v) = b(x,y).$$

More explicitly, if $x_1,\dots,x_g$ is a basis for X and $y_1,\dots,y_g$ is the dual basis of Y we define

$$q_{X\oplus Y}\left(\sum_{i=1}^{g}\alpha_i x_i+\sum_{i=1}^{g}\beta_i y_i\right)=\sum_{i=1}^{g}\alpha_i\beta_i.$$

Definition 2.2.1. Let $x_1,\dots,x_g,y_1,\dots,y_g$ be a symplectic basis of V and let $q\in Q(V)$. The *Arf invariant* of q is defined as

$$\operatorname{Arf}(q)=\sum_{i=1}^{g}q(x_i)q(y_i).$$

A quadratic form is called *even* if $\operatorname{Arf}(q)=0$ and *odd* if $\operatorname{Arf}(q)=1$.

Although this definition seems to depend on the chosen symplectic basis, this is not the case.

Proposition 2.2.2. *The Arf invariant does not depend of the choice of symplectic basis.*

Proposition 2.2.2 was first proven by Arf. The argument we present below is essentially an argument due to Dye, [Dye78]. However, Dye proves a more general statement and his argument can be simplified quite a bit in the case at hand.

Proof. We first observe that since $\operatorname{Sp}(V)$ acts transitively on the set of symplectic bases, it is enough to show that $\operatorname{Arf}(T.q)=Arf(q)$ for all $T\in\operatorname{Sp}(V)$. Further, since $\operatorname{Sp}(V)$ is generated by transvections it is enough to show that $\operatorname{Arf}(T_u.q)=\operatorname{Arf}(q)$ for any transvection T_u.

Let T_u be a transvection. Then we may use (2.2.1) to show that

$$\begin{aligned}q\left(T_u v\right)&=q\left((v+b(u,v)u\right)=\\&=q(v)+b(u,v)q(u)+b(u,v)^2=\\&=q(v)+\big(q(u)+1\big)\,b(u,v),\end{aligned}$$

where the last equality follows since $x^2=x$ for all x in $\mathbb{F}_2$.

Define $\mu=q(u)+1$ and write $u=\sum_{i=1}^{g}\alpha_i x_i+\sum_{i=1}^{g}\beta_i y_i$. We now have

$$\begin{aligned}\operatorname{Arf}(T_u.q)&=\sum_{i=1}^{g}\left[q(x_i)+\mu\beta_i\right]\cdot\left[q(y_i)+\mu\alpha_i\right]=\\&=\sum_{i=1}^{g}q(x_i)q(y_i)+\sum_{i=1}^{g}\mu\alpha_i q(x_i)+\sum_{i=1}^{g}\mu\beta_i q(y_i)+\sum_{i=1}^{g}\mu^2\alpha_i\beta_i= \qquad (2.2.2)\\&=\operatorname{Arf}(q)+\mu\left(\sum_{i=1}^{g}\alpha_i q(x_i)+\beta_i q(y_i)\right)+\sum_{i=1}^{g}\mu\alpha_i\beta_i.\end{aligned}$$

Observe that $\lambda q(v) = q(\lambda v)$ for all $\lambda \in \mathbb{F}_2$ and all $v \in V$. Thus, $\alpha_i q(x_i) = q(\alpha_i x_i)$ and $\beta_i q(y_i) = q(\beta_i y_i)$. By using this observation and applying (2.2.1) repeatedly we now see that

$$\sum_{i=1}^{g} \alpha_i q(x_i) + \beta_i q(y_i) = q(u) + \sum_{i=1}^{g} \alpha_i \beta_i.$$

We now insert this expression into Equation 2.2.2 to see that

$$\begin{aligned}\mathrm{Arf}(T_u.q) &= \mathrm{Arf}(q) + \mu\left(q(u) + \sum_{i=1}^{g} \alpha_i\beta_i\right) + \sum_{i=1}^{g} \mu\alpha_i\beta_i = \\ &= \mathrm{Arf}(q) + \mu q(u) + 2\mu \sum_{i=1}^{g} \alpha_i\beta_i = \\ &= \mathrm{Arf}(q) + \mu q(u).\end{aligned}$$

But $\mu = q(u)+1$ so either $q(u) = 0$ or $\mu = 0$ which gives that $\mu q(u) = 0$. Hence, $\mathrm{Arf}(T_u.q) = \mathrm{Arf}(q)$ as desired. □

The vector space V acts on $Q(V)$ by

$$(v.q)(u) = q(u) + b(v,u).$$

Since b is nondegenerate, this action is free. Specifying values for q on a basis will clearly determine q so $|Q(V)| = 2^{2g}$. Since $|V| = 2^{2g}$ and V acts freely on $Q(V)$ we may now conclude that the action is also transitive.

Definition 2.2.3. If G is a group and X is a set such that G acts on X, then X is said to be a G-*torsor* if, for every pair of elements $x, x' \in X$ there is a unique element $g \in G$ such that $g.x = x'$.

If X is a G-torsor and $g.x = x'$ we say that g is the *ratio* of x' and x and write $g = x'/x$. If G is abelian we instead say that g is the *difference* of x' and x and write $g = x' - x$.

Thus, $Q(V)$ is a V-torsor and if we define $W = V \cup Q(V)$, then W is a $\mathbb{F}_2$-vector space of dimension $2g+1$. Explicitly, the nonobvious additions are given by

$$v + q = q + v := v.q,$$

and

$$q + q' = v,$$

where v is the unique vector in V such that $v.q = q'$ and $v.q' = q$.

The action of the group $\mathrm{Sp}(V)$ on V and $Q(V)$ induces an action of $\mathrm{Sp}(V)$ on the whole of W. Furthermore, we have an $\mathrm{Sp}(V)$-equivariant exact sequence of vector spaces

$$0 \to V \to W \to \mathbb{F}_2 \to 0.$$

Proposition 2.2.4. *The following formulas are satisfied by the Arf invariant.*

(i) $\mathrm{Arf}(q_{X\oplus Y}) = 0$,

(ii) $\mathrm{Arf}(T.q) = \mathrm{Arf}(q)$,

(iii) $\mathrm{Arf}(v.q) = \mathrm{Arf}(q) + q(v)$,

(iv) *the action of* $\mathrm{Sp}(V)$ *on* $Q(V)$ *has two orbits: the* $2^{g-1}(2^g+1)$ *even forms and the* $2^{g-1}(2^g-1)$ *odd forms.*

These properties are well known, see for instance [GH04]. However, since we have not found a complete proof in the literature we provide one here.

Proof. (*i*) Let $V = X \oplus Y$ be an isotropic decomposition of V, let $x_1,\dots,x_g$ be a basis for X and let $y_1,\dots,y_g$ be the corresponding dual basis of Y. The basis $x_1,\dots,x_g,y_1,\dots,y_g$ is then a symplectic basis and we have that

$$\mathrm{Arf}(q_{X\oplus Y}) = \sum_{i=1}^{g} b(x_i,0)b(0,y_i) = 0.$$

Since the Arf invariant does not depend of the choice of symplectic basis, this proves the claim.

(*ii*) As remarked in the proof of Proposition 2.2.2, this is just a reformulation of the fact that the Arf invariant is independent of the choice of symplectic basis.

(*iii*) Clearly, $0.q = q$ and $q(0) = 0$ so $\mathrm{Arf}(0.q) = \mathrm{Arf}(q) + q(0)$. We therefore assume that $v \neq 0$ and set $x_1 = v$. We may now extend x_1 to a symplectic basis $x_1,\dots,x_g,y_1,\dots,y_g$. Then

$$\begin{aligned}\mathrm{Arf}(v.q) &= \sum_{i=1}^{g}(q(x_i)+b(x_i,x_1))(q(y_i)+b(y_i,x_1)) = \\ &= \sum_{i=1}^{g} q(x_i)(q(y_i)+\delta_{i,1}) = \\ &= \sum_{i=1}^{g} q(x_i)q(y_i) + q(x_1) = \\ &= \mathrm{Arf}(q) + q(v),\end{aligned}$$

as claimed.

(*iv*) Note that if $g = 1$, then a form $q_{X\oplus Y}$ corresponding to a isotropic decomposition $V = X \oplus Y$ has the three zeros $(1,0),(0,1)$ and $(0,0)$. Assume that $q_{X\oplus Y}$ has

$$2^{n-2}(2^{n-1}+1),$$

zeros if $g = n-1$. Now let $g = n$ and

$$v = \sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i.$$

We either have $\alpha_n = 0$ or $\alpha_n = 1$.

Suppose first that $\alpha_n = 0$. Then it does not matter if we choose $\beta_n = 0$ or $\beta_n = 1$; in either case the induction hypothesis gives that there are $2^{n-2}(2^{n-1}+1)$ choices for $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}$ such that $q_{X \oplus Y}(v) = 0$. Thus, $q_{X\oplus Y}$ has $2^{n-2}(2^{n-1}+1)$ zeros such that $\alpha_n = 0$.

If $\alpha_n = 1$ on the other hand, we may choose $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}$ however we want in order to obtain a zero of $q_{X\oplus Y}$ but then β_n is determined. Hence, $q_{X\oplus Y}$ has $2^{n-1} \cdot 2^{n-1}$ zeros with $\alpha_n = 1$.

We now add the two cases together to see that $q_{X\oplus Y}$ has

$$2 \cdot 2^{n-2}(2^{n-1}+1) + 2^{n-1} \cdot 2^{n-1} = 2^{n-1}(2^n+1),$$

zeros. We have now shown that $q_{X\oplus Y}$ has $2^{g-1}(2^g+1)$ zeros.

By (*i*) we have that $\mathrm{Arf}(q_{X\oplus Y}) = 0$ and by (*iii*) we have

$$\mathrm{Arf}(v.q_{X\oplus Y}) = \mathrm{Arf}(q_{X\oplus Y}) + q_{X\oplus Y}(v).$$

Since V acts simply transitively on $Q(V)$ we now see that there are $2^{g-1}(2^g+1)$ quadratic forms q such that $\mathrm{Arf}(q) = 0$ and $2^{g-1}(2^g-1)$ quadratic forms such that $\mathrm{Arf}(q) = 1$.

To complete the proof, fix q and consider the action of the transvection T_u. We have

$$\begin{aligned}(T_u.q)(v) = q(T_u v) &= \\ &= q(v + b(v,u)u) = \\ &= q(v) + q(b(v,u)u) + b(v, b(v,u)u) = \\ &= q(v) + b(v,u)q(u) + b(v,u).\end{aligned}$$

Thus, if $q(u) = 1$, then $T_u.q = q$ and if $q(u) = 0$ we have $T_u.q = u.q$. Since V acts simply transitively on $Q(V)$ and $\mathrm{Sp}(V)$ preserves the Arf invariant we now see that $\mathrm{Sp}(V)$ acts transitively on the set of quadratic forms with even resp. odd Arf invariant. □

Corollary 2.2.5. *The following are equivalent*

(i) $\mathrm{Arf}(q) = 0$,

(ii) *q has $2^{g-1}(2^g+1)$ zeros on V,*

(iii) *there is an isotropic decomposition* $V = X \oplus Y$ *such that* q *restricts to zero on* X *and* Y.

Proof. We saw the equivalence of (*i*) and (*ii*) in the proof of Proposition 2.2.4. To see the equivalence of (*i*) and (*iii*), let $V = X \oplus Y$ be an isotropic decomposition and let $q_{X\oplus Y}$ be the corresponding quadratic form. If $\mathrm{Arf}(q) = 0$, then there is some $T \in \mathrm{Sp}(V)$ such that $T.q_{X\oplus Y} = q$. Then q is the quadratic form corresponding to the isotropic decomposition $TX \oplus TY$. □

It is perhaps interesting to remark that because of Corollary 2.2.5, the Arf invariant is sometimes called "the democratic invariant" since it says that $\mathrm{Arf}(q)$ is the value q takes the most times.

Corollary 2.2.6. *The stabilizer* $O(V,q)$ *of* q *in* $\mathrm{Sp}(V)$ *has order*

$$2^{g^2-g+1}(2^{2g-2}-1)\cdot(2^{2g-4}-1)\cdots(2^2-1)\cdot(2^g-1) \quad \textit{if} \,\mathrm{Arf}(q)=0,$$

and

$$2^{g^2-g+1}(2^{2g-2}-1)\cdot(2^{2g-4}-1)\cdots(2^2-1)\cdot(2^g+1) \quad \textit{if} \,\mathrm{Arf}(q)=1.$$

Proof. This is a simple consequence of Proposition 2.2.4 and the orbit-stabilizer theorem. □

Corollary 2.2.7. *Let* $u \in V$ *be a nonzero vector. Then the transvection* T_u *lies in* $O(V,q)$ *if and only if* $q(u) = 1$.

Proof. We saw in the proof of Proposition 2.2.4 that $q(T_u.v) = q(v) + (1 + q(u))b(u,v)$. Thus, if $q(u) = 1$ then $q(T_u.v) = q(v)$. Since b is nondegenerate, there is a vector v such that $b(u,v) \neq 0$. Thus, if $q(u) = 0$ then $q(T_u.v) = q(v) + b(u,v) \neq q(v)$. □

2.2.1. The case $g = 3$. The only case that will be of importantance to us later will be the case where $g = 3$. We shall therefore investigate this case a bit more carefully.

Let $(\mathbb{F}_2^6, b)$ be the standard symplectic space of dimension 6 and let q be the quadratic form defined by

$$q\left(\sum_{i=1}^{3} \alpha_i x_i + \beta_i y_i\right) = \sum_{i=1}^{3} \alpha_i \beta_i.$$

Then q has Arf invariant 0 so, by Corollary 2.2.6, its stabilizer $O(\mathbb{F}_2^6, q)$ has order

$$2^7(2^4-1)(2^2-1)(2^3-1) = 40320 = 8!.$$

This is a first indication that $O(\mathbb{F}_2^6, q)$ in fact is isomorphic to the symmetric group on eight elements, S_8.

By Corollary 2.2.5 we have that q has $2^{3-1}(2^3+1) = 36$ zeros so Corollary 2.2.7 gives that $2^6 - 36 = 28$ nontrivial transvections are contained in $O(\mathbb{F}_2^6, q)$. This is the same as the number of transpositions in S_8 so this gives an idea to how one might cook up the suspected isomorphism.

Recall that S_8 has a presentation given by generators σ_i, $i = 1, \dots, 7$ and relations

$$\begin{aligned} \sigma_i^2 &= \mathrm{id}, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ if } j \neq i \pm 1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{aligned}$$

The generator σ_i corresponds to the transposition of i and $i+1$. We may now define a map ϕ by

$$\begin{aligned} \sigma_1 &\mapsto T_{x_1+y_1}, \\ \sigma_2 &\mapsto T_{x_1+x_2+x_3+y_3}, \\ \sigma_3 &\mapsto T_{x_2+y_2}, \\ \sigma_4 &\mapsto T_{x_1+x_2+x_3+y_1}, \\ \sigma_5 &\mapsto T_{x_3+y_3}, \\ \sigma_6 &\mapsto T_{x_2+y_2+y_3}, \\ \sigma_7 &\mapsto T_{x_1+x_2+x_3+y_1+y_2+y_3}. \end{aligned}$$

It is easily checked that ϕ preserves the relations, so ϕ is a group homomorphism from S_8 to $O(\mathbb{F}_2^6, q)$. However, the only normal subgroups of S_8 are S_8 itself, the alternating group A_8 and the trivial group. Since the image of ϕ clearly contains at least seven elements it follows that ϕ in fact is an isomorphism.

If q and q' are two even quadratic forms, then $O(\mathbb{F}_2^6, q)$ and $O(\mathbb{F}_2^6, q')$ are conjugate since $q' = T.q$ for some $T \in \mathrm{Sp}(\mathbb{F}_2^6)$. Since the index of $O(\mathbb{F}_2^6, q)$ in $\mathrm{Sp}(\mathbb{F}_2^6)$ is 36, we thus obtain 36 copies of S_8. It can also been shown that these are the only ways to embed S_8 into $\mathrm{Sp}(\mathbb{F}_2^6)$, see for instance [CL13].

2.2.2. Aronhold sets. Recall that $W = V \cup Q(V)$ is a $\mathbb{F}_2$-vector space of dimension $2g+1$ and that $Q(V)$ is a V-torsor. Suppose that $A = \{q_1, \dots, q_{2g+1}\}$ is a basis for W with all $q_i \in Q(V)$. An element $w \in W$ can then be expressed as

$$w = \sum_{i=1}^{2g+1} w_i q_i,$$

where $w_i \in \{0,1\}$. If we regard the w_i as integers we may define an integer $0 \le \#_A(w) \le 2g+1$ as

$$\#_A(w) = \sum_{i=1}^{2g+1} w_i.$$

Since $q_i + q_j \in V$ we see that if $q \in Q(V)$, then $\#_A(q)$ is odd.

Definition 2.2.8. A set A is called an Aronhold set if $\mathrm{Arf}(q)$ only depends on $\#_A(q) \mod 4$.

Note that $\#_A(q_1) = \cdots = \#_A(q_{2g+1}) = 1$ so if A is an Aronhold set we must have $\mathrm{Arf}(q_1) = \cdots = \mathrm{Arf}(q_{2g+1})$. We also point out that there is a unique form q_A such that $\#_A(q_A) = 2g+1$, namely the form given by

$$q_A = \sum_{i=1}^{2g+1} q_i.$$

Proposition 2.2.9. *There are Aronhold sets. If $g \equiv 0,1 \mod 4$ then*

$$\mathrm{Arf}(q_1) = \cdots = \mathrm{Arf}(q_{2g+1}) = 0,$$

and if $g \equiv 2,3 \mod 4$ then

$$\mathrm{Arf}(q_1) = \cdots = \mathrm{Arf}(q_{2g+1}) = 1.$$

Furthermore, the group $\mathrm{Sp}(V)$ acts transitively on the collection of Aronhold sets and the stabilizer of an Aronhold set A is the symmetric group of A,

$$\mathrm{Sym}(A) \hookrightarrow O(V, q_A) \hookrightarrow \mathrm{Sp}(V).$$

The proof of this fact is a quite long, but rather elementary computation. The interested reader is referred to [GH04]. There you also find a proof of the following fact.

Lemma 2.2.10. *Let $S = \{q_1, \ldots, q_7\}$ be a set of distinct odd quadratic forms on a 6-dimensional symplectic space. Then S is an Aronhold set if and only if the forms*

$$q_i + q_j + q_k, \quad 1 \le i < j < k \le 7,$$

are all even.

The reason Aronhold sets are important to us is that in the $g = 3$ case, Aronhold sets of odd quadratic forms are the same things as symplectic bases in the following sense.

Proposition 2.2.11. *Let (V, b) be a symplectic space of dimension 6. Then there is a bijection between the set of symplectic bases of V and the set of Aronhold sets of odd quadratic forms of $Q(V)$.*

Proof. We follow the proof given in [DO88].

Recall that we can recover b from a given quadratic form q via relation 2.2.1:

$$b(u, v) = q(u + v) + q(u) + q(v).$$

Let $q_1, \ldots, q_7$ be an ordered Aronhold set of odd quadratic forms. Define elements $v_i \in V$, $i = 1, \ldots, 6$, by

$$v_i = q_i + q_7. \tag{2.2.3}$$

If we take $q = q_7$ in relation 2.2.1 we get

$$b(v_i, v_j) = q_7(v_i + v_j) + q_7(v_i) + q_7(v_j).$$

By Proposition 2.2.4 (*iii*) we have

$$\mathrm{Arf}(q + v) = \mathrm{Arf}(q) + q(v).$$

Since $q_7 + v_i = q_i$ we get that $q_7(v_i) = 0$ for $i = 1, \ldots, 6$. Further, since $q_i + q_j + q_7$ is even if $i \neq j$ we get that $q_7(v_i + v_j) = 1$. Hence, $b(v_i, v_j) = 1$ if $i \neq j$.

Define

$$\begin{array}{llllllllll} x_1 & = & v_1, & x_2 & = & v_2 + v_3, & x_3 & = & v_4 + v_5 \\ y_1 & = & v_1 + \cdots + v_6, & y_2 & = & v_3 + v_4 + v_5 + v_6, & y_3 & = & v_5 + v_6. \end{array}$$

It is easily verified that this is a symplectic basis.

Now suppose that we are given a symplectic basis $x_1, x_2, x_3, y_1, y_2, y_3$ of V. We may now solve the above equations to get six vectors $v_1, \ldots, v_6$ such that $b(v_i, v_j) = 1$ if $i \neq j$. We may now define

$$q\left(\sum_{i=1}^{6} a_i v_i\right) = \sum_{i<j} a_i a_j.$$

Let $u = \sum_{i=1}^{6} a_i v_i$ and $v = \sum_{i=1}^{6} b_i v_i$. Then

$$\begin{aligned} q(u+v) = q\left(\sum_{i=1}^{6} (a_i + b_i) v_i\right) &= \\ = \sum_{i<j} (a_i + b_i)(a_j + b_j) &= \\ = \sum_{i<j} a_i a_j + a_i b_j + a_j b_i + b_i b_j &= \\ = q(u) + \sum_{i \neq j} a_i b_j + q(v) &= \\ = q(u) + b(u, v) + q(v). \end{aligned}$$

Thus, q is a quadratic form of (V,b). A tedious check shows that q has precisely 28 zeros in the vectors $v = \sum_{i=1}^{6} a_i v_i$ with 0, 1, 4 or 5 nonzero coefficients a_i. Thus, q is odd by Corollary 2.2.5, and by Proposition 2.2.4 we have that

$$q'_i = q + v_i, \quad i = 1,\dots,6, \tag{2.2.4}$$

are all odd. Define $q'_7 = q$. We now have seven odd quadratic forms $q'_1,\dots,q'_7$. Note that $q'_i + q'_j + q'_k = q + v_i + v_j + v_k$. Further note that $\mathrm{Arf}(q) = 1$, $q(v_i + v_j) = 1$ if $i \neq j$ and $q(v_i + v_j + v_k) = 1$ if i, j and k are distinct. It follows that if $1 \le i < j < k \le 7$ are distinct and we define $v_7 = 0$, then

$$\begin{aligned}\mathrm{Arf}(q'_i + q'_j + q'_j) &= \mathrm{Arf}(q + v_i + v_j + v_k) = \\ &= \mathrm{Arf}(q) + q(v_i + v_j + v_k) = \\ &= 0.\end{aligned}$$

Thus, $q'_1,\dots,q'_7$ is an Aronhold set by Lemma 2.2.10.

With (2.2.3) and (2.2.4) in mind we see that to check that we have ended up with the Aronhold set we started with, it suffices to show that $q'_7 = q_7$. But it is easily verified, using Proposition 2.2.4 (*iii*), that q_7 has exactly the same zeros as q'_7. Hence, $q_7 = q'_7$ as desired. □

2.3 Curves with symplectic level two structure

Recall that K is an algebraically closed field whose characteristic is different from 2. Let C be a smooth, projective and irreducible curve of genus g over K and let Jac_C be its Jacobian. Recall that the Picard group, $\mathrm{Pic}(C)$, of C is defined as the group of isomorphism classes of line bundles on C under tensor product and that $\mathrm{Pic}^n(C)$ is the subset of line bundles of degree n. Since C is a smooth, projective and irreducible scheme, $\mathrm{Pic}(C)$ is naturally isomorphic (as a graded $\mathbb{Z}$-module) to the group $\mathrm{Cl}(C)$ of divisor classes modulo linear equivalence, see [Har77], Corollary II.6.16. Since we shall only consider group theoretic properties of Jac_C we can make the identifications

$$\mathrm{Jac}_C = \mathrm{Pic}^0(C) = \mathrm{Cl}^0(C).$$

If $D \in \mathrm{Cl}(C)$, we shall denote the corresponding line bundle by $\mathcal{L}(D)$ and we shall use the notation $h^n(D)$ for the dimension of $H^n(C,\mathcal{L}(D))$.

The Jacobian Jac_C has a 2-torsion subgroup

$$\mathrm{Jac}_C[2] := \{v \in \mathrm{Jac}_C | 2v = 0\}.$$

This group is evidently a vector space over the field of two elements, $\mathbb{F}_2$, and it is well known that its dimension is $2g$.

We may define a bilinear form on $\text{Jac}_C[2]$ in the following way. Let u and v be any two elements of $\text{Jac}_C[2]$ and think of them as linear equivalence classes of divisors. Pick a divisor $D \in u$ and a divisor $E \in v$ such that D and E have disjoint support. Since $2u = 2v = 0$ we have $2D = \text{div}(f)$ and $2E = \text{div}(g)$ for some functions f and g. We may now define the *Weil pairing*, b_C, by

$$(-1)^{b_C(u,v)} = \frac{f(E)}{g(D)},$$

where we define $f(E)$ as

$$f(E) := \prod_{P \in C} f(P)^{\text{mult}_P(E)},$$

and make the analogous definition of $g(D)$. There are several things that should be checked here, for instance that the form does not depend on the choices of divisors D and E or the functions f and g and that the Weil pairing is nondegenerate and alternating so that the pair $(\text{Jac}_C[2], b_C)$ is a symplectic vector space of dimension $2g$ over $\mathbb{F}_2$. For these verifications, see for instance [GH78], [ACGH85] or [Mil86].

Definition 2.3.1. A *symplectic level two structure* on a curve C of genus g is an isometry ϕ from the standard symplectic vector space of dimension $2g$ to $(\text{Jac}_C[2], b_C)$. Equivalently, a symplectic level two structure is a choice of an (ordered) symplectic basis $x_1, \dots, x_g, y_1, \dots, y_g$ of $(\text{Jac}_C[2], b_C)$.

Since we shall not talk about other level two structures we shall often just say "level two structure" instead of the more cumbersome "symplectic level two structure".

An isomorphism of curves with level two structures is what one expects it to be. More precisely, two curves with level two structures $(C, x_1, \dots, y_g)$ and $(C', x'_1, \dots, y'_g)$ are isomorphic if there is an isomorphism of curves $\varphi : C \to C'$ such that the induced morphism $\widetilde{\varphi} : \text{Jac}_C[2] \to \text{Jac}_C[2]$ takes one symplectic basis to the other in the sense that

$$\begin{aligned} \widetilde{\varphi}(x_i) &= x'_i, \quad i = 1, \dots g, \\ \widetilde{\varphi}(y_i) &= y'_i, \quad i = 1, \dots g. \end{aligned}$$

We will write (C, φ) and $(C, x_1, \dots, x_g, y_1, \dots, y_g)$ interchangeably, depending on what suits the different situations best.

2.3.1. Genus three curves with level two structure. The main object of interest of this thesis will be the moduli space of genus three curves with level two structure, denoted $\mathcal{M}_3[2]$. This space has dimension 6 and is a degree $|\mathrm{Sp}(\mathbb{F}_2^6)| = 1\,451\,520$ cover of $\mathcal{M}_3$. The space $\mathcal{M}_3[2]$ has two natural subspaces. Firstly, it has the *hyperelliptic locus*, $\mathcal{H}_3[2]$, consisting of curves (C,φ) such that the underlying curve C is hyperelliptic. The space $\mathcal{H}_3[2]$ has dimension 5 and is thus of codimension 1 in $\mathcal{M}_3[2]$. Secondly, if the underlying curve C of (C,φ) is not hyperelliptic, then the canonical linear system $|K_C|$ is very ample and thus gives an embedding $|K_C| : C \hookrightarrow \mathbb{P}^{g-1} = \mathbb{P}^2$. Since $\deg(K_C) = 2g-2 = 4$, we see that the image of C under the canonical embedding is a plane quartic curve. We denote the locus of $\mathcal{M}_3[2]$ consisting of curves whose canonical curve is a plane quartic by $\mathcal{Q}[2]$ and we shall sometimes refer to it as the *plane quartic locus.* The space $\mathcal{Q}[2]$ is a dense open subset of $\mathcal{M}_3[2]$. Thus, the moduli space of genus three curves with level two structure naturally decomposes into the disjoint union

$$\mathcal{M}_3[2] = \mathcal{Q}[2] \coprod \mathcal{H}_3[2],$$

and this decomposition allows us to investigate many questions regarding the full space by investigating the two parts separately.

2.4 Theta characteristics

Again, let C be a smooth curve of genus g over our algebraically closed field K whose characteristic different from 2. We explained above that $\mathrm{Jac}_C[2]$ equipped with the Weil pairing b is a symplectic space over $\mathbb{F}_2$ of dimension $2g$. We shall now explain how this symplectic space and its quadratic forms are connected to the so-called theta characteristics on C.

Definition 2.4.1. Let C be a smooth curve over K and let K_C be its canonical class. An element $\theta \in \mathrm{Pic}(C)$ such that $2\theta = K_C$ is called a theta characteristic.

If θ is a theta characteristic of C and v is an element of $\mathrm{Jac}_C[2]$, then $2(\theta + v) = 2\theta + 2v = 2\theta$ so $\theta + v$ is again a theta characteristic and we see that $\mathrm{Jac}_C[2]$ acts on the set of theta characteristics of C. This might remind us of the way a symplectic vector space V over $\mathbb{F}_2$ acts on its quadratic forms $Q(V)$. In fact, there is a natural identification between the set of theta characteristics of C and the quadratic forms of $\mathrm{Jac}_C[2]$.

Theorem 2.4.2 (The Riemann-Mumford relation)**.** *We may identify the set of theta characteristics of C with the set of quadratic forms on* $\mathrm{Jac}_C[2]$ *(with the Weil pairing) via*

$$q_\theta(v) := h^0(\theta + v) + h^0(\theta) \mod 2.$$

The Arf invariant is given by

$$\mathrm{Arf}(\theta) = h^0(\theta) \mod 2.$$

For a proof, see [Har82] or [ACGH85].

With the Riemann-Mumford relation at hand, we may translate the definitions and results of Section 2.2 to the language of theta characteristics. For instance, we may talk about Aronhold sets of theta characteristics. We also have the following corollary.

Corollary 2.4.3. *If C has genus g, then there are $2^{g-1}(2^g+1)$ even theta characteristics of C and $2^{g-1}(2^g-1)$ odd theta characteristics.*

2.4.1. The genus three case. Since we will mainly be interested in genus three curves, we shall explain in a little more detail what happens in this case. If C is not hyperelliptic, then the canonical system of C gives an embedding into $\mathbb{P}^2$ and the image is a plane quartic curve. We recall the following definition.

Definition 2.4.4. Let C be a plane curve and let $L \subset \mathbb{P}^2$ be a line. Then L is a *genuine bitangent* to C if L is tangent to C in two distinct points and L is a *hyperflex line* if it has contact order 4 in one point of C. If L is either a *genuine bitangent* or a hyperflex line we say that L is a bitangent to C.

Since C is a plane curve, K_C is the restriction of a line L to C and since C has degree 4 we have $K_C = Q_1 + Q_2 + Q_3 + Q_4$ for some, not necessarily distinct, points $Q_1, \ldots, Q_4 \in C$. Thus, an effective theta characteristic must be of the form $\theta = P_1 + P_2$ and such that $2\theta = K_C = 2P_1 + 2P_2$ is the restriction of a line to C. Note that the case $P_1 = P_2$ is allowed. Thus, if P_1 and P_2 are distinct, then P_1 and P_2 must be the points of tangency of a genuine bitangent of C and if $P_1 = P_2 = P$ then P is a hyperflex point, i.e. the point of tangency of a hyperflex line. Thus, there is a natural identification between the set of bitangents of C and the set of effective theta characteristics of C.

If θ is an effective theta characteristic, $\theta = Q_1 + Q_2$, then $h^0(K_C - \theta) = h^0(K_C - Q_1 - Q_2) = h^0(K_C) - 2 = 1$ since K_C is very ample. Thus, the effective theta characteristics of C are all odd. One may show, see for instance [Har77] Chapter IV.2, that a plane quartic has exactly 28 bitangents and Corollary 2.4.3 gives that C has precisely 28 odd theta characteristics, so in fact we have a natural identification between odd theta characteristics and bitangents of C.

In the hyperelliptic case, C will no longer have 28 bitangents but will still have 28 odd theta characteristics. However, K_C can be expressed as the sum of two divisors in the unique g^1_2 of C, see [Har77] Chapter IV.5. Let $Q_1, \ldots, Q_8$

be the ramification points of $g_2^1 : C \to \mathbb{P}^1$. We now find that the divisors $\theta_{i,j} = Q_i + Q_j$, $1 \le i < j \le 8$ are odd theta characteristics of C.

2.5 Points in general position

Recall that our main goal is to study the space $\mathcal{M}_3[2]$ of genus three curves with level two structure (although, admittedly this has not been very apparent this far). Recall also that the space $\mathcal{M}_3[2]$ contains a dense open subset, $\mathcal{Q}[2]$, consisting of plane quartic curves. This space is isomorphic to a space which will be denoted $\mathcal{P}^{2,7}_{\text{gp}}$ which we shall define in this section. The next few sections will be devoted to explaining the isomorphism $\mathcal{Q}[2] \cong \mathcal{P}^{2,7}_{\text{gp}}$.

Definition 2.5.1. Let $\mathbb{P}^2$ denote the projective plane over K and let $(p_1, \ldots, p_r)$ be an ordered tuple of r points in $\mathbb{P}^2$ where $1 \le r \le 7$. We say that the points of the tuple are in *general position* if no three of them lie on a line and no six of them lie on a conic. We shall denote the set of such r-tuples by $\mathbb{P}^{2,r}_{\text{gp}}$.

The projective general linear group PGL(3) acts on $\mathbb{P}^{2,r}_{\text{gp}}$ coordinatewise

$$T.(p_1, \ldots, p_r) = \left(T.p_1, \ldots, T.p_r\right),$$

and this action is free. If $(p'_1, \ldots, p'_r) = T.(p_1, \ldots, p_r)$ for some T in PGL(3) we say that $(p_1, \ldots, p_r)$ and $(p'_1, \ldots, p'_r)$ are *projectively equivalent*. We shall denote the space of r ordered points in the projective plane in general position up to projective equivalence by

$$\mathcal{P}^{2,r}_{\text{gp}} := \mathrm{PGL}(3) \setminus \mathbb{P}^{2,r}_{\text{gp}}.$$

2.6 Del Pezzo surfaces and seven points

The goal of this section is to show that there is an isomorphism between the space $\mathcal{P}^{2,7}_{\text{gp}}$ and the moduli space of so-called *geometrically marked Del Pezzo surfaces* of degree two. As before, we shall work over the algebraically closed field K whose characteristic is not 2.

Definition 2.6.1. A *Del Pezzo* surface is a smooth, rational surface X such that $-K_X$ is ample. The number $(-K_X)^2$ is called the *degree* of the Del Pezzo surface X.

Since K_X is called the canonical class, the class $-K_X$ is often called the *anticanonical class* and the corresponding sheaf ω_X^{-1} is then called the *anticanonical sheaf.* Recall that, by definition, requiring $-K_X$ to be ample is the same as requiring two integers $m, n \ge 1$ and a closed embedding

$$i : X \hookrightarrow \mathbb{P}^m,$$

such that $(\omega_X^{-1})^n$ is the pullback of $\mathcal{O}_{\mathbb{P}^m}(1)$. Another way of putting it is that the global sections of ω_X^{-n} give a closed embedding of X into $\mathbb{P}^m$. It can be shown, see [Kol96], Chapter III.3, that $h^0(X, -mK_X) = \frac{m(m+1)}{2}K_X^2 + 1$ so if ω_X^{-1} would be very ample, as it turns out to be if $K_X^2 \geq 3$, then it gives a closed embedding $X \hookrightarrow \mathbb{P}^{K_X^2}$.

We recall some notions regarding monoidal transformations of surfaces, i.e. blowing up surfaces in a single point. For a more complete treatment, see Chapter V of [Har77].

Let X be a smooth surface, let P be a point of X and let $\pi : \widetilde{X} \to X$ denote the blow-up of X at P. Then $\widetilde{X}$ is also smooth and π restricts to an isomorphism $\widetilde{X} \setminus \pi^{-1}(P) \to X \setminus P$. The inverse image of P under π is an irreducible curve isomorphic to $\mathbb{P}^1$ called an *exceptional curve* and will be denoted by E. The curve E defines a class in $\mathrm{Pic}(\widetilde{X})$, which we also will denote by E, and this class satisfies $E^2 = -1$. Conversely, Castelnuovo's criterion (Theorem V.5.7, [Har77]) tells us that any smooth rational curve D with selfintersection -1 on a surface X occurs as the exceptional curve of a monoidal transformation of another surface.

The map $\pi : \widetilde{X} \to X$ indeuces a natural map $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$. The image of a divisor D under π^* is called the *total transform* of D in $\widetilde{X}$ and is denoted $\pi^* D$. The closure of $\pi^{-1}(C \setminus P)$ in $\widetilde{X}$ is called the *strict transform* of C and denoted $\widetilde{C}$. The maps π^* and

$$\begin{aligned} i : \mathbb{Z} &\to \mathrm{Pic}(\widetilde{X}), \\ 1 &\mapsto E, \end{aligned}$$

give an isomorphism

$$\pi^* \oplus i : \mathrm{Pic}(X) \oplus \mathbb{Z} \to \mathrm{Pic}(\widetilde{X}).$$

We also have a natural projection onto the first factor, $\pi_* : \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(X)$. The maps π^* and π_* satisfy the relations

$$\begin{aligned} \pi^* C \cdot \pi^* D &= C \cdot D, \\ \pi^* C \cdot E &= 0, \\ \pi^* C \cdot B &= C \cdot \pi_* B, \end{aligned}$$

where $C, D \in \mathrm{Pic}(X)$ and $B \in \mathrm{Pic}(\widetilde{X})$. Also, the canonical divisor of $\widetilde{X}$ is given by

$$K_{\widetilde{X}} = \pi^* K_X + E.$$

In particular, we see that $K_{\widetilde{X}}^2 = K_X^2 - 1$. We also recall that

$$\pi^* D = \widetilde{D} + rE,$$

if D is an effective divisor of multiplicity r at P.

The following two results can also be found, stated in slightly greater generality, in [Man74], Chapter IV.24.

Lemma 2.6.2. *Let X be a Del Pezzo surface and let C be an irreducible curve in X with negative self-intersection. Then C is exceptional.*

Proof. Let C be an irreducible curve in X such that $C^2 < 0$. Since $-K_X$ is ample we have that $K_X \cdot C < 0$. The adjunction formula gives

$$2g(C) - 2 = C \cdot (C + K_X) = C^2 + K_X \cdot C < 0,$$

where $g(C)$ denotes the arithmetic genus of C. But C is irreducible so we also have $g(C) \geq 0$ and the only possibility is therefore that $g(C) = 0$. We now conclude that

$$C^2 - (-K_X \cdot C) = -2.$$

Since $C^2 < 0$ and $-(-K_X \cdot C) < 0$ we see that the only possibility is that $C^2 = -1$. Thus, C is a curve of genus 0 with self-intersection -1 and thus exceptional by Castelnuovo's criterion. □

Recall that $p_1, \ldots, p_r$, $1 \leq r \leq 7$ are in general position if no three of them lie on a line and no six of them on a conic.

Theorem 2.6.3. *Let X be a Del Pezzo surface of degree $2 \leq d \leq 7$. Then X is isomorphic to the blow-up of $\mathbb{P}^2$ in $9 - d$ points in general position.*

Proof. Since X is rational, there exists a birational morphism $f : X \to Y$ where Y is a minimal rational surface. The surface Y cannot be a nontrivial ruled surface since it then would have a smooth curve of selfintersection less than -2 and this contradicts Lemma 2.6.2, see [Har77] Proposition V.2.9. Thus, Y is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. If X would itself be equal to $\mathbb{P}^2$ or $\mathbb{P} \times \mathbb{P}^1$, then $K_X^2 = 9$ or $K_X^2 = 8$ which is not the case.

Suppose that $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Since X is not minimal, there is a point P in Y where f^{-1} is not defined. If we define $Y' = \mathrm{Bl}_P Y$, then $f : X \to Y$ can be factored

$$X \xrightarrow{f'} Y' \xrightarrow{\pi} Y,$$

for some birational morphism f', see [Har77] Proposition V.5.3. However, let $p_i : Y = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the projections to the first resp. second factor and define $E_i = \pi^{-1}(p_i^{-1}(p_i(P)))$. Then E_1 and E_2 are exceptional and can be blown down in order to obtain a morphism $Y' \xrightarrow{g} \mathbb{P}^2$, see [Man74] Lemma III.21.3. Then morphism

$$X \xrightarrow{f'} Y' \xrightarrow{g} \mathbb{P}^2,$$

is then a birational morphism.

We may thus assume that there is a birational morphism $f : X \to \mathbb{P}^2$. By Corollary V.5.4 of [Har77] we have that f can be factored as a finite sequence of monoidal transformations

$$X = \mathrm{Bl}_{P_n} X_{n-1} \to X_{n-1} = \mathrm{Bl}_{P_{n-1}} X_{n-2} \to \cdots \to X_1 = \mathrm{Bl}_{P_1} \mathbb{P}^2 \to \mathbb{P}^2.$$

By the above discussion we have that $K_X^2 = \mathbb{K}_{\mathbb{P}^2}^2 - n$. But $K_{\mathbb{P}^2}^2 = 9$ and since X is a Del Pezzo surface of degree d we see that $(-K_X)^2 = K_{\mathbb{P}^2}^2 - n = 9 - n$ so $d = 9 - n$. If one of the blown up points would have lied on an exceptional curve we would get irreducible curves of self-intersection less than -1, which is impossible by Lemma 2.6.2. Hence, X is the blow up of $\mathbb{P}^2$ in $n = 9 - d$ points.

We shall now show the necessity of the points to lie in general position. For simplicity, assume that p_1, p_2 and p_3 lie on a line, D. The general case is completely analogous. Blow up $\mathbb{P}^2$ in p_1, p_2 and p_3 and denote the three corresponding exceptional curves by E_1, E_2 and E_3. We have

$$\begin{aligned}
\widetilde{D}^2 &= (\pi^* D - E_1 - E_2 - E_3)^2 = \\
&= \pi^* D^2 + E_1^2 + E_2^2 + E_3^2 = \\
&= 1 - 1 - 1 - 1 = \\
&= -2.
\end{aligned}$$

Blowing up further points will only decrease $\widetilde{D}^2$ further so $\widetilde{D} \subset \mathrm{Bl}_{p_1,\ldots,p_r} \mathbb{P}^2$ will have self-intersection at most -2. Hence, $\mathrm{Bl}_{p_1,\ldots,p_r} \mathbb{P}^2$ cannot be a Del Pezzo surface by Lemma 2.6.2.

Similarly, if $p_1, \ldots, p_6$ lie on a conic C. Then, after blowing up $p_1, \ldots, p_6$ the strict transform of C will have self-intersection -2 and be irreducible. Thus, the resulting surface cannot be a Del Pezzo surface by Lemma 2.6.2. □

Theorem 2.6.3 allows us to describe the Picard group of a Del Pezzo surface X of degree $2 \leq d \leq 2$. If $r = 9 - d$, then $X = \mathrm{Bl}_{p_1,\ldots,p_r}$ and

$$\mathrm{Pic}(X) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \mathbb{Z}E_r$$

where L is the total transform of a line in $\mathbb{P}^2$ and E_i is the exceptional divisor corresponding to the point p_i. The intersection theory is given by $L^2 = 1$, $E_i^2 = -1$ and $E_i \cdot E_j = E_i \cdot L = 0$ if $i \neq j$. We may also use Theorem 2.6.3 to classify the exceptional curves of Del Pezzo surfaces of degree $2 \leq d \leq 7$ explicitly. Since the only case importantance to us is that of Del Pezzo surfaces of degree 2, we only describe the exceptional curves in this case.

Lemma 2.6.4. *Let $X = \mathrm{Bl}_{p_1,\dots,p_7}\mathbb{P}^2$ be a Del Pezzo surface of degree 2, let $E_1,\dots,$ E_7 be the exceptional curves corresponding to $p_1,\dots,p_7$ and let E be an exceptional curve on X, i.e. an irreducible rational curve of selfintersection -1. Further, let L be the total transform of a line in $\mathbb{P}^2$. Then, either*

(i) *$E = E_i$, $i = 1,\dots,7$, or*

(ii) *$E = L - E_i - E_j$, $1 \le i < j \le 7$, i.e. E is the strict transform of a line through p_i and p_j, or*

(iii) *$E = 2L - \sum_{k=1}^{7} E_k + E_i + E_j$, $1 \le i < j \le 7$, i.e. E is the strict transform of a conic through five of the points $p_1,\dots,p_7$,*

(iv) *$E = 3L - \sum_{k=1}^{7} E_k - E_i$, $i = 1,\dots,7$, i.e. E is the strict transform of a cubic through $p_1,\dots,p_7$ with a double point in p_i.*

In particular, a Del Pezzo surface of degree 2 has exactly 56 exceptional curves.

Proof. Since $-K_X$ is ample we have that $-K_X \cdot E > 0$. Since we also have $E^2 = -1$, the adjunction formula $2g(E)-2 = E\cdot(E+K_X)$ gives that $-K_X\cdot E = 1$.

Let $E = bL + a_1E_1 + \cdots + a_7E_7$. It is easy to see that $K_X = -3L + E_1 + \ldots + E_7$, so we have

$$-K_X \cdot E = 3b + a_1 + \cdots a_7 = 1.$$

We also have

$$E^2 = b^2 - a_1^2 - \cdots - a_7^2 = -1.$$

We rewrite these equalities as

$$a_1 + \cdots a_7 = 1 - 3b,$$

and

$$a_1^2 + \cdots + a_7^2 = b^2 + 1.$$

Recall that the Schwarz inequality states that if x and y are two column vectors in $\mathbb{R}^n$, then $|x^Ty|^2 \le |x|^2 \cdot |y|^2$. If we take $x = (1,1,1,1,1,1,1)$ and $y = (a_1,\dots,a_7)$ we get

$$(a_1 + \cdots + a_7)^2 \le 7\left(a_1^2 + \cdots + a_7^2\right).$$

This now gives that

$$(1-3b)^2 \le 7(b^2+1).$$

This yields $-1 < b < 4$ and one easily checks that the only possible choices for $a_1,\dots,a_7$ are the ones in the above list. □

The converse of Theorem 2.6.3 is also true. We thus have

Theorem 2.6.5. *The blow-up of the projective plane in* $1 \leq r \leq 7$ *points in general position is a Del Pezzo surface of degree* $9-r$.

We shall not prove this fact, but we make some remarks. First, it is clear that the blow-up of the projective plane in any number of points, in general position or not, is a surface which is both smooth and rational. Hence, the only difficulty is showing that the anticanonical class is ample. If $r \leq 6$, one may consider the linear system of cubics passing through $p_1, \ldots, p_6$ and show that it defines a rational map $\mathbb{P}^2 \to \mathbb{P}^{9-r}$ defined away from $p_1, \ldots, p_r$. The composition of this map with the blow-up map $X \to \mathbb{P}^2$ can then be shown to extend to a closed embedding under which ω_X^{-1} is the pullback of $\mathcal{O}_{\mathbb{P}^{9-r}}(1)$, i.e. the anticanonical class is very ample. For a complete discussion along these lines, see [Man74], Chapter IV.24. If $r = 7$ however, the linear system of cubics through $p_1, \ldots, p_7$ gives a two-to-one map from $\mathbb{P}^2$ to $\mathbb{P}^2$ because of the Cayley-Bacharach theorem. Thus, one must instead consider the linear system of sextics through $p_1, \ldots, p_7$ with double points at $p_1, \ldots, p_7$. One may then use the same approach as in [Man74] and with very little extra work show that ω_X^{-1} is base point free and that ω_X^{-2} is very ample. For a slightly different proof, see [Dem76].

We have now described a correspondence between Del Pezzo surfaces of degree $2 \leq d \leq 7$ and tuples of $r = 9-d$ points in general position in $\mathbb{P}^2$. Given r points $p_1, \ldots, p_r$ in general position in $\mathbb{P}^2$ we obtain a Del Pezzo surface X of degree d simply by blowing them up. However, this is not all we get from the blow-up: as remarked earlier, we also get a minimal set of generators for $\mathrm{Pic}(X)$ by letting the first generator L be the total transform of a line in $\mathbb{P}^2$, the second be the first exceptional curve E_1, the third be E_2 and so on up to the $(r+1)$'st generator E_r.

Definition 2.6.6. A *geometrically marked Del Pezzo surface* of degree $d = 9-r$, $2 \leq d \leq 7$, is a pair (X, φ) consisting of a Del Pezzo surface X of degree d and an isometry φ from the standard hyperbolic lattice H_r to $\mathrm{Pic}(X)$ defined via a blow down structure on X by setting

$$\varphi(e_0) = L, \quad \varphi(e_i) = E_i, \quad i = 1, \ldots r.$$

an isomorphism of two geometrically marked Del Pezzo surfaces (X, φ) and (X', φ') is an isomorphism of surfaces $\phi : X \to X'$ such that the diagram

$$\begin{array}{ccc} H_r & \xrightarrow{\varphi'} & \mathrm{Pic}(X') \\ & \searrow^{\varphi} & \downarrow \phi^* \\ & & \mathrm{Pic}(X) \end{array}$$

commutes, i.e. $\varphi = \phi^* \circ \varphi'$.

Equivalently we could define a geometrically marked Del Pezzo surface of degree $d = 9 - r$, $2 \le d \le 7$, as a pair (X, π) consisting of a Del Pezzo surface X of degree d and a blow-up map $\pi : X \to \mathbb{P}^2$ centered in r ordered points $p_1, \ldots, p_r$ (which necessarily are in general position). Two geometrically marked Del Pezzo surfaces (X, π) and (X', π') are then isomorphic if there is an isomorphism of surfaces $\phi : X \to X'$ and an automorphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow[\phi]{} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathbb{P}^2 & \xrightarrow[\varphi]{} & \mathbb{P}^2 \end{array}$$

commutes, and such that $\varphi(p_i) = p_i'$ for $i = 1, \ldots, r$.

Let $\widetilde{\mathcal{DP}}_d$ denote the moduli space of geometrically marked Del Pezzo surfaces of degree d. From the above it is clear that $\widetilde{\mathcal{DP}}_d$ is isomorphic to the space $\mathcal{P}^{2,r}_{\text{gp}}$. This provides the first half of the desired isomorphism $\mathcal{P}^{2,7} \cong \mathcal{Q}[2]$. The second half is to show that $\mathcal{Q}[2]$ is isomorphic to $\widetilde{\mathcal{DP}}_2$.

2.7 Del Pezzo surfaces and plane quartics

In order to explain the connection between geometrically marked Del Pezzo surfaces and plane quartics we shall need some facts about weighted projective spaces. We shall therefore recall some definitions and facts about these objects. For a more thorough treatment, see [Kol96], Section V.1.3.

Let k be a field, $S = k[x_0, \ldots, x_n]$ and let $a_0, \ldots, a_n$ be a nondecreasing sequence of positive integers. We may define a grading on S by setting $\deg x_i = a_i$. The space

$$\mathbb{P}(a_0, \ldots, a_n) = \text{Proj}(S),$$

is called the *weighted projective space* of dimension n with weights $a_0, \ldots, a_n$. In order to somewhat simplify the presentation, we shall always assume that any n of the numbers $a_0, \ldots, a_n$ are relatively prime. All spaces occurring in the sequel are of this form. It can also be shown that any weighted projective space is isomorphic to a weighted projective space of this form so this is in fact not a restriction.

A lot of the properties of ordinary projective spaces have analogues for weighted projective spaces. For instance, we can associate a sheaf to the module $S(m)$, consisting of homogeneous polynomials of (weighted) degree

m, which we will denote by $\mathcal{O}(m)$ in analogy with ordinary projective space. For ordinary projective space the sheaves $\mathcal{O}(m)$ are locally free but now this is only true when m is divisible by all the weights a_i. As a consequence, for (nontrivial) weighted projective spaces, the sheaf $\mathcal{O}(1)$ is never locally free. Another interesting feature of $\mathcal{O}(1)$ is that it has top self intersection

$$\frac{1}{a_0 \cdot a_1 \cdots a_n},$$

when seen as an element in the Picard group of $\mathbb{P}(a_0, \dots, a_n)$.

The canonical sheaf of $\mathbb{P}^n = \mathbb{P}(1, \dots, 1)$ is $\mathcal{O}(-(n+1))$. This generalizes to $\mathbb{P}(a_0, \dots, a_n)$ as

$$\omega_{\mathbb{P}(a_0,\dots,a_n)} = \mathcal{O}\left(-\sum a_i\right),$$

where $\omega_{\mathbb{P}(a_0,\dots,a_n)}$ denotes the dualizing sheaf of $\mathbb{P}(a_0, \dots, a_n)$. While ordinary projective space is smooth, $\mathbb{P}(a_0, \dots, a_n)$ is typically not. At least $\mathbb{P}(a_0, \dots, a_n)$ only has isolated singularities if all the weights are pairwise coprime. If this is the case and H is a smooth hypersurface of (weighted) degree m, where m is divisible by all weights so that $\mathcal{O}(m)$ is locally free, then Proposition II.8.20 of [Har77] applies (to the smooth locus of $\mathbb{P}(a_0, \dots, a_n)$) and we get

$$\omega_H = \omega_{\mathbb{P}(a_0,\dots,a_n)} \otimes \mathcal{O}_{\mathbb{P}(a_0,\dots,a_n)}(m) \otimes \mathcal{O}_H = \mathcal{O}\left(m - \sum a_i\right)|_H.$$

Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a smooth complete intersection of k hypersurfaces of degrees $m_1, \dots, m_k$, where each m_i is divisible by all the weights. Then the anticanonical sheaf of X is given by

$$\omega_X^{-1} = \mathcal{O}_{\mathbb{P}(a_0,\dots,a_n)}\left(\sum_i a_i - \sum_j m_j\right)\Bigg|_X.$$

In particular, the anticanonical sheaf of X is ample if and only if

$$\sum_{i=0}^{n} a_i > \sum_{j=1}^{k} m_j.$$

We are now ready to return to the problem of completing the description of the isomorphism $\mathcal{Q}[2] \cong \mathcal{P}_{\mathrm{gp}}^{2,7}$. We have already shown that $\mathcal{P}_{\mathrm{gp}}^{2,7}$ is isomorphic to the space $\widetilde{\mathcal{DP}}_2$ of geometrically marked Del Pezzo surfaces of degree two. Our aim now is to show that we also have $\widetilde{\mathcal{DP}}_2 \cong \mathcal{Q}[2]$.

Thus, we again work over an algebraically closed field of a characteristic not 2 and we let C be a smooth plane quartic curve over K. Then we have

$$C = V(f),$$

for some homogeneous polynomial f in $K[x_0, x_1, x_2]$. Now consider the polynomial

$$g = x_3^2 - f(x_0, x_1, x_2),$$

in the four variables x_0, x_1, x_2 and x_3. If we give x_0, x_1 and x_2 weight 1 and x_3 weight 2, then g defines a surface X in the weighted projective space $\mathbb{P}(1,1,1,2)$. Note that

$$g_0 = \frac{\partial g}{\partial x_0} = -\frac{\partial f}{\partial x_0}, \qquad g_1 = \frac{\partial g}{\partial x_1} = -\frac{\partial f}{\partial x_1},$$
$$g_2 = \frac{\partial g}{\partial x_2} = -\frac{\partial f}{\partial x_2}, \qquad g_3 = \frac{\partial g}{\partial x_3} = 2x_3.$$

Let $P = (p_0, p_1, p_2, p_3)$ be a point of X. Since C is smooth, $g_0(p_0, p_1, p_2, p_3) = g_1(p_0, p_1, p_2, p_3) = g_2(p_0, p_1, p_2, p_3) = 0$ if and only if $p_0 = p_1 = p_2 = 0$. But if $p_0 = p_1 = p_2 = 0$, then $p_3 \neq 0$ so $g_3(p_0, p_1, p_2, p_3) = 2p_3 \neq 0$ since $\operatorname{char}(K) \neq 2$. Hence, X is smooth.

The surface X has weighted degree 4, which is divisible by all the weights. Further, the sum of the weights is 5 so we conclude that the anticanonical sheaf of X is ample. Hence, $h^0(X, mK_X) = 0$ for all $m \geq 1$ and, in particular, we have $h^0(X, K_X) = h^0(X, 2K_X) = 0$. Recall that the Riemann-Roch theorem for surfaces states that if D is any divisor on a smooth surface S, then

$$h^0(S, D) - h^1(S, D) + h^0(S, K_S - D) = \frac{1}{2} D \cdot (D - K_S) + 1 + h^1(S, K_S).$$

If we take $S = X$ and D to be the zero divisor we get

$$1 - 0 + 0 = 0 + 1 + h^1(X, K_X).$$

We thus see that $h^1(X, K_X) = 1$ and Castelnuovo's rationality criterion now gives that X is rational, see [Har77], Theorem V.6.2.

We have thus shown that the surface $X = V(g) \subset \mathbb{P}(1,1,1,2)$ is smooth, rational and that its anticanonical sheaf is ample. In other words, X is a Del Pezzo surface. From its description as the zero set of the polynomial $g = x_3^2 - f$ we can see that X has an involution ι given by

$$(p_0, p_1, p_2, p_3) \overset{\iota}{\mapsto} (p_0, p_1, p_2, -p_3).$$

If we let $p : X \to \mathbb{P}^2$ denote the degree two covering map

$$(p_0, p_1, p_2, p_3) \mapsto (p_0, p_1, p_2),$$

and recall that $C = V(f)$, then it is easy to see that the fixed point set of ι is exactly $p^{-1}(C)$ and that $p^{-1}(C)$ is isomorphic to C. Using the Riemann-Hurwitz theorem we see that, modulo linear equivalence, we have $K_X =$

$p^*K_{\mathbb{P}^2} + R$ where R is the ramification divisor. By the above we may identify R with the class $p^{-1}(C)$. Since $K_{\mathbb{P}^2} = 3L$, $C = 4L$ and p is a double cover we get

$$\begin{aligned} K_X &= p^*K_{\mathbb{P}^2} + p^{-1}(C) = \\ &= -6p^*L + 4p^*L = \\ &= -2p^*L, \end{aligned}$$

and it follows that $-2K_X = p^{-1}(C)$. The adjunction formula gives

$$\begin{aligned} 4 &= 2g(p^{-1}(C)) - 2 = \\ &= p^{-1}(C)(p^{-1}(C) + K_X) = \\ &= -2K_X(-2K_X + K_X) = \\ &= 2K_X^2, \end{aligned}$$

so $K_X^2 = 2$. Hence, X is a Del Pezzo surface of degree 2 and we have found a way of associating a Del Pezzo surface of degree 2 to a smooth plane quartic.

We shall now explain how to go the other way round. Let Y be a scheme and let $\mathcal{L}$ be an invertible sheaf on Y. For each integer $m \geq 0$, define

$$R_m(\mathcal{L}) = H^0(Y, \mathcal{L}^m).$$

We may now define the *section ring* of $\mathcal{L}$ as the graded ring

$$R(\mathcal{L}) = \oplus_{m \geq 0} R_m(\mathcal{L}).$$

The ring $R(\omega_Y)$ is also called the *canonical ring* of Y and the ring $R(\omega_Y^{-1})$ is also called the *anticanonical ring* of Y.

Lemma 2.7.1. *Let X be a Del Pezzo surface of degree 2. Then the anticanonical ring of X is generated by $R_1(\omega_X^{-1})$ and $R_2(\omega_X^{-1})$.*

The proof of this fact is not very complicated but uses quite a bit of theory which we have little use of elsewhere. The interested reader will find a proof in [Kol96], Chapter III.3.

Theorem 2.7.2. *Let X be a Del Pezzo surface of degree 2. Then $X \cong \mathrm{Proj}(R(\omega_X^{-1}))$ can be described as a surface of degree 4 in $\mathbb{P}(1,1,1,2)$.*

Proof. Since $|-K_X|$ is base point free and $h^0(X, -K_X) = K_X^2 + 1 = 3$ we get a morphism

$$|-K_X| : X \to \mathbb{P}^2.$$

Hence, $R_1(\omega_X^{-1})$ is generated by three elements x, y and z. Since $|-K_X|$ is base point free we have that the subspace of $R_2(\omega_X^{-1})$ generated by $R_1(\omega_X^{-1})$

is six dimensional. Since $h^0(X,-2K_X) = \frac{2(2+1)}{2}K_X^2+1 = 7$ we may use Lemma 2.7.1 to conclude that $R_1(\omega_X^{-1})$ and one element t of $R_2(\omega_X^{-1})$ generates the anticanonical ring of X.

We have that

$$h^0(X,-3K_X) = \frac{3\cdot(3+1)}{2}K_X^2+1 = 13,$$

and there are exactly 13 monomials of degree 3 in x, y, z and t. However, we also have

$$h^0(X,-4K_X) = \frac{4\cdot(4+1)}{2}K_X^2+1 = 21,$$

while there are 22 monomials of degree 4 in x, y, z and t. Thus, there must be a relation

$$g = t^2 - tf_2(x,y,z) - f_4(x,y,z) = 0.$$

Thus, we have that

$$X \cong \mathrm{Proj}(R(\omega_X^{-1})) \cong \mathrm{Proj}(K[x,y,z,t]/(g)),$$

as desired. □

Note that a general surface of degree 4 not contained in a hyperplane in $\mathbb{P}(1,1,1,2)$ is given by a polynomial of the form

$$t^2 - tf_2(x,y,z) - f_4(x,y,z) = 0.$$

However, since the characteristic of K is not 2 we may complete the square and get

$$(t - \frac{1}{2}f_2(x,y,z))^2 - \frac{1}{4}f_2(x,y,z)^2 - f_4(x,y,z) = 0.$$

Thus, after a change of variables we may assume that any such Del Pezzo surface of degree 2 is given by a polynomial of the form $t^2 - f_4(x,y,z) = 0$. We now see that any Del Pezzo surface has an involution ι, which we shall call the *covering involution*. We also see that the fixed point set of the covering involution is isomorphic to a plane quartic curve C and that we have a two-to-one map $p: X \to \mathbb{P}^2$ given by $(x_0, y_0, z_0, t_0) \mapsto (x_0, y_0, z_0)$. Since $X = V(t^2 - f_4(x,y,z))$ is smooth and $C = V(f_4(x,y,z))$, we have that also C is smooth.

Lemma 2.7.3. *Let X be a Del Pezzo surface of degree 2, let ι be its covering involution and let*

$$K_X^{\perp} = \{D \in \mathrm{Pic}(X) | D\cdot K_X = 0\}.$$

Then ι acts on $K_X^{\perp}$ as -1.

Proof. Recall that $|-K_X|$ defines a morphism $\pi : X \to \mathbb{P}^2$. We may thus identify $\mathbb{Z}K_X$ with $\pi^*(\mathrm{Pic}(\mathbb{P}^2))$.

Let $D \in \mathrm{Pic}(X)$ and observe that $D + \iota(D)$ is the pullback of some class $D' \in \mathrm{Pic}(\mathbb{P}^2)$. Thus, we must have $D + \iota(D) = mK_X$ for some integer m. On the other hand, if $D \in K_X^{\perp}$ then $\iota(D \cdot K_X) = \iota(D) \cdot \iota(K_X) = \iota(D) \cdot K_X = 0$ and we must have that $D + \iota(D) \in K_X^{\perp}$. Hence,

$$(D + \iota(D)) \cdot K_X = mK_X^2 = 2m = 0$$

so $m = 0$ and we conclude that $D + \iota(D) = 0$, i.e. $\iota(D) = -D$. □

Proposition 2.7.4. *Let C be a plane quartic curve and let X be the associated Del Pezzo surface of degree 2. Let $p : X \to \mathbb{P}^2$ be the degree 2 morphism ramified along C and let $\iota : X \to X$ be the covering involution. If E is an exceptional curve on X, then $p(E)$ is a bitangent to C. Conversely, the inverse image of any bitangent is the union of two exceptional curves which are conjugate under ι.*

Moreover, $\iota(E) = -K_X - E$ so we have that if E is of type (i) in Lemma 2.6.4 then $\iota(E)$ is of type (iv) and vice versa, and if E is of type (ii) then $\iota(E)$ is of type (iii) and vice versa.

Proof. Let E be an exceptional curve on X. Then $E^2 = -1$ and $E \cdot (-K_X) = 1$. Define $E' = -K_X - E$. Then $E'^2 = -1$ and $E' \cdot (-K_X) = 1$ so E' is also exceptional. Our aim is to show that $\iota(E) = E'$.

Note that $\iota(E') = -K_X - \iota(E)$. Let $D = -K_X - \iota(E) - E = \iota(E') - E$. Then

$$D \cdot K_X = -K_X^2 - \iota(E) \cdot K_X - E \cdot K_X = -2 + 1 + 1 = 0.$$

Thus, $D \in K_X^{\perp}$. By Lemma 2.7.3 we have that ι acts as -1 on $K_X^{\perp}$ so

$$\begin{aligned} -D &= \iota(D) \\ &= -\iota(K_X) - E - \iota(E) = \\ &= -K_X - E - \iota(E) = \\ &= D. \end{aligned}$$

Hence, $2D = 0$. But we know that $\mathrm{Pic}(X)$ is isomorphic to H_7 which is torsion free so we may conclude that $D = 0$. This shows that $\iota(E) = E'$ so the exceptional curves E and E' are conjugate under ι.

We have seen that $p^{-1}(C) \in |-2K_X|$ so $E \cdot p^{-1}(C) = 2$. Hence, E intersects $p^{-1}(C)$ in two points Q_1 and Q_2, which may be the same. Similarly, E' intersects $p^{-1}(C)$ in two points which must be Q_1 and Q_2 since C is pointwise fixed by ι. We thus see that if $Q_1 \neq Q_2$, then $p(E) = p(E')$ intersects C in two points with multiplicity 2 in each and if $Q_1 = Q_2$ then $p(E) = p(E')$

intersects C in a single point with multiplicity 4. In both cases we have $p(E)$ is a bitangent to C.

In Section 2.4.1 we saw that a plane quartic curve has exactly 28 bitangents and Lemma 2.6.4 tells us that X has precisely 56 exceptional curves. By the above we know that these map to bitangents in pairs so we only need to show that no two pairs map to the same bitangent. However, we have already seen that the exceptional curves of types (*i*) and (*ii*) of Lemma 2.6.4 form a set of representatives of the conjugacy classes of ι. It follows from Lemma 2.6.4 that the intersection number of two such curves is at most 1 so two different curves can share at most one point of $p^{-1}(C)$ and can thus not map to the same bitangent. □

As a first corollary we obtain a description of the action ι induces on $\mathbb{P}^2$.

Corollary 2.7.5. *The involution ι acts on L as*

$$\iota(L) = 8L - 3E_1 - \cdots - 3E_7.$$

Thus, ι acts on $\mathbb{P}^2$ as the Cremona transformation given by the linear system of octics through $p_1, \ldots, p_7$ with triple points in each of the points $p_1, \ldots, p_7$.

Proof. We know that $\iota(K_X) = K_X = -3L + E_1 + \cdots + E_7$ and $\iota(E_i) = 3L - E_1 - \cdots E_7 - E_i$. Thus

$$\begin{aligned} -3L + E_1 + \cdots + E_7 = \iota(K_X) = & \\ & = -3\iota(L) + \iota(E_1) + \cdots + \iota(E_7) = \\ & = -3\iota(L) + 21L - 8E_1 - \cdots - 8E_7. \end{aligned}$$

Thus, $\iota(L) = 8L - 3E_1 - \cdots - 3E_7$. □

Recall from Section 2.4 that there is a natural bijection between the set of bitangents of C and the set of odd theta characteristics of C. We thus have the following corollary.

Corollary 2.7.6. *Let C be a plane quartic curve and let X be the associated Del Pezzo surface of degree 2. Then there is a natural two-to-one map from the set of exceptional curves of X to the set of odd theta characteristics of C.*

Recall that a geometrically marked Del Pezzo surface of degree 2 is a pair (X, φ), where X is a Del Pezzo surface of degree 2 and $\varphi : H_7 \to \mathrm{Pic}(X)$ is an isometry such that there is an automorphism α of X such that φ is defined via a blow down structure.

Corollary 2.7.7. *Let (X,φ) be a geometrically marked Del Pezzo surface of degree 2, let $p : X \to \mathbb{P}^2$ be the associated double cover ramified along a smooth plane quartic C and let $E_i = \varphi(e_i)$, $i = 1,\ldots,e_7$, be the exceptional curves coming from the isometry $\varphi : H_7 \to \mathrm{Pic}(X)$. Then there are seven distinct and ordered odd theta characteristics $\theta_1,\ldots,\theta_7$ of C corresponding to the exceptional curves. These are obtained by sending E_i to $p(\pi^{-1}(C) \cap E_i)$.*

In fact, more can be said about the seven odd theta characteristics $\theta_1,\ldots,\theta_7$. The result is apparently due to van Geemen.

Theorem 2.7.8. *The seven ordered odd theta characteristics $\theta_1,\ldots,\theta_7$ of Corollary 2.7.7 form an ordered Aronhold set.*

For a proof of this fact, see [DO88]. This proof takes a detour via so-called Steinerian embeddings. It would be nice to have a proof which is more direct from our current setup, but unfortunately I have not found one.

Thus, to a given geometrically marked Del Pezzo surface X of degree 2 we may associate a plane quartic equipped with an ordered Aronhold set of odd theta characteristics. By Proposition 2.2.11, this is equivalent to giving a plane quartic with a level 2 structure.

Conversely, suppose that we are given a smooth plane quartic curve C equipped with an ordered Aronhold set $\theta_1,\ldots,\theta_7$. We may then consider the double cover $\pi : X \to \mathbb{P}^2$, ramified along C. Each of the theta characteristics θ_i corresponds to a bitangent B_i of C and the inverse image consists of two exceptional curves, $\pi^{-1}(B_i) = E_{i,1} \cup E_{i,2}$.

Suppose that X is the blow-up of $\mathbb{P}^2$ in $p_1,\ldots,p_7$ and that $E_1,\ldots,E_7$ are the corresponding exceptional curves. If ι is the covering involution, then $\iota(E_i) = 3L - E_1 - \cdots - E_7 - E_i$ by Proposition 2.7.4. Define $D_i = \iota(E_i)$. Then $E_i^2 = D_i^2 = -1$, $E_i \cdot E_j = D_i \cdot D_j = 0$ if $i \neq j$, $E_i \cdot D_i = 2$ and $E_i \cdot D_j = 1$ if $i \neq j$. Thus, by properly labeling the exceptional curves $E_{i,j}$, we can make sure that

$$\begin{aligned} E_{i,1} \cdot E_{j,1} = E_{i,2} \cdot E_{j,2} = 0, \qquad & i \neq j \\ E_{i,1} \cdot E_{j,2} = 1, \qquad & i \neq j. \end{aligned}$$

However, we can not distinguish between the two ordered sets $E_{1,1},\ldots,E_{7,1}$ and $E_{1,2},\ldots,E_{7,2}$. In other words, after blowing up seven points and taking the double cover, we can get the curves $E_1,\ldots,E_7$ and their conjugates $\iota(E_1),\ldots,\iota(E_7)$ back and we can separate the two sets, but we can not see which set is which. Luckily, any choice is just as good since they clearly define the same geometric marking on X. This explains the correspondence between $\widetilde{\mathcal{DP}}_2$ and $\mathcal{Q}[2]$.

2.8 Group actions

We have now seen that there are natural maps between the three spaces $\mathcal{P}^{2,7}_{\mathrm{gp}}$, $\mathcal{Q}[2]$ and $\widetilde{\mathcal{DP}}_2$. Each of these spaces has certain groups that act on them in a natural way and we shall now explain how these group actions fit together.

Each element of $\mathcal{P}^{2,7}_{\mathrm{gp}}$ is a projective equivalence class of seven ordered points $p_1,\dots,p_7$ in $\mathbb{P}^2$ and we obtain an action of the symmetric group S_7 simply by permuting the points, $p_i \mapsto p_{\sigma^{-1}(i)}$.

Each element of $\mathcal{Q}[2]$ can be represented by a plane quartic C equipped with a symplectic level 2 structure $\phi : V \to \mathrm{Jac}_C[2]$, where V is the standard symplectic space of dimension $2g$ over $\mathbb{F}_2$. We thus have a natural action of $\mathrm{Sp}(\mathbb{F}_2^6)$ on $\mathcal{Q}[2]$, sending (C,ϕ) to $(C,\phi\circ T^{-1})$.

The elements of $\widetilde{\mathcal{DP}}_2$ are equivalence classes of pairs (X,φ) where X is a Del Pezzo surface of degree 2 and φ is an isometry $H_7 \to \mathrm{Pic}(X)$ which, possibly after an automorphism of X, is given by a blow-down structure. Thus, we might expect that the group of isometries of H_7, $O(H_7)$, acts on $\widetilde{\mathcal{DP}}_2$. This is not the case, since $\varphi\circ\alpha^{-1}$ need not be a geometric marking of X even if φ is. In particular, an element $\alpha \in O(H_7)$ needs to fix the inverse image of the canonical class

$$k = \varphi^{-1}(K_X) = -3e_0 + e_1 + \cdots + e_7,$$

if $\varphi\circ\alpha^{-1}$ shall have any possibility to be a geometric marking. It turns out that this is the only requirement (see [DO88], Proposition V.8) so we find that the stabilizer of k in $O(H_7)$ acts on $\widetilde{\mathcal{DP}}_2$. This group shall be denoted by W. If we define

$$\alpha_1 = e_1 - e_2, \dots, \alpha_6 = e_6 - e_7,\ \alpha_7 = e_0 - e_1 - e_2 - e_3,$$

then W is generated by the elements (see [DO88])

$$s_i(v) = v + b(\alpha_i, v)\alpha_i, \quad i = 1,\dots,7.$$

We have already seen that the canonical involution ι acts trivially on the set of geometric markings. It can be shown that this is the only nontrivial element with this property and that we have an exact sequence of groups (see [DO88])

$$1 \to \langle\iota\rangle \to W \to \mathrm{Sp}(\mathbb{F}_2^6) \to 1.$$

We thus have a natural $\mathrm{Sp}(\mathbb{F}_2^6)$-action also on $\widetilde{\mathcal{DP}}_2$.

So how does these actions fit together? In the case of $\mathcal{Q}[2]$ and $\widetilde{\mathcal{DP}}_2$ one may go through the above constructions to verify that the bijection we have described is in fact equivariant. The S_7-action of $\mathcal{P}^{2,7}_{\text{gp}}$ clearly induces an action on the curve side simply by permuting ordered Aronhold sets. Proposition 2.2.9 describes how this action relates to the $\mathrm{Sp}(\mathbb{F}_2^6)$-action.

One may use the $\mathrm{Sp}(\mathbb{F}_2^6)$-action on $\widetilde{\mathcal{DP}}_2$ to induce a $\mathrm{Sp}(\mathbb{F}_2^6)$-action on $\mathcal{P}^{2,7}_{\text{gp}}$. Abstractly, this action is quite clear: one blows up $\mathbb{P}^2$ in the points $p_1, \dots, p_7$, acts on the resulting exceptional curves by a representative $\alpha \in W$ of the element $\overline{\alpha} \in \mathrm{Sp}(\mathbb{F}_2^6)$ and then blows the result down again. However, we can describe this action more concretely.

Firstly, note that s_i, $i = 1, \dots, 6$, simply acts by transposing the points p_i and p_{i+1}. Since S_7 is generated by these transpositions, see Section 2.2.1, these elements generate an isomorphic copy of $S_7 \subset W$. The element s_7 acts by

$$
\begin{aligned}
s_7(E_1) &= E_1 + (E_0 - E_1 - E_2 - E_3) = E_0 - E_2 - E_3, \\
s_7(E_2) &= E_2 + (E_0 - E_1 - E_2 - E_3) = E_0 - E_1 - E_3, \\
s_7(E_3) &= E_3 + (E_0 - E_1 - E_2 - E_3) = E_0 - E_1 - E_2, \\
s_7(E_4) &= E_4, \\
&\ \ \vdots \\
s_7(E_7) &= E_7.
\end{aligned}
$$

Thus, instead of blowing down E_1 we blow down the exceptional curve corresponding to the line through p_2 and p_3 and similarly for E_2 and E_3. This is exactly the standard quadratic Cremona transformation of $\mathbb{P}^2$ centered in p_1, p_2 and p_3. If we act by an element of PGL(3) so that $p_1 = [1:0:0]$, $p_2 = [0:1:0]$ and $p_3 = [0:0:1]$, then the transformation is given by

$$
s_7([x:y:z]) = [yz : xz : xy].
$$

Thus, $\mathrm{Sp}(\mathbb{F}_2^6)$ acts on $\mathcal{P}^{2,7}_{\text{gp}}$ by permuting the points and performing Cremona transformations in triples of them.

2.8.1. The Geiser involution. Recall that we have two morphisms

$$
\begin{array}{ccc}
X & \xrightarrow{\ p\ } & \mathbb{P}^2_C \\
\downarrow{\scriptstyle \pi} & & \\
\mathbb{P}^2_{p_1,\dots,p_7} & &
\end{array}
$$

where $\mathbb{P}^2_{p_1,\ldots,p_7}$ denotes the projective plane containing the seven points and $\mathbb{P}^2_C$ denotes the projective plane containing the curve C. As a concluding remark, it might be interesting to describe the, possibly singular, curve $\pi(p^{-1}(C))$.

The canonical involution ι of X induces an involution $\tilde{\iota}$ of $\mathbb{P}^2_{p_1,\ldots,p_7}$. By Corollary 2.7.5, this involution is given by the linear system of octic curves through $p_1,\ldots,p_7$ with a triple point in each of the points $p_1,\ldots,p_7$. The involution $\tilde{\iota}$ is known as the *Geiser involution.* In Chapter VII.8 of [RS49], the following alternative description of the Geiser involution is given: consider the net $\mathcal{N}$ of cubics through $p_1,\ldots,p_7$. A point $Q \in \mathbb{P}^2_{p_1,\ldots,p_7} \setminus \{p_1,\ldots,p_7\}$ defines a pencil $\mathcal{P}_Q$ of cubics through $p_1,\ldots,p_7$ and Q. By the Cayley-Bacharach theorem, the pencil $\mathcal{P}_Q$ has nine base points. We already know eight of them so we obtain a ninth point Q'. The Geiser involution takes the point Q to the point Q'. The Geiser involution can be extended to the whole of $\mathbb{P}^2_{p_1,\ldots,p_7}$ by sending p_i to p_i.

It is clear that $\pi(p^{-1}(C))$ must be the fixed point locus of the Geiser involution. We shall now describe this locus more explicitly.

First, we note that a point $Q \in \mathbb{P}^2_{p_1,\ldots,p_7} \setminus \{p_1,\ldots,p_7\}$ is a fixed point if and only if all the members of the pencil $\mathcal{P}_Q$ share the same tangent at Q. By a suitable choice of coordinates, we may assume that $p_1 = [1:0:0]$, $p_2 = [0:1:0]$ and $Q = [0:0:1]$. The pencil $\mathcal{P}_Q$ can then be given as

$$t_0 F_0^Q + t_1 F_1^Q = 0,$$

where

$$F_i^Q = x_2^2(a_0^i x_0 + a_1^i x_1) + x_2(b_0^i x_0^2 + b_{0,1}^i x_0 x_1 + b_1^i x_1) + c_0^i x_0^2 x_1 + c_1^i x_0 x_1^2,$$

for $i = 0$ and 1 are polynomials defining two general members C_0 and C_1 of $\mathcal{P}_Q$. A simple computation shows that the tangent to C_i at Q is given by

$$a_0^i x_0 + a_1^i x_1 = 0.$$

Thus, Q is a fixed point of $\tilde{\iota}$ if and only if there is a nonzero λ such that

$$a_0^1 = \lambda a_0^0, \quad a_1^1 = \lambda a_1^0.$$

We now observe that the curve defined by

$$\lambda F_0^Q - F_1^Q = 0,$$

has a double point at Q and that this curve is the only cubic in $\mathcal{N}$ with a double point at Q. On the other hand, if Q is not a fixed point of $\tilde{\iota}$, then

$(a_0^1, a_1^1) \neq \lambda(a_0^0, a_1^0)$ for all $\lambda \neq 0$ and a simple computation shows that all members of $\mathcal{P}_Q$ are smooth at Q. Hence, the fixed points of the Geiser involution are exactly those points Q such that the pencil $\mathcal{P}_Q$ has a unique member which has a double point at Q. Thus, if the net $\mathcal{N}$ is generated by F_0, F_1 and F_2 then the fixed point locus of $\tilde{\iota}$ is given by

$$\det\begin{pmatrix} \frac{\partial F_0}{\partial x_0} & \frac{\partial F_0}{\partial x_1} & \frac{\partial F_0}{\partial x_2} \\ \frac{\partial F_1}{\partial x_0} & \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_0} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = 0,$$

i.e. a plane sextic curve S. One easily sees that S is singular at $p_1, \dots, p_7$. On the other hand, since $S = \pi(p^{-1}(C))$ we have that the nonsingular model of S has genus 3. Thus, if we let $r_i \geq 2$ denote the order of the singularity at p_i then the genus-degree formula gives

$$3 \leq \frac{(6-1)(6-2)}{2} - \sum_{i=1}^{7} \frac{r_i(r_i-1)}{2} \leq 3.$$

We thus see that $p_1, \dots, p_7$ are the only singularities of S and that they are all double points. The singularity type at p_i depends on the configuration of the points $p_1, \dots, p_7$. More precisely, we have the following.

Proposition 2.8.1. *The curve $\pi(p^{-1}(C))$ is a sextic curve S with double points precisely at $p_1, \dots, p_7$. The singularity at p_i is a node if the cubic curve through $p_1, \dots, p_7$ with a singularity at p_i is nodal, otherwise it is a cusp.*

Proof. The first part is just a restatement of the result of the above discussion. Recall that the strict transform of a curve D is denoted by $\widetilde{D}$. Also, recall Lemma 2.7.4.

Suppose that the cubic C_i through $p_1, \dots, p_7$ with a double point at p_i is nodal. Then E_i and $\iota(E_i) = \widetilde{C}_i$ will intersect in two distinct points. These points are clearly fixed by ι and must therefore be points of $C = \widetilde{S}$. Thus, E_i intersects the strict transform of S in two distinct points, so p_i is a node.

Similarly, if C_i is a cusp we see that E_i and $\iota(E_i) = \widetilde{C}_i$ intersect in a single point with multiplicity two. It follows that E_i and $\widetilde{S}$ intersect in this point with multiplicity two so we have that p_i is a cusp of S. □

3. Cohomology, point counts and representations

When investigating a space, an important invariant is the cohomology. Here we shall discuss general properties of cohomology and its relation to group representations and point counts over finite fields. Later, we shall use this to investigate the spaces $\mathcal{Q}[2]$, $\mathcal{H}_3[2]$ and $\mathcal{M}_3[2]$.

However, before we turn to these considerations we shall explain the setup we shall be working with. This setup will be quite a bit more restrictive than necessary for many of the upcoming discussions but it is general enough for our purposes. See also [Ber08] and [BT07].

3.1 The Frobenius morphism

Let p be a prime number, n a positive integer and let $q = p^n$. We shall denote the finite field $\mathbb{F}_q$ by k, a degree d extension of k by k_d and we shall use the symbol $\overline{k}$ to denote an algebraic closure of k.

Any ring R of characteristic p has a canonical endomorphism φ defined by sending an element x to x^q. This endomorphism is called the Frobenius endomorphism. If we take $R = k_d$, then the points fixed by the Frobenius endomorphism are exactly the points of k. This is especially easy to see in the case where $k = \mathbb{F}_p$ when it is a simple consequence of Fermat's little theorem.

Let X be a scheme over k and let $\{U_i\}$ be an affine cover of X, where $U_i = \mathrm{Spec}(A_i)$ for some k-algebras A_i. In particular, each A_i is a ring of characteristic p and thus has a Frobenius endomorphism φ_i. These endomorphisms induce endomorphisms on the open affine sets $\phi_i : \mathrm{Spec}(A_i) \to \mathrm{Spec}(A_i)$ and we may glue these endomorphisms in order together to obtain an endomorphism $\phi : X \to X$. This endomorphism is called the absolute Frobenius endomorphism. As a map of topological spaces it is the identity, but on sections it takes an element s to s^q.

Now, let X be a variety over k and let $X_{\overline{k}} = X \times_k \mathrm{Spec}(\overline{k})$ be the base change of X to $\overline{k}$. We may then use the same definition as above to obtain a *map* $\phi : X_{\overline{k}} \to X_{\overline{k}}$ which we reasonably could call the absolute Frobenius map. However, this map is the "wrong" thing to consider for several reasons. Most importantly, the absolute Frobenius map is not a morphism of schemes over $\overline{k}$. The reason is that on each open affine $U = \operatorname{Spec}(A) \subset X$, the induced map $\varphi : A \to A$ is not a map of $\overline{k}$-algebras. For this to hold we need $\varphi(r \cdot x) = r \cdot \varphi(x)$ for all $r \in \overline{k}$ and this only holds if $r^q = r$, that is to say only if r is an element of k.

In order to obtain a morphism of $\overline{k}$-schemes, we note that the map

$$F := \phi \times_k \mathrm{id} : X \times_{\overline{k}} \operatorname{Spec}(\overline{k}) \to X \times_k \operatorname{Spec}(\overline{k}),$$

is a $\overline{k}$-scheme endomorphism of $X_{\overline{k}}$. We shall call this morphism the Frobenius morphism, or sometimes only the Frobenius. In contrast to the absolute Frobenius endomorphism on X, the Frobenius endomorphism F on $X_{\overline{k}}$ is not the identity map as a map of topological spaces. This is in fact something we want since we will want the Frobenius to act on $X(\overline{k})$ as an element of $\operatorname{Gal}(\overline{k}/k)$. This, and some other nice properties of the Frobenius, will be discussed a bit more later in this chapter.

3.2 Conjugate tuples

Let X be a k-scheme. In this section we shall introduce some terminology concerning collections of subschemes of $X_{\overline{k}}$ which will be used quite frequently in the remainder of the thesis.

Definition 3.2.1. Let X be a k-scheme and $S \subset X_{\overline{k}}$ be a subscheme. If S is defined over k_d but not over $k_{d'}$ for any d' dividing d we shall say that S is a *strict* k_d*-subscheme*. Equivalently, a strict k_d-subscheme S is a subscheme whose Galois orbit has order d.

A related, but slightly different concept is the following.

Definition 3.2.2. Let X be a k-scheme and let $S \subset X_{\overline{k}}$ be a strict k_d-subscheme. The ordered tuple $(S, FS, \ldots, F^{d-1}S)$ is then called a *conjugate* d*-tuple*. Equivalently, an ordered collection $(S_1, \ldots, S_d)$ of distinct subschemes of $X_{\overline{k}}$ is called a conjugate d-tuple if

$$FS_i = S_{i+1}, \quad i = 1, \ldots, d-1, \quad FS_d = S_1.$$

Thus, a strict k_d-subscheme is not defined over k (unless $d = 1$), but a conjugate d-tuple is by definition defined over k as a set.

We shall now make similar definitions for more general m-tuples of subschemes. Let $\lambda = [1^{\lambda_1}, 2^{\lambda_2}, \ldots, \nu^{\lambda_\nu}]$ be a partition of the positive integer m,

$$m = \sum_{i=1}^{\nu} i \cdot \lambda_i,$$

and define

$$r = \sum_{i=1}^{\nu} \lambda_i.$$

We shall temporarily call an ordered collection $(S_1, \ldots, S_r)$ of subschemes of $X_{\overline{k}}$ *compatible* if it consists of λ_1 strict k-subschemes, λ_2 strict k_2-subschemes, and so on, such that the Galois orbits of each of the subschemes are disjoint from those of the others. We say that the compatible collection has type λ.

Denote the order of the Galois orbit of a subscheme $S \subset X_{\overline{k}}$ by d_S. We obtain a map Λ from the set of compatible collections of subschemes of type λ to the set of m-tuples of subschemes of $X_{\overline{k}}$ by sending a collection $(S_1, \ldots, S_r)$ to

$$(S_1, FS_1 \ldots, F^{d_{S_1}-1} S_1, \ldots, S_r, \ldots, F^{d_{S_r}-1} S_r).$$

Definition 3.2.3. Let X be a k-scheme and let $\lambda = [1^{\lambda_1}, 2^{\lambda_2}, \ldots, \nu^{\lambda_\nu}]$ be a partition of the positive integer m. An ordered collection $S = (S_1, \ldots, S_m)$ of distinct subschemes of $X_{\overline{k}}$ is called a *conjugate λ-tuple* if it is the image of a compatible collection of subschemes of type λ under the map Λ.

Equivalently, S is called a conjugate λ-tuple if it consists of λ_1 conjugate 1-tuples, λ_2 conjugate 2-tuples and so on, where the elements are ordered in a way such that the elements of each λ_i-tuple are consecutive and ordered in the same way as in Definition 3.2.2.

For each positive integer d, we shall use the notation $X(d)$ to denote the set of conjugate d-tuples of of closed points of $X_{\overline{k}}$. From now on we shall say "point" to mean "closed point".

There is a natural bijection from the set $X[k_d]$ of strict k_d-points of $X_{\overline{k}}$ to the set $X(d)$ defined by sending a strict k_d-point x to the conjugate d-tuple

$$\{x, Fx, \ldots, F^{d-1}x\}.$$

For each partition λ of a positive integer m, we define $X(\lambda)$ to be the set of conjugate λ-tuples of closed points of $X_{\overline{k}}$.

If one wants to compute $|X(\lambda)|$, the following lemma is one of the main tools.

Lemma 3.2.4. *Let X be a k-scheme and let $\lambda = [1^{\lambda_1}, 2^{\lambda_2}, \ldots, \nu^{\lambda_\nu}]$ be a partition. Then*

$$|X(\lambda)| = \prod_{i=1}^{\nu} \prod_{j=0}^{\lambda_i - 1} \left[\left(\sum_{d|i} \mu\left(\frac{i}{d}\right) \cdot |X(d)| \right) - i \cdot j \right],$$

where μ is the Möbius function.

3.3 The Lefschetz trace formula, part 1

Although it could perhaps be possible to convince the reader that it is interesting to compute the numbers $|X(\lambda)|$ in their own right, the main motivation for doing so comes from cohomology. The connection is provided by the so-called Lefschetz trace formula. A detailed discussion of this subject goes beyond the scope of this thesis and the treatment presented here will be rather terse. For a more detailed discussion, see [Del77], especially section 4.5, and also [Beh93].

For motivation, recall Lefschetz fixed point theorem from topology. It can be stated in the following way. Let $f : X \to X$ be a continuous map from a triangulable space to itself and let $H^i(f) : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$ be the induced map on (simplicial) cohomology. The Lefschetz fixed point theorem then states that if the Lefschetz number

$$\Lambda_f := \sum_{i\geq 0} (-1)^i \mathrm{Tr}(H^i(f)),$$

is non-zero, then f has at least one fixed point.

Let X be a k-scheme of pure dimension and of finite type over k . Let $\ell \neq p$ be a prime number and let $\mathbb{Q}_\ell$ denote the field of ℓ-adic numbers. It is a highly nontrivial fact that there is a cohomology theory defined for varieties over $\overline{k}$ with coefficients in $\mathbb{Q}_\ell$ satisfying many properties analogous to the singular cohomology of complex algebraic varieties. This cohomology theory is called étale cohomology. For an introduction to this subject, see for instance [Mil80]. Let $H^i_{\text{ét,c}}(X_{\overline{k}},\mathbb{Q}_\ell)$ denote the i'th compactly supported étale cohomology group of $X_{\overline{k}}$ with coefficients in $\mathbb{Q}_\ell$.

Since F is an endomorphism of $X_{\overline{k}}$ we get an induced endomorphism of the $\mathbb{Q}_\ell$-vector space $H^i_{\text{ét,c}}(X_{\overline{k}},\mathbb{Q}_\ell)$, via general functoriality properties of cohomology. We shall denote also this induced endomorphism by F. Our first version of the Lefschetz trace formula now states that

$$|X(1)| = \sum_{i=0}^{\dim X} (-1)^i \cdot \mathrm{Tr}\left(F|H^i_{\text{ét,c}}(X_{\overline{k}},\mathbb{Q}_\ell)\right),$$

where $\mathrm{Tr}\left(F|H^i_{\text{ét,c}}(X_{\overline{k}},\mathbb{Q}_\ell)\right)$ denotes the trace of the induced endomorphism on $H^i_{\text{ét,c}}(X_{\overline{k}},\mathbb{Q}_\ell)$. Thus, by counting the k-points of X we obtain information about the cohomology. In particular, we have a very nice interpretation of the Lefschetz number Λ_F: it is the number of k-points of X.

3.4 Representations

Again, let X be a k-scheme of pure dimension and of finite type over k and suppose that a finite group G is acting on X in a way such that $g : X \to X$ is an automorphism for each $g \in G$.

In order to linearize the action, we pass to cohomology. Again, using general functorial properties of cohomology we have that each $g \in G$ induces an endomorphism of $\mathbb{Q}_\ell$-vector spaces $H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ for each i. Thus, for each i we obtain a linear representation $G \to GL(H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_\ell))$.

Let C be a set of representatives of the irreducible characters of G. For each $\chi \in C$, let V_χ denote the irreducible representation of G corresponding to χ. Then we have

$$H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_\ell) = \bigoplus_{\chi \in C} V_\chi^{r_{i,\chi}},$$

for some non-negative integers $r_{i,\chi}$. We obtain a projection from the vector space $H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ to $V_\chi^{r^i_\chi}$ by (see, for instance, [FH04])

$$\pi_\chi = \dim\left(V_\chi\right) \cdot \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot g. \tag{3.4.1}$$

Remark 3.4.1. To be precise we should really either also assume that all irreducible $\mathbb{Q}_\ell$-representations of G are also absolutely irreducible, i.e. that they remain irreducible over any extension of $\mathbb{Q}_\ell$, or tensor with $\overline{\mathbb{Q}_\ell}$. However, since all irreducible representations we shall be interested in are defined over $\mathbb{Q}$, this point is not particularly important to us.

3.5 The Lefschetz trace formula, part 2

Also in this section we let X be an equidimensional scheme of finite type and let G be a finite group which acts via automorphisms on X. We first point out that our first Lefschetz trace formula can be reformulated as

$$\left|X_{\overline{k}}^F\right| = \sum_{i=0}^{\dim X} (-1)^i \cdot \operatorname{Tr}\left(F | H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right),$$

where $\left|X_{\overline{k}}^F\right|$ is the cardinality of the set of points fixed by the Frobenius morphism.

Since we also have an action of G on X one might ask if there is a similar formula if we replace F by $F \cdot g$ for some g in G. The answer is yes, and it is provided by the second version of Lefschetz trace formula:

$$\left|X_{\overline{k}}^{F \cdot g}\right| = \sum_{i=0}^{\dim X} (-1)^i \cdot \operatorname{Tr}\left(F \cdot g | H^i_{\text{ét,c}}(X_{\overline{k}}, \mathbb{Q}_\ell)\right), \tag{3.5.1}$$

where $\left|X_{\overline{k}}^{F\cdot g}\right|$ is the cardinality of the set of points fixed by the composition of g and the Frobenius.

By combining formulas 3.4.1 and 3.5.1 we obtain the relation

$$\dim\left(V_\chi\right)\cdot\frac{1}{|G|}\sum_{g\in G}\chi(g)\cdot\left|X_{\overline{k}}^{F\cdot g}\right| = \sum_{i=0}^{\dim X}(-1)^i\cdot\mathrm{Tr}\left(F, V_\chi^{r_{i,\chi}}\right). \tag{3.5.2}$$

Thus, by computing $\left|X_{\overline{k}}^{F\cdot g}\right|$ for all possible g we obtain not only cohomological information but also information about the cohomology as a representation of G.

Definition 3.5.1. Let X be an equidimensional scheme of finite type and let G be a finite group which acts via automorphisms on X. A G-equivariant point count of X is a computation of $\left|X_{\overline{k}}^{F\cdot g}\right|$ for each g in G.

We remark that if the action of G is defined over k, then action of G commutes with the Frobenius. We thus see that

$$\left|X_{\overline{k}}^{F\cdot g}\right| = \left|X_{\overline{k}}^{h(F\cdot g)h^{-1}}\right| = \left|X_{\overline{k}}^{F\cdot hgh^{-1}}\right|,$$

and thus that the numbers $\left|X_{\overline{k}}^{F\cdot g}\right|$ only depend on the conjugacy class of g.

4. Counting plane quartics with level structure

We shall now apply the theory discussed in Chapter 3 to the moduli space of smooth plane quartic curves with symplectic level 2 structure, $\mathcal{Q}[2]$. As explained in Chapter 2 we have an action of $\mathrm{Sp}(\mathbb{F}_2^6)$ on $\mathcal{Q}[2]$, so ideally we would make equivariant point counts for this group action. However, this group action is rather subtle and we shall therefore make equivariant point counts for the symmetric group S_7 for which the action can be seen much more directly.

More precisely, recall from Chapter 2 that there is an equivariant bijection from $\mathcal{Q}[2]$ to the moduli space $\mathcal{P}^{2,7}_{\mathrm{gp}}$ of ordered seventuples of points in $\mathbb{P}^2$ in general position. Here we can see the S_7-action as the action which permutes the points of each seventuple.

Before discussing this matter further we shall fix some notation. We shall work over the finite field $k = \mathbb{F}_q$ where $q = p^n$ for some odd prime number p. We shall use k_d to denote a degree d extension of k and $\overline{k}$ to denote an algebraic closure of k. The schemes we shall consider in this chapter will typically be defined over k or k_d. Recall that we say that a subscheme of a k-scheme is a strict k_d-subscheme if it is defined over k_d but not over $k_{d'}$ for any d' dividing d, see Definition 3.2.1. We shall write PGL(3) for the projective general linear group over k. It can be identified with the group of 3×3 invertible matrices with coefficients in k modulo multiplication of nonzero scalars in k. It has cardinality

$$|\mathrm{PGL}(3)| = \frac{(q^3-1)(q^3-q)(q^3-q^2)}{(q-1)} = q^3(q^3-1)(q^2-1).$$

We shall now give an adaptation of an argument which can be found, for instance, in [Ber08] and [Ber09]. Let σ be a permutation of S_7 and let λ be its cycle type. Define S_σ to be the category of seventuples of ordered points $(p_1, \ldots, p_7)$ in $\mathbb{P}^2$ defined over $\overline{k}$ such that $(F \cdot \sigma)(p_i) = p_i$ for $i = 1, \ldots, 7$. The morphisms of S_σ are automorphisms of $\mathbb{P}^2$. For notational purposes we define $S_\sigma^{\overline{k}}$ to be the set of $\overline{k}$-isomorphism classes of S_σ and similarly S_σ^k is the set of k-isomorphism classes of S_σ.

Let $P = (p_1, \dots, p_7)$ be a representative of a point in $(\mathcal{P}_{\mathrm{gp}}^{2,7})^{F\cdot\sigma}$. Then there is an isomorphism between P and $(F\cdot\sigma)P$ and this isomorphism gives an element of S_σ and we find that $|S_\sigma^{\overline{k}}| = |(\mathcal{P}_{\mathrm{gp}}^{2,7})^{F\cdot\sigma}|$, see [KS99].

Let P be an object in S_σ. The sum, over all objects Q in S_σ^k which defines the same object as P in $S_\sigma^{\overline{k}}$, of the reciprocal of the number of k-automorphisms of Q is equal to 1, i.e.

$$\sum_{\substack{[Q]\in S_\sigma^k \\ Q\cong_{\overline{k}} P}} \frac{1}{|\mathrm{Aut}_k(Q)|} = 1.$$

We may thus go between k-isomorphism classes and $\overline{k}$-isomorphism classes via the formula

$$\left|(\mathcal{P}_{\mathrm{gp}}^{2,7})^{F\cdot\sigma}\right| = \sum_{[P]\in S_\sigma^{\overline{k}}} 1 = \sum_{[P]\in S_\sigma^{\overline{k}}} \sum_{\substack{[Q]\in S_\sigma^k \\ Q\cong_{\overline{k}} P}} \frac{1}{|\mathrm{Aut}_k(Q)|} = \sum_{[Q]\in S_\sigma^k} \frac{1}{|\mathrm{Aut}_k(Q)|}.$$

Let $P_1, \dots, P_m$ be representatives of objects in S_σ^k. For each P_i we can act with $\mathrm{Aut}_k(\mathbb{P}^2) = \mathrm{PGL}(3)$ and thus get an orbit in S_σ. The stabilizer of P_i is equal to $\mathrm{Aut}_k(P_i)$ and together the orbits of $P_1, \dots, P_m$ will contain $|\mathbb{P}_{\mathrm{gp}}^{2,7}(\lambda)|$ elements (recall Definition 3.2.3). We thus obtain

$$\left|(\mathcal{P}_{\mathrm{gp}}^{2,7})^{F\cdot\sigma}\right| = \sum_{[Q]\in S_\sigma^k} \frac{1}{|\mathrm{Aut}_k(Q)|} = \frac{|\mathbb{P}_{\mathrm{gp}}^{2,7}(\lambda)|}{|\mathrm{PGL}(3)|}.$$

We may therefore equally well make equivariant point counts for $\mathbb{P}_{\mathrm{gp}}^{2,7}$.

Remark 4.0.2. Since an element of PGL(3) is completely determined by where it takes four points in general position, the stabilizer of an element $P \in S_\sigma$ in $\mathrm{Aut}(\mathbb{P}^2)$ is in fact trivial. However, we felt that not using this fact in the above somewhat clarified the situation.

We remind the reader that we do not need to make a count for each individual element of S_7 since $|(\mathbb{P}_{\mathrm{gp}}^{2,7})^{F\cdot\sigma}| = |(\mathbb{P}_{\mathrm{gp}}^{2,7})^{F\cdot\sigma'}|$ if σ and σ' are in the same conjugacy class of S_7 and that this happens if and only if σ and σ' have the same cycle type. Thus, we only have to make point counts for one element of each of the 15 cycle types

λ
$[7^1]$
$[1^1,6^1]$,
$[2^1,5^1],[1^2,5^1]$
$[3^1,4^1],[1^1,2^1,4^1],[1^3,4^1]$,
$[1^1,3^2],[2^2,3^1]\,[1^2,2^1,3^1],[1^4,3^1]$
$[1^1,2^3],[1^3,2^2],[1^5,2^1]$,
$[1^7]$,

where $[1^{\nu_1},2^{\nu_1},\ldots,7^{\nu_7}]$ means that the permutation has ν_1 one-cycles, ν_2 two-cycles and so on and where we have omitted cycle lengths which do not occur. If σ is a permutation of cycle type λ, we shall use the symbol N_λ to denote the cardinality of $(\mathbb{P}^{2,7}_{\mathrm{gp}})^{F\cdot\sigma}$. Similarly, $\mathcal{N}_\lambda$ denotes the cardinality of $(\mathcal{P}^{2,7}_{\mathrm{gp}})^{F\cdot\sigma}$.

4.1 A few words on the upcoming computations

Our main tool in the upcoming computations will be nothing fancier than the usual principle of inclusion and exclusion. Although this principle is simple enough, the computations tend to become somewhat involved and some general comments might therefore be helpful to the reader.

First of all, in almost all cases it is very hard to compute the numbers N_λ and $\mathcal{N}_\lambda$ directly. Instead, we shall view $\mathbb{P}^{2,7}_{\mathrm{gp}}$ as an open subset of some subset X of the space $\mathbb{P}^{2,7}$ of seven points in the projective plane. This subset X will be chosen so that $|X|$ is easily computed and so that the computation of the cardinality of the complement, $\mathcal{C}$, of $\mathbb{P}^{2,7}$ in X is manageable.

The set $\mathcal{C} \subset X$ will consist of two, not necessarily disjoint, parts: one part, $\mathcal{C}_{\mathrm{lines}}$, corresponding to tuples with at least three points on a line and one part, $\mathcal{C}_{\mathrm{conic}}$, corresponding to tuples with at least six points on a smooth conic. The computation of $|\mathcal{C}|$ will consist of computing $|\mathcal{C}_{\mathrm{lines}}|$, $|\mathcal{C}_{\mathrm{conic}}|$ and $|\mathcal{C}_{\mathrm{lines}} \cap \mathcal{C}_{\mathrm{conic}}|$.

4.2 The case $\lambda = [7^1]$

Throughout this section, λ shall mean the partition $[7^1]$. Since we only need to make the computation for one permutation σ of cycle type λ, we may as well assume that $\sigma^{-1} = (1234567)$ so that F should act on our seventuples by $Fp_i = p_{i+1}$ for $i = 1,\ldots,6$ and $Fp_7 = p_1$.

In this section we shall consider $\mathbb{P}^{2,7}_{\mathrm{gp}}$ as an open subset of $X = \mathbb{P}^{2,7} = (\mathbb{P}^2)^7$. The cardinality $|X(\lambda)|$ is easily computed via Lemma 3.2.4:

$$|X(\lambda)| = q^{14} + q^7 - q^2 - q.$$

We now want to compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\mathrm{gp}}(\lambda)$ in $X(\lambda)$. We decompose $\mathcal{C} = \mathcal{C}_{\mathrm{lines}} \cup \mathcal{C}_{\mathrm{conic}}$ where $\mathcal{C}_{\mathrm{lines}}$ consists of λ-tuples with at least three points on a line and $\mathcal{C}_{\mathrm{conic}}$ consists of λ-tuples with at least six points on a smooth conic.

Lemma 4.2.1. *Let p_1, p_2, p_3 and p_4 be four points in general position and let $l_{i,j}$ denote the line between p_i and p_j. Define $q_1 = l_{1,2} \cap l_{3,4}$, $q_2 = l_{1,3} \cap l_{2,4}$ and $q_3 = l_{1,4} \cap l_{2,3}$. Then q_1, q_2 and q_3 lie in general position.*

Proof. Since p_1, p_2, p_3 and p_4 lie in general position there is an element in $\mathrm{PGL}(3, \overline{k})$ taking them to $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ and $[1:1:1]$. Then the points $\{q_1, q_2, q_3\}$ must be taken to the points $\{[1:1:0], [1:0:1], [0:1:1]\}$ which lie in general position. □

Lemma 4.2.2. *If $(p_0, \dots, p_6)$ is a λ-tuple with three of its points on a line, then all seven points lie on a line defined over k.*

Proof. We consider the indices of the points as elements of $\mathbb{Z}/7\mathbb{Z}$. Suppose that the points of $S = \{p_{i_1}, \dots, p_{i_r}\}$ lie on a line L, where $i_1, \dots, i_r \subset \{0, 1, \dots, 6\}$ and $3 \le r \le 6$. Let $1 \le j \le 6$. The line L is defined by any two points on it and clearly $|S \cap F^j S| < r$. Thus, if S and $F^j S$ have at least two points in common, then $L = F^j L$ and we will have that

$$|S \cup F^j S| = |S| + |F^j S| - |S \cap F^j S| \ge r + r - (r-1) = r + 1,$$

so $r+1$ of the points $p_0, \dots, p_6$ will lie on L. Thus, if we can show this then the first part of the lemma will follow by induction.

To start, we have the following cases:

- If S contains a triple of the form p_i, p_{i+1}, p_{i+2}, then $|S \cap FS| = 2$,
- if S contains a triple of the form p_i, p_{i+2}, p_{i+4}, then $|S \cap F^2 S| = 2$,
- if S contains a triple of the form p_i, p_{i+1}, p_{i+4} then $|S \cap F^3 S| = 2$.

Hence, for all S of a form occurring in the above list we have that at least four of the points lie on a line. The above list contains all possible forms of triples except p_i, p_{i+1}, p_{i+3} and p_i, p_{i-1}, p_{i-3}. If S would happen to be exactly such a triple, then $|S \cap F^j S| = 1$ for $j = 1, \dots, 6$ so we cannot argue in the same way as above.

We assume that p_i, p_{i+1} and p_{i+3} lie on a line and argue by contradiction (the other case is handled in a completely analogous manner). For simplicity, we set $i = 0$ but the general argument is completely analogous. If three of p_0, p_1, p_2 and p_5 lie on a line we are in one of the cases in the above list and are thus done. We therefore assume that this is not the case so that the four points p_0, p_1, p_2 and p_5 lie in general position. Define $l_{0,1}$ as the line between p_0 and p_1, $l_{0,2}$ as the line between p_0 and p_2 and so on. Further, define

$$q_1 = l_{0,1} \cap l_{2,5}, \quad q_2 = l_{0,2} \cap l_{1,5}, \quad q_3 = l_{0,5} \cap l_{1,2}.$$

Since p_0, p_1 and p_3 lie on a line we must have that p_3 lies on $l_{0,1}$. But $F^2 p_0 = p_2$ and $F^2 p_3 = p_5$ so we must have $F^2 l_{0,1} = l_{2,5}$. The line $F^2 l_{0,1} = l_{2,5}$ must contain $F^2 p_1 = p_3$ so we now see that $q_1 = p_3$. By analogous arguments we may show that $q_2 = p_6$ and $q_3 = p_4$.

By Lemma 4.2.1 we now have that p_3, p_4 and p_6 lie in general position. On the other hand, by assumption we have that p_0, p_1 and p_3 lie on a line. Hence, the points $F^3 p_0 = p_3$, $F^3 p_1 = p_4$ and $F^3 p_3 = p_6$ also must lie on a line. This contradiction establishes the first part of the lemma.

Since the seventuple is fixed by F and a line is defined by any two points on it, we conclude that the line containing the seven points must be defined over k. □

Lemma 4.2.2 shows that the set $\mathcal{C}_{\text{lines}}$ is very simple: it simply consists of conjugate seventuples where all seven points lie on a k-line. To compute $|\mathcal{C}_{\text{lines}}|$ we thus first choose a k-line L. There is a $\mathbb{P}^2(k)$ of k-lines in $\mathbb{P}^2$ and

$$|\mathbb{P}^2(k)| = q^2 + q + 1.$$

This line L is isomorphic over k to $\mathbb{P}^1$ and we apply Lemma 3.2.4 to compute the number of λ-tuples on L as

$$|\mathbb{P}^1(\lambda)| = q^7 - q.$$

Hence, we have that

$$|\mathcal{C}_{\text{lines}}| = (q^2 + q + 1) \cdot (q^7 - q).$$

We now turn our attention to the set $\mathcal{C}_{\text{conic}}$ where we have a complete analogue of Lemma 4.2.2. As in the proof of Lemma 4.2.2, we take the indices of the points to lie in $\mathbb{Z}/7\mathbb{Z}$.

Lemma 4.2.3. *If $(p_0, \ldots, p_6)$ is a λ-tuple with six of its points on a smooth conic, then all seven points lie on a smooth conic defined over k.*

Proof. The proof of this fact is much simpler than that of the previous lemma. Suppose that $p_i,\ldots,p_{i+5}$ lie on a smooth conic C. Then $Fp_i = p_{i+1},\ldots,$ $Fp_{i+5} = p_{i+6}$ lie on the smooth conic FC. Hence, C and FC both contain the points $p_{i+1},\ldots,p_{i+5}$. Since a conic is defined by any five points on it we see that $C = FC$ and conclude that all seven points lie on C.

Since the seventuple is fixed by F and a conic is defined by any five points on it, we conclude that C must be defined over k. □

We now see that also $\mathcal{C}_{\text{conic}}$ is very simple. To compute $|\mathcal{C}_{\text{conic}}|$ we first want to choose a smooth conic C over k. There is a $\mathbb{P}^5(k)$ of conics over k and

$$|\mathbb{P}^5(k)| = q^5 + q^4 + q^3 + q^2 + q + 1.$$

However, these are not all smooth. The singular conics are of the following types:

- two k-lines intersecting in a point. There are $\binom{q^2+q+1}{2} = \frac{1}{2}(q^2+q+1)(q^2+q)$ of these.
- double k-lines. There are q^2+q+1 of these.
- two Frobenius conjugate k_2-lines intersecting in a k-point. There is a $\mathbb{P}^2$ of lines in $\mathbb{P}^2$ so Lemma 3.2.4 gives that there are $q^4 - q$ ordered pairs of conjugate k_2-lines. Hence, there are

  $$\frac{1}{2}(q^4 - q),$$

  unordered pairs of conjugate k_2-lines.

Taking away the singular conics we get that there are

$$q^5 - q^2, \tag{4.2.1}$$

smooth conics defined over k. A smooth conic over k is isomorphic over k to $\mathbb{P}^1$. Hence, there are $q^7 - q$ conjugate λ-tuples on C. We conclude that

$$|\mathcal{C}_{\text{conic}}| = (q^5 - q^2)\cdot(q^7 - q).$$

Since a line will intersect a smooth conic in at most 2 points, Lemmas 4.2.2 and 4.2.3 give that $\mathcal{C}_{\text{lines}}$ and $\mathcal{C}_{\text{conic}}$ are disjoint. We can thus compute N_λ as

$$\begin{aligned} N_\lambda = |X(\lambda)| - |\mathcal{C}_{\text{lines}}| - |\mathcal{C}_{\text{conic}}| = \\ = q^{14} - q^{12} - q^8 + q^6. \end{aligned}$$

To obtain $\mathcal{N}_\lambda$ we now simply have to divide by $|\text{PGL}(3)| = q^3\cdot(q^3-1)\cdot(q^2-1)$. This gives us

$$\mathcal{N}_\lambda = q^6 + q^3.$$

4.3 The case $\lambda = [1^1, 6^1]$

Throughout this section we shall let λ denote the partition $[1^1, 6^1]$. We only need to make the computation for one single permutation of this cycle type so we shall do it for $\sigma^{-1} = (123456)(7)$.

As above, we shall consider $\mathbb{P}^{2,7}_{\text{gp}}$ as an open subset of $X = \mathbb{P}^{2,7}$. The cardinality $|X(\lambda)|$ is easily computed via Lemma 3.2.4:

$$|X(\lambda)| = (q^{12} - q^4 - q^3 + q)(q^2 + q + 1).$$

As always, we decompose the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\text{gp}}$ into a union of the two parts $\mathcal{C}_{\text{lines}}$ and $\mathcal{C}_{\text{conic}}$.

Lemma 4.3.1. *If a λ-tuple lies in $\mathcal{C}_{\text{lines}}$, then either*

(A) the six k_6-points of the λ-tuple lie on a k-line or,

(B) the six k_6-points lie on two conjugate k_2-lines, the k_2-lines contain three k_6-points each and these triples are interchanged by the Frobenius, or,

(C) the six k_6-points lie pairwise on three conjugate k_3-lines which intersect in the k-point of the λ-tuple.

Further, the case (A) is disjoint from the other two cases.

Proof. We consider the indices of the k_6-points as elements of $\mathbb{Z}/6\mathbb{Z}$. Let $(p_0, \ldots, p_5, P)$ be an element of $\mathcal{C}_{\text{lines}}$ where $(p_0, \ldots, p_5)$ is a conjugate sixtuple and P is a k-point. Clearly, we either have three of the points $p_0, \ldots, p_5$ on a line or two of the points $p_0, \ldots, p_5$ and the point P on a line (or both).

Suppose that three of $p_0, \ldots, p_5$ lie on a line L. Then this triple has one of the forms

$$S_1 = \{p_i, p_{i+1}, p_{i+2}\}, \quad S_2 = \{p_i, p_{i+1}, p_{i+3}\},$$
$$S_3 = \{p_i, p_{i+1}, p_{i+4}\}, \quad S_4 = \{p_i, p_{i+2}, p_{i+4}\}.$$

For $1 \le i \le 3$ we have $|S_i \cap F^j S_i| < 3$ for $1 \le j \le 5$ and

$$|S_1 \cap F S_1| = |S_2 \cap F^3 S_2| = |S_3 \cap F^3 S_3| = 2,$$

so in these cases we can apply the same argument as in the proof of Lemma 4.2.2 to show that in fact all six points lie on L and we are thus in case (A). However, $|S_4 \cap F^{2j+1} S_4| = 0$ and $|S_4 \cap F^{2j} S_4| = 3$ for all choices of j so L and FL might be different, but at least $L = F^2 L$. We are thus either in case (A) or case (B).

Now suppose that $S = \{p_{i_1}, p_{i_2}, P\}$ lie on a line L. Then $F^j S$ lie on the line $F^j L$. Since P is fixed, we have $|S \cap F^j S| \geq 1$ for all j but sooner or later we must have $|S \cap F^j S| \geq 2$. If $|S \cap FS| = 2$, then $L = FL$ and we therefore must have all seven points on L and we are thus in case (A). If $|S \cap F^2 S| = 2$, then $L = F^2 L$ and we are either in case (A) or (B). Finally, if $|S \cap F^3 S| \geq 2$ then $L = F^3 L$ so we are in case (A) or case (C). The last part of the lemma is obvious. □

We shall denote the set of λ-tuples of type (A) by $\mathscr{A}$, the set of those of type (B) by $\mathscr{B}$ and those of type (C) by $\mathscr{C}$.

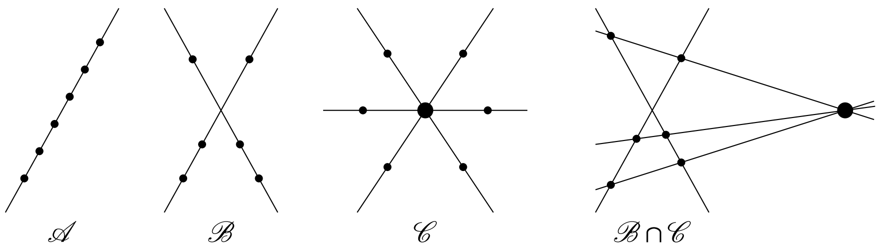

Figure 4.1: Illustration of Lemma 4.3.1.

Case (A) is very simple: we simply choose a k-line, then use Lemma 3.2.4 to choose a conjugate 6-tuple on the line and finally we choose a k-point anywhere in $\mathbb{P}^2$. We thus have

$$|\mathscr{A}| = (q^2 + q + 1)(q^6 - q^3 - q^2 + q)(q^2 + q + 1).$$

To obtain $|\mathscr{B}|$ we first choose a k_2-line, L. By Lemma 3.2.4 there are $q^4 - q$ such lines. The other k_2-line must then be FL. We then choose a k_6-point p_0 on L. The points $p_1 = Fp_0, \ldots, p_5 = F^5 p_0$ will then be the rest of our conjugate sixtuple. By Lemma 3.2.4 (with k_2 as the ground field) there are $q^6 - q^2$ choices. We now have two k_2-lines with three of our six k_6-points on each so all that remains is to choose a k-point anywhere we want in $q^2 + q + 1$ ways. Hence,

$$|\mathscr{B}| = (q^4 - q)(q^6 - q^2)(q^2 + q + 1).$$

To count $|\mathscr{C}|$ we first choose a k-point P in $q^2 + q + 1$ ways. There is a $\mathbb{P}^1$ of lines through P and we want to choose a k_3-line L through P. By Lemma 3.2.4 there are $q^3 - q$ choices. Finally, we choose a k_6-point on L to become p_0. By Lemma 3.2.4 there are $q^6 - q^3$ possible choices. We thus have

$$|\mathscr{C}| = (q^2 + q + 1)(q^3 - q)(q^6 - q^3).$$

All that remains is to compute the size of the intersection $\mathscr{B} \cap \mathscr{C}$. To obtain this, we first choose a pair of conjugate k_2-lines in $\frac{1}{2}(q^4 - q)$ ways. These intersect in a k-point and we choose P away from this point in $q^2 + q$ ways. We then choose a k_3-line through P in $q^3 - q$ ways. This line intersects the two k_2-lines in 2 distinct points which clearly must be defined over k_6. We choose one of them to become p_0 in 2 ways. Thus, in total we have

$$|\mathscr{B} \cap \mathscr{C}| = (q^4 - q) \cdot (q^2 + q) \cdot (q^3 - q).$$

We can now use the principle of inclusion and exclusion to compute $|\mathcal{C}_{\text{lines}}|$ as

$$|\mathcal{C}_{\text{lines}}| = |\mathscr{A}| + |\mathscr{B}| + |\mathscr{C}| - |\mathscr{B} \cap \mathscr{C}|.$$

We now turn our attention to $\mathcal{C}_{\text{conic}}$.

Lemma 4.3.2. *If six of the points of a λ-tuple lie on a smooth conic, then all of the k_6-points of the tuple lie on the conic and the conic is defined over k.*

Proof. We must either have all six k_6-points on the conic C, or five of them and the k-point must lie on the conic. In the former case we are done so we assume that we are in the latter case. Let $S = \{p_{i_1}, \dots, p_{i_5}, P\}$ be the points on the conic. Since P is fixed by the Frobenius we see that $|S \cap FS| = 5$. Since a conic is defined by any five points on it we conclude that $C = FC$ so all seven points lie on C and C is defined over k. □

The computation of $|\mathcal{C}_{\text{conic}}|$ is now easy: we first choose a smooth conic C in $q^5 - q^2$ ways (see Equation 4.2.1) and then use Lemma 3.2.4 to see that we have $q^6 - q^3 - q^2 + q$ ways of choosing a conjugate sixtuple on C. Finally, we choose P anywhere we want in $q^2 + q + 1$ ways. We thus see that

$$|\mathcal{C}_{\text{conic}}| = (q^5 - q^2)(q^6 - q^3 - q^2 + q)(q^2 + q + 1).$$

The only thing that remains is to compute the size of the intersection of $\mathcal{C}_{\text{lines}}$ and $\mathcal{C}_{\text{conic}}$. This intersection was empty in the case $[7^1]$, but here we have nonempty intersection between $\mathcal{C}_{\text{conic}}$ and the set $\mathscr{C}$ (the intersections with the other cases are empty because a smooth conic does not contain three points which lie on a line). The following simple observation will be helpful.

Lemma 4.3.3. *Let $C \subset \mathbb{P}^2$ be a smooth conic over a field K. If there is a point P such that three tangents of C pass through P, then the characteristic of K is 2.*

Proof. The three points of tangency will be in general position, so by a projective change of coordinates they can be transformed to $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ and C will then be given by a polynomial $F = XY + \alpha XZ + \beta YZ$, where $\alpha, \beta \in K^*$. The tangent lines thus become $Y + \alpha Z$, $X + \beta Z$ and $\alpha X + \beta Y$.

Let the coordinates of P be $[a:b:c]$. Since these lines all pass through P, the first tangent equation gives that $b = -\alpha c$ and the second gives $a = -\beta c$. Inserting these expressions into the third tangent equation gives $-2\alpha\beta c = 0$. If $c = 0$, then also $a = b = 0$ which is impossible. Since also α and β are nonzero we see that the only possibility is that the characteristic of K is 2. □

To compute the number $|\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}|$ we begin by choosing a smooth conic C in $q^5 - q^2$ ways and then a k-point P not on C in $q^2 + q + 1 - (q+1) = q^2$ ways. By Lemma 3.2.4 there are $q^3 - q$ strict k_3-lines passing through P. All of these intersect C in two k_3-points since, by Lemma 4.3.3, these lines cannot be tangent to C since the characteristic of k is odd. More precisely, choosing any of the $q^3 - q$ strict k_3-points of C gives a strict k_3-line, and since every such line cuts C in exactly two points we conclude that there are precisely $\frac{1}{2}(q^3 - q)$ strict k_3-lines through P intersecting C in two k_3-points. Thus, the remaining

$$q^3 - q - \frac{1}{2}(q^3 - q) = \frac{1}{2}(q^3 - q),$$

k_3-lines through P will intersect C in two k_6-points. If we pick one of them and label it p_0 we obtain an element in $\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}$. Hence,

$$|\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}| = (q^5 - q^2) q^2 (q^3 - q).$$

By the principle of inclusion and exclusion we now have

$$\begin{aligned} N_\lambda &= |X| - |\mathcal{C}_{\text{lines}}| - |\mathcal{C}_{\text{conic}}| + |\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}| = \\ &= q^{14} - q^{12} - 3q^{11} + 3q^9 + 3q^8 - 3q^6 - q^5 + q^3, \end{aligned}$$

and, after dividing by $|\mathrm{PGL}(3)|$,

$$\mathcal{N}_\lambda = q^6 - 2q^3 + 1.$$

4.4 The case $\lambda = [2^1, 5^1]$

Throughout this section, λ will denote the partition $[2^1, 5^1]$. Here, we are in the rare position of being able to compute N_λ more or less right away. The key is the following observation.

Lemma 4.4.1. *If a conjugate λ-tuple has three points which lie on a line, then all five k_5-points will lie on a k-line. Similarly, if six of the points lie on a smooth conic, then the five k_5-points will lie on that conic and the conic will be defined over k.*

Proof. The proof is very similar to the proofs of Lemmas 4.2.2, 4.2.3, 4.3.1 and 4.3.2 but without any special cases and will therefore be omitted. □

By Lemma 3.2.4, there are $q^{10}+q^5-q^2-q$ conjugate fivetuples whereof $(q^2+q+1)(q^5-q)$ lie on a line. We may thus choose a conjugate fivetuple whose points do not lie on a line in $q^{10}-q^7-q^6+q^3$ ways. This fivetuple defines a smooth conic C. By Lemma 4.4.1, it is enough to choose a conjugate twotuple outside C in order to obtain an element of $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$.

There are q^4-q conjugate twotuples of which q^2-q lie on C. Thus only q^4-q^2 remain. We thus have that

$$\begin{aligned}N_\lambda &= (q^{10}-q^7-q^6+q^3)(q^4-q^2) = \\ &= q^{14}-q^{12}-q^{11}-q^{10}+q^9+q^8+q^7-q^5,\end{aligned}$$

and it follows that

$$\mathcal{N}_\lambda = q^6-q^2.$$

4.5 The case $\lambda = [1^2, 5^1]$

Throughout this section, λ will denote the partition $[1^2, 5^1]$. This case is very similar to the case $[2^1, 5^1]$. In fact, we have a complete analogue of Lemma 4.4.1.

Lemma 4.5.1. *If a conjugate λ-tuple has three points which lie on a line, then all five k_5-points will lie on a k-line. Similarly, if six of the points lie on a smooth conic, then the five k_5-points will lie on that conic and the conic will be defined over k.*

Proof. Omitted since it is very easy and similar to earlier proofs. □

In the previous section we saw that there are $q^{10}-q^7-q^6+q^3$ conjugate fivetuples which do not lie on a line. Such a fivetuple defines a smooth conic C and by Lemma 4.5.1 it is enough to choose two k-points outside C in order to obtain an element of $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$. This can be done in $q^2(q^2-1)$ ways. Hence, we see that

$$\begin{aligned}N_\lambda &= (q^{10}-q^7-q^6+q^3)q^2(q^2-1) = \\ &= q^{14}-q^{12}-q^{11}-q^{10}+q^9+q^8+q^7-q^5,\end{aligned}$$

which in turn gives that

$$\mathcal{N}_\lambda = q^6 - q^2.$$

We note that $\mathcal{N}_{[2^1,5^1]} = \mathcal{N}_{[1^2,5^1]}$.

4.6 The case $\lambda = [3^1, 4^1]$

Throughout this section, λ shall mean the partition $[3^1, 4^1]$. Since we only need to make the computation for one permutation, we shall assume that the Frobenius permutes points p_1, p_2, p_3, p_4 according to (1234) and the three points p_5, p_6, p_7 according to (567).

We shall consider $\mathbb{P}^{2,7}_{\mathrm{gp}}$ as an open subset of $X = \mathbb{P}^{2,7}$. We fist use Lemma 3.2.4 to compute $|X(\lambda)|$ to be

$$|X(\lambda)| = (q^8 - q^2)(q^6 + q^3 - q^2 - q).$$

We now turn to the problem of computing the cardinality of the complement of $\mathbb{P}^{2,7}_{\mathrm{gp}}$ in X.

Lemma 4.6.1. *If a conjugate λ-tuple has three points on a line, then either*

(A) the four k_4-points lie on a k-line, or

(B) the three k_3-points lie on a k-line.

Proof. It is easy to see that if three k_4-points lie on a line, then all four k_4-points lie on that line and even easier to see the corresponding result for three k_3-points.

Suppose that two k_4-points p_i and p_j and a k_3-point P lie on a line L. Since $F^4 p_i = p_i$ and $F^4 p_j = p_j$ we see that $F^4 L = L$. However, $F^4 P = FP \neq P$. Repeating this argument again, with FP in the place of P, shows that also $F^2 P$ lies on L. We are thus in case (B).

If we assume that two k_3-points and a k_4-point lie on a line, then a completely analogous argument shows that all four k_4-points lie on that line. □

To compute $|\mathcal{C}_{\mathrm{lines}}|$ we thus need to compute $|\mathscr{A}|$, $|\mathscr{B}|$ and $|\mathscr{A} \cap \mathscr{B}|$, where $\mathscr{A}$ is the set of λ-tuples of type (A) and $\mathscr{B}$ is the set of λ-tuples of type (B). This is straightforward. For example, to compute $|\mathscr{A}|$ we first choose a k-line L in $q^2 + q + 1$ ways. We then use Lemma 3.2.4 to choose a conjugate fourtuple on L in $q^4 - q^2$ ways. Finally, we choose a conjugate threetuple in $q^6 + q^3 - q^2 - q$ ways. Thus

$$|\mathscr{A}| = (q^2 + q + 1)(q^4 - q^2)(q^6 + q^3 - q^2 - q).$$

Similarly, we find that

$$|\mathscr{B}| = (q^2+q+1)(q^3-q)(q^8-q^2),$$

and

$$|\mathscr{A} \cap \mathscr{B}| = (q^2+q+1)^2(q^4-q^2)(q^3-q).$$

We shall now compute $|\mathcal{C}_{\text{conic}}|$. We have the following simple observation.

Lemma 4.6.2. *If six points of a conjugate λ-tuple lie on a smooth conic C, then all seven points lie on C and C is defined over k.*

Proof. We must consider two cases: the case where one k_4-point may lie outside C and the case where a k_3-point may lie outside C. In both cases, we apply F once and use that a conic is determined by five points to conclude that also the seventh point lies on C. □

Since no three points on a smooth conic lie on a line we conclude, using Lemmas 4.6.1 and 4.6.2, that the intersection $\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}$ is empty. The cardinality $|\mathcal{C}_{\text{conic}}|$ is easily computed to be

$$|\mathcal{C}_{\text{conic}}| = (q^5-q^2)(q^4-q^2)(q^3-q).$$

We now use the principle of inclusion and exclusion to obtain

$$N_\lambda = q^{14} - q^{13} - 3q^{12} + q^{11} + 4q^{10} + 2q^9 - 3q^8 - 3q^7 + q^6 + q^5.$$

Finally, we divide by $|\mathrm{PGL}(3)|$ and get

$$\mathcal{N}_\lambda = q^6 - q^5 - 2q^4 + q^3 + q^2 :$$

4.7 The case $\lambda = [1^1, 2^1, 4^1]$

Throughout this section, λ shall mean the partition $[1^1, 2^1, 4^1]$. Since we only need to make the computation for one permutation, we shall assume that the Frobenius permutes points p_1, p_2, p_3, p_4 according to (1234), switches the two points p_5, p_6 and fixes the point p_7.

We shall consider $\mathbb{P}^{2,7}_{\text{gp}}$ as an open subset of $X = \mathbb{P}^{2,7}$. We fist use Lemma 3.2.4 to compute $|X(\lambda)|$ to be

$$|X(\lambda)| = (q^8-q^2)(q^4-q)(q^2+q+1).$$

We now turn to the problem of computing the cardinality of the complement of $\mathbb{P}^{2,7}_{\text{gp}}$ in X. This will turn out to be far more challenging than in the previous cases, mainly because both 1 and 2 divides 4. We have the following trivial decomposition of $\mathcal{C}_{\text{lines}}$, which we state as a lemma to make the analogy with the previous cases more clear.

Lemma 4.7.1. *If three points of a conjugate λ-tuple lies on a line, then either*

(A) three k_4-points lie on a line, or,

(B) two k_4-points and a k_2-point lie on a line, or,

(C) two k_4-points and the k-point lie on a line, or,

(D) a k_4-point and two k_2-points lie on a line, or,

(E) a k_4-point, a k_2-point and a k-point lies on a line, or,

(F) two k_2-points and the k-point lies on a line.

This decomposition is of course naive and is not very nice to work with since none of the possible intersections are empty. The reader can surely think of many other decompositions which a priori look more promising. However, the more "clever" approaches we have tried have turned out to be quite hard to carry out in practice.

We shall denote the set of λ-tuples in $X(\lambda)$ of type (A) by $\mathscr{A}$ etc.

Observation 4.7.2. (A) If a λ-tuple is of type (A), then in fact all four k_4-points lie on a k-line. Hence, we have

$$|\mathscr{A}| = (q^2+q+1)^2(q^4-q^2)(q^4-q).$$

(B) The case (B) of Lemma 4.7.1 splits into three disjoint subcases. We either have

(a) the four k_4-points and the two k_2-points on a k-line, or,

(b) the two k_4-points and the k_2-point lies on a k_2-line L (and the other two k_4-points and the second k_2-point lies on FL), or,

(c) the four k_4-points and the two k_2-points are intersection points of four conjugate k_4-lines.

The subcases (b) and (c) are illustrated in Figure 4.2 below. The number of λ-tuples of type (a) is easily computed to be $(q^2+q+1)^2(q^4-q^2)(q^2-q)$. To get the number of λ-tuples of type (b), we first choose a k_2-line L in q^4-q ways and then a k_4-point p_1 on L in q^4-q^2 ways. This determines all the four k_4 points since they must be $p_2 = Fp_1$, $p_3 = F^2p_1$ and $p_4 = F^3p_1$. We must now decide if p_5 should lie on L or FL. We then choose a k_2-point on the chosen line. The lines L and FL both contain q^2+1 points defined over k_2 of which precisely one is defined over k (namely the point $L \cap FL$).

Hence, there are q^2 choices for p_5. It now only remains to choose p_7 in one of q^2+q+1 ways. There are thus

$$2(q^4-q)(q^4-q^2)q^2(q^2+q+1),$$

λ-tuples of type (b). It remains to compute the number of λ-tuples of type (c). We first choose a k_2-point p_5 not defined over k in one of q^4-q ways. There are q^4-q^2 lines L strictly defined over k_4 through p_5 and we choose one. We thus get four k_4-lines which intersect in the two k_2-points p_5 and p_6 as well as in four k_4-points. We choose one of these to become p_1 and the labels of the other three points is then given. However, we could as well have chosen the line F^2L and ended up with the same four k_4-points. We therefore must divide by 2. Finally, we choose any of the q^2+q+1 k-points to become p_7. We thus have

$$2(q^4-q)(q^4-q^2)(q^2+q+1),$$

λ-tuples of type (c).

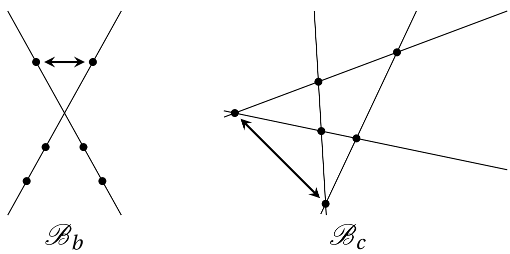

Figure 4.2: Illustration of elements of the sets $\mathscr{B}_b$ and $\mathscr{B}_c$.

(C) The case (C) of Lemma 4.7.1 splits into two disjoint subcases. We either have

(a) the four k_4-points and the k-point on a k-line, or,

(b) two conjugate k_2-lines intersecting in the k-point, each k_2-line containing two of the k_4-points.

These cases are both easily computed. In case (a) we first choose k-line L, then a conjugate fourtuple and a k-point on L and finally a conjugate pair of k_2-points anywhere. This can be done in

$$(q^2+q+1)(q^4-q^2)(q+1)(q^4-q),$$

ways. In case (b) we first choose a k_2-line L not defined over k, then a k_4-point p_4 not defined over k_2 on L and finally a pair of conjugate k_2-points anywhere. There are thus

$$(q^4 - q)^2(q^4 - q^2),$$

λ-tuples of type (b).

(D) Since the line contains two conjugate k_2-points, it is defined over k. It thus follows that all four k_4-points lie on the line. There are thus

$$(q^2 + q + 1)^2(q^4 - q^2)(q^2 - q),$$

λ-tuples of this type.

(E) The case (E) of Lemma 4.7.1 splits into two disjoint subcases. We either have

(a) all seven points on a k-line, or,

(b) two conjugate k_2-lines intersecting om the k-point, each k_2-line containing two of the k_4-points and one of the k_2-points.

The computations are essentially the same as in case (C), the only difference being that we must choose also the k_2-points on the lines. We thus have

$$(q^2 + q + 1)(q^4 - q^2)(q + 1)(q^2 - q),$$

possibilities in case (a), and

$$2(q^4 - q)(q^4 - q^2)q^2,$$

in case (b). The factor 2 in the above expression comes from the choice of placing p_5 either on L or FL, where L has the same meaning as in the computation of case (C).

(F) Here, the two k_2-points and the k-point lie on a k-line L. We thus choose a k-line L, then a conjugate pair of k_2-points and a k-point on L and finally a conjugate fourtuple anywhere. This can be done in

$$(q^2 + q + 1)(q^2 - q)(q + 1)(q^8 - q^2),$$

ways.

As remarked earlier, all the possible intersections of the cases (A)-(F) are nonempty. This means that in total we must compute the cardinalities of 63, a priori possibly different, sets. Fortunately, most of these computations are rather simple and, even more fortunately, many of these sets coincide.

For instance, the quadruple intersections, and higher, will all consist of configurations where all seven points lie on a line. We are therefore confident that the interested reader will be able to make it through the simple, but tedious, computation by him- or herself and we will therefore only describe the computation in one of the more complicated intersections.

Example 4.7.1. Consider the sets $\mathscr{B}$ and $\mathscr{C}$. These cases falls into disjoint unions of subcases as described above and we shall denote the subset of $\mathscr{B}$ consisting of tuples of type (A) by $\mathscr{B}_a$ and so on. We thus have

$$\mathscr{B} = \mathscr{B}_a \sqcup \mathscr{B}_b \sqcup \mathscr{B}_c, \quad \mathscr{C} = \mathscr{C}_a \sqcup \mathscr{C}_b.$$

It is easy to see that $\mathscr{B}_a \cap \mathscr{C}_b$ is empty while $\mathscr{B}_a \cap \mathscr{C}_a$ consists of configurations where all seven points lie on a k-line. There are

$$(q^2+q+1)(q^4-q^2)(q^2-q)(q+1),$$

such λ-tuples.

To compute the cardinality of $\mathscr{B}_b \cap \mathscr{C}_b$ we first choose a k_2-line L in q^4-q ways and then a strict k_4-point p_1 on L in q^4-q^2 ways. We must now decide if p_5 should lie on L or FL. We then choose a strict k_2-point on the chosen line in q^2 ways. We are now in case (B)(b). To end up also in case (C)(b) we have no choice but to put p_7 at the intersection of L and FL. There are thus

$$2(q^4-q)(q^4-q^2)q^2,$$

elements in the intersection $\mathscr{B}_b \cap \mathscr{C}_b$.

As explained in the computation of the cardinality of $\mathscr{B}_c$, there are $2(q^4-q)(q^4-q^2)$ ways to obtain four strict k_4-points and two strict k_2-points which are the intersection points of four conjugate k_4-lines. We must choose such a configuration to be sure to end up in case (B)(c). We now note that there are precisely six lines through pairs of points among the four k_4-points. Four of these lines are of course the four k_4-lines. The remaining two lines are defined over k_2 and therefore intersects in a k-point. To end up also in case (C)(b) we have no choice but to choose p_7 as this intersection point. Therefore, there are

$$2(q^4-q)(q^4-q^2),$$

elements in the intersection $\mathscr{B}_c \cap \mathscr{C}_b$.

By computing the cardinalities of all possible intersections and using the principle of inclusion and exclusion one will find that the cardinality of $\mathcal{C}_{\text{lines}}$ is

$$|\mathcal{C}_{\text{lines}}| = q^{13}+5q^{12}-4q^{10}-5q^9-3q^8+2q^7+q^6+3q^5+q^4-q^3.$$

We shall now compute $|\mathcal{C}_{\text{conic}}|$.

Lemma 4.7.3. *If six points of a conjugate λ-tuple lie on a smooth conic C, then the four k_4-points and the two k_2-points lie on C and C is defined over k.*

Proof. Very similar to the proof of Lemma 4.6.2 and therefore omitted. □

The computation of $|\mathcal{C}_{\text{conic}}|$ is thus very simple. We first choose a smooth conic C defined over k in $q^5 - q^2$ ways, then a conjugate fourtuple on C in $q^4 - q^2$ ways and then a conjugate pair of k_2-points on C in $q^2 - q$ ways. Finally, we may choose a k-point anywhere we want in $q^2 + q + 1$ ways. Hence,

$$|\mathcal{C}_{\text{conic}}| = (q^5 - q^2)(q^4 - q^2)(q^2 - q)(q^2 + q + 1).$$

It only remains to investigate the intersection $\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}$. The reader might be afraid that we shall need to investigate 63 different cases again. However, this is fortunately not the case. Since a smooth conic cannot contain three collinear points, we only have nonempty intersection between $\mathcal{C}_{\text{conic}}$ and the sets $\mathscr{C}_b$ and $\mathscr{F}$.

The intersection $\mathcal{C}_{\text{conic}} \cap \mathscr{C}_b$ is easily handled. We first choose a smooth conic C, then a conjugate fourtuple on C and finally a pair of conjugate k_2-points on C. The k-point is then uniquely defined as the intersection point of the two k_2-lines through pairs of the four k_4-points. We thus have

$$|\mathcal{C}_{\text{conic}} \cap \mathscr{C}_b| = (q^5 - q^2)(q^4 - q^2)(q^2 - q).$$

The same construction as above works for the intersection $\mathcal{C}_{\text{conic}} \cap (F)$ if we remember that we now have some choice for the k-point since it can lie anywhere on the line through the two k_2-points. We thus have

$$|\mathcal{C}_{\text{conic}} \cap \mathscr{F}| = (q^5 - q^2)(q^4 - q^2)(q^2 - q)(q + 1).$$

The only thing that remains to compute is the cardinality of the triple intersection $\mathcal{C}_{\text{conic}} \cap \mathscr{C}_b \cap \mathscr{F}$. This is however a little bit tricky. Since similar difficulties will arise quite frequently also in the future, we shall introduce some terminology that will hopefully clarify the situation and make the discussion less painful.

Definition 4.7.4. Let C be a smooth plane conic defined over k and let P be a k-point not on C. We say that P is on the outside of C if there are two k-tangents to C passing through P. If P is not on the outside of C we say that C is on the inside of C.

In Figure 4.3 we have made an attempt to illustrate Definition 4.7.4 with a real contra complex picture.

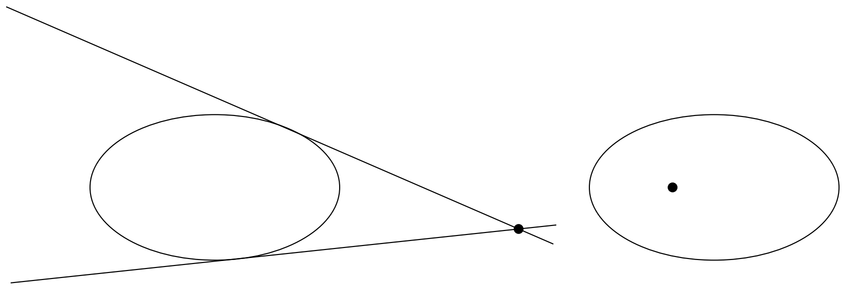

Figure 4.3: Illustration of Definition 4.7.4: a point on the outside of a conic and a point on the inside of a conic.

Observation 4.7.5. A simple computation shows that if a k-point P is on the inside of a smooth conic C, then in fact no k-tangent to C will pass through P. However, there will be two k_2-tangents to C which do pass through P.

We first assume that the k-point is on the outside of a smooth conic C containing the other six points. We first choose C in $q^5 - q^2$ ways. There are $\frac{1}{2}(q+1)q$ ways to choose two k-points P and Q on C. Intersecting the tangents $T_P C$ and $T_Q C$ gives a k-point p_7 which will clearly lie on the outside of C. Hence, there are precisely $\frac{1}{2}(q+1)q$ ways to choose a k-point on the outside of C.

We now want to choose a k-line through p_7 intersecting C in two k_2-points. There are $q+1$ k-lines through p_7 of which two are tangents to C. These tangent lines contain a k-point of C each so there are $q-1$ remaining k-points on C. Picking such a point gives a line through this point, p_7 and one further point on C. We thus see that exactly $\frac{1}{2}(q-1)$ of the k-lines through p_7 intersect C in two k-points. Hence, there are precisely

$$q+1-2-\frac{1}{2}(q-1) = \frac{1}{2}(q-1),$$

k-lines through p_7 which intersect C in two k_2-points. These points are clearly conjugate under F. We label one of them as p_5.

We shall now choose a conjugate pair of k_2-lines through p_7 intersecting C in four k_4-points. There are $q^2 - q$ conjugate pairs of k_2-lines through p_7. By Observation 4.7.5, no k_2-line through p_7 is tangent to C so each k_2-line through p_7 will intersect C in two points. The conic C contains $q^2 - q$ points which are defined over k_2 but not k. Picking such a point gives a line through this point and p_7 as well as one further k_2-point not defined over k. Thus, there are $\frac{1}{2}(q^2 - q)$ lines obtained in this way. Typically, such a line will be defined over k_2 but not k. We saw above that the number of such lines which are defined over k is precisely $\frac{1}{2}(q-1)$. Thus, there are precisely

$$\frac{1}{2}(q^2-q)-\frac{1}{2}(q-1)=\frac{1}{2}(q^2-2q+1),$$

k_2-lines, not defined over k, which intersect C in two k_4-points. Thus, the remaining

$$q^2-q-\frac{1}{2}(q^2-2q+1)=\frac{1}{2}(q^2-1), \tag{4.7.1}$$

k_2-lines must intersect C in two k_4-points. Picking such a line and labeling one of the points p_1 gives a configuration belonging to $\mathcal{C}_{\text{conic}}\cap\mathscr{C}_b\cap\mathscr{F}$ and we thus see that there are

$$\frac{1}{2}(q^5-q^2)q(q+1)(q-1)(q^2-1),$$

such configurations with p_7 on the outside of C.

We now assume that p_7 is on the inside of C. We first choose C in q^5-q^2 ways. Since the number of k-points is q^2+q+1 and $q+1$ of these lie on C the number of k-points not on C is precisely q^2. We just saw that $\frac{1}{2}(q+1)q$ of these lie on the outside of C so there must be

$$q^2-\frac{1}{2}(q+1)q=\frac{1}{2}(q^2-q),$$

k-points which lie on the inside of C.

Since p_7 now lies on the inside of C, every k-line through p_7 will intersect C in two points. Exactly $\frac{1}{2}(q+1)$ will intersect C in two k-points so the remaining $\frac{1}{2}(q+1)$ will intersect C in two conjugate k_2-points. We pick such a pair of points and label one of them p_5.

We now choose a conjugate pair of k_2-lines through p_7 intersecting C in a conjugate fourtuple of k_4-points. The number of k_2-lines, not defined over k, through p_7 is q^2-q. Two of these are tangent to C so, using ideas similar to those above, we see that

$$\frac{1}{2}(q^2-q-2)-\frac{1}{2}(q+1)+2=\frac{1}{2}(q^2-2q+1),$$

of these lines intersect C in points defined over k_2. Hence, the remaining

$$q^2-q-\frac{1}{2}(q^2-2q+1)=\frac{1}{2}(q^2-1), \tag{4.7.2}$$

lines intersect C in two k_4-points which are not defined over k_2. If we pick one of these points to become p_1 we end up with a configuration in $\mathcal{C}_{\text{conic}}\cap\mathscr{C}_b\cap\mathscr{F}$. We thus have

$$\frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q^2-1),$$

such configurations with p_7 on the inside of C. One may note that the expression above actually is the same as the expression when p_7 was on the outside, but this will not always be the case.

We thus have

$$|\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}| = q^{12} + q^{11} - 4q^{10} - 2q^9 + 3q^8 + 4q^7 - 4q^5 + q^3.$$

If we put all the pieces together we get

$$N_\lambda = q^{14} - q13 - 3q^{12} + q^{11} + q^{10} + 2q^9 + 3q^8 - 2q^6 - 5q^5 + 3q^3,$$

and finally also

$$\mathcal{N}_\lambda = q^6 - q^5 - 2q^4 + q^3 - 2q^2 + 3.$$

4.8 The case $\lambda = [1^3, 4^1]$

Throughout this section, λ shall mean the partition $[1^3, 4^1]$. We shall assume that the Frobenius automorphism permutes points p_1, p_2, p_3, p_4 according to (1234) and fixes the k-points p_5, p_6 and p_7.

Define X as the open subscheme of $\mathbb{P}^{2,7}$ consisting of seventuples such that the last three points of the tuple are not collinear. We then have

$$|X(\lambda)| = (q^8 - q^2)(q^2 + q + 1)(q^2 + q)q^2.$$

In other words, we choose any conjugate fourtuple and three k-points which do not lie on a line. We shall consider $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$ as an open subset of $X(\lambda)$ and now turn to the problem of computing the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$ in $X(\lambda)$.

As usual, we shall decompose $\mathcal{C}$ into the subsets $\mathcal{C}_{\text{lines}}$ and $\mathcal{C}_{\text{conic}}$.

Lemma 4.8.1. *If a λ-tuple in $X(\lambda)$ has three points on a line then either*

(A) all four k_4-points lie on a k-line, or,

(B) the k_2-line through p_1 and p_3 intersects the k_2-line through p_2 and p_4 in p_5, p_6 or p_7.

Proof. The three k-points do not lie on a line so at most two of the three points are k-points. If two are k-points, then the line is defined over k and by acting by the Frobenius we see that all four k_4-points lie on the line. Similarly, if all three points are k_4-points then the line will be fixed by the Frobenius and it follows that we are in case (A).

The only remaining case is if we have two k_4-points p_i and p_j and a k-point P on a line L. If $Fp_i = p_j$ or $Fp_j = p_i$ we have that $FL = L$ and it follows that we are in case (A). We thus assume that $Fp_i \neq p_j$ and $Fp_j \neq p_i$. But then we must have $F^2\{p_i, p_j\} = \{p_i, p_j\}$ which shows that $F^2L = L$. Hence, we must be in either case (A) or case (B). □

A nice feature of this decomposition is that the cases (A) and (B) clearly are disjoint and this is in fact one of the reasons we chose X the way we did. However, we will have to pay a price for this choice in the computation of $|\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}|$.

To compute the number of elements in $X(\lambda)$ of type (A) we simply choose a k-line L, a conjugate fourtuple on L and finally place or three k-points in such a ways that they do not lie on a line. This can be done in

$$(q^2+q+1)^2(q^4-q^2)(q^2+q)q^2,$$

ways. To compute the number of elements in $X(\lambda)$ of type (B) we first choose p_5, p_6 or p_7 then a k_2-line L not defined over k through this point. Finally, we choose k_4-point p_1 which is not defined over k_2 on L and make sure that the final two k-points are not collinear with the first. This can be done in

$$3(q^2+q+1)(q^2-q)(q^4-q^2)(q^2+q)q^2,$$

ways. This gives that

$$|\mathcal{C}_{\text{lines}}| = 4q^{12}+6q^{11}+q^{10}-4q^9-5q^8-q^7-q^5.$$

We now turn to investigate $\mathcal{C}_{\text{conic}}$.

Lemma 4.8.2. *If six of the points lie on a smooth conic C, then the four k_4-points must lie on that conic and C must be defined over k.*

Proof. The proof is very similar to earlier proofs and therefore omitted. □

We thus choose a smooth conic C over k and a conjugate fourtuple on C. Then we choose p_5, p_6 or p_7 to possibly not lie on C and place the other two on C. Finally, we place the final k-point anywhere except on the line through the other two k-points. This gives the number

$$3(q^5-q^2)(q^4-q^2)(q+1)q\cdot q^2.$$

However, we have now counted the configurations where all seven points lie on a smooth conic three times. There are

$$(q^5-q^2)(q^4-q^2)(q+1)q(q-1),$$

such configurations and it thus follows that

$$|\mathcal{C}_{\text{conic}}| = 3q^{13}+q^{12}-3q^{11}-2q^{10}-q^9+q^8-q^7+2q^5.$$

We now turn to the intersection $\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}$. Clearly, there is only non-empty intersection with case (B) of Lemma 4.8.1. We begin by choosing one of the k-points p_5, p_6 and p_7 to not lie on the conic C. We call the chosen point P and the remaining two points p_i and p_j where $i < j$. We now have three disjoint possibilities (see Definition 4.7.4):

(*i*) the point P may lie on the outside of C with one of the tangents through P also passing through p_i,

(*ii*) the point P may lie on the outside of C with none of the tangents through P passing through p_i,

(*iii*) the point P may lie on the inside of C.

We consider the three cases (*i*)-(*iii*) separately.

(i). We begin by choosing a smooth conic C in $q^5 - q^2$ ways and a k-point P on the outside of C in $\frac{1}{2}(q+1)q$ ways. By Equation 4.7.1, there are now $q^2 - 1$ ways to choose p_1. Since we require p_i to lie on a tangent to C which passes through P, we only have 2 choices for p_i. Finally, we may choose p_j as any of the q remaining points on C. We thus have

$$3(q^5 - q^2)(q+1)q(q^2 - 1)q,$$

possibilities in this case.

(ii). Again, we begin by choosing a smooth conic C in $q^5 - q^2$ ways and a k-point P on the outside of C in $\frac{1}{2}(q+1)q$ ways and choose p_1 in one of $q^2 - 1$ ways. The point p_i should now not lie on a tangent to C which passes through P so we have $q - 1$ choices. Since the line between P and p_i is not a tangent to C, there is one further intersection point between this line and C. We must choose p_j away from this point and p_i and thus have $q - 1$ possible choices. Hence, we have

$$\frac{3}{2}(q^5 - q^2)(q+1)q(q^2 - 1)(q-1)^2,$$

possibilities in this case.

(iii) We start by choosing a smooth conic C in $q^5 - q^2$ ways and then a point P on the inside of C in $\frac{1}{2}(q^2 - q)$ ways. We continue by using Equation 4.7.2 to see that we can choose p_1 in $q^2 - 1$ ways. We now choose p_i as any of the k-points on C and thus have $q + 1$ choices. Finally, we may choose p_j as any

k-point on C, except in the intersection of C with the line through p_i and P. Hence, we have $q-1$ choices. We thus have

$$\frac{3}{2}(q^5-q^2)(q^2-q)(q^2-1)(q+1)(q-1),$$

possibilities in this case.

We may now add the cases (i)-(iii) to get

$$|\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}| = 3q^{11}-3q^9-3q^5+3q^3.$$

We are now able to compute

$$N_\lambda = q^{14}-q^{13}-3q^{12}+q^{11}+q^{10}+2q^9+3q^8-2q^6-5q^5+3q^3,$$

and, finally,

$$\mathcal{N}_\lambda = q^6-q^5-2q^4+q^3-2q^2+3.$$

We remark that $\mathcal{N}_{[1^3,4^1]} = \mathcal{N}_{[1^1,2^1,4^1]}$.

4.9 The case $\lambda = [1^1,3^2]$

Throughout this section, λ shall mean the partition $[1^1,3^2]$. We shall use the notation a_1, a_2, a_3 for first conjugate threetuple and b_1, b_2, b_3 for the second. The k-point will be denoted by p_7.

We shall consider $\mathbb{P}^{2,7}_{\text{gp}}$ as a subset of $X = \mathbb{P}^{2,7}$. The cardinality $|X(\lambda)|$ is easily computed using Lemma 3.2.4:

$$|X(\lambda)| = (q^6+q^6-q^2-q)(q^6+q^3-q^2-q-3)(q^2+q+1).$$

We shall now compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$ in $X(\lambda)$.

Lemma 4.9.1. *Suppose that a λ-tuple has three points on a line. Then either*

(A) the points a_1, a_2 and a_3 lie on a k-line, or,

(B) the points a_1, a_2 and a_3 are the intersection points of a conjugate triple of k_3-lines with each of the lines containing one of the points b_1, b_2 and b_3, or,

(C) the points b_1, b_2 and b_3 are the intersection points of a conjugate triple of k_3-lines with each of the lines containing one of the points a_1, a_2 and a_3, or,

(D) the points b_1, b_2 and b_3 lie on a k-line, or,

(E) *the point p_7 is the intersection of three conjugate k_3-lines with each of the k_3-lines containing one of the points a_1, a_2 and a_3 and one of the points b_1, b_2 and b_3.*

Proof. This is just a tedious case by case check. We shall let i, j and l be integers such that $1 \le i < j \le 3$ and $1 \le l \le 3$.

If a_1, a_2 and a_3 lie on a line, then we are obviously in case (A). Now suppose that a_i, a_j and b_l lie on a line L. If $FL = L$ we will be in cases (A) and (D), otherwise we will be in case (B). If a_i, a_j and p_7 lie on a line L, then $\{a_i, a_j, p_7\} \cap F\{a_i, a_j, p_7\}$ will contain at least 2 elements and we thus have $L = FL$. Hence, we must be in case (A). If a_i, b_j and p_7 lie on a line L, then we will be in cases (A) and (D) if $FL = L$ and in case (E) otherwise.

The remaining cases are handled entirely analogously, the only difference being that the a_i's and b_j's switch roles. □

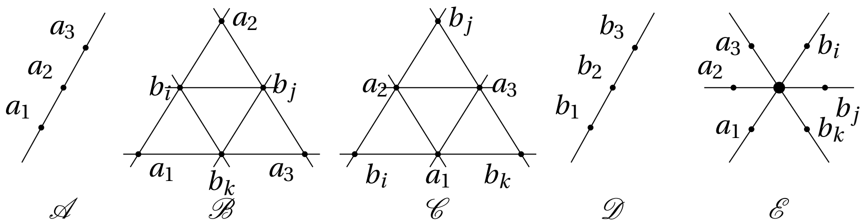

Figure 4.4: Illustration of Lemma 4.9.1.

At first glance, this situation looks frighteningly similar to the case of the partition $[1^1, 2^1, 4^1]$ where we had to handle 63 cases. Fortunately, one quickly sees that most higher intersections are empty so the situation is not too bad.

We shall denote the subset of $X(\lambda)$ consisting of λ-tuples of type (A) by $\mathscr{A}$ etc.

4.9.1. The cases (A) and (D). The cardinalities of $\mathscr{A}$ and $\mathscr{D}$ are obviously the same so we only make the computation for $\mathscr{A}$. We thus choose a k-line L, a conjugate k_3-tuple a_1, a_2, a_3 on L, a conjugate k_3-tuple b_1, b_2, b_3 anywhere except equal to the other k_3-tuple and, finally, a k-point anywhere. We thus get

$$|\mathscr{A}| = |\mathscr{D}| = (q^2+q+1)^2(q^3-q)(q^6+q^3-q^2-q-3).$$

4.9.2. The cases (B) and (C). The cardinalities of $\mathscr{B}$ and $\mathscr{C}$ are of course also the same. To compute $|\mathscr{B}|$ we first choose a conjugate threetuple of lines, L, FL, F^2L, which do not intersect in a point. There are $q^6+q^3-q^2-q$ conjugate threetuples of lines of which $(q^2+q+1)(q^3-q)$ intersect in a point. There are thus $q^6-q^5-q^4+q^3$ possible threetuples. We give the label a_1 to the point $L\cap FL$ which determines the labellings of the other two intersection points. We must now choose if b_1 should lie on L, FL or F^2L and then place b_1 on the chosen line. There are $3(q^3-1)$ ways to do this. Finally, we choose any k-point. We thus have

$$|\mathscr{B}| = |\mathscr{C}| = 3(q^6-q^5-q^4+q^3)(q^3-1)(q^2+q+1).$$

4.9.3. The case (E). We first choose a k-point p_7 anywhere and then a conjugate threetuple of lines, L, FL, F^2L through p_7. We then choose a point a_1 somewhere on L in q^3 ways. We must now decide if b_1 should lie on L, FL or F^2L and then pick a point on the chosen line in one of q^3-1 ways. We thus see that

$$|\mathscr{E}| = 3(q^2+q+1)(q^3-q)q^3(q^3-1).$$

We must now compute the cardinalities of the different intersections.

4.9.4. Intersections. Firstly, note that the intersection between $\mathscr{A}$ and $\mathscr{B}$ is empty. Secondly, the size of the intersection of $\mathscr{A}$ and $\mathscr{C}$ is equal to that of the intersection of $\mathscr{B}$ and $\mathscr{D}$. To obtain $|\mathscr{A}\cap\mathscr{C}|$ we first choose a conjugate threetuple of lines, L, FL, F^2L, which do not intersect in a point and label the intersection $L\cap FL$ by b_1. We then choose a k-line L' and thus get three k_3-points $L'\cap L$, $L'\cap FL$ and $L'\cap F^2L$. We label one of these by a_1. If we now choose any k-point to become p_7 we obtain a configuration of both type (A) and (C). We thus see that

$$|\mathscr{A}\cap\mathscr{C}| = |\mathscr{B}\cap\mathscr{D}| = 3(q^6-q^5-q^4+q^3)(q^2+q+1)^2.$$

When we consider the intersection between $\mathscr{A}$ and $\mathscr{D}$ we must distinguish between the cases where the two threetuples lie on the same line and when they do not. A simple computation then gives

$$|\mathscr{A}\cap\mathscr{D}| = (q^2+q+1)^2(q^3-q)(q^3-q-3)+(q^2+q+1)^2(q^2+q)(q^3-q)^2.$$

We continue by observing that $|\mathscr{A}\cap\mathscr{E}| = |\mathscr{D}\cap\mathscr{E}|$. To compute $|\mathscr{A}\cap\mathscr{E}|$ we first choose a k-point p_7 and then a conjugate k_3-tuple of lines L, FL, F^2L through p_7. We continue by choosing a k-line L' not through p_7 in one of q^2

ways and then label $L' \cap L$, $L' \cap FL$ or $L' \cap F^2L$ by b_1. Finally, we choose one of the remaining $q^3 - 1$ points of L to become a_1. Hence,

$$|\mathscr{A} \cap \mathscr{E}| = |\mathscr{D} \cap \mathscr{E}| = 3(q^2 + q + 1)(q^3 - q)q^2(q^3 - 1).$$

The sets $\mathscr{B}$ and $\mathscr{C}$ do not intersect and neither do the sets $\mathscr{C}$ and $\mathscr{D}$. Hence, there are only two intersections left, namely the one between $\mathscr{B}$ and $\mathscr{E}$ and the one between $\mathscr{C}$ and $\mathscr{E}$. These have equal cardinalities. To compute $|\mathscr{B} \cap \mathscr{E}|$ we first choose a conjugate threetuple of lines, L, FL, F^2L, which do not intersect in a point and label the intersection $L \cap FL$ by b_1. We then choose a k-point p_7. The lines between p_7 and the points b_1, b_2 and b_3 intersect the lines L, FL and F^2L in three points and if we label one of them by a_1 we obtain a configuration of both type (B) and (E). We thus have

$$|\mathscr{B} \cap \mathscr{E}| = |\mathscr{C} \cap \mathscr{E}| = 3(q^6 - q^5 - q^4 + q^3)(q^2 + q + 1).$$

There is only one triple intersection, namely between $\mathscr{A}$, $\mathscr{D}$ and $\mathscr{E}$. To compute the size of this intersection we first choose a k-point p_7 and then a conjugate threetuple of lines, L, FL, F^2L through p_7. We then choose a k-line L' not through p_7 and label the intersection $L \cap L'$ by a_1. Finally, we choose another k-line L'' and label one of the intersections $L'' \cap L$, $L'' \cap FL$ and $L'' \cap F^2L$ by b_1. This shows that

$$|\mathscr{A} \cap \mathscr{D} \cap \mathscr{E}| = 3(q^2 + q + 1)(q^3 - q)q^2(q^2 - 1).$$

We now turn to the computation of $|\mathcal{C}_{\text{conic}}|$. Clearly, if six points of a λ-tuple lies on a smooth conic C, then both of the conjugate threetuples must lie on C and C must be defined over k. Hence, to obtain $|\mathcal{C}_{\text{conic}}|$ we only have to choose a smooth k-conic C, two conjugate threetuples on C and a k-point anywhere. Hence,

$$|\mathcal{C}_{\text{conic}}| = (q^5 - q^2)(q^3 - q)(q^3 - q - 3)(q^2 + q + 1).$$

Since the sets $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ all require three of the k_3-points to lie on a line, they will have empty intersection with $\mathcal{C}_{\text{conic}}$. This is however not true for the set $\mathscr{E}$. To obtain such a configuration we first choose a smooth conic C and a k-point p_7. Now choose a k_3-point a_1 on C which is not defined over k in $q^3 - q$ ways. Since both C and p_7 are defined over k we know that any tangent to C which passes through p_7 must either be defined over k_2 (or k). Hence, the line through p_7 and a_1 will intersect C in a_1 and one point more. We label this point with b_1, b_2 or b_3. We thus have

$$|\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}| = 3(q^5 - q^2)q^2(q^3 - q).$$

We may now use the principle of inclusion and exclusion to obtain

$$N_\lambda = q^{14} - 2q^{13} - 3q^{12} - 7q^{11} + 20q^{10} + 21q^9 + 11q^8 - \\ - 28q^7 - 39q^6 - 5q^5 + 10q^4 + 21q^3,$$

and, after dividing by |PGL(3)|,

$$\mathcal{N}_\lambda = q^6 - 2q^5 - 2q^4 - 8q^3 + 16q^2 + 10q + 21.$$

4.10 The case $\lambda = [2^2, 3^1]$

Throughout this section, λ shall mean the partition $[2^2, 3^1]$. We shall use the notation p_1, p_2, p_3 for the conjugate threetuple and a_1, a_2 and b_1, b_2 for the two conjugate twotuples.

We shall consider $\mathbb{P}^{2,7}_{\text{gp}}$ as a subset of $X = \mathbb{P}^{2,7}$. The cardinality $|X(\lambda)|$ is easily computed using Lemma 3.2.4:

$$|X(\lambda)| = (q^6 + q^6 - q^2 - q)(q^4 - q)(q^4 - q - 2).$$

We shall now compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$ in $X(\lambda)$.

Lemma 4.10.1. *If a λ-tuple has three of its points on a line, then either*

(A) the three k_3-points lie on a k-line, or,

(B) the four k_2-points lie on a k-line.

Further, if six points lie on a smooth conic C, then all seven points lie on C and C is defined over k.

Proof. If three k_3-points lie on a line, then we must be in case (A). If two k_3-points p_i and p_j and a k_2-point p_l lie on a line L, then $|\{p_i, p_j, p_l\} \cap F^2\{p_i, p_j, p_l\}| = 2$ so $F^2L = L$. It now follows that we must have all three k_3-points on L. If two k_2-points and a k_3-point p_i lie on a line L, then $F^2L = L$ and $F^4L = L$ so F^2p_i and $F^4p_i = Fp_i$ lie on L. Hence, we are in case (A). Finally, if three k_2-points lie on a line L, then two of the points must be conjugate so $FL = L$. It now follows that also the fourth k_2-point must lie on L.

If the threetuple lies on C, then C contains a conjugate threetuple and a conjugate twotuple. These are fixed, as sets, by the Frobenius so C is fixed since a conic is defined by five points. If C only contains two of the three k_3-points, say p_i and p_j, then one must be mapped to the other via the Frobenius. Since the other four points are two pairs of conjugate k_2-points we see that C must be fixed by F. Hence, Fp_i and Fp_j both lie on C so all seven points must lie on C. □

We thus see that this case is surprisingly simple. In particular, the sets $\mathcal{C}_{\text{lines}}$ and $\mathcal{C}_{\text{conic}}$ are disjoint. We shall denote the subset of $X(\lambda)$ consisting of λ-tuples of type (A) by $\mathscr{A}$ and the subset consisting of λ-tuples of type (B) by $\mathscr{B}$. We have

$$|\mathscr{A}| = (q^2+q+1)(q^3-q)(q^4-q)(q^4-q-2),$$

and

$$|\mathscr{B}| = (q^2+q+1)(q^2-q)(q^2-q-2)(q^6+q^3-q^2-q).$$

Also, the cardinality of the intersection is easily computed to be

$$|\mathscr{A} \cap \mathscr{B}| = (q^2+q+1)^2(q^3-q)(q^2-q)(q^2-q-2).$$

This allows us to compute $|\mathcal{C}_{\text{lines}}|$. We also see that

$$|\mathcal{C}_{\text{conic}}| = (q^5-q^2)(q^3-q)(q^2-q)(q^2-q-2).$$

Putting these pieces together gives

$$N_\lambda = q^{14} - q^{13} - 3q^{12} + 3q^{11} + 4q^{10} - 2q^9 - 5q^8 - q^7 + 5q^6 + q^5 - 2q^4,$$

and, finally,

$$\mathcal{N}_\lambda = q^6 - q^5 - 2q^4 + 3q^3 + q^2 - 2q.$$

4.11 The case $\lambda = [1^2, 2^1, 3^1]$

Throughout this section, λ shall mean the partition $[1^2, 2^1, 3^1]$. We shall use the notation a_1, a_2, a_3 for the conjugate threetuple, b_1, b_2 for the conjugate twotuple and use p_6 and p_7 to denote the two k-points.

We shall consider $\mathbb{P}^{2,7}_{\text{gp}}$ as a subset of $X = \mathbb{P}^{2,7}$. The cardinality $|X(\lambda)|$ is easily computed using Lemma 3.2.4:

$$|X(\lambda)| = (q^6+q^6-q^2-q)(q^4-q)(q^2+q+1)(q^2+q).$$

We shall now compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$ in $X(\lambda)$.

Lemma 4.11.1. *If three points of a conjugate λ-tuple lie on a line, then either*

(A) the three k_3-points lie on a k-line, or

(B) the two k_2-points lie and one of the k-points lie on a k-line.

Further, if six of the points lie on a smooth conic C, then C is defined over k and contains the three k_3-points and the two k_2-points.

Proof. The proof is very similar to the proof of Lemma 4.10.1 and is therefore omitted. □

We shall, as usual, denote the set of λ-tuples of type (A) by $\mathscr{A}$ and the set of those of type (B) by $\mathscr{B}$. The λ-tuples of type (B) are of two different kinds: those where the line through the two k_2-points passes through p_6 and those where it passes through p_7. We denote the set of tuples of the first kind by $\mathscr{B}_6$ and the set of those of the second kind by $\mathscr{B}_7$.

The cardinalities of the different sets are now easily computed. We have

$$|\mathscr{A}| = (q^2+q+1)^2(q^3-q)(q^4-q)(q^2+q),$$

and

$$|\mathscr{B}_6| = |\mathscr{B}_7| = (q^2+q+1)(q^2-q)(q+1)(q^6+q^3-q^2-q)(q^2+q).$$

We now turn to the double intersections. We have

$$|\mathscr{A}\cap\mathscr{B}_6| = |\mathscr{A}\cap\mathscr{B}_7| = (q^2+q+1)^2(q^3-q)(q^2-q)(q+1)(q^2+q),$$

and

$$|\mathscr{B}_6\cap\mathscr{B}_7| = (q^2+q+1)(q^2-q)(q+1)q(q^6+q^3-q^2-q).$$

Finally, we compute the cardinality of the intersection of all three sets

$$|\mathscr{A}\cap\mathscr{B}_6\cap\mathscr{B}_7| = (q^2+q+1)^2(q^3-q)(q^2-q)(q+1)q.$$

This now allows us to compute $\mathcal{C}_{\text{lines}}$.

To compute $|\mathcal{C}_{\text{conic}}|$ we begin by choosing a smooth conic C over k. We then choose a conjugate threetuple and a conjugate pair of k_2-points on C. Then, we choose either p_6 or p_7 and place the chosen point on C. Finally, we place the remaining point anywhere we want. We thus obtain the number

$$2(q^5-q^2)(q^3-q)(q^2-q)(q+1)(q^2+q).$$

However, in the above we have counted the configurations where all seven points lie on the conic twice. We thus have to take away

$$(q^5-q^2)(q^3-q)(q^2-q)(q+1)q,$$

in order to obtain $|\mathcal{C}_{\text{conic}}|$.

It only remains to compute the cardinality of the intersection $\mathcal{C}_{\text{lines}}\cap\mathcal{C}_{\text{conic}}$. We only have nonempty intersection between the set $\mathcal{C}_{\text{conic}}$ and the set $\mathscr{B}$. To compute the cardinality of this intersection, we only have to make sure to choose the point p_6 (resp. p_7) on the line through the two k_2-points. Hence, we have

$$|\mathscr{B}_6 \cap \mathcal{C}_{\text{conic}}| = |\mathscr{B}_7 \cap \mathcal{C}_{\text{conic}}| = (q^5 - q^2)(q^3 - q)(q^2 - q)(q+1)^2,$$

and, therefore,

$$|\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}| = 2(q^5 - q^2)(q^3 - q)(q^2 - q)(q+1)^2.$$

This gives us

$$N_\lambda = q^{14} - 3q^{13} - 2q^{12} + 7q^{11} + 4q^{10} - 5q^9 - 8q^8 + q^7 + 7q^6 - 2q^4,$$

and, finally,

$$\mathcal{N}_\lambda = q^6 - 3q^5 - q^4 + 5q^3 - 2q.$$

4.12 The case $\lambda = [1^4, 3^1]$

This case is, perhaps surprisingly, rather simple. Throughout this section, λ shall mean the partition $[1^4, 3^1]$. We shall denote the four k-points by p_1, p_2, p_3 and p_4 and denote the conjugate threetuple by a_1, a_2, a_3.

We shall consider $\mathbb{P}^{2,7}_{\text{gp}}$ as a subset of $X \subset \mathbb{P}^{2,7}$ consisting of seventuples of points with the first four in general position. The cardinality $|X(\lambda)|$ is easily computed to be

$$|X(\lambda)| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(q^6 + q^3 - q^2 - q).$$

We shall now compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$ in $X(\lambda)$.

Lemma 4.12.1. *If a tuple in $X(\lambda)$ has three points on a line, then the three points of the conjugate threetuple lies on a k-line. If six of the points lie on a smooth conic C, then the conjugate threetuple lies on C and C is defined over k.*

Proof. The proof is similar to those of Lemmas 4.9.1 and 4.11.1 and is therefore omitted. □

Thus, $\mathcal{C}_{\text{lines}}$ and $\mathcal{C}_{\text{conic}}$ are disjoint.

To compute the size of $\mathcal{C}_{\text{lines}}$, we only need to place the four k-points in general position, choose k-line L and place the conjugate k_3-tuple on L. Thus,

$$|\mathcal{C}_{\text{lines}}| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(q^2 + q + 1)(q^3 - q).$$

To compute $|\mathcal{C}_{\text{conic}}|$, we first choose a smooth conic C defined over k and then a conjugate threetuple on C. We then choose one of the points p_1, p_2, p_3 and p_4 to possibly not lie on C. Call this point P. We then place the other three points on C. These three point define three lines which, in total, contain $(q+1)+q+(q-1)=3q$ points. As long as we choose P away from these points, the four k-points will be in general position. We thus obtain

$$4(q^5-q^2)(q^3-q)(q+1)q(q-1)(q^2-2q+1).$$

However, we have counted the seventuples with all seven points on a smooth conic four times. We thus need to take away

$$3(q^5-q^2)(q^3-q)(q+1)q(q-1)(q-2).$$

This now gives

$$N_\lambda = q^{14}-5q^{13}+9q^{12}-q^{11}-16q^{10}+6q^9+11q^8+11q^7-15q^6-11q^5+10q^4,$$

and, after dividing by $|\mathrm{PGL}(3)|$,

$$\mathcal{N}_\lambda = q^6-5q^5+10q^4-5q^3-11q^2+10q.$$

4.13 The case $\lambda = [1^1, 2^3]$

Throughout this section, λ shall mean the partition $[1^1, 2^3]$. We shall denote the three conjugate pairs of k_2-points by a_1, a_2, b_1, b_2 and c_1, c_2 and the k-point by p_7.

We shall consider $\mathbb{P}^{2,7}_{\text{gp}}$ as a subset of $X \subset \mathbb{P}^{2,7}$ consisting of seventuples of points such that the first six points has no three on a line. To compute $|X(\lambda)|$ we first choose any conjugate pair a_1, a_2 in q^4-q ways. The second pair b_1, b_2 may be chosen anywhere except on the line through a_1 and a_2, so there are $q^4-q-(q^2-q)=q^4-q^2$ choices. When we pick the final pair c_1, c_2 we must stay away from the k-line through a_1 and a_2 and the k-line through b_1 and b_2. We must also stay away from the four k_2-lines through pairs of these points. We thus have

$$q^4-q-2(q^2-q)-4(q^2-2)=q^4-6q^2+q+8,$$

choices. Finally, we place p_7 anywhere. Hence

$$|X(\lambda)| = (q^4-q)(q^4-q^2)(q^4-6q^2+q+8)(q^2+q+1).$$

We shall now compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\text{gp}}(\lambda)$ in $X(\lambda)$.

The following observation is almost trivial, but we state it as a lemma to accentuate the similarities with the other cases.

Lemma 4.13.1. *If three points of a λ-tuple in X lie on a line, then either*

(A) two conjugate k_2-points and the k-point lie on a k-line, or,

(B) two conjugate k_2-lines, containing two k_2-points each, intersect in the k-point.

Further, if six of the points lie on a smooth conic C, then the three conjugate twotuples must lie on C and C is defined over k.

The set $\mathscr{A}$ of tuples of type (A) naturally decomposes into three equally large, but not disjoint, subsets:

- the set $\mathscr{A}_a$ where a_1, a_2 and p_7 lie on a k-line,
- the set $\mathscr{A}_b$ where b_1, b_2 and p_7 lie on a k-line, and
- the set $\mathscr{A}_c$ where c_1, c_2 and p_7 lie on a k-line.

Similarly, the set $\mathscr{B}$ consisting of tuples of type (B) consists of six disjoint and equally large subsets:

- the two sets $\mathscr{B}_{a_1,b_i}$ where the line through a_1 and b_i also passes through p_7,
- the two sets $\mathscr{B}_{a_1,c_i}$ where the line through a_1 and c_i also passes through p_7, and
- the two sets $\mathscr{B}_{b_1,c_i}$ where the line through b_1 and c_i also passes through p_7.

The cardinalities of these sets are easily computed to be

$$|\mathscr{A}_a| = |\mathscr{A}_b| = |\mathscr{A}_c| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8)(q + 1),$$

and

$$|\mathscr{B}_{a_1,b_i}| = |\mathscr{B}_{a_1,c_i}| = |\mathscr{B}_{b_1,c_i}| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8).$$

To compute the intersection of $\mathscr{A}_a$ and $\mathscr{A}_b$ we note that if we place the three pairs of k_2-points such that no three lie on a line, then the line through a_1 and a_2 and the line through b_1 and b_2 will intersect in a k-point. By choosing this point as p_7 we obtain an element of $\mathscr{A}_a \cap \mathscr{A}_b$. Since the other two intersections of this type are entirely analogous, we see that

$$|\mathscr{A}_a \cap \mathscr{A}_b| = |\mathscr{A}_a \cap \mathscr{A}_c| = |\mathscr{A}_b \cap \mathscr{A}_c| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8).$$

The intersection $\mathscr{A}_a \cap \mathscr{B}_{b_1,c_1}$ (and its analogues) is slightly more complicated. We first choose a conjugate twotuple b_1, b_2 and then a conjugate pair of k_2-points c_1, c_2 which does not lie on the line through b_1 and b_2. We now only have one choice for p_7. We choose a k-line L through p_7. There are two possibilities: either L will pass through the intersection point P of the line through b_1 and c_2 and the line through b_2 and c_1 or it will not. If L passes through P, then we have q^2-q possible choices for a_1 and a_2 on L. Otherwise, we only have q^2-q-2 choices. Hence

$$
\begin{aligned}
&|\mathscr{A}_a \cap \mathscr{B}_{b_1,c_i}| = |\mathscr{A}_b \cap \mathscr{B}_{a_1,c_i}| = |\mathscr{A}_c \cap \mathscr{B}_{a_1,b_i}| = \\
&= (q^4-q)(q^4-q^2)(q^2-q) + (q^4-q)(q^4-q^2)q(q^2-q-2).
\end{aligned}
$$

Since the sets of the decomposition of $\mathscr{B}$ are disjoint, this finishes the discussion of intersection between two sets. Clearly, the intersections $\mathscr{A}_a \cap \mathscr{A}_b$, $\mathscr{A}_a \cap \mathscr{A}_c$ and $\mathscr{A}_b \cap \mathscr{A}_c$ do not intersect any of the sets in the decomposition of $\mathscr{B}$. Hence, there is only one nonempty triple intersection, namely $\mathscr{A}_a \cap \mathscr{A}_b \cap \mathscr{A}_c$. A computation very similar to the one for $|\mathscr{A}_a \cap \mathscr{B}_{b_1,c_1}|$ gives

$$
\begin{aligned}
|\mathscr{A}_a \cap \mathscr{A}_b \cap \mathscr{A}_c| = {}& 2(q^4-q)(q^4-q^2)(q^2-q-2)+ \\
&+ (q^4-q)(q^4-q^2)(q-3)(q^2-q-4).
\end{aligned}
$$

We may now compute $|\mathcal{C}_{\text{lines}}|$.

The turn has now come to $\mathcal{C}_{\text{conic}}$. Since no three points of a smooth conic can lie on a line, we shall obtain an element of $\mathcal{C}_{\text{conic}}$ simply by choosing a smooth conic C, three conjugate pairs on C and, finally, a k-point anywhere. We thus have

$$
|\mathcal{C}_{\text{conic}}| = (q^5-q^2)(q^2-q)(q^2-q-2)(q^2-q-4)(q^2+q+1).
$$

We shall now compute the cardinality of the intersection between $\mathcal{C}_{\text{lines}}$ and $\mathcal{C}_{\text{conic}}$. The intersections with the cases $\mathscr{A}_a$, $\mathscr{A}_b$, and $\mathscr{A}_c$ are easily handled: we simply choose a smooth conic with three conjugate pairs on it and then place p_7 on the line to the right conjugate pair. We thus get

$$
\begin{aligned}
&|\mathscr{A}_a \cap \mathcal{C}_{\text{conic}}| = |\mathscr{A}_b \cap \mathcal{C}_{\text{conic}}| = |\mathscr{A}_c \cap \mathcal{C}_{\text{conic}}| = \\
&= (q^5-q^2)(q^2-q)(q^2-q-2)(q^2-q-4)(q+1).
\end{aligned}
$$

The intersections with the sets $\mathscr{B}_{a_1,b_i}$, $\mathscr{B}_{a_1,c_i}$ and $\mathscr{B}_{b_1,c_i}$ are perhaps even simpler: once we have chosen our conic C and our conjugate pairs we have only one choice for p_7. Hence,

$$
\begin{aligned}
&|\mathscr{B}_{a_1,b_i} \cap \mathcal{C}_{\text{conic}}| = |\mathscr{B}_{a_1,c_i} \cap \mathcal{C}_{\text{conic}}| = |\mathscr{B}_{b_1,c_i} \cap \mathcal{C}_{\text{conic}}| = \\
&= (q^5-q^2)(q^2-q)(q^2-q-2)(q^2-q-4).
\end{aligned}
$$

An analogous argument shows that

$$|\mathscr{A}_a \cap \mathscr{A}_b \cap \mathcal{C}_{\text{conic}}| = |\mathscr{A}_a \cap \mathscr{A}_c \cap \mathcal{C}_{\text{conic}}| = |\mathscr{A}_b \cap \mathscr{A}_c \cap \mathcal{C}_{\text{conic}}| =$$
$$= q^5 - q^2)(q^2 - q)(q^2 - q - 2)(q^2 - q - 4).$$

The remaining intersections are quite a bit harder than the previous ones. Recall that Definition 4.7.4 states that a k-point is on the outside of C if it is the intersection point of two k-tangents to C and on the inside of C otherwise. To simplify the discussion, we shall make the computation for $\mathscr{A}_a \cap \mathscr{B}_{b_1,c_1} \cap \mathcal{C}_{\text{conic}}$, but the other intersections of this type are completely analogous and have the same size.

We first consider the case when p_7 is on the outside of C. There are $q+1$ lines through p_7. Of these, precisely two are tangents and $\frac{1}{2}(q-1)$ intersect C in k-points. Thus, the remaining $\frac{1}{2}(q-1)$ lines will intersect C in two conjugate k_2-points. We thus pick one of these lines and label one of the intersection points by a_1.

Picking a k_2-point not defined over k on C will typically define a k_2-line through p_7 which is not defined over k. However, some of these choices will give k-lines and we saw above that the number of such k-lines is $\frac{1}{2}(q-1)$. Thus, the number of k_2-lines, not defined over k, intersecting C in two k_2-points is

$$\frac{1}{2}(q^2 - q) - \frac{1}{2}(q-1) = \frac{1}{2}(q^2 - 2q + 1).$$

We pick one such line, label one of the intersection points b_1 and the other intersection point c_1. This gives us a configuration of the desired type. Hence, the number of tuples in $\mathscr{A}_a \cap \mathscr{B}_{b_1,c_1} \cap \mathcal{C}_{\text{conic}}$ with p_7 on the outside of C is

$$\frac{1}{2}(q^5 - q^2)(q+1)q(q-1)(q^2 - 2q + 1).$$

We now turn to the case when p_7 is on the inside of C. Of the $q+1$ lines defined over k which passes through p_7, $\frac{1}{2}(q+1)$ will now intersect C in k-points and equally many in conjugate k_2-points. We thus pick a line that intersect C in two conjugate k_2-points and label one of them by a_1.

We now want to pick a k_2-line through p_7 which is not defined over k and which intersect C in two k_2-points which are not defined over k. To obtain such a line we pick a k_2-point which is not defined over k on C. However, two such points define tangents to C which pass through p_7 and $\frac{1}{2}(q+1)$ of the lines obtained in this way are actually defined over k. We thus have

$$\frac{1}{2}(q^2 - q - 2) - \frac{1}{2}(q+1) = \frac{1}{2}(q^2 - 2q - 3),$$

choices. We thus pick such a line a then label the intersection points by b_1 and c_1. Hence, the number of tuples in $\mathscr{A}_a \cap \mathscr{B}_{b_1,c_1} \cap \mathcal{C}_{\text{conic}}$ with p_7 on the inside of C is

$$\frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q^2-2q-3).$$

This finishes the computation of $|\mathscr{A}_a \cap \mathscr{B}_{b_1,c_i} \cap \mathcal{C}_{\text{conic}}|$, $|\mathscr{A}_b \cap \mathscr{B}_{a_1,c_i} \cap \mathcal{C}_{\text{conic}}|$ and $|\mathscr{A}_c \cap \mathscr{B}_{a_1,b_i} \cap \mathcal{C}_{\text{conic}}|$.

The only remaining intersection is $\mathscr{A}_a \cap \mathscr{A}_b \cap \mathscr{A}_c \cap \mathcal{C}_{\text{conic}}$ which we shall handle in a way similar to that above. Fortunately, much of the work has already been done. To start, if p_7 is on the outside of C, then there are $\frac{1}{2}(q-1)$ lines though p_7 which are defined over k and intersect C in conjugate pairs of k_2-points. Thus, there are

$$(q-1)(q-3)(q-5),$$

ways to pick three lines and label the intersection points with a_1 and a_2, b_1 and b_2 and c_1 and c_2. Hence, the number of λ-tuples in $\mathscr{A}_a \cap \mathscr{A}_b \cap \mathscr{A}_c \cap \mathcal{C}_{\text{conic}}$ with p_7 on the outside of C is

$$\frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q-3)(q-5).$$

Similarly, if p_7 lies on the inside of C we have seen that there are $\frac{1}{2}(q+1)$ lines through p_7 which are defined over k and which intersect C in a pair of conjugate k_2-points. Thus, there are

$$(q+1)(q-1)(q-3),$$

ways to pick three lines and label the intersection points with a_1 and a_2, b_1 and b_2 and c_1 and c_2. Hence, the number of λ-tuples in $\mathscr{A}_a \cap \mathscr{A}_b \cap \mathscr{A}_c \cap \mathcal{C}_{\text{conic}}$ with p_7 on the inside of C is

$$\frac{1}{2}(q^5-q^2)(q+1)q(q+1)(q-1)(q-3).$$

We may now add everything up to obtain

$$\begin{aligned} N_\lambda = {} & q^{14}-3q^{13}-7q^{12}+21q^{11}+15q^{10}-36q^9-21q^8+ \\ & +12q^7+36q^6-q^5-24q^4+7q^3, \end{aligned}$$

and, after dividing by $|\text{PGL}(3)|$,

$$\mathcal{N}_\lambda = q^6-3q^5-6q^4+19q^3+6q^2-24q+7.$$

4.14 The case $\lambda = [1^3, 2^2]$

Throughout this section, λ shall mean the partition $[1^3, 2^2]$. We shall denote the k-points by p_1, p_2 and p_3 and the two conjugate pairs of k_2-points by a_1, a_2 and b_1, b_2.

We shall consider $\mathbb{P}^{2,7}_{\mathrm{gp}}$ as a subset of $X \subset \mathbb{P}^{2,7}$ consisting of seventuples of points such that the first five points lie in general position. Thus, the points p_1, p_2, p_3, a_1 and a_2 should always be chosen such that no three lie on a line. To compute $|X(\lambda)|$ we first choose three k-points in general position in $(q^2+q+1)(q^2+q)q^2$ ways. The pair a_1, a_2 should now be chosen away from the k-lines through the points p_1, p_2 and p_3. There are $3q$ such lines and each of these contain q^2-q points defined over k_2 but not k. Since two k-lines intersect in a k-point, none of these k_2-points will lie on two of these lines. Hence, there are $q^4 - q - 3q(q^2-q) = q^4 - 3q^3 + 3q^2 - q$ choices for a_1 and a_2. Finally, we can choose b_1 and b_2 as any conjugate pair we want except a_1, a_2. We thus have

$$|X(\lambda)| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q^4-q-2).$$

We shall now compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\mathrm{gp}}(\lambda)$ in $X(\lambda)$.

As in the previous section we shall now make an almost trivial observation but state it as a lemma to accentuate the similarities with the other cases.

Lemma 4.14.1. *If three points of a λ-tuple in X lie on a line, then either*

(A) the line through b_1 and b_2 also passes through p_1, p_2 or p_3, or

(B) the points b_1 and b_2 lie on the line through a_1 and a_2, or

(C) a line through a_1 and one of the points p_1, p_2 and p_3 also contains b_1 or b_2.

If a smooth conic C contains six of the points, then C contains both the conjugate k_2-pairs and C is defined over k.

We shall denote the set of elements in $X(\lambda)$ of type (A) by $\mathscr{A}$, the set of elements of type (B) by $\mathscr{B}$ and the set of elements of type (C) by $\mathscr{C}$. The set $\mathscr{A}$ decomposes as a union of the sets $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ consisting of tuples with the line through b_1 and b_2 passing through p_1, p_2 and p_3, respectively. Similarly, the set $\mathscr{C}$ is the union of the six sets $\mathscr{C}_{i,j}$, $i = 1,2$, $j = 1,2,3$, where $\mathscr{C}_{i,j}$ contains all tuples such that a_1, b_i and p_j lie on a line.

The cardinalities of the above sets are easily computed to be

$$|\mathscr{A}_i| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q+1)(q^2-q),$$
$$|\mathscr{B}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q^2-q-2),$$

and

$$|\mathscr{C}_{i,j}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q^2-1).$$

The cardinality of $\mathscr{A}_i \cap \mathscr{A}_j$, $i \neq j$, is also easily computed:

$$|\mathscr{A}_i \cap \mathscr{A}_j| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q^2-q).$$

There are only nonempty intersection between the set $\mathscr{A}_i$ and the set $\mathscr{C}_{j,l}$ if $l \neq i$. We then place the first five points in general position and choose a k-line through p_l which does not pass through p_i in q ways. This gives a tuple of the desired form. We thus see that

$$|\mathscr{A}_i \cap \mathscr{C}_{j,l}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)q.$$

We also have nonempty intersection between the sets $_{1,i}$ and the set $_{2,j}$ where $i \neq j$. Such a configuration is actually given by specifying the first five points in general position since we must then take b_1 as the intersection point of the line between a_1 and p_i and the line between a_2 and p_j and similarly for b_2. Hence,

$$|\mathscr{C}_{1,i} \cap \mathscr{C}_{2,j}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q).$$

Since the set $\mathscr{B}$ cannot intersect any of the other sets, because this would require a_1 and a_2 to lie on a line through one of the k-points, it is now time to turn to the triple intersections.

Since p_1, p_2 and p_3 do not lie on a line we have that the intersection of $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ is empty. We thus only have two types of triple intersections, namely $\mathscr{A}_i \cap \mathscr{A}_j \cap \mathscr{C}_{r,s}$ and $\mathscr{A}_i \cap \mathscr{C}_{1,j} \cap \mathscr{C}_{2,s}$ where, of course, i, j and s are assumed to be distinct.

An element of $\mathscr{A}_i \cap \mathscr{A}_j \cap \mathscr{C}_{r,s}$ is specified by choosing the fist five points in general position. The point b_r must then be chosen as the intersection point of the line between p_i and p_j and the line between a_1 and p_s and similarly for Fb_r. We thus have

$$|\mathscr{A}_i \cap \mathscr{A}_j \cap \mathscr{C}_{r,s}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q).$$

To compute the cardinality of the intersection $\mathscr{A}_i \cap \mathscr{C}_{1,j} \cap \mathscr{C}_{2,s}$ we first choose two k-points p_j and p_s. We then choose a conjugate pair of k_2-lines through each of these points. The intersections of these lines give four k_2-points which we only have one way to label with a_1, a_2, b_1 and b_2. We remark that the four points p_j, p_s, a_1 and a_2 clearly are in general position. We must now place the point p_i somewhere on the line L through b_1 and b_2. The line through p_i and p_s intersects L in one k-point and the line through a_1 and a_2 intersects L in another. Thus, we have $q-1$ choices for p_i. We thus see that

$$|\mathscr{A}_i \cap \mathscr{C}_{1,j} \cap \mathscr{C}_{2,s}| = (q^2+q+1)(q^2+q)(q^2-q)^2(q-1).$$

This completes the investigation of $\mathcal{C}_{\text{lines}}$.

We now turn our attention to $\mathcal{C}_{\text{conic}}$. We first choose one of the points p_1, p_2 and p_3 to possibly lie outside our smooth conic C. We shall call the chosen point P. We then place the other two points and the two k_2-pairs on C. Finally, we must place P somewhere to make p_1, p_2, p_3, a_1 and a_2 lie in general position. Hence, we must choose P away from the line through the two other k-points and away from the line through a_1 and a_2. This gives us

$$(q^5-q^2)(q+1)q(q^2-q)(q^2-q-2)(q^2-q).$$

However, in the above we have counted the configurations where all seven points lie on C three times. We must therefore take away

$$2\cdot(q^5-q^2)(q+1)q(q-1)(q^2-q)(q^2-q-2),$$

in order to obtain $\mathcal{C}_{\text{conic}}$.

The intersection $B\cap\mathcal{C}_{\text{conic}}$ is empty but the intersections of $\mathcal{C}_{\text{conic}}$ with the other sets in the decomposition of $\mathcal{C}_{\text{lines}}$ are not. Recall that a k-point is said to be on the outside of C if it is the intersection point of two k-tangents to C and on the inside of C otherwise (see Definition 4.7.4).

To compute $|\mathscr{A}_i\cap\mathcal{C}_{\text{conic}}|$ we shall first assume that p_i lies on the outside of C. Of the $q+1$ lines through p_i which are defined over k we have that 2 are tangent to C and $\frac{1}{2}(q-1)$ intersect C in two k-points. Thus, there are $\frac{1}{2}(q-1)$ lines left which must intersect C in a pair of conjugate k_2-points. We pick such a line and label the intersection points by b_1 and b_2 in one of two ways. We shall now place the other two k-points on C. There are $\frac{1}{2}(q+1)q$ ways to choose two k-points on C of which $\frac{1}{2}(q-1)$ pairs lie on a k-line through p_i. There are thus $\frac{1}{2}(q^2+1)$ pairs which do not lie on a line through p_i and, since there are two ways to label each pair, we thus have q^2+1 choices for the two k-points. Finally, we shall place a_1 and a_2 somewhere on C but we have to make sure that the points p_1, p_2, p_3, a_1 and a_2 are in general position. Since the lines between p_i and the other two k-points intersect C only in k-points, the only thing that might go wrong when choosing a_1 and a_2 is that the line through a_1 and a_2 might also go through p_i. As seen above, there are exactly $q-1$ choices for a_1 and a_2 for which this happens, so the remaining $q^2-q-(q-1)=q^2-2q+1$ choices will give a configuration of the desired type. We thus have that the number of elements in $\mathscr{A}_i \cap \mathcal{C}_{\text{conic}}$ such that p_i lies on the outside of C is

$$\frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q^2+1)(q^2-2q+1).$$

We now assume that p_i lies on the inside of C. We proceed similarly to the above. First we observe that the number of k-lines through p_i is $q+1$ of which half intersect C in two k-points and half intersect C in conjugate pairs of k_2-points. We choose a line which intersects C in two conjugate k_2-points and label the intersection points by b_1 and b_2. We now choose a k-point p_j on C in one of $q+1$ ways. The line through p_i and p_j intersects C in another k-point and we choose the final k-point away from this intersection point and p_j. Finally, we shall place the points a_1 and a_2 on C in a way so that the points p_1, p_2, p_3, a_1 and a_2 are in general position. As above, the only thing that might go wrong is that the line through a_1 and a_2 might go through p_i and there are precisely $q+1$ choices for a_1 and a_2 for which this happens. Thus, there are $q^2-q-(q+1)=q^2-2q-1$ valid choices for a_1 and a_2. Hence, there are

$$\frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q+1)(q-1)(q^2-2q-1),$$

elements in $\mathscr{A}_i \cap \mathcal{C}_{\text{conic}}$ such that p_i lies on the in side of C.

To compute the intersection $\mathscr{C}_{i,j} \cap \mathcal{C}_{\text{conic}}$ we note that if we place p_j outside of C and then choose two k-points on C and two conjugate k_2-points a_1 and a_2 on C such that p_1, p_2, p_3, a_1 and a_2 are in general position, then we must choose b_i as the other intersection point of C with the line through a_1 and p_j. We may thus use constructions analogous to those above and to see that there are

$$\frac{1}{2}(q^5-q^2)(q+1)q(q^2+1)(q^2-2q+1),$$

elements in $\mathscr{A}_{i,j} \cap \mathcal{C}_{\text{conic}}$ with p_j on the outside of C and

$$\frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q-1)(q^2-2q-3),$$

elements with p_j on the inside of C. Note that, in the latter case, the factor corresponding to the choice of a_1 and a_2 is two less than the corresponding factor for the intersection $\mathscr{A}_i \cap \mathcal{C}_{\text{conic}}$. This is because in this case we need to choose a_1 and a_2 away from the two points with tangents passing through p_j in order to make the line through a_1 and p_j intersect C in two points.

We may now put all these pieces together to obtain

$$N_\lambda = q^{14} - 7q^{13} + 9q^{12} + 21q^{11} - 29q^{10} - 32q^9 + 19q^8 + \\ + 44q^7 + 8q^6 - 41q^5 - 8q^4 + 15q^3,$$

and, after dividing by $|\mathrm{PGL}(3)|$,

$$\mathcal{N}_\lambda = q^6 - 7q^5 + 10q^4 + 15q^3 - 26q^2 - 8q + 15.$$

4.15 The case $\lambda = [1^5, 2^1]$

Throughout this section, λ shall mean the partition $[1^5, 2^1]$. We shall denote the k-points by p_1, p_2, p_3, p_4 and p_5 and the points of the conjugate pair of k_2-points by a_1 and a_2.

We shall consider $\mathbb{P}^{2,7}_{\mathrm{gp}}$ as a subset of $X \subset \mathbb{P}^{2,7}$ consisting of seventuples of points such that the first five points lie in general position. Thus, the points p_1, p_2, p_3, p_4 and p_5 should always be chosen such that no three lie on a line. To compute $|X(\lambda)|$ we first choose the first four k-points in general position in $(q^2+q+1)(q^2+q)q^2(q^2-2q+1)$ ways. The six k-lines through pairs of these points together contain

$$(q+1) + 2q + 3(q-2) = 6q - 5,$$

points, so there are $q^2+q+1-(6q-5) = q^2-5q+6$ choices for p_5. Finally, we choose any conjugate twotuple we want. Hence

$$|X(\lambda)| = (q^2+q+1)(q^2+q)q^2(q^2-2q+1)(q^2-5q+6)(q^4-q).$$

We shall now compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\mathrm{gp}}(\lambda)$ in $X(\lambda)$.

The choice of X makes the number of cases we have to investigate very few.

Lemma 4.15.1. *If three points of a conjugate λ-tuple in $X(\lambda)$ lie on a line, then a_1 and a_2 lie on a line passing through one of the k-points. If six of the points of a conjugate λ-tuple lie on a smooth conic C, then C is defined over k and contains a_1 and a_2.*

There are

$$(q+1) + q + (q-1) + (q-2) + (q-3) = 5q - 5,$$

k-lines passing through p_1, p_2, p_3, p_4 or p_5 (or possibly two of them). Each of these lines contains $q^2 - q$ conjugate twotuples and no conjugate twotuple lie on two such lines. We thus have

$$|\mathcal{C}_{\text{lines}}| = (q^2+q+1)(q^2+q)q^2(q^2-2q+1)(q^2-5q+6)(5q-5)(q^2-q).$$

To compute the cardinality of $\mathcal{C}_{\text{conic}}$ we first choose p_1, p_2, p_3, p_4 or p_5 to possibly lie outside C and call the chosen point P. Then, we choose four k-points and a conjugate twotuple on C. Finally, we choose P away from the six lines through pairs of the other four k-points. We thus get

$$5(q^5-q^2)(q+1)q(q-1)(q-2)(q^2-q)(q^2-5q+6).$$

In the above we have counted the λ-tuples with all seven points on a conic five times. We therefore must take away

$$4(q^5-q^2)(q+1)q(q-1)(q-2)(q-3)(q^2-q),$$

in order to obtain $|\mathcal{C}_{\text{conic}}|$.

To compute the size of the intersection $\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}$ we shall decompose this set into a disjoint five subsets $\mathscr{A}_i$, $i = 1, \dots, 5$, where $\mathscr{A}_i$ consists of those tuples where p_i does not lie on the conic C through the other six points. Each of the sets $\mathscr{A}_i$ is then be decomposed further into a union of the sets $\mathscr{A}_i^{\text{out}}$ and $\mathscr{A}_i^{\text{in}}$ where $\mathscr{A}_i^{\text{out}}$ consists of those tuples with p_i on the outside of C and $\mathscr{A}_i^{\text{in}}$ consists of those with p_i on the inside of C. Finally, we shall decompose $\mathscr{A}_i^{\text{out}}$ into three disjoint subsets:

- the set $\mathscr{A}_{i,0}^{\text{out}}$ consisting of λ-tuples such that the tangent lines to C passing through p_i does not pass through any of the other points of the λ-tuple,
- the set $\mathscr{A}_{i,1}^{\text{out}}$ consisting of λ-tuples such that exactly one of the tangent lines to C passing through p_i passes through one of the other points of the λ-tuple,
- the set $\mathscr{A}_{i,2}^{\text{out}}$ consisting of λ-tuples such that both the tangent lines to C passing through p_i passes through another point of the λ-tuple.

To compute $|\mathscr{A}_i^{\text{out}}|$, we first choose a smooth conic C defined over k in $q^5 - q^2$ ways and then a point p_i outside C in $\frac{1}{2}(q+1)q$ ways. As seen many times before, there are exactly $\frac{1}{2}(q-1)$ lines through p_i which are defined over k and which intersect C in a conjugate pair of points. We pick such a line and label the points a_1 and a_2 in one of two ways. From this point on, the computations are a little bit different for the three subsets of $\mathscr{A}_i^{\text{out}}$.

4.15.1. The subset $\mathscr{A}_{i,0}^{\text{out}}$. We shall now pick the other four k-points of the λ-tuple. Since we should not pick points whose tangent passes through p_i, we have $q-1$ choices for the first point. For the second point, we should stay away from the tangent points, the first point and the other intersection point of C and the line through p_i and the first point. Hence, we have $q-3$ choices. In a similar way, we see that we have $q-5$ choices for the third point and $q-7$ for the fourth. Hence,

$$|\mathscr{A}_{i,0}^{\text{out}}| = \frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q-1)(q-3)(q-5)(q-7).$$

4.15.2. The subset $\mathscr{A}_{i,1}^{\text{out}}$. We begin by choosing one of the four k-points to lie on a tangent to C passing through C and then we pick the tangent it should lie on. For the first of the remaining three points we now have $q-1$ choices and, similarly to the above case, we have $q-3$ choices for the second and $q-5$ for the third. Thus,

$$|\mathscr{A}_{i,1}^{\text{out}}| = 4\cdot 2\cdot \frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q-1)(q-3)(q-5).$$

4.15.3. The subset $\mathscr{A}_{i,2}^{\text{out}}$. We begin by choosing two of the four k-points to lie on tangents to C passing through C and then we pick which point should lie on which tangent. For the first of the remaining two points we now have $q-1$ choices and we then have $q-3$ choices for the second. Thus,

$$|\mathscr{A}_{i,2}^{\text{out}}| = \binom{4}{2}\cdot 2\cdot \frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q-1)(q-3).$$

It thus only remains to compute $|\mathscr{A}_i^{\text{in}}|$. We thus choose a smooth conic C defined over k in q^5-q^2 ways and then a point p_i on the inside of C in $\frac{1}{2}(q^2-q)$ ways. We have already seen that there now is $\frac{1}{2}(q+1)$ lines passing through p_i which are defined over k and which intersect C in a conjugate pair of points. We thus pick such a line and label the intersection points by a_1 and a_2. Since any k-line through p_i will intersect C in precisely two points, we have $(q+1)(q-1)(q-3)(q-5)$ choices for the remaining four k-points of the λ-tuple. We thus see that

$$|\mathscr{A}_i^{\text{in}}| = \frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q-1)(q-3)(q-5).$$

We may now add everything up to obtain

$$\begin{aligned} N_\lambda = {} & q^{14}-15q^{13}+89q^{12}-251q^{11}+299q^{10}-24q^9-109q^8 \\ & --84q^7-80q^6+359q^5-200q^4+15q^3, \end{aligned}$$

and, finally,

$$\mathcal{N}_\lambda = q^6 - 15q^5 + 90q^4 - 265q^3 + 374q^2 - 200q + 15.$$

4.16 The case $\lambda = [1^7]$

Throughout this section, λ shall mean the partition $[1^7]$. We shall denote the seven k-points by $p_1, \ldots, p_7$. Since almost all points we shall be interested in this section will be defined over k we shall use the word "point" to mean "k-point". At the few occasions where we shall consider other kinds of points we shall of course be careful to specify this. In a similar manner, the word "line" will mean "k-line" and "conic" shall always mean "conic defined over k".

We shall consider $\mathbb{P}^{2,7}_{\mathrm{gp}}$ as a subset of $X \subset \mathbb{P}^{2,7}$ consisting of seventuples of points such that the first four points lie in general position. Thus, the points p_1, p_2, p_3 and p_4 and should always be chosen such that no three lie on a line. We thus have

$$|X(\lambda)| = (q^2+q+1)(q^2+q)q^2(q^2-2q+1)(q^2+q-3)(q^2+q-4)(q^2+q-5).$$

We shall now compute the cardinality of the complement $\mathcal{C}$ of $\mathbb{P}^{2,7}_{\mathrm{gp}}(\lambda)$ in $X(\lambda)$.

Remark 4.16.1. The reader might wonder why we do not, as in the previous two sections, require the first five points to lie in general position. Indeed, it is fairly easy to place five points in general position (in fact, we did this in the beginning of the previous section). However, with five points we have much less control over how lines between pairs of the points intersect. For instance, depending on the choice of the fifth point we may or may not have three of these lines intersecting in one point. In the previous two sections this has not mattered since the final two points have been defined over k_2 but not k and these intersection points are k-points. This is the explanation to why we have not cared about these points before, but now have to deal with them in one way or another.

The following definition will be quite convenient.

Definition 4.16.2. If a and b are two points in $\mathbb{P}^2$, then the line through a and b shall be denoted $\langle a, b\rangle$.

4.16.1. The set $\mathcal{C}_{\mathrm{lines}}$. The set $\mathcal{C}_{\mathrm{lines}}$ decomposes into a union of three sets.

Lemma 4.16.3. *If three points of a λ-tuple in $X(\lambda)$ lie on a line then either*

(A) *one of the lines* $\langle p_i, p_j\rangle$, $1 \le i < j \le 4$, *contains one of the points* p_5, p_6 *and* p_7, *or*

(B) *one of the lines* $\langle p_i, p_j\rangle$, $5 \le i < j \le 7$, *contains one of the points* p_1, p_2, p_3 *and* p_4, *but* $\{p_5, p_6, p_7\} \cap \mathscr{S} = \emptyset$, *or*

(C) *the three points* p_5, p_6 *and* p_7 *lie on a line and which does not pass through* p_1, p_2, p_3 *or* p_4.

Remark 4.16.4. The added requirement $\{p_5, p_6, p_7\} \cap \mathscr{S} = \emptyset$ in cases (B) and (C) is, of course, not really necessary and may seem strange. However, removing these conditions lead to quite horrible intersections and making the cases (B) and (C) slightly more complicated is therefore a comparatively small price to pay in order to make cases disjoint.

We shall denote the set of tuples of type (A) by $\mathscr{A}$, the set of tuples of type (B) by $\mathscr{B}$ and the set of tuples of type (C) by $\mathscr{C}$. It will also be convenient to denote the union of the six lines $\langle p_i, p_j\rangle$, $1 \le i < j \le 4$ by $\mathscr{S}$. We note that $\mathscr{S}$ contains

$$6(q-2)+4+3 = 6q-5,$$

k-points (of which four are the points p_1, p_2, p_3 and p_4). Also, note that the set $\mathscr{A}$ decomposes further into a union of the three sets $\mathscr{A}_5$, $\mathscr{A}_6$ and $\mathscr{A}_7$ consisting of tuples with p_5, p_6 resp. p_7 in $\mathscr{S}$. Similarly, $\mathscr{B}$ decomposes into a union of the three sets $\mathscr{B}_{5,6}$, $\mathscr{B}_{5,7}$ and $\mathscr{B}_{6,7}$ where $\mathscr{B}_{i,j}$ consists of the tuples where the line through p_i and p_j also passes through p_1, p_2, p_3 or p_4. The sets $\mathscr{B}_{i,j}$ decompose even further into four sets $\mathscr{B}^r_{i,j}$ consisting of tuples such that the line $\langle p_i, p_j\rangle$ passes through p_r where p_r is either p_1, p_2, p_3 or p_4.

4.16.2. The set $\mathscr{A}$. A typical element of $\mathscr{A}_i$ is illustrated in Figure 4.5 above. The computation of $|\mathscr{A}_i|$ is simple: we first choose p_i as any point in $\mathscr{S}$ and then place the remaining two points anywhere. Hence

$$|\mathscr{A}_i| = \underbrace{(q^2+q+1)(q^2+q)q^2(q^2-2q+1)}_{=|\mathrm{PGL}(3)|}(6q-9)(q^2+q-4)(q^2+q-5).$$

Very similar computations give that

$$|\mathscr{A}_i \cap \mathscr{A}_j| = |\mathrm{PGL}(3)| \cdot (6q-9)(6q-10)(q^2+q-5),$$

and

$$|\mathscr{A}_1 \cap \mathscr{A}_2 \cap \mathscr{A}_3| = |\mathrm{PGL}(3)| \cdot (6q-9)(6q-10)(6q-11).$$

This allows us to compute $|\mathscr{A}|$ using the principle of inclusion and exclusion:

$$|\mathscr{A}| = |\mathrm{PGL}(3)| \cdot (18q^5 - 99q^4 + 252q^3 - 414q^2 + 417q - 180).$$

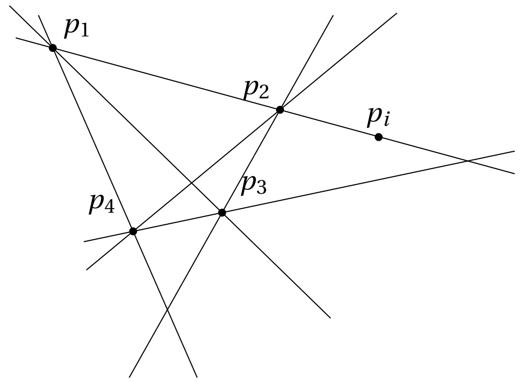

Figure 4.5: A typical element of $\mathscr{A}_i$.

4.16.3. The set $\mathscr{B}$. A typical element of $\mathscr{B}^r_{i,j}$ is illustrated in Figure 4.6. To obtain an element of $\mathscr{B}^r_{i,j}$ we first place p_1, p_2, p_3 or p_4 in general position. There are $q+1$ lines through p_r of which 3 are contained in $\mathscr{S}$. We choose $\langle p_i, p_j\rangle$ as one of the remaining $q-2$ lines. Note that $\langle p_i, p_j\rangle$ will not pass through any of the points

$$q_1 = \langle p_1, p_4\rangle \cap \langle p_2, p_3\rangle, \quad q_2 = \langle p_2, p_4\rangle \cap \langle p_1, p_3\rangle, \quad q_3 = \langle p_3, p_4\rangle \cap \langle p_1, p_2\rangle. \tag{4.16.1}$$

Hence, $\langle p_i, p_j\rangle$ will intersect $\mathscr{S}$ in p_r and three further points. There are thus $q-3$ ways to choose p_i and then $q-4$ ways to choose p_j. Finally, there are

$$|\mathbb{P}^2 \setminus \mathscr{S}| - 2 = q^2 + q + 1 - (6q-5) - 2 = q^2 - 5q + 4,$$

possibilities for the seventh point. We thus have

$$|\mathscr{B}^r_{i,j}| = |\mathrm{PGL}(3)| \cdot (q-2)(q-3)(q-4)(q^2-5q+4).$$

However, we have counted some tuples several times. To begin with, the points of $\mathscr{B}^r_{5,6} \cap \mathscr{B}^r_{5,7} \cap \mathscr{B}^r_{6,7}$ have been counted three times. There are

$$|\mathscr{B}^r_{5,6} \cap \mathscr{B}^r_{5,7} \cap \mathscr{B}^r_{6,7}| = |\mathrm{PGL}(3)| \cdot (q-2)(q-3)(q-4)(q-5),$$

of these.

Further, the sets $\mathscr{B}^r_{i,j_1}$ and $\mathscr{B}^s_{i,j_2}$ will intersect for $r \neq s$ and $j_1 \neq j_2$. A typical element is illustrated in Figure 4.7 below.

It is very easy to compute $|\mathscr{B}^r_{i,j_1} \cap \mathscr{B}^s_{i,j_2}|$. We begin by choosing p_1, p_2, p_3 and p_4 in general position and then choosing p_i outside $\mathscr{S}$ in q^2-5q+6 ways. This gives us two lines $\langle p_i, p_r\rangle$ and $\langle p_i, p_s\rangle$ which intersect $\mathscr{S}$ in four

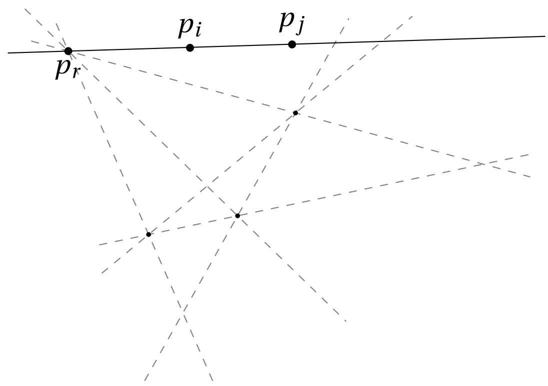

Figure 4.6: A typical element of $\mathscr{B}^r_{i,j}$.

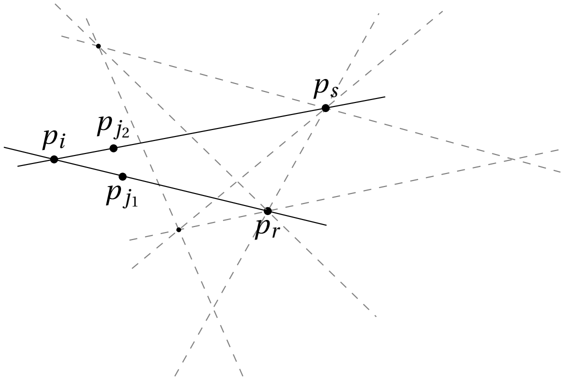

Figure 4.7: A typical element of $\mathscr{B}^r_{i,j_1} \cap \mathscr{B}^s_{i,j_2}$.

points each. We choose p_{j_1} on $\langle p_i, p_r\rangle$ away from p_i and $\mathscr{S}$ in $q-4$ ways and similarly for p_{j_2}. This gives

$$|\mathscr{B}^r_{i,j_1} \cap \mathscr{B}^s_{i,j_2}| = |\mathrm{PGL}(3)| \cdot (q^2 - 5q + 6)(q-4)^2.$$

Finally, we must compute the cardinality of the triple intersection $\mathscr{B}^r_{5,6} \cap \mathscr{B}^s_{5,7} \cap \mathscr{B}^t_{6,7}$. A typical element of the intersection is illustrated in Figure 4.8.

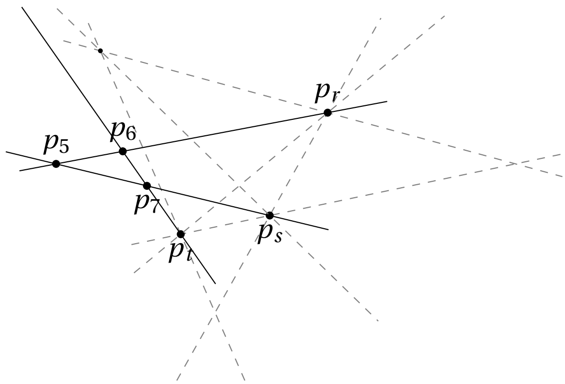

Figure 4.8: A typical element of $\mathscr{T}^{r,s,t}$.

This is where we have to pay for the extra requirement that p_5, p_6 and p_7 should not be in $\mathscr{S}$. We shall view $\mathscr{B}^r_{5,6} \cap \mathscr{B}^s_{5,7} \cap \mathscr{B}^t_{6,7}$ as an open subset of the set $\mathscr{T}^{r,s,t}$ consisting of tuples such that

- the line $\langle p_5, p_6\rangle$ passes through p_r, $\langle p_5, p_7\rangle$ passes through p_s and $\langle p_6, p_7\rangle$ passes through p_t but,
- we allow p_5, p_6 and p_7 to lie in $\mathscr{S}$, but,
- we do not allow the lines $\langle p_i, p_j\rangle$, $5 \le i < j \le 7$ to be contained in $\mathscr{S}$.

The set $\mathscr{T}^{r,s,t}$ can be decomposed into a union of three subsets $\mathscr{T}^{r,s,t}_i$, $i = 5,6,7$, consisting of those tuples with p_i in $\mathscr{S}$. We then get

$$|\mathscr{B}^r_{5,6} \cap \mathscr{B}^s_{5,7} \cap \mathscr{B}^t_{6,7}| = |\mathscr{T}^{r,s,t}| - |\mathscr{T}^{r,s,t}_5| - |\mathscr{T}^{r,s,t}_6| - |\mathscr{T}^{r,s,t}_7| +$$
$$+ |\mathscr{T}^{r,s,t}_5 \cap \mathscr{T}^{r,s,t}_6| + |\mathscr{T}^{r,s,t}_5 \cap \mathscr{T}^{r,s,t}_7| + |\mathscr{T}^{r,s,t}_6 \cap \mathscr{T}^{r,s,t}_7| - |\mathscr{T}^{r,s,t}_5 \cap \mathscr{T}^{r,s,t}_6 \cap \mathscr{T}^{r,s,t}_7|,$$

by the principle of inclusion and exclusion.

The computation of either of these cardinalities is quite simple. We begin with $\mathscr{T}^{r,s,t}$. To obtain such a tuple, we begin by choosing a line L_r through p_r in $q-2$ ways. We shall then choose a line L_s through p_s. There are however two cases that may occur. Typically, the intersection point $p_5 =$

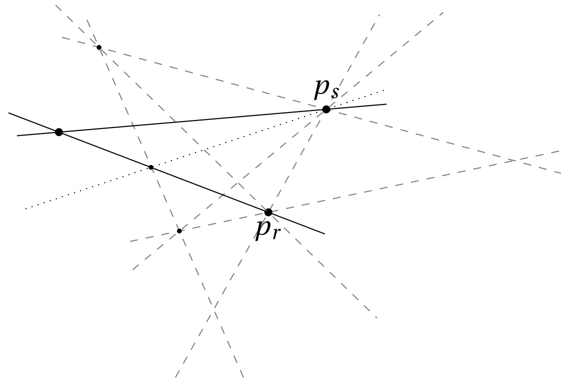

Figure 4.9

$L_r \cap L_s$ will lie outside $\mathscr{S}$ but for one choice of L_s it will lie in $\mathscr{S}$. The situation is illustrated in Figure 4.9.

There are $q-3$ ways to choose L_s so that $L_r \cap L_s$ lies outside $\mathscr{S}$. When we choose the line L_t through p_t we must make sure that L_t is not contained in $\mathscr{S}$ and that L_t does not pass through $L_r \cap L_s$, since we want to end up with three distinct intersection points. We thus have $q-3$ choices. On the other hand, if we choose L_s as the one line making the intersection point $L_r \cap L_s$ lie in $\mathscr{S}$ we only need to make sure that L_t is not contained in $\mathscr{S}$ and we thus have $q-2$ choices. Hence, we see that

$$|\mathscr{T}^{r,s,t}| = |\mathrm{PGL}(3)| \cdot \big((q-2)(q-3)^2 + (q-2)^2\big).$$

The computation of $|\mathscr{T}_i^{r,s,t}|$, $i = 5,6,7$, is even simpler. To simplify the discussion, we describe the computation for $i = 5$. We then begin by choosing a line L_r through p_r in $q-2$ ways. The line L_s through p_s is then completely determined since we must have $p_5 \in \mathscr{S}$. This gives us $q-2$ choices for the final line L_t through p_t. Hence,

$$|\mathscr{T}_i^{r,s,t}| = |\mathrm{PGL}(3)| \cdot (q-2)^2.$$

We now turn to the computation of $|\mathscr{T}_i^{r,s,t} \cap \mathscr{T}_j^{r,s,t}|$, $5 \le i < j \le 7$. These cardinalities are clearly the same for all i and j so we make the computation of $|\mathscr{T}_{5,6}^{r,s,t}|$ in order to make the discussion simpler. As above, we begin by choosing a line L_r through p_r in $q-2$ ways. Since p_5 must lie in $\mathscr{S}$ we have only one choice for L_s. Since $p_6 = L_s \cap L_t$ we see that we now have precisely one choice for L_t also. Hence,

$$|\mathscr{T}_i^{r,s,t} \cap \mathscr{T}_j^{r,s,t}| = |\mathrm{PGL}(3)| \cdot (q-2).$$

The computation of $|\mathscr{T}_5^{r,s,t} \cap \mathscr{T}_6^{r,s,t} \cap \mathscr{T}_7^{r,s,t}|$ is extremely simple: once the four points p_1, p_2, p_3 and p_4 have been placed in general position, there is precisely one such tuple. The situation is illustrated in Figure 4.10. The proof of this fact is a simple exercise in linear algebra.

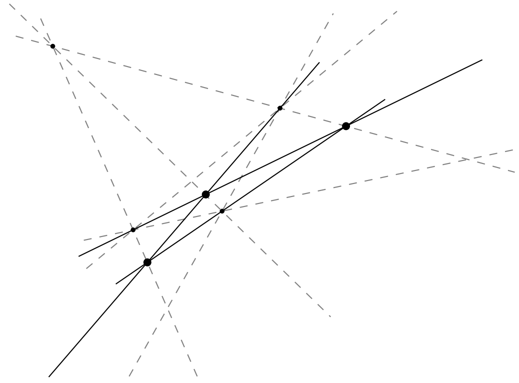

Figure 4.10: The only element in $\mathscr{T}_5^{r,s,t} \cap \mathscr{T}_6^{r,s,t} \cap \mathscr{T}_7^{r,s,t}$.

This allows us to compute

$$|\mathscr{B}_{5,6}^r \cap \mathscr{B}_{5,7}^s \cap \mathscr{B}_{6,7}^t| = |\mathrm{PGL}(3)| \cdot (q^3 - 10q^2 + 32q - 33),$$

and, finally,

$$|\mathscr{B}| = |\mathrm{PGL}(3)| \cdot (12q^5 - 212q^4 + 1504q^3 - 5320q^2 + 9296q - 6360).$$

4.16.4. The set $\mathscr{C}$. Recall the definition of the three points q_1, q_2 and q_3 from Equation 4.16.1. Using these three points we may decompose $\mathscr{C}$ into a disjoint union of the following subsets:

- $\mathscr{C}_{\langle q_r, q_s \rangle}$ consisting of those tuples of $\mathscr{C}$ where p_5, p_6 and p_7 lie on the line $\langle q_r, q_s \rangle$, $1 \le r < s \le 3$, and,
- $\mathscr{C}_{q_r}$ consisting of those tuples of $\mathscr{C}$ with p_5, p_6 and p_7 on a line through q_r, $1 \le r \le 3$, which does not pass through any of the other q_i, and
- $\mathscr{C}_{\mathrm{gen}}$ consisting of those tuples of $\mathscr{C}$ with p_5, p_6 and p_7 on a line which does not pass through q_1, q_2 or q_3.

We shall count the cardinalities of these sets separately. We begin by $\mathscr{C}_{\langle q_1, q_2 \rangle}$. The line $\langle q_r, q_s \rangle$ contains $q+1$ points of which four lie in S. There are thus $q-3$ choices for p_5, $q-4$ choices for p_6 and $q-5$ choices for p_7. Hence,

$$|\mathscr{C}_{\langle q_r, q_s \rangle}| = |\mathrm{PGL}(3)| \cdot (q-3)(q-4)(q-5).$$

We continue with $|\mathscr{C}_{q_r}|$. There are $q+1$ lines through q_r of which two are contained in $\mathscr{S}$ and two are the lines through the other two q_i. Hence, there are $q-3$ choices for a line L though q_r. The line L intersect S in five points so we have $q-4$ choices for p_5, $q-5$ choices for p_6 and $q-6$ choices for p_7. We conclude that

$$|\mathscr{C}_{q_r}| = |\mathrm{PGL}(3)| \cdot (q-3)(q-4)(q-5)(q-6).$$

To compute $|\mathscr{C}_{\mathrm{gen}}|$ we shall begin by choosing a line L which does not pass through any of the points p_1, p_2, p_3, p_4, q_1, q_2 and q_3. There are q^2+q+1 lines in $\mathbb{P}^2$, of which $q+1$ passes through p_i, $i=1,2,3,4$. There are exactly two lines through each pair of these points so there are

$$q^2+q+1-4(q+1)+6 = q^2-3q+3,$$

lines which does not pass through p_1, p_2, p_3 and p_4. Of the $q+1$ lines through q_i, $i=1,2,3$, precisely two have been removed above and the line $\langle q_i, q_j\rangle$ passes through both q_i and q_j. Hence, we have

$$q^2-3q+3-3(q-1)+3 = q^2-6q+9,$$

choices for L.

The line L intersects S in six points. We therefore have $q-5$ choices for p_5, $q-6$ choices for p_6 and $q-7$ choices for p_7. Hence,

$$|\mathscr{C}_{\mathrm{gen}}| = |\mathrm{PGL}(3)| \cdot (q^2-6q+9)(q-5)(q-6)(q-7).$$

We now the different cardinalities together to obtain

$$|\mathscr{C}| = |\mathrm{PGL}(3)| \cdot (q^5-21q^4+173q^3-693q^2+1338q-990).$$

Finally, we may add $|\mathscr{A}|$, $|\mathscr{B}|$ and $|\mathscr{C}|$ together to obtain

$$|\mathcal{C}_{\mathrm{lines}}| = |\mathrm{PGL}(3)| \cdot (31q^5-332q^4+1929q^3-6427q^2+11051q-7530).$$

4.16.5. The set $\mathcal{C}_{\mathrm{conic}}$. Here we have two different cases

(A) six points lie on a smooth conic C with one of the points p_1, p_2, p_3 or p_4 possibly outside C,

(B) six points lie on a smooth conic C with one of the points p_5, p_6 or p_7 possibly outside C.

We shall denote the set of tuples of type (A) by $\mathscr{A}$ and the set of tuples of type (B) by $\mathscr{B}$.

To obtain an element of $\mathscr{A}$ we first choose one of the points p_1, p_2, p_3 and p_4 and call it P. Then we choose a smooth conic C in $q^5 - q^2$ ways and place all of the seven points except P on C in

$$(q+1)q(q-1)(q-2)(q-3)(q-4),$$

ways. There are three lines through pairs of points in $\{p_1, p_2, p_3 p_4\} \setminus \{P\}$ which together contain $3q$ points. These lines do not contain p_5, p_6 and p_7 so we have

$$q^2 + q + 1 - 3q - 3 = q^2 - 2q - 2,$$

choices for P. Multiplying everything together we obtain

$$N_{\mathscr{A}} := 4(q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)(q^2 - 2q - 2),$$

which is almost $|\mathscr{A}|$ except that we have counted the tuples where all seven points lie on C four times.

To obtain an element of $\mathscr{B}$ is even simpler. We first choose p_5, p_6 and p_7 and call the chosen point P. We then choose a smooth conic C and place all but the chosen points on C. Finally, we place P anywhere in $\mathbb{P}^2$ except at the six chosen points. In this way we obtain the number

$$N_{\mathscr{B}} := 3(q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)(q^2 + q - 5),$$

which is almost equal to $|\mathscr{B}|$ except that we have counted the tuples with all seven points on C three times.

We now want to compute the number of tuples with all seven points on a smooth conic C. We thus choose a smooth conic C and place all seven points on it in

$$N_7 := (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)(q-5),$$

ways. We thus have

$$\begin{aligned}|\mathcal{C}_{\text{conic}}| &= N_{\mathscr{A}} + N_{\mathscr{B}} - 6N_7 = \\ &= |\mathrm{PGL}(3)| \cdot (7q^5 - 74q^4 + 288q^3 - 517q^2 + 446q - 168).\end{aligned}$$

4.16.6. The intersection $\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}$. We introduce the filtration $\mathscr{F}_3 \subset \mathscr{F}_2 \subset \mathscr{F}_1 = \mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}$ where

- the set $\mathscr{F}_1$ consists of tuples such that at least one line contain three points of the tuple,

- the set $\mathscr{F}_2$ consists of tuples such that at least two lines contain three points of the tuple,
- the set $\mathscr{F}_3$ consists of tuples such that at least three lines contain three points of the tuple.

The strategy will be to compute the numbers:

$$\begin{aligned}
N_1 &= |\mathscr{F}_1| + |\mathscr{F}_2| + |\mathscr{F}_3|,\\
N_2 &= |\mathscr{F}_2| + 2|\mathscr{F}_3|,\\
N_3 &= |\mathscr{F}_3|,
\end{aligned}$$

and thereby obtain the desired cardinality.

Since the points p_1, p_2, p_3 and p_4 are assumed to constitute a frame, we must do things a little bit differently depending on whether the point not on the conic is one of these four or not. We therefore make further subdivisions.

4.16.7. The subsets with p_5, p_6 or p_7 not on the conic. We shall denote the subsets in question by $\mathscr{F}_i^{5,6,7}$ and, similarly

$$\begin{aligned}
N_1^{5,6,7} &= |\mathscr{F}_1^{5,6,7}| + |\mathscr{F}_2^{5,6,7}| + |\mathscr{F}_3^{5,6,7}|,\\
N_2^{5,6,7} &= |\mathscr{F}_2^{5,6,7}| + 2|\mathscr{F}_3^{5,6,7}|,\\
N_3^{5,6,7} &= |\mathscr{F}_3^{5,6,7}|.
\end{aligned}$$

To compute $N_1^{5,6,7}$, we first choose one of the points p_5, p_6 or p_7 to be the point P not on the smooth conic C and call the remaining two points p_i and p_j. We then choose C in $q^5 - q^2$ ways and choose two points among $\{p_1, p_2, p_3, p_4, p_i, p_j\}$ and call them a_1 and a_2. There are $(q+1)q$ ways to place a_1 and a_2 on C and there are then $q-1$ ways to place P on the line $\langle a_1, a_2\rangle$. Finally, we place the remaining four points on C in $(q-1)(q-2)(q-3)(q-4)$ ways. Multiplying everything together we obtain

$$N_1^{5,6,7} := 3 \cdot \binom{6}{2} \cdot (q^5 - q^2)(q+1)q(q-1)^2(q-2)(q-3)(q-4).$$

In order to compute $N_2^{5,6,7}$, we first choose one of the points p_5, p_6 or p_7 to be the point P not on the smooth conic C and call the remaining two points p_i and p_j. We then choose C in $q^5 - q^2$ ways and choose two unordered pairs of unordered points among $\{p_1, p_2, p_3, p_4, p_i, p_j\}$. This can be done in $\frac{1}{2} \cdot \binom{6}{4} \cdot \binom{4}{2}$ ways. We call the points of the first pair a_1 and a_2 and those of the second b_1 and b_2. There are $(q+1)q(q-1)(q-2)(q-3)(q-4)$

ways to place $\{p_1, p_2, p_3, p_4, p_i, p_j\}$ on C and the point P is then completely determined as $P = \langle a_1, a_2\rangle \cap \langle b_1, b_2\rangle$. Thus

$$N_2^{5,6,7} = 3 \cdot \frac{1}{2} \cdot \binom{6}{4} \cdot \binom{4}{2} \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4).$$

The computation of $N_3^{5,6,7}$ is slightly more complicated since we need to subdivide into two subcases depending on if P is on the outside or on the inside of C. We call the two corresponding numbers $N_{3,out}^{5,6,7}$ and $N_{3,in}^{5,6,7}$.

To compute $N_{3,out}^{5,6,7}$ we first choose one of the points p_5, p_6 or p_7 to be the point P not on the smooth conic C. We proceed by choosing the smooth conic C in $q^5 - q^2$ ways and then the point P on the outside of C in $\frac{1}{2}(q+1)q$ ways. We now place p_1 at one of the $q-1$ points of C whose tangent does not pass through P and choose one of the remaining 5 points as the other intersection point in $C \cap \langle p_1, P\rangle$. There are now four remaining points p_i, p_j, p_k and p_l to place on C. We place p_i at one of the $q-3$ remaining points of C whose tangent does not pass through P and choose one of the remaining three points as the other intersection point in $C \cap \langle p_i, P\rangle$. There are now two points p_r and p_s to place on C. We place p_r at one of the $q-5$ possible points and the point p_s is then determined. We thus have

$$N_{3,out}^{5,6,7} = 3 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q+1)q \cdot (q-1) \cdot 5 \cdot (q-3) \cdot 3 \cdot (q-5).$$

We proceed by computing $N_{3,in}^{5,6,7}$. We first choose one of the points p_5, p_6 or p_7 to be the point P not on the smooth conic C. We proceed by choosing the smooth conic C in $q^5 - q^2$ ways and then the point P on the outside of C in $\frac{1}{2}(q-1)q$ ways.

We now place p_1 at one of the $q+1$ points of C whose tangent does not pass through P and choose one of the remaining 5 points as the other intersection point in $C \cap \langle p_1, P\rangle$. There are now four remaining points p_i, p_j, p_k and p_l to place on C. We place p_i at one of the $q-1$ remaining points of C and choose one of the remaining three points as the other intersection point in $C \cap \langle p_i, P\rangle$. There are now two points p_r and p_s to place on C. We place p_r at one of the $q-3$ possible points and the point p_s is then determined. We now see that

$$N_{3,in}^{5,6,7} = 3 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q-1)q \cdot (q+1) \cdot 5 \cdot (q-1) \cdot 3 \cdot (q-3).$$

The subsets with p_1, p_2, p_3 or p_4 not on the conic.

We shall denote the subsets in question by $\mathscr{F}_i^{1,2,3,4}$ and, similarly

$$\begin{aligned} N_1^{1,2,3,4} &= |\mathscr{F}_1^{1,2,3,4}| + |\mathscr{F}_2^{1,2,3,4}| + |\mathscr{F}_3^{1,2,3,4}|, \\ N_2^{1,2,3,4} &= |\mathscr{F}_2^{1,2,3,4}| + 2|\mathscr{F}_3^{1,2,3,4}|, \\ N_3^{1,2,3,4} &= |\mathscr{F}_3^{1,2,3,4}|. \end{aligned}$$

In order to compute $N_1^{1,2,3,4}$, we first choose one of the points p_1, p_2, p_3 or p_4 to be the point P not on the smooth conic C and call the remaining three points p_r, p_s and p_t. We continue by choosing a smooth conic C in $q^5 - q^2$ ways.

We first assume that P lies on a line $\langle a_1, a_2\rangle$ where $\{a_1, a_2\} \subset \{p_5, p_6, p_7\}$. We therefore choose the two points in 3 ways and call the remaining point p_i. We then place a_1 and a_2 on C in $(q+1)q$ ways. We continue by choosing the three points p_r, p_s and p_t on C in $(q-1)(q-2)(q-3)$ ways. The lines $\langle p_r, p_s\rangle$, $\langle p_r, p_t\rangle$ and $\langle p_s, p_t\rangle$ intersect the line $\langle a_1, a_2\rangle$ in three distinct points so there are $q-4$ ways to choose the point P on $\langle a_1, a_2\rangle$ but away from these three points and a_1 and a_2. Finally, we place p_i at one of the $q-4$ remaining points of C. Multiplying everything together we get

$$4 \cdot 3 \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)^2.$$

We now assume that P lies on a line $\langle a, b\rangle$ with $a \in \{p_1, p_2, p_3, p_4\}$ and $b \in \{p_5, p_6, p_7\}$. We thus first choose a as one of the remaining three points in $\{p_1, p_2, p_3, p_4\}$ and the point b as one of the points $\{p_5, p_6, p_7\}$ and place a and b on C in one of $(q+1)q$ ways. We then place the remaining two points, p_r and p_s, of $\{p_1, p_2, p_3, p_4\}$ on C in $(q-1)(q-2)$ ways. The line $\langle p_r, p_s\rangle$ intersects $\langle a, b\rangle$ in a point outside of C so there is $q-2$ ways to choose P on $\langle a, b\rangle$ but away from this intersection point and a and b. Finally, we place the remaining two points of $\{p_5, p_6, p_7\}$ on C in one of $(q-3)(q-4)$ ways. Multiplying everything together we obtain

$$4 \cdot 3 \cdot 3 \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)^2(q-3)(q-4).$$

We now add the two above together to get

$$N_1^{1,2,3,4} = 24q^3(q-2)(q-3)(q-4)(2q-5)(q+1)(q^2+q+1)(q-1)^2.$$

To compute $N_2^{1,2,3,4}$, we first choose one of the points p_1, p_2, p_3 or p_4 to be the point P not on the smooth conic C and call the remaining three points p_r, p_s and p_t. We continue by choosing a smooth conic C in $q^5 - q^2$ ways.

We first assume that P lies on two lines $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$ where $\{a_1, b_1\} \subset \{p_5, p_6, p_7\}$ and $\{a_2, b_2\} \subset \{p_r, p_s, p_t\}$. We now choose two points among $\{p_5, p_6, p_7\}$ in three ways and choose two points among $\{p_r, p_s, p_t\}$ in three ways and rename the remaining two points to p_u and p_v. There are now two possible ways to label the four chosen points a_1, a_2, b_1 and b_2 in such a way that $\{a_1, b_1\} \subset \{p_5, p_6, p_7\}$ and $\{a_2, b_2\} \subset \{p_r, p_s, p_t\}$ and we choose one of them. We then place the four points a_1, a_2, b_1 and b_2 on the conic C in $(q+1)q(q-1)(q-2)$ ways. The point P is now given as $P = \langle a_1, a_2\rangle \cap \langle b_1, b_2\rangle$ and no matter how we place p_u and p_v, the three lines $\langle p_r, p_s\rangle$, $\langle p_r, p_t\rangle$ and $\langle p_s, p_t\rangle$ will not go through P. We can now multiply everything together to obtain

$$4 \cdot 3 \cdot 3 \cdot 2 \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4).$$

The other possibility is that P lies on two lines $\langle a_1, a_2\rangle$ and $\langle a_3, b\rangle$ where $\{a_1, a_2, a_3\}$ is the set $\{p_5, p_6, p_7\}$ and $b \in \{p_r, p_s, p_t\}$. We thus choose b in three ways and rename the remaining two points in $\{p_r, p_s, p_t\}$ to p_u and p_v. From now on, we must differentiate between when P is on the outside and on the inside of C.

First, we choose P on the outside of C in $\frac{1}{2}(q+1)q$ ways. We then choose b as a point on C whose tangent does not pass through P in $q-1$ ways. We then choose one of the points p_5, p_6 and p_7 to become the second intersection point in $C \cap \langle b, P\rangle$. Then, we place the remaining two points among $\{p_5, p_6, p_7\}$ on C such that the line through them passes through P in $q-3$ ways. There are now $q-5$ ways to choose p_u and p_v such that the line $\langle p_u, p_v\rangle$ will pass through P. Thus, the remaining $(q-3)(q-4)-(q-5) = q^2 - 8q + 17$ choices must give p_u and p_v such that none of the lines $\langle p_r, p_s\rangle$, $\langle p_r, p_t\rangle$ and $\langle p_s, p_t\rangle$ will contain P. We may now multiply everything together to obtain

$$4 \cdot (q^5 - q^2) \cdot 3 \cdot \frac{1}{2}(q+1)q \cdot (q-1) \cdot 3 \cdot (q-3) \cdot (q^2 - 8q + 17).$$

Now we choose P on the inside of C in one of $\frac{1}{2}(q-1)q$ ways. We then choose b as a point on C whose tangent does not pass through P in $q+1$ ways. We then choose one of the points p_5, p_6 and p_7 to become the second intersection point in $C \cap \langle b, P\rangle$. Then, we place the remaining two points among $\{p_5, p_6, p_7\}$ on C such that the line through them passes through P in $q-1$ ways. There are now $q-3$ ways to choose p_u and p_v such that the line $\langle p_u, p_v\rangle$ will pass through P. Thus, the remaining $(q-3)(q-4)-(q-3) = (q-3)(q-5)$ choices must give p_u and p_v such that none of the lines $\langle p_r, p_s\rangle$, $\langle p_r, p_t\rangle$ and $\langle p_s, p_t\rangle$ will contain P. We may now multiply everything together to obtain

$$4 \cdot (q^5 - q^2) \cdot 3 \cdot \frac{1}{2}(q-1)q \cdot (q+1) \cdot 3 \cdot (q-1) \cdot (q-3)(q-5).$$

We may now add everything together to get

$$N_2^{1,2,3,4} = 36q^3(q+1)(q^2+q+1)(5q^3-37q^2+82q-60)(q-1)^2.$$

Finally, we need to compute $N_3^{1,2,3,4}$. We begin by choosing one of the points p_1, p_2, p_3 or p_4 to be the point P not on the smooth conic C and call the remaining three points p_r, p_s and p_t. We continue by choosing a smooth conic C in $q^5 - q^2$ ways.

Here, we only have the possibility that P lies on three lines $\langle a_1, b_1\rangle$, $\langle a_2, b_2\rangle$ and $\langle a_3, b_3\rangle$ where $\{a_1, a_2, a_3\} = \{p_5, p_6, p_7\}$ and $\{b_1, b_2, b_3\} = \{p_r, p_s, p_t\}$. However, we must take care of the case that P is on the outside of C and the case that P is on the inside of C separately. We call the corresponding numbers $N_{3,out}^{1,2,3,4}$ and $N_{3,in}^{1,2,3,4}$.

We begin by computing $N_{3,out}^{1,2,3,4}$. We thus choose the point P as a point on the outside of C in $\frac{1}{2}(q+1)q$ ways. We begin by placing p_5 at one of the points of C whose tangent does not pass through P in $q-1$ ways. We label the second intersection point of $C \cap \langle p_5, P\rangle$ with p_r, p_s or p_t and call the remaining two points p_u and p_v. We then place p_6 at one of the $q-3$ remaining points of C whose tangent does not pass through P and then choose one of the points p_u and p_v to become the other intersection point of $C \cap \langle p_6, P\rangle$. Finally, we place p_7 at one of the remaining $q-5$ points and label the other point of $C \cap \langle p_7, P\rangle$ in the only possible way. We thus have

$$N_{3,out}^{1,2,3,4} = 4 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q+1)q \cdot (q-1) \cdot 3 \cdot (q-3) \cdot 2 \cdot (q-5).$$

We now turn to computing $N_{3,in}^{1,2,3,4}$. We thus choose the point P as a point on the inside of C in $\frac{1}{2}(q-1)q$ ways. We begin by placing p_5 at one of the points of C whose tangent does not pass through P in $q+1$ ways. We label the second intersection point of $C \cap \langle p_5, P\rangle$ with p_r, p_s or p_t and call the remaining two points p_u and p_v. We then place p_6 at one of the $q-1$ remaining points of C and then choose one of the points p_u and p_v to become the other intersection point of $C \cap \langle p_6, P\rangle$. Finally, we place p_7 at one of the remaining $q-3$ points and label the other point of $C \cap \langle p_7, P\rangle$ in the only possible way. We now see that

$$N_{3,in}^{1,2,3,4} = 4 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q-1)q \cdot (q+1) \cdot 3 \cdot (q-1) \cdot 2 \cdot (q-3).$$

We may now add $N_{3,out}^{1,2,3,4}$ and $N_{3,in}^{1,2,3,4}$ to get

$$N_3^{1,2,3,4} = 192q^3(q+1)(q^2+q+1)(q^2-3q+3)(q-1)^2.$$

Finally, we may now use the principle on inclusion and exclusion to see that

$$|\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}| = |\text{PGL}(3)| \cdot (93q^4 - 1245q^3 + 6195q^2 - 13470q + 10737).$$

This gives us that

$$\begin{aligned} N_\lambda &= |\mathcal{C}_{\text{lines}}| + |\mathcal{C}_{\text{conic}}| - |\mathcal{C}_{\text{lines}} \cap \mathcal{C}_{\text{conic}}| = \\ &= |\text{PGL}(3)| \cdot (q^6 - 35q^5 + 490q^4 - 3485q^3 + 13174q^2 - 24920q + 18375), \end{aligned}$$

and thus

$$\mathcal{N}_\lambda = q^6 - 35q^5 + 490q^4 - 3485q^3 + 13174q^2 - 24920q + 18375.$$

4.17 Summary of computations

We have summarized the results in Table 4.1. Let

$$H^i_{\text{ét,c}}(\mathcal{Q}[2], \mathbb{Q}_\ell) = \bigoplus_{\lambda \vdash 7} V_\lambda^{r_{i,\lambda}},$$

be the decomposition of $H^i_{\text{ét,c}}(\mathcal{Q}[2], \mathbb{Q}_\ell)$ into irreducible S_7-representations, where V_λ is the irreducible representation corresponding to λ and where $r_{i,\lambda}$ is its multiplicity in $H^i_{\text{ét,c}}(\mathcal{Q}[2], \mathbb{Q}_\ell)$. Similarly, let s_λ denote the irreducible character of S_7 corresponding to the partition λ. If we use the results of Table 4.1 and apply the Leftschetz trace formula, Equation 3.5.1, we obtain the sum

$$\sum_{\lambda \vdash 7} \sum_{i=0}^{6} (-1)^i \text{Tr}(F | V_\lambda^{r_{i,\lambda}}) s_\lambda,$$

as given in Equation 4.17.1. Another way to express this sum is given by Equation 4.17.2 where the coefficient of q^i is a virtual representation of S_7.

λ	$\lvert \mathcal{Q}[2]^{F\cdot\sigma_\lambda} \rvert$
$[7]$	q^6+q^3
$[6,1]$	q^6-2q^3+1
$[5,2]$	q^6-q^2
$[5,1^2]$	q^6-q^2
$[4,3]$	$q^6-q^5-2q^4+q^3+q^2$
$[4,2,1]$	$q^6-q^5-2q^4+q^3-2q^2+3$
$[4,1^3]$	$q^6-q^5-2q^4+q^3-2q^2+3$
$[3^2,1]$	$q^6-2q^5-2q^4-8q^3+16q^2+10q+21$
$[3,2^2]$	$q^6-q^5-2q^4+3q^3+q^2-2q$
$[3,2,1^2]$	$q^6-3q^5+5q^3-q^2-2q$
$[3,1^4]$	$q^6-5q^5+10q^4-5q^3-11q^2+10q$
$[2^3,1]$	$q^6-3q^5-6q^4+19q^3+6q^2-24q+7$
$[2^2,1^3]$	$q^6-7q^5+10q^4+15q^3-26q^2-8q+15$
$[2,1^5]$	$q^6-15q^5+90q^4-265q^3+374q^2-200q+15$
$[1^7]$	$q^6-35q^5+490q^4-3485q^3+13174q^2-24920q+18375$

Table 4.1: The S_7-equivariant point count of $\mathcal{Q}[2]$. We use σ_λ to denote any permutation in S_7 of cycle type λ.

$$
\begin{aligned}
\sum_{\lambda\vdash 7}\sum_{i=0}^{6}(-1)^i \operatorname{Tr}(F|V_\lambda^{r_{i,\lambda}})s_\lambda = &\left(q^6-q^5-q^3+4\,q^2-6\,q+6\right)s_7+\\
&-\left(6\,q^5-18\,q^4+48\,q^3-120\,q^2+198\,q-138\right)s_{6,1}-\\
&-\left(14\,q^5-56\,q^4+196\,q^3-616\,q^2+1064\,q-714\right)s_{5,2}+\\
&+\left(60\,q^4-270\,q^3+705\,q^2-1140\,q+810\right)s_{5,1^2}-\\
&-\left(14\,q^5-42\,q^4+196\,q^3-616\,q^2+1008\,q-756\right)s_{4,3}+\\
&+\left(175\,q^4-1050\,q^3+3465\,q^2-6230\,q+4445\right)s_{4,2,1}+\\
&+\left(20\,q^4-320\,q^3+1120\,q^2-1940\,q+1480\right)s_{4,1^3}+\\
&+\left(63\,q^4-336\,q^3+1176\,q^2-2184\,q+1596\right)s_{3^2,1}+\\
&+\left(21\,q^4-252\,q^3+1134\,q^2-2205\,q+1617\right)s_{3,2^2}+\\
&+\left(35\,q^4-630\,q^3+2905\,q^2-5915\,q+4410\right)s_{3,2,1^2}-\\
&-\left(60\,q^3-480\,q^2+1065\,q-810\right)s_{3,1^4}-\\
&-\left(84\,q^3-434\,q^2+910\,q-756\right)s_{2^3,1}+\\
&-\left(42\,q^3-350\,q^2+896\,q-700\right)s_{2^2,1^3}+\\
&+\left(48\,q^2-156\,q+132\right)s_{2,1^5}+\\
&+\left(q^2-3\,q+5\right)s_{1^7}.
\end{aligned}
\tag{4.17.1}
$$

$$
\begin{aligned}
& s_7 q^6 - \\
& -\left(s_7 + 6\, s_{6,1} + 14\, s_{5,2} + 14\, s_{4,3}\right) q^5 + \\
& +\left(18\, s_{6,1} + 56\, s_{5,2} + 60\, s_{5,1^2} + 42\, s_{4,3} + 175\, s_{4,2,1} 20\, s_{4,1^3} + 63\, s_{3^2,1} + 21\, s_{3,2^2} + 35\, s_{3,2,1^2}\right) q^4 - \\
& -\left(s_7 + 48\, s_{6,1} + 196\, s_{5,2} + 270\, s_{5,1^2} + 196\, s_{4,3} + 1050\, s_{4,2,1} + 320\, s_{4,1^3} + 336\, s_{3^2,1} + 252\, s_{3,2^2} + 630\, s_{3,2,1^2} + 60\, s_{3,1^4} + 84\, s_{2^3,1} + 42\, s_{2^2,1^3}\right) q^3 + \\
& +\left(4\, s_7 + 120\, s_{6,1} + 616\, s_{5,2} + 705\, s_{5,1^2} + 616\, s_{4,3} + 3465\, s_{4,2,1} + 1120\, s_{4,1^3} + 1176\, s_{3^2,1} + 1134\, s_{3,2^2} + 2905\, s_{3,2,1^2} + 480\, s_{3,1^4} + 434\, s_{2^3,1} + 350\, s_{2^2,1^3} + 48\, s_{2,1^5} + s_{1^7}\right) q^2 - \\
& -\left(6\, s_7 + 198\, s_{6,1} + 1064\, s_{5,2} + 1140\, s_{5,1^2} + 1008\, s_{4,3} + 6230\, s_{4,2,1} + 1940\, s_{4,1^3} + 2184\, s_{3^2,1} + 2205\, s_{3,2^2} + 5915\, s_{3,2,1^2} + 1065\, s_{3,1^4} + 910\, s_{2^3,1} + 896\, s_{2^2,1^3} + 156\, s_{2,1^5} + 3\, s_{1^7}\right) q + \\
& +6\, s_7 + 138\, s_{6,1} + 714\, s_{5,2} + 810\, s_{5,1^2} + 756\, s_{4,3} + 4445\, s_{4,2,1} + 1480\, s_{4,1^3} + 1596\, s_{3^2,1} + 1617\, s_{3,2^2} + 4410\, s_{3,2,1^2} + 810\, s_{3,1^4} + 756\, s_{2^3,1} + 700\, s_{2^2,1^3} + 132\, s_{2,1^5} + 5\, s_{1^7}
\end{aligned}
\tag{4.17.2}
$$

5. The hyperelliptic locus

In Chapter 4 we made S_7-equivariant point counts of the space $\mathcal{Q}[2]$ of smooth plane quartics with a symplectic level 2 structure. The space $\mathcal{Q}[2]$ is a dense open subset of of the space $\mathcal{M}_3[2]$ of smooth curves of genus three with a symplectic level 2 structure. The complement of $\mathcal{Q}[2]$ in $\mathcal{M}_3[2]$ is the space $\mathcal{H}_3[2]$ of smooth hyperelliptic curves with a symplectic level two structure. Thus, in order to extend our computations to the whole of $\mathcal{M}_3[2]$, we need to deal with the hyperelliptic locus.

5.1 A description of the hyperelliptic locus

Luckily, there already exist a description of $\mathcal{H}_g[2]$ which is suitable for our purposes. We shall only briefly discuss it here.

Recall that a hyperelliptic curve C of genus g is determined, up to isomorphism, by $2g+2$ distinct points on $\mathbb{P}^1$, up to projective equivalence. The $2g+2$ point are the branch points of the hyperelliptic map $C \to \mathbb{P}^1$. If $g = 3$ we see that we need 8 points, $Q_1, \dots, Q_8$. Let $P_1, \dots, P_8$ be the corresponding ramification points in C. If we order the 8 points we obtain the 28 odd theta characteristics of the curve as the classes $P_i + P_j$ for $1 \leq i < j \leq 8$, as explained in Section 2.4.1. We also have the following theorem which can be found in [DO88], Theorem VIII.1.

Theorem 5.1.1. *Each irreducible component of $\mathcal{H}_g[2]$ is isomorphic to the moduli space $\mathcal{M}_{0,2g+2}$ of $2g+2$ ordered points on the projective line.*

In the genus 3 case we can describe the situation as follows. A point

$$(\mathbb{P}^1, Q_1, \dots, Q_8) \in \mathcal{M}_{0,8},$$

gives a genus 3 curve C by taking the double cover of $\mathbb{P}^1$ ramified over the points $Q_1, \dots, Q_8$ and we get an ordered Aronhold set by taking

$$A = \{P_1 + P_8, P_2 + P_8, \dots, P_7 + P_8\}.$$

To see that this is an Aronhold set, use Lemma 2.2.10 and [ACGH85] Appendix B.32-33. Recall that, by Lemma 2.2.11, ordered Aronhold sets correspond to symplectic level 2 structures. Reordering the points gives other ordered Aronhold sets, and thus other symplectic level 2 structures, but we cannot get all possible Aronhold sets simply by permuting the points. We get the other symplectic level 2 structures of C by acting on A with $\mathrm{Sp}(\mathbb{F}_2^6)$. This suggests that we have an irreducible component of $\mathcal{H}_g[2]$ corresponding to each of the conjugacy classes of S_8 in $\mathrm{Sp}(\mathbb{F}_2^6)$, that is $|\mathrm{Sp}(\mathbb{F}_2^6)|/|S_8| = 36$ components in total where each component is an isomorphic copy of $\mathcal{M}_{0,8}$.

Dolgachev and Ortland, [DO88], also pose the question whether the irreducible components of $\mathcal{H}_g[2]$ also are the connected components, or in other words if $\mathcal{H}_g[2]$ is smooth. In the complex case, the question was answered positively by Tsuyumine in [Tsu91] and later by a shorter argument by Runge in [Run97]. Using the results of [And03], the argument of Runge carries over word for word to an algebraically closed field of positive characteristic different from 2.

Theorem 5.1.2. *If $g \geq 2$, then each irreducible component of $\mathcal{H}_g[2]$ is also a connected component.*

We have a natural action of S_{2g+2} on the space $\mathcal{M}_{0,2g+2}$. Since different orderings of the points correspond to different symplectic level 2 structures, S_{2g+2} sits naturally inside $\mathrm{Sp}(\mathbb{F}_2^{2g})$ and, in fact, for $g = 3$ and for even g it is a maximal subgroup, see [Dye79]. With Theorems 5.1.1 and 5.1.2 at hand, the following slight generalization of a corollary in [DO88] (p.145) is clear.

Corollary 5.1.3. *Let $g \geq 2$ and let $X_{[\tau]} = \mathcal{M}_{0,2g+2}$ for each left coset $[\tau] \in \mathcal{T} := \mathrm{Sp}(\mathbb{F}_2^{2g})/S_{2g+2}$. Then*

$$\mathcal{H}_g[2] \cong \coprod_{[\tau]\in\mathcal{T}} X_{[\tau]},$$

and the group $\mathrm{Sp}(\mathbb{F}_2^{2g})$ acts transitively on the set of connected components $X_{[\tau]}$ of $\mathcal{H}_g[2]$. In particular, there are

$$\frac{\left|\mathrm{Sp}(\mathbb{F}_2^{2g})\right|}{\left|S_{2g+2}\right|} = \frac{2^{g^2}\left(2^{2g}-1\right)\left(2^{2g-2}-1\right)\cdots\left(2^2-1\right)}{(2g+2)!},$$

connected components of $\mathcal{H}_g[2]$.

Remark 5.1.4. As pointed out in [Run97], the argument of the corollary stated in [DO88] is not quite correct in full generality as it is given there. However, it is enough to prove the result for $g = 3$ and for even g, and in [Run97] it is explained how to obtain the full result.

Let us now, once and for all, choose a set T of representatives of $\mathrm{Sp}(\mathbb{F}_2^{2g})/S_{2g+2}$. If we denote the elements of $X_{[\mathrm{id}]}$ by x, then any element in

$X_{[\tau]}$ can be written as τx for some $x \in X_{[\mathrm{id}]}$. Let α be any element of $\mathrm{Sp}(\mathbb{F}_2^{2g})$. Then

$$\alpha\tau = \tau'\sigma,$$

for some $\sigma \in S_{2g+2}$ and some $\tau' \in T$. Since the Frobenius commutes with the action of $\mathrm{Sp}(\mathbb{F}_2^{2g})$,

$$F \cdot \alpha(\tau x) = \tau x,$$

if and only if

$$F \cdot (\tau'\sigma x) = \tau'(F \cdot \sigma x) = \tau x.$$

But the Frobenius acts on each of the components of $\mathcal{H}_g[2]$ so we see that $F \cdot \alpha(\tau x) = \tau x$ if and only if $\tau' = \tau$ and $F\sigma x = x$.

We now translate the above observation into more standard representation theoretic vocabulary. Define a class function ψ on S_{2g+2} by

$$\psi(\sigma) = |X_{\mathrm{id}}^{F\cdot\sigma}|,$$

and define a class function $\hat{\psi}$ on $\mathrm{Sp}(\mathbb{F}_2^{2g})$ by setting

$$\hat{\psi}(\alpha) = |\mathcal{H}_g[2]^{F\cdot\alpha}|,$$

for any $\alpha \in \mathrm{Sp}(\mathbb{F}_2^6)$. By the above observation we have that

$$\hat{\psi}(\alpha) = \sum_{\tau \in T} \widetilde{\psi}(\tau^{-1}\alpha\tau),$$

where

$$\widetilde{\psi}(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \in S_{2g+2}, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\hat{\psi}$ is the class function ψ induced from S_{2g+2} up to $\mathrm{Sp}(\mathbb{F}_2^{2g})$. Thus, to make an S_{2g+2}-equivariant point count of $\mathcal{H}_g[2]$ we can make an S_{2g+2}-equivariant point count of $\mathcal{M}_{0,2g+2}$ and then use the representation theory of S_{2g+2} and $\mathrm{Sp}(\mathbb{F}_2^{2g})$ in order to first induce the class function up to $\mathrm{Sp}(\mathbb{F}_2^{2g})$ and then restrict it down again to S_{2g+2}. Once this is done, we can obtain the S_{2g+1}-equivariant point count by restricting from S_{2g+2} to S_{2g+1}. Of course, we shall only be interested in the case $g = 3$.

To summarize, we are interested in the class function

$$\hat{\psi} = \mathrm{Ind}_{S_8}^{\mathrm{Sp}(\mathbb{F}_2^6)} \psi,$$

which we will later restrict to S_8 and then restrict further to S_7. The remainder of this chapter will be devoted the investigation of the function $\hat{\psi}$ and its restrictions.

5.2 An equivariant count of eight points on the line

Using Lemma 3.2.4, this computation is trivial since all we need to do is to compute the number of λ-tuples of $\mathbb{P}^1$ for each partition of λ of 8 and then divide by $|PGL(2)|$

$$|\mathrm{PGL}(2)| = \frac{(q^2-1)(q^2-q)}{q-1} = q(q^2-1),$$

in order to obtain $|\mathcal{M}_{0,8}^{F\cdot\sigma}|$, where σ is a permutation in S_8 of cycle type λ. Since the computations present no difficulties whatsoever, we simply give the results for each of the 22 partitions of 8 rightaway. The results are given in Table 5.1. We also mention that the equivariant Poincaré polynomials of $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ have been computed for all $n \geq 3$ in [Get95].

5.3 The representations of S_8 and $\mathrm{Sp}(\mathbb{F}_2^6)$

In order to complete the S_8-equivariant point count of $\mathcal{H}_3[2]$ we need to understand the representation theory of S_8 and $\mathrm{Sp}(\mathbb{F}_2^6)$ and how the two relate. We may then use Frobenius reciprocity to complete the computation.

Lemma 5.3.1 (Frobenius reciprocity)**.** *Let H be a subgroup of the finite group G and let χ and χ' be characters on H and G respectively. Then*

$$(\mathrm{Ind}_H^G \chi, \chi')_G = (\chi, \mathrm{Res}_H^G \chi')_H,$$

where $(-,-)$ denotes the standard inner product on characters.

For a proof, see [FH04], Corollary 3.20.

The character table of $\mathrm{Sp}(\mathbb{F}_2^6)$ can be found in [CCN+85] and can be seen in Table 5.4. For symmetric groups there is a well developed theory, see for instance [FH04]. We give the character table for S_8 in Table 5.5.

In order to use Lemma 5.3.1, we want to determine in which conjugacy class of $\mathrm{Sp}(\mathbb{F}_2^6)$ each element of S_8 lies. The first step is the following lemma.

Lemma 5.3.2. *Two elements of S_8 which are not conjugate in S_8 are not conjugate as elements in $\mathrm{Sp}(\mathbb{F}_2^6)$.*

Proof. See Tables 5.4 and 5.5 for the notation used below.

The first observation is that if $\phi : S_8 \hookrightarrow \mathrm{Sp}(\mathbb{F}_2^6)$ is an injective homomorphism and σ is an element of S_8, then σ and $\phi(\sigma)$ will have the same order. Thus, the only thing we will have to worry about is that two elements of the same order but different conjugacy classes in S_8 become conjugate in $\mathrm{Sp}(\mathbb{F}_2^6)$.

We now see that the conjugacy class $[1^8]$ of S_8 ends up in the class $1A$ of $\mathrm{Sp}(\mathbb{F}_2^6)$, the class $[5,1^3]$ ends up in the class $5A$, the class $[7,1]$ ends up in $7A$, the class $[5,2,1]$ in $10A$ and the class $[5,3]$ in $15A$.

The remainder of the proof is a tedious verification that the conjugacy classes of different orders in S_8 remain non-conjugate in $\mathrm{Sp}(\mathbb{F}_2^6)$ using the character tables of $\mathrm{Sp}(\mathbb{F}_2^6)$ and S_8 and some simple linear algebra. We therefore only illustrate the method in a simple case which is easily done by hand.

Suppose that two nonconjugate elements of order 2 both lie in the conjugacy class $2A$ of $\mathrm{Sp}(\mathbb{F}_2^6)$. The restriction of the character χ_2 to S_8 then takes the value -5 on the conjugacy classes of both these elements, the value 7 on the class $[1^8]$ and the value 0 on the class $[7,1]$. Thus, using Table 5.5, $\mathrm{Res}_{S_8}^{\mathrm{Sp}(\mathbb{F}_2^6)} \chi_2$ can be seen to not be irreducible on S_8 but at least it should be expressable as a sum of irreducible representations. There is only one sum of irreducible representations on S_8 which takes the value 7 on $[1^8]$ and -5 on two different conjugacy classes of order 2, namely $s_8 + 6s_{1^8}$. However, this character takes the value 7 on the class $[7,1]$. Hence, $2A$ cannot contain two elements which are nonconjugate in S_8 but conjugate in $\mathrm{Sp}(\mathbb{F}_2^6)$. □

5.3.1. Representation sudoku. In order to identify in which conjugacy class of $\mathrm{Sp}(\mathbb{F}_2^6)$ each class of S_8 ends up, we are going to play a little game I like to call representation sudoku since the arguments very much resemble those one uses when solving a sudoku puzzle. In order to follow the arguments, the reader should be prepared to look up Tables 5.4 and 5.5 repeatedly. Table 5.2 may also be of use to keep track of the results.

In the proof of Lemma 5.3.2 we identified the classes of $\mathrm{Sp}(\mathbb{F}_2^6)$ containing $[1^8]$, $[5,1^3]$, $[5,2,1]$, $[5,3]$ and $[7,1]$. It is also easy to see that $[2,1^6]$ must lie in the class $2A$ since $[2,1^6]$ has a centralizer of order 1440 in S_8 and $2A$ is the only class of order 2 of $\mathrm{Sp}(\mathbb{F}_2^6)$ which has a centralizer whose order is a multiple of 1440. Similarly, one can see that the class $[3,1^5]$ must lie in $3A$. These facts will have the role of the given numbers in a sudoku.

We now start solving. The classes $[4,1^4]$, $[4,2,1^2]$ and $[4,2^2]$ all end up in the class $[2^2,1^4]$ when they are squared. If we look in Table 5.4 we see that of the five classes of order 4 there are precisely three classes ($4B$, $4C$ and $4E$) which end up in the class $2C$. We conclude that the classes $[4,1^4]$, $[4,2,1^2]$ and $[4,2^2]$ must correspond to the classes $4B$, $4C$ and $4E$ (but we cannot see which is which at this point) and that the class $[2^2,1^4]$ must lie in the class $2C$.

The class $[4^2]$ squares to $[2^4]$. We know that $[4^2]$ does not lie in $4B$, $4C$ or $4E$ so it either must lie in $4A$ or $4D$. But both $4A$ and $4D$ square to $2B$. Thus, the class $[2^4]$ must lie in $2B$. We have now identified the classes lying in $2A$, $2B$ and $2C$ so we may conclude that $[2^3,1^2]$ must lie in $2D$.

We now consider the class $[3,2^2,1]$. When squared it ends up in the class $[3,1^5]$, which we know lies in $3A$, and its cube lies in the class $[2^2,1^4]$ which lies in $2C$. The only class of $\mathrm{Sp}(\mathbb{F}_2^6)$ which squares to $3A$ and whose cube lies in $2C$ is the class $6D$. We conclude that $[3,2^2,1]$ lies in $6D$.

Similarly, the class $[3,2,1^3]$ squares to $[3,1^5]$ and its cube lies in the class $[2,1^6]$. Since $[3,1^5]$ lies in the class $3A$ and $[2,1^6]$ lies in $2A$ we may conclude that $[3,2,1^3]$ lies in the class $6A$ since this class is the only class whose square lies in $3A$ and whose cube lies in $2A$.

We now turn to the classes $[6,2]$, $[6,1^2]$ and $[3^2,2]$. These classes all square to the class $[3^2,1^2]$ but their cubes are $[2^4]$, $[2^3,1^2]$ and $[2,1^6]$. There are two triples of classes of order 6 in $\mathrm{Sp}(\mathbb{F}_2^6)$ which square to the same class of order 3. One of these triples squares to $3A$ which we know corresponds to $[3,1^5]$. The other triple squares to $3C$ so we may conclude that $[3^2,1^2]$ lies in $3C$. Since we also know that $[2^4]$, $[2^3,1^2]$ and $[2,1^6]$ lie in $2B$, $2D$ and $2A$ we may also see that $[6,2]$ lies in $6F$, $[6,1^2]$ lies in $6G$ and that $[3^2,2]$ lies in $6E$.

Since the class $[4,3,1]$ squares to $[3,2^2,1]$, which lies in the class $6D$, we may conclude that $[4,3,1]$ either must lie in $12A$ or $12B$.

We have now identified 16 of 22 classes completely and we have some additional information about the classes which are yet to be identified.

Observation 5.3.3. The restriction of the character χ_2 to S_8 takes the value 7 on the class $[1^8]$ and the value -5 on the class $[2,1^6]$. Thus, it either must be $s_{2,1^6}$ or $6s_{1^8}+s_8$. However, we excluded the latter possibility in the proof of Lemma 5.3.2 so we have $\mathrm{Res}_{S_8}^{\mathrm{Sp}(\mathbb{F}_2^6)}\chi_2 = s_{2,1^6}$.

Now consider the class $[4,1^4]$. Since it squares to $[2^2,1^4]$, which lies in $2C$, and has a centralizer of order 96 we must have that it lies in $4B$ or $4C$. But since $s_{2,1^6}([4,1^4]) = -3$, $\chi_2(4B) = 1$ and $\chi_2(4C) = -3$ we see that $[4,1^4]$ must lie in $4C$.

Similarly, $[4^2]$ squares to $[2^4]$ so $[4^2]$ must lie in $4A$ or $4D$. However, $s_{2,1^6}([4^2]) = -1$ gives that $[4^2]$ must lie in $4D$. Note that this also gives us that the class $4A$ cannot contain any element of S_8. Since there are only two classes of order 8 in $\mathrm{Sp}(\mathbb{F}_2^6)$ and the square of $8A$ lies in $4A$ we must have that $[8]$ lies in $8B$.

We have already seen that the class $[4,3,1]$ lies in $12A$ or $12B$. Now, since we have seen that $[4,1^4]$ lies in $4C$, we may conclude that $[4,3,1]$ lies in $12B$.

It now only remains to identify the classes $[4,2^2]$ and $[4,2,1^2]$. To do this we must make yet another observation.

Observation 5.3.4. The restriction of the character χ_3 to S_8 has dimension 15 and must thus be a sum of the 6 irreducible characters on S_8 of dimension at most 15. Since we know the values of $\mathrm{Res}_{S_8}^{\mathrm{Sp}(\mathbb{F}_2^6)}\chi_3$ on 20 out of 22 classes, some simple linear algebra allows us to deduce that $\mathrm{Res}_{S_8}^{\mathrm{Sp}(\mathbb{F}_2^6)}\chi_3 = s_{1^8}+s_{2^4}$.

We have already seen that $[4,2^2]$ either must lie in $4B$ or $4E$. Since $s_{1^8} + s_{2^4}([4,2^2]) = -3$ we conclude that $[4,2^2]$ lies in $4B$. Hence, we also get that $[4,2,1^2]$ lies in $4E$. The results are summarized in Table 5.2.

We now have all the information we need to make inductions and restrictions of representations between S_8 and $\mathrm{Sp}(\mathbb{F}_2^6)$, see Table 5.2. Recall that we denote the class function $|\mathcal{H}_3[2]^{F\cdot\sigma}|$ by $\hat{\psi}(\sigma)$. Using Table 5.1 we may express ψ as a linear combination

$$\psi = \sum_{\lambda \vdash 8} f_\lambda(q) s_\lambda,$$

where, due to the cohomological interpretation, each $f_\lambda(q)$ is a polynomial in q with integer coefficients. One then uses Tables 5.2, 5.4, 5.5 and Frobenius reciprocity in order to induce each of the representations s_λ up to $\mathrm{Sp}(\mathbb{F}_2^6)$. By linearity, this determines $\hat{\psi} = \mathrm{Ind}_{S_8}^{\mathrm{Sp}(\mathbb{F}_2^6)} \psi$. Finally, we may use Table 5.2 to restrict the induced representation down to S_8. Finally, we choose to do the first step of the reduction, namely to S_8, by evaluating $\hat{\psi}$ on each conjugacy class of S_8

$$|\mathcal{H}_3[2]^{F\cdot\sigma}| = \mathrm{Res}_{S_8}^{\mathrm{Sp}(\mathbb{F}_2^6)} \mathrm{Ind}_{S_8}^{\mathrm{Sp}(\mathbb{F}_2^6)} \psi(\sigma).$$

The results are shown in Table 5.3.

λ	$\lvert\mathcal{M}_{0,8}^{F\cdot\sigma_\lambda}\rvert$
$[8]$	$\left(q^2+1\right)q^3$
$[7,1]$	$\left(q+1\right)\left(q^2+q+1\right)\left(q^2-q+1\right)$
$[6,2]$	$q\left(q-1\right)\left(q^3+q-1\right)$
$\left[6,1^2\right]$	$q\left(q+1\right)\left(q^3+q-1\right)$
$[5,3]$	$q\left(q-1\right)\left(q+1\right)\left(q^2+1\right)$
$[5,2,1]$	$q\left(q-1\right)\left(q+1\right)\left(q^2+1\right)$
$\left[5,1^3\right]$	$q\left(q-1\right)\left(q+1\right)\left(q^2+1\right)$
$\left[4^2\right]$	$q\left(q^4-q^2-4\right)$
$[4,3,1]$	$\left(q-1\right)q^2\left(q+1\right)^2$
$\left[4,2^2\right]$	$\left(q-1\right)\left(q-2\right)\left(q+1\right)q^2$
$\left[4,2,1^2\right]$	$\left(q-1\right)\left(q+1\right)q^3$
$\left[4,1^4\right]$	$\left(q-1\right)\left(q-2\right)\left(q+1\right)q^2$
$\left[3^2,2\right]$	$q\left(q-1\right)\left(q^3-q-3\right)$
$\left[3^2,1^2\right]$	$q\left(q+1\right)\left(q^3-q-3\right)$
$\left[3,2^2,1\right]$	$q\left(q-1\right)\left(q-2\right)\left(q+1\right)^2$
$\left[3,2,1^3\right]$	$\left(q+1\right)q^2\left(q-1\right)^2$
$\left[3,1^5\right]$	$q\left(q-1\right)\left(q-2\right)\left(q-3\right)\left(q+1\right)$
$\left[2^4\right]$	$\left(q-2\right)\left(q-3\right)\left(q+2\right)\left(q^2-q-4\right)$
$\left[2^3,1^2\right]$	$q\left(q-2\right)\left(q+1\right)\left(q^2-q-4\right)$
$\left[2^2,1^4\right]$	$q\left(q-1\right)\left(q+1\right)\left(q-2\right)^2$
$\left[2,1^6\right]$	$q\left(q-1\right)\left(q-2\right)\left(q-3\right)\left(q-4\right)$
$\left[1^8\right]$	$\left(q-2\right)\left(q-3\right)\left(q-4\right)\left(q-5\right)\left(q-6\right)$

Table 5.1: The S_8-equivariant point count of $\mathcal{M}_{0,8}$. We use σ_λ to denote any permutation in S_8 of cycle type λ.

Conj. class in S_8	Conj. class in $\mathrm{Sp}(\mathbb{F}_2^6)$
$[8]$	$8B$
$[7,1]$	$7A$
$[6,2]$	$6F$
$[6,1^2]$	$6G$
$[5,3]$	$15A$
$[5,2,1]$	$10A$
$[5,1^3]$	$5A$
$[4^2]$	$4D$
$[4,3,1]$	$12B$
$[4,2^2]$	$4B$
$[4,2,1^2]$	$4E$
$[4,1^4]$	$4C$
$[3^2,2]$	$6E$
$[3^2,1^2]$	$3C$
$[3,2^2,1]$	$6D$
$[3,2,1^3]$	$6A$
$[3,1^5]$	$3A$
$[2^4]$	$2B$
$[2^3,1^2]$	$2D$
$[2^2,1^4]$	$2C$
$[2,1^6]$	$2A$
$[1^8]$	$1A$

Table 5.2: The conjugacy classes in S_8 and their corresponding conjugacy classes in $\mathrm{Sp}(\mathbb{F}_2^6)$.

λ	$\lvert\mathcal{H}_3[2]^{F\cdot\sigma_\lambda}\rvert$
$[8]$	$2q^5+2q^3$
$[7,1]$	$q^5+q^4+q^3+q^2+q+1$
$[6,2]$	$3q^5+3q^3-6q^2-3q^4+3q$
$[6,1^2]$	$q^5+q^4+q^3-q$
$[5,3]$	q^5-q
$[5,2,1]$	q^5-q
$[5,1^3]$	q^5-q
$[4^2]$	$4q^5-16q-4q^3$
$[4,3,1]$	$2q^5+2q^4-2q^3-2q^2$
$[4,2^2]$	$6q^5+12q^2-12q^4-6q^3$
$[4,2,1^2]$	$2q^5-2q^3$
$[4,1^4]$	$2q^5-4q^4-2q^3+4q^2$
$[3^2,2]$	$q^5-q^4-q^3-2q^2+3q$
$[3^2,1^2]$	$3q^5+3q^4-3q^3-12q^2-9q$
$[3,2^2,1]$	$2q^5-2q^4-6q^3+2q^2+4q$
$[3,2,1^3]$	$4q^5-4q^4-4q^3+4q^2$
$[3,1^5]$	$6q^5-30q^4+30q^3+30q^2-36q$
$[2^4]$	$12q^5+48q-60q^3+336q^2-48q^4-576$
$[2^3,1^2]$	$4q^5-8q^4-20q^3+24q^2+32q$
$[2^2,1^4]$	$8q^5-32q^4+24q^3+32q^2-32q$
$[2,1^6]$	$16q^5-160q^4+560q^3-800q^2+384q$
$[1^8]$	$36q^5-720q^4+5580q^3-20880q^2+37584q-25920$

Table 5.3: The S_8-equivariant point count of $\mathcal{H}_3[2]$. We use σ_λ to denote any permutation in S_8 of cycle type λ.

	1A	2A	2B	2C	2D	3A	3B	3C	4A	4B	4C	4D	4E	5A	6A	6B	6C	6D	6E	6F	6G	7A	8A	8B	9A	10A	12A	12B	12C	15A
cent.	1451 520	23 040	4 608	1 536	384	2 160	648	108	384	192	192	128	32	30	144	144	72	48	36	36	12	7	16	16	9	10	24	24	12	15
p pow.	–	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*B*	*C*	*C*	*B*	*C*	*A*	*AA*	*AB*	*BB*	*AC*	*CA*	*CB*	*CD*	*A*	*A*	*D*	*B*	*AA*	*DB*	*DC*	*CA*	*AA*
conj. cl.	1*A*	2*A*	2*B*	2*C*	2*D*	3*A*	3*B*	3*C*	4*A*	4*B*	4*C*	4*D*	4*E*	5*A*	6*A*	6*B*	6*C*	6*D*	6*E*	6*F*	6*G*	7*A*	8*A*	8*B*	9*A*	10*A*	12*A*	12*B*	12*C*	15*A*
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	-5	-1	3	-1	4	-2	1	3	1	-3	-1	1	2	-2	2	2	0	1	-1	-1	0	-1	1	1	0	-2	0	0	-1
χ_3	15	-5	7	3	-1	0	-3	3	-1	-3	1	3	1	0	-2	-2	1	0	1	1	-1	1	1	-1	0	0	0	-2	-1	0
χ_4	21	9	-3	1	-3	6	3	0	5	-1	3	1	-1	1	0	0	3	-2	0	0	0	0	-1	1	0	-1	2	0	-1	1
χ_5	21	-11	5	5	-3	6	3	0	1	-3	-3	1	1	1	-2	2	-1	2	-2	2	0	0	-1	-1	0	-1	0	0	1	1
χ_6	27	15	3	7	3	9	0	0	3	1	5	-1	1	2	3	3	0	1	0	0	0	-1	1	-1	0	0	1	-1	0	-1
χ_7	35	-5	3	-5	3	5	-1	2	7	-1	-1	-1	-1	0	1	-3	3	1	-2	0	0	0	1	1	-1	0	-1	-1	1	0
χ_8	35	15	11	7	3	5	-1	2	-1	5	1	3	1	0	3	-1	-1	1	0	2	0	0	-1	1	-1	0	-1	1	-1	0
χ_9	56	-24	-8	8	0	11	2	2	0	4	-4	0	0	1	-3	1	-2	-1	0	-2	0	0	0	0	-1	1	1	-1	0	1
χ_{10}	70	-10	-10	6	-2	-5	7	1	2	2	2	2	-2	0	-1	-1	-1	3	-1	-1	1	0	0	0	1	0	-1	-1	-1	0
χ_{11}	84	4	20	4	4	-6	3	3	4	0	0	4	0	-1	-2	2	-1	-2	1	-1	1	0	0	0	0	-1	0	0	1	-1
χ_{12}	105	-35	1	5	1	15	-3	-3	5	-1	-5	1	-1	0	1	1	1	-1	1	1	1	0	1	-1	0	0	-1	1	-1	0
χ_{13}	105	25	-7	9	1	0	6	3	-3	-3	-3	-3	1	0	4	-4	2	0	1	-1	1	0	-1	-1	0	0	0	0	0	0
χ_{14}	105	5	17	-3	-7	0	6	3	-3	3	-1	1	-1	0	2	2	2	0	-1	-1	-1	0	1	-1	0	0	0	2	0	0
χ_{15}	120	40	-8	8	0	15	-6	0	0	-4	4	0	0	0	1	1	-2	-1	-2	-2	0	1	0	0	0	0	-1	1	0	0
χ_{16}	168	40	8	8	8	6	6	-3	0	0	0	0	0	-2	-2	2	2	2	1	-1	-1	0	0	0	0	0	0	0	0	1
χ_{17}	189	21	-3	-11	-3	9	0	0	9	1	1	1	1	-1	-3	-3	0	1	0	0	0	0	-1	-1	0	1	1	1	0	-1
χ_{18}	189	-51	-3	13	-3	9	0	0	-3	1	1	-3	1	-1	-3	-3	0	1	0	0	0	0	1	1	0	-1	1	1	0	-1
χ_{19}	189	-39	21	1	-3	9	0	0	-3	-5	-1	1	-1	-1	3	3	0	1	0	0	0	0	-1	1	0	1	1	-1	0	-1
χ_{20}	210	50	2	2	-6	15	3	0	-2	2	2	-2	-2	0	-1	-1	-1	-1	2	2	0	0	0	0	0	0	-1	-1	1	0
χ_{21}	210	10	-14	10	2	-15	-6	3	6	-2	-2	-2	-2	0	1	1	-2	1	1	1	-1	0	0	0	0	0	1	1	0	0
χ_{22}	216	-24	24	8	0	-9	0	0	0	-4	4	0	0	1	-3	-3	0	-1	0	0	0	-1	0	0	0	1	-1	1	0	1
χ_{23}	280	-40	-8	-8	8	10	10	1	0	0	0	0	0	0	2	-2	-2	-2	-1	1	-1	0	0	0	1	0	0	0	0	0
χ_{24}	280	40	24	8	0	-5	-8	-2	0	4	-4	0	0	0	1	-3	0	-1	-2	0	0	0	0	0	1	0	1	-1	0	0
χ_{25}	315	-45	-21	3	3	0	-9	0	-5	3	3	3	-1	0	0	0	3	0	0	0	0	0	-1	-1	0	0	0	0	1	0
χ_{26}	336	-16	16	-16	0	6	-6	0	0	0	0	0	0	1	2	-2	-2	2	2	-2	0	0	0	0	0	-1	0	0	0	1
χ_{27}	378	-30	-6	2	-6	-9	0	0	6	2	2	-2	2	-2	3	3	0	-1	0	0	0	0	0	0	0	0	-1	-1	0	1
χ_{28}	405	45	-27	-3	-3	0	0	0	-3	-3	-3	5	1	0	0	0	0	0	0	0	0	-1	1	1	0	0	0	0	0	0
χ_{29}	420	20	4	-12	4	0	-3	3	-4	0	0	-4	0	0	-4	4	1	0	-1	1	1	0	0	0	0	0	0	0	-1	0
χ_{30}	512	0	0	0	0	-16	8	-4	0	0	0	0	0	2	0	0	0	0	0	0	0	1	0	0	-1	0	0	0	0	-1

Table 5.4: The character table of $\mathrm{Sp}(\mathbb{F}_2^6)$ as found in [CCN+85]. The irreducible characters $\chi_1, \ldots, \chi_{30}$ are listed by increasing dimension. The conjugacy classes are denoted by their order followed by a letter to be able to distinguish between conjugacy classes of the same order, e.g. $2C$ denotes the third class of order 2. The top row contains the order of the centralizers. The row "*p* pow." tells which conjugacy class the class ends up in after raising it to a prime power dividing its order where the prime divisors are listed in increasing order. For instance, the AC above $6D$ denotes the fact that an element of $6D$ raised to the power 2 lies in $3A$ and an element raised to the power 3 lies in $2C$.

cent.	40320	1440	192	96	384	360	36	24	36	36	96	16	32	12	32	30	10	15	12	12	7	8
	$[1^8]$	$[2,1^6]$	$[2^2,1^4]$	$[2^3,1^2]$	$[2^4]$	$[3,1^5]$	$[3,2,1^3]$	$[3,2^2,1]$	$[3^2,1^2]$	$[3^2,2]$	$[4,1^4]$	$[4,2,1^2]$	$[4,2^2]$	$[4,3,1]$	$[4^2]$	$[5,1^3]$	$[5,2,1]$	$[5,3]$	$[6,1^2]$	$[6,2]$	$[7,1]$	$[8]$
s_8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$s_{7,1}$	7	5	3	1	-1	4	2	0	1	-1	3	1	-1	0	-1	2	0	-1	1	-1	0	-1
$s_{6,2}$	20	10	4	2	4	5	1	1	-1	1	2	0	2	-1	0	0	0	0	-1	1	-1	0
$s_{6,1^2}$	21	9	1	-3	-3	6	0	-2	0	0	3	-1	-1	0	1	1	-1	1	0	0	0	1
$s_{5,3}$	28	10	4	2	-4	1	1	1	1	1	-2	0	-2	1	0	-2	0	1	-1	-1	0	0
$s_{5,2,1}$	64	16	0	0	0	4	-2	0	-2	-2	0	0	0	0	0	-1	1	-1	0	0	1	0
$s_{5,1^3}$	35	5	-5	-3	3	5	-1	1	2	2	1	-1	1	1	-1	0	0	0	0	0	0	-1
s_{4^2}	14	4	2	0	6	-1	1	-1	2	-2	-2	0	2	1	2	-1	-1	-1	0	0	0	0
$s_{4,3,1}$	70	10	2	-2	-2	-5	1	-1	1	1	-4	0	0	-1	-2	0	0	0	1	1	0	0
$s_{4,2^2}$	56	4	0	4	8	-4	-2	0	-1	1	0	0	0	0	0	1	-1	1	1	-1	0	0
$s_{4,2,1^2}$	90	0	-6	0	-6	0	0	0	0	0	0	2	0	0	2	0	0	0	0	0	-1	0
$s_{4,1^4}$	35	-5	-5	3	3	5	1	1	2	-2	-1	-1	-1	-1	-1	0	0	0	0	0	0	1
$s_{3^2,2}$	42	0	2	0	-6	-6	0	2	0	0	0	-2	0	0	2	2	0	-1	0	0	0	0
$s_{3^2,1^2}$	56	-4	0	-4	8	-4	2	0	-1	-1	0	0	0	0	0	1	1	1	-1	-1	0	0
$s_{3,2^2,1}$	70	-10	2	2	-2	-5	-1	-1	1	-1	4	0	0	1	-2	0	0	0	-1	1	0	0
$s_{3,2,1^3}$	64	-16	0	0	0	4	2	0	-2	2	0	0	0	0	0	-1	-1	-1	0	0	1	0
$s_{3,1^5}$	21	-9	1	3	-3	6	0	-2	0	0	-3	-1	1	0	1	1	1	1	0	0	0	-1
s_{2^4}	14	-4	2	0	6	-1	-1	-1	2	2	2	0	-2	-1	2	-1	1	-1	0	0	0	0
$s_{2^3,1^2}$	28	-10	4	-2	-4	1	-1	1	1	-1	2	0	2	-1	0	-2	0	1	1	-1	0	0
$s_{2^2,1^4}$	20	-10	4	-2	4	5	-1	1	-1	-1	-2	0	-2	1	0	0	0	0	1	1	-1	0
$s_{2,1^6}$	7	-5	3	-1	-1	4	-2	0	1	1	-3	1	1	0	-1	2	0	-1	-1	-1	0	1
s_{1^8}	1	-1	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	1	-1	1	-1	1	1	-1

Table 5.5: The character table of S_8. The irreducible character corresponding to the partition λ is denoted s_λ and the conjugacy class of elements of cycle type λ is denoted by λ. The first row contains the order of the centralizers of the different conjugacy classes.

6. Assembling the equivariant point count of the total space

In Chapter 4 we made an S_7-equivariant point count of the space of plane quartics with a symplectic level 2 structure, $\mathcal{Q}[2]$. This space is a dense open subset of the space $\mathcal{M}_3[2]$ of genus three curves with a symplectic level 2 structure. The complement of $\mathcal{Q}[2]$ in $\mathcal{M}_3[2]$ is the space $\mathcal{H}_3[2]$ consisting of hyperelliptic curves of genus 3 and a symplectic level 2 structure. In Chapter 5 we made an S_8-equivariant point count of $\mathcal{H}_3[2]$.

Since we only have an S_7-equivariant point count for $\mathcal{Q}[2]$ we want to use our S_8-equivariant point count of $\mathcal{H}_3[2]$ to obtain an S_7-equivariant point count also for this space. This is easily done using the Pieri formula, see [FH04]. In our special case, this formula states that

$$\mathrm{Res}_{S_n}^{S_{n+1}} s_\nu = \sum_{\lambda} s_\lambda,$$

where the sum is over all λ which can be obtained from ν by deleting a box in the Young diagram of ν and where s_λ is the irreducible character of S_n corresponding to the partition λ. Using this formula and the results in Table 5.3 we obtain the S_7-equivariant point count of $\mathcal{H}_3[2]$. The results are displayed in Table 6.1.

We may now add the results of Table 4.1 and Table 6.1 to obtain the S_7-equivariant point count of $\mathcal{M}_3[2]$. The results are shown in Table 6.2. As in the case of $\mathcal{Q}[2]$, let

$$H^i_{\text{ét,c}}(\mathcal{M}_3[2], \mathbb{Q}_\ell) = \bigoplus_{\lambda \vdash 7} V_\lambda^{r_{i,\lambda}},$$

be the decomposition of $H^i_{\text{ét,c}}(\mathcal{M}_3[2], \mathbb{Q}_\ell)$ into irreducible representations of S_7, where V_λ is the irreducible representation corresponding to λ and where $r_{i,\lambda}$ is its multiplicity in $H^i_{\text{ét,c}}(\mathcal{M}_3[2], \mathbb{Q}_\ell)$. Equations 6.0.1 and 6.0.2 show the sum

$$\sum_{\lambda \vdash 7} \sum_{i=0}^{6} (-1)^i \mathrm{Tr}\left(F | V_\lambda^{r_{i,\lambda}}\right) s_\lambda,$$

in two slightly different forms.

In [Loo93], Looijenga proves that the Poincaré-Serre polynomial of $\mathcal{M}_3$ is

$$1 + t^2u^2 + t^6u^{12}.$$

The coefficient of s_7 in Equation 6.0.1 is $q^6+q^5+1-2q^4+2q^3-2q^2+2q$ so if we compare this expression with Loojenga's result we see that the part $q^6 + q^5 + 1$ must correspond to $\mathrm{Sp}(\mathbb{F}_2^6)$-invariant classes while the terms $-2q^4 + 2q^3 - 2q^2 + 2q$ must correspond to classes which are S_7-invariant but not $\mathrm{Sp}(\mathbb{F}_2^6)$-invariant.

λ	$\vert\mathcal{H}_3[2]^{F\cdot\sigma_\lambda}\vert$
$[7]$	$q^5+q^4+q^3+q^2+q+1$
$[6,1]$	$q^5+q^4+q^3-q$
$[5,2]$	q^5-q
$[5,1^2]$	q^5-q
$[4,3]$	$2q^5+2q^4-2q^3-2q^2$
$[4,2,1]$	$2q^5-2q^3$
$[4,1^3]$	$2q^5-4q^4-2q^3+4q^2$
$[3^2,1]$	$3q^5+3q^4-3q^3-12q^2-9q$
$[3,2^2]$	$2q^5-2q^4-6q^3+2q^2+4q$
$[3,2,1^2]$	$4q^5-4q^4-4q^3+4q^2$
$[3,1^4]$	$6q^5-30q^4+30q^3+30q^2-36q$
$[2^3,1]$	$4q^5-8q^4-20q^3+24q^2+32q$
$[2^2,1^3]$	$8q^5-32q^4+24q^3+32q^2-32q$
$[2,1^5]$	$16q^5-160q^4+560q^3-800q^2+384q$
$[1^7]$	$36q^5-720q^4+5580q^3-20880q^2+37584q-25920$

Table 6.1: The S_7-equivariant point count of $\mathcal{H}_3[2]$. We use σ_λ to denote any permutation in S_7 of cycle type λ.

λ	$\lvert\mathcal{M}_3[2]^{F\cdot\sigma_\lambda}\rvert$
$[7]$	$q^6+q^5+q^4+2q^3+q^2+q+1$
$[6,1]$	$q^6+q^5+q^4-q^3-q+1$
$[5,2]$	$q^6+q^5-q^2-q$
$[5,1^2]$	$q^6+q^5-q^2-q$
$[4,3]$	$q^6+q^5-q^3-q^2$
$[4,2,1]$	$q^6+q^5-2q^4-q^3-2q^2+3$
$[4,1^3]$	$q^6+q^5-6q^4-q^3+2q^2+3$
$[3^2,1]$	$q^6+q^5+q^4-11q^3+4q^2+q+21$
$[3,2^2]$	$q^6+q^5-4q^4-3q^3+3q^2+2q$
$[3,2,1^2]$	$q^6+q^5-4q^4+q^3+3q^2-2q$
$[3,1^4]$	$q^6+q^5-20q^4+25q^3+19q^2-26q$
$[2^3,1]$	$q^6+q^5-14q^4-q^3+30q^2+8q+7$
$[2^2,1^3]$	$q^6+q^5-22q^4+39q^3+6q^2-40q+15$
$[2,1^5]$	$q^6+q^5-70q^4+295q^3-426q^2+184q+15$
$[1^7]$	$q^6+q^5-230q^4+2095q^3-7706q^2+12664q-7545$

Table 6.2: The S_7-equivariant point count of $\mathcal{M}_3[2]$. We use σ_λ to denote any permutation in S_7 of cycle type λ.

$$
\begin{aligned}
\sum_{\lambda\vdash 7}\sum_{i=0}^{6}(-1)^i \operatorname{Tr}\left(F|V_\lambda^{r_{i,\lambda}}\right)s_\lambda = &\left(q^6+q^5-2\,q^4+2\,q^3-2\,q^2+2\,q+1\right)s_7-\\
&-\left(24\,q^4-60\,q^3+90\,q^2-90\,q+48\right)s_{6,1}-\\
&-\left(70\,q^4-224\,q^3+420\,q^2-532\,q+294\right)s_{5,2}-\\
&-\left(15\,q^4-195\,q^3-495\,q^2-615\,q+345\right)s_{5,1^2}-\\
&-\left(28\,q^4-154\,q^3+392\,q^2-518\,q+252\right)s_{4,3}-\\
&-\left(70\,q^4-700\,q^3+2205\,q^2-3255\,q+1855\right)s_{4,2,1}+\\
&+\left(80\,q^3-600\,q^2+1060\,q-580\right)s_{4,1,1,1}+\\
&+\left(210\,q^3-756\,q^2+1113\,q-672\right)s_{3,3,1}-\\
&-\left(21\,q^4-147\,q^3+609\,q^2-1113\,q+651\right)s_{3,2^2}+\\
&+\left(280\,q^3-1610\,q^2+2975\,q-1890\right)s_{3,2,1^2}+\\
&+\left(15\,q^3-165\,q^2+510\,q-345\right)s_{3,1^4}+\\
&+\left(14\,q^3-196\,q^2+434\,q-252\right)s_{2^3,1}+\\
&+\left(14\,q^3-154\,q^2+392\,q-308\right)s_{2^2,1^3}-\\
&-\left(12\,q^2-54\,q+54\right)s_{2,1^5}+\\
&+q s_{1^7}
\end{aligned} \tag{6.0.1}
$$

$$
\begin{aligned}
& s_7 q^6 + \\
& + s_7 q^5 - \\
& - \left(2 s_7 + 24 s_{6,1} + 70 s_{5,2} + 15 s_{5,1^2} + 28 s_{4,3} + 70 s_{4,2,1} + 21 s_{3,2^2}\right) q^4 + \\
& + \left(2 s_7 + 60 s_{6,1} + 224 s_{5,2} + 195 s_{5,1^2} + 154 s_{4,3} + 700 s_{4,2,1} + 80 s_{4,1^3} + 210 s_{3^2,1} + 147 s_{3,2^2} + 280 s_{3,2,1^2} + 15 s_{3,1^4} + 14 s_{2^3,1} + 14 s_{2^2,1^3}\right) q^3 - \\
& - \left(2 s_7 + 90 s_{6,1} + 420 s_{5,2} + 495 s_{5,1^2} + 392 s_{4,3} + 2205 s_{4,2,1} + 600 s_{4,1^3} + 756 s_{3^2,1} + 609 s_{3,2^2} + 1610 s_{3,2,1^2} + 165 s_{3,1^4} + 196 s_{2^3,1} + 154 s_{2^2,1^3} + 12 s_{2,1^5}\right) q^2 + \\
& + \left(2 s_7 + 90 s_{6,1} + 532 s_{5,2} + 615 s_{5,1^2} + 518 s_{4,3} + 3255 s_{4,2,1} + 1060 s_{4,1^3} + 1113 s_{3^2,1} + 1113 s_{3,2^2} + 2975 s_{3,2,1^2} + 510 s_{3,1^4} + 434 s_{2^3,1} + 392 s_{2^2,1^3} + 54 s_{2,1^5} + s_{1^7}\right) q + \\
& + s_7 - 48 s_{6,1} - 294 s_{5,2} - 345 s_{5,1^2} - 252 s_{4,3} - 1855 s_{4,2,1} - 580 s_{4,1^3} - 672 s_{3^2,1} - 651 s_{3,2^2} - 1890 s_{3,2,1^2} - 345 s_{3,1^4} - 252 s_{2^3,1} - 308 s_{2^2,1^3} - 54 s_{2,1^5}
\end{aligned}
\tag{6.0.2}
$$

References

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