Direct Images of Locally Constant Sheaves on Complements to Plane Line Arrangements

Iara Cristina Alvarinho Gonçalves
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Abstract

On the complement $X = \mathbb{C}^2 - \bigcup_{i=1}^{n} L_i$ (where each $L_i$ is a line passing through the origin) to a plane line arrangement $\bigcup_{i=1}^{n} L_i \subset \mathbb{C}^2$, a locally constant sheaf of complex vector spaces is given by a multi-index $\alpha \in \mathbb{C}^n$. Using the description of Mac-Pherson and Vilonen ([11] and [12]) we obtain a criterion for the irreducibility of the direct image $R^j_* L_\alpha$ as a perverse sheaf, where $j : X \rightarrow \mathbb{C}^2$ is the canonical inclusion.
Sammanfattning

På komplementet till ett centralt linjearrangemang i komplexa planet $X = \mathbb{C}^2 - \cup L_i$ (där $L_i$, $i = 1, ..., n$ är linjer genom origo och $\cup L_i$ själva linje-arrangemanget), är en lokalt konstant kärve av komplexa vektorrum av dimension 1 $L_\alpha$ bestämd av ett multi-index $\alpha \in \mathbb{C}^n$. Med hjälp av MacPhersons och Vilonens beskrivning av perversa kärvar (11 och 12) ges i denna avhandling ett kriterium i termer av $\alpha$ för när $Rj_* L_\alpha$ är irreducibel som pervers kärve, där $j : X \to \mathbb{C}^2$ är den kanoniska inklusionen.
Abstract

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1. Introduction

The aim of this work is to analyze the irreducible factors in a composition series of a perverse direct image of a rank 1 locally constant sheaf, $L_a$. To define $L_a$, consider a central line arrangement $\bigcup_{i=1}^{n} L_i \subset \mathbb{C}^2$. A locally constant sheaf (of complex vector spaces) is given by a representation of $\pi_1(\mathbb{C}^2 - \bigcup_{i=1}^{n} L_i)$ which means that to each multi-index $a = (a_1, \ldots, a_n)$ there is associated such a sheaf $L_a$ ($a_i$ is the result of the action of a loop around $L_i$). The main result is the establishment of a criterion for the irreducibility in terms of the action of the fundamental group:

**Theorem 1.0.1.** The perverse sheaf $Rj_\ast L_a$, where $j : \mathbb{C}^2 - \bigcup_{i=1}^{n} L_i \to \mathbb{C}^2$, is irreducible if, and only if, both of the following conditions are satisfied:

- $a_i \neq 1$, for all $i = 1, \ldots, n$;
- $\prod_{i=1}^{n} a_i \neq 1$.

The chapter 2, after some basic preliminaries, starts with a presentation of the classification of locally constant sheaves of rank 1 on $\mathbb{C}^2 - \bigcup_{i=1}^{n} L_i$ and give correspondence to multi-indices in $\mathbb{C}^n$ (section 2.1.3). Similarly, to each $\alpha \in \mathbb{C}$, there is a locally constant sheaf $L_\alpha$ on $\mathbb{C}^* = \mathbb{C} - \{0\}$.

In the section 2.2.2 the definition of perverse sheaves (according to [2]) is given. After that, considering the inclusion $j : \mathbb{C}^* \to \mathbb{C}$, a computation of $j_\ast L_\alpha$ and $j_\ast L_a$ is presented. This is expanded to the computation of the irreducible factors of $Rj_\ast L_a$.

The results for the case of sheaves in $\mathbb{C}^*$ are known, but they appear as a background and introduction for the results presented in the next chapter, concerning perverse direct image of a rank 1 locally constant sheaf in $\mathbb{C}^2 - \bigcup_{i=1}^{n} L_i$, $i = 1, \ldots, n$. These results represent a new contribution to this subject.

The chapter 2 starts with a description of a certain algebraically defined category $\mathcal{C}(F, G; T)$, introduced by R. Macpherson and K. Vilonen ([11]). The objects of this category are the pairs $(A, B)$ together with diagrams:

\[
\begin{array}{ccc}
FA & \xrightarrow{T_A} & GA \\
\downarrow & & \downarrow \\
& B & \\
\end{array}
\]
where $A$ and $B$ are objects of categories $\mathcal{A}$ and $\mathcal{B}$ (respectively), $F$, left exact, and $G$, right exact, are functors between these two categories and $T$ is natural transformation from $F$ to $G$. In [11] and [17] the notions of kernel, cokernel and irreducible objects of $\mathcal{C}(F,G;T)$ are described without all details of the proofs. In the present work a detailed description of this notions is given and, in order to do so, some assumptions not explicitly present in the original article had to be made.

Then, a result from [12], that establishes the equivalence between the category $\mathcal{C}(F,G;T)$ and the category of perverse sheaves, is applied to a concrete situation. Consider the space $\mathbb{C}^2$ and let $S$ be a closed stratum: $S = \bigcup_i L_i, \ i = 1, \ldots, n$, where each $L_i$ is a line passing through the origin. By noticing that the action of $\pi_1(\mathbb{C}^2 - S)$ on a one-dimensional vector space is given by a linear character, we have a correspondence between local systems in $\mathbb{C}^2 - S$ and multi-indices $a = (a_1, \ldots, a_n)$ in $\mathbb{C}^n$. Let $\mathcal{L}_a$ denote the rank 1 locally constant sheaf associated to $a$. The main result consists on deciding in which conditions the direct image of $\mathcal{L}_a$ via the maps

$$
\mathbb{C}^2 - S \xrightarrow{j_1} \mathbb{C}^2 - \{0\} \xrightarrow{j_2} \mathbb{C}^2
$$

is irreducible. The first step is to translate the information of $p_j^1_! \mathcal{L}_a$ into a diagram of $\mathcal{C}(F,G;T)$. The second step is the computation of the direct image of a locally constant sheaf $\mathcal{L}_a$ in $\mathbb{C}^2 - \{S\}$ (the complement of the closed stratum), and then of the direct image of the perverse sheaf $p_j^1 \mathcal{L}_a$ in $\mathbb{C}^2 - \{0\}$. The final result is reached by gluing these structures coming from different strata. Finally, the theorem presented earlier is stated and proved.
2. Introduction to Machinery

2.1 Notation

In this section we will introduce the conventions and notations we are going to use, as well as give some results that we need.

2.1.1. Sheaves and Operations As general reference for sheaves on topological spaces we will use \[7\]. We will only consider sheaves for which the sections are complex vector spaces.

Let $X$ be a topological space, $U$ an open subset of $X$ and $F$ the closed complement. We will denote by $j$ the inclusion of $U$ in $X$ and by $i$ the inclusion of $F$ in $X$. Let $Sh(X)$ (respectively $Sh(U)$ and $Sh(F)$) denote the category of sheaves on $X$ (respectively on $U$ and $F$).

Then there are the following basic functors that relate the categories above:

- $j^! : Sh(U) \rightarrow Sh(X)$: extension by 0 (exact);
- $j^* : Sh(X) \rightarrow Sh(U)$: restriction (exact, also denoted by $j^!$);
- $j_* : Sh(U) \rightarrow Sh(X)$: direct image (left exact);
- $i^* : Sh(X) \rightarrow Sh(F)$: restriction (exact);
- $i_* : Sh(F) \rightarrow Sh(X)$: direct image (exact, also denoted by $i^!$);
- $i^! : Sh(X) \rightarrow Sh(F)$: sections with support in $F$ (left exact).

It is easy to verify that between these functors we have the relations (see\[7\], \[2\], pp.43):

\[ j^* i_* = 0, \quad i^* j_* = 0, \quad i^! j_* = 0 \]
\[ i^* i_* = 1d, \quad i^! i_* = 1d, \quad j^* j_* = 1d, \quad j^* j^! = 1d \quad (2.1.1) \]
2.1.2. Abelian Categories  Recall that an abelian category is a category in which all hom-sets are abelian groups and the composition is bilinear, all finite limits and colimits (in particular kernels and cokernels) exist and have certain good properties, some of which we will describe below. As general reference on abelian categories we use [15].

We will later be concerned with kernels and cokernels, in abelian categories different from module categories, and present then their definition. Let $\mathcal{C}$ be an abelian category, $A, B \in \mathcal{C}$ and $f$ the morphism $A \to B$.

**Definition 2.1.1.** We say that $k : S \to A$ is a kernel for $f$ when firstly $f \circ k = 0$, and secondly, $k$ has the property that any morphism $g : M \to A$, with $f \circ g = 0$, factors uniquely through $k$(see the diagram below).

![Diagram of kernel](image)

Now we describe the dual notion of cokernel:

**Definition 2.1.2.** We say that $p : B \to C$ is a cokernel for $f$ when firstly $p \circ f = 0$, and secondly, $k$ has the property that any morphism $h : B \to N$ with $h \circ f = 0$ factors uniquely through $p$.

![Diagram of cokernel](image)

**Definition 2.1.3.** We say that the morphism $f : A \to B$ is an injection (or monomorphism) if $\ker(f) = 0$ and a surjection (or epimorphism) if $\coker(f) = 0$. If $f$ is a injection we say that $A$ is a subobject of $B$, and if $f$ is a surjection that $B$ is a quotient object of $A$.

**Remark 2.1.4.** By abuse of notation we are going to use the notions of kernel and cokernel both for objects and morphisms.

Also important for abelian categories are the notions of image and co-image ([15], pp.129):
Definition 2.1.5. The image of a morphism \( f : A \to B \) is a subobject \( g : B' \to B \) of \( B \), such that \( f \) factors through \( g \) and, furthermore, if \( g_1 : B_1 \to B \) is another subobject of \( B \), such that \( f \) factors through \( g_1 \), then \( B' \subseteq B_1 \), i.e. there exists a morphism \( h : B' \to B_1 \) such that \( g_1 \circ h = g \).

Definition 2.1.6. The co-image of a morphism \( f : A \to B \) is a quotient object \( q : A \to C \) such that there is a map \( f_C : C \to B \) with \( f = f_C \circ q \). Furthermore, for any epimorphism \( h : A \to D \), for which there is a map \( f_D : D \to B \) with \( f = f_D \circ h \), there is a unique map \( t : D \to C \) such that both \( q = t \circ h \) and \( f_D = f_C \circ t \).

Recall that the fundamental defining property of an abelian category is:

\[
\text{cok}(\ker(f) \to A) \cong \ker(B \to \text{cok}(f)),
\]

which for categories of modules over a ring reduces to one of the isomorphism theorems: \( A/\ker(f) \cong \text{im}(f) \).

In addition we will have use for some results concerning morphisms and exact functors that will be applied later. Recall that monomorphisms are morphisms whose kernel is 0, and epimorphisms morphisms whose cokernel is 0.

Lemma 2.1.7. (see [15], pp.7) Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be morphisms in an abelian category \( C \):

1. if \( f \) and \( g \) are monomorphisms, then \( g \circ f \) is also a monomorphism;
2. if \( g \circ f \) is a monomorphism, then \( f \) is a monomorphism;
3. if \( f \) and \( g \) are epimorphisms, then \( g \circ f \) is also an epimorphism;
4. if \( g \circ f \) is a epimorphism, then \( g \) is an epimorphism.

The following definition is also standard (see [9], pp.201): let \( A \) and \( B \) be abelian categories. The additive functor \( T : A \to B \) is called left exact, when it preserves kernels, right exact when it preserves cokernels, and exact if it preserves both kernels and cokernels.

2.1.3. The Monodromy Representation of Locally Constant Sheaves

A general reference for this section is [7], in particular pp. 223-225. We recall that we only consider sheaves with sections that are finite-dimensional vector spaces over \( \mathbb{C} \). There is a direct equivalence between complex representations of the fundamental group and locally constant
sheaves, through the monodromy representation on stalks. This representation is defined in the following way. First recall that a locally constant sheaf on a simply connected and path-connected space is constant. Let then \( L \) be a locally constant sheaf on \( X \). For a fix point \( x_0 \in X \), consider an element \( \gamma \in \pi_1(X, x_0) \), represented by a path \( \gamma(t), t \in [0, 1] \). Clearly \( L_{x_0} = L_{\gamma(t)}, 0 \leq t < 1 \) and similarly \( L_{x_0} = L_{\gamma(t)}, 0 < t \leq 1 \). Then \( v \in L_{x_0} \) may first be identified with \( w \in L_{\gamma(1/2)} \) using the first identification, and then this element with an element \( v' \in L_{x_0} \) using the second identification. The element \( \gamma v := v' \) will not depend on the homotopy class of \( \gamma(t) \), the map \( v \mapsto v' \) is linear and hence this gives us a representation of \( \pi_1(X, x_0) \) on \( L_{x_0} \).

**Proposition 2.1.8.** Let \( X \) be a path-connected topological space and \( \pi_1(X, x_0) \) be its fundamental group with base point \( x_0 \). The monodromy representation of \( \pi_1(X, x_0) \) on the stalk \( L_{x_0} \), defines an equivalence between the category of local systems \( L \) on a space \( X \) and the category of finite dimensional complex representations of the fundamental group of \( X \).

We will now introduce the sheaves that we are going to work with later.

### 2.1.3.1 The complement to a point in the complex line

In this case the fundamental group of \( \mathbb{C}^* = \mathbb{C} - \{0\} \) is

\[
\pi_1(\mathbb{C}^*) = \{n\gamma, n \in \mathbb{Z}\} \cong \mathbb{Z},
\]

where \( \gamma \) is a loop going around 0 once starting in \( x_0 = 1 \).

Hence the proposition means the following:

**Corollary 2.1.9.** Locally constant sheaves \( L \) of rank 1 on \( \mathbb{C}^* \) are classified up to isomorphism by the element \( \alpha \in \mathbb{C} \), such that for the monodromy representation

\[
\gamma e = \alpha e \tag{2.1.3}
\]

where \( L_1 = \mathbb{C} e \) is the stalk at \( 1 \in \mathbb{C}^* \).

Denote by \( L_\alpha \) a rank 1 sheaf corresponding to the representation (2.1.3).

The sections of \( L_\alpha \) on an open neighborhood of the type \( U = U(x, r) = \{z \in \mathbb{C}^*: |z - x| < r\} \) will be

\[
L_\alpha(U) = \begin{cases} 
\mathbb{C} e, & \text{if } \pi_1(U) = 1 \iff 0 \notin \bar{U} \\
\ker(\gamma - 1), & \text{if } \pi_1(U) = \mathbb{Z} \iff 0 \in \bar{U}
\end{cases}
\]

This sheaf occurs naturally. Consider the multi-valued analytic function defined by a power \( x^\beta, \beta \in \mathbb{C} \). It defines a sheaf \( E_\beta \) with sections
\[ \mathcal{E}_\beta(U) := \{Ax^\beta, A \in \mathbb{C}, \text{ such that } Ax^\beta \text{ is defined as a single-valued analytic function on } U \}. \]

Then a quick calculation of the monodromy of the analytical function \( x^\beta \), gives that

\[ \mathcal{E}_\beta \cong \mathcal{L}_\alpha, \]

where \( \alpha = e^{2\pi i \beta} \).

### 2.1.3.2 The complement in the plane to a line configuration

Now consider a union of lines \( L_i, i = 1, \ldots, n \), through the origin in \( \mathbb{C}^2 \), given by equations \( z_i = a_i x_1 + b_i x_2 = 0 \). Let \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{C}^n \), be a multi-index, and define the corresponding function

\[ z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}, \]

which is a multi-valued analytic function defined in the complement to the union \( \bigcup_{i=1}^n L_i \), and hence defines, as in the previous example, a locally constant rank 1 sheaf, \( \mathcal{E}_\beta \), with sections

\[ \mathcal{E}_\beta(U) := \{Az^\beta, A \in \mathbb{C}, \text{ such that } Az^\beta \text{ is defined as a single-valued analytic function on } U \}. \]

The fundamental group of the complement to the line configuration has been known for a long time (see [12]). Choose a complex plane \( M \) that is transversal to all the \( L_i \) and does not contain the origin, and a base point \( x_0 \in M \). Then the intersection \( L_i \cap M \) is a single point \( P_i \), and we let \( \Gamma_i \) be a loop in \( M \) around \( P_i \), starting and ending in \( x_0 \). These loops generate the fundamental group, which has the following presentation:

\[ \pi_1(\mathbb{C}^2 \setminus \bigcup_{i=1}^n L_i) = \langle \Gamma_1, \ldots, \Gamma_n \rangle / R, \]

where \( R \) is the group generated by the (cyclic) relations

\[ \Gamma_1 \Gamma_2 \ldots \Gamma_n = \Gamma_2 \ldots \Gamma_n \Gamma_1 = \Gamma_n \Gamma_1 \ldots \Gamma_{n-1}. \]

**Corollary 2.1.10.** Locally constant sheaves \( \mathcal{L} \) of rank 1 on \( \mathbb{C}^2 \setminus \bigcup_{i=1}^n L_i \) are classified up to isomorphism by the element \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \), such that for the monodromy representation

\[ \Gamma_i e = \alpha_i e \quad (2.1.4) \]

where \( \mathcal{L}_{x_0} = \mathbb{C}e \) is the stalk at \( x_0 \).

**Proof.** Given a locally constant sheaf \( \mathcal{L} \) we know that it corresponds to the representation of the fundamental group on the stalk \( \mathcal{L}_{x_0} = \mathbb{C}e \). Since the sheaf has rank 1, the action is given by \( \Gamma_i e = \alpha_i e \mid i = 1, \ldots, n \), for some \( \alpha_i \in \mathbb{C} \),
By commutativity of multiplication of complex numbers the relations in \( R \) do not add any extra information: 
\[
\Gamma_1 \cdots \Gamma_n e = \cdots = \Gamma_i e = \cdots \Gamma_{i-1} e = \prod_{j=1}^n \alpha_j e.
\]
This last argument also shows that conversely, any multi-index \( \alpha \in \mathbb{C}^n \) defines a representation of the fundamental group, and hence a locally constant sheaf.

We denote the locally constant sheaf correspondent to \( \alpha \in \mathbb{C}^n \) by \( L_\alpha \). By calculating the monodromy of the multivalued analytical function \( x^\beta \), it is easy to see that
\[
\mathcal{E}_\beta \cong L_\alpha, \quad \alpha_i = e^{2\pi i \beta_i}, \ i = 1, \ldots, n.
\]

## 2.2 Perverse sheaves

In this section we present a very general view of some of the definitions and results that we will need, that will later be applied in a more concrete way. Our general reference is [2].

### 2.2.1. The category of complexes and the derived category

We will consider the category of complexes \( K(\mathcal{A}) \) (respectively, \( K^+ (\mathcal{A}) \)) with morphisms up to homotopy built from objects in the category \( \mathcal{A} \) of sheaves on a topological space. Let \( Q \) be the class of \( K(\mathcal{A}) \) (\( K^+(\mathcal{A}) \)) consisting of all quasi-isomorphisms. The category obtained by formally inverting the class \( Q \) of quasi-isomorphisms is the (bounded below) derived category \( D(\mathcal{A}) \) (\( D^+(\mathcal{A}) \)) of \( \mathcal{A} \) (see [7], pp. 430-435).

Any bounded below complex \( A^\bullet \) admits a quasi-isomorphism \( f : A^\bullet \to J^\bullet \) into a bounded below complex of injective objects, \( I^\bullet \) (an injective resolution of \( A^\bullet \), that is a right resolution whose all elements are injective)(see [7], pp.40). By working with injective resolutions the derived category becomes more manageable. The definition of derived functors is an example.

**Definition 2.2.1.** ([16], pp.14) Suppose that \( F : \mathcal{A} \to \mathcal{B} \) is a left exact functor between abelian categories \( \mathcal{A} \) and \( \mathcal{B} \). Let \( A^\bullet \) be a complex of \( \mathcal{A} \) and \( A^\bullet \cong I(A^\bullet) \) an injective resolution. Then define \( RF(A^\bullet) := F(I(A^\bullet)) \). This establishes a functor
\[
RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})
\]
called a right derived functor of \( F \). The i-th right derived functor
\[
R^i F : D^+(\mathcal{A}) \to \mathcal{B}
\]
is defined by \( R^i F = H^i \circ RF \).
If $A \in K(A)$, the shifted complex $A[1]$ is defined to be the complex that in degree $n$ is $A^{n+1}$, and has differentials that are those of $A$ multiplied by $-1$. If $f : A \rightarrow B$ is a chain map then the mapping cone is the complex $M := A[1] \oplus B$ with differential $d_M = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}$. It sits in a sequence

$$A \xrightarrow{f} B \rightarrow M,$$

which can be continued by a map $M \rightarrow A[1] \rightarrow B[1] \rightarrow \ldots$, and therefore often written as

$$A \xrightarrow{f} B \rightarrow M^+ \rightarrow \ldots$$

We will denote the mapping cone as above by $\text{Cone}(f)$.

Any sequence in the derived category that is homotopy equivalent to a sequence of type (2.2.1) in $K(A)$ is called a distinguished triangle. This descends to the derived category $D(A)$, where again a distinguished triangle is a sequence that is isomorphic to one of the type (2.2.1).

These two features, the translation functor $A \mapsto A[1]$ and the set of distinguished triangles, make $D(A)$ into a triangulated category (see the axioms that have to be satisfied in [2]).

2.2.2. Definition of perverse sheaves

We start by giving an overview of the main elements related to the definition of perverse sheaves. The definition of perverse sheaves can be presented in different ways according to the properties we want to explore, but every definition demands a certain level of abstraction and a rather complex previous technical work. We are going to present the definition from [2].

Let $X$ be a topological space and $D = D_X$ the derived category of sheaves on it. The definition of perverse sheaves is based on the concept of a $t$-structure, which comprehends a triangulated category $D$, two full subcategories $p^D \leq 0$ and $p^D \geq 0$ and perverse truncation functors, $\tau^0_p : D \rightarrow p^D \leq 0$ and $\tau^0_p : D \rightarrow p^D \geq 0$. The category of perverse sheaves $\mathcal{M}(X)$ in the case $D = D_X$ is a full subcategory of $D$, corresponding to its heart (or coeur): $\mathcal{M}(X) := p^D \leq 0(X) \cap p^D \geq 0(X)$.

$\mathcal{M}(X)$ turns out to be an abelian category and therefore we can find kernels and cokernels. Let $f : Q^* \rightarrow R^*$ be a map of perverse sheaves. The kernel and cokernel are defined through the perverse truncation functors as: $\text{ker}(f) = (\tau^0_p (\text{Cone}(f))^*)[-1]$ and $\text{coker}(f) = \tau^0_p (\text{Cone}(f))^*$. In $\mathcal{M}(X)$ is also defined a cohomological functor $H^0 := \tau^0_p \tau^0_p$. We will now describe this more precisely. Note that the most important of the above concepts is the concept of truncation—using it the others may be defined.
2.2.2.1 t-category

Definition 2.2.2. ([4], pp.125) A t-category is a triangulated category $D$, with two strictly full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ of the category $D$, such that, by setting $D^{\leq n} = D^{\leq 0}[n]$ and $D^{\geq n} = D^{\geq 0}[n]$, one has the following properties:

- $\text{Hom}(X^*; Y^*) = 0$ if $X^* \in D^{\leq 0}$ and $Y^* \in D^{\geq 1}$;
- $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$;
- for any object $X^* \in D$, there is a distinguished triangle
  \[ A^* \to X^* \to B^* \to A^*[+1] \]

with the object $A^*$ in $D^{\leq 0}$ and the object $B^*$ in $D^{\geq 1}$.

We say that $(D^{\leq 0}, D^{\geq 0})$ is a t-structure over $D$. Its heart is the full subcategory $C = D^{\leq 0} \cap D^{\geq 0}$.

In the above definition $A^* := \tau_{\leq 0} X^*$ and $B^* := \tau_{\geq 1} X^*$. More generally, truncation functors may be defined as $\tau_{\leq n} X^* := (\tau_{\leq n}(X^*[-n]))[n]$ and $\tau_{\geq n} X^* := (\tau_{\geq n}(X^*[-n]))[n]$. For these functors we have the following proposition.

Proposition 2.2.3. ([2], pp.29) The inclusion of $D^{\leq n}$ in $D$ admits a right adjoint $\tau_{\leq n}$, and the inclusion of $D^{\geq n}$ admits a left adjoint $\tau_{\geq n}$. For every $X^*$ in $D$, there exists a unique morphism $d \in \text{Hom}^1(\tau_{\geq 1} X^*, \tau_{\leq 0} X^*)$ such that

\[ \tau_{\leq 0} X^* \to X^* \to \tau_{\geq 1} X^* \xrightarrow{d} \]

is a distinguished triangle. Apart from isomorphism, this is the unique distinguished triangle $(A^*, X^*, B^*)$ with $A^*$ in $D^{\leq 0}$ and $B^*$ in $D^{\geq 1}$.

The simplest example of a t-structure on $D_X$ is the one induced by $\tau^{\leq 0}$ and $\tau^{\geq 0}$ being the ordinary truncation operators on complexes. In this case the heart is just the category of sheaves on $X$. By using shifted truncation operators one gets a t-structure with a heart that is the the category of sheaves on $X$, but now shifted to a fix degree. We call this the shifted trivial t-structure. The idea of using perversities is now to shift differently according to the different strata in a stratification of $X$. 

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2.2.2.2 Perversities

We will only use a certain perversity, but it makes the construction clearer if we introduce the concept more generally.

**Definition 2.2.4.** Let $X$ be a topological space and $\Sigma$ a stratification of $X$. A perversity is a map $p : \Sigma \to \mathbb{Z}$ (to each stratum an integer is associated).

**Example 2.2.1.** Some important perversities are given by:

- $m = (0, 0, 1, 1, 2, 2, 3, \ldots)$, lower middle perversity (the one we are going to use in our work);
- $n = (0, 1, 1, 2, 2, 3, 3, \ldots)$, upper middle perversity.

2.2.2.3 Gluing t-categories using perversities

(For the following construction see [2], pp. 48-58.) The truncation in the category $D_X$ has to be understood inductively from the truncation $\tau_U$, associated to a t-structure in the category $D_U$, and $\tau_F$ in $D_F$, where $U \subset X$ is an open subset and $F = X \setminus U$ the closed complement. Consider $(D_U^{\leq 0}, D_U^{\geq 0})$ a t-structure on $D_U$ and $(D_F^{\leq 0}, D_F^{\geq 0})$ a t-structure on $D_F$. Then define:

$$D^{\leq 0} := \{ K \in D \mid j^* K \in D_U^{\leq 0} \text{ and } i^* K \in D_F^{\leq 0} \}$$

$$D^{\geq 0} := \{ K \in D \mid j^* K \in D_U^{\geq 0} \text{ and } i^! K \in D_F^{\geq 0} \}$$

**Proposition 2.2.5.** $(D^{\leq 0}, D^{\geq 0})$ is a t-structure over $D$.

We say that we glue the t-structures over $U$ and $F$.

Now, given a perversity, the idea is to build the t-structure inductively, starting with $U_1 = S_0$ and $F_1 = S_1$ ($S_0, S_1 \in \Sigma$), and using the t-structures on $D_{U_1}$ and $D_{F_1}$, that are trivial t-structures that are shifted according to the perversities $p_{U_1}$ and $p_{F_1}$, to build a t-structure on $X \setminus X_2 = S_0 \cup S_1$. In the next step one considers $X \setminus X_3 = X \setminus X_2 \cup (X_2 \setminus X_3)$, where one, by the previous inductive step, has a t-structure on $D_{U_2}$, with $U_2 = X \setminus X_2$ and uses the trivial t-structure on $D_{F_2}$, with $F_2 = X_2 \setminus X_3 = S_2$ shifted according to $p_{S_2}$.

We now describe this equivalently, just using the strata.

**Lemma 2.2.6.** The subcategory $^p D^{\leq 0}(X)$ (resp. $^p D^{\geq 0}(X)$) of $D(X)$ is the subcategory given by the complexes $K^*$ (resp. $K^*$ in $D^+(X)$) such that for each stratum $S$, being $i_S$ the inclusion of $S$ in $X$, one has $H^n(i_S^* K^*) = 0$ for $n > p(S)$ (resp. $H^n(i_S^! K^*) = 0$ for $n < p(S)$).

**Definition 2.2.7.** The category $\mathcal{M}(p, X)$ of $p$-perverse sheaves over $X$ is the category $^p D^{\leq 0} \cap ^p D^{\geq 0}$ (the heart of the t-structure $(^p D^{\leq 0}, ^p D^{\geq 0})$).
2.2.2.4 Truncation

We just describe the truncation for the case of two strata, an open $U$ and a closed $F$. Let $j : U \to X$ and $i : F \to X$. To these strata we assume that we have perversities, where $p_U$ is the perversity of $U$ and $p_F$ the perversity of $F$. For $K^\bullet$ on $U$, $\tau^U_{\geq i}(K^\bullet)$ in $D_U$ is given by the shifted trivial truncation $\tau^U_{\geq i} := \tau_{\geq i+p_U}(K^\bullet)$. For $L^\bullet$ on $F$, $\tau^F_{\geq i}(L^\bullet)$ is given by $\tau^F_{\geq i} := \tau_{\geq i+p_F}(L^\bullet)$ in $D_F$. Similarly with the other truncations.

To get the perverse truncation which is associated to the glued t-structure:

1. start with an object $X^\bullet$ in $D$, and choose $Y^\bullet$ in a way so that we have the distinguished triangle

   $$(Y^\bullet, X^\bullet, j_* \tau^U_{\geq 1} j^* X^\bullet);$$

2. then we define $A^\bullet$ such that the following is also a distinguished triangle

   $$(A^\bullet, Y^\bullet, i_* \tau^F_{\geq 1} i^* Y^\bullet);$$

3. finally we define $B^\bullet$ so that we have the third distinguished triangle (using the composition $A^\bullet \to Y^\bullet \to X^\bullet$ as the first map)

   $$(A^\bullet, X^\bullet, B^\bullet).$$

Clearly all of this constructions may be done using mapping cones. It is not difficult to prove from Propositions 2.2.2 and 2.2.3 that $A^\bullet$ is in $D^{\leq 0}$ and $B^\bullet$ in $D^{\geq 1}$, and so the perverse truncation is determined as $A^\bullet = \tau^{\leq 0}_p X^\bullet$ and $B^\bullet = \tau^{\geq 1}_p X^\bullet$.

2.2.2.5 The relation between the perverse truncation and the kernel and cokernel

In the case of the trivial truncation it is easy to relate it with the kernel and cokernel of a morphism in the heart, i.e. a map between sheaves. There is an equivalent result for the perverse truncation and thus this gives a way of computing the kernel and cokernel for a morphism of complexes in the derived category.

Let $Z^\bullet$ be the mapping cone of the morphism $f : Q^\bullet \to R^\bullet$, between two perverse sheaves. Then we have maps

$$Q^\bullet \to R^\bullet \to Z^\bullet$$

and

$$Z^\bullet[-1] \to Q^\bullet \to R^\bullet.$$
Following the results of [2], pp.27-31, we have that:

\[ \ker(f) = (\tau_{\leq -1} Z^*)[-1], \quad \text{coker}(f) = \tau_{\geq 0} Z^*, \]

or more precisely the kernel of \( f \) is the composition

\[ (\tau_{\leq -1} Z^*)[-1] \to Z^*[-1] \to Q^*, \]

and the cokernel is the composition

\[ R^* \to Z^* \to \tau_{\geq 0} Z^*. \]

For our purposes we rewrite this as

1. \[ \ker(f) = (\tau_{\leq -1} Z^*)[-1] = (\tau_{\leq 0} Z^*[-1][1][-1])[-1] = \tau_{\leq 0} (Z^*[-1]) \] (2.2.2)
2. \[ \text{coker}(f) = \tau_{\geq 0} Z^* = (\tau_{\geq 1} (Z^*[-1]))[1] \] (2.2.3)

In particular, one should note that the kernel and cokernel are possible to calculate in terms of the truncation operators.

**Remark 2.2.8.** Let \( Q^* \) and \( R^* \) be complexes that are different from zero only in degree 0. Therefore the mapping cone is the complex \( Z^* : Q^* \to R^* \). Let us apply the definitions of kernel and cokernel given above to this \( Z^* \), but using the normal truncation, which corresponds to the perversity \( p \equiv 0 \). \( Z^*[-1] \) will be a complex different from zero in degrees 0, 1 and the result of \( \tau_{\leq 0} (Z^*[-1]) \) will be precisely \( \ker(f) \) (in degree 0). The result of \( \tau_{\geq 1} (Z^*[-1]) \) is, according to the definition of (trivial) truncation functors, \( \text{coker}(f) \), in degree 1. After the last shifting, \( (\tau_{\geq 1} (Z^*[-1]))[1] \), the result will be \( \text{coker}(f) \) in degree 0. Hence the truncation functor gives the kernel and cokernel for the trivial t-structure.

### 2.3 Locally constant rank 1 sheaves on \( \mathbb{C}^* \)

The subject of the thesis is the irreducibility of certain direct images of locally constant sheaves in the derived category. We will now present the easiest such result, using the previous machinery to prove (most of) the following well known result. The irreducibility follows from general results in [2], but we will instead prove it as a consequence of the algebraic description of the category of perverse sheaves in the next chapter (see Example 3.2.1).

Recall the sheaves \( L_\alpha \) on \( \mathbb{C}^* \) from Section 2.1.3. Consider the inclusion \( j : \mathbb{C}^* \to \mathbb{C} \). Recall that \( j_! \) is exact. Denote by \( S \) the skyscraper sheaf at 0.
Proposition 2.3.1. 1. If $\alpha \neq 1$ then

$$j_! L_\alpha \cong R^0 j_* L_\alpha \cong Rj_* L_\alpha$$

are isomorphic and irreducible perverse sheaves in $D_C$.

2. If $\alpha = 1$, and so $C := L_1$ is the constant sheaf on $\mathbb{C}^*$, there are short exact sequences of perverse sheaves

$$S[-1] \hookrightarrow j_! C \twoheadrightarrow R^0 j_* C$$

and

$$R^0 j_* C \hookrightarrow Rj_* C \twoheadrightarrow S[-1].$$

Furthermore $S[-1]$ and $R^0 j_* C$ are irreducible perverse sheaves.

The proof proceed in several steps. First we will (for completeness sake) calculate the sheaves involved through describing their stalks, then we will use the technique for calculating kernels and cokernels in the preceding section from [2].

2.3.1. Stalks  We denote the constant sheaf $L_1$ on $\mathbb{C}^*$ by $C$.

Lemma 2.3.2. We have that:

1. If $\alpha \neq 1$ then $j_! L_\alpha \cong Rj_* L_\alpha$.

2. $$(R^0 j_* C)_p = \mathbb{C}, \text{ for all } p \in \mathbb{C}.$$ $$(R^1 j_* C)_p = \begin{cases} 0, & \text{if } p \neq 0 \\ \mathbb{C}, & \text{if } p = 0 \end{cases}$$

Proof. If $p \neq 0$, clearly $(j_! L_\alpha)_p = (j_* L_\alpha)_p$. Since $j_!$ is exact it has no higher derived images. Also, if $p \neq 0$, there is a sequence of contractible open neighborhoods $p \in V \subset \mathbb{C}^*$ that converge to $p$, and so $\lim_{V \ni p} H^1(V, L_\alpha) = 0$. Hence $(R^1 j_* C)_p = 0$. Finally, for $p = 0$, we calculate $H^i(V \setminus \{0\}, L_\alpha)$, $i \geq 0$, for a disk $V \subset \mathbb{C}$ centered around 0. Define the two open contractible subsets, $U_1 = V \setminus \mathbb{R}_+$ and $U_2 = V \setminus \mathbb{R}_-$, such that:

- $V \setminus \{0\} = U_1 \cup U_2$;
- The intersection $U_1 \cap U_2$ has two components $O_1$ and $O_2$;
- $O_1 \cap O_2 = \emptyset$. 

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$U_1$ and $U_2$ is then a cover for $V \setminus \{0\}$. Let $U = \{U_1, U_2\}$. We have that

$$H^i(V \setminus \{0\}, \mathcal{L}_\alpha) = \check{H}^i(U, \mathcal{L}_\alpha),$$

where the Čech complex (see [3], pp. 89-112), is as in the diagram below. Let $s$ and $t$ be sections of $\mathcal{L}_\alpha$ on $U_1$ and $U_2$ respectively. The Čech differential, $d$, from the global sections of $U_1$ and $U_2$ to the global sections of $O_1$ and $O_2$, looks as follows:

$$\Gamma(U_1, \mathcal{L}_\alpha) \oplus \Gamma(U_2, \mathcal{L}_\alpha) \quad d \quad \Gamma(O_1, \mathcal{L}_\alpha) \oplus \Gamma(O_2, \mathcal{L}_\alpha)$$

$$s \quad t \quad (s, t) \quad d' \quad (s, s) \quad (-t, -\gamma t) \quad (s - t, s - \gamma t)$$

since $\Gamma(U_1 \cap U_2, \mathcal{L}_\alpha) = \Gamma(O_1, \mathcal{L}_\alpha) \oplus \Gamma(O_2, \mathcal{L}_\alpha)$. It is easy to observe that each one of the global sections is equal to the vector space $\mathbb{C}$. Since $\mathbb{C} = \Gamma(U_2, \mathcal{L}_\alpha)$, we can identify it with $\Gamma(O_1, \mathcal{L}_\alpha)$ and with $\Gamma(U_1, \mathcal{L}_\alpha)$. In the relation between $\Gamma(U_2, \mathcal{L}_\alpha)$ and $\Gamma(O_2, \mathcal{L}_\alpha)$ however, we have to take in account the fact of completing a loop around 0 and so the effect of the generator $\gamma$ of the fundamental group of $\mathbb{C}^*$ in the differential above. Clearly, if $\alpha \neq 1$, $\gamma t = \alpha t \neq t$ and then $d'$ is an isomorphism, and both the kernel and cokernel are zero. So $H^i(V \setminus \{0\}, \mathcal{L}_\alpha) = 0$ for all $i$, and so also $(R^i j_* \mathcal{L}_\alpha)_0 = 0$. This concludes the proof of (1). If $\alpha = 1$, then both the kernel and the cokernel of $d'$ are isomorphic to $\mathbb{C}$, and this concludes the proof of (2).

**Remark 2.3.3.** In the case where $i : \{0\} \to \mathbb{C}$ is the inclusion of a locally closed subspace and $\mathbb{C}(0)$ is the constant sheaf on $\{0\}$, the result of applying $i_! = i_*$ to $\mathbb{C}(0)$ is the skyscraper sheaf, $S$, at 0:

$$(i_!(\mathbb{C}(0)))_p = (S)_p = \begin{cases} 0, & p \in \mathbb{C}^* \\ \mathbb{C}, & p = 0 \end{cases}$$

The results above then says that, while when $\alpha \neq 1$, $j_* \mathcal{L}_\alpha \cong R j_* \mathcal{L}_\alpha$, for $\alpha = 1$, we have the short exact sequence

$$j_! \mathcal{C} \hookrightarrow R^0 j_* \mathcal{C} \twoheadrightarrow S. \quad (2.3.3)$$

**2.3.2. The cohomology groups of the mapping cone** We just know $R j_* \mathcal{L}_\alpha^*$, by the cohomology of its stalks, but we can observe that it is a complex different from zero only in degrees 0 and 1 represent it as:

$$R j_* \mathcal{L}_\alpha^* : \ldots \to 0 \to E^0 \to E^1 \to 0 \to \ldots \quad (2.3.4)$$

We will consider the mapping cone in two situations:
The mapping cone $M^*_T$ is a complex different from zero only in degrees -1 and 0, given by the sequence:

$$M^*_T: \cdots \rightarrow d^{-3} \rightarrow 0 \rightarrow d^{-2} \rightarrow j_! C \rightarrow d^{-1} R^0 j_* C \rightarrow d^0 \rightarrow 0 \rightarrow \cdots$$

The mapping cone $M^*$ is a complex different from zero in degrees -1, 0 and 1, given by the sequence:

$$M^*: \cdots \rightarrow 0 \rightarrow j_! C \rightarrow d^{-1} E^0 \rightarrow d^0 E^1 \rightarrow d^1 \rightarrow 0 \rightarrow \cdots$$

The sequence of complexes (2.3.6) can now be described by the following diagram:

Recalling the result of the preceding section, and using the long cohomological sequence, associated to the mapping cone, we get the following description.

**Lemma 2.3.4.** The cohomology groups of the mapping cone are

$$H^n(M^*_T) = \begin{cases} S, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$H^n(M^*) = \begin{cases} S, & n = 0, 1 \\ 0, & n \neq 0, 1 \end{cases}$$
**2.3.3. Kernel and cokernel for the morphism** \( \theta \)  

In this section we will prove the first part of Proposition 2.3.1 (2), in the following form. Observe that there is a quasi-isomorphism \( M_T^*[-1] \cong S[-1] \), as a consequence of Lemma 2.3.4. Hence sequence (2.3.1) follows from the following result.

**Lemma 2.3.5.** The morphism \( \theta : j_! C \to R^0 j_* C \) has kernel \( M_T^*[-1] \) and cokernel 0.

**Proof.** Our complex \( M_T^*[-1] \) is

\[
M_T^*[-1] : \cdots \xrightarrow{d^{-2}} 0 \xrightarrow{d^{-1}} j_! C \xrightarrow{d^0} R^0 j_! C \xrightarrow{d^1} 0 \xrightarrow{d^2} \cdots
\]

The cohomology groups are (see Lemma 2.3.4):

\[
H^n(M_T^*[-1]) = \begin{cases} 
S, & n = 1 \\
0, & n \neq 1
\end{cases}
\]

We want to apply the construction described in Section 2.2.2.5 to our complex \( M_T^*[-1] \). So, \( X \) will correspond to \( M_T^*[-1] \), the open set is \( U = \mathbb{C}^* \), with \( p(\mathbb{C}^*) = 0 \) and the closed set is \( F = \{0\} \), with \( p(\{0\}) = 1 \) (see Section 2.2.2.2).

We start by determining \( Y^* \). As \( j_* \) is an exact functor we know that, for every complex \( F^* \), \( H^i(j_* F^*) = j_* H^i(F^*) \), for all \( i \in \mathbb{Z} \). Being obvious that \( H^i(M_T^*[-1])_p = 0 \) for every \( p \neq 0 \), we have that: \( j_* H^i(M_T^*[-1]) \cong 0 \Leftrightarrow H^i(j_* M_T^*[-1]) \cong 0 \Rightarrow j_* M_T^*[-1] \cong 0 \Rightarrow \tau^{\geq 1}(j_* M_T^*[-1]) \cong 0 \). Therefore \( Y^* = M_T^*[-1] \).

Now we want to construct \( A^* \). First we observe that \( i_* \tau^{\geq 1} j_* M_T^*[-1] = i_* \tau^{\geq 2} i_* M_T^*[-1] \), because \( i_* (M_T^*[-1]) \) is in \( D_F \) and \( p(\{0\}) = 1 \). By applying \( i_* \) to the complex \( M_T^*[-1] \), we get that \( i_* (j_! C) = 0 \) (see relations in (2.1.1)) and \( i_* (j_* C) = i_* C = S \), therefore:

\[
i^* (M_T^*[-1]) : \cdots \xrightarrow{d^{-1}} 0 \xrightarrow{d^0} S \xrightarrow{d^1} 0 \xrightarrow{d^2} \cdots
\]

and

\[
H^n(i^* M_T^*[-1]) = \begin{cases} 
S, & n = 1 \\
0, & n \neq 1
\end{cases}
\]

From this we get: \( \tau^{\geq 2} i^* M_T^*[-1] = 0 \Rightarrow i_* \tau^{\geq 2} i_* M_T^*[-1] = 0 \). Therefore \( A^* = Y^* = X^* \), implying that \( B^* = 0 \).

Finally we conclude that the kernel \( A^* \) is just the complex \( M_T^*[-1] \) and that the cokernel \( B^* \) is zero. Thus the morphism \( \theta \) is a surjection:

\[
\begin{array}{ccccccc}
R^0 j_* C & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
j_! C & \rightarrow & j_! C & \rightarrow & R^0 j_* C & \rightarrow & 0
\end{array}
\]

\( \square \)
Note that this is precisely the opposite result to what we get for the trivial t-structure, when the kernel is taken as for morphisms between sheaves and it is the cokernel that is \( S \), and the map is injective.

### 2.3.4. Kernel and cokernel for the morphism \( \theta_R \)

In this section we will prove the remaining part of Proposition\(^{2.3.1}\)(2). Again since by the quasi-isomorphism \( M^*_T[-1] \cong S[-1] \) (Lemma\(^{2.3.4}\)), the exactness of the sequence (2.3.2) will be shown to follow from the next result.

**Lemma 2.3.6.** For the morphism \( \theta_R : j_!C \to Rj_*C \), the kernel is \( M^*[-1] \) and the cokernel is \( B^*[1] \), where \( B^* \) is the complex given by \( B^* = i_* \tau^{\geq 2} i^* M^*[-1] \), with cohomology groups

\[
H^n(B^*) = \begin{cases} 
S, & n = 2 \\
0, & n \neq 2 
\end{cases}
\]

**Proof.** Our complex \( M^*[-1] \) comes from the morphism (2.3.6):

\[
M^*[-1] : \cdots & \to d^{-2} & \to d^{-1} & \to j_!C & \to d^0 & \to E_0 & \to d^1 & \to E_1 & \to d^2 & \to 0 & \to d^3 & \to \cdots
\]

The cohomology groups are (see Lemma\(^{2.3.4}\)):

\[
H^n(M^*[-1]) = \begin{cases} 
S, & n = 1, 2 \\
0, & n \neq 1, 2 
\end{cases}
\]

Like before \( X^* = M^*[-1] \). The method used in the previous situation to determine \( Y^* \) can be replicated: since \( j_! \tau^{\geq 1} j^* M^*[-1] = 0 \), we get the same result, \( Y^* = M^*[-1] \). Note that

\[
i_* \tau^{\geq 1} i^* M^*[-1] = i_* \tau^{\geq 2} i^* M^*[-1].
\]

Then \( A^* \) is defined by the distinguished triangle

\[
(A^*, Y^*, i_* \tau^{\geq 1} i^* Y^*) = (A^*, M^*[-1], i_* \tau^{\geq 2} i^* M^*[-1]),
\]

and consequently, by uniqueness of distinguished triangles, we must have \( B^* = i_* \tau^{\geq 2} i^* M^*[-1] \). We now calculate the cohomology groups of \( B^* \).

The functors \( i^* \) and \( i_* \) are exact functors. Applying these functors to \( H^n(M^*[-1]) \) will not affect it because they represent, respectively, the restriction to the closed set, \( \{0\} \), and the direct image to the set \( C \); given that \( H^n(M^*[-1])_p = 0 \) for \( p \neq 0 \), the cohomology groups remain the same, being only affected by the truncation functor. Then we get

\[
H^n(B^*) = \begin{cases} 
S, & n = 2 \\
0, & n \neq 2 
\end{cases}
\]
Recalling that $coker(\theta_R) = \tau_p^{\geq 1}(M^*[\cdot - 1])[1]$ (see [2.2.3]) we can finally say that $coker(\theta_R) = B^*[1]$.

To understand $ker(\theta)$ we observe the long exact sequence of cohomology groups associated to the distinguished triangle $(A^*, M^*[\cdot - 1], B^*)$:

$$
\ldots \rightarrow H^{-1}(B^*) \rightarrow H^0(A^*) \rightarrow H^0(M^*[\cdot - 1]) \rightarrow H^0(B^*) \rightarrow H^1(A^*)
\rightarrow H^1(M^*[\cdot - 1]) \rightarrow H^1(B^*) \rightarrow H^2(A^*) \rightarrow H^2(M^*[\cdot - 1]) \rightarrow H^2(B^*)
\rightarrow H^3(A^*) \rightarrow H^3(M^*[\cdot - 1]) \rightarrow \ldots
\leftrightarrow
\ldots \rightarrow 0 \rightarrow H^0(A^*) \rightarrow 0 \rightarrow H^1(A^*) \rightarrow S \rightarrow 0 \rightarrow H^2(A^*) \rightarrow S \rightarrow 0 \rightarrow H^3(A^*) \rightarrow 0 \rightarrow \ldots
$$

By observation of the sequence above, we have:

- $H^n(A^*) = 0$, for $n \leq 0$;
- $H^n(A^*) = 0$, for $n \geq 3$;
- $H^1(A^*) \cong H^1(M^*[\cdot - 1])$ therefore $H^1(A^*) = S$;
- $H^2(M^*[\cdot - 1]) \cong H^2(B^*)$, implying that $H^2(A^*) = 0$.

Then we can present the cohomology groups for $A^*$:

$$
H^n(A^*) = \begin{cases} 
S, & n = 1 \\
0, & n \neq 1 
\end{cases}
$$

We observe that, apart from a shifting in the degrees, $A^*$ has the same cohomology groups as the mapping cone $M^*_T$ (see Lemma [2.3.4]). \qed

To conclude the proof of Proposition [2.3.1] we want to understand the relation between the two morphisms whose kernels and cokernels we have computed. The relation between the morphisms $\theta$ and $\theta_R$, and their kernel and cokernels, are described in the following diagram:

$$
\text{ker}(\theta) \xrightarrow{d^{-1}} j_! \mathcal{C} \xrightarrow{\theta} R^0j_* \mathcal{C} \xrightarrow{d^1} 0 \\
\downarrow \eta_{-1} \quad \quad \downarrow \eta_0 \quad \quad \downarrow \eta_1 \quad \quad \downarrow \eta_2 \\
A^* \xrightarrow{\delta^{-1}} j_! \mathcal{C} \xrightarrow{\theta_R} Rj_* \mathcal{C} \xrightarrow{\delta^1} B^*[1] \tag{2.3.7}
$$

We claim that $\text{ker}(\theta) : j_! \mathcal{C} \rightarrow R^0j_* \mathcal{C}$ is also a kernel for the morphism $\theta_R$ and therefore $A^* = \text{ker}(\theta_R) = \text{ker}(\theta)$.

We first observe that in the commutative square in the left:
• the morphism \( \eta_0 \) is an isomorphism;
• the composition \( \eta_0 \circ d^{-1} \) is injective;
• the composition \( \partial^{-1} \circ \eta_{-1} \) will also be injective;
• the morphism \( \eta_{-1} \) is injective (recall Lemma 2.1.7).

From this follows that \( \eta_{-1} \) is a quasi-isomorphism, since the cone of the morphism \( \eta_{-1} \) is

\[
\text{Cone}^\bullet(\eta_{-1}) : \ldots \rightarrow 0 \rightarrow j_! \mathcal{C} \xrightarrow{\Gamma^{-1}} R^0 j_* \mathcal{C} \oplus j_! \mathcal{C} \xrightarrow{\Gamma^0} R^0 j_* \mathcal{C} \xrightarrow{\Gamma^1} 0 \rightarrow \ldots
\]

which is then easily seen to have zero cohomology in all degrees. Consequently \( \ker(\theta) \) and \( A^\bullet \) are quasi-isomorphic, and \( \ker(\theta) \) is isomorphic, in the derived category, to \( \ker(\theta_R) \). As a consequence, \( j_! \mathcal{C} / \ker(\theta) \cong R^0 j_* \mathcal{C} \hookrightarrow R j_* \mathcal{C} \) equals the kernel of \( R j_* \mathcal{C} \twoheadrightarrow B^\bullet[1] \). This means that the sequence (2.3.2) is exact, and this finishes the proof of Proposition 2.3.1.
3. Direct Images on Lines
Configurations in $\mathbb{C}^2$

In this chapter we will apply the results of [10], [11], [12] and [17]. We will start by describing an algebraically defined category, $\mathcal{C}(F,G; T)$, which gives a description of the category of perverse sheaves, and a detailed proof about how its irreducible objects are constructed ([17]). Then, a particular case of a $\mathcal{C}(F; G; T)$ category is presented. Following [10] and [11], we consider the situation of a topological stratified space and the case of a complex analytic manifold. Finally we use the more specific results of [12] and rewrite them in the case when the singular space is given by $S = \bigcup_i L_i$, $i = 1, \ldots, n$, where each $L_i$ is a line passing through the origin. For this example we determine the conditions in which the direct images of locally constant rank 1 sheaves, with a given action of $\pi_1(\mathbb{C}^2 - \bigcup_i L_i)$, are irreducible perverse sheaves.

3.1 The Category $\mathcal{C}(F, G; T)$

The definition of the category of perverse sheaves as a subcategory of $D_X$ is less suited for calculations and for our work it will be more clear to use a different construction. We will describe a category $\mathcal{C}(F,G; T)$ that is equivalent to the category of perverse sheaves, $\mathcal{M}(X)$. The idea is to show that the category of perverse sheaves is glued from the category of perverse sheaves on an open strata of maximal dimension and the category of perverse sheaves on the complement.

We present the initial definitions as stated in [11]: consider two categories $A$ and $B$, two functors $F$ and $G$ from $A$ to $B$, and a natural transformation $T$ from $F$ to $G$. Symbolically $F,G : A \to B$ and $F \overset{T}{\to} G$. We define the category $\mathcal{C}(F,G; T)$ to be the category whose objects are pairs $(A; B) \in \text{Obj } A \times \text{Obj } B$ together with a commutative triangle

$$
\begin{array}{ccc}
FA & \xrightarrow{T_A} & GA \\
m & \searrow & \nearrow n \\
B & & \\
\end{array}
$$
and whose morphisms are pairs \((a, b) \in \text{Mor}\, A \times \text{Mor}\, B\) such that

\[
F A \xrightarrow{T_A} GA \quad \xrightarrow{m} \quad \xrightarrow{n} \quad B \xrightarrow{b} Ga
\]

\[
FA' \xrightarrow{T'_A} GA' \quad \xrightarrow{m'} \quad \xrightarrow{n'} \quad B' \xrightarrow{b'}
\] 

(3.1.1)

**Proposition 3.1.1.** (see [11], pp. 405-407) If \(A\) and \(B\) are abelian categories and if \(F\) is right exact and if \(G\) is left exact then the category \(C(F, G; T)\) is abelian and the functors taking \((A, B) \mapsto A\) and \((A, B) \mapsto B\) from \(C(F, G; T)\) to \(A\) and \(B\) are exact.

In the following subsection we are going to present the construction of the kernel and cokernel for a morphism \((a, b)\) in the category \(C(F, G; T)\). We are also going to prove that the objects presented actually represent a kernel and a cokernel, according to the definition. This forms an expanded version of the descriptions on [11].

### 3.1.1. Kernel

In an abelian category every morphism has a kernel and a cokernel. We are going to show that for a morphism \((a, b)\) in \(C(F, G; T)\) (as in Diagram (3.1.1)) the kernel is

\[
F(\text{ker}(a)) \longrightarrow G(\text{ker}(a)) \quad \text{ker}(b)
\] 

(3.1.2)

(We will shortly define the maps in this diagram.)

In the category \(\mathcal{A}\), consider the morphism \(\xrightarrow{a} A'\). There is a natural inclusion

\[
\xrightarrow{0} \quad \xrightarrow{a} \quad \xrightarrow{A} \quad \xrightarrow{A'}
\]

\(\text{ker}(a)\)

Applying a functor \(L:\)

\[
L(\text{ker}(a)) \longrightarrow L(A) \xrightarrow{La} L(A')
\] 

(3.1.3)
we observe that there is a canonical morphism: \( L(\ker(a)) \to \ker(L(a)) \). If \( L \) is a left exact functor then (see Section 2.1.2), this morphism is an isomorphism, thus \( G(\ker(a)) \cong \ker(G(a)) \).

Hence we have the following commutative diagram

\[
\begin{array}{ccc}
F(\ker(a)) & \xrightarrow{ij} & \ker(F(a)) \\
\downarrow T & & \downarrow T \\
G(\ker(a)) & \xrightarrow{i} & F(A) \\
\end{array}
\]

First we want to show that the morphisms described in Diagram 3.1.2 actually exist. One part is to see that \( m \circ i \circ j \) factors through \( \ker(b) \to B \) (as described in the following diagram). To do this it is enough to check that \( b \circ m \circ i \circ j = 0 \), by the definition of kernels.

\[
\begin{array}{ccc}
F(\ker(a)) & \xrightarrow{ioj} & F(A) \\
\downarrow m & & \downarrow b \\
\ker(b) & \xrightarrow{B} & B' \\
\end{array}
\]

From the diagram we can see that it is possible to define a morphism \( F(\ker(a)) \to \ker(b) \): according to Diagram 3.1.3, we can say that the composition \( Fa \circ i \circ j = 0 \). And if we compose it with \( m' \), clearly the result is also zero. On the other side, and because the diagram is commutative, we know that \( b \circ m \circ i \circ j = 0 \) as well. And in this way we described the wanted correspondence.

It remains to prove that the map \( \ker(b) \to G(A) \) factors through \( G(\ker(a)) \cong \ker(G(a)) \). We look at a detail of the diagram

\[
\begin{array}{ccc}
\ker(b) & \xrightarrow{n} & G(A) \\
\downarrow 0 & & \downarrow Ga \\
\ker(G(a)) & & \\
\end{array}
\]

If we apply the composition \( n' \circ b \) to \( \ker(b) \) the result will, obviously, be zero. Since \( n' \circ b = Ga \circ n \) (recall 3.1.1) we first notice that, by the property of kernel of \( G(a) \), the morphism \( n \), between \( \ker(b) \) and \( G(A) \), can be factorized through \( \ker(G(a)) \). As \( \ker(G(a)) \cong G(\ker(a)) \), we have shown that
n : \text{ker}(b) \to G(A) \text{ factors through } G(\text{ker}(a)). \text{ We saw that the diagram of compositions } \text{(3.1.2)} \text{ is valid.}

Using the definition, we are going to prove that \text{(3.1.2)} is a kernel. Let X, Y, Z and W be objects in a category \( \mathcal{C} \). We say that \((W, h)\) is a kernel for the morphism \( f : X \to Y \) if \( h : W \to X \) is a monomorphism with \( fh = 0 \) and such that any morphism \( g : Z \to X \) with \( fg = 0 \) factors through \( h \), \( g = hg' \).

\[
\begin{array}{c}
Z \xrightarrow{g} X \xrightarrow{f} Y \\
W \xleftarrow{h}
\end{array}
\]

Let \((A'', B'')\) be an object in \( \mathcal{C}(F, G; T) \) and \((a'', b'') : (A'', B'') \to (A, B)\) a morphism in \( \mathcal{C}(F, G; T) \) such that:

\[
\begin{array}{ccc}
A'' & \xrightarrow{a''} & A \\
\downarrow & & \downarrow \\
F(a'') & \xrightarrow{(0,0)} & F(A)
\end{array}
\quad
\begin{array}{ccc}
B'' & \xrightarrow{b''} & B \\
\downarrow & & \downarrow \\
G(b'') & \xrightarrow{(a,b)} & G(A)
\end{array}
\]

Let \( i_{FA}, i_{GA} \) and \( i_B \) denote the inclusions:

- \( i_{FA} : F(\text{ker}(a)) \hookrightarrow F(A) \)
- \( i_{GA} : G(\text{ker}(a)) \hookrightarrow G(A) \)
- \( i_B : \text{ker}(b) \hookrightarrow B \)

In order to make the diagrams understandable, we show only the objects that include the pairs \( (A, B), (A'', B''), (\text{ker}(a), \text{ker}(b)) \) and their relations.
We observe that \((F \alpha'', G \alpha'', b'')\) factor through \((i_{FA}, i_{GA}, i_B)\), all the conditions are satisfied and we proved that (3.1.2) is the kernel in this case.

### 3.1.2. Cokernel

We are going to show that, for the present situation, the cokernel is the pair \((cok(a), cok(b))\) together with the commutative triangle

\[
\begin{array}{ccc}
F(cok(a)) & \rightarrow & G(cok(a)) \\
\downarrow m_c & & \downarrow n_c \\
cok(b) & & \\
\end{array}
\quad \text{(3.1.4)}
\]

First we want to prove that the maps in the following diagram exist:

\[
\begin{array}{ccc}
G(A) & \rightarrow & G(cok(a)) \\
\downarrow T_A & & \downarrow T_A \\
F(A') & \rightarrow & F(cok(a)) \\
\downarrow m' & & \downarrow m_c \\
B' & \rightarrow & cok(b) \\
\end{array}
\]

All horizontal maps exist canonically. \(T_A\) and \(T_A\) are clear to exist by the description of the category. It remains to explain \(m_c\) and \(n_c\).

To describe the cokernel we have to follow a process similar to the one of the previous subsection. As \(A\) and \(B\) are abelian categories we have that:

- every morphism has a kernel and a cokernel;
- given a morphism \(M \xrightarrow{f} N\), in the canonical factorization \(M \rightarrow coim(f) \rightarrow im(f) \rightarrow N\), the map \(coim(f) \rightarrow im(f)\) is an isomorphism.

From morphism \(a: A \rightarrow A'\) we get the sequence

\[
ker(a) \rightarrow A \rightarrow coim(a) \xrightarrow{\simeq} im(a) \rightarrow A' \rightarrow cok(a),
\]

giving rise to the diagram

\[
\begin{array}{ccc}
F(ker(a)) & \xrightarrow{\gamma} & F(A) & \xrightarrow{\delta} & F(im(a)) & \xrightarrow{\epsilon} & F(A') & \xrightarrow{\xi} & F(cok(a)) \\
\downarrow T & & \downarrow T & & \downarrow T & & \downarrow m' & & \downarrow T \\
G(ker(a)) & \xrightarrow{\gamma'} & G(A) & \xrightarrow{\delta'} & G(im(a)) & \xrightarrow{\epsilon'} & G(A') & \xrightarrow{\xi'} & G(cok(a)) \\
\downarrow m & & \downarrow n & & \downarrow \delta & & \downarrow m_c & & \downarrow \xi \\
B & \xrightarrow{b} & B' & \rightarrow & cok(b) \\
\end{array}
\]

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Remark 3.1.2. The morphism \( \zeta \) is a surjection because \( F \) is a right exact functor and therefore takes surjections to surjections (see Section 2.1.2).

Having a close look at a detail of the diagram

\[
\begin{array}{c}
F(A) \xrightarrow{\delta} F(A') \xrightarrow{\zeta} F(cok(a)) \\
\downarrow m \downarrow m' \downarrow \downarrow \\
B \xrightarrow{b} B' \xrightarrow{\psi} cok(b)
\end{array}
\]

we observe that the composition \( \psi \circ b \circ m = 0 \). Because the diagram is commutative we know that \( \psi \circ m' \circ \delta \) is also 0. Then, naturally it exists a morphism 0 from \( F(A) \) to \( cok(b) \) and hence, by the definition of cokernel, the morphism between \( F(cok(a)) \) and \( cok(b) \) exists. The map \( m_c \) is then defined.

Now we want to consider the morphism \( n_c \). Observe the diagram

\[
\begin{array}{c}
G(A) \xrightarrow{\delta'} G(A') \xrightarrow{\zeta'} G(cok(a)) \\
\uparrow n \uparrow n' \uparrow \uparrow \\
B \xrightarrow{b} B' \xrightarrow{\psi'} cok(b)
\end{array}
\]

Observe that \( \zeta' \circ \delta' \circ n = 0 \). Because we are in a commutative diagram \( \zeta' \circ n' \circ b = 0 \) as well. So, there is a natural morphism 0 from \( B' \) to \( G(cok(a)) \). And again, by the definition of cokernel, we have the desired morphism between \( cok(b) \) and \( G(cok(a)) \). The map \( n_c \) is finally defined.

Using the definition, we are going to prove that Diagram (3.1.4) represents a cokernel. Let \( X, Y, Z \) and \( K \) be objects in category \( C \). We say that \( (K, p) \) is a cokernel for the morphism \( f : X \rightarrow Y \) if \( p : Y \rightarrow K \) is an epimorphism with \( pf = 0 \) and such that any morphism \( g : Y \rightarrow Z \) with \( gf = 0 \) factors through \( p \), such that \( g = g'p \).

\[
\begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{g} Z \\
\uparrow 0 \downarrow \uparrow \\
\downarrow p \downarrow \\
K \xrightarrow{g'}
\end{array}
\]

Let \((A'', B'')\) be an object in \( C(F, G; T) \) and \((a', b') : (A', B') \rightarrow (A'', B'')\) a morphism in \( C(F, G; T) \) such that:

\[
\begin{array}{c}
A' \xrightarrow{a'} A'' \\
B' \xrightarrow{b'} B''
\end{array}
\]
and

\[
\begin{array}{c}
(A, B) \\
\downarrow (a, b) \quad \downarrow (a', b') \\
(A', B') \\
\downarrow \quad \downarrow \\
(0, 0) \\
\end{array}
\]

Let \( s_{FA'} \), \( s_{GA'} \) and \( s_{B'} \) denote the epimorphisms:

- \( s_{FA'} : F(A') \hookrightarrow F(\text{cok}(a)) \)
- \( s_{GA'} : G(A') \hookrightarrow G(\text{cok}(a)) \)
- \( s_{B'} : B' \hookrightarrow \text{cok}(b) \)

Due to the amount of relations represented by arrows we only present in the diagrams for the pairs \((A', B')\), \((A'', B'')\) and \((\text{cok}(a), \text{cok}(b))\).

The maps \((Fa', Ga', b')\) factor through \((s_{FA'}, s_{GA'}, s_{B'})\), the relations are the required ones, \((3.1.4)\) is a cokernel for the described morphisms.

### 3.1.3. Irreducible Objects

We want to study the irreducible objects in the category \( \mathcal{C}(F, G; T) \). In order to do that we have to describe functors defined in \([17]\), that allow us to go from the category \( \mathcal{C}(F, G; T) \) to the categories \( \mathcal{A} \) and \( \mathcal{B} \). In the category \( \mathcal{C}(F, G; T) \) we have the restriction functors \(|A| : \mathcal{C}(F, G; T) \rightarrow \mathcal{A}\) and \(|B| : \mathcal{C}(F, G; T) \rightarrow \mathcal{B}\) which are defined as follows. Let \( N \in \text{Obj}(\mathcal{C}(F, G; T)) \) be given by \((A, B) \in \text{Obj}(\mathcal{A}) \times \text{Obj}(\mathcal{B})\) and factorization \( FA \xrightarrow{m} B \xrightarrow{n} GA\). Then we define \( N|A = A \) and \( N|B = B\).

We also have an inclusion functor \( \mathcal{B} \rightarrow \mathcal{C}(F, G; T) \). The functor \(|\mathcal{A}\) has a left and a right adjoints \( \hat{F} \) and \( \hat{G} \) which are given, respectively, by:
Finally there is a functor $\hat{T}: A \to \mathcal{C}(F,G; T)$ which is given by

We are going to assume that the functors $F$ and $G$ have the property:

- $A \neq 0 \implies FA \neq 0$;
- $A \neq 0 \implies GA \neq 0$.

Let $(A, B)$ and $(A', B')$, together with the correspondent commutative triangles, be objects of $\mathcal{C}(F,G; T)$ and let $(a, b): (A, B) \to (A', B')$ be a morphism in $\mathcal{C}(F,G; T)$. Since $\mathcal{C}(F,G; T)$ is an abelian category, the morphism $(a, b)$ is injective if and only if $\ker(a, b) = 0$.

The kernel of the morphism $(a, b)$ is zero when

$$F(\ker(a)) \to G(\ker(a)) = 0 \to 0$$

That is, when $F(\ker(a)) = 0$, $G(\ker(a)) = 0$ and $\ker(b) = 0$. According with the assumption above, $F(\ker(a)) = G(\ker(a)) = 0$ indicates that $\ker(a) = 0$. Hence:

**Corollary 3.1.3.** The morphism $(a, b): (A, B) \to (A', B')$ is injective if and only if the morphisms $a: A \to A'$ and $b: B \to B'$ are injective.

We are interested in the irreducible objects of $\mathcal{C}(F,G; T)$.

**Definition 3.1.4.** The object $N$ of $\mathcal{C}(F,G; T)$ is irreducible if and only if one of the following conditions is satisfied:

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• for every injection $M \to N$, we have: $M = N$ or $M = 0$;

• for every surjection $N \to P$, we have: $P = N$ or $P = 0$.

According to [17]:

**Proposition 3.1.5.** All the irreducible objects in $C(F;G;T)$ are either of the form $\hat{T}(L)$, where $L$ is irreducible in $A$, or of the form

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
L & \to & 0
\end{array}
\]

(3.1.5)

where $L$ is irreducible in $B$.

We give a detailed proof below, expanding the presentation in [17].

**Proof. Part 1:** first we are going to show that these two kind of objects are irreducible.

First case: Suppose that there exists an object $(A,B)$ such that $(a,b) : (A,B) \to \hat{T}(L)$ is an injection:

\[
\begin{array}{ccc}
FA & \xrightarrow{T_A} & GA \\
\downarrow Fa & & \downarrow Ga \\
B & \xrightarrow{b} & GL \\
\downarrow T_L & & \downarrow GL \\
FL & \xleftarrow{im(T_L)} & GL
\end{array}
\]

As the morphism $(a,b)$ is injective we know that both $a$ and $b$ are injective. In the category $\mathcal{A}$ we have $a : A \to L$ with $a$ injective and $L$ irreducible, so it means that either $A = 0$ or $A = L$. Then we have two possibilities:

1. $A = 0$:
The square with vertex $B, O, GL$ and $im(T_L)$ is commutative. The composition $(Ga \circ n)$ from $B$ to $GL$ is zero. Therefore the composition $(n' \circ b)$ is also zero. According to Lemma 2.1.7 point 1, $(n' \circ b)$ is an injection and for the result to be zero we must have $B = 0$.

\[2.A = L:\]

\[\begin{array}{c}
FL \\
\downarrow F a \\
\downarrow \downarrow \\
B \\
\downarrow b \\
GL \\
\end{array} \quad \begin{array}{c}
T_L \\
\downarrow m \\
\downarrow \downarrow \\
0 \\
\end{array} \quad \begin{array}{c}
GL \\
\downarrow Ga \\
\downarrow \downarrow \\
F L \\
\end{array}
\]

The square with vertex $FL, B, im(T_L)$ and $FL$ is commutative. In this case $Fa$ and $Ga$ are isomorphisms The composition $(m' \circ Fa)$ gives the same result that the composition $(b \circ m)$. As $m'$ is a surjection and $Fa$ an isomorphism, the composition $(m' \circ Fa)$ is a surjection. By Lemma 2.1.7 point 4, we can say that $b$ is an epimorphism, that is, a surjection. But, by hypothesis, $b$ was an injection. That means that $b$ is an isomorphism and therefore $B \equiv im(T_L)$.

We showed that if $(a, b) : (A, B) \to \hat{T}(L)$ is an injection then $(A, B) = (0, 0)$ or $(A, B) = \hat{T}(L)$, proving that $\hat{T}(L)$ is irreducible.

Second case: Suppose that there exists an object $(A, B)$ such that $(a, b) : (A, B) \to (0, L)$ is an injection:

\[\begin{array}{c}
FA \\
\downarrow Fa \\
\downarrow \downarrow \\
B \\
\downarrow b \\
0 \\
\end{array} \quad \begin{array}{c}
T_A \\
\downarrow \\
\downarrow \downarrow \\
GA \\
\end{array} \quad \begin{array}{c}
GA \\
\downarrow Ga \\
\downarrow \downarrow \\
0 \\
\end{array} \quad \begin{array}{c}
\downarrow \downarrow \\
L \quad \downarrow \downarrow \\
\end{array}
\]

As before, we know that both $a$ and $b$ are injective. In the category $A$ we have $a : A \to 0$ with $a$ injective so we can say that $a \equiv 0$. By other side, in $B$ we have $b : B \to L$ with $b$ injective and $L$ irreducible in $B$, so it means that either $B = 0$ or $B = L$. Therefore we can say that
\((A, B) = (0, 0)\) or \((A, B) = (0, L)\)

So, the object in Diagram (3.1.5) is irreducible.

**Part 2:** now we are going to prove that the only irreducible objects in \(C(F; G; T)\) are the ones described above.

Suppose

\[
\begin{array}{ccc}
FA & \xrightarrow{T_A} & GA \\
\downarrow{m} & & \downarrow{n} \\
B & &
\end{array}
\]

is an irreducible object. Let \((A, im(m))\) be an object of \(C(F; G; T)\) and \((p, q)\) a morphism such that \((p, q) : (A, im(m)) \to (A, B)\).

The morphism \((p, q)\) is an injection because:

- \(p : A \to A\) is an isomorphism;
- \(im(m) \in B\) and therefore the morphism \(q : im(m) \to B\) is an injection.

As \((A, B)\) is irreducible, we know that \((A, im(m))\) is either \((0, 0)\) or \((A, B)\).

\((A, im(m)) = (A, B)\): In this case the morphism \((p, q)\) is an isomorphism and then \((A, B)\) is

\[
\begin{array}{ccc}
FA & \xrightarrow{T_A} & GA \\
\downarrow{m} & & \downarrow{n} \\
 im(m) & &
\end{array}
\]

Let \(K\) be the kernel of the morphism \(im(m) \xrightarrow{n} GA\). There is an obvious injective morphism from the object \((0, K)\) to \((A, im(m))\). There are two possibilities:

- \(K \neq 0\): \((0, K)\) would be a subobject of \((A, im(m))\), different from itself or from \((0, 0)\), contradicting the initial assumption of irreducibility of \((A, im(m))\);
- \(K = 0\): this means that the morphism \(n\) is injective; therefore \(im(m) \cong im(T_A)\).

In this way we prove that \((A, B) = (A, im(T_A))\).

We have left to prove that \(A\) is irreducible in \(\mathcal{A}\). Suppose that there exists \(A'\) in \(\mathcal{A}\) such that the morphism \(a : A' \to A\) is an injection and \(A' \neq 0, A\).
Consider \((a, b)\) the morphism between \((A', \text{im}(T_A'))\) and \((A, \text{im}(T_A))\):

\[
\begin{array}{ccc}
FA' & \xrightarrow{T_A} & GA' \\
\downarrow{F a} & & \downarrow{G a} \\
FA & \xrightarrow{m} & GA \\
\downarrow{m'} & & \downarrow{n'} \\
nm(T_A') & \xrightarrow{n'} & \text{im}(T_A) \\
\downarrow{b} & & \downarrow{b} \\
FA & \xrightarrow{T_A} & GA \\
\downarrow{m} & & \downarrow{m} \\
\text{im}(T_A) & \xrightarrow{n} & \text{im}(T_A) \\
\end{array}
\]

Because \(G\) is a left exact functor we have \(G(\ker(a)) \approx \ker(G(a))\). As \(\ker(a) = 0\) we know that \(G(\ker(a)) = 0\) and therefore \(\ker(G(a)) = 0\), meaning that \(G(a)\) is injective.

According to Lemma 2.1.7 point 1, the composition \(G(a) \circ n'\) is injective and as the diagram is commutative \(n \circ b\) has to be also a monomorphism. By Lemma 2.1.7 point 2, we know that \(b\) is an injective morphism.

This tells us that \((a, b)\) is also injective. And now we got a contradiction, because \((A, \text{im}(T_A))\) is irreducible and we could define an injective morphism from \((A', \text{im}(T_A'))\) to \((A, \text{im}(T_A))\) with \((A', \text{im}(T_A')) \neq (A, \text{im}(T_A))\) and \((A', \text{im}(T_A')) \neq 0\).

We conclude then that our assumption saying that the object \(A\) is not irreducible in \(\mathcal{A}\) isn’t correct.

\((A, \text{im}(m)) = (0, 0)\): It means that \(F(A) = 0\), \(G(A) = 0\) and \(\text{im}(m) = 0\). But, because \(p\) is an isomorphism the object \((A, B)\) is given by

\[
\begin{array}{ccc}
0 & \xrightarrow{p} & 0 \\
\downarrow & & \downarrow \\
B & & B
\end{array}
\]

where \(B\) has to be an irreducible object in \(\mathcal{B}\) (so that the initial supposition, of the irreducibility of \((A, B)\), holds.)

\[\square\]

### 3.2 The equivalence between \(\mathcal{M}(X)\) and \(\mathcal{C}(F, G; T)\)

We relate the categories \(\mathcal{M}(X)\) and \(\mathcal{C}(F, G; T)\) through a construction that associates perverse sheaves in an open set to local systems in the space defined by the closed strata. We construct \(\mathcal{M}(X)\) from \(\mathcal{M}(X - S)\), by induction
on the strata of $X$. An object of $\mathcal{M}(X)$ is an object $A^* \in \mathcal{M}(X - S)$ together with a commutative triangle

\[
\begin{array}{ccc}
FA^* & \xrightarrow{T_A} & GA^* \\
\downarrow m & & \downarrow n \\
B & & B
\end{array}
\]

$\mathcal{M}(X)$ is obtained by gluing the information collected from the smaller spaces. We are going to quickly present some fundamental concepts and give a general overview of this construction (for more details see [10], [11] and [12]).

3.2.1. Construction  Now we are going to apply the theory of $C(F,G;T)$ to a concrete situation. For completeness, we include the definitions of the functors $F$, $G$ and $T$.

Let $\mathcal{S}$ be a stratification of $X$ and $S$ a closed contractible stratum of dimension $2d$. Let $T_S$ be the tubular neighborhood of $S$, $\pi_S : T_S \to S$ the projection and $\rho_S$ a function measuring the distance to $S$.

Definition 3.2.1. ([11, Definition 4.1.]) The link bundle $\pi : L \to S$ and the normal slice bundle $\pi' : D \to S$ are defined as follows: for a small enough positive valued function $\epsilon : S \to \mathbb{R}$,

\[
L = \{ x \in T_S \mid \rho_S(x) = \epsilon(\pi_S(x)) \}
\]

\[
D = \{ x \in T_S \mid \rho_S(x) \leq \epsilon(\pi_S(x)) \}.
\]

The maps $\pi$ and $\pi'$ are restrictions of $\pi_S$.

Definition 3.2.2. ([10, Définition], pp.444) Let $L$ be the link bundle of $S$, $K$ a closed subset of $L$, $\kappa : K \to L$ an inclusion and $\gamma : L - K \to L$ the inclusion of the open complement. We say that $K$ is a perverse link bundle if:

- $R^i(\pi \circ \kappa)_* A^* | K \cong 0$ for all $i \geq -(\dim S)/2$ and all $A^* \in \mathcal{M}(X - S)$;
- $R^i \pi_* \gamma! A^* | L - K \cong 0$ for all $i < -(\dim S)/2$ and all $A^* \in \mathcal{M}(X - S)$;

Remark 3.2.3. A perverse link bundle always exists.

Let $F$ be the functor that sends $A^*$ to $R^{-1}(\pi \circ \kappa)_* (A^* | K)[-\dim S/2]$ and $G$ the functor that sends $A^*$ to $R^0 \pi_* \gamma! A^* | (L - K)[-\dim S/2]$ and $T$ be induced by $\delta : \kappa_* (A^* | K) \to \gamma!(A^* | L - K)[1]$.  

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Proposition 3.2.4. (Théorème 1) The category $\mathcal{M}(X)$ is equivalent to the full subcategory $\tilde{\mathcal{C}}$ of $\mathcal{C}(F,G;T)$ whose objects satisfy the following condition: in the factorization $FA^* \rightarrow B \rightarrow GA^*$, the cokernel of the morphism with extremity $B$ is a local system over $S$ (or, in an equivalent way, the kernel of the morphism with origin in $B$ is a local system). The equivalence is given by the functor $E : \mathcal{M}(X) \rightarrow \mathcal{C}(F,G;T)$ that sends $E^* \in \mathcal{M}(X)$ to $(E^* | X - S, R^0\pi^*\varphi_! (E^* | D - K)(-(dimS)/2))$ (where $\varphi : D - K \rightarrow D$ denotes the inclusion).

We can present the relation between $\mathcal{M}(X)$ and $\mathcal{C}(F,G;T)$ in a more specific way, the complex analytic case (see [11], Section 5).

Let $X$ be a complex manifold with a given Whitney stratification and $S$ a stratum of complex dimension $d$. Denote

\[ \Lambda_S = T^*_S X \quad \text{and} \quad \tilde{\Lambda}_S = \Lambda_S - \bigcup_{R \neq S} T^*_R X, \]

where $T^*_S X$ is the cotangent bundle of $X$. We define:

- the right exact functor $\psi : \mathcal{M}(X - S) \rightarrow \{\text{local systems in } \tilde{\Lambda}_S\}$, called the nearby cycles;
- the left exact functor $\psi_c : \mathcal{M}(X - S) \rightarrow \{\text{local systems in } \tilde{\Lambda}_S\}$, called the nearby cycles with compact support;
- a functor $\Phi : \mathcal{M}(X) \rightarrow \{\text{local systems in } \tilde{\Lambda}_S\}$, called the vanishing cycle;
- a natural transformation $var : \psi \rightarrow \psi_c$, called the variation.

In order to give a more concrete description of our functors $F$ and $G$, we need some more definitions. Consider the fiber bundle $\pi : D \rightarrow \tilde{\Lambda}_S$, a sub-bundle $L \subset D$, a further sub-bundle $\mathcal{L} \subset L$ and the inclusions $\kappa : \mathcal{L} \hookrightarrow L$ and $\gamma : L - \mathcal{L} \hookrightarrow L$. We define (see [11], Section 5):

\[ FP^* = \psi(P^*) = R^{-d-1}\pi^*\kappa_!\kappa^*P^* \quad GP^* = \psi_c(P^*) = R^{-d}\pi^*\gamma_!\gamma^*P^* \quad (3.2.1) \]

By taking $A = \mathcal{M}(X - S)$, $B = \{\text{local systems in } \tilde{\Lambda}_S\}$, $T = var$, $p : \tilde{\Lambda}_S \rightarrow S$ as the projection and $F$ and $G$ like above, we have the information to construct a category $\mathcal{C}(F,G;T)$.

The next two propositions give us the desired equivalence (for $m$, $n$ as in Diagram 3.1.1).
Proposition 3.2.5. ([11 Prop.5.2.]) If S is contractible, then there are unique natural transformations $\alpha : \psi_c \to \psi$ for any $\alpha \in \pi_1(\tilde{\Lambda}S)$ such that, for any $P^* \in \mathcal{M}(X)$, the local system structure on $\Phi(P^*)$ is given by $\mu_a - 1 = m \circ I_\alpha \circ n$, where $\mu$ is the monodromy of $\alpha$ on $\Phi(P^*)$.

Proposition 3.2.6. ([11 Prop.5.3.]) The category $\mathcal{M}(X)$ is equivalent to the subcategory $\mathcal{C}$ of the category $\mathcal{C}(F,G;T)$ whose objects satisfy the condition that in the factorization $FP^* \to B \to GP^*$ the monodromy $\mu_a$ on $B$, for any $\alpha \in \pi_1(p^{-1}(x))$, $x \in S$, is given by $\mu_a - 1 = m \circ I_\alpha \circ n$.

In the case which we will study, $I_\alpha = Id$ (see [12], pp.91).

Before describing a more concrete situation we are going to present some definitions and results that will be important for our work.

3.2.1.1 The functors $\pi_j P^*$, $\pi_j P^*$ and $\pi_j P^*$

In the category $\mathcal{C}(F,G;T)$ there are three fundamental functors, $\hat{F}, \hat{T}$ and $\hat{G}$, that allow us to extend an object in a category $\mathcal{A}$ to an object in $\mathcal{C}(F,G;T)$ (see Section 3.1.3). These functors have their correspondents in $\mathcal{M}(X)$. Recall the definitions of $j_!, j_*, j^!, i_!, i^*$ and $i^!$ given earlier.

Definition 3.2.7. Let $C, C_U$ and $C_F$ be the hearts of the t-categories $D, D_U$ and $D_F$ and $\epsilon$ the inclusion of $C, C_U$ and $C_F$ in $D, D_U$ and $D_F$. Let $T$ be any of the functors $j_!, j^!, j_*, i^!, i_*$ and $i^!$. $\pi T$ is defined as $H^0 \circ T \circ \epsilon$.

We have the following equivalent definition:

Definition 3.2.8. Let $A$ be an object of the category of perverse sheaves $\mathcal{M}(X)$. We define $\pi T(A) = \tau_{\geq p} T(A) = H^p(T(A))$.

Definition 3.2.9. The functor $\pi_j P^*$ is given by $im(\pi_j P^* \to \pi_j P^*)$.

Now we are in conditions of describing the relations referred above.

Proposition 3.2.10. (see [11], pp.418) The functors $\pi_j P^*$, $\pi_j P^*$ and $\pi_j P^*$ correspond, respectively, to the functors $\hat{F}, \hat{T}$ and $\hat{G}$, and can be understood as factorizations of $T : FP^* \to GP^*$:

- $\pi_j P^* : FP^* \xrightarrow{id} FP^* \xrightarrow{T} GP^*$;
- $\pi_j P^* : FP^* \xrightarrow{\pi_j P^*} i_m(T) \xrightarrow{i_m} GP^*$;
- $\pi_j P^* : FP^* \xrightarrow{T} GP^* \xrightarrow{id} GP^*$.
We can describe the morphism between the functors $p_j^*P$ and $p_j^*P^*$ in the following, obviously commutative, diagram:

\[
\begin{align*}
FP^* & \xrightarrow{\cong} FP^* \\
\downarrow{id} & \downarrow{T} \\
FP^* & \rightarrow FP^* \\
\downarrow{T} & \downarrow{id} \\
GP^* & \xrightarrow{\cong} GP^*
\end{align*}
\]

Through this diagram is immediate to see the definition of $p_j^*P^*$ is the image of $p_j^*P \rightarrow p_j^*P^*$.

**Example 3.2.1.** Recall the sheaf $\mathcal{L}_\alpha$ in $\mathbb{C}^*$, defined in Section 2.1.3. $\mathcal{L}_\alpha$ is a perverse sheaf, therefore the computation of $Rj^*\mathcal{L}_\alpha$ corresponds to the computation of $p_j^*\mathcal{L}_\alpha$ and can be represented by the following diagram:

\[
p_j^*\mathcal{L}_\alpha : \quad \mathbb{C} = \mathcal{L}_{(1)} \xrightarrow{\alpha^{-1}} \mathcal{L}_{(1)} = \mathbb{C}
\]

According to the value of $\alpha$ we have two possibilities:

- $\alpha \neq 1$: the diagram corresponds to an irreducible object, by Proposition 3.1.5

- $\alpha = 1$: the diagram does not represent an irreducible object, thus we can define its composition series

\[
\begin{align*}
\mathbb{C} & \xrightarrow{0} \mathbb{C} \\
\downarrow{0} & \downarrow{0} \\
0 & \rightarrow 0
\end{align*}
\]

which corresponds to the decomposition in Proposition 2.3.1.
3.2.2. Main Result

3.2.2.1 General view

We will give the calculations in the case which interest us, following the steps of [12].

Consider the space $\mathbb{C}^2$ and the stratification $\{0\} \subset S \subset \mathbb{C}^2$, where $S = \bigcup_{i=1}^{n} S_i$ is a curve given by the equation $y^n = x^n$ and each component is given by $S_i = \{x + \epsilon_i^n y = 0\}$, a line passing through the origin. This is a Whitney stratification. In this phase we will give a description of $M(\mathbb{C}^2 - \{0\})$. According to the previous subsection, we define $\Lambda = T^*_S \mathbb{C}^2 \cup \overline{T^*_S \mathbb{C}^2} - \{0\} \cup T^*_0 \mathbb{C}^2$.

The standard hermitian metric allows us to identify $T^*_i S_i \mathbb{C}^2$ (the cotangent bundle) with $T S_i \mathbb{C}^2$ (the tangent bundle). Let $U_i$ be a neighborhood of the zero-section of $T^*_i S_i \mathbb{C}^2$ and $V_i$ a neighborhood of $S_i$ in $\mathbb{C}^2$. By the tubular neighborhood theorem we know that there exists a diffeomorphism between $U_i$ and $V_i$. If $A$ is a local system in $\mathbb{C}^2 - S$, for each $\tilde{\Lambda}_S_i$, we can define a local system $\tilde{A}_i$. $\tilde{A}_i$ is obtained by the pullback of the following maps between fundamental groups:

$$\pi_1(\tilde{\Lambda}_S_i) \xrightarrow{\cong} \pi_1(U_i - S_i) \xrightarrow{\cong} \pi_1(V_i - S_i) \to \pi_1(\mathbb{C}^2 - S).$$

Noticing that $\tilde{\Lambda}_S_i = S_i \times \mathbb{C}^*$ and considering the projection $\pi : \tilde{\Lambda}_S_i \to S_i$, we observe that $\tilde{\Lambda}_S_i$ is a trivial $\mathbb{C}^*$-bundle over $S_i$. Thus, we get an isomorphism $\pi_1(\tilde{\Lambda}_S_i) \cong \pi_1(S_i) \times \pi_1(\mathbb{C}^*)$, by choosing a section of the bundle. From now on we will assume that this section and the base point are fixed. Let $\gamma_i$ be the generator of $\pi_1(\mathbb{C}^*)$ in $\pi_1(\tilde{\Lambda}_S_i)$ and $\Gamma_i$ be the image of $\gamma_i$ in $\pi_1(\mathbb{C}^2 - S)$. If $A$ is a local system in $\mathbb{C}^2 - S$, then $A$ is a perverse sheaf in $\mathbb{C}^2 - S$, $A \in M(\mathbb{C}^2 - S)$.

Recalling the definition of the previous subsection we get:

**Lemma 3.2.11.** ([12, Lemma 1.1]) We have $\psi(A) = \psi_c(A) = \tilde{A}_i$ on $\tilde{\Lambda}_S_i$ and the variation map $\text{var} : \tilde{A}_i \to \tilde{A}_i$ is given by $\text{var}(a) = \gamma_i(a) - a$.

From this result we can build a category $Q_\Lambda$, consisting of the pair $(\tilde{A}_i, B_i)$ (where $B_i$ is a $\pi_1(S_i)$-module), and a commutative diagram

$$\tilde{A}_i \xrightarrow{\text{var}} \tilde{A}_i$$

$$\downarrow \quad \downarrow$$

$$B_i \quad B_i$$

And hence we have the proposition:

**Proposition 3.2.12.** ([12, Proposition 1.2.] The category $M(\mathbb{C}^2 - \{0\})$ is equivalent to $Q_\Lambda$.}
We know that the monodromy defines a functor from local systems on \( \mathbb{C}^2 - S \) to finite dimensional representations of \( \pi_1(\mathbb{C}^2 - S) \) on the vector space \( \mathbb{C} \).e. Now we want to apply this.

A representation of \( \pi_1(\mathbb{C}^2 - S) \) is given by \( \langle \Gamma_1, \ldots, \Gamma_n \rangle / R \), where \( R \) is the group generated by the relations \( \Gamma_1 \Gamma_2 \ldots \Gamma_n = \Gamma_2 \ldots \Gamma_n \Gamma_1 = \Gamma_n \Gamma_1 \ldots \Gamma_{n-1} \). Observing that an action of \( \pi_1(\mathbb{C}^2 - S) \) on a vector space \( \mathbb{C} \).e is given by \( \{ \Gamma_i.e = a_i.e \mid i = 1, \ldots, n \} \) and that \( (\Gamma_1, \ldots, \Gamma_n).e = (\Gamma_i, \ldots, \Gamma_{i-1}).e = \prod_{j=1}^n a_i.e \), we can say that there is a correspondence between local system of rank 1 in \( \mathbb{C}^2 - S \) and multi-indices \( a = (a_1, \ldots, a_n) \) in \( \mathbb{C}^n \) (note that the relations are automatically satisfied). We denote the locally constant sheaf correspondent to \( a \) by \( L_a \).

The pull-back of \( L_a \) to \( V_i - S_i \subset \mathbb{C}^2 - S \) will be a locally constant sheaf of rank 1 and corresponds to a representation of \( \pi_1(V - S_i) = \pi_1(S_i) \times \pi_1(\mathbb{C}^*) \) on a one-dimensional complex space, \( \tilde{A}_i \cong \mathbb{C} \). The action of \( \Gamma_i \in \pi_1(\mathbb{C}^*) \) is given by multiplication by \( a_i : \Gamma_i(e) = a_i.e \).

Hence, by Lemma 3.2.11 the variation map is given by multiplication by \( a_i - 1 : \text{var}(e) = \Gamma_i(e) - e = a_i.e - e = (a_i - 1)e \). Thus, in our case, the variation is either an isomorphism, if \( a_i \neq 1 \), or 0, if \( a_i = 1 \).

### 3.2.2.2 Main Result: irreducible perverse sheaves

In particular, considering the inclusions

\[ \mathbb{C}^2 - S \xrightarrow{j_1} \mathbb{C}^2 - \{0\} \xrightarrow{j_2} \mathbb{C}^2 \]

and applying the results of Proposition 3.2.10 we know that \( p^i_j \tilde{A}_i \) can be represented as the following diagram

\[
\begin{array}{c}
\tilde{A}_i = \mathbb{C} \\
\downarrow p_i = a_i - 1 \\
B_i = \mathbb{C}
\end{array} \xrightarrow{a_i - 1} \begin{array}{c}
\tilde{A}_i = \mathbb{C} \\
\downarrow q_i = \text{id}
\end{array}
\]

(3.2.2)

Note that \( B_i \) is determined by the condition that the diagram corresponds to a direct image \( p^i_j \), since the map \( B_i \to \tilde{A}_i \) is the identity.

**Proposition 3.2.13.** \( p^i_j \mathcal{L}_a \) is irreducible in \( \mathcal{M}(\mathbb{C}^2 - \{0\}) \) if, and only if, \( a_i \neq 1 \), for all \( i = 1, \ldots, n \).

**Proof.** By Proposition 3.2.12 this means that \( p^i_j \mathcal{L}_a \) corresponds to the set of all diagrams [3.2.2] and by Proposition 3.1.5 this is irreducible exactly when \( a_i - 1 \neq 0 \), for all \( i = 1, \ldots, n \).
Now we want to extend $j_! \mathcal{L}_a = j_! j_*^2 (j_*^1 \mathcal{L}_a)$. First we need the following two lemmas from [12].

**Lemma 3.2.14.** [12, Lemma 2.2. and Lemma 2.3.] Given an object $P^*$ in $\mathcal{M}(\mathbb{C}^2 - \{0\})$, described by a set of diagrams (3.2.2) (so corresponding to a sheaf $\mathcal{L}_a$), we have:

- $\psi(P^*) = \text{cok}(A \xrightarrow{(p_1, \ldots, p_n)} B_1 \oplus \ldots \oplus B_n)$;
- $\psi_c(P^*) = \ker(B_1 \oplus \ldots \oplus B_n \xrightarrow{(q_1, \ldots, q_n)} A)$.

The map $\text{var} : \psi(P^*) \rightarrow \psi_c(P^*)$ is given by

$$\text{var}(b_ke_k) = (-\sum_{i=1}^{k+m} p_i a_{i-1} \ldots a_{k+1} b_ke_i) + \prod_{i=1}^{n} a_i b_ke_k - b_ke_k.$$

First all the indices are considered as integers modulo $n$. Secondly, although it was not referred in the original paper [12] we have to make 2 assumptions. If the sequence $a_{i-1} \ldots a_{k+1}$ is an increasing sequence (in indices) we consider that it is equal to 1. In the same sequence, if we have only two elements such that $a_{i-1} = a_{k+1}$, then we say that $a_{i-1} a_{k+1} = a_{k+1}$.

The maps

$$\oplus B_i = \mathbb{C}^n \rightarrow \psi(\mathcal{L}_a) \xrightarrow{\text{var}} \psi_c(\mathcal{L}_a) \rightarrow \mathbb{C}^n = \oplus B_i$$

will be given by the matrix $M$. Let $d_{ij}$ $(i, j = 1, \ldots, n)$ represent an entry of $M$, then:

$$d_{ij} = \begin{cases} (1 - a_r) \prod_{i=1}^{r-1} a_i \prod_{i=c+1}^{n} a_i, & \text{if } i < j \\ (-1 + \prod_{i=1}^{n} a_i)/a_c, & \text{if } i = j \\ (1 - a_r) \prod_{i=c+1}^{r-1} a_i, & \text{if } i > j \end{cases} \quad (3.2.3)$$

Here $c$ represents the number of the column and $r$ the number of the row. In the products, when the upper bound is less than the lower bound, we consider that the result is equal to 1.

**Example 3.2.2.** For $m = n = 4$, the matrix $M_{4 \times 4}$ is given by

$$
\begin{pmatrix}
-1 + a_2 a_3 a_4 & (1 - a_1) a_3 a_4 & (1 - a_1) a_4 & 1 - a_1 \\
1 - a_2 & -1 + a_1 a_3 a_4 & (1 - a_2) a_1 a_4 & (1 - a_2) a_1 \\
(1 - a_3) a_2 & 1 - a_3 & -1 + a_1 a_2 a_4 & (1 - a_3) a_1 a_2 \\
(1 - a_4) a_2 a_3 & (1 - a_4) a_3 & 1 - a_4 & -1 + a_1 a_2 a_3 
\end{pmatrix}
$$

The map $\text{var}$ is defined between spaces of dimension $n$, $\text{var} : \mathbb{C}^n \rightarrow \mathbb{C}^n$. $\psi(P^*)$ and $\psi_c(P^*)$ have dimension $n-1$, but through the right identifications it’s possible to show that both spaces are isomorphic to $\mathbb{C}^{n-1}$, with basis
We can then consider a map \( \var : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1} \). This map can be understood as a matrix \( M' \), based in the matrix above, but where we eliminate the first column and the last row. If \( \det(M') \neq 0 \), we may conclude that \( \var : \psi(P^*) \to \psi_c(P^*) \) is an isomorphism.

Now we prove the following result:

**Theorem 3.2.15.** The perverse sheaf \( R j_! L_\alpha \), where \( j : C^2 \cup \bigcup_{i=1}^n L_i \to C^2 \), is irreducible if, and only if, both of the following conditions are satisfied:

- \( a_i \neq 1 \), for all \( i = 1, \ldots, n \);
- \( \Pi_{i=1}^n a_i \neq 1 \).

In order to prove this theorem we are going to use the lemma:

**Lemma 3.2.16.** Let \( M' \) be the matrix obtained from \( M \), excluding the first column and the last row. Then, \( \det(M') = (-1)^{n-1}(-1 + a_1)(-1 + \Pi_{i=1}^n a_i)^{n-2} \). Hence, \( \var \) is an isomorphism if \( a_1 \neq 1 \) and \( \Pi_{i=1}^n a_i \neq 1 \).

**Remark 3.2.17.** The fact of having the factor \((-1 + a_1)\) in the expression of the determinant results of our choice of coordinates to define the matrix \( M' \). By choosing to exclude a different row or column, another \( a_i \) would be present in the expression of \( \det(M') \). Obviously, this change would not affect the other factors present in the expression of \( \det(M') \).

**Proof.** (of 3.2.16) We are going to prove this result by induction in the number of columns (and rows) of \( M \):

- \( k = 3 \):
  \[
  M_{3 \times 3} = \begin{bmatrix}
  -1 + a_2 a_3 & (1 - a_1) a_3 & 1 - a_1 \\
  1 - a_2 & -1 + a_1 a_3 & (1 - a_2) a_1 \\
  (1 - a_3) a_2 & 1 - a_3 & -1 + a_1 a_2 
  \end{bmatrix}
  \]

  It is easy to see the the determinant of \( M'_{2 \times 2} = (-1 + a_1)(1 - a_1 a_2 a_3) \).

- \( k = n \): assume that
  \[
  \det(M'_{n-1 \times n-1}) = (-1)^{n-1}(-1 + a_1)(-1 + \Pi_{i=1}^n a_i)^{n-2}
  \]

- \( k = n + 1 \): we want to show that
  \[
  \det(M'_{n+1 \times n+1}) = (-1)^{n}(-1 + a_1)(-1 + \Pi_{i=1}^{n+1} a_i)^{n-1}
  \]

  First notice that the only entries of \( M \) that don’t correspond to products are the ones in the main diagonal, the ones in the diagonal below the main diagonal and the entry in the top right, \( d_{11} \). We start by comparing the last two columns of matrices the \( M_{n \times n} \) and \( M_{n+1 \times n+1} \). Through the description
of \( d_{ij} \) in (3.2.3) it is easy to see that they are given by

\[
M_{n \times n} =
\begin{bmatrix}
\ldots & (1 - a_1) a_n & 1 - a_1 \\
\ldots & (1 - a_2) a_1 a_n & (1 - a_2) a_1 \\
\ldots & \ldots & \ldots \\
\ldots & -1 + (\prod_{i=1}^{n-1} a_i)/(a_{n-1}) & (1 - a_{n-1}) \prod_{i=1}^{n-2} a_i \\
\ldots & 1 - a_n & -1 + (\prod_{i=1}^{n} a_i)/(a_n)
\end{bmatrix}
\]

\[
M_{n+1 \times n+1} =
\begin{bmatrix}
\ldots & (1 - a_1) a_{n+1} & 1 - a_1 \\
\ldots & (1 - a_2) a_1 a_{n+1} & (1 - a_2) a_1 \\
\ldots & \ldots & \ldots \\
\ldots & -1 + (\prod_{i=1}^{n+1} a_i)/(a_{n}) & (1 - a_{n}) \prod_{i=1}^{n-1} a_i \\
\ldots & 1 - a_{n+1} & -1 + (\prod_{i=1}^{n+1} a_i)/(a_{n+1})
\end{bmatrix}
\]

As observed, for each one of the matrices, with the exception of the last two rows, the last two columns only differ by a multiplicative factor. This means that, in the computation of \( \det M' \), in both cases, if we decide to expand along the last row of \( M' \), all the determinants, with the exception of the last 2, are going to be zero, because we are going to be working with matrices in which one column is multiple of the other.

Comparing the matrix \( M'_{n-1 \times n-1} \) and the matrix \( M'_{n \times n} \) (excluding its last row), we notice that the last column is exactly identical and the column before that, only differs in one element: the multiplication is by \( a_{n+1} \) instead of \( a_n \). From (3.2.3) we can see, that:

- for all \( d_{ij} \), where \( i < j \), the only difference is that we add \( a_{n+1} \) to the product (with the exception of the last column, mentioned above);
- for all \( d_{ij} \), where \( i = j \), the product in the second term has one more element, \( a_{n+1} \);
- for all \( d_{ij} \), where \( i > j \), their value only depends on the row and column we are in, so, unless for the last row or column (that we are not considering), there are no differences between \( M'_{n-1 \times n-1} \) and \( M'_{n \times n} \).

Now, in \( M'_{n \times n} \), define \( \bar{a}_n = a_n a_{n+1} \). We can finally proceed to the computation of \( M'_{n \times n} \). We have seen that we have only two minors to consider, associated to the following entries:

- \( d_{n,n-1} \): we observe that, taking the definition of \( \bar{a}_n \) above, this minor has exactly the same structure as \( M'_{n-1 \times n-1} \), therefore the determinant is the same;
- \( d_{n,n} \): again, using \( \bar{a}_n \), we see that the structure of this minor differs from \( M'_{n-1 \times n-1} \) only on its last column, where all the entries are multiplied by \( a_{n+1} \), therefore its determinant is given by \( a_{n+1} \det (M'_{n-1 \times n-1}) \).
In conclusion, replacing now $\tilde{a}_n$ by $a_na_{n+1}$ (thus the upper bound in the product in the expression of $\det(M'_{n-1\times n-1})$ has to be changed to $n+1$):

\[
\det(M'_{n\times n}) = (-1)^{n+n-1}d_{n,n-1}\det(M'_{n-1\times n-1}) + (-1)^{n+n}d_{n,n-1}\det(M'_{n-1\times n-1}) = \\
= (-1)^{n-1}(1 + (\prod_{i=1}^{n+1} a_i)/a_n)(-1)^{n-1}(1 + a_1)(-1 + \prod_{i=1}^{n+1} a_i)^{n-2} + \\
+ (-1)^{2n}(1 - a_n)(\prod_{i=1}^{n-1} a_i)a_{n+1}(-1)^{n-1}(1 + a_1)(-1 + \prod_{i=1}^{n+1} a_i)^{n-2} = \\
= (((-1)^{n-1}(1 + a_1)(-1 + \prod_{i=1}^{n+1} a_i)^{n-2}) \\
[1 - a_1a_2\ldots a_{n-1}a_{n+1} + a_{n+1}(1 - a_n)a_1a_2\ldots a_{n-1}] = \\
= ((-1)^{n}(1 + a_1)(-1 + \prod_{i=1}^{n+1} a_i)^{n-1})
\]

\[
\frac{1}{n} - a_1a_2\ldots a_{n-1}a_{n+1} + a_{n+1}(1 - a_n)a_1a_2\ldots a_{n-1} \\
= ((-1)^{n}(1 + a_1)(-1 + \prod_{i=1}^{n+1} a_i)^{n-1})
\]

\[\square\]

**Proof.** (of Theorem 3.2.15) Assume that the numerical conditions of the theorem hold. Then, by Proposition 3.2.13, $P^* = p_j^*L_a$ is irreducible, and described by the set of diagrams 3.2.2. Hence, according to Lemma 3.2.14, $p_j^*P^*$ is described by the diagram

\[
\begin{array}{c}
\psi(P^*) \\
\downarrow \text{var} \\
\Phi(P^*) \\
\downarrow \\
\psi_c(P^*)
\end{array}
\]

where the map \textit{var} is an isomorphism, by the determinant calculation in Lemma 3.2.16. This is irreducible by Proposition 3.1.5 since it is of the form $\hat{\Gamma}(L)$.

Assume that $p_j^*L_a$ is irreducible. If, say $1 \equiv a_1$, then $p_j^*L_a$ is not irreducible by Proposition 3.2.13 and, since $p_j^2$ is exact (see [2, Proposition 1.4.16]), not $p_j^*L_a$ either. Hence $a_i \neq 1$, $i = 1,\ldots, n$. If then $\prod_{i=1}^{n} a_i = 1$, \textit{var} is not an isomorphism, and again, by Proposition 3.1.5, $p_j^*L_a$ is not irreducible (note that we know by Lemma 3.2.14 that $\psi(P^*)$ is non-zero).

Finally we note that $j$ is an affine morphism between affine varieties and so $Rj_*$ is t-exact and $p_j^* = Rj_*$ (see [4, Corollary 5.2.17]). \[\square\]

**Remark 3.2.18.** By Riemann-Hilbert correspondence, this relates to an assertion of D-modules, which was treated by Abebaw-Bøgvad in [1], Theorem 1.3.
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Matematiska Institutionen, Stockholms Universitet, 106 91 Stockholm
E-mail address: iara@math.su.se