

On a class of power ideals

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Abstract

In this paper we study the class of power ideals generated by the k^n forms $(x_0 + \xi^{g_1} x_1 + \dots + \xi^{g_n} x_n)^{(k-1)d}$ where ξ is a fixed primitive k^{th} -root of unity and $0 \leq g_j \leq k-1$ for all j . For $k=2$, by using a \mathbb{Z}_k^{n+1} -grading on $\mathbb{C}[x_0, \dots, x_n]$, we compute the Hilbert series of the associated quotient rings via a simple numerical algorithm. We also conjecture the extension for $k > 2$. Via Macaulay duality, those power ideals are related to schemes of fat points with support on the k^n points $[1 : \xi^{g_1} : \dots : \xi^{g_n}]$ in \mathbb{P}^n . We compute Hilbert series, Betti numbers and Gröbner basis for these 0-dimensional schemes. This explicitly determines the Hilbert series of the power ideal for all k : that this agrees with our conjecture for $k > 2$ is supported by several computer experiments.

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1. Introduction

We denote by $S = \bigoplus_{i \geq 0} S_i$ the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ with *standard* gradation, namely S_d is the \mathbb{C} -vector space of homogeneous polynomials, or *forms*, of degree d .

Definition 1.1. A homogeneous ideal $I \subset S$ is called a **power ideal** if generated by some powers $L_1^{d_1}, \dots, L_m^{d_m}$ of linear forms and $\text{span}(L_1, \dots, L_m) = S_1$.

This class of ideals received recently a considerable attention in the mathematical literature thanks to the connections with the theories of fat points, e.g. see [1; 2], and of Cox rings and box splines, see [3] for a complete survey about that connections.

In this article, we want to consider a special class of power ideals depending on three positive indices and recently introduced in connection with a Waring problem for polynomial rings; see [4]. For any triple (n, k, d) of positive integers and with a primitive k^{th} -root of unity ξ , we consider the homogeneous ideal $I_{n,k,d}$ generated by the k^n powers $(x_0 + \xi^{g_1} x_1 + \dots + \xi^{g_n} x_n)^{(k-1)d}$ where $0 \leq g_j \leq k-1$ for all $j = 1, \dots, n$. We denote the quotient ring as $R_{n,k,d} := \mathbb{C}[x_0, \dots, x_n]/I_{n,k,d}$ and with $[R_{n,k,d}]_j$ its homogeneous component of degree j .

In [4], the authors needed to focus on the homogeneous part of degree kd of that quotient rings and their result about that is the following.

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Theorem 1.2 ([4], Corollary 10). $[R_{n,k,d}]_{kd} = 0$, i.e. for any triple (n, k, d) , the power ideal $I_{n,k,d}$ contains all forms of degree kd .

In this article, we continue the study of the family of ideals $I_{n,k,d}$ and their quotient rings $R_{n,k,d}$. The main goal is to determine the Hilbert series of $R_{n,k,d}$.

In Section 2 we introduce a \mathbb{Z}_k^{n+1} -grading on $R_{n,k,d}$. It is the main tool for our first investigation on those power ideals and, as a first consequence, we get a minimal set of generators for the ideal $I_{n,k,d}$. In Section 3.1 we focus on the $k = 2$ case. We determine the Hilbert series for the quotient ring $R_{2,n,d}$. One consequence is that $[R_{2,n,d}]_{2d-1} = 0$, which strengthens Theorem 1.2 in the $k = 2$ case. In Section 3.2, we consider the $k > 2$ case and we conjecture the extension of our results in the $k = 2$ case.

By Macaulay duality, it is possible to relate the Hilbert function of power ideals to the Hilbert function of schemes of fat points. In Section 4, we investigate how to apply our results on our class of power ideals to determine the Hilbert function of the corresponding schemes of fat points supported on the k^n points of type $[1 : \xi^{i_1} : \dots : \xi^{i_n}] \in \mathbb{P}^n$, where ξ is a k^{th} root of unity and $0 \leq i_j \leq k-1$, for all $j = 1, \dots, n$. In particular, we get the following result and we check, with the support of a computer, that the Hilbert function attained for the schemes of fat points coincides, via Macaulay duality, with the conjecture in Section 3.2 on the Hilbert function of the power ideals.

Theorem 1.3. Let $I_k^{(d)}$ be the ideal of the scheme of fat points of multiplicity d with support on the k^n points of type $[1 : \xi^{g_1} : \dots : \xi^{g_n}] \in \mathbb{P}^n$ where ξ is a primitive k^{th} -root of unity and $0 \leq g_j \leq k-1$ for all $j = 1, \dots, n$. Then, we have that the Hilbert series of the quotient ring $S/I_k^{(d)}$ is given by

$$\text{HS} \left(S/I_k^{(d)}; t \right) = \frac{1 + \sum_{i=1}^n (-1)^i \beta_{i, kd+k(i-1)} t^{kd+k(i-1)}}{(1-t)^{n+1}},$$

where Betti numbers are given by

$$\beta_{i, kd+k(i-1)} = \binom{d+i-2}{i-1} \binom{d+n-1}{n-i}, \quad \text{for } i = 1, \dots, n.$$

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2. Multicyclic gradation

Let $\mathbb{Z}_k = \{[0]_k, [1]_k, \dots, [k-1]_k\}$ be the cyclic group of integers modulo k . Let ξ be a primitive k^{th} -root of unity and observe that, for any $\nu \in \mathbb{Z}_k$, the complex number ξ^ν is well-defined.

We often use a small abuse of notation denoting a class of integer modulo k simply with its smallest representative $\mathbb{N} = \{0, 1, 2, \dots\}$. For instance, given a vector $g = (g_0, \dots, g_n) \in \mathbb{Z}_k^{n+1}$, we define the **weight** of \mathbf{g} as the sum $\text{wt}(\mathbf{g}) := \sum_{j=0}^n g_j \in \mathbb{N}$, where the g_j 's are considered as natural numbers between 0 and

$k - 1$. Clearly, we get that the weight of a vector \mathbf{g} is always non-negative and equal to zero if and only if \mathbf{g} is the zero vector.

Example 2.1. Let $n = 5, k = 3$. Considering $\mathbf{g} = ([2]_3, [4]_3, [1]_3, [7]_3, [6]_3) \in \mathbb{Z}_3^6$, we have that the weight of \mathbf{g} is $\text{wt}(\mathbf{g}) = 2 + 1 + 1 + 1 + 0 = 5$.

Keeping in mind this abuse of notation, we consider, for each vector $\mathbf{g} = (g_0, \dots, g_n) \in \mathbb{Z}_k^{n+1}$, the polynomial

$$\phi_{\mathbf{g}} := \left(\sum_{i=0}^n \xi^{g_i} x_i \right)^D, \text{ where } D := (k - 1)d.$$

Let $I_{n,k,d}$ be the ideal generated by all $\phi_{\mathbf{g}}$, with $\mathbf{g} \in \{0\} \times \mathbb{Z}_k^n$. It is homogeneous with respect to the standard gradation, but it is also homogeneous with respect to the \mathbb{Z}_k^{n+1} -gradation we are going to define.

Consider the projection $\pi_k : \mathbb{N} \rightarrow \mathbb{Z}_k$ given by $\pi_k(n) = [n]_k$. For any vector $\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{N}^{n+1}$, we define the **multicyclic degree** as follows.

Given a monomial $\mathbf{x}^{\mathbf{a}} := x_0^{a_0} \dots x_n^{a_n}$, we set

$$\text{mcdeg}(\mathbf{x}^{\mathbf{a}}) := \pi_k^{n+1}(\mathbf{a}) = ([a_0]_k, \dots, [a_n]_k).$$

Thus, combining this multicyclic degree with the standard gradation, we get the multigradation on the polynomial ring S given by

$$S = \bigoplus_{i \in \mathbb{N}} S_i = \bigoplus_{i \in \mathbb{N}} \bigoplus_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} S_{i,\mathbf{g}}, \text{ where } S_{i,\mathbf{g}} := S_i \cap S_{\mathbf{g}};$$

where, for any $i_1, i_2 \in \mathbb{N}$ and $\mathbf{g}_1, \mathbf{g}_2 \in \mathbb{Z}_k^{n+1}$, we have that

$$S_{i_1, \mathbf{g}_1} \cdot S_{i_2, \mathbf{g}_2} \subseteq S_{i_1+i_2, \mathbf{g}_1+\mathbf{g}_2}.$$

Remark 2.2. For $\mathbf{0} := (0, \dots, 0)$, we get obviously that $S_{\mathbf{0}} = \mathbb{C}[x_0^k, \dots, x_n^k]$, and then, for any $i \in \mathbb{N}$,

$$S_{i, \mathbf{0}} \neq 0 \text{ if and only if } i = jk \text{ for some } j \in \mathbb{N},$$

in such a case

$$\dim_{\mathbb{C}} S_{jk, \mathbf{0}} = \binom{n+j}{n}.$$

Lemma 2.3. Let $i \in \mathbb{N}$ and $\mathbf{g} \in \mathbb{Z}_k^{n+1}$. Then,

$$S_{i,\mathbf{g}} \neq 0 \text{ if and only if } i - \text{wt}(\mathbf{g}) = jk, \text{ for some } j \in \mathbb{N}.$$

In this case,

$$\dim_{\mathbb{C}} S_{i,\mathbf{g}} = \binom{n+j}{n}.$$

Proof. Given a monomial $\mathbf{x}^{\mathbf{a}}$ with $i = \text{deg}(\mathbf{x}^{\mathbf{a}})$, consider $\mathbf{g} = \pi_k^{n+1}(\mathbf{a})$. If we interpret \mathbf{g} as a vector of natural numbers, we have that $\mathbf{x}^{\mathbf{a}-\mathbf{g}} \in S_{i-\text{wt}(\mathbf{g}), \mathbf{0}}$. Hence,

$$\dim_{\mathbb{C}} S_{i,\mathbf{g}} = \dim_{\mathbb{C}} S_{i-\text{wt}(\mathbf{g}), \mathbf{0}} = \binom{n+j}{n}.$$

□

Now, let $\mathcal{G}_{k,n,i}$ denote the set set of all multicycles satisfying the two equivalent conditions of Lemma 2.3, i.e.

$$\mathcal{G}_{k,n,i} := \{\mathbf{h} \in \mathbb{Z}_k^{n+1} \mid i - \text{wt}(\mathbf{h}) \in k\mathbb{N}\} = \{\mathbf{h} \in \mathbb{Z}_k^{n+1} \mid S_{i,\mathbf{h}} \neq 0\}.$$

Turning back to our ideal, since we can write $S_D = \bigoplus_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} S_{D,\mathbf{g}}$, we can represent the generator $\phi_{\mathbf{0}} = (x_0 + \dots + x_n)^D$ of $I_{n,k,d}$ as

$$\phi_{\mathbf{0}} = \sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \psi_{\mathbf{g}}, \text{ where } \psi_{\mathbf{g}} \in S_{D,\mathbf{g}}.$$

Clearly, if $\psi_{\mathbf{g}} \neq 0$ then $\mathbf{g} \in \mathcal{G}_{k,n,D}$, but one can also check that actually

$$\psi_{\mathbf{g}} \neq 0 \iff \mathbf{g} \in \mathcal{G}_{k,n,D}.$$

In fact, for $\mathbf{g} \in \mathcal{G}_{k,n,D}$, we have that,

$$\psi_{\mathbf{g}} = \sum_{\substack{\mathbf{d}: d_0 + \dots + d_n = D \\ \pi_k^{n+1}(d_0, \dots, d_n) = \mathbf{g}}} \binom{D}{d_0, \dots, d_n} \mathbf{x}^{\mathbf{d}} \neq 0.$$

In the following example, we make this construction more explicit.

Example 2.4. Consider the case $k = 2, n = 2, d = D = 4$ and $\phi_{\mathbf{0}} = (x_0 + x_1 + x_2)^4$. We have

$$\begin{aligned} \psi_{(0,0,0)} &= x_0^4 + 6x_0^2x_1^2 + 6x_0^2x_2^2 + x_1^4 + 6x_1^2x_2^2 + x_2^4; \\ \psi_{(1,0,0)} &= \psi_{(0,1,0)} = \psi_{(0,0,1)} = \psi_{(1,1,1)} = 0; \\ \psi_{(1,1,0)} &= 4x_0^3x_1 + 12x_0x_1x_2^2 + 4x_0x_1^3; \\ \psi_{(1,0,1)} &= 4x_0^3x_2 + 12x_0x_1^2x_2 + 4x_0x_2^3; \\ \psi_{(0,1,1)} &= 4x_1x_2^3 + 12x_0^2x_1x_2 + 4x_1x_2^3. \end{aligned}$$

Actually, since $(1,0,0) \notin \mathcal{G}_{2,2,4}$, we already saw that $\psi_{(1,0,0)} = 0$, and similarly for $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$.

Lemma 2.5. For any $\mathbf{g} \in \mathbb{Z}_k^{n+1}$, one has

$$\phi_{\mathbf{g}} = \sum_{\mathbf{h} \in \mathcal{G}_{k,n,D}} \xi^{\langle \mathbf{g}, \mathbf{h} \rangle} \psi_{\mathbf{h}};$$

and conversely,

$$\psi_{\mathbf{g}} = k^{-n-1} \sum_{\mathbf{h} \in \mathbb{Z}_k^{n+1}} \xi^{-\langle \mathbf{g}, \mathbf{h} \rangle} \phi_{\mathbf{h}}.$$

Proof. From the definition, we have

$$\begin{aligned} \phi_{\mathbf{g}} &= \left(\sum_{i=0}^n \xi^{g_i} x_i \right)^D = \sum_{d_0 + \dots + d_n = D} \binom{D}{d_0, \dots, d_n} \prod_{l=0}^n \xi^{g_l d_l} x_l^{d_l} = \\ &= \sum_{d_0 + \dots + d_n = D} \binom{D}{d_0, \dots, d_n} \xi^{\langle \mathbf{g}, \mathbf{d} \rangle} \mathbf{x}^{\mathbf{d}}. \end{aligned}$$

Now, consider for each $\mathbf{d} = (d_0, \dots, d_n)$ the vector $\pi_k^{n+1}(\mathbf{d}) = \mathbf{h} \in \mathbb{Z}_k^{n+1}$. Since ξ is a k^{th} root of unity, we have $\xi^{\langle \mathbf{g}, \mathbf{d} \rangle} = \xi^{\langle \mathbf{g}, \mathbf{h} \rangle}$. Thus, indeed

$$\phi_{\mathbf{g}} = \sum_{\mathbf{h} \in \mathcal{G}_{k,n,D}} \xi^{\langle \mathbf{g}, \mathbf{h} \rangle} \sum_{\substack{d_0 + \dots + d_n = D \\ \pi_k^{n+1}(\mathbf{d}) = \mathbf{h}}} \binom{D}{d_0, \dots, d_n} \mathbf{x}^{\mathbf{d}} = \sum_{\mathbf{h} \in \mathcal{G}_{k,n,D}} \xi^{\langle \mathbf{g}, \mathbf{h} \rangle} \psi_{\mathbf{h}}.$$

For the second part of the statement, we consider the following equality which follows from the already proved first part. For any $\mathbf{m} \in \mathbb{Z}_k^{n+1}$,

$$\sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \xi^{-\langle \mathbf{g}, \mathbf{m} \rangle} \phi_{\mathbf{g}} = \sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \sum_{\mathbf{h} \in \mathcal{G}_{k,n,D}} \xi^{\langle \mathbf{g}, \mathbf{h} \rangle - \langle \mathbf{g}, \mathbf{m} \rangle} \psi_{\mathbf{h}}.$$

On the right hand side, we have

$$\begin{cases} \text{if } \mathbf{m} = \mathbf{h} : & \sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \psi_{\mathbf{h}} = k^{n+1} \psi_{\mathbf{h}}; \\ \text{if } \mathbf{m} \neq \mathbf{h} : & \sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \xi^{\langle \mathbf{g}, \mathbf{h} - \mathbf{m} \rangle} \psi_{\mathbf{h}} = \sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \xi_0^{g_0} \dots \xi_n^{g_n} \psi_{\mathbf{h}} = 0. \end{cases}$$

□

Thus $\{\psi_{\mathbf{g}}\}_{\mathbf{g} \in \mathcal{G}_{k,n,D}}$ is a set of nonzero polynomials with distinct multicyclic degree and consequently linearly independent. In other words, we have proved the following proposition.

Proposition 2.6. $I_{n,k,d}$ is minimally generated by $\{\psi_{\mathbf{g}}\}_{\mathbf{g} \in \mathcal{G}_{k,n,D}}$.

Consequently, the next theorem counts the number of minimal generators of the ideal $I_{n,k,d}$.

Theorem 2.7. With k, n, d and $D = (k-1)d$ as above,

$$\begin{aligned} |\mathcal{G}_{k,n,D}| &= \\ &= \sum_{i \geq 0} \sum_{\nu_2, \dots, \nu_{k-1} \geq 0} \binom{n+1}{D - ki - \sum_{j=1}^{k-1} (j-1)\nu_j} \binom{D - ki - \sum_{j=1}^{k-1} (j-1)\nu_j}{\nu_2, \dots, \nu_{k-1}, D - \sum_{j=2}^{k-1} j\nu_j} = \\ &= \sum_{i, \nu_2, \dots, \nu_{k-1} \geq 0} \binom{n+1}{\nu_2, \dots, \nu_{k-1}, D - ki - \sum_{j=2}^{k-1} j\nu_j, n+1 - D + ki + \sum_{j=2}^{k-1} (j-1)\nu_j}. \end{aligned}$$

In particular, if $k = 2$, then this number of generators equals $\sum_{i \geq 0} \binom{n+1}{d-2i}$.

Proof. We shall count the number of $\mathbf{g} \in \mathcal{G}_{k,n,D}$ by means of the *partition* of \mathbf{g} ; namely, given $\mathbf{g} = (g_0, \dots, g_n) \in \mathbb{Z}_k^{n+1}$ we define $\text{part}(\mathbf{g}) := (\#\{i \mid g_i = 0\}, \dots, \#\{i \mid g_i = k-1\}) = (\nu_0, \dots, \nu_{k-1})$.

First, note that for any such $\text{part}(\mathbf{g})$,

$$\sum_{j=0}^{k-1} \nu_j = n+1 \quad \text{and} \quad \sum_{j=0}^{k-1} j\nu_j = \text{wt}(\mathbf{g}) = D - ik$$

for some $i \in \mathbb{N}$. Solving for ν_0 and ν_1 , we find that indeed

$$\nu_1 = D - ki - \sum_{j=2}^{k-1} j\nu_j,$$

and

$$\nu_0 = n + 1 - D + ki + \sum_{j=2}^{k-1} (j-1)\nu_j.$$

Thus, the multinomial coefficient in the statement counts precisely the number of ways to choose ν_2 entries g_j 's of \mathbf{g} for which $g_j = 2$, ν_3 entries for which $g_j = 3$, et cetera for all $j = 0, \dots, k-1$. The theorem now follows, by noting that the multinomial coefficients indeed are non-zero only for the i , ν_2 , ..., and ν_{k-1} , such that also ν_1 and ν_0 are non-negative. \square

Example 2.8. For $k = 4, d = 3, n = 2$ we get that the number of minimal generators is

$$\binom{3}{0,3,0,0} + \binom{3}{0,1,2,0} + \binom{3}{1,1,0,1} + \binom{3}{2,0,1,0} + \binom{3}{0,0,1,2} = 1 + 3 + 6 + 3 + 3 = 16 = 4^2.$$

This means that the generators $\phi_{\mathbf{g}}$ are minimal.

On the other hand, for $k = 4, d = 2, n = 3$, we get

$$\begin{aligned} \binom{4}{0,1,3,0} + \binom{4}{0,2,0,2} + \binom{4}{1,1,1,1} + \binom{4}{2,0,2,0} + \binom{4}{3,0,0,1} + \binom{4}{0,0,2,2} + \binom{4}{1,0,0,3} \\ = 4 + 6 + 24 + 6 + 4 + 6 + 4 = 54 < 64 = 4^3. \end{aligned}$$

Hence, with that numerical assumptions, the generators $\phi_{\mathbf{g}}$ are not minimal.

Theorem 2.9. *If $k = 2$, the generators $\{\phi_{\mathbf{g}}\}_{\mathbf{g} \in 0 \times \mathbb{Z}_2^n}$ are linearly independent if and only if $n + 1 \leq d$.*

Proof. $\{\psi_{\mathbf{g}}\}$ is linearly independent, and they are $\sum_{i \geq 0} \binom{n+1}{d-2i}$ many. This sum equals 2^n if and only if $n + 1 \leq d$. \square

3. Hilbert function of the power ideal $I_{n,k,d}$

In order to simplify the notation, when there is no ambiguity, we denote $I := I_{n,k,d}$ and $R := R_{n,k,d} = S/I$ with the multicycling gradation described in the previous section, $R = \bigoplus_{i \in \mathbb{N}} \bigoplus_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} R_{i,\mathbf{g}}$.

Definition 3.1. For $0 \leq i \leq d$ and given a vector $\mathbf{h} \in \mathbb{Z}_k^{n+1}$, let

$$\begin{aligned} \mu_{i,\mathbf{h}} : D_{i,\mathbf{h}} := \bigoplus_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} S_{i,\mathbf{h}-\mathbf{g}} &\longrightarrow S_{i+D,\mathbf{h}}, \\ (\dots, f_{\mathbf{g}}, \dots) &\longmapsto \sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} f_{\mathbf{g}} \psi_{\mathbf{g}}. \end{aligned}$$

be the map given by the multiplication by each $\psi_{\mathbf{g}} \in S_{D,\mathbf{g}}$.

Remark 3.2. In order to work with relevant examples, we shall assume always that $i + D - \text{wt}(\mathbf{h}) \in k\mathbb{Z}$ whence $S_{i+D,\mathbf{h}} \neq 0$. We may also observe that, under this assumption, we have the following equivalence

$$i - \text{wt}(\mathbf{h} - \mathbf{g}) \in k\mathbb{Z} \iff D - \text{wt}(\mathbf{g}) \in k\mathbb{Z};$$

in other words, again from the properties of this multicyclic gradation explained in the previous section, we have

$$S_{i,\mathbf{h}-\mathbf{g}} \neq 0 \iff \psi_{\mathbf{g}} \neq 0.$$

Thus, it makes sense to study the injectivity of the $\mu_{i,\mathbf{h}}$'s and it will be the crucial step for our computations.

Lemma 3.3. *Given $0 \leq i \leq d$ and $\mathbf{h} \in \mathbb{Z}_k^{n+1}$, if $i + D - \text{wt}(\mathbf{h}) \in k\mathbb{N}$ and $\text{wt}(\mathbf{h}) \leq (k-1)(d-i)$, we have*

$$\dim(D_{i,\mathbf{h}}) \leq \dim(S_{i+D,\mathbf{h}});$$

with equality if $\text{wt}(\mathbf{h}) = (k-1)(d-i)$.

Proof. Under these assumptions, we have that $D_{i,\mathbf{h}}$ is simply S_i ; thus,

$$\dim_{\mathbb{C}} D_{i,\mathbf{h}} = \binom{n+i}{n};$$

moreover, we may observe that, for some integer $m \geq 0$,

$$km = i + D - \text{wt}(\mathbf{h}) \geq i + D - (k-1)(d-i) = ki;$$

hence, $i + D - \text{wt}(\mathbf{h}) = k(i+j)$ for some $j \geq 0$ and

$$\dim(S_{i+D,\mathbf{h}}) = \binom{n+i+j}{n}.$$

□

For any $0 \leq i \leq d$ and $\mathbf{h} \in \mathbb{Z}_k^{n+1}$, the image of the map $\mu_{i,\mathbf{h}}$ is simply the part of multicycling degree (i, \mathbf{h}) of our ideal I . These maps will be the main tool in our computations regarding the Hilbert function of I and its quotient ring R . By Remark 3.10 and Lemma 3.3, it makes sense to ask if $\mu_{i,\mathbf{h}}$ is injective whenever $\text{wt}(\mathbf{h}) \leq (k-1)(d-i)$ and $i + D - \text{wt}(\mathbf{h}) \in k\mathbb{Z}$: in that case, the dimension of $I_{i+D,\mathbf{h}}$ in degree i is simply the dimension of $D_{i,\mathbf{h}} = S_i$. On the other hand, again by Lemma 3.3, one could hope that $\mu_{i,\mathbf{h}}$ is surjective in all the other cases, i.e. $R_{i+D,\mathbf{h}} = 0$.

This is true for $k=2$ as we are going to prove in the next section.

3.1. The $k=2$ case

In this case, $D = (k-1)d = d$. Moreover, as we said in Remark 3.10, we shall consider only the maps $\mu_{i,\mathbf{h}}$ such that $i + d - \text{wt}(\mathbf{h})$ is even.

Lemma 3.4. *In the same notation as above, we have:*

1. $\mu_{d,\mathbf{0}}$ is bijective;
2. $\mu_{i,\mathbf{h}}$ is injective if $\text{wt}(\mathbf{h}) \leq d-i$;
3. $\mu_{i,\mathbf{h}}$ is surjective if $\text{wt}(\mathbf{h}) \geq d-i$.

Proof. (1) The map $\mu_{d,\mathbf{0}}$ is surjective from the Theorem 1.2 and it is also injective because we are in the limit case of Lemma 3.3, i.e. where the dimensions of the source and the target are equal.

(2) Given a monomial M with $M \in S_{d+i,\mathbf{h}}$, there exists a monomial M' such that $MM' \in S_{2d,\mathbf{0}}$; indeed, it is enough to consider the monomial $\mathbf{x}_{\mathbf{h}}$ to get $\text{mcdeg}(\mathbf{x}_{\mathbf{h}}M) = \mathbf{0}$ and then we can multiply for any monomial with the right degree to get degree equal to $2d$ and multicyclic degree equal to $\mathbf{0}$. Hence, the injectivity of $\mu_{i,\mathbf{h}}$ follows from (1).

(3) If $\text{wt}(\mathbf{h}) = (d-i)$, we are in the limit case of Lemma 3.3 and then, from injectivity of $\mu_{i,\mathbf{h}}$, it follows also the surjectivity. Instead, the case $\text{wt}(\mathbf{h}) > (d-i)$ follows from the previous one because, given any monomial M with $M \in S_{n,\mathbf{h}}$ and $n - \text{wt}(\mathbf{h}) = 2m$, then M is a product of a monomial M' with $M' \in S_{n-2m,\mathbf{h}}$. □

We let $\text{HF}(R, i)$ denote the *Hilbert function* of $R = S/I$ computed in degree i , i.e. $\text{HF}(R, i) := \dim_{\mathbb{C}}(S_i) - \dim_{\mathbb{C}}(I_i)$, and with $\text{HS}(R; t)$ the *Hilbert series* defined as $\text{HS}(R; t) := \sum_{i \in \mathbb{N}} \text{HF}(R, i)t^i$.

Lemma 3.5. *In the same notation as above, we have:*

1. if $i < d$, $I_i = 0$;
2. if $i = j + d$ with $j \geq 0$, $R_{i, \mathbf{h}} \neq 0$ if and only if

$$\mathbf{h} \in \mathcal{H}_j := \{\mathbf{h}' \mid i - \text{wt}(\mathbf{h}') \in 2\mathbb{N}, \text{wt}(\mathbf{h}') < d - j, \text{wt}(\mathbf{h}') \leq n + 1\};$$

moreover, if $\mathbf{h} \in \mathcal{H}_j$, then

$$\dim_{\mathbb{C}} R_{i, \mathbf{h}} = \dim_{\mathbb{C}} S_{i, \mathbf{h}} - \binom{n+j}{n}.$$

Proof. Since I has generators in degree d , then $I_i = 0$ for all $i < d$.

Consider now $i = d + j$ for some $j \geq 0$. Since $R_i = \bigoplus_{\mathbf{h} \in \mathbb{Z}_k^{n+1}} R_{i, \mathbf{h}}$, we focus on the dimension of each summand $R_{i, \mathbf{h}}$. Fix $\mathbf{h} \in \mathbb{Z}_k^{n+1}$.

We have seen that $I = (\psi_{\mathbf{g}} \mid \mathbf{g} \in \mathcal{G}_{2, n, D})$; hence, $I_{i, \mathbf{h}} = \text{Im}(\mu_{j, \mathbf{h}})$.

By Lemma 3.4, for $\text{wt}(\mathbf{h}) \geq d - j$, we know that $\mu_{j, \mathbf{h}}$ is surjective and then $I_{i, \mathbf{h}} = S_{i, \mathbf{h}}$; consequently, $R_{i, \mathbf{h}} = 0$. Moreover, by Lemma 2.3, we need to consider only $\mathbf{h} \in \mathbb{Z}_k^{n+1}$ such that $i - \text{wt}(\mathbf{h}) \in 2\mathbb{N}$ otherwise $S_{i, \mathbf{h}} = 0$ and consequently, $R_{i, \mathbf{h}} = 0$. Thus, we just need to consider \mathbf{h} in the set \mathcal{H}_j defined in the statement.

By Lemma 3.4, in that numerical assumptions, $\mu_{j, \mathbf{h}}$ is injective and then

$$\dim_{\mathbb{C}} I_{i, \mathbf{h}} = \sum_{\mathbf{g} \in \mathbb{Z}_k^{n+1}} \dim_{\mathbb{C}} S_{j, \mathbf{h} - \mathbf{g}} = \dim_{\mathbb{C}} S_j = \binom{n+j}{n},$$

or equivalently,

$$\dim_{\mathbb{C}} R_{i, \mathbf{h}} = \dim_{\mathbb{C}} S_{i, \mathbf{h}} - \binom{n+j}{n}.$$

□

Theorem 3.6. *The Hilbert function of the quotient ring R is given by:*

1. if $i < d$, $\text{HF}(R; i) = \binom{n+i}{n}$;
2. if $i = j + d$ with $j \geq 0$,

$$\text{HF}(R; i) = \sum_{\mathbf{h} \in \mathcal{H}_j} \dim_{\mathbb{C}} R_{i, \mathbf{h}} = \sum_{\substack{h < d-j \\ i-h \in 2\mathbb{N}}} \binom{n+1}{h} \left(\binom{n + \frac{i-h}{2}}{n} - \binom{n+j}{n} \right)$$

Proof. For $i < d$ it is trivial.

Consider $i = j + d$ with $j \geq 0$. First, we may observe that, by Lemma 3.5, whenever $\mathbf{h} \in \mathcal{H}_j$, the dimension of $R_{i, \mathbf{h}}$ depends only on the weight of \mathbf{h} . Indeed, considering $\mathbf{h} \in \mathcal{H}_j$ and denoting $h := \text{wt}(\mathbf{h})$, we get, by Lemma 2.3,

$$\dim_{\mathbb{C}} R_{i, \mathbf{h}} = \dim_{\mathbb{C}} S_{i-h, \mathbf{0}} - \binom{n+j}{n} = \binom{n + \frac{i-h}{2}}{n} - \binom{n+j}{n}.$$

To conclude our proof, we just need to observe that, fixed a weight h , we have exactly $\binom{n+1}{h}$ vectors $\mathbf{h} \in \mathbb{Z}_k^{n+1}$ with that weight. □

Corollary 3.7. $R_{2d-1} = 0$.

Proof. $R_{2d-1, \mathbf{h}} \neq 0$ if and only if $\text{wt}(\mathbf{h})$ is odd and $\text{wt}(\mathbf{h}) < 1$, so never. \square

In the following example, we explicit our algorithm in a particular case in order to help the reader in the comprehension of the theorem.

Example 3.8. Let's take $n + 1 = 4$, i.e. $S = \mathbb{C}[x_0, \dots, x_3]$, and $d = 5$. We compute the Hilbert function of the quotient $R = S/I_{2,3,5}$ where

$$I_{2,3,5} = ((x_0 \pm x_1 \pm x_2 \pm x_3)^5).$$

For $i < 5$, we have

$$\text{HF}(R; i) = \binom{3+i}{3}.$$

For $i = 5$ ($j = 0$), we have that $\mathcal{H}_0 = \{\mathbf{h} \mid \text{wt}(\mathbf{h}) = 1, 3\}$, hence

$$\begin{aligned} \text{HF}(R; 5) &= \sum_{\text{wt}(\mathbf{h})=1} \dim_{\mathbb{C}} R_{5, \mathbf{h}} + \sum_{\text{wt}(\mathbf{h})=3} \dim_{\mathbb{C}} R_{5, \mathbf{h}} = \\ &= \binom{4}{1} (\dim_{\mathbb{C}}(S_{4, \mathbf{0}}) - 1) + \binom{4}{3} (\dim_{\mathbb{C}}(S_{2, \mathbf{0}}) - 1) = \\ &= 4(10 - 1) + 4(4 - 1) = 36 + 12 = 48. \end{aligned}$$

For $i = 6$ ($j = 1$), we have that $\mathcal{H}_1 = \{\mathbf{h} \mid \text{wt}(\mathbf{h}) = 0, 2\}$, hence

$$\begin{aligned} \text{HF}(R; 6) &= \dim_{\mathbb{C}} R_{6, \mathbf{0}} + \sum_{\text{wt}(\mathbf{h})=2} \dim_{\mathbb{C}} R_{6, \mathbf{h}} = \\ &= (\dim_{\mathbb{C}}(S_{6, \mathbf{0}}) - 4) + \binom{4}{2} (\dim_{\mathbb{C}}(S_{4, \mathbf{0}}) - 4) = \\ &= (20 - 4) + 6(10 - 4) = 16 + 36 = 52. \end{aligned}$$

For $i = 7$ ($j = 2$), we have that $\mathcal{H}_2 = \{\mathbf{h} \mid \text{wt}(\mathbf{h}) = 1\}$, hence

$$\text{HF}(R; 7) = \sum_{\text{wt}(\mathbf{h})=1} \dim_{\mathbb{C}} R_{7, \mathbf{h}} = \binom{4}{1} (\dim_{\mathbb{C}}(S_{6, \mathbf{0}}) - 10) = 4(20 - 10) = 40.$$

For $i = 8$ ($j = 3$), we have that $\mathcal{H}_3 = \{\mathbf{0}\}$, hence

$$\text{HF}(R; 8) = \dim_{\mathbb{C}} R_{8, \mathbf{0}} = \dim_{\mathbb{C}}(S_{8, \mathbf{0}}) - 20 = 35 - 20 = 15.$$

For $i \geq 9$ ($j \geq 4$), we can easily see that $\mathcal{H}_j = \emptyset$. Thus, the Hilbert function is

i	0	1	2	3	4	5	6	7	8	9
$\text{HF}(R; i)$	1	4	10	20	35	48	52	40	15	-

With the following theorem, we are going to work on our result in order to compute more explicitly the Hilbert series in cases with small number of variables.

Theorem 3.9. *The Hilbert series of $R_{2,1,d}$ is given by $(1 - 2t^d + t^{2d})/(1 - t)^2$.
The Hilbert series of $R_{2,2,d}$, for $d \geq 2$ is given by*

$$\begin{aligned} \text{HS}(R_{2,2,d}; t) &= \frac{(1 - 4t^d + dt^{2d-1} + 3t^{2d} - dt^{2d+1})}{(1 - t)^3} = \\ &= \sum_{i=0}^{d-1} \binom{i+2}{2} t^i + \sum_{i=0}^{d-2} \left(\binom{d+i+2}{2} - 4 \binom{i+2}{2} \right) t^{d+i}. \end{aligned}$$

The Hilbert series of $R_{2,3,d}$, for $d \geq 3$ is given by

$$\begin{aligned} \text{HS}(R_{2,3,d}; t) &= \\ &= \frac{(1 - 8t^d + \binom{d}{2} t^{2d-2} + 4dt^{2d-1} - (d^2 - 7)t^{2d} - 4dt^{2d+1} + \binom{d+1}{2} t^{2d+2})}{(1 - t)^4} = \\ &= \sum_{i=0}^{d-1} \binom{i+3}{3} t^i + \sum_{i=0}^{d-3} \left(\binom{d+i+3}{3} - 8 \binom{i+3}{3} \right) t^{d+i} + \binom{d+1}{2} t^{2d-2}. \end{aligned}$$

Proof. Case $n + 1 = 2$. Simply, we have a complete intersection and it follows that the Hilbert series is $(1 - 2t^d + t^{2d})/(1 - t)^2$.

Case $n + 1 = 3$. From Lemma 3.4, we have that $[I_{2,2,d}]_{d+j} = S_j[I_{2,2,d}]_d$ for any $0 \leq j \leq d - 3$ since $\text{wt}(\mathbf{h}) \leq d - 3$ for all possible \mathbf{h} . Since $2d - 2$ is even, we get that $\text{wt}(\mathbf{h})$ should be even and then, $\text{wt}(\mathbf{h}) \leq 2 = d - (d - 2)$; thus, we get injectivity also in this degree. Now, from Theorem 3.6, we get that $\dim_{\mathbb{C}}([R_{2,2,d}]_{d+j}) = \dim_{\mathbb{C}}(S_{d+j}) - \#(\mathcal{H}_j) \cdot \binom{n+j}{n}$.

In our numerical assumption, it is clear that, for $0 \leq i \leq d - 3$, \mathcal{H}_i is exactly the half of all possible vectors in \mathbb{Z}_2^{n+1} , i.e. $\#(\mathcal{H}_i) = 2^n$; hence,

$$\text{HS}(R_{2,2,d}; t) = \sum_{i=0}^{d-1} \binom{i+2}{2} t^i + \sum_{i=0}^{d-2} \left(\binom{d+i+2}{2} - 4 \binom{i+2}{2} \right) t^{d+i}.$$

A simple calculation shows that $(1 - t)^3 \text{HS}(R_{2,2,d}; t) = (1 - 4t^d + dt^{2d-1} + 3t^{2d} - dt^{2d+1})$.

Case $n + 1 = 4$. From Lemma 3.4, since $\text{wt}(\mathbf{h}) \leq 4$ for all possible \mathbf{h} , we get that $[I_{2,3,d}]_{d+i} = S_i[I_{2,3,d}]_d$ for all $0 \leq i \leq d - 4$. Moreover, since $2d - 3$ is odd, we get that $\text{wt}(\mathbf{h})$ should be odd and consequently $\text{wt}(\mathbf{h}) \leq 3 = d - (d - 3)$; hence, we have injectivity also in this degree. Moreover, for all $0 \leq i \leq d - 3$, we get that \mathcal{H}_i is half of all possible vectors in \mathbb{Z}_2^{n+1} , i.e. \mathcal{H}_i has cardinality equal to 2^n .

Now, we just miss to compute the dimension of $[R_{2,3,d}]_{2d-2}$. By definition, the vectors $\mathbf{h} \in \mathcal{H}_{d-2}$ have to be odd, since $2d - 2$ is odd, and to satisfy the condition $\text{wt}(\mathbf{h}) < 2$; thus, we get only $\mathbf{h} = \mathbf{0}$ and $\#(\mathcal{H}_{d-2}) = 1$. Thus, by Theorem 3.6,

$$\begin{aligned} \dim_{\mathbb{C}}([R_{2,3,d}]_{2d-2}) &= \dim_{\mathbb{C}}([R_{2,3,d}]_{2d-2, \mathbf{0}}) = \dim_{\mathbb{C}}(S_{2d-2, \mathbf{0}}) - \binom{3+d-2}{3} = \\ &= \binom{d+2}{3} - \binom{d+1}{3} = \binom{d+1}{2}. \end{aligned}$$

Putting together our last observations, we get

$$\text{HS}(R_{2,3,d}; t) = \sum_{i=0}^{d-1} \binom{i+3}{3} t^i + \sum_{i=0}^{d-3} \left(\binom{d+i+3}{3} - 8 \binom{i+3}{3} \right) t^{d+i} + \binom{d+1}{2} t^{2d-2}.$$

A simple calculation shows that

$$(1-t)^4 \text{HS}(R_{2,3,d}; t) = \left(1 - 8t^d + \binom{d}{2} t^{2d-2} + 4dt^{2d-1} - (d^2 - 7)t^{2d} - 4dt^{2d+1} + \binom{d+1}{2} t^{2d+2} \right).$$

□

Remark 3.10. From the proof of Theorem 3.9, we can say something more also about the Hilbert series of $R_{2,n,d}$ even for more variables.

Assuming $d \geq n$, by using the same ideas as in the theorem above, we get that for all $0 \leq j \leq d - n$, the $(d+j)^{\text{th}}$ -coefficient of our Hilbert series is equal to

$$\text{HF}(R_{2,n,d}; d+j) = \binom{n+d+j}{n} - 2^n \binom{n+j}{n}.$$

Moreover, we get that, for any $d \geq 2$, $\mathcal{H}_{d-2} = \{\mathbf{0}\}$ and consequently,

$$\begin{aligned} \text{HF}(R_{2,n,d}; 2d-2) &= \dim_{\mathbb{C}}([R_{2,n,d}]_{2d-2}) = \dim_{\mathbb{C}}([R_{2,n,d}]_{2d-2, \mathbf{0}}) = \\ &= \dim_{\mathbb{C}}(S_{2d-2, \mathbf{0}}) - \binom{n+d-2}{n} = \binom{n+d-1}{n} - \binom{n+d-2}{n} = \\ &= \binom{n+d-2}{n-1}. \end{aligned}$$

Similarly, we have that, for any $d \geq 3$, $\mathcal{H}_{d-3} = \{h \in \mathbb{Z}_k^{n+1} \mid \text{wt}(h) = 1\}$, thus

$$\begin{aligned} \text{HF}(R_{2,n,d}; 2d-3) &= \dim_{\mathbb{C}}([R_{2,n,d}]_{2d-3}) = \sum_{\text{wt}(h)=1} \dim_{\mathbb{C}}([R_{2,n,d}]_{2d-3, h}) = \\ &= (n+1) \left[\dim_{\mathbb{C}}(S_{2d-2, \mathbf{0}}) - \binom{n+d-2}{n} \right] = \\ &= (n+1) \binom{n+d-2}{n-1}. \end{aligned}$$

Conjecture 1. $R_{2,n,d}$ is level algebra, i.e. $\text{Soc}(R_{2,n,d}) = [R_{2,n,d}]_{2d-2}$.

If so, from Remark 3.10, we would have that $\text{Soc}(R_{2,d,n})$ has dimension $\binom{n+d-2}{n-1}$.

3.2. The $k > 2$ case.

We would like to generalize our results for the cases $k > 2$. Inspired by Lemma 3.3, we conjecture the following behavior of the maps $\mu_{i, \mathbf{h}}$.

Conjecture 2. In the same notation as Definition 3.1, we have

1. $\mu_{i, \mathbf{h}}$ is injective if $\text{wt}(\mathbf{h}) \leq (k-1)(d-i)$;
2. $\mu_{i, \mathbf{h}}$ is surjective if $\text{wt}(\mathbf{h}) \geq (k-1)(d-i)$.

Following the same ideas as Lemma 3.5, from Conjecture 2 we would get the following results.

Conjecture 3. *In the same notation as above, we have*
if $i = j + D$ with $j \geq 0$, $R_{i,\mathbf{h}} \neq 0$ if and only if

$$\mathbf{h} \in \mathcal{H}_j := \{\mathbf{h}' \mid i - \text{wt}(\mathbf{h}') \in k\mathbb{N}, \text{wt}(\mathbf{h}') < d - j, \text{wt}(\mathbf{h}') \leq (k-1)(n+1)\};$$

moreover, if $\mathbf{h} \in \mathcal{H}_j$, then

$$\dim_{\mathbb{C}} R_{i,\mathbf{h}} = \dim_{\mathbb{C}}(S_{i,\mathbf{h}}) - \binom{n+j}{n}.$$

Proposition 3.11. *Conjecture 2 \implies Conjecture 3.*

Proof. Follow the proof of Theorem 3.6. □

Remark 3.12. From these conjectures, it would follow a direct generalization of the algorithm described in Example 3.8 to compute the Hilbert function of the quotient rings R . Trivially, we already know that, for $i < D$, since the ideal I has generators only in degree D ,

$$\text{HF}(R; i) = \binom{n+i}{n}.$$

For the cases $i = D + j$ with $j \geq 0$, from Conjecture 3, we would have

$$\text{HF}(R; i) = \sum_{\substack{h < (k-1)(d-j) \\ i-h \in k\mathbb{N}}} N_h \left(\binom{n + \frac{i-h}{k}}{n} - \binom{n+j}{n} \right);$$

where N_h is simply the number of vectors $\mathbf{h} \in \mathbb{Z}_k^{n+1}$ of weight $\text{wt}(\mathbf{h}) = h$. In order to compute the numbers N_h we may look at the following formula,

$$\sum_{h=0}^{(k-1)(n+1)} N_h x^h = (1 + x + \dots + x^{k-1})^{n+1} = \left(\frac{1-x^k}{1-x} \right)^{n+1};$$

from there, expanding the right hand side, we get, for all $h = 0, \dots, (k-1)(n+1)$,

$$N_h = \sum_{s=0}^{\lfloor \frac{h}{k} \rfloor} (-1)^s \binom{n+1}{s} \binom{n+h-ks}{n}.$$

Remark 3.13. From the conjectures, we would get also the extension of Corollary 3.7 in the $k > 2$ case, i.e.

$$[R_{k,n,d}]_{kd-1} = 0.$$

Indeed, with the same notation as above, let's take $j = d-1$. Thus, to compute the Hilbert function of the quotient in position $kd-1$ we should compute the set \mathcal{H}_{d-1} , i.e. the set of $\mathbf{h} \in \mathbb{Z}_k^{n+1}$ satisfying the following conditions:

$$kd-1 - \text{wt}(\mathbf{h}) \in k\mathbb{Z}, \quad \text{wt}(\mathbf{h}) < (k-1)(d-d+1) = k-1.$$

From the first condition, we get that $\text{wt}(\mathbf{h}) \in (k-1) + k\mathbb{Z}_{\geq 0}$ which is clearly in contradiction with the second condition above. Thus, \mathcal{H}_{d-1} is empty and $\text{HF}(R; kd-1) = 0$.

Example 3.14. Let's give one explicit example of the computations in order to clarify the algorithm.

We consider the following parameters: $k = 4$, $n = 2$, $d = 8$. Thus we have $D = 24$. Let's compute, for example, the Hilbert function of the corresponding quotient ring in degree $i = 28$, i.e. $j = 4$. Via the support of a computer algebra software, as CoCoA5 [5] or Macaulay2 [6] and the implemented functions involving Gröbner basis, one can see that

$$\text{HF}(R; 28) = 195.$$

Let's apply our algorithm to compute the same number. First, we need to write down the vector N where, for $l = 0 \dots (k-1)(n+1)$, $N_l := \#\{\mathbf{h} \in \mathbb{Z}_k^{n+1} \mid \text{wt}(\mathbf{h}) = l\}$. In our numerical assumptions we have

$$N = (N_0, \dots, N_9) = (1, 3, 6, 10, 12, 12, 10, 6, 3, 1).$$

Now, we need to compute the vector H where we store all the possible weights for the vectors $\mathbf{h} \in \mathcal{H}_4$, i.e. all the number $0 \leq h \leq 9$ s.t. the following numerical conditions hold,

$$28 - h \in 4\mathbb{Z}, \quad h < (k-1)(d-j) = 12;$$

thus, $H = (H_0, H_1, H_2) = (0, 4, 8)$. Hence, we can finally compute $\text{HF}(R; 28, \mathbf{h})$ for each $\mathbf{h} \in \mathcal{H}_4$. From our formula, it is clear that those numbers depend only on the weight of \mathbf{h} ; thus, we just need to consider each single element in the vector H .

Assume $\text{wt}(\mathbf{h}) = 0$. We get,

$$R_0 := \text{HF}(R; 28, \mathbf{0}) = \dim_{\mathbb{C}} S_{28, \mathbf{0}} - \binom{n+j}{n} = 36 - 15 = 21;$$

Similarly, we get: if $\text{wt}(\mathbf{h}) = 4$,

$$R_4 := \text{HF}(R; 28, \mathbf{h}) = \dim_{\mathbb{C}} S_{24, \mathbf{0}} - \binom{n+j}{n} = 28 - 15 = 13;$$

and, if $\text{wt}(\mathbf{h}) = 8$,

$$R_8 := \text{HF}(R; 28, \mathbf{h}) = \dim_{\mathbb{C}} S_{20, \mathbf{0}} - \binom{n+j}{n} = 21 - 15 = 6.$$

Now, we are able to compute the Hilbert function in degree 28.

$$\begin{aligned} \text{HF}(R; 28) &= N_{H_0} R_{H_0} + N_{H_1} R_{H_1} + N_{H_2} R_{H_2} = \\ &= 21 + 12 \cdot 13 + 3 \cdot 6 = 21 + 156 + 18 = 195. \end{aligned}$$

Algorithm 3.15. We show the algorithm implemented by using CoCoA5 programming language, see [5]. As we have seen in the previous section, in the case $k > 2$, the algorithm is just conjectured. However, as we will see in Section 4.11, we made several computer experiments supporting our conjectures. Here is the CoCoA5 script of our algorithm based on Theorem 3.6 and Remark 3.12.

```

-- 1) Input parameters K, N, D;
K := ;
N := ;
D := ;
DD :=(K-1)*D;

-- HF is the vector representing the Hilbert function
-- of the quotient ring;
HF :=[];

-- 2) Input vector NN where NN[I] counts the number of vectors
-- in ZZ^{n+1} modulo K of weight I;
Foreach H In 0..((N+1)*(K-1)) Do
  M := 0;
  Foreach S In 0..(Div(H,K)) Do
    M := M+(-1)^S*Bin(N+1,S)*Bin(N+H-K*S,N);
  EndForeach;
  Append(Ref NN,M);
EndForeach;

-- 3) Compute the Hilbert Function:
-- in degree <DD:
Foreach L In 0..(DD-1) Do
  Append(Ref HF,Bin(N+L,N));
EndForeach;

-- in degree =DD,...,K*D-1:
Foreach J In 0..(D-2) Do
  I:=DD+J;
  H:=[];
  M:=0;

  Foreach S In 0..I Do
    If Mod(I-S,K)=0 Then
      If S<(K-1)*(D-J) Then
        If S<(K-1)*(N+1)+1 Then
          Append(Ref H,S);
          M:=M+1;
        EndIf;
      EndIf;
    EndIf;
  EndForeach;

  HH:=0;
  If M>0 Then
    Foreach S In 1..M Do
      HH:=HH+NN[H[S]+1]*(Bin(N+Div(I-H[S],K),N)-Bin(N+J,N));
    EndForeach;
  EndIf;

```

```

Append(Ref HF,HH);
EndForeach;

```

```

-- 4) Print the Hilbert function:
HF;

```

Remark 3.16. In the $k = 2$ case, our algorithm, which is proved to be true by Theorem 3.6, works very fast even with large values of n and d , e.g. $n, d \sim 300$; cases that the computer algebra softwares, by involving the computation of Gröbner basis, cannot do in a reasonable amount of time and memory.

As regards the $k > 2$ case, with the support of computer algebra software Macaulay2 and its implemented function to compute Hilbert series of quotient rings, we have checked that our numerical algorithm produces the right Hilbert function for two and three variables for low k and d . Moreover, in Section 4, we study the schemes of fat points related to our power ideals and our results on their Hilbert series, will support Conjecture 3 in much more cases. With the support of the computer algebra software CoCoA5, we have checked that the conjectured algorithm gives the correct Hilbert function for all

$$n + 1 = 3, 4, 5, \quad k = 3, 4, 5 \text{ and } d \leq 150.$$

4. Hilbert function of ξ -points in \mathbb{P}^n

As we said in the introduction, there is a close connection between power ideals and many different theories of mathematics. In this section, we see how our results can give important informations on particular arrangement of *fat points* in projective spaces. In particular, we consider schemes of fat points with support on the k^n points of type $[1 : \xi^{g_1} : \dots : \xi^{g_n}] \in \mathbb{P}^n$ where ξ is a fixed primitive k^{th} -root of unity and $0 \leq g_i \leq k - 1$ for all $i = 1, \dots, n$. Thanks to our results in Section 3.1 and Section 3.2, we have been able to completely understand these schemes of fat points in terms of generators, Hilbert series and Betti numbers.

For any point P in the projective space \mathbb{P}^n we associate the prime ideal $\wp \subset \mathbb{C}[x_0, \dots, x_n]$ which consists of the ideal of all homogeneous polynomials vanishing at the point P , namely of all the hypersurfaces passing through the point P . A **fat point** supported at P is the non-reduced 0-dimensional scheme associated to some power \wp^d of the prime ideal. That scheme is usually denoted with dP and consists of all homogeneous polynomials such that all differentials of degree $\leq d - 1$ vanish at the point P . From a geometrical point of view, it is the ideal of all hypersurfaces of \mathbb{P}^n which are singular at P with multiplicity d .

In general, a **scheme of fat points** $\mathbb{X} = dP_1 + \dots + dP_g$ is the 0-dimensional scheme in \mathbb{P}^n associated to the ideal $I^{(d)} = \wp_1^d \cap \dots \cap \wp_g^d$ where the \wp_i 's are the ideals associated to the points P_i 's for all $i = 1, \dots, g$, respectively. That ideal is, from an algebraic point of view, the d^{th} -symbolic power of the ideal $I = \wp_1 \cap \dots \cap \wp_g$.

Macaulay duality

The relation between power ideals and fat points is given by the *Macaulay duality* or *Apolarity Lemma*. For all positive integer d , we consider the power

ideal $I_d = (L_1^d, \dots, L_g^d) \subset S = \mathbb{C}[x_0, \dots, x_n]$ where $L_i = a_0^{(i)}x_0 + \dots + a_n^{(i)}x_n$, for all $i = 1, \dots, g$. We associate to each linear form L_i the projective points $P_i = [a_0^{(i)} : \dots : a_n^{(i)}] \in \mathbb{P}^n$ and its associated prime ideal \wp_i . Let $I = \wp_1 \cap \dots \cap \wp_g$.

The Macaulay duality connects the Hilbert function of the quotients $R_d = S/I_d$ with the Hilbert function of the schemes of fat points associated to the symbolic powers of I , see [1] or [7].

Theorem 4.1 (MACAULAY DUALITY). *For all $m \geq d$, we have that*

$$\text{HF}(I^{(d)}, m) = \text{HF}(R_{m-d+1}, m).$$

4.1. *The $k = 2$ case.*

We begin by considering our class of power ideals in the $k = 2$ case, where the generators of the ideal I_d are the d^{th} -powers of the 2^n linear forms of type $L = x_0 \pm x_1 \pm \dots \pm x_n$. In Section 3.1, we have described a easy algorithm to compute the Hilbert function of the quotient rings $R_d = S/I_d$, thus, via Macaulay duality, we can apply our computations to get the Hilbert function of schemes of fat points supported at all (± 1) -points of \mathbb{P}^n , namely the 2^n points of the type $[1 : \pm 1 : \dots : \pm 1]$. We will see later that the results for these arrangement of points can be directly extended to the $k > 2$ case.

Proposition 4.2. *Let $I^{(d)}$ be the ideal associated to the scheme of d -fat points supported on the (± 1) -points of \mathbb{P}^n . Then,*

$$\text{HF}(S/I^{(d)}, m) = \begin{cases} \binom{n+m}{n} & \text{for } m \leq 2d - 1 \\ \binom{n+2d}{n} - \binom{d+n-1}{n-1} & \text{for } m = 2d \\ \binom{n+2d+1}{n} - (n+1)\binom{d+n-1}{n-1} & \text{for } m = 2d + 1 \\ 2^n \binom{n+d-1}{n} & \text{for } m \geq 2d + n - 2 \end{cases}$$

Proof. By Corollary 3.7, we know that $\text{HF}(R_{m-d+1}, m) = 0$ for all m satisfying the inequality $m \geq 2(m-d+1) - 1$ or, equivalently $m \leq 2d - 1$; moreover, by Remark 3.10, we have that $\text{HF}(R_{d+1}, 2d) = \binom{n+d-1}{n-1}$, $\text{HF}(R_{d+2}, 2d+1) = (n+1)\binom{n+d-1}{n-1}$ and $\text{HF}(R_{m-d+1}, m) = \binom{n+m}{n} - 2^n \binom{n+d-1}{n}$ for $m \leq 2(m-d+1) - n$, or equivalently, $m \geq 2d + n - 2$. By Macaulay duality, we are done. \square

Remark 4.3. This result tell us that the ideal $I^{(d)}$ is generated in degree $\geq 2d$ and, in particular, with $\binom{d+n-1}{n-1}$ generators in degree $2d$. Thanks to the geometrical meaning of the symbolic power $I^{(d)}$, we can easily find the generators.

We may observe that we have exactly n pairs of hyperplanes which split our 2^n points. Namely, for any variable except x_n , we can consider the hyperplanes

$$H_i^+ = \{x_i + x_n = 0\} \text{ and } H_i^- = \{x_i - x_n = 0\}, \text{ for all } i = 0, \dots, n-1.$$

It is clear that, for all i , half of our (± 1) -points lie on H_i^+ and half on H_i^- . Consequently, we have n quadrics passing through our points exactly once, i.e. $Q_i = H_i^+ H_i^- = x_i^2 - x_n^2$, for all $i = 0, \dots, n-1$.

Now, we want to find the generators of $I^{(d)}$, hence we want to find hyper-surfaces passing through our points with multiplicity d and we can consider,

for example, all the *monomials* of degree d constructed with these quadrics $\mathcal{Q}_0, \dots, \mathcal{Q}_{n-1}$, i.e. the degree $2d$ forms

$$\mathcal{G}_1 := \mathcal{Q}_0^d, \mathcal{G}_2 := \mathcal{Q}_0^{d-1} \mathcal{Q}_1, \mathcal{G}_3 := \mathcal{Q}_0^{d-1} \mathcal{Q}_2, \dots, \mathcal{G}_N := \mathcal{Q}_{n-1}^d,$$

where $N = \binom{n+d-1}{n-1}$. We can actually prove that they generate the part of degree $2d$ of $I^{(d)}$ as a \mathbb{C} -vector space. Since the number of \mathcal{G}_i 's is equal to the dimension of $[I^{(d)}]_{2d}$ computed in Proposition 4.2, it is enough to prove the following statement.

Claim. The \mathcal{G}_i 's are linearly independent over \mathbb{C} .

Proof of the Claim. We prove it by double induction over the number of variables n and the degree d . For two variables, i.e. $n = 1$, we have that the dimension of $[I^{(d)}]_{2d}$ is equal to 1 for all d and then, $\mathcal{G}_1 = \mathcal{Q}_0^d$ is the unique generator. For $n > 1$, we consider first the $d = 1$ case. Assume to have a linear combination

$$\alpha_0 \mathcal{Q}_0 + \dots + \alpha_{n-1} \mathcal{Q}_{n-1} = \alpha_0(x_0^2 - x_n^2) + \dots + \alpha_{n-1}(x_{n-1}^2 - x_n^2) = 0.$$

Specializing on the hyperplane $H_0^- = \{x_0 = x_n\}$, we reduce the linear combination in one variable less and, by induction, we have $\alpha_i = 0$ for all $i = 1, \dots, n-1$; consequently, also $\alpha_0 = 0$.

Assume to have a linear combination for $d \geq 2$, namely

$$\begin{aligned} \alpha_1 \mathcal{G}_1 + \alpha_2 \mathcal{G}_2 + \dots + \alpha_N \mathcal{G}_N &= \\ &= \alpha_1(x_0^2 - x_n^2)^d + \alpha_2(x_0^2 - x_n^2)^{d-1}(x_1^2 - x_n^2) + \dots + \alpha_N(x_0^2 - x_n^2)^d = 0. \end{aligned}$$

By specializing again on the hyperplane $H_0^- = \{x_0 = x_n\}$, we get a linear combination in the same degree but with one variable less and, by induction over n , we have that $\alpha_i = 0$ for all i where the definition \mathcal{G}_i doesn't involve $(x_0^2 - x_n^2)^d$.

Thus, we remain with a linear combination of type

$$(x_0^2 - x_n^2) [\alpha_0 \mathcal{Q}_0^{d-1} + \alpha_1 \mathcal{Q}_0^{d-2} \mathcal{Q}_1 + \dots + \alpha_m \mathcal{Q}_{n-1}^{d-1}] = 0;$$

by induction over d , we are done. \square

Hence, we can consider the ideal $J_d = (x_0^2 - x_n^2, \dots, x_{n-1}^2 - x_n^2)^d$. It is clearly contained in $I^{(d)}$ but, *a priori*, it could be smaller.

In order to show that the equality holds and that $I^{(d)}$ is minimally generated by the \mathcal{G}_i 's, we start by studying the Hilbert series of the ideal J_d .

Lemma 4.4. *Let $T_d = \mathbb{C}[x_0, \dots, x_n]/J_d$, where $J_d = (x_0^2 - x_n^2, \dots, x_{n-1}^2 - x_n^2)^d$, then the Hilbert series is*

$$\text{HS}(T_d; t) = \frac{1 + \sum_{i=1}^n (-1)^i \beta_i t^{2d+2(i-1)}}{(1-t)^{n+1}},$$

where $\beta_i := \beta_{i, 2d+2(i-1)} = \binom{d+i-2}{i-1} \binom{d+n-1}{n-i}$, for all $i = 1, \dots, n$, and the multiplicity is $e(T_d) = 2^n \binom{d+n-1}{n}$.

Proof. The quotient T_d is a 1-dimensional Cohen-Macaulay ring and x_n is a non-zero divisor. Thus, we have that T_d and the quotient $T_d/(x_n)$ have the same Betti numbers; moreover, we have that

$$T_d/(x_n) = \mathbb{C}[x_0, \dots, x_{n-1}]/(x_0^2, \dots, x_{n-1}^2)^d,$$

and the resolution of those quotients are very well known. The quotient ring $\mathbb{C}[x_0, \dots, x_n]/(x_0, \dots, x_{n-1})^d$ has a *pure resolution* of type $(d, d+1, \dots, d+n-1)$ and its Betti numbers and multiplicity are expressed with an explicit formula, see Theorem 4.1.15 in [8].

Thus, $T_d/(x_n)$ has a *pure resolution of type* $(2d, 2d+2, 2d+4, \dots, 2d+2(n-1))$, i.e.

$$\dots \longrightarrow S(-2d-4)^{\beta_{3,2d+4}} \longrightarrow S(-2d-2)^{\beta_{2,2d+2}} \longrightarrow S(-2d)^{\beta_{1,2d}} \longrightarrow 0,$$

where S is the graded polynomial ring $\mathbb{C}[x_0, \dots, x_{n-1}]$ and $S(-i)$ is its i^{th} -shifting, i.e. $[S(-i)]_j := S_{j-i}$. Moreover, the Betti numbers and the multiplicity of the quotient are given by the following formulas,

$$\begin{aligned} \beta_i &:= \beta_{i,2d+2(i-1)} = (-1)^{i+1} \prod_{j \neq i} \frac{d+j-1}{j-i} = \\ &= \frac{(-1)^{i+1} d(d+1) \cdots (d+i-2)}{(-1)^{i-1} (i-1)!} \cdot \frac{(d+i) \cdots (d+n-1)}{(n-i)!} = \\ &= \binom{d+i-2}{i-1} \binom{d+n-1}{n-i}; \\ e(T_d) &= \frac{1}{n!} \prod_{i=1}^n (2d+2(i-1)) = 2^n \binom{d+n-1}{n}. \end{aligned}$$

From the Betti numbers, we can easily get the Hilbert series of $T_d = S/J_d$,

$$\text{HS}(T_d; t) = \frac{1 + \sum_{i=1}^n (-1)^i \beta_i t^{2d+2(i-1)}}{(1-t)^{n+1}}.$$

□

Corollary 4.5. *Let $T_d = \mathbb{C}[x_0, \dots, x_n]/J_d$, where $J_d = (x_0^2 - x_n^2, \dots, x_{n-1}^2 - x_n^2)^d$, then*

$$\text{HF}(T_d, m) = \begin{cases} \binom{n+m}{n} & \text{for } m \leq 2d-1 \\ \binom{n+2d}{n} - \binom{d+n-1}{n-1} & \text{for } m = 2d \\ \binom{n+2d+1}{n} - (n+1) \binom{d+n-1}{n-1} & \text{for } m = 2d+1 \\ 2^n \binom{n+d-1}{n} & \text{for } m \gg 0 \end{cases}$$

Proof. The values of the Hilbert function for $m \leq 2d+1$ follow directly by extending the Hilbert series computed in Lemma 4.4, recalling that $\frac{1}{(1-t)^{n+1}} = \sum_{i \geq 0} \binom{n+i}{n} t^i$. Moreover, since T_d is a 1-dimensional CM ring, we have that its Hilbert function is eventually constant and equal to the multiplicity. □

Now, we are able to complete our study of the ideal of fat points with support on the (± 1) -points in \mathbb{P}^n and prove the Theorem 1.3 for those points.

Theorem 4.6. *Let $I^{(d)}$ be the ideal associated to the scheme of fat points of multiplicity d and support on the 2^n points $[1 : \pm 1 : \dots : \pm 1] \in \mathbb{P}^n$. The generators are given by the monomials of degree d made with the n quadrics $\mathcal{Q}_i = x_i^2 - x_n^2$, for all $i = 0, \dots, n-1$, and the Hilbert series is*

$$\text{HS}\left(S/I^{(d)}; t\right) = \frac{1 + \sum_{i=1}^n (-1)^i \beta_i t^{2d+2(i-1)}}{(1-t)^{n+1}},$$

where the Betti numbers are given by

$$\beta_i := \beta_{i, 2d+2(i-1)} = \binom{d+i-2}{i-1} \binom{d+n-1}{n-i}, \quad \text{for } i = 1, \dots, n.$$

Proof. Let's write $I^{(d)} = J_d + J$ where $J_d = (\mathcal{Q}_0, \dots, \mathcal{Q}_{n-1})^d$. From Lemma 4.4, it is enough to show that $J = 0$. We consider the quotient $T_d = S/(I^{(d)} + (x_n)) = \mathbb{C}[x_0, \dots, x_{n-1}]/((x_0^2, \dots, x_{n-1})^d + \bar{J})$ and the exact sequence

$$0 \longrightarrow \text{Ann}(x_n) \longrightarrow S/I^{(d)} \xrightarrow{\cdot x_n} S/I^{(d)} \longrightarrow T_d \longrightarrow 0.$$

Consequently, we get

$$\text{HS}(T_d; t) = (1-t)\text{HS}(S/I^{(d)}; t) + \text{HS}(\text{Ann}(x_n); t).$$

Since $S/I^{(d)}$ is 1-dimensional ring, we have that $\text{HS}(S/I^{(d)}; t) = \frac{h(t)}{(1-t)}$ and the multiplicity is given by $e(S/I^{(d)}) = h(1)$. Thus, the multiplicity of T_d is given by

$$e(T_d) = h(1) + \text{HS}(\text{Ann}(x_n); 1) \geq e(S/I^{(d)}) = 2^n \binom{d+n-1}{n}; \quad (1)$$

moreover, the equality holds if and only if x_n is a non-zerodivisor of T_d . On the other hand, we have that $T_d = \mathbb{C}[x_0, \dots, x_{n-1}]/(x_0^2, \dots, x_{n-1})^d + \bar{J}$ and consequently, by Lemma 4.4, we have

$$e(T_d) \leq e(\mathbb{C}[x_0, \dots, x_{n-1}]/(x_0^2, \dots, x_{n-1})^d) = 2^n \binom{d+n-1}{n}; \quad (2)$$

where equality holds if and only if $\bar{J} = 0$. From (1) and (2), we can conclude that

- x_n is a non-zerodivisor for $T_d = S/I^{(d)}$;
- $\bar{J} = 0$.

Now, let's assume $J \neq 0$ and take a non-zero element $f \in J$ of minimal degree in J . Then, since $\bar{J} = 0$, we get that $f = x_n \cdot g$, for some g , thus we have $x_n \cdot g = 0$ in T_d . This contradicts that x_n is a non-zerodivisor in T_d , since $g \notin J$ because of minimality of f in J and $g \notin J_d$ because f is not. \square

Remark 4.7. In the last decades, the study of the behavior between *symbolic* and *regular powers* of homogeneous ideals involved many mathematicians and different areas. By definition, we always have the inclusion $I^m \subset I^{(m)}$, but the

equality is not always true. Consequently, people started to study *containment problems*, as in [9] and [10]. In [11], the author showed that for any $c < n$, there exists an ideal of points in \mathbb{P}^n such that $I^{(m)} \not\subset I^r$ for some $m > cr$. In [12], there is a list of open conjectures regarding this containment problems. The authors showed also that all the conjectures hold in case of equality between symbolic and regular powers $I^{(m)} = I^m$ for any m .

Our ideals of points in \mathbb{P}^n satisfy always the equality between symbolic and regular powers; consequently, they satisfy all the conjectures listed in [12].

Even from the point of view of *Gröbner basis*, our result is very useful. Fixed an ordering on the variables, a Gröbner basis for the ideal I is simply a set of generators such that their *initial terms* generate the *initial ideal* $\text{in}(I)$; see e.g. [13]. We recollect these properties in the following.

Corollary 4.8. *Let $I^{(d)}$ be the ideal of fat points of multiplicity d supported on the (± 1) -points of \mathbb{P}^n . Then, we have the equality between $I^{(d)} = I^d$. Moreover, for any ordering such that $x_n > x_i$ for all $i = 0, \dots, n-1$, the set of generators given in Theorem 4.6 is actually a Gröbner basis for $I^{(d)}$.*

Proof. It follows directly from Theorem 4.6, since we have that

$$I = I^{(1)} = (x_0^2 - x_k^2, \dots, x_{n-1}^2 - x_n^2).$$

Moreover, considering the \mathcal{G}_i 's, i.e. the set of generators obtained by taking all the possible monomial of degree d in the quadrics $x_i - x_n$, for all $i = 0, \dots, n-1$, we have that their leading terms generate the initial ideal, i.e. they are a Gröbner basis. Indeed, we clearly have the inclusion

$$(\text{in}(\mathcal{G}_i)) \subset \text{in}(I);$$

but, we also have that the left hand side is exactly $(\text{in}(\mathcal{G}_i)) = (x_0^2, \dots, x_{n-1}^2)^d$, which has the same Hilbert function of I , as we have seen in the proof of Theorem 4.6, and consequently the same Hilbert function of $\text{in}(I)$. Hence, the equality holds. \square

4.2. The $k > 2$ case.

Let ξ be a k^{th} -root of unity and consider the ideal $I_k^{(d)}$ corresponding to the scheme of fat points of multiplicity d and support on the k^n ξ -points of type $[1 : \xi^{g_1} : \dots : \xi^{g_n}] \in \mathbb{P}^n$ with $0 \leq g_i \leq k-1$, for all $i = 1, \dots, n$.

In Section 3.2, we have considered the power ideals $I_{n,k,d}$ related to those points where the powers were only multiples of $(k-1)$. Thus, we cannot hope to get the Hilbert series of our scheme of fat points directly from our previous results on the Hilbert series of $R_{n,k,d} = S/I_{n,k,d}$. However, we can easily observe the following,

$$\text{HF}(I^{(d)}, kd-1) = \text{HF}(R_{n,k,d}, kd-1);$$

from Remark 3.13, we get that, assuming true the Hilbert function of $R_{k,d}$ conjectured, the ideal $I_k^{(d)}$ should be generated *at least* in degree kd . Thus, inspired by the $k=2$ case, we can actually claim that $I_k^{(d)}$ is nonzero in degree kd . Indeed, we have that, for any variable x_0, \dots, x_{n-1} , we can consider the k hyperplanes

$$H_i^0 = \{x_i - x_n = 0\}, \quad H_i^1 = \{x_i - \xi x_n = 0\}, \dots, \quad H_i^{k-1} = \{x_i - \xi^{k-1} x_n = 0\};$$

these hyperplanes divide the k^n points in k distinct groups of k^{n-1} points; thus, their products give a set of degree k forms which vanish with multiplicity 1 at each point, i.e.

$$\mathcal{Q}_i = H_i^0 \cdot H_i^1 \cdots H_i^{k-1} = x_i^k - x_n^k, \text{ for all } i = 0, \dots, n-1.$$

Consequently, we get

$$J_{k,d} = (\mathcal{Q}_0, \mathcal{Q}_1, \dots, \mathcal{Q}_{n-1})^d \subset I_k^{(d)}.$$

Now, by using the same ideas as for the $k = 2$ case, we can get the analogous of Lemma 4.4 and Theorem 4.6 for all $k \geq 2$ and consequently we get the following general result.

Theorem 4.9. *Let $I_k^{(d)}$ be the ideal associated to the scheme of fat points of multiplicity d and support on the k^n ξ -points $[1 : \xi^{g_1} : \dots : \xi^{g_n}] \in \mathbb{P}^n$ for $0 \leq g_i \leq k-1$. The generators are given by the monomials of degree d made with the n forms of degree k $\mathcal{Q}_i = x_i^k - x_n^k$, for all $i = 0, \dots, n-1$ and the Hilbert series is*

$$\text{HS} \left(S/I_k^{(d)}; t \right) = \frac{1 + \sum_{i=1}^n (-1)^i \beta_i t^{kd+k(i-1)}}{(1-t)^{n+1}},$$

where the Betti numbers are given by

$$\beta_i := \beta_{i, kd+k(i-1)} = \binom{d+i-2}{i-1} \binom{d+n-1}{n-i}, \text{ for } i = 1, \dots, n.$$

Remark 4.10. Moreover, similarly as for Corollary 4.8, we have that

- $I_k^{(d)} = I_k^d$,
- the set of generators given in the theorem above, is a Gröbner basis.

Remark 4.11. Since we have explicitly computed the Hilbert series of ξ -points in \mathbb{P}^n , by using again Macaulay duality, we can go back to look at the Hilbert series of the power ideals $I_{n,k,d}$. In particular, we can check that our Conjecture 3 holds in a lot of cases.

Let $R_{n,k,d}$ be the quotient ring $S/I_{n,k,d}$ where $I_{n,k,d}$ is the power ideal generated by all the $(x_0 + \xi^{g_1}x_1 + \dots + \xi^{g_n}x_n)^{(k-1)d}$ with $0 \leq g_i \leq k-1$ for all $i = 1, \dots, n$; and let $I_k^{(d)}$ be the ideal associated to the scheme of fat points of multiplicity d and support on the ξ -points of \mathbb{P}^n .

Now, we have seen in Section 3.2 that, since $I_{n,k,d}$ is generated in degree $(k-1)d$ and generate the whole space in degree $kd-1$, the Hilbert function of $R_{n,k,d}$ has to be computed only in the degrees $i = (k-1)d + j$, with $j = 0, \dots, d-2$. In that degrees, by Macaulay duality, we get

$$\text{HF}(R_{n,k,d}; i) = \text{HF} \left(I_k^{(j+1)}; i \right).$$

From Theorem 4.9, we can explicitly compute this Hilbert function, i.e. for all $j = 0, \dots, d-2$,

$$\begin{aligned} \text{HF}(R_{n,k,d}; i) &= \tag{3} \\ &= \sum_{\substack{s \in \mathbb{N} \\ s \leq \frac{k-1}{k}(d-j)}} (-1)^{s+1} \binom{n + (k-1)(d-j) - ks}{n} \binom{j+s-1}{s-1} \binom{j+n}{n-s}. \end{aligned}$$

In Section 3.2, we conjectured an extension of our formula for the Hilbert series of the quotient $R_{n,k,d}$ based on a \mathbb{Z}_k^{n+1} -grading on the polynomial ring. We may recall the formula conjectured: for all $j = 0, \dots, d-2$,

$$\text{HF}(R_{n,k,d}; i) = \sum_{\substack{h < (k-1)(d-j) \\ i-h \in k\mathbb{N}}} N_h \left(\binom{n + \frac{i-h}{k}}{n} - \binom{n+j}{n} \right); \quad (4)$$

where N_h is simply the number of vectors $\mathbf{h} \in \mathbb{Z}_k^{n+1}$ of weight $\text{wt}(\mathbf{h}) = h$, see Remark 3.12. In order to show that formula (4) is right and then to prove Conjecture 3, we should show that the right hand side is equal to the right hand side of formula (3).

Proposition 4.12. *Assuming $n = 1$, i.e. in the two variables case, the formulas (3) and (4) are equal and Conjecture 3 is true.*

Proof. For any k and d , the unique non-zero addend is the one for $s = 1$; thus,

$$(3) = 1 + (k-1)(d-j) - k.$$

Now, we look at formula (4). First of all we may observe that, for $n = 1$, the number of vectors in \mathbb{Z}_k^2 with fixed weight h can be computed very easily, indeed

$$N_h = \begin{cases} h+1 & \text{for } 0 \leq h \leq k-1; \\ 2k - (h+1) & \text{for } k \leq h \leq 2(k-1). \end{cases}$$

Thus, any $i = (k-1)d + j$ can be written as $ck + r$ for some positive integers c, r with $0 \leq r \leq k-1$ and then, we get

$$\begin{aligned} (4) &= N_r(1+c-(j+1)) + N_{r+k}(1+(c-1)-(j+1)) = \\ &= (r+1)(1+c-(j+1)) + (k-r-1)(1+(c-1)-(j+1)) = \\ &= \cancel{(r+1)c} + r + 1 - \cancel{(r+1)(j+1)} + kc - \cancel{(r+1)c} - kj - k + \cancel{(r+1)(c+1)}; \end{aligned}$$

moreover, recalling that $i = ck + r = (k-1)d + j$, we finally get

$$(4) = 1 + (k-1)d + j - kj - k = 1 + (k-1)(d-j) - k.$$

□

Remark 4.13. With similar, but longer and more intricate arguments as for Proposition 4.2, we have been able to check also the case $n+1 = 3$. Unfortunately, we have been not able to prove that the two expressions given in (3) and (4) give the same numerical value for any possible parameters (k, n, d) . With the support of a computer, by implementing with the CoCoA5 language those formulas, we have been able to check all the cases with $n, k \leq 20$, $d \leq 150$.

Algorithm 4.14. Here is the implementation of the formula (3) by using CoCoA language, for the formula (4), we have used the algorithm described in Algorithm 3.15.

```

-- 1) Input of the parameters K, N, D;
K := ;
N := ;
D := ;
DD := (K-1)*D;

-- HF will be the vector containing the relevant part of
-- the Hilbert function, i.e. from (K-1)D to KD-2;
HF := [];

-- 2) Compute the Hilbert function;
Foreach J In 0..(D-2) Do
  B := 0;
  KK := (K-1)*(D-J)/K;
  Foreach S In 1..N Do
    If S <= KK Then
      B :=
      B+(-1)^(S+1)*Bin(N+(K-1)*(D-J)-K*S,N)*Bin(J+S-1,S-1)*Bin(J+N,N-S);
    EndIf;
  EndForeach;

  Append(Ref HF , B );
EndForeach;

-- 3) Print the Hilbert function;
HF;

```

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