ON FINITE UNIONS OF SUBMODULES

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Abstract

This paper is concerned with finite unions of ideals and modules. The first main result is that if $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$ is a covering of a module N by submodules N_i, such that all but two of the N_i are intersections of strongly irreducible modules, then $N \subseteq N_k$ for some k. The special case when N is a multiplication module is considered. The second main result generalizes earlier results on coverings by primary submodules. In the last section unions of cosets is studied.

1. Introduction

The Prime avoidance lemma is a well-known and very useful tool in commutative algebra. In it's usual textbook form it reads as follows:

Let I be an ideal, and let $P_1, P_2, \ldots, P_n$ be prime ideals in a commutative ring. Suppose for each P_i there is an element in I which is not in P_i. Then there is in I an element which is in none of the P_i.

In other words, if each P_i can be avoided by some element in I, then there is an element in I which avoids them all. For example the lemma is often used in problems concerning zero-divisors (the set of zero-divisors being a union of prime ideals). Alternatively, we can formulate the Prime avoidance lemma in the positive form:

Suppose $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$. *Then* $I \subseteq P_k$ *for some* k.

This can be generalized in several ways. One can modify the conditions on the P_i (for example two of them are allowed to be non-prime). One can consider modules rather than just ideals. Also, one can consider cosets of ideals or modules.

First, let us recall the usual proof of the Prime avoidance lemma. Assuming that the covering $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$ by prime ideals has been reduced so that no P_i may be excluded from the union, it has to be proved that $n = 1$. So suppose $n > 1$. Then there is an element x which lies in I and all P_i, $i > 1$, but outside P_1. Indeed to obtain such an element just multiply together elements $x_k \in I \cap P_k \setminus P_1$ for $k > 1$. This is where it is used that P_1 is prime. Now, there is also an element y, which lies inside I and P_1 but outside the other prime ideals. The sum $x + y$ gives the contradiction: $x + y \in I$ but $x + y$ lies outside all P_i.

This is the most common proof, and a careful look at the argument reveals that it suffices that $n - 2$ of the P_i are prime. Note that multiplication by elements is used in the proof, so it is not obvious how to generalize to modules.

In my first work on this topic [6], I considered ideals and cosets of ideals. I made a survey of some known results and made some improvements. As a starting-point, I used what I call McCoys lemma, after Neil H. MCoy (see [12]):

If $I = I_1 \cup I_2 \cup \cdots \cup I_s$ *where no* I_k *may be omitted, then any* I_k *contains the intersection of the remaining* I_j. So that for example $I_2 \cap I_3 \cap \cdots \cap I_s = I_1 \cap I_2 \cap \cdots \cap I_s$.

Following Stephen McAdam ([11]), I called $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_s$ an *efficient* covering by ideals if no I_k can be omitted. I will use this terminology also in this paper. Note that it follows immediately from McCoys lemma, that in an efficient covering by more than one ideal, no ideal can be prime. The Prime avoidance lemma follows from this, and this way of arguing is more easily adapted to the case of modules.

McCoy showed that, if the covering is efficient then some power I^n is contained in the intersection $I_1 \cap I_2 \cap \cdots \cap I_s$ (he also showed an analoguos result for non-abelian groups). He used this to generalize the Prime avoidance lemma from prime ideals to radical ideals (i.e intersections of prime ideals). In fact he showed that it suffices that all but two of the ideals are radical.

In [6] I discussed the problem of finding the smallest possible n such that $I^n \subseteq I_1 \cap I_2 \cap \cdots \cap I_s$. This problem was solved completely by Jörgen Backelin, [3].

A covering by two ideals $I \subseteq I_1 \cup I_2$ can never be efficient, as is easily seen. A covering by three ideals $I \subseteq I_1 \cup I_2 \cup I_3$ may be efficient, but then $I_k \cap I \big/ I \cap I_1 \cap I_2 \cap I_3$ is a set of only two elements. In fact, in the general case of an efficient covering by n ideals, the set $I \big/ I \cap I_1 \cap I_2 \cap \cdots \cap I_n$ is always a finite set. The same holds if I is efficiently covered by cosets of the I_i. This follows from [13] Lemma 5.2 and is also proved in [7].

In [7] I returned to the subject, this time with focus on finite unions $M = M_1 \cup M_2 \cup \cdots \cup M_n$ of submodules. I used a module-analogue of McCoys lemma and proved a few theorems given conditions on the $(M_i : M)$ and the radicals $\sqrt{M_i : M}$. The main result was that if $M = M_1 \cup M_2 \cup \cdots \cup M_n$ is an efficient union, and all the M_i are primary submodules of M, then the ideals $\sqrt{M_i : M}$ are all equal to one and the same maximal ideal of the ring ([7] Theorem 9).

Similar results have been obtained by Chin-Pi Lu, [10], and S.E. Atani & Ü. Tekir, [2]. Lu considers submodules of a module M and coverings $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ where all but possibly two of the N_k are prime submodules of M and where $(N_i : M) \nsubseteq (N_j : M)$, for all $i \neq j$. Then $N \subseteq N_k$ for some k. This has as a nice application a theorem on the torsion submodule of a finitely generated module over a Noetherian ring ([10] Theorem 3.6).

The avoidance-theorem of Lu was generalized in [2] to the situation where the N_i (with the possible exception of two) are primary submodules of M and $\sqrt{N_i : M} \nsubseteq \sqrt{N_j : M}$, for all $i \neq j$. I will return to this below in Section 5 and generalize the avoidance-theorems of Lu and Atani-Tekir.

The avoidance-theorem of Lu has also been generalized by Ahmad Khaksari, who considers a countable union of prime submodules (see [9]).

I would like to mention also the work of P. Quartararo & H.S. Butts, [14]. They call an ideal a *u-ideal* if it can not be efficiently covered by more than one ideal and a ring a *u-ring* if every ideal in the ring is a u-ideal. They prove, among other things, that invertible ideals are u-ideals (we will return to this fact in Section 4) and they give the following complete description of u-rings. A ring R is a u-ring if and only if for each maximal ideal $\mathfrak{m}$ either the residue field $R/\mathfrak{m}$ is infinite or the localization $R_{\mathfrak{m}}$ is a Bézout ring (which in the local case means that the ideals are totally ordered). Further research of this kind has been made by F. Callialp & Ü. Tekir [4].

I have now found it possible to improve the presentation and simultaneously extend some of the earlier results. Thus this paper contains both a survey, or recapitulation, of known results, and also some new results. In order to make the paper self-contained I begin with the very basic ideas.

2. Basic ideas.

The most basic idea is the simple fact that the sum of an element inside an ideal and an element outside the same ideal is again an element outside the ideal. And the same holds for modules. This is all you need to prove

Lemma 1. *Suppose* $N \subseteq N_1 \cup N_2$. *Then* $N \subseteq N_1$ *or* $N \subseteq N_2$.

So a covering by only two modules is never efficient. *In the sequel* N *and* N_i *etc. are always assumed to be submodules of some module* M *over a commutative ring* R

Following the basic idea, we can prove the following, which generalizes the McCoy lemma mentioned above.

Proposition 2. *Suppose* $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_r \cup N'_1 \cup N'_2 \cup \cdots \cup N'_s$. *Then* $N \cap N_1 \cap N_2 \cap \cdots \cap N_r \subseteq \bigcup_{i=1}^{s} N'_i$ *or* $N \cap N'_1 \cap N'_2 \cap \cdots \cap N'_s \subseteq \bigcup_{i=1}^{r} N_i$.

Proof. We give an indirect proof and suppose there is an $x \in N \cap N_1 \cap N_2 \cap \cdots \cap N_r \setminus \bigcup_{i=1}^{s} N'_i$ and a $y \in N \cap N'_1 \cap N'_2 \cap \cdots \cap N'_r \setminus \bigcup_{i=1}^{r} N_i$. Then $x + y \in N$ but, by the basic idea, $x + y$ is outside every N_i and every N'_i.

Usually we will use this in the following special case.

Corollary 3. *Suppose* $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 2$. *Then* $N \subseteq N_2 \cup N_3 \cup \cdots \cup N_s$ *or* $N \cap N_2 \cap N_3 \cap \cdots \cap N_s \subseteq N_1$.

Proof. If $N \cap N_2 \cap N_3 \cap \cdots \cap N_s \not\subseteq N_1$, then according to Proposition 2, $N \cap N_1 \subseteq N_2 \cup N_3 \cup \cdots \cup N_s$. But $N \setminus N_1 \subseteq N_2 \cup N_3 \cup \cdots \cup N_s$. Thus $N \subseteq N_2 \cup N_3 \cup \cdots \cup N_s$.

Now it is time to introduce some terminology.

Definition. Let $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$ be a covering of a module N by modules N_i. If a certain N_k can not be omitted (i.e. if $N \not\subseteq \bigcup_{i \neq k} N_i$), then we shall say that N_k is *necesary*. Thus the covering is efficient if and only if every N_k is necessary.

Let us reformulate Corollary 3.

Corollary 4. *Suppose* $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 2$ *and suppose that* N_1 *is necessary. Then* $N \cap N_2 \cap N_3 \cap \cdots \cap N_s \subseteq N_1$.

Thus a necessary module N_k contains the intersection of N and the remaining N_i.

3. Strongly irreducible and pseudo-radical modules and ideals.

To obtain more substantial results, we impose conditions on the submodules, or on the module M itself. We recall the definition of a strongly irreducible submodule.

Definition. A submodule N of M is said to be *strongly irreducible* (in M) if whenever $N \supseteq N_1 \cap N_2$, then $N \supseteq N_1$ or $N \supseteq N_2$ holds.

An example immediately presents itself, namely the prime ideals of a commutative ring. The term strongly irreducible ideal seems to have been coined by W. Heinzer-L. Ratliff-D. Rush. See [8] for a treatise of strongly irreducible ideals. Strongly irreducible modules appear in [1].

Proposition 5. *Suppose* $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 2$ *where* N_1 *is strongly irreducible. Then* $N \subseteq N_2 \cup N_3 \cup \cdots \cup N_s$ *or* $N \subseteq N_1$. *In particular, if* N_1 *is necessary, then* $N \subseteq N_1$.

Proof. If $s = 2$, this is just Lemma 1. Suppose $s \geq 3$ and that $N \not\subseteq N_2 \cup N_3 \cup \cdots \cup N_s$. Then, by Corollary 3, $N_1 \supseteq N \cap N_2 \cap N_3 \cap \cdots \cap N_s$. Now, if $N_1 \not\supseteq N$, then, say, $N_1 \supseteq N_2$ and hence $N \subseteq N_1 \cup N_3 \cup N_4 \cup \cdots \cup N_s$. Now if $s = 3$ the proof is complete (again by Lemma 1). Otherwise, we can go on deleting one of $N_3, N_4, \ldots, N_s$.

We can extend this result to a larger class of modules, which I choose in this paper to call pseudo-radical modules.

Definition. A submodule of M, which is the intersection of a family of strongly irreducible submodules of M will be called a *pseudo-radical* submodule.

Recall that an ideal I is said to be radical if $I = \sqrt{I}$, i.e. if I is an intersection of prime ideals. Thus radical ideals are pseudo-radical. This motivates the terminology.

Proposition 6. *Suppose* $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 2$, *where* N_1 *is a pseudo-radical submodule of* M. *Then* $N \subseteq N_2 \cup N_3 \cup \cdots \cup N_s$ *or* $N \subseteq N_1$. *In particular if* N_1 *is necessary, then* $N \subseteq N_1$.

Proof. Suppose $N \not\subseteq N_2 \cup N_3 \cup \cdots \cup N_s$ and say $N_1 = \bigcap_{j \in J} Q_j$, where the Q_j are strongly irreducible. Then for each $j \in J$ we have $N \subseteq Q_j \cup N_2 \cup N_3 \cup \cdots$

$\cup N_s$ and hence $N \subseteq Q_j$ follows from Proposition 5. But since this holds for all $j \in J$, we actually have $N \subseteq N_1$.

Corollary 7. *Suppose $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 2$ is an efficient covering. Then none of the N_i is pseudo-radical.*

I think it is worth-while to formulate Proposition 6 in the special case of rings.

Corollary 8. *Suppose $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_s$ is a covering of an ideal I by ideals $I_1, I_2, \ldots, I_s$ of a commutative ring. Further assume that I_1 is a pseudo-radical ideal. Then $I \subseteq I_1$ or $I \subseteq I_2 \cup I_3 \cup \cdots \cup I_s$. In particular, if I_1 is necessary, then $I \subseteq I_1$.*

Thus in an efficient covering of an ideal by more than two ideals, none of the ideals may be pseudo-radical (in particular they can not be prime). Corollary 7 is all we need to prove a generalization of the Prime avoidance lemma.

Theorem 9. *Suppose $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, where all but possibly two of the submodules N_i are pseudo-radical. Then $N \subseteq N_k$ holds for some k.*

Proof. After having reduced the covering to an efficient covering no pseudo-radical N_i is left. This follows from Corollary 7. Thus N is covered by the union of just two of the N_i, and hence by some N_k alone.

Example. Suppose $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_s$, where all but possibly two of the I_k are pseudo-radical. Then $I \subseteq I_k$ for some k. For radical ideals this was proved by McCoy, [12].

Examples. In a valuation ring every ideal is clearly strongly irreducible (the ideals being linearly ordered). In the ring of integers, Z, every ideal generated by a prime power is strongly irreducible. Consider the ring $Z \times Z$ and the ideal $I = (4) \times (9) = ((4) \times (1)) \cap ((1) \times (9))$. Then I is pseudo-radical, but not radical (and not strongly irreducible). We have $\sqrt{I} = (2) \times (3) = ((2) \times (1)) \cap ((1) \times (3))$.

4. Coverings in multiplication modules.

First recall the definition of a multiplication module.

Definition. M is said to be a multiplication module if for every submodule N there is an ideal I such that $N = IM$. Note that it clearly follows that $N = (N :_R M)M$.

Next recall the definition of prime and primary submodules.

Definition. A submodule N of M is said to be a *prime submodule* of M if whenever $rx \in N$ where $r \in R$, $x \in M$, either $x \in N$ or $rM \subseteq N$. A submodule N of M is said to be a *primary submodule* of M if whenever $rx \in N$ where $r \in R$, $x \in M$, either $x \in N$ or $r^n M \subseteq N$ for some n.

It is not difficult to prove that if N is a prime submodule of M, then $(N : M)$ is a prime ideal. Note also that $(N : M) = (N : x)$ for any $x \in M \setminus N$, if N is prime and that $\sqrt{N : M} = \sqrt{N : x}$ for any $x \in M \setminus N$, if N is primary.

Prime modules are not necessarily strongly irreducible, but it is well known and easy to see that *prime submodules of multiplication modules are strongly irreducible*. Indeed let N be a prime submodule of a multiplication module M and suppose $N \supseteq N_1 \cap N_2$. Then $(N : M) \supseteq (N_1 : M) \cap (N_2 : M)$ and hence, say, $(N : M) \supseteq (N_1 : M)$, where we use that $(N : M)$ is a prime ideal. Therefore $N = (N : M)M \supseteq (N_1 : M)M = N_1$.

Thus for multiplication modules we have.

Proposition 10. *Let M be a multiplication module and suppose we have $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$ a covering by submodules. Suppose that all but possibly two of the N_i are intersections of prime submodules. Then $N \subseteq N_k$ for some k.*

Proof. This follows from Theorem 9, since in this case intersections of prime submodules are pseudo-radical.

Quartararo and Butts showed that invertible ideals are u-ideals. Following their ideas we can prove the following.

Proposition 11. *Suppose $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$ where N is a finitely generated multiplication module such that* $\operatorname{Ann} N = 0$. *Then $N \subseteq N_k$ for some k.*

Proof. We use induction on s. There are ideals $J_1, J_2, \ldots, J_s$ such that $N \cap N_i = J_i N$. Thus $N = J_1 N \cup J_2 N \cup \cdots \cup J_s N$. We may assume that this union is reduced to an efficient union and now we want to prove that $s = 1$. Suppose $s \geq 2$. Then $N \neq J_1 N$ and we obtain from Corollary 4 that $J_2 N \cap J_3 N \cap \cdots \cap J_s N \subseteq J_1 N$ and hence $J_2 J_3 \cdots J_s N \subseteq J_1 N$. Thus $(J_1 + J_2 J_3 \cdots J_s)N \neq N$, whence it follows that $J_1 + J_2 J_3 \cdots J_s \neq R$ and hence that $J_1 + J_k \neq R$ for some $k \geq 2$. Let us say $J_1 + J_2 \neq R$ and let $\mathfrak{m}$ be a maximal ideal containing $J_1 + J_2$. We have $N = (J_1 + J_2)N \cup J_3 N \cup \cdots \cup J_s N$, whence by the induction hypothesis $N = (J_1 + J_2)N$ and hence $N = \mathfrak{m}N$. But since N is finitely generated it would follow that N is 0 after localizing at $\mathfrak{m}$. But this is impossible since $\operatorname{Ann} N = 0$.

Since invertible ideals are finitely generated multiplication ideals with zero annihilator, it follows that invertible ideals are u-ideals.

Multiplication modules are indeed very special modules. In the general case we need stronger conditions on the submodules than just being prime. In what follows we shall look at conditions on the $(N_i : N)$ or $(N_i : M)$ and generalize the avoidance-theorems of [10] and [2].

5. Conditions on the ideals $(N_i : N)$ and an avoidance-theorem for primary submodules.

We need a preparatory lemma before being able to prove the avoidance-theorem for primary submodules.

Lemma 12. *Suppose $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 3$, where N_1 and N_2 are necessary and let $a \in R$ be such that $aN \subseteq N_3 \cap N_4 \cap \cdots \cap N_s$. Then $a^2 N \subseteq N_1 \cap N_2 \cap \cdots \cap N_s$.*

Proof. We have $N \subseteq (N_1 \cap N + N_2 \cap N) \cup N_3 \cup N_4 \cup \cdots \cup N_s$ so, by Corollary 4, $N \cap N_3 \cap N_4 \cap \cdots \cap N_s \subseteq N_1 \cap N + N_2 \cap N$. Thus $a^2 N \subseteq a(N_1 \cap N + N_2 \cap N)$. Now $a(N_1 \cap N) \subseteq N_1 \cap aN \subseteq N \cap N_1 \cap N_3 \cap N_4 \cap \cdots \cap N_s$ but $N_2 \supseteq N \cap N_1 \cap N_3 \cap N_4 \cap \cdots \cap N_s$, since N_2 is necessary. Thus $a(N_1 \cap N) \subseteq N \cap N_1 \cap N_2 \cap \cdots \cap N_s$. Similarly $a(N_2 \cap N) \subseteq N \cap N_1 \cap N_2 \cap \cdots \cap N_s$. Thus $a^2 N \subseteq N_1 \cap N_2 \cap \cdots \cap N_s$.

We now prove a theorem similar of McCoys lemma.

Proposition 13. *Suppose $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 3$, where N_1 and N_2 are both necessary. Then $\sqrt{N_1 : N} \cap \sqrt{N_2 : N} \supseteq \bigcap_{i=3}^{s} \sqrt{N_i : N}$. In other words $\bigcap_{i=3}^{s} \sqrt{N_i : N} = \bigcap_{i=1}^{s} \sqrt{N_i : N}$.*

Proof. Take any $a \in \bigcap_{i=3}^{s} \sqrt{N_i : N}$. Then for some m we have $a^m N \subseteq \bigcap_{i=3}^{s} N_i$, and hence, by Lemma 12, $a^{2m} N \subseteq \bigcap_{i=1}^{s} N_i$. Thus $a \in \bigcap_{i=1}^{s} \sqrt{N_i : N}$.

Corollary 14. *Suppose $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 3$, is an efficient covering. Suppose also that $\sqrt{N_1 : N}$ is a prime ideal. Then $\sqrt{N_1 : N} \supseteq \sqrt{N_i : N}$ holds for at least two values of $i \neq 1$.*

Proof. By Proposition 13, we have $\sqrt{N_1 : N} \supseteq \bigcap_{i=3}^{s} \sqrt{N_i : N}$. Thus, since $\sqrt{N_1 : N}$ is prime, $\sqrt{N_1 : N} \supseteq \sqrt{N_k : N}$, some $k \geq 3$. Again, by Proposition 13, $\sqrt{N_1 : N} \supseteq \bigcap_{i \geq 2, i \neq k} \sqrt{N_i : N}$. Thus also $\sqrt{N_1 : N} \supseteq \sqrt{N_i : N}$ for an $i \geq 2$ different from k.

Proposition 15. *Suppose $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 3$, where $\sqrt{N_i : N}$ are prime for $i = 1, 2, \ldots, s-2$. Suppose also that for each $i \leq s-2$, the inclusion $\sqrt{N_i : N} \supseteq \sqrt{N_j : N}$ holds for at most one $j \neq i$. Then $N \subseteq N_k$, some k.*

Proof. According to Corollary 14, the covering is not efficient, so we may delete one of the N_i. As long as $s \geq 3$ we can go on deleting one of the N_i without violating the hypotheses. Thus N is contained in the union of two of the N_i, and hence, by Lemma 1, $N \subseteq N_k$, some k.

Note that we have so far not supposed that the N_i are primary. This condition is needed when we focus on $(N_i : M)$ rather than $(N_i : N)$, as in the following theorem which generalizes the main result in [2].

Theorem 16. *Suppose $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_s$, $s \geq 3$, where N_i is a primary submodule of M for $i \leq s-2$. Suppose also that for each $i \leq s-2$ the inclusion $\sqrt{N_i : M} \supseteq \sqrt{N_j : M}$ holds for at most one $j \neq i$. Then $N \subseteq N_k$, some k.*

Proof. Assume that $N \not\subseteq N_k$ for all $k \leq s-2$ and take $x_k \in N \setminus N_k$ for $k \leq s-2$. Then $\sqrt{N_k : M} = \sqrt{N_k : x_k} = \sqrt{N_k : N}$ for $k \leq s-2$. Also $\sqrt{N_k : M} \subseteq \sqrt{N_k : N}$ for $k = s-1, s$. Thus the hypotheses of Proposition 15 are fulfilled and hence $N \subseteq N_k$ for some k.

Remark. It follows from [7] Proposition 7 that when N_i is a primary submodule, then $\sqrt{N_i : M}$ is actually a maximal ideal. As an alternative to the hypotheses in Theorem 16 we could therefore suppose that the N_i are primary for $i \leq s-1$ with the $\sqrt{N_i : M}$, $i \leq s-1$ all different.

6. A few notes on cosets of ideals and modules.

In this section, we will use the words *efficient* and *necessary* in their obvious interpretation for cosets. McCoys lemma can be generalized to cosets in different ways. We shall use the following.

Proposition 17. *Suppose $N \subseteq (N_1 + a_1) \cup (N_2 + a_2) \cup \cdots \cup (N_s + a_s)$ and suppose $N_1 + a_1$ is necessary. Then $N_1 \supseteq N \cap \bigcap_{i=2}^{s} N_i$. In particular if N_1 is strongly irreducible, then $N \subseteq N_1$ (in which case $a_1 \in N_1$) or $N_1 \supseteq N_k$, some $k > 1$.*

Proof. Take any $x \in N \cap \bigcap_{i=2}^{s} N_i$ and a $y \in N \cap (N_1 + a_1) \setminus \bigcup_{i=2}^{s} N_i + a_i$. Then $x + y \in N \setminus \bigcup_{i=2}^{s} N_i + a_i$. Thus $x + y \in N_1 + a_1$ and hence $x \in N_1$.

Theorem 18. *Suppose $N \subseteq (N_1 + a_1) \cup (N_2 + a_2) \cup \cdots \cup (N_s + a_s)$, where all N_i are strongly irreducible and where the modules minimal among N_1, $N_2, \ldots, N_s$ each occur only once. Then $(N, a_k) \subseteq N_k$, some k.*

Proof. We may restrict the covering to an efficient one, and after having done so we are supposed to show that $n = 1$. Assume $n > 1$. Let N_k be one of the minimal submodules among the N_i. Then, by Proposition 17, either

$N \subseteq N_k$, $a_k \in N_k$ and there is nothing more to prove, or there is a $j \neq k$ such that $N_k = N_j$ which would contradict the hypothesis. Thus $(N, a_k) \subseteq N_k$ some k.

As a corollary we obtain the following result of Gilmer ([5] Lemma 2).

Corollary 19. Suppose $I \subseteq (P_1 + a_1) \cap (P_2 + a_2) \cap \cdots \cap (P_r + a_r)$, where the P_i are pairwise distinct prime ideals. Then $(I, a_k) \subseteq P_k$ holds for some k.

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