Essays on the Economics of Networks Under Incomplete Information

Theodoros Rapanos

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Abstract
Social networks constitute a major channel for the diffusion of information and the formation of attitudes in a society. Introducing a dynamic model of social learning, the first part of this thesis studies the emergence of socially influential individuals and groups, and identifies the characteristics that make them influential. The second part uses a Bayesian network game to analyse the role of social interaction and conformism in the making of decisions whose returns or costs are ex ante uncertain.

Keywords: social networks, social learning, social influence, incomplete information, Bayesian game.

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Under Incomplete Information
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Essays on the Economics of Networks
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Theodoros Rapanos

Department of Economics
Stockholm University
Στην οικογένειά μου
Uppsatser om ekonomi av nätverk under ofullständig information

Sammanfattning på svenska

I denna avhandling används nätverksteori för att studera kommunikation och beslutsfattande under ofullständig information. Den består av tre forskningspapper/kapitel.

Kapitel 1 – Hur skapas en opinionsledare: expertis vs. popularitet

Kapitel 2 – Påtryckningsgrupper, experter och allmänheten: en nätverk-modell av politisk inflyttande

I denna artikel studeras utvecklingen av politiska övertygelser genom användandet av en nätverksmodell av socialt inflytande. Agenterna i modellen delar med sig av sin information och diskuterar sina åsikter med sina nätverkskontakter och uppdaterar därmed deras egen övertygelse. Trots att informationen som kommer från bättre informerade agenter ceteris paribus erhåller en större vikt, ser och kommuniserar individer informationen genom deras ideologiska prisma. Papperet studerar också nätverk med individer eller grupper som är inte intresserade av att lära sig eller att utbyta information, men snarare att marknadsföra sina åsikter till andra agenter. Särdragen som gör sådana grupper inflytelserika identifieras och analyseras.

Kapitel 3 – Konformism under ofullständig information

Acknowledgments

Although the present thesis features my name, the help of certain people has been instrumental in completing it, each one contributing in their own way.

First and foremost, I would like to thank my supervisors, my co-authors, and the people who have contributed most directly in writing my thesis. Yves Zenou has been my thesis advisor and a wonderful co-author. Working with him has been a most educational and inspirational experience.

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The help I received from Elias Papaioannou, Andreas Madestam, and Anna Tompsett during my job market year is deeply appreciated. I am also thankful to the administrative staff of our Department, and especially Anne Jensen, Anita Karlsson, Audra Mozuraitiene, Gunilla Thunberg and Ingela Arvidsson for their help and hospitality.
I would like also to thank Ana Maria Ceh, André Richter, Jakob Almerud, Paola Di Casola, and Spiros Sichlimiris. Ana is one of the most tenacious persons I have met. Also being my occasional officemate, I had the opportunity to benefit from her suggestions and her insight, that is not limited to mathematical issues. André was one of first persons that I met when I came to Stockholm, and has been a steadfast *fika* companion ever since. Constantly managing to maintain a positive attitude in some miraculous (to me) way, the discussions I had with him encouraged me to keep pressing on in difficult times. Jakob became one of my best friends during my time in Stockholm, and redefined the concept of officemate that I had in my mind. He has been patient enough to endure my attempts to construct meaningful sentences while conversing in Swedish, and his practicality and efficiency have been an inspiration to me. Together with André, he was a founding member of our cinephilia film club that provided us with short but relaxing cultural breaks amid the frenzy of research and deadlines. Paola is an outstanding researcher, and a wonderful person. Our regular communication about the latest research developments and upcoming seminars helped me discover research areas and topics the mere existence of which I could not have even imagined before. I was also lucky to have met Spiros, who has an admirable ability of keeping his calm demeanour under pressure and adversities. He has been one of my closest friends during the last six years, and his advice has been truly valuable.

During my time as a visiting researcher at Stanford University, I was fortunate enough to meet a new co-author, but more importantly, a very good friend. The endless discussions Constantine Yannelis and I had provided me with inspiration, and motivated my thought.

Spending time with Leda Pateli was an interesting and thought-provoking experience; most likely I had more debates with her than with all the other persons I met during the last six years combined. With Luca Fachinello we had many fascinating and intense discussions, and spent together several stressful hours during the tedious job market process. My wall-climbing
adventures with Paola Montero Ledezma and Yangzhou Yuan have been a much needed distraction from the world of theorems and equations, but also a reminder that failing just provides you with additional motivation to keep trying and succeed.

Many thanks to the people who shared some of their time with me during my time in Stockholm and Stanford: Alberto Vesperoni, Anders Österling, Andreas Bjerre-Nielsen, Egle Karmaziene, Glenn Magerman, Kristoffer Milonas, Paolo Bonomolo, Sara Fogelberg, Tamara Sobolevskaia, Tanneli Mäkinen, Valentina Gavazza, and many others.

My family have stood by me from the moment when I decided to pursue my academic aspirations. Vassilis, Sophia, Grigoris, and Froso have been truly supportive, and have shown understanding beyond any measure. Maria Vosnaki’s support has been crucial, and she has been there for me every day, despite being physically so far away.

I would like to close this note by getting back to the first person I mentioned, Yves, but this time in his capacity as my supervisor. His support has been instrumental in completing my thesis. Yves kept believing in me more than I did myself. It is very likely this thesis would have never come to be, if it weren't for him.

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My inspiration for writing this thesis arose from the two concepts that make up its title: incomplete information and network theory.

First, incomplete information and suboptimal information acquisition are generally recognised as fundamental sources of inefficiency in many economic and social phenomena. Acquiring thus a deeper understanding of their causes and consequences can help us remedy some of the problems they create.

Second, the allure and fascination that network theory has exerted on me ever since I got acquainted with it has been also been a decisive factor. Interpersonal, social, and economic networks play a vital role in communication and the transmission of information, beliefs, attitudes, and values; a role that has been reinforced and has come to the limelight with the rise of the digital social media in the recent years.

The present thesis consists of three essays, each constituting a chapter of this book, and focusing on aspects of information acquisition and economic decisions, formation of political beliefs, and social interactions respectively.

**Chapter 1 – What makes an opinion leader: expertise versus popularity**

This paper studies learning based on information obtained through social or professional networks. Building on the framework first proposed by DeGroot (1974), agents repeatedly update their beliefs by weighting the information acquired from their peers. The innovation lies in the introduction of dynamically updated weights. This allows agents to weight a contact with poor information little at first, but more later on, if that contact has in the mean-

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time gathered better information from other, more knowledgeable agents. The main finding is that individuals’ social influence depends on both their popularity (as captured by eigenvector centrality) and their expertise (as captured by information precision) in a simple and intuitively appealing way. It is moreover shown that even completely uninformed agents can contribute to social learning, and that under some network structures, providing certain agents with better information could actually lead society to worse assessments. The paper also discusses how the relationship between expertise and popularity in a network affects the learning process.

Chapter 2 – Lobbies, experts, and the public: a network model of political influence

This paper studies the evolution of political beliefs using a network model of social learning. Agents communicate their information, discuss their opinions with their peers in the network, and update their beliefs accordingly. Although information originating from better-informed agents receives ceteris paribus a larger weight, individuals receive and communicate information through their ideological prism. The paper also studies networks with individuals or groups of individuals who are not interested in learning or exchanging of information, but rather in promoting their own views to other agents. The features that make such groups influential are identified and discussed.

Chapter 3 – Conformism under incomplete information

As a large body of literature in sociology and economics has shown, social interaction induces conformism, and as it has been observed, behaviours deviating from the social norm tend to be punished. Although conformism has been studied in a network setup before, this is one of the first papers to examine conformism under incomplete information, and the first to provide and discuss thoroughly a comprehensive theoretical framework. Social interaction is modelled as a Bayesian network game, which is the natural
setup for analysing decisions whose potential returns or costs are ex ante uncertain (e.g. education, crime). We establish existence and uniqueness of the equilibrium, characterise the optimal decisions, and examine conditions under which policy interventions can be welfare-improving.
Chapter 1
What makes an opinion leader: expertise vs popularity

This paper studies learning based on information obtained through social or professional networks. Building on the framework first proposed by DeGroot (1974), agents repeatedly update their beliefs by weighting the information acquired from their peers. The innovation lies in the introduction of dynamically updated weights. This allows agents to weight a contact with poor information little at first, but more later on, if that contact has in the meantime gathered better information from other, more knowledgeable agents. The main finding is that individuals’ social influence depends on both their popularity (as captured by eigenvector centrality) and their expertise (as captured by information precision) in a simple and intuitively appealing way. It is moreover shown that even completely uninformed agents can contribute to social learning, and that under some network structures, providing certain agents with better information could actually lead society to worse assessments. The paper also discusses how the relationship between expertise and popularity in a network affects the learning process.
1.1 Introduction

Acquisition and aggregation of information is a critical part of the decision making process of individuals, firms, organisations, and governments. As a rule, however, obtaining access to the primary sources of information may be quite difficult, if it is feasible at all. Consequently, most information reaches agents through secondary sources or contacts, who may have themselves acquired it indirectly. Social and professional networks arise thus as a major channel of information diffusion in a society.

The present paper focuses on identifying some key determinants of social influence. What are the characteristics of the individuals that get to lead public opinion? Are the beliefs of the experts, or those of the most popular agents that have a greater influence in a society? Is it the most popular individual who should be entrusted with passing on information or raising awareness about an issue in the public?

In order to provide some answers to the above questions, this paper builds on a seminal model of social influence introduced by DeGroot (1974). The simple but compelling idea behind this model is that agents update their beliefs by repeatedly communicating with their neighbours, and weighting their information. There is, however, an important innovation introduced here: Agents are no longer assumed to assign fixed weights to their peers, as in the standard model. Instead, they update these weights every period, adapting them to reflect the information their peers gain access to. Using this richer setup, it is shown that each agent’s social influence stems from two components: a position-driven, popularity, and an information-driven one, expertise. This new approach enables a social planner to design targeted policy interventions based on the above characteristics, and evaluate their impact. The rest of the introduction provides a short overview of the relevant literature, and discusses the aforementioned points more thoroughly.

Individuals use information acquired through their networks in various
facets of their lives. The key role that social contacts can play in job search was documented in the landmark work by Granovetter (1973), and more recent evidence from empirical studies seems to corroborate, if not strengthen this finding.\(^1\) Consumers often seek the advice of friends who have a deeper knowledge or a better understanding of the relevant area before deciding to buy a new computer or car. Even in everyday consumption decisions, information transmitted through social networks can be crucial (see, for example, Moretti, 2011). The importance of social networks as information transmission mechanisms has grown over the last years due to the emergence of the digital social media. According to the American Press Institute, 44% of Americans use online social media as a news source, and this climbs to 88% in the age group 18 to 34 (two thirds of which on a daily basis).\(^2\) Moreover, as recent evidence suggests, people use online networks to exchange information on a broad variety of topics, ranging from health issues and the use of medication (Lefebvre and Bornkessel, 2013) to immigration decisions and life in a new country (Dekker and Engbersen, 2014).

Information aggregation—the process of combining information collected from various sources—is also an important stage of the decision making process, and has occupied researchers and scholars at least since the time of Condorcet (1785). Although conceptually a distinct process, information aggregation is, in practice, inextricably tied to information acquisition. As recent findings indicate (see for example Choi, Gale, and Kariv, 2008; Chandrasekhar, Larreguy, and Xandri, 2015; Grimm and Mengel, 2015), people are not in a position to accurately track back to their original source all pieces of information communicated to them, and as a result they are bound to treat already known information as new.

There are two main paradigms in the literature, often referred to as fully

\(^1\) See Ioannides and Datcher Loury (2004) for a comprehensive survey of the relevant literature.

rational or Bayesian learning, and boundedly rational or naïve learning respectively. In practice though this distinction is not always straightforward, since several models encompass elements of both approaches. The main idea behind the Bayesian benchmark is that agents are fully rational, and aggregate information in an optimal way any information that becomes available to them. Whenever possible, Bayesian agents hold some prior—potentially subjective—beliefs that satisfy Kolmogorov’s probability axioms. Taking advantage of the information that comes to their attention, they make use of Bayes’ rule to update their priors, and form some posterior beliefs regarding the unknown parameters of interest. As (Bayesian) consistency would suggest, under some regularity conditions agents in large networks will eventually converge to the same beliefs and/or actions, presumably the optimal ones.

The Bayesian approach hinges on the assumption that agents possess the mental capacity to optimally extract and aggregate information in the aforementioned way, or at least along similar lines. Although in some cases this assumption may be plausible, empirical evidence suggests that even in very small and simple network structures this may not be true when repeated interaction is involved (Choi et al., 2008). In fact, observations from a recent field experiment (Chandrasekhar et al., 2015) are compatible with the assumption that virtually all subjects exhibit a non-Bayesian behaviour. In some cases, it may be actually challenging even for the modeler to apply fully Bayesian analysis, especially if the assumption of common knowledge of the network structure is relaxed.

The present work adheres to the paradigm of boundedly rational learning. In particular, agents are assumed to update their beliefs through a so-called average-based updating process, first introduced by DeGroot (1974). In his

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3 See, for example, Theorem 1 in DeMarzo, Vayanos, and Zwiebel (2003). Nevertheless, factors such as the existence of disconnected or weakly connected network components (Gale and Kariv, 2003), excessively influential agents (Acemoğlu, Dahleh, Lobel, and Ozdaglar, 2010) or highly unbalanced network structures (Mossel, Sly, and Tamuz, 2015) can lead society astray.
seminal study, the author proposes a method for how a group of individuals, such as a committee, exchange opinions and update their beliefs about the value of some unknown parameter they wish to estimate. The process is simple and intuitively appealing: Each member of the group assigns some weight (a degree of “trust”) to each other member, and in every period they revise their opinion by taking a weighted average of the beliefs of their peers. The weights can be subjective, and thus not all members need to agree on them. Trust is not necessarily reciprocal, and some members may even disregard completely the opinion of some other members of the committee. In modern network theory terminology, this communication structure can be seen as a directed and weighted network. As shown by DeGroot, under some regularity assumptions (for example, any piece of information should be able to flow through the entire network, irrespectively of its origin), a consensus among the members of the group will be attained.

Based on DeGroot's model, DeMarzo et al. (2003) study the formation of (political) opinions in the presence of what they refer to as persuasion bias. People in this type of models fail to account for the repetition of the information they receive, either because they are not aware of the entire network structure, or because they lack the cognitive ability or the time required to fully track the path that information has followed via the network before reaching them. Hence, due to the influence of some prominently positioned individuals, their beliefs may be driven away from both the true value of the parameter, and the society’s initial average beliefs.

Golub and Jackson (2010) maintain the same framework, but introduce a rigorous network-theoretic framework, and a more standard networks-based approach. Their main finding is that persuasion bias will be present even in arbitrarily large societies. Only if the influence of prominent individuals goes to zero as the network grows can society learn efficiently. This result shows that persuasion bias is not, in general, remedied by large numbers, suggesting thus that the intuition behind Condorcet’s auspicious finding may no longer apply if some members of the society receive disproportionately
high attention.

The present work can be seen as an addition to the literature on average-based social learning, since it retains the spirit underlying the DeGroot model as well as its main idea: agents revise their beliefs by simply weighting the beliefs of their peers. It differs, however, from the existing literature in a key aspect: the introduction of dynamically updated weights, that enable agents to adjust the degree of trust they assign to their contacts to the flow of information. Dynamic weights capture the intuitive idea that individuals may initially assign a low weight to the belief of a peer who is uninformed or possesses low-quality information, but which they can subsequently increase, if that peer acquires information from contacts who are considered experts in the field, or simply have access to more accurate information.

The main findings are the following. First, it is shown that in the present model of learning, as under DeGroot learning, agents’ beliefs will over time converge to a stable consensus in strongly connected networks.

Second, again similarly to the DeGroot model, the influence of each agent’s initial beliefs in the formation of the consensus belief can be obtained explicitly. Unlike, however, DeGroot learning, the present aggregation process is no longer a “black box”, since the determinants of each agent’s influence can be explicitly calculated. In particular, it is shown that it can be attributed to three components: the agent’s *popularity* (expressed as his or her eigenvector centrality in the network), the agent’s *expertise* (expressed as the precision of the information he or she possesses), and a parameter that captures how information is distorted by the network, and which is hence common for all agents in a given network. Interestingly, and perhaps surprisingly enough, relaxing the assumption of fixed weights not only does not increase the informational requirements for the calculation of social influences on behalf of the modeler or the social planner, but in fact it facilitates analysis in cases where the latter is not fully aware of the network structure.

Third, it is shown that agents with low initial expertise (i.e. low-quality in-
formation), and even agents possessing no information at all initially, can play a crucial role in the learning process. This is a compelling feature that facilitates the study of networks where the majority of information originates from a minority of individuals. This is a common empirical observation in network analysis (see, for example, Galeotti and Goyal, 2010, and the references therein). Moreover, it is a prediction that is in line with findings from the Bayesian strand of literature (see, for example, Mueller-Frank, 2013), but perhaps also with common intuition: individuals without any information or knowledge of their own are often in a position to affect public opinion by propagating information or opinions that they have acquired from more knowledgable contacts. Hence, although such agents merely act as conduits for the transmission of information, their contribution can be significant, especially if they are centrally located or have direct access to expert agents. In contrast, under DeGroot learning, agents without any information would either be completely ignored, since they would be given a zero initial weight that they would carry over forever thereon, or would be given a positive but largely arbitrary weight, based on their neighbours’ assessment of their future access to information. In the present model such issues are overcome by allowing for weights to vary over time.

Finally, the breakdown of social influence into an information-driven, a popularity-driven, and a (common) network-driven component allows room for the evaluation of policy interventions.

The rest of the paper is structured as follows: Section 1.2 introduces a dynamic model of average-based social learning; the belief-updating process is presented and motivated. Section 1.3 studies the dynamics of the new process, establishes convergence of beliefs, and presents the main theorem of the paper, which is used in Section 1.4 to study the efficiency of target policy interventions. Section 1.5 concludes. Section 1.A of the Appendix discusses an extension of the main model, while the proofs of the propositions and theorems presented in the paper have been deferred to Section 1.B.
1.2 The model

1.2.1 The network

Consider a society consisting of a finite number of individuals who would like to gather more information about a parameter of interest or form an opinion regarding an issue they need to make a decision on. The pattern of communication among the agent is captured by a network $\mathcal{G}$. As in the rest of the literature on average-based updating, it will be assumed that the agents are only interested in estimating the true value of the unknown parameter, and stick to the stipulated updating process; they do not seek to maximize their social influence, nor do they have something to gain from propagating a particular belief.

A network is modelled as a—potentially directed—graph $\mathcal{G} = \langle \mathcal{N}, \mathcal{E} \rangle$, where the set of vertices or nodes, $\mathcal{N} := \{1, 2, \ldots, n\}$, represents a set of agents who can potentially interact with each other, and the set of edges $\mathcal{E} \subseteq \mathcal{N}^2$ represents the links among them. In many applications of network theory it is more convenient to represent a network using a matrix $G := [g_{ij}]_{(i,j) \in \mathcal{N}^2} \in \{0,1\}^{n \times n}$, where $g_{ij} := 1$ if there is a directed edge from node $i$ to node $j$ (i.e. agent $i$ is linked to agent $j$), and $g_{ij} := 0$ otherwise. In network theory terminology, matrix $G$ is referred to as the adjacency matrix of network $\mathcal{G}$.

In the analysis in the following sections of this paper, the adjacency matrix $G$ will represent the pattern of communication and transmission of information across the network. Consider any pair of agents $(i, j) \in \mathcal{N}^2$. A link from agent $i$ to agent $j$, denoted by $(i, j) \in \mathcal{E}$ or $g_{ij} = 1$, has the interpretation that agent $i$ has access to agent $j$’s belief. It shall be then said that agent $i$ observes, pays attention to, or listens to agent $j$, or in network theory terminology, agent $i$ is an in-neighbour of agent $j$. Equivalently, it can be said that agent $j$ receives attention from, or is an out-neighbour of agent $i$.\footnote{This terminology has its roots in the drawing of networks as graphs (see, for example, networks A and B in Example 1.1). A directed link from agent $i$ to agent $j$ means that $i$ gives}
Two important remarks are in order at this point. First, as the above discussion suggests, attention may not be reciprocal: the fact that agent $i$ can observe agent $j$’s belief does not necessarily imply that agent $j$ is in a position to observe agent $i$’s belief. Hence the adjacency matrix $G$ will be, in general, non-symmetric. Second, it is reasonable to assume that every agent can observe himself.\footnote{In the terminology introduced above, this implies that every agent is an out-neighbour of himself.} The diagonal of matrix $G$ will thus consist of ones, that is, $g_{ii} = 1$ for all $i \in \mathcal{N}$. This assumption will be maintained throughout this paper and will not be stated explicitly again.

The set of all agents that agent $i$ pays attention to (that is, all the out-neighbours of agent $i$) in network $G$ constitutes the \textit{out-neighbourhood} of agent $i$, and is denoted by $D_G(i)$. Using mathematical notation, for any $i \in \mathcal{N}$,

$$D_G(i) := \{j \in \mathcal{N} | g_{ij} = 1\}.$$ 

Notice that the out-neighbourhood of any agent is a non-empty set, since $i \in D_G(i)$ for every $i \in \mathcal{N}$.

1.2.2 Popularity: the eigenvector centrality measure

Eigenvector centrality was first proposed by Bonacich (1972) as a measure of influence, prestige, or popularity in a network. It captures the idea that \textit{what makes an agent important in a network is how well-connected this agent is to other important agents}. More specifically, each agent’s eigenvector centrality is a weighted average of the eigenvector centralities of his or her \textit{in-neighbours}. That is, an individual is considered more influential if he or she \textit{receives} attention from influential individuals. A formal definition is provided below.
Definition 1.1: Eigenvector Centrality

Consider a strongly connected network $G = (\mathcal{N},\mathcal{E})$ with adjacency matrix $G$. The eigenvector centrality profile of network $G$ is defined as the positive left eigenvector of matrix $G$, that is, as a vector $c := [c_i]_{i \in \mathcal{N}} > 0$ satisfying

$$c^T G = \rho_G c^T$$

normalised so that

$$\|c\|_1 := \sum_{i=1}^{n} |c_i| = 1,$$

(1.1)

where $\rho_G$ is the spectral radius of adjacency matrix $G$, and $\| \cdot \|_1$ denotes the vector 1-norm. The eigenvector centrality of agent $i \in \mathcal{N}$ is given by the element $c_i \in [0,1]$.

In other words, eigenvector centrality simply states that the importance of each agent is proportional to that of his or her out-neighbours, that is, the agents that he or she has access to. The eigenvector centralities of the agents in two simple networks are given below. The size of the nodes in the figures has been adjusted to represent their eigenvector centrality.

Example 1.1.

The networks shown in Figure 1.1 are strongly connected. A directed link (arrow) indicates that the agent at its origin observes the information of the agent at its target; hence information flows opposite to the direction of the arrows.6

In network 1.1.a, agents 1, 3, 5, and 6 are each given attention by three peers. Agent 1 is nevertheless less important than agent 5 under the eigenvector centrality measure. The reason is that, although agents 1 and 5 have in common two peers that pay attention to them, namely, agents 2 and 3, the third in-neighbor of agent 5 (i.e. agent 6) is more important than the third in-neighbor of agent 1 (i.e. agent 4). By the same token, agent 3 is more important than
Example 1.1: Eigenvector centralities

![Network 1.1.a](image1.png) ![Network 1.1.b](image2.png)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$c_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.167</td>
</tr>
<tr>
<td>2, 4</td>
<td>0.129</td>
</tr>
<tr>
<td>3</td>
<td>0.198</td>
</tr>
<tr>
<td>5, 6</td>
<td>0.188</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$i$</th>
<th>$c_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 5</td>
<td>0.094</td>
</tr>
<tr>
<td>2, 6</td>
<td>0.173</td>
</tr>
<tr>
<td>3</td>
<td>0.319</td>
</tr>
<tr>
<td>4</td>
<td>0.146</td>
</tr>
</tbody>
</table>

agent 5 because agent 1, who listens to agent 3, is more important than agent 2, who listens to agent 5.

In network 1.1.b, notice that agents 1 and 5 are equally important since the only peer that pays attention to them is agent 2, and hence they both derive all their prestige or popularity from this agent. The same holds true for agents 2 and 6, who are given attention only by agent 3.

Eigenvector centrality can potentially be problematic as a measure of influence because it is self-referential, and as such, it may not be always well-

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6 Observe that while attention can be reciprocal, this need not be the case. To keep graphs as simple as possible, and since all agents are assumed to observe themselves, self-loops have not been drawn.
defined. Nevertheless, the assumption of strong connectedness is sufficient to guarantee that there is one and only one positive eigenvector associated with matrix $G$ (see Section 1.B.1 in the Appendix for a proof). Bonacich and Lloyd (2001) and Jackson (2008) provide a motivation for the use of eigenvector centrality as a measure of influence or prestige, and propose alternative measures that can be used in the cases that the former is not well-defined.

1.2.3 Beliefs and expertise

The present paper studies the role of social networks in the shaping and evolution of the beliefs of the individuals in a society. Agents would like to estimate an unknown state of the world, $\theta^*$, in order to take a decision or make a choice in an efficient way. Agents' initial beliefs are denoted by $b_i(0)$ for each agent $i \in N$, and are updated based on information received through the network. In each round, agents ask their out-neighbours for their beliefs, as well as an assessment of how precise or accurate these beliefs are. Then they update their own beliefs by weighting the information they receive based on their peers' expertise. The belief of agent $i$ after $t$ rounds of communication, where $t \in \{0, 1, 2, \ldots\}$, will be denoted by $b_i(t) \in B$.

Depending on the context, beliefs can be represented as a percentage, a value (expressed as a number), a set of values (expressed as a vector) or even as more general objects, such as functions or probability distributions. In fact, the model discussed here can admit as beliefs any objects that are

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7 To see this, notice that if $\rho_{G}^{-1}G$ is interpreted as a linear mapping, then $c$ can be seen as a fixed point. There are mappings with no real (non-zero) fixed points, mappings with real but non-positive fixed points, and mappings with more than one non-negative fixed points. Any of the above would be problematic as a measure of centrality.

8 Recall that eigenvectors are unique up to the relative magnitude of their entries. Since $c$ is an eigenvector of matrix $G$, any positive multiple of $c$ is also an eigenvector of $G$, and contains exactly the same information about $G$ as $c$ does; hence it can be also considered a vector of eigenvector centralities. Normalisation (2.1) serves only to pin down agents' centralities in a unique way, and to facilitate the definition of some measures introduced in the sections that follow. Normalising eigenvector $c$ with respect to its 2-norm, so that $\|c\|_2 := (\sum_{i=1}^{n} |c_i|^2)^{1/2} = 1$, is also quite common. The results in this paper are not affected by that particular choice.
members of some convex subset $B$ of a linear space. In the simplest case, these initial beliefs are equal to some unbiased signal $s_i$ about the true state $\theta^*$. These signals will be assumed to be independent, but not identically distributed.

It will be also assumed that agents assign a degree of certainty, or precision, to their beliefs, which will be referred to as expertise. This is a non-negative number that expresses how much they trust the information they possess. Agents are assumed to start with some initial expertise, which changes as they communicate with their neighbours and exchange opinions. Hence, after a round of communication, the expertise of individuals who observe experts (that is, agents with relatively high precision) should be expected to increase more than that of individuals who do not have access to such individuals.

Initial expertise on the topic of interest may differ across agents for a variety of reasons, such as access to more trustworthy first-hand information, experience, or education, to mention some. In modelling terms, expertise can be captured by some appropriate parameter or statistic, depending on the application. Consider the case in which the initial belief of each agent $i$ is equal to some noisy signal $s_i$ that he or she receives about the true value of the parameter in question. If this signal is normally distributed, the precision associated with it can be defined as some sufficient statistic for the variance of the agent’s signal-generating distribution, as shown in the example below.

**Example 1.2.**

Consider a group of prospective investors who would like to predict the future price movement of the stock of an import company, say, Harry Lime & Co. This

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9 A constructive approach can be helpful in understanding why this requirement is sufficient for the process to be well-defined. Firstly, the space of beliefs $B$ must have a basic structure, in order for the process of aggregation of beliefs to be both well-defined, and conceptually meaningful. This structure can be imposed by assuming that this space is abelian under addition. Yet, since the updating process to be discussed below will be based on the averaging of beliefs, with the weights being real numbers, the operation of scalar multiplication needs to be defined; hence the linear space. The convexity assumption simply guarantees that the object resulting from the updating process will still be some valid belief.
will depend on the company’s quarterly earnings report, to be announced in the near future. Assume that the company’s profits were in fact equal to $v^*$, but investors do not have access to this information yet. Instead, they each observe some noisy signal $v_i$ about profits, with $v_i \sim N\left(v^*, \sigma_i^2\right)$. These signals are independent from each other ($v_i \perp v_j$), and potentially heteroscedastic ($\sigma_i \neq \sigma_j$ for $i \neq j$). Smaller variances reflect that some investors may follow closer the developments in the imports sector, or may have access to inside information.

In this example, investors' initial beliefs can be assumed to be equal to the signals they observe, $b_i(0) := v_i$, and their expertise can be defined as the inverse of the variance of their signal, $\pi_i(0) := \frac{1}{\sigma_i^2}$.

A different parametrisation can be used in the case that the unknown parameter is a probability.

**Example 1.3.**

A group of economists planning to attend a conference are reviewing their travel options. The most economical option would be to fly the local airline, Carcosa Air, but a severe delay could cause them to miss their connection flight. In fact such delays occur with probability $p^*$, unknown to the prospective travellers. Before making a decision, the economists communicate with their friends in order to exchange information.

In this example, agents' initial expertise (precision), $\pi_i(0)$, is represented by the number of times they have flown Carcosa Air in the past, while their initial belief about the probability of a severe delay, $b_i(0)$, can be taken to be the percentage of their past flights that were delayed. The probability of delay $p^*$, which is the parameter of interest, can be seen as the (unknown) success probability of a draw from a Bernoulli distribution with support $\Omega = \{0 \text{ (on time)}, 1 \text{ (delayed)}\}$.

1.2.4 The canonical average-based updating process

It is useful to begin by presenting the canonical average-based updating process, due to DeGroot (1974). As described above, agents start with some ini-
tial beliefs that they update by consulting with their out-neighbours. Before
the communication process begins, each agent $i \in \mathcal{N}$ assigns a weight $\gamma_{ij} \in [0, 1]$ to each out-neighbour $j \in D_G(i)$, including himself, so that $\sum_{j=1}^{n} \gamma_{ij} = 1$.

These weights reflect the relevance or trustworthiness of the opinion of each
neighbour; they may be derived from an objective formula or simply be some—potentially subjective—assessment of the informational value contained in each belief. In DeMarzo et al. (2003) the weight $\gamma_{ij}$ is referred to as the direct influence of agent $j$ on agent $i$. If $j \not\in D_G(i)$, meaning that agent $i$ cannot observe agent $j$, the corresponding weight is set equal to zero: $\gamma_{ij} := 0$.

The DeGroot model can generically admit any weights $\gamma_{ij} \in [0, 1]$. Of particular interest, though, is the case in which the weights that agents assign to their out-neighbours are consistent, in the sense that they are equal to relative expertise of each neighbour.\(^{10}\) It is also a quite plausible choice for the agents in the absence of information about the network structure beyond their out-neighbours. In that case, the direct influence of agent $j$ on agent $i$ (the weight that agent $i$ assigns to agent $j$) will be given by

$$
\gamma_{ij} = \frac{g_{ij} \pi_j(0)}{\sum_{k=1}^{n} g_{ik} \pi_k(0)}.
$$

Once the weights have been set, the communication process begins. In every period $t \in \{1, 2, \ldots\}$ each agent $i$ observes the beliefs of his or her out-neighbours $j \in D_G(i)$, and revises his or her beliefs accordingly. In particular, the new belief of agent $i$ will be a weighted average of the previous-period beliefs of his or her out-neighbours

$$
b_i(t) = \sum_{j=1}^{n} \gamma_{ij} b_j(t-1)
$$

or, using matrix notation,

$$
b(t) = \Gamma b(t-1)
$$

\(^{10}\) Notice that under an appropriate definition of expertise, the first-period weights will be the ones prescribed by Bayes’ rule, as in Examples 1.2 and 1.3.
where \( \mathbf{\Gamma} := \left[ \varphi_{ij} \right]_{(i,j) \in N^2} \) is the matrix of weights, and \( \mathbf{b}(t) := \left[ b_i(t) \right]_{i \in N} \) is the belief profile in period \( t \). Observe that \( \mathbf{\Gamma} \) is, by definition, a row stochastic matrix.\(^{11}\) Notice that in the context of Example 1.2, the belief of each agent \( i \) after the first round of communication, \( b_i(1) \), will be a sufficient statistic for the mean of \( v_i \), given the information that agent \( i \) has access to through his or her neighbours.

By iterating on process (1.3) we can express the belief profile in period \( t \) as a function of the initial beliefs

\[
\mathbf{b}(t) = \mathbf{\Gamma}^t \mathbf{b}(0).
\]

It follows that the cumulative weight assigned by agent \( i \) to agent \( j \) following \( t \) rounds of communication, will be given by the \((i,j)\)-th element of matrix \( \mathbf{\Gamma}^t \), denoted by \( \varphi_{ij}(t) \). The limiting belief profile can be calculated then as a function of the matrix of weights, \( \mathbf{\Gamma} \), and the initial belief profile, \( \mathbf{b}(0) \), as

\[
\lim_{t \to +\infty} \mathbf{b}(t) = \lim_{t \to +\infty} \mathbf{\Gamma}^t \mathbf{b}(0).
\]

The limiting belief of agent \( i \) will be therefore given by

\[
\lim_{t \to +\infty} b_i(t) = \sum_{j=1}^{n} \lim_{t \to +\infty} \varphi_{ij}(t) b_j(0). \tag{1.4}
\]

A version of the main result in DeGroot (1974), adapted to the context of the present paper, is given below.

---

\(^{11}\) A non-negative square matrix \( \mathbf{A} \in \mathbb{R}^{n \times n} \) is said to be row stochastic if the elements of each of its rows sum up to 1, that is, if \( \mathbf{A} \mathbf{1}_n = \mathbf{1}_n \), where \( \mathbf{1}_n \) is an \( n \)-dimensional vector of ones. This is why such matrices are often called right stochastic. Similarly, a non-negative square matrix \( \mathbf{A} \) is called column stochastic or left stochastic if its columns sum up to 1, that is, if \( \mathbf{1}_n^\top \mathbf{A} = \mathbf{1}_n^\top \).
**Proposition 1.1: Reaching a Consensus (DeGroot 1974)**

Assume that $\mathcal{G}$ is strongly connected and aperiodic, and agents follow the average-based updating process described by expression (1.3). Then, for all $i, j \in \mathcal{N}$, the cumulative weight assigned by agent $i$ to agent $j$ in the limit is given by

$$\overline{\gamma}_j^{(\infty)} = \lim_{t \to +\infty} \gamma_{ij}(t), \tag{1.5}$$

where $\overline{\gamma}_j^{(\infty)}$ is the $j$-th element of the left eigenvector $\overline{\gamma}^{(\infty)}$ of matrix $\overline{\Gamma}$. In DeMarzo et al. (2003), this limiting weight is referred to as the *social influence* of agent $j$.

The above proposition readily gives rise to three remarks. First, if the stipulated conditions are met, each agent $i$ will have the same limiting influence on all other agents in the network, and hence the term *social influence* of agent $i$. Indeed, as expression (1.5) suggests

$$\lim_{t \to +\infty} \gamma_{ij}(t) = \lim_{t \to +\infty} \gamma_{jh}(t) \quad \text{for all } i, j, h \in \mathcal{N}.$$ 

Notice, of course, that different agents will have in general different social influences, that is, $\overline{\gamma}_j^{(\infty)} \neq \overline{\gamma}_k^{(\infty)}$ for $j \neq k$. As (1.5) implies, all rows of matrix $\overline{\Gamma}^t$ will be identical in the limit, and more specifically

$$\lim_{t \to +\infty} \overline{\Gamma}^t = 1_n (\overline{\gamma}^{(\infty)})^T.$$ 

Second, it follows from expressions (1.4) and (1.5) that all agents will have the same limiting beliefs, or as it shall be said hence forth, they will reach a *consensus*\textsuperscript{12}. In particular, for any $i \in \mathcal{N}$ it will hold

$$\lim_{t \to +\infty} b_i(t) = \sum_{j=1}^{n} \overline{\gamma}_j^{(\infty)} b_j(0).$$

Third, observe the vector of social influences, $\overline{\gamma}^{(\infty)}$, can be seen as a vector

\textsuperscript{12}For a formal definition of *consensus*, see Definition 1.3.
1.2.5 The dynamic average-based updating process

Motivation and preliminaries

In some cases though it would be more reasonable to assume that the weights agents' assign to their neighbours are not constant but rather change based on how reliable the second-hand information the latter have access to is. Consider, for example, a person who has to decide whether to accept or turn down a job offer, and asks the opinions of his friends and colleagues. It may be the case that one of them used to work in the past for the firm making the offer, and she has thus some partial but perhaps outdated information about it. Yet she may still be in contact with her former colleagues at the firm, whom she may contact. It would make therefore sense for the prospective employee to place some rather moderate weight on the opinion of his friend, but increase after the latter has consulted with her more informed contacts. Similar arguments could apply, among other, in the case of a prospective buyer of a house in a neighbourhood he has never lived in, a student who has to decide whether to pursue post-graduate education or work in the industry, and a first-time traveller to a holiday destination.

More generally, assume that an agent \( i \in N \) has to decide how to weight the opinions of his out-neighbours \( j \in D_G(i) \). It could be the case that one of them, agent \( j \), is less well-informed compared to other out-neighbours of agent \( i \), but she is able to observe a third agent, \( k \), who is an expert in the issue in question, and whom agent \( i \) cannot directly observe. In that case it would make sense for agent \( i \) to assign a small weight to agent \( j \) in the

\[ \text{From a technical (although not a conceptual) point of view, it essentially coincides with the stationary distribution of a homogeneous and aperiodic Markov chain with transition matrix} \ F. \text{This is a standard result from Markov chains theory; for a more detailed discussion, see, for example, Section 5 in DeGroot (1974) and the references therein. Moreover, a similar, but more general approach is used to analyse the dynamic average-based updating process introduced in this paper, so the reader is referred to Section 1.3 for a motivation and a more technical analysis.} \]
first round of communication, but a larger one in the subsequent rounds, since by then agent $j$’s belief will have incorporated information from her better-informed friend, agent $k$. Analysing this problem using the canonical average-based updating process discussed above is not possible since matrix of direct influences $\Gamma$ has been assumed to be fixed across time.

Another implication of the fixed-weights assumption is that agents with no reliable first-hand information (that is, zero initial expertise) will be completely ignored, and thus will play no role in shaping public opinion. It would be reasonable though to consider that such agents can have a significant, albeit indirect, influence by simply spreading information they acquired from their out-neighbours. This case is of particular interest when it comes to agents who occupy a prominent position in the network, but rely on the their peers for information on a topic.

The aforementioned issues could in principle be addressed within the canonical average-based updating model. For example, agents could be assumed to weight their neighbours based on the precision of the information that the latter are expected to receive in future periods. Such an approach though would be highly problematic: not only would this increase distortion due to inappropriate weighting of information, but also the weights used would have to be quite arbitrary; using some “correct” or “objective” weights would require advance knowledge of the information precision of one’s neighbours, and that of the neighbours of their neighbours, and so on, which would effectively translate into a requirement for full knowledge of the network structure. The updating process to be introduced in this section, instead, addresses the above problems by allowing for weights that vary over time.

Another important question that the model proposed here can help us answer, is what makes an agent influential in a network. In the framework used in our analysis there can be two sources of influence: network position, or \textit{popularity}, and information precision, or \textit{expertise}. It is not straightforward however how these attributes combine to determine the social influence of
each agent. Under what conditions would a relatively badly informed or non-expert, yet centrally positioned agent, be more influential than an expert who is not in the spotlight? How much more influential would the former be? Up to what extent can people with a high degree of knowledge or specialisation in an area rely on their expertise to stir public opinion, disregarding social networking? Although the canonical model does not provide direct answers to the above questions, its dynamic counterpart introduced in this section provides a more suitable framework to study these issues.

In order for this to be achieved, we need to take a step back, and study how the weights assigned to neighbours’ opinions, or the the direct influences \( \gamma_{ij} \) as called above, are determined in the first place. That is, we need to decompose the matrix of direct influences \( \Gamma \) into a part depending only on the information or knowledge of the agents possess initially ("expertise"), and a part that depends only on the position of the agents in the network ("centrality" or "popularity").

Consider now the set of all agents who have access to some non-trivial information in period \( t = 1 \), either directly or through their immediate out-neighbours. Formally,

\[
A^o := \{ i \in N \mid \pi_j(0) > 0 \text{ for at least one } j \in D_G(i) \}.
\]

(1.6)

The following assumption is invoked to facilitate the presentation of some of the results in the present paper.

**Assumption 1.1.** *Every agent in the network has direct or indirect access to some information in the first round of communication, that is, \( A^o = N \).*

In other words, every agent \( i \) has at least one out-neighbour \( j \) (who may potentially be agent \( i \) himself/herself) who receives an informative signal \( (\pi_j(0) > 0) \). The purpose this assumption serves is to keep technicalities and notation at a minimum, and does not qualitatively affect our results.\(^{14}\)

\(^{14}\)In most practical applications it would be reasonable to assume that agents have direct or indirect access to some information, even arbitrarily inaccurate, about the value of the un-
Timing

With the technicalities in order, the model can be now introduced. Each agent \( i \in \mathcal{N} \) starts with initial belief \( b_i(0) \) to which he assigns precision \( \pi_i(0) \). The belief profile of the agents at the beginning of each period \( t \in \{1, 2, \ldots\} \) is denoted by \( b(t-1) \), and the corresponding precisions with \( \pi(t-1) \). The timing of the updating process, and how beliefs and precisions are updated every period \( t \), are summarised below.

1. The \( t \)-th round of communication takes place. Agent \( i \) collects from each out-neighbour \( j \) a report of that neighbour’s previous-period belief and precision (expertise), that is, a pair \( (b_j(t-1), \pi_j(t-1)) \in \mathcal{B} \times \mathbb{R}_+ \) for every \( j \in \mathcal{D}_G(i) \).

2. Agent \( i \) updates the weight he or she assigns to each neighbour \( j \) (i.e. the direct influence of agent \( j \) on agent \( i \)) \( \gamma_{ij}(t-1) \) to \( \gamma_{ij}(t) \), according to expression (1.8a).

3. Agent \( i \) updates his or her belief \( b_j(t-1) \) according to (1.7). The new belief, \( b_j(t) \), is the weighted average of the beliefs \( b_j(t-1) \) reported by agent \( i \)'s out-neighbours, using the new weights \( \gamma_{ij}(t) \) calculated in stage [2] above.

4. Agent \( i \) calculates the precision of his or her updated belief as shown in expression (1.9). The new precision \( \pi_i(t) \) is simply the sum of precisions of his or her out-neighbours (including own-precision) reported in stage [1] above.

Hence, “beliefs in period \( t \)” or “expertise in period \( t \)”, \( b(t) \) and \( \pi(t) \) respectively, shall refer to the belief and precision profiles of the agents at the end updating process, and after all communication has taken place in period \( t \).
The process

As discussed above in this section, in the initial period, \( t = 0 \), agents hold beliefs \( b(0) \), with the corresponding precisions being \( \pi(0) \). In the first round of communication, agents exchange information in the form \((b_j(0), \pi_j(0))\) in the manner described in Section 1.2.4. Agent \( i \)'s updated belief in period 1, \( b_i(1) \), will then be a weighted average of the beliefs reported by his or her neighbours, with the weight \( \gamma_{ij}(1) \) assigned to each belief being equal to its relative initial precision. In the notation introduced above, and provided that agent \( i \) has access to nontrivial information in period \( t = 1 \) \((i \in A^0)\), his or her beliefs will be formed as

\[
b_i(1) = \sum_{j=0}^{n} \gamma_{ij}(1) b_j(0) = \sum_{j=0}^{n} \frac{g_{ij}\pi_j(0)}{\sum_{k=0}^{n} g_{ik}\pi_k(0)} b_j(0).
\]

Agents who do no have access to information are simply assumed to maintain their initial, arbitrary beliefs \((b_i(1) = b_i(0))\). Up to this point the process is almost identical to the standard average-based updating process à la DeGroot presented in Section 1.2.4. The difference lies in the assumption that, under the current process, agents update not just their beliefs per se, but also the corresponding precisions. Precision \( \pi_i(1) \) that agent \( i \) places to his or her updated belief after the first round of communication, \( b_i(1) \), will be assumed to be simply the sum of the initial expertise of all his or her out-neighbours, including agent \( i \)'s own initial expertise, \( \pi_i(0) \):

\[
\pi_i(1) = \sum_{j=0}^{n} g_{ij}\pi_j(0).
\]

This updating process is repeated ad infinitum. In the second round of communication, agent \( i \) inquires with his or her out-neighbours \( j \in \mathcal{D}_G(i) \) about their new beliefs and expertise, \((b_j(1), \pi_j(1))\), and based on these reports revises his or her belief once more. The new weights assigned to each belief reported are calculated based on neighbours’ updated precisions, \( \pi_j(1) \); hence the precision given to the new belief \( b_i(2) \) will be the sum of these precisions.
In general, the belief of any agent $i \in \mathcal{N}$ in period $t \in \{1, 2, \ldots\}$ (that is, after $t$ rounds of communication) will be given by

$$b_i(t) = \sum_{j=0}^{n} \gamma_{ij}(t) b_j(t-1), \quad (1.7)$$

where

$$\gamma_{ij}(t) := \begin{cases} 
    \frac{g_{ij} \pi_j(t-1)}{\sum_{k=0}^{n} g_{ik} \pi_k(t-1)} & \text{if } i \in A^o \\
    \delta_{i,j} & \text{if } i \not\in A^o
\end{cases} \quad (1.8a)$$

$$\delta_{i,j}$$

denotes the relative weight that agent $i \in \mathcal{N}$ places on the belief reported by agent $j \in \mathcal{N}$ in the $t$-th round of communication, and $\delta_{i,j}$ is the Kronecker delta.

As expression (1.8a) implies, unlike the standard average-based updating (DeGroot) process presented in the previous section, direct influences in this model will not be constant over time—hence the time index $t$. Notice of course that if agent $i$ does not observe agent $j$, then $\gamma_{ij}(t) = 0$ for every $t \in \mathbb{N}$, since $g_{ij} := 0$.

Dynamic weights $\gamma_{ij}(t)$ are a consequence of precisions being updated every period. After the $t$-th round of communication has taken place, the aggregate precision $\pi_i(t)$ attached by each agent $i$ to his or her new belief $b_i(t)$ will be assumed to be simply the sum of the precisions of the beliefs reported by his or her neighbours that period

$$\pi_i(t) = \sum_{j=0}^{n} g_{ij} \pi_j(t-1). \quad (1.9)$$

Updating rule (1.7) can be expressed in matrix form as

$$b(t) = \Gamma(t) b(t-1), \quad (1.10)$$

where $\Gamma(t) := [\gamma_{ij}(t)]_{(i,j) \in \mathcal{N}^2}$ is the matrix of direct influences in period $t$. Similarly, agents’ aggregate precisions in period $t$ can be written in vector form
as follows

\[ \pi(t) = G \pi(t - 1) \]

or as a vector-valued function \( \pi(t) \), with \( \pi : \mathbb{N} \rightarrow \mathbb{R}_+^n \), and

\[ \pi(t) = G^t \pi(0) \]  

(1.11)

for any given vector of initial precisions \( \pi(0) \).

Now the dynamic updating process introduced in this paper can be defined formally.

**Definition 1.2: The Dynamic Average-based Updating Process**

Agents are said to follow the dynamic average-based updating process if their updated beliefs after each round of communication equal a weighted average of the beliefs reported by their out-neighbours (including themselves), where the weight assigned to each belief is equal to the relative aggregate precision associated with it. Formally, this process is described by expressions (1.7), (1.8), (1.9), and (1.10), for every \( t \in \mathbb{N} \).

It would be useful at this point to express the belief profile \( b(t) \), in any period \( t \in \{1, 2, \ldots\} \), as a function of the initial belief profile \( b(0) \). Through recursive backward substitutions, expression (1.10) can be written as

\[ b(t) = \Gamma(t) \Gamma(t - 1) \cdots \Gamma(1) b(0), \]  

(1.12a)
or in the more compact form\(^{15}\)

\[
b(t) = \prod_{\tau=1}^{t} \Gamma(\tau) b(0). \tag{1.12b}
\]

The cumulative influence, or simply influence \(w_{ij}(t)\) of agent \(j\) on agent \(i\) after \(t\) rounds of communication is defined as the \((i,j)\)-th element of matrix \(W(t)\), where

\[
W(t) := \prod_{\tau=1}^{t} \Gamma(\tau). \tag{1.13}
\]

Hence the belief updating process given by expressions (1.12a, 1.12b) can be written as

\[
b(t) = W(t) b(0). \tag{1.14}
\]

In any period \(t\), the element \(w_{ij}\) of matrix \(W(t)\) expresses how much agent \(i\)’s belief has been affected by agent \(j\)’s initial belief over the course of all past periods.

### 1.2.6 A note on backward matrix products

Note that although the updating rule stipulated by expressions (1.12) may be reminiscent of an inhomogeneous, or as it is sometimes called, a non-stationary Markov chain, it is in fact a different process. First, from a conceptual point of view, the process described in this paper is very different from an inhomogeneous Markov chain. Observe that, unlike a Markov chain, the dynamic average-based updating process is entirely deterministic. Moreover, the elements of the matrix of direct influences, \(\Gamma(t)\), represent weights, and not transition probabilities, as the elements of a Markov matrix \(M(t)\)

---

\(^{15}\)Since matrix multiplication is generally non-commutative, the order of multiplications in the product prescribed by the product operator \(\prod\) is not uniquely defined. In this paper, however, it shall be used to denote the so-called backward matrix product, that is

\[
\prod_{\tau=1}^{t} X(\tau) := X(t) X(t-1) \cdots X(2) X(1)
\]

for some \(X \in \mathbb{R}^{n \times n}\). A brief discussion of these products, as well as some additional references, are provided in Section 1.2.6.
do. Consequently, the object being updated is a vector of beliefs $b(t)$, not a probability distribution $p(t)$ as in the case of a Markov chain. Hence, although $p(t)$ is by definition a stochastic vector, this will not be true in general for belief profile $b(t)$.

Second, from a technical perspective, recall that the dynamics of an inhomogeneous Markov chain are captured by what is often referred to as a forward product of stochastic matrices, that is, a product of the form $M(1)M(2)\cdots M(t)$. The distribution in period $t$ will be thus given by

$$p(t) = p(0)M(1)M(2)\cdots M(t).$$

Recall, however, from expression (1.12a), that the dynamic average-based updating process is described by a backward product of the form

$$b(t) = \Gamma(t)\Gamma(t-1)\cdots \Gamma(1)b(0).$$

As non-commutativity of matrix multiplication would suggest, these two processes are different both in dynamics and in asymptotics. Hence, the resulting beliefs (or “marginal distributions”, if it is a Markov chain) in any time period will be in general different under each process ($b(t) \neq p(t)$ for $t \in \{1, 2, \ldots\}$), even if the starting points and all transition matrices are identical ($b(0) = p(0)$ and $\Gamma(t) = M(t)$). Backward products, moreover, depend more heavily on the first matrix in the sequence, $\Gamma(1)$, than forward products do on $M(1)$.

Technically, the canonical average-based updating process à la DeGroot is

16Unfortunately, although the literature on forward matrix products is quite rich, there is a dearth of studies on backward products, perhaps due their more limited applications (namely studying aspects of Markov Decision Processes, and distributed algorithms, aside from DeGroot-type updating). Chatterjee and Seneta (1977), (Seneta, 1981, chapter 4.6), and Leizarowitz (1992) provide some sufficient conditions for convergence; Anthonisse and Tijms (1977) and Federgruen (1981) study the rate of convergence of such sequences. The author of the present paper is not aware of any work providing an explicit formula for the limit of convergent sequences of backward products, analogous to those we have for forward products (see, for example, Isaacson and Madsen, 1976, Theorem V.4.7). The proofs in the present paper are based on results derived in the aforementioned papers, as well as on certain results from the Markov chains literature that do not depend on the direction of the matrix product.
also described by a backward product (and should be thought of as such). Since, however, the matrix of direct influences is constant over time ($\Gamma(t) := \bar{\Gamma}$), the standard results for homogeneous Markov chains can be used in that case.

### 1.3 Information exchange dynamics and convergence of beliefs

#### 1.3.1 Reaching a consensus in the dynamic model

This section studies the conditions under which a common belief arises in the network. Although the analysis is asymptotic, it may still be a very good approximation of the finite belief and influences dynamics, especially in the cases where convergence is fast.

**Definition 1.3: Consensus**

It is said that the dynamic average-based updating process in network $\mathcal{G}$ leads to a consensus if for any initial belief profile $b(0) \in \mathcal{B}^n$, and any vector of initial precisions $\pi(0) \in \mathbb{R}_+^n$, it holds

$$\lim_{t \to +\infty} (b_i(t) - b_j(t)) = 0 \quad \text{for all } (i, j) \in \mathcal{N}^2. \quad (1.15)$$

If moreover there exists some belief $b^{(\infty)} \in \mathcal{B}$ such that

$$\lim_{t \to +\infty} b_i(t) = b^{(\infty)} \quad \text{for all } i \in \mathcal{N}$$

the consensus shall be called definitive, and $b^{(\infty)}$ will be referred to as the consensus belief. Otherwise, the consensus will be called oscillatory.

As expression (1.15) suggests, a consensus is reached if after any—potentially arbitrarily large—number of communication rounds, all agents end up holding the same beliefs. Notice that this does not imply necessarily that these
beliefs will be constant over time; it could be the case that all agents change their beliefs synchronously every period, or more accurately, keep oscillating indefinitely among a number of different beliefs. Interestingly enough though, it turns out that the dynamic average-based updating process discussed here cannot lead to oscillatory consensuses. The following result is an immediate application of Theorem 1 in Chatterjee and Seneta (1977).

**Proposition 1.2: Stable Beliefs in the Limit**

Consider a strongly connected network \( G \), and suppose that all agents in \( N \) reach a consensus by following the dynamic average-based updating process stipulated in Definition 1.2. Then this consensus must be definitive.

Hence, if beliefs end up being identical across agents, they will also be constant over time. For the remainder of the paper, the qualifier *definitive* will be omitted; since there can be no oscillatory consensuses in the current setup, it should be clear that the term *consensus* must refer to a definitive consensus.

The result below establishes convergence of the dynamic average-based updating process.

**Proposition 1.3: Convergence**

Consider a strongly connected network \( G = \langle N, E \rangle \), and assume that agents follow the dynamic average-based updating process. Then there exists a unique stochastic vector \( w^{(\infty)} := [w_i^{(\infty)}]_{i \in N} \) such that

\[
\lim_{t \to +\infty} w_{ij}(t) = w_j^{(\infty)}
\]

for all \((i, j) \in N^2\). The limiting weight \( w_j^{(\infty)} \) is called the *social influence* of agent \( j \). It follows that the agents in \( G \) will reach a consensus, with the
consensus belief \( b^{(\infty)} \in B \) given by

\[
b^{(\infty)} = \sum_{j=1}^{n} w_j^{(\infty)} b_j(0). \tag{1.17}
\]

Proposition 1.3 demonstrates that if agents follow the communication process introduced in Definition 1.3, they will all end up having the same belief about the state of the world \( \theta^* \). It is moreover shown that the consensus belief will be a weighted average of the agents’ initial beliefs, with constant weights. This implies that, asymptotically, the impact of each initial opinion \( b_i(0) \) will be the same on all agents in the network, irrespectively whether they can directly observe agent \( i \) or not.

As discussed in Section 1.2.6, there is no general formula for the limit described in (1.16), and consequently the consensus beliefs given by expression (1.17). In the present case, however, under Assumption 1.1 discussed in the beginning of this section, an explicit formula for the social influence of each agent can be obtained by using some “direction-free” results from matrix algebra.\(^{18}\)

**Theorem 1.1: The Determinants of Social Influence**

Assume that network \( G \) is strongly connected, Assumption 1.1 is satisfied, and agents follow the dynamic average-based updating process stipulated in Definition 1.2. Then the social influence of each agent \( i \) is equal to the product of his or her eigenvector centrality and his or her relative initial precision, adjusted by a network effects multiplier, that is

\[
w_i^{(\infty)} = \alpha \epsilon_i \sigma_i \bar{\pi}_i(0) \tag{1.18}
\]

where

\(^{17}\) A vector \( y \) is said to be a stochastic or probability vector if it is non-negative, and its elements sum up to 1, that is if \( y \in [0,1]^n \) and \( \sum_{i=1}^{n} y_i = 1. \)

\(^{18}\) The case in which this assumption fails is conceptually very similar, and is discussed in Section 1.A of the Appendix.
\(c_i\) is agent \(i\)'s popularity (eigenvector centrality)
\[
\tilde{\pi}_i(0) := \frac{\pi_i(0)}{\sum_{j=1}^{n} \pi_j(0)}
\]
is agent \(i\)'s relative initial expertise (precision), and
\[
\alpha_{c,\pi} := \alpha(c, \pi(0)) = \frac{\sum_{j=1}^{n} c_j \pi_j(0)}{\sum_{j=1}^{n} \pi_j(0)}
\]
is the centrality–expertise dispersion parameter, a scalar common for all agents in network \(G\) that captures the distortion in agents' influences induced by the lack of direct universal communication (incompleteness of the network).

As this result demonstrates, the social influence of each agent under the dynamic average-based updating process depends only on his or her position in the network, as captured by eigenvector centrality (popularity), and on his or her relative initial expertise (that is, how precise their information is relative to that of the other agents). These two effects are disentangled in a clear and straightforward way. The parameter \(\alpha_{c,\pi}\) captures the relationship between popularity and initial expertise in the network, as discussed in detail in Section 1.3.3. Observe also that the expertise-driven component of social influence is determined by the relative initial expertise (that is, precision of the information) of each agent; any changes in absolute expertise matter only inasmuch as they alter relative precisions.

**Example 1.4.** Let us return to the setup introduced in Example 1.3. In this example, agents' initial expertise or precision, \(\pi_i(0)\), can be taken to be the number of times they have flown Carcosa Air in the past, while their initial belief about the probability of a severe delay, \(b_i(0)\), can be the percentage of their past flights that were delayed.\(^{19}\)

The communication pattern among the economists is presented in Figure 1.4.

\(^{19}\)Recall that the probability of delay \(p^*\) can be seen as the (unknown) success probability of a draw from a Bernoulli distribution with support \(\Omega = \{0 \text{ (on time)}, 1 \text{ (delayed)}\}\). Then the dynamic average-based updating process described in Definition 1.2 would be the optimal information aggregation process in the absence of persuasion bias. In fact, expression (1.10) can be derived from the application of Bayes rule under the assumption that every agent \(i\) has a Beta(\(\beta, \delta\)) prior distribution for the probability of a delay, \(p^*\), where \(\beta = b_i(0)\pi_i(0)\) and \(\delta = (1 - b_i(0))\pi_i(0)\). Beta distribution is emerges here because it is the conjugate prior of the Bernoulli distribution; for a more in-depth discussion, see Pham-Gia (2004).
Nodes representing agents with a higher degree of expertise are depicted in darker colours. Agents’ popularity ($c_i$), initial relative and absolute expertise ($\overline{\pi}_i(0)$ and $\pi_i(0)$), as well as their social influence at the consensus ($w_i^{(\infty)}$), are given in Table 1.4.

Example 1.4: Social influence

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.4.png}
\caption{Figure 1.4}
\end{figure}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$i$ & $c_i$ & $\overline{\pi}_i(0)$ & $\pi_i(0)$ & $w_i^{(\infty)}$ \\
\hline
1 & 0.094 & 0.222 & (14) & 0.154 \\
2 & 0.173 & 0.111 & (7) & 0.142 \\
3 & 0.319 & 0.048 & (3) & 0.112 \\
4 & 0.146 & 0.206 & (13) & 0.221 \\
5 & 0.094 & 0.270 & (17) & 0.188 \\
6 & 0.173 & 0.143 & (9) & 0.183 \\
\hline
$\sum$ & 1 & 1 & (63) & 1 \\
\hline
\end{tabular}
\caption{Table 1.4}
\end{table}

In the small society of this example, the individual with the highest expertise is agent 5, while the most popular one is, by far, agent 3. Yet the most influential one is agent 4, who ranks third in expertise, and just fourth in popularity. As
this example suggests, in general it is the agent who has the “appropriate” (for that society) mixture of expertise and popularity that gets to influence public opinion the most. Agent 5 is the expert here, but she is rather at the margin of social attention, and this limits her influence. Agent 3, on the other hand, is the most popular agent in the network, yet he is considered to be quite ignorant regarding the topic in question. As a result, his initial belief will be heavily discounted by the other agents.

The following remark follows directly from expression (1.18).

**Remark 1.1.** The expertise-driven component of social influence is determined by the relative initial expertise (that is, precision of the information) of each agent; any changes in absolute expertise matter only inasmuch as they alter relative precisions.

The result in Theorem 1.1 presents interest from both a theoretical and an empirical point of view. Not only it is straightforward to see whether important agents derive their influence from their position in the network or the information they possess, but it is also easy to see how a small change in the information precision of some agent, or a rewiring of his links, would affect his social influence as well as the consensus beliefs. This could have direct implications on how some social planner could intervene in order to facilitate or disrupt the flow of information in a network.

### 1.3.2 Constant versus dynamic weights and the role of uninformed agents

Another interesting observation is that, under the dynamic average-based updating process, even agents without any credible initial information can affect consensus beliefs. This is because although their initial expertise may be zero, they can affect the beliefs of their neighbours in subsequent periods by passing on second-hand information. Hence, despite that their own social influence will be zero, they will affect the social influences of the other agents, possibly unevenly, through their effect on eigenvector centralities.
This is a compelling finding, since as empirical literature has shown, information often originates from a small number of individuals. In many cases, acquiring first-hand information may be costly in terms of time and effort. Hence, a large number of people prefer to obtain their information indirectly, through a minority of expert or well-informed individuals, something that Galeotti and Goyal (2010) called the law of the few. A prominent example are online communities such as network forums. It is therefore important to understand the role that the initially ignorant agents play in the information diffusion process once they have learned from the experts.

This is not straightforward, however, under the canonical average based updating process, since weights are assumed to be constant. Hence, if agents with no information receive zero initial weight from their neighbours, this will be carried over ad infinitum. As a result they will be completely ignored, and their presence will have no impact on the consensus beliefs or the social influence of the other agents. The following example shows how the predictions of the dynamic model introduced in this paper can differ from those of the baseline DeGroot model.

**Example 1.5.** Let us revisit the group of investors introduced in Example 1.2. Recall that they are interested in forecasting as accurately as possible the profits of an imports firm, Harry Lime & Co., before they are officially announced. Towards this, they communicate with their contacts, and exchange information.

Investors 1 and 5 have been following the imports industry closely, so they are the experts in this example. Investor 2, on the contrary, has no information at all about the developments in HLC or the imports industry in general, and hence relies entirely on second-hand information from her contacts.

Investors’ popularity ($c_i$), initial relative and absolute expertise ($\bar{\pi}_i(0)$ and $\pi_i(0)$), and their social influence at the consensus under the dynamic updating process ($\gamma_i^{(\infty)}$) and the baseline DeGroot model ($\gamma_i^{(\infty)}$) are given in Table 1.5. In Figure 1.5(i) node size represents popularity, while a darker colour represents higher expertise.
Example 1.5: The role of ignorant agents

![Diagram](image)

Figure 1.5(i)

Table 1.5

<table>
<thead>
<tr>
<th></th>
<th>(\pi_i(0))</th>
<th>(\bar{\pi}_i(0))</th>
<th>(c_i)</th>
<th>(w_i^{(\infty)}) (dynamic)</th>
<th>(\gamma_i^{(\infty)}) (DeGroot)</th>
<th>(c_i)</th>
<th>(w_i^{(\infty)}) (dynamic)</th>
<th>(\gamma_i^{(\infty)}) (DeGroot)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.254 (1.8)</td>
<td></td>
<td>0.167</td>
<td>0.239</td>
<td>0.229</td>
<td>0.162</td>
<td>0.208</td>
<td>0.229</td>
</tr>
<tr>
<td>2</td>
<td>0 (0)</td>
<td></td>
<td>0.129</td>
<td>0</td>
<td>0</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>3</td>
<td>0.211 (1.5)</td>
<td></td>
<td>0.198</td>
<td>0.237</td>
<td>0.271</td>
<td>0.241</td>
<td>0.257</td>
<td>0.271</td>
</tr>
<tr>
<td>4</td>
<td>0.141 (1.0)</td>
<td></td>
<td>0.129</td>
<td>0.103</td>
<td>0.113</td>
<td>0.162</td>
<td>0.116</td>
<td>0.113</td>
</tr>
<tr>
<td>5</td>
<td>0.254 (1.8)</td>
<td></td>
<td>0.188</td>
<td>0.270</td>
<td>0.229</td>
<td>0.194</td>
<td>0.248</td>
<td>0.229</td>
</tr>
<tr>
<td>6</td>
<td>0.171 (1.0)</td>
<td></td>
<td>0.188</td>
<td>0.150</td>
<td>0.157</td>
<td>0.241</td>
<td>0.171</td>
<td>0.157</td>
</tr>
<tr>
<td>Σ</td>
<td>1 (7.1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Under the dynamic updating process, the agents whose initial belief has the highest social influence \( w_i^{(\infty)} \) is agent 5. On the contrary, agent 2 has zero social influence: since she has zero expertise \( (\pi_2(0) = 0) \), her initial belief (whatever this may be) will have no impact on the consensus belief.\(^{20}\) Despite that, agent 2 cannot be ignored, since she has some role to play in the information diffusion process, and hence in the formation of the consensus belief. In a network without agent 2, the most influential agent would be agent 3, ranks just third if agent 2 is present. The reason is that although she is completely uninformed, she can learn from other agents who possess better information (including her neighbour agent 5, who is one of the leading experts in this network). This way she can contribute in propagating their views, increasing thus their influence. Indeed, without agent 2, agents 1 and 5 would be less popular, and hence less influential. Notice that agent 2 was chosen to be one of the least popular agents in the network; the error from ignoring a more popular agent on the grounds of being initially ignorant could be significantly larger.

This is not the case, however, in the baseline DeGroot model. If the time-constant direct influences \( \gamma_{ij} \) are assumed to be the optimal first-period weights, as given by expression (1.2), the direct influence of agent 2 will be zero in all periods. This not only implies that her initial opinion would be completely ignored, but also that she will keep being ignored, even when she would have accumulated knowledge from her neighbours. As shown in Table 1.5, the presence of uninformed agents such as investor 2 can be completely ignored in the DeGroot model.

Figure 1.5(ii) shows the evolution of investor 2’s direct influence on her neigh-

\(^{20}\)Zero expertise in this case would imply that the signal of the “ ignorant” agent originates from a normal distribution with infinite variance. Although this would not be a well-defined probability distribution, it can be still used as a “starting point” to model cases where the signal is uninformative, or the agent is missing prior information. In the literature, such distributions are often called improper priors. Using them should not be a problem as long as they are not over-interpreted (as, for example, representing total ignorance), and the posterior they give rise to is a proper distribution (see, for example, Robert, 2007, chapters 1.5 and 3.5). Alternatively, zero precision could be thought of as an arbitrarily low positive precision, due to a signal from a normal distribution with very high variance. Although these two interpretations are not equivalent, treating them as such would be harmless given their limited use in the context of this simple example.
bours over time. In the beginning, her relative expertise is zero, and hence receives no attention from investors 1 and 3 ($\gamma_{12}(1) = \gamma_{52}(1) = 0$). Yet, in the next period, investor 1 becomes interested in her belief since it potentially contains information from agents that investor 1 cannot directly observe (in this case, investor 3). The same applies to investor 5. Hence, in the dynamic model, agents adjust the weights they assign to investor 2’s belief in order to account for the second-hand information she possesses in periods $t > 1$.

There are two remarks worth bringing forth before concluding the discussion of the above example. First, an alternative approach using the DeGroot model would be to let agents account for possible information that an initially uninformed neighbour may obtain later on, and assign a positive fixed weight to him. This would nevertheless have to be quite arbitrary, and would require at least some partial knowledge about the uninformed agent’s neighbours, or even about the neighbours of his neighbours, and so on. An educated guess would work in this case, but if agents are assumed to trust their neighbours reported expertise, as they trust their reported beliefs, the dynamic-weights process could be a more intuitive approach for them to follow.

Second, the above example shows that under DeGroot updating, decomposing an agent’s social influence into a popularity-driven and an expertise-driven part is not as straightforward, at least with respect to the measures used in the dynamic approach (eigenvector centrality and information precision). Agents 1 and 5 have the same expertise, and despite the fact that agent 5 is more popular than agent 1, they end up having the same social influence. This suggests that the determinants of social influence in the DeGroot model could be more difficult to pin down.
1.3.3 Network-induced distortion

The benchmark

Consider the case in which every agent can directly observe every other agent in the society. In networks theory terminology, this is often referred to as the complete network. Although trivial from an analytical perspective, it will serve as the benchmark in this paper: all agents have immediate access to all information available in the network, and there is practically no persuasion bias.

It can be readily seen from Theorem 1.1 that in the case of a complete network it will hold that $\alpha_{c,\pi^o} = n$ and $c_i = \frac{1}{n}$ for all agents $i \in \mathcal{N}$. It follows then that every agent's social influence will be equal to his or her true relative expertise, that is, $w_i^{(\infty)} = \tilde{\pi}_i(0)$.21

The general case

However, the communication pattern dictated by an incomplete network, in combination with the failure of agents to account for repetitions of information, induces a distortion on the distribution of social influences. In expression (1.18), this network-induced distortion of agent $i$’s influence is captured by

$$d_i := \alpha_{c,\pi^o} c_i.$$ 

Agents with $d_i > 1$ enjoy disproportionately higher social influence than the one justified by their expertise, while agents with $d_i < 1$ see their influence weighted down compared to complete network benchmark. Similarly therefore to the DeGroot model, learning in the present setup is suboptimal in general, and the consensus beliefs do not constitute a sufficient statistic for the initial information in the network. There is, however, a specific class of

---

21 If signals are generated from certain exponential-family distributions (e.g. normal, binomial), the consensus belief in the complete network under the dynamic average-based updating process is the optimal one, in the sense that it is an efficient estimator for the true state of the world (see also Section 1.4).
networks that induce optimal aggregation of information.

**Corollary 1.1.** In a strongly connected network $\mathcal{G}$, the dynamic average-based updating process leads to the benchmark consensus beliefs, irrespectively of the initial beliefs $b(0)$ or the initial precisions $\pi(0)$, if and only if all agents are equally popular, that is, if and only if $c_i = \frac{1}{n}$ for all agents $i \in \mathcal{N}$.

The above statement implies that the benchmark beliefs are generally reached only if $\alpha_{c,\pi^o} = n$. Although the above condition is rather restrictive, it is milder than its counterpart under the canonical DeGroot model. The dynamic model introduced in this paper presupposes that agents are slightly more sophisticated, since they update the weights they attach to their peers. This relaxes the condition required for the vector of social influences, $w^{(\infty)}$, to be the optimal one.

Notice also that the scalar $\alpha_{c,\pi^o}$ has a nice intuitive interpretation: it captures the distortion in network $\mathcal{G}$, that is, it can be readily seen that

$$\alpha_{c,\pi^o} = \sum_{i \in \mathcal{N}} d_i.$$

Yet, since this measure depends on the size of the network, it would be more meaningful to scale it by the number $n$ of agents in the network:

$$\bar{\alpha}_{c,\pi^o} := \frac{1}{n} \alpha_{c,\pi^o}.$$

The following statement follows form the definition of $\pi_{c,\pi^o}$.

**Proposition 1.4: Expertise/Popularity Pattern and Distortion**

The following relationship exists between the scaled centrality–expertise dispersion parameter, $\bar{\alpha}_{c,\pi^o}$, and the covariance between popularity and

---

22 As shown by DeMarzo et al. (2003, Theorem 2), DeGroot learning leads to the correct beliefs if and only if the matrix of direct influences, $\Gamma$, is “balanced”, that is, if and only if $\sum_{j=1}^{n} \delta_{ij} \gamma_{jj} = 1$ for all agents $i \in \mathcal{N}$.
expertise in \( \mathcal{G} \), \( \text{Cov}[c, \pi(0)] \):

\[
\begin{align*}
\text{Cov}[c, \pi(0)] > 0 & \iff \overline{\alpha}_{c, \pi^o} < 1 \quad (1.19a) \\
\text{Cov}[c, \pi(0)] = 0 & \iff \overline{\alpha}_{c, \pi^o} = 1 \quad (1.19b) \\
\text{Cov}[c, \pi(0)] < 0 & \iff \overline{\alpha}_{c, \pi^o} > 1. \quad (1.19c)
\end{align*}
\]

Based on the above result, three different patterns of expertise–popularity allocation emerge:

**Pattern A** \((\overline{\alpha}_{c, \pi^o} < 1)\) : An \( \overline{\alpha} \) smaller than 1 suggests that more centrally positioned agents will on average possess more precise information. An example is a star network with the central agent having higher expertise than the peripheral agents. In that sense, \( \overline{\alpha} \) can be a measure of how the network affects “inequality” (in terms of influence): networks with smaller \( \overline{\alpha} \) reinforce the influence of agents who would anyway be influential due to their high precision.

**Pattern B** \((\overline{\alpha}_{c, \pi^o} = 1)\) : There is zero correlation between agents’ position in the network and the precision of their signals. This will be the case if all agents have the same popularity, such as for example, in a circle, a line, or a complete network. Another case that induces \( \overline{\alpha} = 1 \) is the one in which all agents have the same initial expertise, irrespectively of the structure of the network. Note that \( \overline{\alpha} = 1 \) can arise even in cases in which agents differ from each other both in popularity and expertise.

**Pattern C** \((\overline{\alpha}_{c, \pi^o} > 1)\) : In this case it is the less central agents who possess on average more precise information. A star network with the agent in the centre having less precise information than the average precision in the network is such an example.

Corollary 1.1 implies that networks with nontrivial distortion (Patterns A and C) cannot lead to optimal aggregation of information, in the sense that consensus beliefs will not constitute an efficient estimator of the state of the world.
1.4 Efficiency of learning and some policy implications

The analysis in the above section establishes a straightforward relationship between the characteristics of the individuals in a network, in terms of expertise and popularity, and their social influence. It moreover quantifies the degree to which the network distorts information as the latter flows through it: the opinions of popular individuals get to be heard more, and hence receive more attention than what their informational value would justify. Conversely, the opinions of some less popular but better informed agents are underweighted compared to the optimum.

A question that arises naturally following the preceding discussion is whether some network configurations, or some information allocation patterns, favour learning more than others. The answer to this has direct policy implications. Assume, for example, that providing information to individuals is costly. What are the characteristics of the individual or the organisation that a policy maker should target in order to better inform society about the benefits of a new technology, or the right measures to help prevent a disease? Conversely, if the purpose is to disrupt the flow of information, and create confusion in a criminal organisation, which member should be given false information?

In order to be able to provide answers to such questions, some measure of efficiency of the learning process should be introduced. An obvious candidate would be the expected deviation of the consensus belief from the true value of the parameter of interest, \( \theta^* \). To keep facilitate the discussion, and keep formulas simple, it will be assumed for the rest of this section that the unknown state of the world is a real scalar, \( \theta^* \in \mathbb{R} \), and hence the belief space is a subset of the real line, \( B \subseteq \mathbb{R} \).\(^{23}\) Let the initial belief \( b_i(0) \) of each agent \( i \) be equal to the realisation of a signal \( s_i \), as discussed in Section 1.2. All

\(^{23}\)The findings in this section can be shown to be qualitatively similar in the case that the unknown state of the world is represented by a finite vector, \( \theta^* \in \mathbb{R}^m \) for some \( m \in \mathbb{N} \). It is less straightforward to see whether this holds true for more complicated objects, such as probability distributions.
agents’ signals can be stacked into a vector $s := [s_i]_{i \in \mathcal{N}}$. Then the consensus bias in network $\mathcal{G}$ is defined as

$$
\text{Bias}^{(\infty)}_{\mathcal{G}}(\theta^*, s) := \mathbb{E}[b^{(\infty)} - \theta^*].
$$

Hence, any norm of $\text{Bias}^{(\infty)}_{\mathcal{G}}(\theta^*, s)$, or simply its absolute value, in case that $\theta^*$ is a scalar, could potentially serve as an efficiency measure. It turns out, however, that if the signals that agents receive before the beginning of the communication process are unbiased, so will be the consensus belief as well.

**Lemma 1.1.** Let agents form their initial beliefs $b(0)$ as the realisations of some independent signals, stacked into a vector $s$. Assume that these signals are unbiased, so that $\mathbb{E}[s_i] = \theta^*$ for all agents $i \in \mathcal{N}$. Then, the consensus belief under the dynamic average-based updating process will be unbiased, that is

$$
\text{Bias}^{(\infty)}_{\mathcal{G}}(\theta^*, s) = 0,
$$

or equivalently

$$
\mathbb{E}[b^{(\infty)}] = \theta^*.
$$

Notice that the above lemma applies to the baseline DeGroot as well as the dynamic average-based updating process. An important remark is that the consensus belief will be equal to the true value of the parameter in question only in expectation.

Interestingly enough, a utility-based approach can help us motivate a more meaningful measure for the efficiency of the learning process. Assume that agents in the network intend to use the information they acquired through this communication process to take a decision. Examples could be, among other, the decision to pursue university education, to buy a product or a service (Example 1.3), or invest in a financial asset (Examples 1.2 and 1.5). Such choices will in general depend on each agent’s preferences or constraints (income, time, credit constraints, etc). For the purpose of the present analysis, though, it will be simpler to focus on a very basic case: individuals
wish to estimate the unknown parameter \( \theta^* \) as accurately as possible, since any deviations would be costly. The payoff of each agent \( i \) can be expressed then by the additive inverse of a quadratic loss function,

\[
u_i(\theta^*, b^{(\infty)}) = -(b^{(\infty)} - \theta^*)^2.
\]

It follows then that the expected utility of agent \( i \) before the realisation of the signals will be captured by a widely used statistical measure: (the opposite of) the mean squared error of \( b^{(\infty)} \),

\[
E[u_i(\theta^*, b^{(\infty)})] = -E[b^{(\infty)} - \theta^*]^2 = -\text{MSE}[b^{(\infty)}|\theta^*].
\]

It is well known though that if an estimator is unbiased, its mean squared error collapses to its variance (see, for example, Greene, 2008, Definition C.4). Hence it holds that

\[
E[u_i(\theta^*, b^{(\infty)})] = -\text{Var}[b^{(\infty)}].
\]

Maximising, therefore, expected social welfare in that case would amount to minimising the variance of (the estimator of) the consensus belief.\(^{24}\)

**Definition 1.4: Quality of Assessments and Efficiency of Learning**

Consider two disjoint sets of agents, \( \mathcal{N} \) and \( \mathcal{N}' \), with the communication pattern within each population described by a strongly connected network, \( \mathcal{G} \) and \( \mathcal{G}' \) respectively. Assume that before communication begins, agents receive some independent, noisy, but unbiased signals about a common true state of the world, \( \theta^* \). These signals follow distributions \( f_i \) for all \( i \in \mathcal{N} \), and \( f_j' \) for all \( j \in \mathcal{N}' \). Denote the consensus belief in each network by \( b^{(\infty)} \) and \( b'^{(\infty)} \) respectively. Then \( b^{(\infty)} \) will be said to be a better assessment than \( b'^{(\infty)} \) (or equivalently \( b'^{(\infty)} \) to be a worse assessment

\(^{24}\)In the case that that the unknown state of the world is a vector, \( \theta^* \), the formulas for bias and mean squared error are generalised as: \( \text{Bias}^{(\infty)}[\theta^*, s] := E[b^{(\infty)} - \theta^*] \), and \( \text{MSE}[b^{(\infty)}|\theta^*] = E[\text{tr}((b^{(\infty)} - \theta^*)(b^{(\infty)} - \theta^*)^T)] \). It can be seen then that the expected utility of agent \( i \) will be given by \( E[u_i(\theta^*, b^{(\infty)})] \geq -\text{tr}[V[b^{(\infty)}]] \), where \( V[b^{(\infty)}] \) is the variance–covariance matrix of \( b^{(\infty)} \).
than $b^{(\infty)}$ if
\[ \text{Var}[b^{(\infty)}] < \text{Var}[b^{(\infty)\prime}]. \]

If, moreover, it holds that
\[ \sum_{i \in \mathcal{N}} \pi_i(0) \leq \sum_{j \in \mathcal{N}'} \pi_j(0), \]
then the learning process in network $\mathcal{G}$ with initial expertise distribution $\pi(0)$, will be relatively more efficient than the learning process in network $\mathcal{G}'$ with initial expertise distribution $\pi'(0)$.

The first part of the above definition states that a consensus belief is preferable to another if its ex ante (before the realisation of the signals) variance around its mean is smaller. The second part enables us to compare the efficiency of the learning process across different networks. In particular, the learning process is said to be more efficient in network $\mathcal{G}$ compared to network $\mathcal{G}'$ if agents in the former reach a better assessment, while having access to the same or worse information (as measured by initial aggregate precision) than the agents in the latter. This may be due to differences in the network structure, or different distribution of initial information across agents, or both (cf. findings in Section 1.3.3).

Having introduced the appropriate framework, we can now study some of the issues raised above. Interestingly enough, providing an agent with better information can lead to a worse assessment, and decrease social welfare. This is demonstrated through an example.

**Example 1.6.** Let us return to Example 1.4, where the agents in Network B would like to estimate the probability of a delay that would cause them to miss their connection flight. In this setup, the precision of an agent’s signal is captured by the number of times this agent has drawn an observation from the true distribution. Hence, agents who have flown Carcosa Air more times in the past have better initial information, or a less noisy signal about the true
probability of such delays. Let \( \pi_i(0) \) denote the number of times that each agent has travelled with this airline, which can be interpreted as their “expertise” in this example.

Carcosa Air believe that prospective consumers overestimate the probability of delay. In order to remedy that in the light of the upcoming conference, the company decides to alter this perception by providing better information to the consumers (for example, offer a free flight to some agent). Agent 3 is the most popular economist in that network, which may, intuitively, suggest that he is the most suitable candidate for that purpose.

As shown, however, in Table 1.6(i), the learning process will become less efficient, since it will lead to a worse assessment, despite the improvement in aggregate initial expertise. That is, the ex ante expected error will increase although, ceteris paribus, one of the agents gets access to better information.

The above finding may seem quite surprising at a first glance. Some intuition can be provided perhaps based on Remark 1.1, and the fact that an increase in an agent’s initial expertise implies a decrease in all other agents’ relative initial expertise, and hence their social influence. More specifically, improving agent \( i \)'s information gives rise to two effects. First, it decreases the variance of that agent’s signal, and hence the aggregate variance in the network. It follows from Definition 1.4 that this effect tends to improve the assessment. There is, however, a second-order effect: as expression (1.18) implies, increasing an agent’s expertise makes him or her ceteris paribus more influential. If agent \( i \) was already more influential than it would be justified by his or her initial expertise, providing that agent with better information could increase network distortion. This is because other agents, who may possess better information than agent \( i \) even after his or her signal improves, will see their social influence decrease further below the optimum. This was the case in Example 1.6 above with agent 3.
Example 1.6: More information can hurt

\[
\begin{array}{cccc}
\text{measure} & \text{before} & \text{after} \\
& \text{additional info} & \text{additional info} \\
distortion (\alpha_{c,\pi^e}) & 4.914 & 4.812 \\
scaled distortion (\pi_{c,\pi^e}) & 1.229 & 1.203 \\
\text{MSE}[b^{(\infty)}|p^*] & 0.018309 & 0.018310 \\
\text{MSE}_{after} - \text{MSE}_{before} & 6.633 \times 10^{-7} > 0 \\
\end{array}
\]

Table 1.6(i)

<table>
<thead>
<tr>
<th>(i)</th>
<th>(c_i)</th>
<th>(\bar{\pi}_i(0))</th>
<th>(\pi_i(0))</th>
<th>(w_i^{(\infty)})</th>
<th>(\bar{\pi}_i'(0))</th>
<th>(\pi_i'(0))</th>
<th>(w_i'^{(\infty)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.094</td>
<td>0.222</td>
<td>(14)</td>
<td>0.154</td>
<td>0.219</td>
<td>(14)</td>
<td>0.149</td>
</tr>
<tr>
<td>2</td>
<td>0.173</td>
<td>0.111</td>
<td>(7)</td>
<td>0.142</td>
<td>0.109</td>
<td>(7)</td>
<td>0.137</td>
</tr>
<tr>
<td>3</td>
<td>0.319</td>
<td>0.048</td>
<td>(3)</td>
<td>0.112</td>
<td>0.063</td>
<td>(4)</td>
<td>0.144</td>
</tr>
<tr>
<td>4</td>
<td>0.146</td>
<td>0.206</td>
<td>(13)</td>
<td>0.221</td>
<td>0.203</td>
<td>(13)</td>
<td>0.213</td>
</tr>
<tr>
<td>5</td>
<td>0.094</td>
<td>0.270</td>
<td>(17)</td>
<td>0.188</td>
<td>0.266</td>
<td>(17)</td>
<td>0.181</td>
</tr>
<tr>
<td>6</td>
<td>0.173</td>
<td>0.143</td>
<td>(9)</td>
<td>0.183</td>
<td>0.141</td>
<td>(9)</td>
<td>0.176</td>
</tr>
</tbody>
</table>

\[
\sum \begin{array}{c}
1 \\
(63) \\
1 \\
(64) \\
1 \\
\end{array}
\]

Table 1.6(ii)
A more general result is given below.

**Proposition 1.5: Conditions for Welfare-improving Policy Interventions**

Consider a strongly connected network $G$, where agents update their beliefs according to dynamic average-based updating process. Assume that agents’ signals are independent and follow a normal distribution, $s_i \sim N\left(\theta^*, \sigma_i^2\right)$ where $\pi_i(0) := \frac{1}{\sigma_i}$, or a binomial distribution, $s_i \sim Bin(m_i, \theta^*)$ where $\pi_i(0) := m_i$. Then a small increase in agent $i$’s expertise leads to a better assessment if and only if

$$c_i < 2 \frac{\sum_{j=1}^n c_j^2 \pi_j(0)}{\sum_{j=1}^n c_j \pi_j(0)}. \tag{1.20}$$

The above proposition shows that better information leads to better assessments only if it is not given to excessively popular agents. Otherwise, information may be overweighted, increasing thus distortion.

### 1.5 Conclusions

The present work contributes to the literature on boundedly rational social learning by proposing a variant of the DeGroot model that accounts for the determinant of social influence. Under the canonical average-based updating process, agents revise their beliefs by weighting the opinions of their peers; the framework introduced in this paper enables them to revise the weights too. Although the introduction of this new element constitutes a step towards a more rational, Bayesian approach, the updating process remains unambiguously naïve: agents’ updated beliefs are still just weighted averages of those of their neighbours, and do not account for possible repetitions of information. As a result, the simple and intuitively appealing

---

25 In Examples 1.2 and 1.5 agents signals are drawn from normal distribution, while in Examples 1.3 and 1.6 signals are observations from a binomial distribution.
mechanism behind the standard DeGroot model is retained. Moreover, empirical observations, such as agents’ inability to account for the repetition of information, and the subsequent emergence of persuasion bias, emerge in the model as well.

At the same time, however, the richer structure introduced in this paper provides us with a deeper insight into the determinants of social influence. In particular, as shown in Theorem 1.1, each agent’s social influence is driven by two components: their popularity, as captured by their eigenvector centrality, and their expertise, as captured by the relative precision of their initial beliefs. The magnitude of the network effects is determined by a network-specific component, common for all agents.

Apart from providing an better understanding of the origins of social influence, the above result has further important implications. As it suggests, even agents with very little or no expertise at all can contribute to social learning: although their direct influence will be initially zero, they may play an important part in the information diffusion process, and even end up being the king-makers. Ignoring the presence of such agents would lead to miscalculation of the other agents’ influence, since the popularity of the uninformed agents’ peers would be underestimated. Hence it may be a more suitable tool to analyse network where the majority of information originates from a small number of experts.

Furthermore, the breakdown of social influence into its primary constituents has significant policy implications. First, the amount of information that a social planner needs to have in order to estimate the agents’ influences is lower than under the DeGroot model. An assessment of an agent’s relative expertise (information precision) and popularity (eigenvector centrality) suffices to get a rough measure of their social influence. Although these informational requirements are still quite strong, they are much milder than the corresponding requirement in the DeGroot model (complete knowledge of the network structure). Second, social influence is described by a mathematically simple formula, expressed in terms of agents’ popularity and ex-
pertise. This enables the use of comparative statics, and facilitates the design and evaluation of targeted policy interventions. Interestingly enough, it turns out some interventions may have an adverse effect even if they increase the aggregate information that is available in the network. A mistargeted information campaign, for example, could lead society to more inaccurate estimates of the true state of the world.

The present paper studies only the asymptotic behaviour of the learning process. In cases where convergence is fast, this can be a good approximation of the evolution of the short-term dynamics of the model. If, however, convergence of beliefs is slow, as it could be under the presence of homophily (Golub and Jackson, 2012), the finite dynamics of the model become more relevant. In the DeGroot model, the speed of convergence is captured by the subdominant eigenvalue of the matrix of direct influences. In the case with dynamically updated weights though, as Federgruen (1981) suggests, this is not as straightforward.

Finally, it would be interesting to bring the model proposed by this paper to the data, and test its predictions against those of the standard DeGroot model in different setups. Given, however the high informational requirement of such a project, the relevant data would ideally be collected through a field lab experiment similar to the one conducted by Chandrasekhar et al. (2015).

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APPENDIX

1.A Determining social influence without Assumption 1.1

Intuitively, if network $\mathcal{G}$ is strongly connected, after some rounds of communication, every agent will have gained access to some piece of non-trivial information. The first period when this is true can be considered as the new “initial” period, where Assumption 1.1 is now satisfied. Then the results in this paper that hinge on that assumption (such as Theorem 1.1) will still hold, albeit adapted to the new time index, and with some limitations, as discussed in this section.

A generalisation of Assumption 1.1 will be useful in order to demonstrate this formally, namely, that every agent $i$ in the network is in possession of non-trivial information ($\pi(t) > 0$), or can obtain it directly from an out-neighbor $j$ ($\pi_j(t) > 0$) in the $t$-round of communication, or earlier. Formally:

**Assumption 1.2.** There exists some $t \in \mathbb{N}$ such that for every agent $i \in \mathcal{N}$ there is some agent $j \in \mathcal{D}_\mathcal{G}(i)$ with $\pi_j(t) > 0$.

Strong connectedness of network $\mathcal{G}$ implies that if agents follow the dynamic average-based updating process there will be some $\tilde{t} \in \mathbb{N}$ such that this assumption holds for all $t \geq \tilde{t}$. Denote the smallest such $\tilde{t}$ with $\mathcal{I}$. Consider now a dynamic averaged-based updating process in network $\mathcal{G}$ with initial beliefs $b'(0) := b(\mathcal{I})$ and initial precisions $\pi'(0) := \pi(\mathcal{I})$. Notice that this process satisfies Assumption 1.1. According therefore to Definition 1.2, the matrix of direct influences under that process across time can be described by a sequence $\{\Gamma'(t)\}_{t \in \mathbb{N}}$ with

$$\Gamma'(t) = \left[\left((G'^t \pi'(t) 1_n^\top) \circ I_n\right)^{-1} \left[G \circ I_n (G^{t-1} \pi'(0))^\top\right]\right].$$

(1.21)
It follows that agents’ beliefs in every period $t \in \mathbb{N}$ will be given by

$$b'(t) = W'(t)b'(0).$$

(1.22)

where

$$W'(t) := \Gamma'(t) \Gamma'(t-1) \cdots \Gamma'(1)$$

is the matrix of (cumulative) influences in period $t$ (cf. expressions (1.12a) and (1.13) in Section 1.2.5).

**Proposition 1.6: Consensus Beliefs and Social Influence**

Assume that network $G$ is strongly connected, and agents follow the dynamic average-based updating process stipulated in Definition 1.2. Denote the first period in which Assumption 1.2 holds with $\bar{t}$. Then the following statements are true:

- $b(t) = b'(t - \bar{t})$ for $t \in \{\bar{t} + 1, \bar{t} + 2, \ldots\}$,
- $\Gamma(t) = \Gamma'(t - \bar{t})$ for $t \in \{\bar{t} + 1, \bar{t} + 2, \ldots\}$,
- $b^{(\infty)} = b'^{(\infty)}$,
- $w^{(\infty)} = (W(\bar{t}))^T w'^{(\infty)}$,

where $b'(t)$, $\Gamma'(t)$, and $W'(t)$, denote respectively the belief profile, the matrix of direct influences, and the matrix of cumulative influences in period $t$ under the dynamic average-based updating process with initial beliefs $b'(0) := b(\bar{t})$, and initial precisions $\pi'(0) := \pi(\bar{t})$.

Yet notice that the above result comes with a caveat: Although agents’ beliefs and direct influences in the two processes coincide (adjusting for the shift in time), this does not carry over to their (cumulative) influences, that is, in general it will be $W'(t) \neq W(t + \bar{t})$. In fact, even asymptotically, social influences will differ ($w'(\infty) \neq w(\infty)$). The reason is that the modified process captures the evolution of beliefs (or influences) only after period $\bar{t}$. Before that period, there will still be agents without access to any information (zero
precision), who, by definition, cannot have any influence on the society.\textsuperscript{26} Under the modified process, however, every agent starts with access to some information. As a result, the social influences calculated under that process will overestimate the influence of the agents who were initially uninformed under the original process, and underestimate the social influence of the informed agents. An adjustment is therefore required in order to capture the social influence of each agent based on the initial belief profile $b(0)$, and not the starting belief profile $b'(0) = b(\tilde{t})$ of the modified process.

**Corollary 1.2.** Assume that agents follow the dynamic average based updating process. Then the social influence of each agent $i \in \mathcal{N}$ is given by

$$w_i^{(\infty)} = 0 \quad \text{for all } i \notin A^\circ,$$

while

$$w_i^{(\infty)} = \alpha_{c,\pi'} \sum_{j=1}^{n} c_j \tilde{\pi}_j(0) w_{ji}(\tilde{t}) \quad \text{for } i \in A^\circ,$$

where

$\mathcal{A}^\circ := \left\{ i \in \mathcal{N} \mid \pi_j(0) > 0 \text{ for at least one } j \in \mathcal{D}_G(i) \right\}$ (see also (1.6) and the related discussion),

$c_j$ is agent $j$’s popularity (eigenvector centrality) in network $G$,

$\tilde{\pi}_j(0)$ is agent $i$’s relative initial expertise (precision) under the modified process, and

$$\alpha_{c,\pi'} := \frac{\sum_{j=1}^{n} \pi_j(0)}{\sum_{j=1}^{n} c_j \pi_j(0)}$$ is the centrality–expertise dispersion parameter under the modified process (see Theorem 1.1 and Section 1.3.3 for a thorough explanation).

\textsuperscript{26}Yet such agents may play a key role in spreading information, and shaping the consensus beliefs (see the discussion in Section 1.3.2, and especially Example 1.5).
1.B Proofs

1.B.1 Existence and uniqueness of eigenvector centrality

We begin by establishing that adjacency matrix $G$, and hence its transpose, $G^T$, is non-negative and irreducible; thus the Perron–Frobenius theorem applies (see Section B.B.1). Non-negativity holds true by definition, since $G \in \{0,1\}^{n \times n}$, while irreducibility follows from Lemma A.1 and the assumption that network $G$ is strongly connected.

It can now be readily shown that eigenvector centrality is a well defined measure, that is, it exists and it is unique in any strongly connected network $G$. To establish existence, notice that [B.PF.1] suggests that $\rho_G$ will be an eigenvalue of $G^T$, and hence, by [B.PF.2], $c$ will be the Perron vector of matrix $G^T$. It will therefore be a positive vector, and thus meaningful as a measure of centrality, since it will not contain any negative or non-real entries. Uniqueness follows from the fact that the Perron vector is the only positive eigenvector of $G$ (see [B.PF.3]).

1.B.2 Proof of Proposition 1.3

Consider first the case in which Assumption 1.1 holds. As suggested by expressions (1.12), the dynamic average-based updating process can be viewed as the backward product of a series of stochastic matrices $\{\Gamma(t)\}_{t \in \mathbb{N}}$. If the aforementioned sequence converges to a primitive matrix, then the corollary of Theorem 5 in Chatterjee and Seneta (1977) guarantees convergence of the dynamic average-based updating process, and hence the existence of a limiting vector $w^{(\infty)}$; uniqueness follows directly from Theorem 3 in that paper. This implies that consensus obtains, and as suggested by Proposition 1.2, it should be a definitive one.

In order, therefore, to prove Proposition 1.3 in this case, it suffices to show
that

\[ \Gamma^{(\infty)} := \lim_{t \to +\infty} \Gamma(t) \]

is a primitive matrix. It follows from Definition 1.2 that\(^{27}\)

\[ \Gamma(t) = \left( (G^t \pi(0) 1_n^T) \circ I_n \right)^{-1} \left[ G \circ I_n (G^{t-1} \pi(0))^T \right]. \]

The assumption that agents are out-neighbours of themselves implies that network \(G\) is aperiodic (see Perkins, 1961, Theorem 1). Then, according to Lemma A.1, adjacency matrix \(G\) must be primitive. Hence, with the help of Lemma B.1, the above limit can be written as

\[ \Gamma^{(\infty)} = \left[ \lim_{t \to +\infty} \left( \frac{G}{\rho_G} \right)^t \pi(0) 1_n^T \right] \circ I_n \left[ \frac{G}{\rho_G} \circ I_n \left( \lim_{t \to +\infty} \left( \frac{G}{\rho_G} \right)^{t-1} \pi(0) \right)^T \right] \]

\[ = \left[ \left( \frac{c^T \pi(0) 1_n^T}{c^T p} \right) \circ I_n \right]^{-1} \left[ \frac{G}{\rho_G} \circ I_n \left( \frac{c^T \pi(0)}{c^T p} \right) \right] \]

\[ = \left( \frac{c^T \pi(0)}{c^T p} \right)^{-1} \left[ (p 1_n^T) \circ I_n \right]^{-1} \left( \frac{c^T \pi(0)}{c^T p} \right) \left[ \frac{G}{\rho_G} \circ (1_n p^T) \right] \]

\[ = \left[ (p 1_n^T) \circ I_n \right]^{-1} \frac{G}{\rho_G} \left[ (1_n p^T) \circ I_n \right] \]

by [B.H.6]

\[ = \left[ (p 1_n^T) \circ I_n \right]^{-1} \left[ (1_n p^T) \circ I_n \right] \]

by [B.H.8]

Hence each element of matrix \(\Gamma^{(\infty)}\) will be of the form \(\gamma_{ij}^{(\infty)} = \frac{p_i}{p_i \delta_{ij}} \). Since \(p\) is the Perron vector of a primitive matrix, \(p_i > 0\) for all \(i \in \mathbb{N}\) (see Proposition B.1). Then it can be readily seen that matrix \(\Gamma^{(\infty)}\) will be primitive if and only if matrix \(G\) is.

Let us examine now the case in which Assumption 1.1 fails. Since adjacency matrix \(G\) is primitive, by statement (8.3.16) in Meyer (2001) the must exist some \(\hat{t} \in \mathbb{N}\) such that \(G^\hat{t}\) is a positive matrix for all \(t \geq \hat{t}\). Then, it follows from expression (1.11) that

\[ \pi(\hat{t}) = G^\hat{t} \pi(0) > 0_n \]

since \(\pi(0) \geq 0_n\).

---

\(^{27}\)See also expression (1.23) in Section 1.B.3.
Consider now a dynamic averaged-based updating process in network $G$ with initial beliefs $b'(0) := b(\hat{t})$ and initial precisions $\pi'(0) := \pi(\hat{t})$. In every period $t$, the matrix of direct influences under that process will be given by a sequence $\{\Gamma'(t')\}_{t' = \hat{t} + t}^{\infty}$ where
\[
\Gamma'(t') = \left[ (G^{t'} \pi(\hat{t}) 1_n^T) \circ I_n \right]^{-1} \left[ G \circ 1_n (G^{t'-1} \pi(\hat{t}))^T \right]
\]
for all $t' \in \{\hat{t} + 1, \hat{t} + 2, \ldots\}$. As discussed in the first part of this proof, whether such a process converges (albeit not its limit per se) depends only on the primitivity of matrix $\Gamma'(t')$. Notice that the sequence of belief profiles $\Gamma'(t')$ under the modified process is simply a subsequence of the respective series $\{\Gamma(t)\}_{t \in \mathbb{N}}$, with $\Gamma'(t') = \Gamma(t)$. It follows thus that if the former converges, the latter should do so as well. Yet notice that Assumption 1.1 is satisfied for the modified process ($\pi_i(\hat{t}) > 0$ for all agents $i \in \mathcal{N}$ is a sufficient condition for this), and hence its convergence follows from the result established above.

1.B.3 Proof of Theorem 1

Recall that if Assumption 1.1 is satisfied, agents’ beliefs in period $t$ (that is, after $t$ rounds of communication) can be written in matrix format as
\[
b(t) = \Gamma(t) b(t - 1). \tag{1.10}
\]
Then the matrix of agents’ direct influences will be given by
\[
\Gamma(t) = \left[ (G^t \pi(0) 1_n^T) \circ I_n \right]^{-1} \left[ G \circ 1_n (G^{t-1} \pi(0))^T \right], \tag{1.23}
\]
where $A \circ B$ denotes the Hadamard product of matrices $A$ and $B$. Then the matrix of cumulative influences defined in (2.7) can be written as
\[
W(t) = \prod_{\kappa=1}^{t} \Gamma(\kappa)
\]

---

28 The Hadamard product of two equidimensional matrices is the matrix of the products of their respective elements. For a formal definition as well as some properties that are used in the proofs of the statements in this paper, see Section B.A in the Appendix.
\[
\prod_{k=1}^{t} \left[ \left( G^k \pi(0) 1_n^\top \right) \circ I_n \right]^{-1} \left[ G \circ 1_n \left( G^{k-1} \pi(0) \right) 1_n^\top \right]
\]

\[
= \prod_{k=1}^{t} \left[ \left( G^k \pi(0) 1_n^\top \right) \circ I_n \right]^{-1} G \left[ \left( G^{k-1} \pi(0) 1_n^\top \right) \circ I_n \right]^{-1} G
\]

\[
\times \left[ \left( G^{t-2} \pi(0) 1_n^\top \right) \circ I_n \right] \cdots \left[ \left( G^2 \pi(0) 1_n^\top \right) \circ I_n \right]^{-1} G \left[ \left( G \pi(0) 1_n^\top \right) \circ I_n \right]^{-1} G \left[ \left( \pi(0) 1_n^\top \right) \circ I_n \right]
\]

\[
\times \left[ \left( \pi(0) 1_n^\top \right) \circ I_n \right]^{-1} G \left[ \left( \pi(0) 1_n^\top \right) \circ I_n \right]
\]

where we have used the properties of Hadamard product discussed in Section B.A of the Appendix, and \( D_{\pi(0)} := \text{diag}(\pi_1(0), \pi_2(0), \ldots, \pi_n(0)) \) is a diagonal matrix with the elements of vector \( \pi(0) \) on its main diagonal.

Now we can use Lemma B.1 to obtain an expression for the social influence of the agents. From (B.4) it follows that

\[
\lim_{t \to +\infty} W(t) = \lim_{t \to +\infty} \left\{ \left[ \left( \frac{G}{\rho_G} \right)^t \pi(0) 1_n^\top \right] \circ I_n \right\}^{-1} \left( \frac{G}{\rho_G} \right)^t D_{\pi(0)}
\]

\[
= \left[ \lim_{t \to +\infty} \left( \frac{G}{\rho_G} \right)^t \pi(0) 1_n^\top \right] \circ I_n \right\}^{-1} \lim_{t \to +\infty} \left( \frac{G}{\rho_G} \right)^t D_{\pi(0)}
\]

\[
= \left[ \left( \frac{c}{p} \pi(0) 1_n^\top \right) \circ I_n \right]^{-1} \frac{p c^\top}{c^\top p} D_{\pi(0)}
\]

\[
= \left( \frac{c}{p} \pi(0) 1_n^\top \right) \circ I_n \right]^{-1} p c^\top D_{\pi(0)}
\]

\[
= (c^\top \pi(0))^{-1} \left[ (p 1_n^\top) \circ I_n \right]^{-1} p c^\top D_{\pi(0)}
\]

by [B.H.6]
\[
\begin{align*}
&= (c^\top \pi(0))^{-1} D_p^{-1} p c^\top D_{\pi(0)} \\
&= (c^\top \pi(0))^{-1} 1_n c^\top D_{\pi(0)} \\
&= (c^\top \pi(0))^{-1} 1_n \left( c^\top \circ \pi(0)^\top \right) \\
&= 1_n \left( c^\top \pi(0) \right)^{-1} \left( c \circ \pi(0) \right)^\top \\
&= 1_n \left( \frac{1}{n} \pi(0) \right) \left( c \circ \frac{\pi(0)}{1_n \pi(0)} \right)^\top \\
&= 1_n \alpha_{c,\pi^o} \left( c \circ \pi(0) \right)^\top,
\end{align*}
\]

where \( D_{\pi(0)} := \text{diag}(\pi(0)) \), \( \alpha_{c,\pi^o} := \frac{1}{c^\top \pi(0)} = \frac{\sum_{j=1}^n \pi_j(0)}{\sum_{i=1}^n c_i \pi_i(0)} \) is a scalar that captures the effects of the network on social influence, and \( \tilde{\pi}(0) := \frac{\pi(0)}{1^\top \pi(0)} \) is the vector of relative initial precisions of the agents in network \( G \).

### 1.B.4 Proof of Corollary 1.1

Consensus beliefs \( b^{(\infty)} \) will be the optimal ones for every initial belief profile \( b(0) \), and any initial expertise profile \( \pi(0) \) only if \( w_i^{(\infty)} = \pi(0)_i \). Then it follows from Theorem 1.1 that

\[
w_i^{(\infty)} = \alpha_{c,\pi^o} c_i \tilde{\pi}_i(0) = \tilde{\pi}_i(0),
\]

which implies that \( c_i = \frac{1}{\alpha_{c,\pi^o}} \), or \( c_i = \frac{1}{n} \) due to normalisation (2.1).

### 1.B.5 Proof of Proposition 1.4

Notice that in this case the sampling fraction relevant for calculating covariance is the entire network population. Then the standard formula for covariance (see, for example, Tam, 1985) yields

\[
\text{Cov}[c,\pi(0)] = \frac{1}{n} \sum_{i=1}^n \left( c_i - \frac{1}{n} \sum_{j=1}^n c_j \right) \left( \pi_i(0) - \frac{1}{n} \sum_{j=1}^n \pi_j(0) \right)
\]
\[
\begin{align*}
&= \frac{1}{n} \left( \sum_{i=1}^{n} c_i \pi_i - \frac{1}{n} \sum_{i=1}^{n} \pi_i - \frac{1}{n} \sum_{i=1}^{n} \pi_i(0) + \frac{1}{n^2} \sum_{i=1}^{n} \pi_i(0) \right) \\
&= \frac{1}{n} \left( \sum_{i=1}^{n} c_i \pi_i - \frac{1}{n} \sum_{i=1}^{n} \pi_i \right), \quad (1.24)
\end{align*}
\]

where the second equation holds true since \(\sum_{i=1}^{n} c_i = 1\). Then from (1.24) it follows that

\[
\text{Cov}[c, \pi(0)] > 0 \iff \sum_{i=1}^{n} c_i \pi_i > \frac{1}{n} \sum_{i=1}^{n} \pi_i > 0 \iff \frac{1}{n} \sum_{i=1}^{n} \pi_i < 1 \iff \bar{\alpha}_{c, \pi} < 1.
\]

1.B.6 Proof of Lemma 1.1

Recall from expression (1.17) that the consensus belief will be

\[
b^{(\infty)} = \sum_{i=1}^{n} w_i^{(\infty)} b_i(0).
\]

The \textit{ex ante} expectation of the consensus belief (that is, before the signals are realised, and the prior beliefs are formed) will be

\[
E[b^{(\infty)}] = E \left[ \sum_{i=1}^{n} w_i^{(\infty)} b_i(0) \right] \\
= \sum_{i=1}^{n} w_i^{(\infty)} E[b_i(0)] \\
= \sum_{i=1}^{n} w_i^{(\infty)} \theta^* \\
= \theta^*.
\]

1.B.7 Proof of Proposition 1.5

From Theorem 1.1, the \textit{ex ante} variance of the consensus belief \(b^{(\infty)}\) is given by

\[
\text{Var}[b^{(\infty)}] = \text{Var} \left[ \sum_{j=1}^{n} w_j^{(\infty)} b_j(0) \right]
\]
\[ \alpha^2_{c, \pi^0} = \frac{1}{(\sum_{j=1}^{n} c_j \pi_j(0))} \sum_{j=1}^{n} c_j^2 (\bar{\pi}_j(0))^2 \text{Var}[b_j(0)] \]

\[ = \left( \frac{\sum_{j=1}^{n} \pi_j(0)}{\sum_{j=1}^{n} c_j \pi_j(0)} \right)^2 \sum_{j=1}^{n} c_j^2 \left( \frac{\pi_j(0)}{\sum_{j=1}^{n} \pi_j(0)} \right)^2 \text{Var}[b_j(0)] \]

\[ \text{Var}[b^{(\infty)}] = \frac{1}{(\sum_{j=1}^{n} c_j \pi_j(0))^2} \sum_{j=1}^{n} c_j^2 (\pi_j(0))^2 \text{Var}[b_j(0)]. \quad (1.25) \]

Let us consider first the case where \( s_i \sim N(\theta^*, \pi_i(0)^{-1}) \). Then for the initial belief of each agent \( i \in N \) it holds \textit{ex ante} that \( \text{Var}[b_i(0)] = \frac{1}{\pi_i(0)} \), and hence it follows from expression (1.25) above that

\[ \text{Var}[b^{(\infty)}] = \frac{\sum_{j=1}^{n} c_j^2 \pi_j(0)}{(\sum_{j=1}^{n} c_j \pi_j(0))^2}. \quad (1.26) \]

We can now calculate how \( \text{Var}[b^{(\infty)}] \) changes with \( \pi_i(0) \):

\[ \frac{d\text{Var}[b^{(\infty)}]}{d\pi_i(0)} = \frac{d}{d\pi_i(0)} \left[ \frac{\sum_{j=1}^{n} c_j^2 \pi_j(0)}{(\sum_{j=1}^{n} c_j \pi_j(0))^2} \right] \]

\[ = \frac{1}{(\sum_{j=1}^{n} c_j \pi_j(0))^4} \left[ c_i^2 \left( \sum_{j=1}^{n} c_j \pi_j(0) \right)^2 - 2c_i \sum_{j=1}^{n} c_j \pi_j(0) \sum_{j=1}^{n} c_j^2 \pi_j(0) \right] \]

\[ = \frac{1}{(\sum_{j=1}^{n} c_j \pi_j(0))^3} \left[ c_i \left( \sum_{j=1}^{n} c_j \pi_j(0) \right) - 2 \sum_{j=1}^{n} c_j^2 \pi_j(0) \right] \quad (1.27) \]

Then using (1.27) we can show that

\[ \frac{d\text{Var}[b^{(\infty)}]}{d\pi_i(0)} > 0 \iff c_i > 2 \frac{\sum_{j=1}^{n} c_j^2 \pi_j(0)}{\sum_{j=1}^{n} c_j \pi_j(0)}. \]
which gives condition (1.20) if the signals are normally distributed.

Consider now the case with \( s_i \sim Bin(\pi_i(0), \theta^*) \). It will then hold that \( \pi_i(0) := q_i + r_i \), where \( q_i \) is the number of successes and \( r_i \) the number of failures in the observations drawn. Assume that, based on this signal, the initial beliefs \( b_i(0) \) of each agent \( i \in \mathcal{N} \) are formed as

\[
b_i(0) := \frac{q_i}{q_i + r_i} = \frac{q_i}{\pi_i(0)}.
\]

If the expertise \( \pi_i(0) \) of each agent (the “sample size”) is known, then it follows that

\[
\text{Var}[b_i(0)] := \text{Var}\left[\frac{q_i}{q_i + r_i}\right] = \frac{1}{\pi_i(0)} \text{Var}[q_i] = \frac{\theta^*(1 - \theta^*)}{\pi_i(0)}.
\]

Substituting this into expression (1.25) gives

\[
\text{Var}[b^{(\infty)}] = \theta^*(1 - \theta^*) \frac{\sum_{j=1}^{n} c_j^2 \pi_j(0)}{(\sum_{j=1}^{n} c_j \pi_j(0))^2}.
\]

(1.28)

The rest of the proof is similar to the case with \( s_i \sim N(\theta^*, \frac{1}{\pi_i(0)}) \), since expression (1.28) is proportional to (1.26).

1.B.8 Proof of Proposition 1.6

The first statement in Proposition 1.6 follows directly from the definition of the modified dynamic average-based updating process.

Notice now that Assumption 1.1 is satisfied for the modified dynamic average-based updating process, and hence the agents’ direct influences on each other can be written in matrix form as in expression (1.21). Then the second statement follows from the definition of the modified dynamic average-based updating process.

To prove the third statement, observe that sequence \( \{b'(t)\}_{t \in \mathcal{N}} \) describing the beliefs under modified dynamic average-based updating process is a subsequences of its counterparts under the original dynamic average-based
updating process, \( \{b(t)\}_{t \in \mathcal{N}} \), with a finite number of elements omitted (the first 7). Hence if these sequences converge (and this is guaranteed by Proposition 1.3; see also its proof), they must tend towards the same limit.

According to expression (1.22), the consensus beliefs of the modified process is given by

\[
b'(\infty) = (w'(\infty))^\top b'(0).
\]

Notice that this is in terms of the initial belief profile of the modified process, \( b'(0) \). This process though is an artificial construct to facilitate the exposition of our result in case Assumption 1.1, so there in general there will be no reason to be particularly interested in \( w'(\infty) \). Recall that, by definition, the social influence of an agent is the impact of his or her initial beliefs \( b_i(0) \) on the consensus belief, and that is the real object of interest. Hence the above equality can be written as

\[
b'(\infty) = (w'(\infty))^\top W(\tilde{t}) b(0),
\]

since

\[
b'(0) := b(\tilde{t}) = W(\tilde{t}) b(0).
\]

The fourth statement follows then directly from expression (1.29).

1.B.9 Proof of Corollary 1.2

Since Assumption 1.1 is satisfied for the modified dynamic average-based updating process, Theorem 1.1 applies, and the social influence of each agent \( i \in \mathcal{N} \) is given by

\[
w'_i(\infty) = \alpha_{c,\pi^\circ} c_i \tilde{\pi}'_i(0)
\]

Then Corollary 1.2 follows from the fourth statement in Proposition 1.6.
Chapter 2
To be efficient, a nation must develop specialists in discovering, transmitting, and analyzing popular opinion, just as it develops specialists in everything else [...] Because some voters can be influenced, specialists in influencing them appear.

– Anthony Downs, *An economic theory of political action in a democracy* (1957)

2

Lobbies, experts, and the public:
a network model of political influence

The present paper studies the evolution of political beliefs using a network model of social learning. Agents communicate their information, discuss their opinions with their peers in the network, and update their beliefs accordingly. Although information originating from better-informed agents receives *ceteris paribus* a larger weight, individuals receive and communicate information through their ideological prism. The paper also studies networks with individuals or groups of individuals who are not interested in learning or exchanging of information, but rather in promoting their own views to other agents. The features that make such groups influential are identified and discussed.
2.1 Introduction

More often than not, important political and economic issues are brought into public debate in a way that incorporates both normative and positive aspects. Would a trade partnership agreement between the United States and the European Union be beneficial for both sides? On each side of the Atlantic, which groups are going to gain and which groups are going lose from it? How are British citizens going to be affected if the United Kingdom decides to withdraw from the European Union? Providing well-substantiated answers, or even justified opinions to such questions requires a deep understanding of the topic of interest, some degree of knowledge of the relevant background, and familiarity with the tools used to analyse such issues. It follows thus that the opinion of people considered experts in those fields, such as a Professor of International Economics, specialising in the EU–US relationships, or an analyst at a trade policy institute, carry a fairly high weight in the society.

This paper introduces a network model of social learning to study social influence and the evolution of political beliefs. In particular, it allows agents to weight any information they receive based on both objective and on subjective criteria. It also examines the characteristics that make an individual (or a group of individuals) influential in a society. The main focus of the analysis is on cases in which there exists closed groups of individuals who are more interested in promoting their views rather than gathering information and improving their understanding of the issue.

The above questions, like most real-world political issues, are ideologically loaded to a greater or lesser extent. This affects the heed that the public pays to the opinions of experts via two channels. Firstly, individuals in a society have different values and ideologies, and may be negatively or positively predisposed towards certain opinions. At the same time, they understand that the beliefs of the experts are not (and perhaps cannot be) completely objective, and are expressed from their point of view. This relates to the
concept of selective attention. As Taber and Lodge (2006) show, individuals are largely prepared to ignore, or even actively counter information and arguments that come contrast their beliefs on political issues. This seems to be motivated by the ideological background and the prior attitudes of the individuals, without the degree of bias alleviated (but rather exacerbated) as subjects become more sophisticated.

Nevertheless, *ceteris paribus*, individuals would be expected to place more trust on the opinions of experts (or someone who talks regularly to experts) than an otherwise identical person, who’s a non-expert, and does not have access to expert opinions. Using a sample of U.S. citizens, Page, Shapiro, and Dempsey (1987) find a strongly positive and statistically significant correlation between the opinions of experts, as communicated through the television, and the shift in the public opinion over the same period.

Social networks have traditionally been a major channel of communication, interaction, political dialogue, and exchange of beliefs and opinions. The close relationship between social ties and the emergence of similar ideological and political views and actions among individuals has been observed by the researchers since longtime ago (Lazarsfeld, Berelson, and Gaudet, 1948), and is well documented (see, for example, Huckfeldt and Sprague, 1987; Jackson, 2008; Cohen and Malloy, 2014). Yet the extent to which this interaction actually shapes individuals' views and behaviour may not be as clear to infer, especially in the presence of political homophily. Homophily is loosely defined as the empirically observed tendency of individuals to associate more with people who are similar with them. Verbrugge (1977) was one of the first to provide evidence of homophily on the basis of political beliefs.¹ It could be hence argued that it is not necessarily the interaction in a network that influences the political attitudes of the individuals, but rather that the network itself is the result of those attitudes.

A more recent study by Conover, Ratkiewicz, Francisco, Gonçalves, Menczer, ¹ It is interesting that this seems to be the case even among individuals that identify themselves as “Independents”.
and Flammini (2011) sheds new light on some aspects of political homophily. The researchers gathered and analysed a large set of data from Twitter, a popular microblogging and social networking service, in order to study the evolution and communication of political opinions in the eve of the United States mid-term election of 2010. More specifically, the data collected concerned two networks, referred to as the mention and the retweet networks respectively. The former captures the pattern of communication across the individuals, that is messages addressed directly (replies) or potentially indirectly (mentions) to other users. The latter represents the pattern of reposting and spreading of others’ opinions. Interestingly enough, although the mention network is dominated by a giant component, the retweet network is characterised by a high degree of homophily-based clustering. This suggests that people tend to selectively disregard opinions that are ideologically distant from their own opinions, even if they have access to them. Furthermore, it seems to provide additional evidence in favour of the findings of the literature on “motivated skepticism” discussed above. It is less of an indication of homophily-based formation of networks though, since the paper does not delve rigorously into studying the direction of causality.

Over the last years there have been several research attempts explicitly seeking to disentangle the effect of within-network interaction on sculpturing political beliefs from the pre-existence of such beliefs that may have potentially led to the formation of the network in the first place. The Using data from surveys among public policy students, Lazer, Rubineau, Chetkovich, Katz, and Neblo (2010) provide further evidence in favour of the contribution of social (albeit not the “professional”) ties in the formation of political attitudes. Political homophily may be present, but the findings suggest that its role is limited, and networks are formed based primarily on other types of homophily.

In a related study, Algan, Do, Le Chapelain, and Zenou (2015) find that friendship networks of students at an elite French university play an important role in forming their political opinions, and shaping their moral values.
On average, convergence in political opinions due to peer effects amounts to 8% of the pre-existing mean difference (or 11% of its standard deviation) within the first six months. These networks have been put together exogenously, and thus the results of the paper account purely for networks effects, and are independent of any additional effects that may arise due to homophily in the network formation process. Hence empirical evidence seem to corroborate the key function that social perform in shaping political beliefs, beyond any role homophily may play. The above effects are in fact larger for pairs of students less likely to become friends endogenously.

The present work studies how the communication structure in a society contributes to the formation of attitudes and beliefs. To that end, it extends a well-known model of social learning in networks, introduced by DeGroot (1974). The potential of this model in studying public opinion formation was first brought to the limelight by DeMarzo et al. (2003). The authors show that since agents are not able to track the path that every message follows in the network, they may take into account the same piece of information more than once. Hence they are susceptible to persuasion bias, resulting in the beliefs of the more central agents receiving disproportionately high attention. It is moreover shown that before reaching a long-run consensus, every agent’s position on a range of different issues will collapse to a single point on “left-right” line, even if they start with ideologically different views on different issues.

As argued in the sections to follow, the hybrid model discussed in the present paper is in general more suitable for studying the evolution of political beliefs compared both to the baseline DeGroot model, and its dynamic variant introduced in Chapter 1 of the present thesis. In fact, it will be shown that both these paradigms can be obtained as special cases of the more general framework used in this paper. The main advantage of this social learning process is that it allows agents to take into account expertise when deciding how to weight the opinions of their peers. Note that, in principle, this can be accommodated by the DeGroot model by properly adjusting the weights.
agents assign to their peers’ opinions. This would make it hard though to identify the contribution of each determinant (expertise versus subjective or exogenous factors) in the formation of the public opinion. There is however another, perhaps more essential drawback in this approach. Weights in the DeGroot model remain fixed throughout the updating process, and consequently they cannot reflect the changes in the quality of information that agents possess. The introduction of dynamic weights enables us to model exactly this accumulation of knowledge and expertise. It can moreover account for initially ignorant agents, who listen to the opinions of their neighbours, learn from them, and can subsequently pass this knowledge on.

The paper proceeds as follows: Section 2.2 introduces a model of weighted dynamic updating and provides some asymptotic results. Section 2.3 studies the short-run dynamics of weighted dynamic updating. With the assumption of strong connectedness relaxed, Section 2.4 studies how the beliefs of the society and the influence of closed groups is formed. Section 2.5 concludes. The proofs of the statements in this paper are presented in Appendix 2.A.

2.2 The model

2.2.1 The agents

Society is modelled as a finite set of agents $\mathcal{N}$ with cardinality $n$. In order to maximise their utility, agents in the present model need to estimate an unknown parameter, representing their ideal opinion on an issue (e.g. minimum wage, affirmative action, or abortion legislation) or their optimal action (e.g. elections, referendum, or voting in a committee). The unknown parameter specific to agent $i$, denoted by $\vartheta_i(\theta^*, \chi_i)$, is a function of two random variables: the state of the world $\theta^*$, common to all agents, and his or her type $\chi_i$, representing that agent’s idiosyncratic characteristics. Before the communication process begins, the state of the world is realised. Although agents cannot directly observe the realised value $\theta^*$, they each receive a noisy but unbiased private signal about $\theta^*$; agents’ signals
are assumed to be independent from each other. They also privately observe their type $\chi_i \in \mathcal{X}$, which is drawn from a joint probability distribution $f(\chi) : \mathcal{X}^n \rightarrow \mathbb{R}_+$ that is common knowledge.

Based on their signal and their type, agents start with some initial beliefs, which they revise as they communicate with their peers, and gather information from other agents. Technically, beliefs are quite generic and can be represented by numbers, vectors, or even distributions. Formally, the initial beliefs of agent $i \in \mathcal{N}$ will be denoted by $b_i(0) \in \mathcal{B}$, where $\mathcal{B}$ is some convex subset of a linear space. For example, the subjective probability of an event, a vector expressing the ideological position of an individual on a series of issues on a scale from 1 to 10, the evaluation of a policy implemented by the government, and the ideal distribution of wealth in a society, are all admissible as beliefs in this model.

Apart from his or her initial beliefs, each agent $i \in \mathcal{N}$ is also characterised by his or her (initial) expertise, denoted by $\pi_i(0)$. Expertise represents the informativeness of the signal that the agent receives before the communication process begins. It is a measure of the quality of information that agents possess initially, and it is quantified as a non-negative real number. Depending on the distributional assumptions in the model, it can coincide with the inverse of the variance of the signal of each agent, or the sample size of the realised outcomes of the state-of-the-world generating distribution that each agent has observed \textit{ex ante}, or any other measure that captures the idea of better or more precise initial information.

The pattern of communication and transmission of information between agents is represented by a network. Technically, this communication network is modelled as a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where the set of vertices or nodes, $\mathcal{N} := \{1, 2, \ldots, n\}$, represents a set of agents who can potentially interact with each other, and the set of edges $\mathcal{E} \subseteq \mathcal{N}^2$ represents the links among them. In many applications of network theory it is more convenient to represent a network using an \textit{adjacency matrix} $\mathbf{G} := [g_{ij}]_{(i,j) \in \mathcal{N}^2} \in \{0,1\}^{n \times n}$, where $g_{ij} := 1$ if there is a directed edge from node $i$ to node $j$ (i.e. agent $i$
is linked to agent \( j \), and \( g_{ij} := 0 \) otherwise.

A link from agent \( i \) to agent \( j \), \( g_{ij} = 1 \), has the interpretation that agent \( i \) has access to agent \( j \)'s belief. It shall be then said that agent \( i \) observes, pays attention to, or listens to agent \( j \), or in network theory terminology, agent \( i \) is an in-neighbour of agent \( j \). Equivalently, it can be said that agent \( j \) receives attention from or is an out-neighbour of agent \( i \).\(^2\)

There are three main issues that need to be addressed before using the model introduced in Chapter 1 to study the formation of political opinions. First, the existence of a unique, true state of the world is seldom the case when it comes to political debate. Since ideology plays an important role, the updating process should be modified in order to allow for (partially) subjective weighting of the opinions of one's neighbours. Second, the finite-horizon dynamics, and the time required for a consensus to be attained under the dynamic average-based process, may be of primary importance. This is especially true in cases in which individuals need to act or make decisions based on their beliefs before a consensus has been reached. Third, there may exist groups or individuals whose target may be to shift opinion towards their preferred positions. Such agents keep to themselves and do not pay any attention to outsiders. Hence the case of a weakly connected network arises naturally, and needs to studied as well. Each section in this paper addresses one the aforementioned issues, starting with the present one.

### 2.2.2 The network

The framework introduced in Chapter 1 enables agents to revise the weights they assign to their peers, accounting thus for the fact that individuals can learn from their peers and become more knowledgeable. It needs, however, some adaptation in order to be made suitable for the study of the formation of political beliefs. In particular, a more advanced definition of a network is

\(^2\) This terminology has its roots in the drawings of networks as graphs. A directed link from agent \( i \) to agent \( j \) means that agent \( i \) gives attention to agent \( j \); hence \( j \) is an out-neighbour of \( i \). This of course implies that \( j \) receives attention from \( i \); hence \( i \) is an in-neighbour of \( j \).
Chapter 2. Lobbies, Experts, and the Public

Definition 2.1: Weighted Network

Consider a network \( G = \langle N, E \rangle \), and let \( z_{ij} \) be a non-negative number associated with every link \((i, j) \in E\). Let scalars \( z_{ij} \) be referred to as weights, and \( E^Z := \{(i, j; z_{ij}) \mid i, j \in E, z_{ij} \in \mathbb{R}_+ \} \) denote the set of links together their respective weights. Then \( G^Z = \langle N, E^Z \rangle \) shall be called a \( z \)-weighted network. In matrix format, it will be represented by a non-negative matrix of weights, \( Z := [z_{ij}]_{(i, j) \in N^2} \), where \( z_{ij} := 0 \) if \( g_{ij} = 0 \).

A weighted network \( G^Z \) is said to be strongly connected if there exists a directed path comprised of links with positive weight from every node to every other node in \( G^Z \).

The set of all agents in \( G \) that agent \( i \) has access and assigns a positive weight to (that is, all the out-neighbours \( j \) of agent \( i \) with \( z_{ij} > 0 \)) constitutes the out-neighbourhood of agent \( i \) in \( G^Z \), and is denoted by \( D^z_G(i) \). Using mathematical notation, for any \( i \in N \)

\[
D^z_G(i) := \{ j \in N | z_{ij} > 0 \}.
\]

In many setups, a weighted network is more suitable than an unweighted one for representing the communication pattern among individuals in a society. In a wide range of issues, factors such as ideology, personal feelings, and other exogenously determined characteristics matter as much as expertise does in determining the amount of attention individuals pay on the opinion of their peers. Political dialogue, and the exchange of political beliefs and opinions is perhaps one of the most characteristic such cases.

Note that although \( g_{ij} = 0 \) implies \( z_{ij} = 0 \), the inverse need not be true, that is, \( g_{ij} = 1 \) does not necessarily imply \( z_{ij} > 0 \). This suggests that an agent may choose to completely ignore some peer, even if the former has access to the
information or the beliefs of that peer. For the analysis in this section to be valid though, the network represented by matrix $Z$ should still be strongly connected and aperiodic. That is, agents are assumed not to “cancel out” any link $(i,j)$ required for information originating from any agent to reach any other agent in the network by assigning zero weight to it ($z_{ij} = 0$).\(^3\) The discussion of the case in which this fails, giving rise to a weakly connected network, has been deferred to Section 2.4.

The following technical assumption that will be useful in the sections to follow.

**Assumption 2.1.** For every agent $i \in \mathcal{N}$ there exists some agent $j \in D^c_G(i)$ such that $\pi_j(0) > 0$.

Assumption 2.1 states that every agent $i$ has at least one out-neighbour $j$ with $z_{ij} > 0$ (or potentially be agent $i$ himself) who is not totally ignorant about the issue in question. The purpose that this assumption serves is to keep technicalities and notation at a minimum, and does not qualitatively affect our results. From a practical point of view it is not particularly restrictive, since in most applications it would be reasonable to assume that agents have some direct or indirect access to some information, even arbitrarily imprecise.\(^4\) For the remainder of this Chapter, Assumption 2.1 will be assumed to hold, and it will not be explicitly reiterated in the theorems and propositions to follow.

\(^3\) In technical terms, matrix $Z$ should be irreducible and primitive (see Section B.B of the Mathematical Appendix). This is the exact condition required for the canonical DeGroot model to converge (see Proposition 1.1 in Chapter 1).

\(^4\) A sufficient, although not necessary, condition for this to hold is that every agent places positive precision to his or her initial belief, that is, $\pi(0) > 0$. A case similar to the one in which Assumption 2.1 fails is discussed in detail in Appendix 1.A in Chapter 1. As it is shown, as long as the rest of the assumptions hold, the results will be qualitatively (albeit not necessarily quantitatively) similar.
2.2.3 Some network centrality measures

Finally, before proceeding with the main part of this section, it would be useful to present two centrality measures, weighted eigenvector centrality and Bonacich centrality.

Eigenvector centrality, first proposed by Bonacich (1972), captures the idea that what makes an agent important in a network is how well-connected this agent is to other important agents. More specifically, each agent’s eigenvector centrality is a weighted average of the eigenvector centralities of the agents who pay attention to him or her. An individual is thus given a higher eigenvector centrality score if he or she receives attention from individuals who have themselves a high score.

**Definition 2.2: Weighted Eigenvector Centrality**

Consider a strongly connected and potentially weighted network $G^z = \langle N, E^z \rangle$ with weighted adjacency matrix $Z$. The **weighted eigenvector centrality profile** of network $G^z$ is defined as the positive left eigenvector of $Z$, that is, as a vector $\tilde{c} := [\tilde{c}_i]_{i \in N} \in \mathbb{R}_+^n$ satisfying

$$\tilde{c}^\top Z = \rho_Z \tilde{c}^\top$$

normalised so that

$$\|\tilde{c}\|_1 := \sum_{i=1}^n |\tilde{c}_i| = 1,$$

(2.1)

where $\rho_Z$ is the spectral radius of adjacency matrix $Z$, and $\|\cdot\|_1$ denotes the vector 1-norm. The **eigenvector centrality** of agent $i \in N$ is given by the element $\tilde{c}_i \in [0, 1]$.

The rationale behind weighted eigenvector centrality is similar to that of unweighted eigenvector centrality, but takes into account the weights that agents assign to their peers. Hence a well-connected agent may, for example,
receive a low score if he or she ignores or underweights his or her contacts who are considered important (i.e. who receive a high score).

A different, but closely related measure was introduced by Bonacich (1987), and emerges often in the analysis of networks, and social influence in particular.\(^5\)

**Definition 2.3: Bonacich Centrality**

Consider a network \(G = \langle N, E \rangle\). The \(q\)-weighted Katz-Bonacich centrality of each agent \(i\) is given by the \(i\)-th element of the non-negative vector \(c^B := [c_i^B]_{i \in N}\) defined as

\[
c^B := \left( I_n - \frac{1}{\beta} G \right)^{-1} q
\]

where \(\beta \in (0, \rho_G^{-1})\) is a scaling factor that captures the importance of network effects, and \(q \in \mathbb{R}_n^+\) expresses the base value of paying attention to each agent.

Bonacich centrality assigns a score on each agent not only based on the sum of scores of his or her immediate neighbours, but also on that of his or her second-degree neighbours (i.e. neighbours of neighbours), and third-degree neighbours, and so on. The scaling factor \(\beta\) expresses the degree by which the contribution of neighbours of a higher degree is discounted; the larger \(\beta\) is, the higher the value of having direct access to an agent with high score.

### 2.2.4 A weighted dynamic average-based updating process

Now a generalised version of the dynamic average-based updating process that accounts for subjective weighting of beliefs can be introduced. Conceptually, the weight \(z_{ij}\) associated with each link \((i, j)\) captures the subjective

\(^5\) See also Jackson (2008, Chapter 2.2.4), or de Martí and Zenou (2015) for the weighted variant.
importance that agent $i$ assigns to agent $j$’s belief. Technically, the process is very similar to the dynamic average-based updating process discussed in detail in Chapter 1, and thus the analysis in this section will not enter into details that have already been discussed. As it will be seen, the difference lies in the use of a weighted network instead of an unweighted one, and the subsequent replacement of adjacency matrix $G$ by the matrix of subjective weights, $Z$. As discussed below in this section, an additional assumption, that of aperiodicity, is required in order for the generalised process to converge.

Agents’ interaction with their peers is represented by weighted network $G^Z$. As discussed in Section 2.2.1, in the initial period, $t = 0$, each agent holds some initial belief $b_i(0) \in B$, to which he or she assigns a precision or expertise, $\pi_i(0) \in \mathbb{R}_+$. The model is quite general, and can admit as beliefs any object that belongs to a linear space, for example real or complex numbers, vectors, or probability distributions.

Agents update their initial beliefs through communication with their neighbours. In each round, they ask their out-neighbours for their initial beliefs $b_i(0)$, as well as an assessment $\pi_i(0)$ of how precise or accurate these beliefs are. Then they update their own beliefs by weighting the information they receive based on their peers’ expertise. The belief of agent $i$ after $t$ rounds of communication, where $t \in \{0, 1, 2, \ldots\}$, will be denoted by $b_i(t) \in B$.

The timing of the communication process is the following:

**in the ex ante stage**

[i] Agent $i$ privately observes his or her type, $\chi_i$, and updates his or her beliefs about the type of the other agents.

[ii] Agent $i$ observes a private and noisy signal about the value of the parameter of interest, $\vartheta_i$, and forms his or her initial belief, $b_i(0)$.

**in each round of communication**

[1] The $t$-th round of communication takes place. Agent $i$ collects from
each out-neighbour \( j \) a report of his or her previous period beliefs and precision (expertise), that is, a pair \((b_j(t-1), \pi_j(t-1))\) \(\in \mathcal{B} \times \mathbb{R}_+\) for every \( j \in D_{G}^Z(i) \).

[2] Agent \( i \) updates the weight he or she assigns to each neighbour \( j \) (i.e. the direct influence of agent \( j \) on agent \( i \)) \( \gamma_{ij}(t-1) \) to \( \hat{\gamma}_{ij}(t) \), according to expression (2.3)

[3] Agent \( i \) updates his or her belief \( b_j(t-1) \) according to (2.2). The new belief, \( b_j(t) \), is the weighted average of the beliefs \( b_j(t-1) \) reported by agent \( i \)'s out-neighbours, using the new weights \( \hat{\gamma}_{ij}(t) \) calculated in stage [2] above.

[4] Agent \( i \) calculates the precision of his or her updated belief as shown in expression (2.3). The new precision \( \hat{\pi}_i(t) \) is simply the sum of precisions of his or her out-neighbours (including own-precision) reported in stage [1] above.

As suggested by the description of the process above, the expression “beliefs in period \( t \)” or “expertise in period \( t \)”, \( b(t) \) and \( \pi(t) \) respectively, shall refer to the belief and precision profiles of the agents at the end updating process, and after all communication has taken place in period \( t \).

Communication takes place at the beginning of each period \( t \in \mathbb{N}^* \), with the agents observing the beliefs and precisions of their out-neighbours. Then, they revise their beliefs based on the information collected. More specifically, their updated beliefs in period \( t-1 \) emerge as a weighted average of the period-\( t \) beliefs of their out-neighbours, and this process continues ad infinitum. What differentiates this process from the one introduced in the previous chapter is that the weight assigned to each neighbour no longer needs to be equal to the relative expertise of that agent, but it can also depend on other, potentially exogenous factors. Formally, the beliefs of agent \( i \) in period \( t \) are given by

\[
b_i(t) = \sum_{j=1}^{n} \hat{\gamma}_{ij}(t) \ b_j(t-1). \tag{2.2}
\]
where
\[ \hat{\gamma}_{ij}(t) := \frac{z_{ij}\hat{\pi}_j(t-1)}{\sum_{k=1}^{n} z_{ik}\hat{\pi}_k(t-1)} \] (2.3)
denotes the direct influence of agent \( j \) on agent \( i \) in period \( t \) (that is, the relative weight that agent \( i \) places on the belief of agent \( j \) in the \( t \)-th round of communication), and
\[ \hat{\pi}_i(t) = \sum_{j=1}^{n} z_{ij}\hat{\pi}_j(t-1). \]
is agent \( i \)'s subjective accumulated precision or expertise in period \( t \). As it will be argued in this section, depending on the setup, subjective accumulated expertise can be interpreted either as the overall precision of information that agent \( i \) believes to possess in period \( t \), or how certain this agent is that his or her beliefs in that period are the “correct” ones.

In the analysis that follows, it will be useful to rewrite updating rule (2.2) in matrix format as
\[ \mathbf{b}(t) = \hat{\Gamma}(t)\mathbf{b}(t-1), \] (2.4)
where \( \mathbf{b}(t) := [b_i(t)]_{i \in \mathcal{N}} \) and \( \hat{\Gamma}(t) := [\hat{\gamma}_{ij}(t)]_{(i,j) \in \mathcal{N}^2} \) is the matrix of direct influences in period \( t \). Iterating backwards, agents’ beliefs in period \( t \) can be written as a function of their initial beliefs, \( \mathbf{b}(0) \):
\[ \mathbf{b}(t) = \hat{\Gamma}(t)\hat{\Gamma}(t-1)\cdots\hat{\Gamma}(1)\mathbf{b}(0) =: \prod_{\tau=1}^{t} \hat{\Gamma}(\tau)\mathbf{b}(0). \] (2.5)

The cumulative influence, or simply the influence of agent \( j \) on agent \( i \) in period \( t \), denoted by \( \hat{\omega}_{ij}(t) \), captures the overall influence, direct and indirect, that agent \( j \) has had on agent \( i \). Expressed differently, \( \hat{\omega}_{ij}(t) \) represents the degree to which agent \( j \)'s initial belief has contributed in shaping agent \( j \)'s belief in period \( t \). Formally, \( \hat{\omega}_{ij}(t) \) is the \((i,j)\)-th element of the matrix \( \hat{W}(t) \)

---

6 A hat (\( \hat{\cdot} \)) has been used to denote that the measures refer to a weighted network, and hence the qualifier subjective.
defined as
\[
\hat{W}(t) := \prod_{\tau=1}^{t} \hat{\Gamma}(\tau). \quad (2.6)
\]

Using this definition, (2.5) can be written as
\[
b(t) = \hat{W}(t) b(0). \quad (2.7)
\]

Similarly, agents’ subjective accumulated precisions in period \( t \) can be stacked into a vector \( \hat{\pi}(t) \) such that
\[
\hat{\pi}(t) = Z \hat{\pi}(t-1) = Z^t \pi(0) \quad (2.8)
\]

A rigorous definition of this process is given below.

**Definition 2.4: A Weighted Dynamic Average-based Updating Process**

Let \( Z \) be a row-stochastic matrix of subjective weights. Agents will be said to follow the weighted dynamic average-based updating process, or simply weighted dynamic updating (WDU), if their updated beliefs after each round of communication equal a weighted average of the beliefs reported by their out-neighbours (including themselves), where the weight assigned to the belief of each neighbour \( j \) is proportional to agent \( j \)’s subjective accumulated expertise, \( \hat{\pi}_j(t-1) \), adjusted by the subjective weight \( z_{ij} \) attached to agent \( j \) by agent \( i \). Agents’ beliefs in period \( t \in \{1, 2, \ldots\} \) are given by
\[
b(t) = \hat{\Gamma}(t) b(t-1) \quad (2.4)
\]

and their accumulated precisions by
\[
\hat{\pi}(t) = Z \hat{\pi}(t-1) = Z^t \pi(0). \quad (2.8)
\]

WDU is similar to the dynamic average-based updating process discussed in Chapter 1 of the present thesis, but with the adjacency matrix \( G \) replaced by the matrix of subjective weights \( Z \). Recall that the elements \( z_{ij} \) of matrix \( Z \)
are non-negative, and express some (potentially exogenous and subjective) weight that agent $i$ gives to the opinion of each out-neighbour $j$.

**Proposition 2.1: Social Influence Under Weighted Dynamic Updating**

Assume that $\mathcal{G}^z$ is strongly connected and aperiodic, and agents follow the weighted dynamic average-based updating process as stipulated in Definition 2.4. Then the social influence of each agent is equal to the product of his or her weighted eigenvector centrality and his or her initial precision, adjusted by a network effects multiplier

$$\widehat{w}_i^{(\infty)} = \alpha \widehat{c}\tilde{\pi}(0)$$

where

- $\widehat{c}$ is the *weighted* eigenvector centrality of agent $i$
- $\tilde{\pi}_i(0) := \frac{\pi_i(0)}{\sum_{j=1}^{n} \pi_j(0)}$ is agent $i$’s relative initial precision, and
- $\alpha := \frac{\sum_{j=1}^{n} \pi_j(0)}{\sum_{j=1}^{n} \hat{c}_j \pi_j(0)}$ is the *centrality–expertise dispersion parameter*, a scalar that captures the distortion in agents’ influences induced by the $z$–weighted network.

This proposition is a generalisation of the famous result due to DeGroot (1974), as adapted to the social networks framework by DeMarzo et al. (2003) and Golub and Jackson (2010). In the canonical average-based updating model developed in the aforementioned literature, agents may assign any —potentially arbitrary— weight to their peers. These weights (“direct influences”), though, remain fixed, and are not updated during the communication process. In the present setup, direct influences are determined partly by subjective criteria, and partly by more objective criteria, namely the precision reported by each out-neighbour. Since the latter changes over time according to expression (2.8), so will direct influences.

**Remark 2.1.** Consider the weighted dynamic updating process (WDU) stipulated in Definition 2.4.
• Assume that all agents \( i \in \mathcal{N} \) assign equal subjective weights to their out-neighbours, that is, \( z_{ij} = z > 0 \) if agent \( i \) observes agent \( j \), and \( z_{ij} = 0 \) otherwise, for all \((i,j) \in \mathcal{N}^2\). Then WDU collapses to the dynamic average-based updating process.

• Assume that all agents \( i \in \mathcal{N} \) have equal initial expertise, so that \( \pi_i(0) = \bar{\pi} \), with \( \bar{\pi} \in \mathbb{R}_{++} \), and that the matrix of subjective weights \( \mathbf{Z} \) is right-stochastic (Scaling II). Then WDU collapses to the canonical average-based (DeGroot) updating process.

The above remark demonstrates that both the canonical (DeGroot) and the dynamic average-based updating processes can be recovered as special cases of WDU. By setting, for example, \( \pi(0) = 1_n \), and normalising the weights so that each row of \( \mathbf{Z} \) sums up to 1, the model reduces to the canonical DeGroot model, while assuming that the agents’ direct influences on their peers depend solely on their accumulated expertise gives the dynamic average-based updating model introduced in Chapter 1 of the present thesis.

Notice that one of the conditions required for Proposition 2.1 to hold is aperiodicity of weighted network \( \mathcal{G}^2 \). This is necessary for primitivity of the matrix of direct influences \( \mathbf{\Gamma}(t) \), which in turn guarantees the convergence of the weighted dynamic updating process. This is a condition required for both the canonical (DeGroot), and the dynamic (Chapter 1) average-based updating processes to converge. In the latter though we get it for free (\( g_{ii} = 1 \) for some \( i \in \mathcal{N} \) guarantees aperiodicity of network \( \mathcal{G} \)), and therefore it is not mentioned explicitly in the statement of Theorem 1.1.

### 2.2.5 Scaling the matrix of subjective weights \( \mathbf{Z} \)

The process introduced in this paper is abstract and quite generic, and technically it can admit any non-negative, non-zero matrix of subjective weights. Imposing, nevertheless, a particular structure, and scaling matrix \( \mathbf{Z} \) accordingly, can help us make the process more intuitive, and more suitable for studying the evolution of beliefs. The following are just two of the various
ways in which matrix $Z$ can be scaled towards this end:

**Scaling I:** Subjective self-weight of the agents is normalised to 1 ($z_{ii} = 1$ for all $i \in \mathcal{N}$).

**Scaling II:** The matrix of subjective weights $Z$ is row-stochastic ($\sum_{j=1}^{n} z_{ij} = 1$ for all $i \in \mathcal{N}$).

*Scaling I* is perhaps the most natural and intuitive normalisation of the matrix of subjective weights $Z$. First, it provides the modeler with a consistent scale to quantify the weights individuals assign to their out-neighbours. If a different scaling is used for some agents, they may artificially appear to be more or less influential than they truly are. This would be the result of aggregating beliefs originating from individuals whose precision was reported on different scales.

Second, subjective weights $z_{ij}$ under this scaling can be interpreted as the exogenous amount of trust that agents place on their out-neighbours, relative to themselves. Assume, for example, that this trust depends on the political ideology of the each agent $i \in \mathcal{N}$, expressed as a vector $v_i \in [0, 1]^m$. Then agents may choose to pay less attention to the opinions of individuals who are ideologically further from their own position, so that

$$z_{ij} := 1 - \frac{1}{m} \| v_i - v_j \|_k$$

for every $(i, j) \in \mathcal{N}^2$, and some $k \in \mathbb{N} \cup \{+\infty\}$. Although in this example it will hold that $z_{ij} \in [0, 1]$, in other setups it may well be that $z_{ij} > 1$ for some $j \in \mathcal{N}$. In many cases, for instance, individuals may assign a higher weight to the opinions of other agents (family members, political or religious leaders, or popular culture icons) compared to the weights they assign to their own opinions, even if these other agents are not considered experts on the topic in question.

Adopting either of the aforementioned scalings, imposes some fundamental structure on the updating process, that can be summarised in the following
properties:

- In a society in which there is only one type of agent (i.e. all agents share the same ideology and the same objectives),\(^7\) and agents differ only up to their expertise, they should place more trust on the opinions of agents with relatively higher expertise.

- In a society in which everybody is equally expert, each agent should assign a higher weight to the opinions of the peers that are ideologically closer or whose beliefs are more relevant to him or her.

In general, though, agents will differ both in type and in expertise. Hence, the relative weight that each agent will assign to opinion of each of his or her peers will depend on the interaction of these two factors, as it is shown in the next section. An important assumption in this model is that although agents may learn and become more experts in a topic, their type is immutable throughout the communication process.

### 2.2.6 Some comments on WDU

Under the dynamic average-based updating process, the sole determinant of the direct influence of an agent \(i\) on another agent, say \(j\), is the relative expertise of agent \(i\), that is, agent \(i\)'s (perceived) information quality. Individuals have been assumed to be (boundedly) rational, and update their beliefs in a fairly sophisticated way. As discussed in Chapter 1, there are two important underlying assumptions in order for this process to be meaningful.

First, accumulated expertise, that is, (perceived) information quality, is the sole determinant of weights (“direct influences”). Personal feelings such as sympathy or antipathy towards another person, ideological differences,

\(^7\) This may be reminiscent of the renowned Jury Theorem due to Condorcet (1785, Third Section). Note, however, that if agents are allowed to exchange opinions, and the communication pattern is governed by an incomplete network, this theorem may fail: even if agents are more likely to be right than wrong, increasing the size of the network does not ensure convergence to the right decision with almost absolute certainty, as in the case of the classic theorem.
prejudice, or any assessment of a person’s trustworthiness stemming from factors other than the possession of information or expertise are not relevant for the learning process. Second, and perhaps most importantly, there exists a unique, common state of the world that agents wish to estimate.

In many cases, the above assumptions do not constitute a big abstraction from reality. In these cases, the dynamic average-based updating process can be suitable for studying social learning and the evolution of beliefs. Consumers interested in gathering information about some features of a product or a service, and investors who would like to update their expectations about some fundamentals or figures that have not become public yet, are some examples discussed in the previous chapter. In other cases, however, one or both of the above assumptions may fail. Political dialogue and the formation of public opinion is a very characteristic such example. As discussed above, empirical evidence are in line with the perception that individuals pay limited attention to, or even disregard, ideologically opposite opinions, even if they have access to them. This suggests that factors other than expertise, or even perceived expertise, are involved in the belief formation process.

Furthermore, quite often, even under complete information, it may not be clear which decision is the best one or even if a universally “correct” decision exists at all. In fact, in many economic and political issues, disagreement is not limited to the normative aspects of a policy, but extends even to its positive ones (the debate on the relationship between the minimum wage and unemployment is a prominent example). Yet, even if a statement (for example, about the effectiveness of a particular policy) can be evaluated in a well-identified, generally acceptable way, disagreement about what its objectives may still be there. In short, the true value of the parameter that agents seek to estimate may differ across agents.

In the dynamic average-based updating process, however, unlike the canonical DeGroot model, agents aggregate information in a semi-Bayesian way,\footnote{For certain prior distributions within the exponential family (e.g. normal, beta), and disre-}
which leaves no room for subjective weighting of information. The introduction of a weighted network addresses exactly this issue.

2.3 Time to consensus and short-run dynamics

2.3.1 The speed of convergence to consensus

As discussed in the previous section of this paper, society will, in the very long run, approach a consensus where the social influence of each agent is given by (2.9). Whether consensus beliefs constitutes a good approximation of the beliefs that agents hold in a specific period of time \( t \) depends from how far in the future the period of interest is, and the speed of convergence of the agents’ beliefs \( b(t) \) towards their consensus value, \( b(\infty) \). It is straightforward to see that, ceteris paribus, societies in which agents’ initial beliefs are closer to each other will, be closer to agreement at any point in time compared to societies in which the initial differences in beliefs are large.

Perhaps a more interesting question, in the light of the analysis in this paper, would be: How does the communication structure in a society, as captured by the weighted network, affects the time required for a consensus to be attained? In other words, given any initial belief profile \( b(0) \), what are the characteristics of the communication network that encourage or prevent agents from approaching towards a consensus sooner? Intuitively, societies in which communication links are dense (\( g_{ij} = 1 \) for many pairs of agents \((i,j)\)), and agents are not overly stubborn or self-centered (i.e. ratios \( z_{ij} / z_{ii} \) are not very low) should be able to reach a consensus faster than fragmented societies, where individuals largely ignore the opinions of oth-
ers. The analysis in this section formalises these concepts.

As shown by Anthonisse and Tijms (1977), if a sequence of stochastic matrices, \( \{\check{\Gamma}(t)\}_{t \in \mathbb{N}} \) converges to a primitive matrix, then its backward product, \( \hat{W}(t) \), converges elementwise to \( 1_n (\hat{w}^{(\infty)})^T \) at a geometric rate. That is, there exists some \( a \in (0, 1] \) such that

\[
\left| \hat{w}^{(t)}_{ij} - \hat{w}^{(\infty)}_j \right| = O(a^t)
\]

for all \( (i, j) \in \mathbb{N}^2 \). In the particular case of the weighted dynamic average-based updating process discussed in this paper, though, a stronger result can be established.

**Theorem 2.1: The Speed of Convergence Under WDU**

Assume that network \( G^z \) is strongly connected and aperiodic. Then the period-\( t \) influence of each agent \( j \in \mathcal{N} \) on any agent \( i \in \mathcal{N} \), \( \hat{w}^{(t)}_{ij} \), converges linearly over time \( t \) to his or her long-run social influence, \( \hat{w}^{(\infty)}_j \), at rate \( \rho_z^{-1} |\lambda_2| \). Formally,

\[
\lim_{t \to +\infty} \sup \left| \frac{\hat{w}^{(t+1)}_{ij} - \hat{w}^{(\infty)}_j}{\hat{w}^{(t)}_{ij} - \hat{w}^{(\infty)}_j} \right| = \frac{|\lambda_2|}{\rho_z}.
\]

where \( |\lambda_2| \) is the modulus of the subdominant eigenvalue(s) of matrix \( Z \), and \( \rho_z \) is its spectral radius.

The above result can be seen as a generalisation of a similar result by De-Marzo et al. (2003) for the canonical DeGroot model (see also Jackson, 2008, Chapter 8), since it covers the case of dynamically-updated weights as well. An intuition on why the speed of convergence depends on the eigenvalue with the largest modulus can be borrowed from the literature on Markov chains: according to a well-known finding, stochastic matrices with subdominant eigenvalues close to 1 are nearly uncoupled.\(^9\) Although

\(^9\) The converse holds true as well, but that may require values extremely close to 1 for this to
the network is strongly connected, allowing opinions originating from any agent to reach any other agent, this channel of communication may in practice be very thin. A network with two or more clusters of agents paying attention mainly within their own group, and very limited attention to outsiders, is such an example. In algebraic terms, such a network is represented by a matrix with one or more subdominant eigenvalues with modulus close to one.

In practice, this may result in disagreement persisting for very long or even almost indefinitely. This give raise to political polarisation, as it is the case with the retweet network in Conover et al. (2011). This can be exacerbated if the formation of the communication network _per se_ is based on closeness of beliefs. Using a model of an undirected network where agents follow the DeGroot updating process, Golub and Jackson (2012) analyse thoroughly how the presence of homophily can delay consensus, especially in large societies. In the context of the weighted dynamic updating process discussed in this paper, Theorem 2.1 has an additional interesting implication.

**Corollary 2.1.** Assume that network \( G^2 \) is strongly connected and aperiodic, and agents follow the weighted dynamic average-based updating process. Then the speed of convergence of the beliefs to their consensus value depends only upon the weighted communication structure imposed by \( G^2 \), and it is independent from the initial distribution of expertise over the agents, \( \pi(0) \).

This result may not be very intuitive: one could have argued that the learning process should converge faster in networks in which the more central or popular agents are better informed. Yet the difference of the present approach from the baseline DeGroot model is exactly that it allows agents to take into account not only the information transmitted by their peers, but also its precision. Hence, even if an expert agent is located in the periphery of the network, and is rather secluded, his or her opinion will reach the entire network after a given number of periods, together with its expertise be a sufficient condition for near-uncoupling (see Hartfiel and Meyer, 1998).
“tag”. In other words, high-quality information will not be underweighted because it is communicated indirectly. The conclusions, for example, of a study conducted by researchers at a top university will be still taken into account even if it is communicated through the press or through other, less prestigious researchers. As a result, any delays that occur in this model are caused by “bottlenecks” due to the network per se.

2.3.2 Social impact

In networks where convergence to the asymptotic vector of social influences is rapid, consensus beliefs can be a good approximation of the short-run beliefs of society in the short run. It may be the case, however, that convergence towards the consensus is slow, or that agents need to take some action based on beliefs that are still evolving. In such cases, the consensus values may not constitute a good proxy for the overall influence that each agent has had on the society over the course of time. In fact, some agents who are very influential during those critical, “transition” periods, can have a larger impact on the society compared to agents who end up being influential only in the distant future. A different measure may be therefore needed in order to capture this form of influence or impact.

**Definition 2.5: Social Impact**

Consider a real-valued, non-negative sequence \( V := \{v(t)\}_{t \in \mathbb{N}} \). Assume that \( V \) is bounded from above, and that \( v(t_k) > 0 \) for at least one \( t_k \in \mathbb{N} \). Then the impact of agent \( j \) on agent \( i \) under valuation stream \( V \), denoted by \( \xi_{ij}(V) \), is defined as the \((i,j)\)-th element of the right-stochastic matrix

\[
\Xi(V) := \lim_{T \to +\infty} \sum_{t=1}^{T} \overline{v}_T(t) W(t),
\]
where
\[
\tilde{v}_T(t) := \begin{cases} 
\frac{v(t)}{\sum_{\tau=1}^{T} v(\tau)} & \text{if } v(\tau) > 0 \text{ for some } \tau \in \{1,\ldots,T\} \\
0 & \text{otherwise.}
\end{cases}
\]

Moreover, given a stochastic vector \(s\),\(^{10}\) the \textit{social impact of agent } \(j\) \textit{under valuation stream } \(V\) \textit{and significance profile } \(s\), or \((V,s)\)-social impact in short, denoted by \(\xi^s_j(V)\), is defined as the \(j\)-th element of the stochastic vector
\[
\xi^s(V) := (\Xi(V))^T s.
\]

The social impact of agent \(j\), \(\xi^s_j(V)\), measures the aggregate impact that agent \(j\) has had on the society. Sequence \(V\) represents the valuation attached to each period \(t\). More specifically, \(\xi_{ij}(V)\), as defined above, captures the \(V\)-weighted average influence \(u_{ij}(t)\) of agent \(j\) on agent \(i\) throughout the belief formation process. The significance profile, \(s\), expresses the relative importance of the opinion of each agent. A natural choice would be to set \(s = \frac{1}{n}1_n\), assigning thus equal weight to all agents. In that case, the social impact of an agent \(j\) would simply be his or her average impact on the members of the society, that is, \(\xi^s_j(V) = \frac{1}{n} \sum_{i \in N^t} \xi_{ij}(V)\).

In several cases, however, influencing certain agents may be more crucial than influencing others. Consider, for example, the targeting of an election campaign. Swing voters — that is, voters who are either undecided or are thinking of abstaining, even if they have some preference — can play a crucial role in close races. Influencing such voter groups could be one of the main objectives of the campaign, which can be modelled by assigning them a higher significance \(s\).

At the same time, there may be groups who are ineligible to vote, such as non-citizens, minors, or felons on probation. It would be meaningful thus to

\(^{10}\)A vector \(y\) is called \textit{stochastic} or \textit{probability vector} if it is non-negative, and its elements sum up to 1, that is if \(y \in [0,1]^n\) and \(\sum_{i=1}^{n} y_i = 1\).
assign to these groups a very low, or potentially zero significance $s_i$. Hence, even if some agents have a high impact on such groups, their social impact may be low, since their influence will be heavily discounted when it comes to the electoral issue.

A question that may arise from the above discussion is whether it would be simpler to just ignore such agents, exclude them from the network under study when it comes to issues in which their significance $s_i$ is considered to be zero. Disregarding those agents, though, could lead to highly misleading conclusions. Notice even agents with zero significance can have a large social impact. Individuals who do not have the right to vote, for example, may well have family members or friends who do, or may be influencing society through the media.

Under certain assumptions though, social influence may be a meaningful measure, even in finite horizon.

**Remark 2.2.** Assume that agents follow the dynamic average-based updating process, and consider a valuation stream $V = \{v(t)\}_{t \in \mathbb{N}}$, with $v(t) = \bar{v} \in \mathbb{R}^{++}$ for all $t \in \mathbb{N}$. Then the impact of each agent $j$ on any other agent $i$ under valuation stream $V$ coincides with agent $j$’s social influence, that is,

$$\xi_{ij}(V) = \hat{w}_j^{(\infty)}$$

for all $i, j \in \mathcal{N}$.

The above remark shows that, despite being an asymptotic measure, social influence can be a good characterisation of agents’ importance in cases in which beliefs in all periods matter, or the future is not heavily discounted. Consider now the standard, fixed-weight DeGroot process. Assume that all periods have the same value $\bar{v}$, but future periods are discounted at a constant rate. Then the social impact measure defined above is closely related to a well-known measure of network centrality.

**Remark 2.3.** Assume that agents follow the canonical average-based (DeG-
root) updating process (i.e. \( Z \) is row-stochastic, and \( \pi(0) = \bar{\pi}_1 \)), and consider a valuation stream \( V = \{ \beta^{t-1} \}_{t \in \mathbb{N}} \) with \( \beta \in (0,1) \). Then the \((V, s)\)-social impact of agent \( j \) coincides with his or her \( s \)-weighted Katz-Bonacich centrality of parameter \( \beta \), that is, the \( j \)-th element of the vector
\[
\xi^s(V) = (I_n - \beta \mathbf{1})^{-1}s.
\]

2.4 Lobbies and public influence

All the results in this paper so far hinge on assumption that the network of interest is strongly connected. It is straightforward to extend the analysis above to disconnected networks, as long as every isolated component is strongly connected. It would be interesting though to study the belief formation process in weakly connected networks. Such a configuration may arise if some agents, or groups of agents, do not have any access, direct or indirect, to the beliefs some other agents in the network. There is, however another reason for which the communication structure in a society may be described better by a weakly connected network: some agents may deliberately choose to ignore some of their neighbours. These “persuaders”, as called by Downs (1957), are not interested in gathering information and resolving their uncertainty about some unknown \( a \) but rather in shifting public opinion towards their cause.

In either case, the flow of information in the network is disrupted, and the initial beliefs of some agents may never reach some parts of the society. The absence of strong connectedness can have important implications for the belief formation process, and the influence of the agents in the society. Indeed, as it will be seen, the conclusions reached in this section differ both quantitatively and qualitatively from the ones discussed in the preceding sections. In order however to study such networks, some additional concepts and tools are required.
2.4.1 Definitions and preliminaries

Consider a network $G = \langle N, E \rangle$, and a subset $A$ of $N$. The induced sub-network $G[A] = \langle A, E[A] \rangle$ (or the restriction of $G$ to $A$) is the network that consists of all the agents (vertices) in subset $A$, and those and only those links (edges) in set $E$ that begin from and end to agents in $A$. Links in $E$ that involve any agents outside $A$ are not members of $E[A]$. In matrix notation, $G[A]$ is represented by a submatrix $A$ of the adjacency matrix $G$ that is obtained by removing all rows and columns of $G$ that correspond to agents outside $A$.

Consider a weighted network $G = \langle N, E \rangle$. A subnetwork $G[A]$ induced by set $A \subseteq N$ will be called closed if the members of subset $A$ pay attention only to members of set $A$, that is, if $z_{ij} = 0$ for all $(i, j) \in A \times \{N \setminus A\}$.

A special type of closed subnetworks are of particular interest when studying political influence and the determination of the opinions of the society.

**Definition 2.6: Lobbies and the Public**

Consider a weighted network $G = \langle N, E \rangle$. Subnetwork $G[L_k]$ induced by set $L_k \subseteq N$ is said to constitute a lobby if the following conditions are satisfied:

[i] $G[L_k]$ is closed (i.e. $z_{ij} = 0$ for all $(i, j) \in L_k \times \{N \setminus L_k\}$)

[ii] $G[L_k]$ is strongly connected

[iii] $G[L_k]$ is aperiodic (e.g. $z_{ii} > 0$ for at least some agent $i \in L_k$).

Denote the set of all agents who do not belong in a lobby with $P$. Subnetwork $G^P$, induced by set $P$, shall be referred to as the public.

Lobby $L_k$ can be represented by a submatrix of $Z$, to be denoted by $L_k$. Assume that there are $\ell \in \mathbb{N}$ lobbies in the society. Denote the family of all lobbies in $N$ with $\mathcal{L} := \{L_1, L_2, \ldots, L_\ell\}$, and the corresponding principal
It follows from Definition 2.6 that $L$ is a block diagonal matrix of the form

$$L = \begin{bmatrix} L_1 & & \\ & L_2 & \\ & & \ddots \\ & & & L_{\ell} \end{bmatrix}.$$  

Notice that the case of a strongly connected, aperiodic weighted network discussed above can be seen as a network consisting of one big lobby that includes all agents in the society. The analysis in the present paper focuses on networks that contain at least one lobby.\(12\)

Since network $G^Z$ is weakly connected, by Lemma A.1 its adjacency matrix $Z$ can be written as a block triangular matrix. More specifically, by appropriately labelling the players, it can be brought into its canonical form\(13\)

$$Z = \begin{bmatrix} P & P_L \\ O & L \end{bmatrix},$$

where $P$ and $L$ are square matrices, and $O$ is a matrix of zeros.

Submatrix $P$ represents the attention that the members of the public give to fellow members of the public, and submatrix $P_L$ expresses the attention that the members of the public give to lobbyists. The latter can be written as a block matrix of the form $P_L = [P_{L_1} \ldots P_{L_{\ell}}]$, where block $P_{L_k} := [z_{ij}]_{(i,j) \in P \times L_k}$

\(11\) Formally, a lobby has been defined as subnetwork, so $L_k$ should be referred to as “the set of agents in lobby $G[L_k]$” or “the set of vertices in subnetwork $G[L_k]$”. In order to economise on the amount of terminology employed in this paper, and since there is no danger of confusion, the term “lobby” will be (ab)used to refer both to the subnetwork, $G[L_k]$, and to the set of agents $L_k$ comprising it. Similarly, both set $P$, and the induced subnetwork, $G[P]$, will be referred to as “the public”.

\(12\) Although this may appear to be a rather restrictive assumption, the majority of the remaining possible network configurations can be studied as special cases of the two configurations studied in this paper (strongly connected, aperiodic network, and weakly connected network with at least one lobby; see Appendix). The only case that is not subsumed in the present analysis is the one in which condition [iii] in Definition 2.6 fails. This would imply though that every period, agents in the lobbies keep “forgetting” or discarding all the information that they have accumulated in the past.

\(13\) For a formal definition of the canonical form of a reducible matrix, see Definition B.3 in the Mathematical Appendix.
captures the weight that every agent $i \in \mathcal{P}$ assigns to every agent $j \in \mathcal{L}_k$.

2.4.2 Characteristics of the lobbies

Agents in a lobby may have different initial beliefs and expertise. How efficient the exchange of opinions among the members of the lobby and the learning process are depends heavily on the structure of the communication network. Intuitively, agents who have better access to the opinions of their fellow lobbyists, and are relatively more receptive to different views, will learn faster, and their opinion will be more representative of the opinions in the lobby than that of other, more self-absorbed agents. It may be therefore useful to define a measure that quantifies this concept of having access to other lobby member, and/or being receptive to their ideas.

**Definition 2.7: Relative Openness**

In a strongly connected, aperiodic, and potentially weighted network $\mathcal{G}^z = \langle \mathcal{N}, \mathcal{E}^z \rangle$, the **relative openness** or receptiveness of each agent $i$ is captured by the $i$-th element of the Perron vector (i.e. the positive right eigenvector) of the weighted adjacency matrix $Z$, that is, a vector $\mathbf{r} := [r_i]_{i \in \mathcal{N}} > \mathbf{0}_n$ satisfying

$$Z \mathbf{r} = \rho_z \mathbf{r}$$

normalised so that

$$\|\mathbf{r}\|_1 := \sum_{i=1}^{n} |r_i| = 1,$$

where $\rho_z$ is the spectral radius of weighted adjacency matrix $Z$, and $\| \cdot \|_1$ denotes the vector 1-norm. Vector $\mathbf{r}$ is called the **openness profile of lobby** $\mathcal{L}_k$.

This measure can be interpreted as some form of “outwards” centrality. As discussed in Section 2.2, eigenvector centrality quantifies the concept of how
important an agent is due to receiving attention or being listened to by other important agents. Relative openness expresses the concepts that an agent is an important as a source of information because he or she has access to other agents who could potentially be important sources of second-hand information. Four remarks are in order at this point.

First, a similarly to eigenvector centrality, relative openness is a not an absolute measure, and hence the qualifier “relative”. It can be thus used for comparing the receptiveness or the access to information an agent relative to the other agents in the same lobby, but not for comparisons across lobbies. Two agents in different lobbies may have the same relative openness, but due to different link densities, one may have access to more second-hand information that the other.

Second, relative openness is a network feature, hence it does not imply that agents with high relative openness actually have access to more precise or “better” information; but rather that these agents have access to the beliefs of agents who observe a large number or some important agents. Whether these agents are actually experts or not is orthogonal to that. Yet, if all agents had equal initial expertise, agents with higher openness would have access to more precise information.

Third, exactly because relative openness is entirely a network-specific feature, its relationship between eigenvector centrality is not ex ante pre-defined. Central or popular agents may be open and reciprocate attention, or may self-absorbed and pay attention only to a few selected peers, or anything in between. A measure capturing this correlation is the centrality–openness dispersion parameter of lobby $\mathcal{L}_k$, defined as

$$
\phi_{\mathcal{L}_k} := \frac{1}{\hat{c}_{\mathcal{L}_k} r_{\mathcal{L}_k}} = \sum_{h \in \mathcal{L}_k} \hat{c}_{\mathcal{L}_k,h} r_{\mathcal{L}_k,h}
$$

where $\hat{c}_{\mathcal{L}_k}$ and $r_{\mathcal{L}_k}$ are the eigenvector centrality and relative openness profiles of lobby $\mathcal{L}_k$. Similarly to $\alpha_{\mathcal{L}_k}$, a smaller value of $\phi_{\mathcal{L}_k}$ implies a higher concentration of openness towards the central agents, while a higher value
suggests that less central agents are on average more open.

Fourth, notice that relative openness has been defined only for strongly connected networks (or subnetworks, such as lobbies). Unfortunately, attempting to extend this measure to weakly connected networks fails. Conceptually, since lobbyists do not have access to the information of the public or other lobbies, assigning some relative openness to them can be ambiguous, especially if there are more than one lobbies in the society. This in turn renders relative openness of the members of the public problematic as well. Recall that an agent’s openness depends on the openness of his or her out-neighbours, which in this case may be lobby members, without well-defined openness themselves.\footnote{Technically, the Perron-Frobenius Theorem does not apply to reducible matrices. Hence there may be several different “relative openness” profiles for such networks, some of which may contain non-positive elements.}

It may be therefore unclear how to measure the members’ of the public access to information or “openness”. In order to address this issue, a different measure needs to be introduced.

**Definition 2.8: Exposure**

Consider a weighted network $G^Z = \langle N, E^Z \rangle$. The exposure of agent $i \in P$ to lobby $L_k$ is given by the $i$-th element of the non-negative vector $\varepsilon_{L_k} = [\varepsilon_{L_k,i}]_{i \in P}$ defined as

$$\varepsilon_{L_k} := \left( I_{n(P)} - \beta P P \right)^{-1} P \rho_{L_k} r_{L_k}$$

(2.10)

where $\beta \in (0, \rho_{P}^{-1})$ is a scaling factor that captures the importance of network effects, and $r_{L_k}$ is the relative openness profile of lobby $L_k$.

The above measure is a variant of Katz-Bonacich centrality, and captures the direct and indirect exposure of the members of the public to the opinions of a given lobby. Equivalently, it can be interpreted as the overall access
that the members of the public have to the beliefs of that lobby. Studying formula (2.10) more thoroughly reveals that this “exposure” or “access” is established via three channels. First, members of the public may directly pay attention to some lobbyists, and this is captured by matrix $P_{\mathcal{L}_k}$.

Second, they may have access to beliefs of some lobbyists indirectly, by observing some other member of that lobby. The more open to opinions the observed lobbyists are (as captured by relative openness $r_{\mathcal{L}_k}$), the higher is the exposure to their lobby they provide to the members of the public observing them. This is why the attention given to each lobbyist, as captured by adjacency submatrix $P_{\mathcal{L}_k}$, is weighted by the relative openness of that lobbyist, expressed by vector $r_{\mathcal{L}_k}$. Intuitively, the assessment of an agent in a lobby who has good contact with his or her fellow lobbyists should be more informative compared to the opinion of a more self-absorbed agent in the same lobby.

Third, members of the public may be exposed to the beliefs of the agents of a lobby even if they do not have any direct access to any member of that lobby; it suffices that they have an out-neighbour who has access to that lobby, or that some out-neighbour has an out-neighbour who does, and so on. Second-order access is discounted by a factor of $\rho_Z^{-1}$, third-order access is discounted by $\rho_Z^{-2}$, and $n$-th order access by $\rho_Z^n$. This is the idea behind the Katz-Bonacich centrality measure, and mathematically is represented by term $\left( I_{n(p)} - \beta PP \right)^{-1}$ in formula (2.10).  

\footnote{To see this, notice that this term constitutes a Neumann series:}

$$
\left( I_{n(p)} - \beta PP \right)^{-1} = \sum_{i=0}^{\infty} \beta^n PP.
$$

with $\beta = \rho_Z$. In linear algebra literature, $\left( I_{n(p)} - \frac{1}{\rho_Z} PP \right)^{-1}$ is often referred to as the fundamental matrix.
2.4.3 Asymptotic influence in weakly connected networks

With all the necessary concepts in place, the main result of this section can be now introduced.

**Theorem 2.2: Lobbies and Asymptotic Influence**

Consider a weakly connected, weighted network $\mathcal{G}^Z = (\mathcal{N}, \mathcal{E}^Z)$, with $\mathcal{N} = \{P, \{L_1, \ldots, L_\ell\}\}$. Assume that the agents follow the weighted dynamic updating process described in Definition 2.4. Then for each agent $i \in \mathcal{N}$ there exists a unique asymptotic belief $b_i^{(\infty)} \in \mathcal{B}$ such that

$$b_i^{(\infty)} := \lim_{t \to +\infty} b_i(t).$$

Each lobby $L_k$, for $k \in \{1, 2, \ldots, \ell\}$, will reach a within-lobby consensus. The asymptotic belief of each agent $i \in L_k$ will be given by

$$b_{L_k}^{(\infty)} := \lim_{t \to +\infty} b_i(t) = \sum_{j \in L_k} \tilde{w}[L_k]_{j}^{(\infty)} b_j(0),$$

where

$$\tilde{w}[L_k]_{j}^{(\infty)} = \alpha_{L_k} \tilde{c}_{L_k,j} \tilde{\pi}_{L_k,j}(0),$$

is the within-lobby influence of agent $j \in L_k$, with $\sum_{j \in L_k} \tilde{w}[L_k]_{j}^{(\infty)} = 1$, and $\tilde{c}_{L_k,j}$ is agent $j$’s within-lobby popularity, that is, agent $j$’s eigenvector centrality in subnetwork $\mathcal{G}^Z[L_k]$.

$$\tilde{\pi}_{L_k,j}(0) := \frac{\pi_j(0)}{\sum_{h \in L_k} \pi_h(0)}$$

is agent $j$’s relative initial expertise (precision) within lobby $L_k$, and

$$\alpha_{L_k} := \frac{\sum_{h \in L_k} \pi_h(0)}{\sum_{h \in L_k} \tilde{c}_{L_k,h} \pi_h(0)}$$

is the within-lobby centrality–expertise dispersion parameter, a scalar that is common for all agents in lobby $L_k$, that captures the network effects on the communication pattern in $L_k$. 

The public will not reach a consensus. The asymptotic belief $b_i^{(\infty)}$ of each agent $i \in \mathcal{P}$ will be given by

$$b_i^{(\infty)} := \lim_{t \to +\infty} b_i(t) = \sum_{k=1}^{\ell} \sum_{j \in \mathcal{L}_k} \hat{w}_{ij}^{(\infty)} b_j(0),$$

(2.12)

where

$$\hat{w}_{ij}^{(\infty)} = \frac{\phi_{\mathcal{L}_k}^{\alpha_{\mathcal{L}_k}} \pi_{\mathcal{L}_k}^{0} \varepsilon_{\mathcal{L}_k,i}}{\sum_{q=1}^{\ell} \phi_{\mathcal{L}_q}^{\alpha_{\mathcal{L}_q}} \pi_{\mathcal{L}_q}^{0} \varepsilon_{\mathcal{L}_q,i}}$$

(2.13)

is the asymptotic influence of agent $j \in \mathcal{L}_k$ on agent $i \in \mathcal{P}$, $\pi_{\mathcal{L}_k}^{0} := \sum_{h \in \mathcal{L}_k} \pi_{h}^{0}(0)$ is the initial expertise of lobby $\mathcal{L}_k$, and $\phi_{\mathcal{L}_k} := \frac{1}{\xi_{\mathcal{L}_k} r_{\mathcal{L}_k}}$ is the centrality–openness dispersion parameter in lobby $\mathcal{L}_k$.

The above theorem presents two main results. First, it asserts that beliefs under WDU converge, even in weakly connected networks. Second, it provides an explicit formula for the agents' limiting belief profile. It can thus be seen as a generalisation of Theorem 2.1. Its implications, however, about the opinions that agents settle down to, and well as their influence on the society are in general different.

Firstly, recall that in the case of a strongly connected, aperiodic network, WDU leads to a consensus, and the social influence of each agent represents the contribution of the initial belief of that agent to the consensus beliefs. If the network though is not strongly connected, a given agent's initial beliefs no longer have the same influence on all other agents; hence the non-transitive social influence of agent $j$ has been replaced by the transitive term asymptotic influence of agent $j \in \mathcal{N}$ on agent $i$.

The above observation leads to the second remark: since agents are influenced asymmetrically by their peers, no consensus will be reached, and in
general they will end up having different limiting beliefs. The term consensus beliefs is no longer meaningful in this more general setup, and hence the term asymptotic or limiting belief profile is used instead.

Third, recall that according to Definition 2.6, agents in a weakly connected network such as the one described above can be sorted into two mutually exclusive and collectively exhaustive categories: members of the public, or members of a lobby (“lobbyists”). Their affiliation has fundamentally different implications concerning their asymptotic beliefs, and asymptotic influence on their peers.

Study of the lobbies leads to a rather familiar result: the members of each lobby will eventually converge towards a consensus, and maintain this belief indefinitely. Each agent’s asymptotic influence will be uniform across all of his or her peers within the lobby, and hence the term within-lobby consensus. To better understand why this is the case, observe that each lobby is insulated from the influence of any outsider, and constitutes by definition a strongly connected, aperiodic network. Hence it can be thought of as a small society itself, and the results from the previous sections, such as Proposition 2.1 carry over directly. Since, moreover, there is no communication across different lobbies, each agent can influence only agents within his or her lobby.

It is perhaps not as straightforward to see that, asymptotically, the members of the public will be unable to influence not just lobbyists, but even other members of the public. As implied by formula (2.13), the asymptotic influence of a member of the public on any other agent will be zero.16 Although this may not very intuitive, it is a consequence of the constant exposure of the members of the public to the opinions of the lobbyists. Figuratively, it is like a runaway greenhouse effect: all members of the public every period are, directly of indirectly, subject to the influence of some lobbyists.

---

16 Observe that no particular assumptions have been made about the structure of subnetwork $P$: it can be strongly connected, weakly connected, or even consist of several disconnected subnetworks.
They do not, however, receive back any attention from lobby members. In each round of communication, a positive amount of information originating from lobby members makes it to the public, who incorporate it into their previous-period beliefs. As a result, lobbies’ beliefs will gradually take over, and eventually they will completely crowd out the initial beliefs of the public. Table 2.1 summarises how agents influence each other based on their affiliation.

Table 2.1: The asymptotic influence of agent \( j \) on agent \( i \) \((k \neq q)\)

<table>
<thead>
<tr>
<th>( i \in \mathcal{P} )</th>
<th>( \mathcal{L}_k )</th>
<th>( \mathcal{L}_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{P} )</td>
<td>0</td>
<td>( \hat{w}_{ij}^{(\infty)} )</td>
</tr>
<tr>
<td>( \mathcal{L}_k )</td>
<td>0</td>
<td>( \hat{w}[\mathcal{L}_k_j]^{(\infty)} )</td>
</tr>
<tr>
<td>( \mathcal{L}_q )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Although, in general, there will be disagreement among the individuals in a weakly connected society, under some particular communication structures a consensus will emerge. The statement that follows addresses this case.

**Corollary 2.2.** If there exists a single lobby in the network, so that \( \mathcal{N} = \{\mathcal{P}, \mathcal{L}_1\} \), a consensus will be reached in the society, with the beliefs of every member of the public converging towards the consensus belief \( b^{(\infty)} \) obtained among the members of the lobby:

\[
\lim_{t \to +\infty} b_i(t) = b^{(\infty)} = \sum_{j \in \mathcal{L}_1} \hat{w}[\mathcal{L}_1_j]^{(\infty)} b_j(0)
\]

for all agents \( i \in \mathcal{N} \).

Note that this holds true even if there is no public, \( \mathcal{P} = \emptyset \), and thus the network consists only of one lobby, \( \mathcal{N} = \mathcal{L}_1 \). This leads in fact to the case of a strongly connected, aperiodic network discussed in the previous sections. If there are agents outside the lobby, their asymptotic beliefs will be
determined entirely by the consensus beliefs of the lobby.

2.4.4 Influence of the lobbies

The analysis in this section has provided us with some insight on the belief updating process and the characteristics that make agents influential in a weakly connected network. It would be arguably interesting though to study how the presence of a non-trivial number of lobbies, with potentially different structure and beliefs, affects the formation of public opinion. As discussed above, expressions (2.12) and (2.13) stipulate the asymptotic influence of agent $j \in \mathcal{L}_k$ on agent $i \in \mathcal{P}$, that is, the extent to which the initial beliefs of agent $j$ have shaped the asymptotic beliefs of agent $i$. It would more interesting, and perhaps more meaningful, though to study the contribution of each lobby, rather than that of each agent, in shaping the asymptotic beliefs of the public. In other words, once the members of a lobby have reached a consensus and settled down to some belief, to what degree are they able to pass on this belief to the public? A more careful examination of Theorem 2.2 can help us provide an answer to this question.

**Corollary 2.3.** The asymptotic belief of each member of the public $i \in \mathcal{P}$ is given by

$$b_i^{(\infty)} = \sum_{k=1}^{\ell} \hat{w}_{i,L_k}^{(\infty)} b_j^{(\infty)}$$

where

$$\hat{w}_{i,L_k}^{(\infty)} = \frac{\phi_{L_k}}{\alpha_{L_k}} \pi_{L_k} \mathcal{E}_{L_k,i}$$

is the influence of lobby $L_k$ on agent $i \in \mathcal{P}$.

The above corollary establishes that the asymptotic beliefs of the agents in the public depend solely on the relative influence of each lobby on them. More specifically, they are formed as a weighted average of the within-lobby
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consensus beliefs of different lobbies. A question therefore that arises naturally at this point is: how is the relative influence of each lobby determined? What are the characteristics of the lobbies that are predicted to be more influential? As discussed in Section 2.2.6, in the case of a strongly connected network, a consensus is reached in the society. The social influence of an agent then can be measured naturally as the contribution of that agent’s initial belief to the consensus belief. In the case of a weakly connected network studied in this section, the members of the public will end up with different beliefs asymptotically. As a result, the measure of social influence is no longer well-defined. In order to be able proceeding with our analysis, an alternative measure of the influence of a lobby on the society needs to be formally introduced.

**Definition 2.9: Public Influence of a Lobby**

Consider a weakly connected, weighted network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, with $\mathcal{N} = \{P, \mathcal{L}_1, \cdots, \mathcal{L}_\ell\}$, and let $s := [s_i]_{i \in P}$ be a stochastic vector. The public influence of $\mathcal{L}_k$ under significance profile $s$, or simply the $s$-public influence of lobby $\mathcal{L}_k$, denoted by $\tilde{w}_{P,\mathcal{L}_k}(s)$, is the weighted average asymptotic influence of lobby $\mathcal{L}_k$ on the members of the public, given by

$$
\tilde{w}_{P,\mathcal{L}_k}(s) := \sum_{i \in P} s_i \tilde{w}_{i,\mathcal{L}_k}^{(\infty)}.
$$

The public influence of a lobby is simply a weighted average of the influence of that lobby on the members of the public. The exogenous weights $s$ capture the relative significance that each member of the public has in the context of each case study.\(^{17}\)

Now the factors that determine the relative influence of each lobby can be formally studied. The following proposition provides some insight into some

\(^{17}\text{Recall the discussion following Definition 2.5 in Section 2.3.1.}\)
of them.

Proposition 2.2: Determinants of the Public Influence of a Lobby

- An increase in the attention that a lobby receives from any member of the public increases, ceteris paribus, the public influence of that lobby. Controlling for other characteristics of the agents, the gains are larger for the lobby if the additional attention is directed towards lobbyists with high relative openness $r_{L_k,I}$.

- An increase in the aggregate initial expertise of a lobby, $\pi_{L_k}$, induces an increase in the public influence of that lobby, ceteris paribus.

- The public influence of a lobby increases, ceteris paribus, as the relative initial expertise of its more central members increases.

- Among otherwise identical lobbies, more influential are those in which less central agents have higher openness.

- Larger lobbies are more influential than smaller ones, ceteris paribus. Yet an increase in the size of a lobby has an ambiguous effect on its public influence.

The first statement in the above proposition is quite straightforward and intuitive. An increase in the attention that the public pays to a lobby, say $L_k$, can be achieved in two ways: members of the public already observing some agents in lobby $L_k$ can increase the weights that they assign to them, and members of the public that were ignoring that lobby can start paying attention to some of its members. Either change would increase the lobby’s public influence by increasing the row sums of submatrix $P_{L_1}$.

The second statement is equally unsurprising. Since expertise is one of the parameters that agents take into account when assigning weights to their neighbours, higher expertise should translate into larger influence. Observe that public influence is a function of the aggregate expertise of each lobby.
At consensus there is a common belief shared by all lobby members, and this has emerged as a result of the interaction of the beliefs and the expertise of all the agents in the lobby. Hence, all other things equal, it is the aggregate amount of information available in the first place that matters.

The third statement suggests that lobbies in which experts receive a high amount of attention from other lobby members, tend to be more influential in the society, other things being equal of course. To understand the reason behind this, observe that networks where expertise is spread out towards more peripheral agents are more likely to reach a consensus that is far from the many agents’ initial beliefs. In a lobby $\mathcal{L}_k$, this correlation is captured by a relatively high centrality–expertise dispersion parameter $\alpha_{\mathcal{L}_k}$. Intuitively, central agents in such lobbies will not be able fully translate their advantageous network position into influence, since they lack the expertise required to vest their opinion with a high weight. Expert agents, on the other hand, face the opposite problem: their opinion carries a high weight, but they are not central enough counter the persuasion bias due to the central agents, and make their views heard.

On the contrary, influential agents in lobbies in which centrality and expertise are in line (relatively small $\alpha_{\mathcal{L}_k}$), tend to concentrate both these influence-inducing characteristics in their person. They are thus less prone to be influenced by others, and more likely to avoid large shifts in their beliefs. Although less influential agents in such lobbies may experience more dramatic changes in their beliefs, this will have a rather limited impact, exactly because these agents are less influential. As a result, they will adopt the prevalent opinions soon, and all the agents in that lobby will be communicating a (more or less) common belief to the public.

Although the rationale behind the fourth statement is a bit different, the

\[18\]

Moreover, as the reader may have noticed, this statement is weaker than the one above: it does not assert that increasing the openness of the less central agents makes a lobby more influential. The reason is that in order for an agent to become more open, he or she needs to establish more links; links that affect the centrality not only of the agents that they link to, but also of their neighbours, and the neighbours of their neighbours, and so on. Hence,
main idea is the same as above: lobbies whose members communicate for long enough to the public beliefs that are different from those in which they finally settle down to, tend to be less influential than lobbies that persist on expressing consistent beliefs throughout the communication process. As suggested by expression (2.11), less central agents (i.e., those with relatively small $c_i$) tend to have smaller within-lobby influence. Consequently, the within-lobby consensus belief is expected to be further away from their beliefs, and closer to those of the central agents, *ceteris paribus*. This is independent of their relative openness. In lobbies, however, with a small centrality–openness parameter $\phi_L$, less central lobbyists tend to be less receptive and more self-absorbed as well (i.e., they have a relatively small $r_i$). As a result, not only will they remain close to their initial opinion for a long time before converging towards the consensus, but moreover they will be communicating those beliefs to the public during that time. Hence, during that period, they may be in fact shifting the public away from the eventual consensus belief of their lobby, instead of towards it. This is especially true if their initial beliefs happen to be very different from those of the central agents.

The two statements discussed above lead to an interesting observation.

**Corollary 2.4.** In order for a lobby to be publicly influential, the agents with high within-lobby influence should be the experts. Yet they may not need to be the more suitable ones to communicate the lobby’s views to the public. This role very likely should be undertaken by some agent who has a more spherical knowledge of the different views and opinions initially present in the lobby, at least those of the influential agents.

Let us now discuss briefly the idea behind the fifth statement. If two lobbies are in all aspects identical, but one is larger in size, it will be also more influential, since it will be receiving more attention from the public. Interestingly
enough, however, this does not imply that lobbies can increase their public influence by simply expanding. An enlargement could of course increase the lobby influence, but it could reduce it as well. One reason for this ambiguity is that the effect depends highly on the characteristics of the new agent, such as his or her expertise, and how this relates to the characteristics of the other agents and his or her network position. On the one hand, increasing the size of a lobby increases the attention that it receives from the public, as well as its initial expertise, effects that tend to increase its influence. On the other hand, especially in smaller lobbies, the addition of new members may increase the centrality–expertise dispersion parameter, reducing thus the influence of the lobby. Moreover its effect through the centrality–openness parameter is in the first place unclear.

2.5 Conclusions

The present paper studies communication and the transmission of information and opinions in a network setting. In particular, it focuses on how factors such as ideology, expertise, and popularity interact, and contribute to the formation of individuals' beliefs. Its main contribution is that it provides a comprehensive theoretical framework that enables us to study the formation and evolution of (potentially subjective) beliefs in a society. In order to achieve this, it introduces a generalised version of the DeGroot model (1974), one of the most widely-used frameworks for boundedly rational learning in networks. The proposed model of weighted dynamic updating possesses two features that make it more fit for studying the shaping of the public opinion.

First, in many areas, and especially in the political dialogue, the amount of trust (weights) that individuals place on the opinions of their peers depends on both “subjective” criteria, such as ideology or values, and on “objective” criteria, such as expertise in or knowledge about the topic in question. Weights in the model proposed in this paper take into account both criteria
to a degree that may vary across agents. This implies that unlike the baseline DeGroot model, weights will be, in general, dynamic: although values may be considered more rigid, the distribution of expertise in the network changes with communication and the exchange of information. As a result, the “expertise” part, and hence the weights themselves, will not be constant.

Second, the analysis in this paper accounts for the role of closed groups agents who do not communicate with agents outside their group, either because they do not have access to them or simply because they choose to ignore them. This allows us to study belief formation in the presence of “lobbies”, that is, parties, interest groups, political zealots, or any group interested more in promoting some opinions in the society, and less in learning or adapting their views. Some of the characteristics that make such groups influential are pinpointed and analysed.

In particular, it is shown that lobbies will be more influential if they manage to attract more attention from the public, and appear to be more sophisticated by having expert and well-informed individuals in the lobby. The former effect is larger though if the attention of the public is directed not towards the most influential members in the lobby, but rather those individuals who have access to the influential members, and are well-informed of their beliefs. The magnitude of the latter effect, on the other hand, is larger if expertise and centrality coincide, that is, if the experts of the lobby are those who receive the most attention within the lobby, directly or indirectly. Theory hence suggests that lobbies benefit from having experts in their ranks, especially if it appears that their contribution in shaping the lobby’s belief is vital. The ideal spokesperson or public figure of the lobby is not necessarily one of them, but rather a member who fits the first description above. Another interesting finding is that enlargement can be harmful for a lobby: if the profile of the new members is incompatible with that of the existing members, their addition could alter unfavourably the balances described above.

Another issue studied in the present paper is time to consensus. It is found
that the results of the existing literature on the canonical, fixed-weight De-Groot model DeMarzo et al. (2003); Jackson (2008); Golub and Jackson (2012) carry over to the more general case of weighted dynamic updating. Namely, it is shown that the rate of convergence of the dynamic process is geometric, and depends on the modulus of the subdominant (“second-largest”) eigenvalue of the weighted adjacency matrix. In order keep track of the influence the agent in finite time, a new measure of social impact is proposed. Under some commonly used parameter values, this measure coincides with well-known measures of network prestige, such as the Katz–Bonacich centrality and DeGroot social influence.

One vital assumption underlying our analysis is that agents are truthful when reporting their beliefs and expertise, or equivalently that their neighbours have complete access to this information. This assumption is not an exclusive feature of the present model, but it is commonplace in social learning models with self-reporting or observable actions, either Bayesian (e.g. Gale and Kariv, 2003; Acemoğlu et al., 2010; Mueller-Frank, 2013; Mossel et al., 2015) or naïve (e.g. DeMarzo et al., 2003; Golub and Jackson, 2010, 2012). It would be thus interesting to explore the implications of relaxing this assumption. In the first instance, agents would have an incentive to act strategically and misreport their beliefs or the precision of the information they possess. Since however their peers are also rational, they will predict it and discount for it. Hence if the true initial beliefs and precisions can be seen draws from a known distribution, the aforementioned assumption may be a severe constraint. This is merely a conjecture though, and it remains yet to be studied formally.
APPENDIX

2.A Proofs

2.A.1 Proof of Proposition 2.1

Notice that agents’ beliefs in period $t$ can be written in matrix format as

$$b(t) = \hat{\Gamma}(t)b(t-1),$$

where

$$\hat{\Gamma}(t) = \left[(Z^t\pi(0)1_n^T) \circ I_n\right]^{-1}Z \circ I_n(Z^{t-1}\pi(0))^T.$$

Then

$$\hat{W}(t) = \prod_{\kappa=1}^{t} \hat{\Gamma}(\kappa)$$

$$= \prod_{\kappa=1}^{t} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \circ I_n(Z^{\kappa-1}\pi(0))^T$$

$$= \prod_{\kappa=1}^{t} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(Z^{\kappa-1}\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$\times \left[(Z^{t-2}\pi(0)1_n^T) \circ I_n\right] \cdots \left[Z\pi(0)1_n^T \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$= \prod_{\kappa=1}^{t} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$= \prod_{\kappa=1}^{t-1} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$= \prod_{\kappa=1}^{t-1} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$= \prod_{\kappa=1}^{t-1} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$= \prod_{\kappa=1}^{t-1} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$= \prod_{\kappa=1}^{t-1} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$= \prod_{\kappa=1}^{t-1} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$

$$= \prod_{\kappa=1}^{t-1} \left[(Z^\kappa\pi(0)1_n^T) \circ I_n\right]^{-1}Z \left[(\pi(0)1_n^T) \circ I_n\right]^{-1}Z$$
\[
\left( Z^t \pi(0) 1^T_n \right) \odot I_n \right]^{-1} Z^t \mathbf{D}_{\pi(0)}
\]  

(2.14)

where we have used the properties of Hadamard product discussed in Section B.A of the Appendix, and \( \mathbf{D}_{\pi(0)} := \text{diag}(\pi_1(0), \pi_2(0), \ldots, \pi_n(0)) \) is a diagonal matrix with the elements of vector \( \pi(0) \) on its main diagonal.

Now we can use Lemma B.1 to obtain an expression for the social influence of the agents. From (B.4) it follows that

\[
\lim_{t \to +\infty} \mathbf{W}(t) = \lim_{t \to +\infty} \left\{ \left[ \left( \frac{Z}{\rho_Z} \right)^t \pi(0) 1^T_n \right) \odot I_n \right]^{-1} \left( \frac{Z}{\rho_Z} \right)^t \mathbf{D}_{\pi(0)} \right\}
\]

\[
= \left[ \lim_{t \to +\infty} \left( \frac{Z}{\rho_Z} \right)^t \pi(0) 1^T_n \right) \odot I_n \right]^{-1} \lim_{t \to +\infty} \left( \frac{Z}{\rho_Z} \right)^t \mathbf{D}_{\pi(0)}
\]

\[
= \left[ \left( \mathbf{r} \mathbf{c}^T \pi(0) 1^T_n \right) \odot I_n \right]^{-1} \mathbf{r} \mathbf{c}^T \mathbf{D}_{\pi(0)}
\]

\[
= \left( \mathbf{c}^T \pi(0) \right)^{-1} \mathbf{r} \mathbf{c}^T \mathbf{D}_{\pi(0)}
\]

(2.15)

where \( \mathbf{D}_{\pi(0)} := \text{diag}(\pi(0)) \), \( \alpha_{c,\pi} := \frac{1^T \pi(0)}{\mathbf{c}^T \pi(0)} = \frac{\sum_{i=1}^n \pi_i(0)}{\sum_{i=1}^n \pi_i(0)} \) is a scalar that captures the effects of the network on social influence, and \( \mathbf{\bar{c}}(0) := \frac{\pi(0)}{1^T \pi(0)} \) is the vector of relative initial precisions of the agents in network \( G^Z \).
2.A.2 Proof of Theorem 2.1

Since matrix $Z$ is diagonalizable, it can be written as a spectral decomposition of the form

$$Z = \sum_{i=1}^{n} \lambda_i Q_i,$$

where $\lambda_i$, for $i = 1, 2, \ldots, n$, are the eigenvalues of matrix $Z$, in descending order according to their modulus ($\rho_Z = |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$), and the spectral projectors $Q_i$ are idempotent and orthogonal to each other.\(^{19}\)

Then (2.9) can be written as

$$\hat{W}(t) = \left[ \sum_{k=1}^{n} \lambda_k Q_k \right]^t \pi(0) 1_n^T \circ I_n \left[ \sum_{k=1}^{n} \lambda_k Q_k \right]^{-1} D_{\pi(0)} = \left[ \sum_{k=1}^{n} \lambda_k Q_k \pi(0) 1_n^T \circ I_n \right]^{-1} \sum_{k=1}^{n} \lambda_k^t Q_k D_{\pi(0)}. \quad (2.16)$$

Recall that by the Perron-Frobenius theorem it holds that $\lambda_1 = \rho_Z$. Moreover, since network $G^Z$ is aperiodic, the matrix of subjective weights $Z$ must be primitive, and hence $\lambda_1 = \rho_Z$ is a simple eigenvalue. Then according to expression (7.2.12) in Meyer (2001, Chapter 7.2) it holds that

$$Q_1 = \frac{r c^T}{c^T r},$$

where $r$ is the Perron vector of matrix $Z$. Hence expression (2.16) can be written as

$$\hat{W}(t) = \left[ \left( \rho_Z^t \frac{r c^T}{c^T r} + \sum_{k=2}^{n} \lambda_k^t Q_k \right) \pi(0) 1_n^T \circ I_n \right]^{-1} \left( \rho_Z^t \frac{r c^T}{c^T r} + \sum_{k=2}^{n} \lambda_k^t Q_k \right) D_{\pi(0)} = \left[ \rho_Z^t \frac{r c^T}{c^T r} + \sum_{k=2}^{n} \lambda_k^t Q_k \left( \pi(0) 1_n^T \circ I_n \right)^t \right]^{-1} \left( \rho_Z^t \frac{r c^T}{c^T r} + \sum_{k=2}^{n} \lambda_k^t Q_k \right) D_{\pi(0)}.$$
Then the \((i,j)\)-th element of matrix \(\hat{W}(t)\) can be written as

\[
\hat{w}_{ij}(t) = \frac{r_i \tilde{c}_j \pi_j(0) + \tilde{c}^T \mathbf{r} \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t q_{ij}^{(k)} \pi_j(0)}{r_i \tilde{c}^T \pi(0) + \tilde{c}^T \mathbf{r} \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0)}
\]

\[
= \frac{\Psi_{ij}^{(0)}(0) + \frac{1}{r_i} \tilde{c}^T \pi(0) \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0)}{1 + \frac{1}{r_i} \tilde{c}^T \pi(0) \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0)}
\]

\[
= \frac{\hat{w}_j^{(\infty)} + \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t q_{ij}^{(k)} \pi_j(0)}{1 + \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0)},
\]

where \(\psi_i := \frac{1}{r_i} \tilde{c}^T \pi(0)\) and \(\frac{\Psi_{ij}^{(0)}(0)}{\tilde{c}^T \pi(0)} = \hat{w}_j^{(\infty)}\) from (2.9). It follows then that

\[
\hat{w}_{ij}(t) - \hat{w}_j^{(\infty)} = \frac{\hat{w}_j^{(\infty)} + \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t q_{ij}^{(k)} \pi_j(0) - \hat{w}_j^{(\infty)} - \hat{w}_j^{(\infty)} \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0)}{1 + \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0)}
\]

\[
= \frac{\psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t (q_{ij}^{(k)} \pi_j(0) - \hat{w}_j^{(\infty)} \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0))}{1 + \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0)}.
\]

The definitions below will help economise on the expressions to follow:

\[
A^{(k)} := q_{ij}^{(k)} \pi_j(0) - \hat{w}_j^{(\infty)} \sum_{h=1}^{\infty} q_{ih}^{(k)} \pi_h(0)
\]

\[
B^{(k)} := \sum_{h=1}^{n} q_{ih}^{(k)} \pi_h(0)
\]

Then

\[
\frac{\hat{w}_{ij}(t+1) - \hat{w}_{ij}^{(\infty)}}{\hat{w}_{ij}(t) - \hat{w}_{ij}^{(\infty)}} = \frac{\psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t A^{(k)} + B^{(k)}}{1 + \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t A^{(k)}}
\]

\[
= \frac{\sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t A^{(k)} + \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t B^{(k)}}{1 + \psi_i \sum_{k=2}^{n}(\frac{\lambda_k}{p_Z})^t A^{(k)}}
\]
The rate of convergence of sequence \( \{ \widetilde{w}_{ij}(t) \}_{t \in \mathbb{N}} \) is given by
\[
\lim_{t \to +\infty} \frac{|\widetilde{w}_{ij}(t + 1) - w_{j}^{(\infty)}|}{|\widetilde{w}_{ij}(t) - \widetilde{w}_{j}^{(\infty)}|} = \frac{|\lambda_2|}{\rho Z} \in (0,1).
\]

It follows thus that \( \{ \widetilde{w}_{ij}(t) \}_{t \in \mathbb{N}} \) converges linearly to \( \widetilde{w}_{j}^{(\infty)} \), for all \( i, j \in \mathcal{N} \).

2.A.3 Proof of Theorem 2.2

The following lemma will be useful in proving Theorem 2.2.

**Lemma 2.1.** Consider a non-negative, reducible matrix \( G \), written in its canonical form
\[
G = \begin{bmatrix}
P_p & P_L \\
O & L
\end{bmatrix}
\]
where \( P_p, L \) are square block matrices, and \( O \) is a matrix of zeros (see Definition B.3).

Then
\[
\lim_{t \to +\infty} \left( \frac{G}{\rho Z} \right)^t = \begin{bmatrix}
O & P_L^{(\infty)} \\
O & L^{(\infty)}
\end{bmatrix} \quad (2.17)
\]
where

\[ P_L^{(\infty)} = \left( \rho_G I_m - P_P \right)^{-1} P_L^{(\infty)}, \]  

(2.18)

\[ L^{(\infty)} = \begin{bmatrix}
    r_{l_1} c_{L_1}^T & 0 & \cdots \\
    c_{L_1}^T r_{l_1} & 0 & \cdots \\
    0 & \ddots & 0 \\
    0 & \cdots & r_{l_\ell} c_{L_\ell}^T \\
    c_{L_\ell}^T r_{l_\ell} & 0 & \cdots 
\end{bmatrix}, \]  

(2.19)

and \( c_{L_k}, r_{L_k} \), are respectively the left and right Perron eigenvectors of matrix \( L_k \), for \( k = 1, 2, \ldots, \ell \).

**Proof.** The above lemma is the generalisation of a well-known proposition in linear algebra on the limit of reducible Markov matrices with irreducible, primitive ergodic classes.\(^{20}\) The proof starts by observing that \( G \) is a block triangular matrix, and thus it will hold that

\[ \left( \frac{G}{\rho_G} \right)^t = \begin{bmatrix}
    \left( \frac{P_P}{\rho_G} \right)^t \\
    0 \\
    \left( \frac{L}{\rho_G} \right)^t
\end{bmatrix}. \]

In order, therefore, to prove this lemma, it suffices therefore to establish convergence of each block of matrix \( G \) in expression (B.3), divided by spectral radius \( \rho_G \), to the corresponding block of the limit matrix in expression (2.17). First, it is well known in the matrix algebra literature (see, for example, Meyer, 2001, chapter 10.7) that if \( \rho_G > \rho_{p_P} \), then

\[ \lim_{t \to +\infty} \left( \frac{P_P}{\rho_G} \right)^t = O. \]

Notice that because \( P_P \) is a block triangular matrix, its spectrum coincides with the union of the spectra of its diagonal blocks (see, for example, Karlin and Taylor, 1975, Appendix 1.C). Since matrix \( G \) is in its canonical form,

\(^{20}\)See, for example, Meyer (2001), Chapter 8.4. The difference in the present case is that neither matrix \( G \) nor its blocks \( L_k \) are necessarily stochastic. This poses some challenges, but they can be dealt with using a different set of results from the linear algebra literature.
each diagonal block of $P_p$ is either irreducible or zero. The leading right eigenvector of matrix $G$ can be partitioned as $r = [r_{P1}, \ldots, r_{Pp}, \ldots, r_{L1}, \ldots, r_{L\ell}]$, so that

$$
\begin{bmatrix}
P_{p,11} & P_{p,12} & \cdots & P_{p,1p} \\
O & P_{p,22} & \cdots & P_{p,2p} \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & P_{p,pp}
\end{bmatrix}
\begin{bmatrix}
P_{L,11} & P_{L,12} & \cdots & P_{L,1\ell} \\
P_{L,21} & P_{L,22} & \cdots & P_{L,2\ell} \\
\vdots & \vdots & \ddots & \vdots \\
P_{L,p1} & P_{L,p2} & \cdots & P_{L,p\ell}
\end{bmatrix}
= \rho_G
\begin{bmatrix}
r_p \\
r_2 \\
\vdots \\
r_p
\end{bmatrix}
\begin{bmatrix}
r_1 \\
r_2 \\
\vdots \\
r_\ell
\end{bmatrix}
$$

(2.20)

with the blocks $r_{Pj}$ (for $j = 1, 2, \cdots, p$), and $r_{Lk}$ (for $k = 1, 2, \cdots, \ell$) of the right eigenvector $r$ being conformable with the corresponding blocks $P_{p,ij}$, and $P_{L,ik}$ of matrix $G$ respectively. Then for any $i \in \{1, \ldots, p\}$ the following equality will be true:

$$
\sum_{j=1}^{p} P_{p,ij} r_{Pj} + \sum_{k=1}^{\ell} P_{L,ik} r_{Lk} = \rho_G r_{P_i}.
$$

(2.21)

Because all blocks $P_{p,ij}$ and $P_{L,ik}$ are non-negative, the above expression readily implies that

$$
P_{p,ii} r_{P_i} \leq \rho_G r_{P_i}
$$

(2.22)

for every $i \in \{1, \ldots, p\}$. Since $P_p$ is a submatrix of $G$, it must be that $\rho_{P_p} \leq \rho_G$. Now assume towards a contradiction that $\rho_{P_p} = \rho_G$. Since each diagonal block $P_{p,ii}$ is irreducible ($G$ is in its canonical form), and $r_{P_i}$ is non-negative ($r$ is the leading eigenvector of a non-negative matrix), inequality (2.22) together with Lemma 2.4 in Zijm (1984) imply that

$$
P_{p,ii} r_{P_i} = \rho_G r_{P_i}
$$

(2.23)
for every $i \in \{1, \ldots, p\}$. Yet recall that $G$ is weakly connected, and thus for every $i \in \{1, \ldots, m\}$ it must be either $P_{L,ij} \neq 0$ for some $j \in \{1, \ldots, \ell\}$, or $P_{P,ij} \neq 0$ for some $j \in \{1, \ldots, m\}$ with $j > i$. Expression (2.21) implies then that
\[
P_{P,ii} r_i < \rho_G r_i
\]
for every $i \in \{1, \ldots, m\}$, which contradicts expression (2.23). It must therefore be the case that $\rho_{M_{P,ii}} < \rho_G$, implying that $\rho_{M_P} < \rho_G$; convergence of $\left(\rho_G^{-1} P_P \right)^t$ to $0$ follows.

Next, it is shown that $\left(\rho_G^{-1} L_k \right)^t$ converges to $L_{k}^{(\infty)}$. Since $L$ is a block diagonal matrix, it suffices to show that each block $\left(\rho_G^{-1} L_k \right)^t$, converges to $\frac{r_k c_k^T}{c_k^T r_k}$, with $k = 1, \ldots, \ell$. Expression (2.20) readily implies that
\[
L_k r_L = \rho_G r_L k,
\]
and thus $\rho_G$ is the spectral radius of each block $L_k$, for $k = 1, \ldots, \ell$. Then, given that each $L_k$ is primitive, and since it has been just shown that $\rho_G = \rho(L_k)$, Lemma B.1 establishes convergence of $\left(\rho_G^{-1} L_k \right)^t$ to $L_{k}^{(\infty)}$.

It remains to prove convergence of $\tilde{P}_L(t)$ to $P_{L}^{(\infty)}$. Through iterative expansions it can be seen that
\[
\tilde{P}_L(t) = \sum_{\tau=1}^{t-1} \left( \frac{P_P}{\rho_G} \right) \left( \frac{L}{\rho_G} \right)^{t-\tau}
\]
and hence
\[
\lim_{t \to +\infty} \tilde{P}_L(t) = \sum_{\tau=1}^{+\infty} \left( \frac{P_P}{\rho_G} \right) \left( \frac{L}{\rho_G} \right)^{t-\tau}
\]
where $\lim_{t \to +\infty} \left( \frac{L}{\rho_G} \right)^{t-\tau} = L_{k}^{(\infty)}$ was shown above. Moreover, since $\rho_{M_P} < \rho_G$, the above sum constitutes a well-known sequence in linear algebra, known
as the Neumann series,\(^{21}\) with its limit given by

\[
\lim_{t \to +\infty} \sum_{\tau=1}^{t} \left( \frac{P}{\rho} \right)^{\tau-1} = \left( I_{n(P)} - \frac{P}{\rho} \right)^{-1}. 
\]

This completes the proof of Lemma 2.1. ■

We can now proceed with the proof of Theorem 2.2. Applying Lemma B.1 from Appendix B to expression (2.14) gives

\[
\lim_{t \to +\infty} W(t) = \left( \frac{Z}{\rho} \right)^{t} \pi(0)_{n} \circ I_{n}^{-1} \left( \frac{Z}{\rho} \right)^{t} D_{\pi(0)}
\]

where we have used Lemma 2.1, vector \( \pi(0) \) has been partitioned to blocks \( \pi_{P}(0) \) and \( \pi_{L}(0) \) representing the initial precisions of the agents in sets \( P \) and \( L \) respectively (notice that they are conformable with \( P_{L}^{(\infty)} \) and \( P_{P}^{(\infty)} \) respectively), and \( D_{x} \) is a diagonal matrix with the elements of vector \( x \) on its main diagonal. Now expression (2.24) can be written as

\[
\lim_{t \to +\infty} W(t) = \begin{bmatrix}
P_{L}^{(\infty)} \pi_{L}(0) & 1_{n(P)}^T \\
L^{(\infty)} \pi_{L}(0) & 1_{n(L)}^T
\end{bmatrix} \circ I_{n}
\]

\[
\begin{bmatrix}
P_{L}^{(\infty)} D_{\pi_{L}(0)} \\
L^{(\infty)} D_{\pi_{L}(0)}
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
P_{L}^{(\infty)} \pi_{L}(0) & 1_{n(P)}^T \\
L^{(\infty)} \pi_{L}(0) & 1_{n(L)}^T
\end{bmatrix}
\]

\[
\pi_{L}(0)_{n} \circ I_{n}^{-1} \\
\pi_{L}(0)_{n} \circ I_{n}^{-1}
\]

\[
\left( \frac{Z}{\rho} \right)^{t} \pi_{L}(0)_{n} \circ I_{n}^{-1}
\]

\[
\left( \frac{Z}{\rho} \right)^{t} D_{\pi_{L}(0)}
\]

\[
\begin{bmatrix}
P_{L}^{(\infty)} D_{\pi_{L}(0)} \\
L^{(\infty)} D_{\pi_{L}(0)}
\end{bmatrix}
\]

\[
\left( \frac{Z}{\rho} \right)^{t} \pi_{L}(0)_{n} \circ I_{n}^{-1}
\]

\[
\left( \frac{Z}{\rho} \right)^{t} D_{\pi_{L}(0)}
\]

\[
\text{Named after German mathematician Carl Gottfried Neumann (1832–1925). For a proof of convergence and some properties of this series, see Meyer (2001), Chapter 7.10.}
\[ \begin{bmatrix}
O & P_L^{(\infty)} D_{\pi_L(0)} \\
O & L^{(\infty)} D_{\pi_L(0)}
\end{bmatrix}
\times
\begin{bmatrix}
O & P_L^{(\infty)} D_{\pi_L(0)} \\
O & L^{(\infty)} D_{\pi_L(0)}
\end{bmatrix}
= \begin{bmatrix}
O & \left( P_L^{(\infty)} \pi_L(0) 1^T_{n(\mathcal{P})} \right) \circ I_{n(\mathcal{P})}^{-1} P_L^{(\infty)} D_{\pi_L(0)} \\
O & \left( L^{(\infty)} \pi_L(0) 1^T_{n(\mathcal{L})} \right) \circ I_{n(\mathcal{L})}^{-1} L^{(\infty)} D_{\pi_L(0)}
\end{bmatrix}.\]

Substituting for \( P_L^{(\infty)} \) from expression (2.18) in Lemma 2.1 yields
\[
\lim_{t \to +\infty} W(t) = W^{(\infty)} = \begin{bmatrix}
O & W_{\mathcal{P} \mathcal{L}}^{(\infty)} \\
O & W_{\mathcal{L} \mathcal{L}}^{(\infty)}
\end{bmatrix}
\]
where
\[
W_{\mathcal{P} \mathcal{L}}^{(\infty)} := \left( \left( \rho_Z I_m - P_P \right)^{-1} P_L^{(\infty)} \pi_L(0) 1^T_{n(\mathcal{P})} \circ I_{n(\mathcal{P})} \right)^{-1} \left( \rho_Z I_m - P_P \right)^{-1} P_L^{(\infty)} D_{\pi_L(0)},
\]
\[
W_{\mathcal{L} \mathcal{L}}^{(\infty)} := \left( L^{(\infty)} \pi_L(0) 1^T_{n(\mathcal{L})} \circ I_{n(\mathcal{L})} \right)^{-1} L^{(\infty)} D_{\pi_L(0)}.
\]
Observe that the asymptotic influence of any member of the public is zero, not only on lobby members, but also on other members of the public. Although the former is quite intuitive (since lobbies are, by definition, self absorbed), the latter may be more subtle. Yet it is the intuition is the same, since it also an immediate consequence of the weakly connected structure of the network. Under DWU (or any process subsumed into it, such as DeGroot updating) agents fail to account for repetitions in the information they receive, that is, they are susceptible to persuasion bias, as termed by DeMarzo et al. (2003). As a result, more persistent opinions will prevail. While each lobby is practically insulated from external influence from the rest of the society, the members of the public always pay some attention to beliefs originating from some lobby, directly or indirectly. Hence with every exchange of opinions, lobbies’ beliefs get gradually more and more incorporated into those of the public, until they completely phase out the public’s members initial beliefs.

In order, thus, to study the asymptotic influence of the lobbies on the public, it is necessary to examine first how beliefs are formed within each lobby.
The \((2,2)\)-th block of matrix \(W^{(\infty)}\), denoted by \(W^{(\infty)}_{\mathcal{LL}}\), captures the social influence of lobby members \(i \in \mathcal{L}\) on other lobby members. Notice that block \(W^{(\infty)}_{\mathcal{LL}}\) is a block diagonal matrix itself, with each block \(L_k\) corresponding to a different lobby, \(\mathcal{L}_k\). This implies that agents in any given lobby can only influence members of their own lobby, since, by definition, lobbyists do not pay attention to outsiders. Using expression (2.19) from Lemma 2.1 gives

\[
W^{(\infty)}_{\mathcal{LL}} := \left[ \left( L^{(\infty)} \pi_{L^{(0)}} \right) \circ I_{n(L)} \right]^{-1} L^{(\infty)} D \pi_{L^{(0)}}
\]

\[
= \left[ \begin{array}{cccc}
\pi_{L^{(0)}} & 0 \\
0 & \pi_{L^{(0)}}
\end{array} \right]^{-1} \left[ \begin{array}{cccc}
\pi_{L^{(0)}} & 0 \\
0 & \pi_{L^{(0)}}
\end{array} \right]^{-1}
\]

\[
= \left[ \begin{array}{cccc}
\pi_{L^{(0)}} & 0 \\
0 & \pi_{L^{(0)}}
\end{array} \right]^{-1} \left[ \begin{array}{cccc}
\pi_{L^{(0)}} & 0 \\
0 & \pi_{L^{(0)}}
\end{array} \right]^{-1}
\]

Observe that each lobby constitutes a strongly connected, aperiodic subnetwork, so intuitively Proposition 2.1 should apply. Indeed, notice that each diagonal block of the above matrix has the same form as the right-hand side of expression (2.15) in the proof of that proposition in Section 2.A.1. It follows that the above expression can be written as

\[
W^{(\infty)}_{\mathcal{LL}} = \left[ \begin{array}{cccc}
1_{n(L_1)} \alpha_{L_1} \left( \hat{c}_{L_1} \circ \pi_{L_1^{(0)}} \right)^T & 0 \\
0 & 1_{n(L_\ell)} \alpha_{L_\ell} \left( \hat{c}_{L_\ell} \circ \pi_{L_\ell^{(0)}} \right)^T
\end{array} \right]
\]

This proves the first part of Theorem 2.2. We proceed now to prove the second part. The \((1,2)\)-th block of matrix \(W^{(\infty)}\), denoted by \(W^{(\infty)}_{\mathcal{LP}}\), gives the asymptotic influence \(\hat{w}_ij^{(\infty)}\) of each agent \(j \in \mathcal{L}\) on each agent \(i \in \mathcal{P}\). From
expression (2.18) in Lemma 2.1 it follows that

\[ W_{p_2}^{(\infty)} := \left( \left( \rho_z I_{n(p)} - P_p \right)^{-1} P_{\mathcal{L}} L^{(\infty)} \pi_{\mathcal{L}}(0) I_{n(p)}^T \right) \circ I_{n(p)} \right]^{-1} \times \]

\[ \times \left( \rho_z I_{n(p)} - P_p \right)^{-1} P_{\mathcal{L}} L^{(\infty)} D_{\pi_{\mathcal{L}}}(0) \]

\[ = \left( \rho_z I_{n(p)} - P_p \right)^{-1} P_{\mathcal{L}} \begin{bmatrix} r_{l_1} \varepsilon_{l_1}^{T} \\ c_{l_1} \varepsilon_{l_1}^{T} \end{bmatrix} \begin{bmatrix} \cdots \\ 0 \end{bmatrix} \begin{bmatrix} \pi_{\mathcal{L}_1}(0) \\ \vdots \\ \pi_{\mathcal{L}_\ell}(0) \end{bmatrix} \right]^{-1} \times \]

\[ \times \left( \rho_z I_{n(p)} - P_p \right)^{-1} P_{\mathcal{L}} \begin{bmatrix} r_{l_1} \varepsilon_{l_1}^{T} \\ c_{l_1} \varepsilon_{l_1}^{T} \end{bmatrix} \begin{bmatrix} \cdots \\ 0 \end{bmatrix} \begin{bmatrix} D_{\pi_{\mathcal{L}_1}(0)} \\ \vdots \\ D_{\pi_{\mathcal{L}_\ell}(0)} \end{bmatrix} \]

\[ = \left( I_{n(p)} - \frac{1}{\rho_z} P_p \right)^{-1} P_{\mathcal{L}_1} \begin{bmatrix} c_{l_1} \varepsilon_{l_1}^{T} \pi_{\mathcal{L}_1}(0) \\ \vdots \\ c_{l_\ell} \varepsilon_{l_\ell}^{T} \pi_{\mathcal{L}_\ell}(0) \end{bmatrix} \begin{bmatrix} r_{l_1} \varepsilon_{l_1}^{T} \\ c_{l_1} \varepsilon_{l_1}^{T} \end{bmatrix} \begin{bmatrix} D_{\pi_{\mathcal{L}_1}(0)} \\ \vdots \\ D_{\pi_{\mathcal{L}_\ell}(0)} \end{bmatrix} \]

\[ = \left( I_{n(p)} - \frac{1}{\rho_z} P_p \right)^{-1} \left( \sum_{q=1}^{\ell} P_{\mathcal{L}_q} r_{L_q} \frac{c_{l_q} \varepsilon_{l_q}^{T} \pi_{\mathcal{L}_q}(0)}{c_{l_q} \varepsilon_{l_q}^{T} r_{L_q}} \right) I_{n(p)} \circ I_{n(p)} \right]^{-1} \times \]

\[ \times \left( I_{n(p)} - \frac{1}{\rho_z} P_p \right)^{-1} P_{\mathcal{L}_1} r_{L_1} \cdots \left( I_{n(p)} - \frac{1}{\rho_z} P_p \right)^{-1} P_{\mathcal{L}_\ell} r_{L_\ell} \right] \times \]

\[ \begin{bmatrix} c_{l_1} \varepsilon_{l_1}^{T} \pi_{\mathcal{L}_1}(0) \\ \vdots \\ c_{l_\ell} \varepsilon_{l_\ell}^{T} \pi_{\mathcal{L}_\ell}(0) \end{bmatrix} \]

\[ \times \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \]
Chapter 2. Lobbies, Experts, and the Public

It can be seen from the last equality that the \((i,j)\)-th element of matrix \(W_\infty^{(L)}\), expressing the asymptotic influence of lobbyist \(j \in L_k\) on member of the public \(i \in P\) will be a scalar of the form

\[
\hat{w}_{ij}^{(\infty)} = \frac{c_{k,j}^T \pi_{L_k}^{(0)} \varepsilon_{L_k,i}}{c_{L_k} \tau_{L_k}}.
\]

where we have used formula (2.11) proved in the first part of this proof, and the definitions of \(\alpha_{L_q} := \sum_{h \in L_q} \pi_h^{(0)}\), \(\phi_{L_q} := \frac{1}{c_{L_q}^T \tau_{L_q}}\), and \(\pi_{L_q}^{(0)} := \sum_{h \in L_q} \pi_h^{(0)}\).

2.A.4 Proof of Corollary 2.2

If there is only one lobby, then expression (2.13) (which gives the asymptotic influence of each lobbyist \(j \in L_1\) on a member of the public \(i \in P\)) becomes

\[
\hat{w}_{ij}^{(\infty)} = \frac{\phi_{L_1} \pi_{L_1}^{(0)} \varepsilon_{L_1,i}}{\alpha_{L_1}} \hat{w}[L_1]^{(\infty)} = \hat{w}[L_1]^{(\infty)},
\]
and substituting this into expression (2.12) gives
\[ b_i^{(\infty)} = \sum_{j \in \mathcal{L}_1} \hat{w}[\mathcal{L}_1]^{(\infty)}_j b_j(0) = b^{(\infty)}_{\mathcal{L}_1} \]
for all agents \( i \in \mathcal{P} \), where \( b^{(\infty)}_{\mathcal{L}_1} \) is the within-lobby consensus in \( \mathcal{L}_1 \) (see Theorem 2.2).

2.A.5 Proof of Corollary 2.3

Substituting expression (2.13) into expression (2.12) in Theorem 2.2 gives
\[ b_i^{(\infty)} = \sum_{k=1}^{\ell} \sum_{j \in \mathcal{L}_k} \sum_{q=1}^{\ell} \frac{\phi_{\mathcal{L}_k}}{\alpha_{\mathcal{L}_k}} \pi_{\mathcal{L}_k, i}^{\alpha \mathcal{L}_k} \sum_{j \in \mathcal{L}_k} \hat{w}[\mathcal{L}_k]^{(\infty)}_j b_j(0) \]

\[ = \sum_{k=1}^{\ell} \left[ \sum_{q=1}^{\ell} \frac{\phi_{\mathcal{L}_q}}{\alpha_{\mathcal{L}_q}} \pi_{\mathcal{L}_q, i}^{\alpha \mathcal{L}_q} \hat{w}[\mathcal{L}_q]^{(\infty)}_j b_j(0) \right] \]

\[ = \sum_{k=1}^{\ell} \sum_{q=1}^{\ell} \frac{\phi_{\mathcal{L}_q}}{\alpha_{\mathcal{L}_q}} \pi_{\mathcal{L}_q, i}^{\alpha \mathcal{L}_q} b^{(\infty)}_{\mathcal{L}_k}, \]

which proves Corollary 2.3.

2.A.6 Proof of Proposition 2.2

From Definition 2.9 and Corollary 2.3 it follows that the public influence of lobby \( \mathcal{L} \) can be written as
\[ \hat{w}^{(\infty)}_{\mathcal{P}, \mathcal{L}_k}(s) = \sum_{h \in \mathcal{P}} \left[ \frac{\phi_{\mathcal{L}_k}}{\alpha_{\mathcal{L}_k}} \pi_{\mathcal{L}_k, h}^{\alpha \mathcal{L}_k} \sum_{q=1}^{\ell} \frac{\phi_{\mathcal{L}_q}}{\alpha_{\mathcal{L}_q}} \pi_{\mathcal{L}_q, h}^{\alpha \mathcal{L}_q} \right] \]
\begin{equation}
\frac{\text{d} \hat{w}_{P,L_k}^{(\infty)}(s)}{\text{d} z_{ij}} = \frac{\phi_{L_k}}{\alpha_{L_k} \Pi_{L_k}^{0}} \sum_{h \in P} \left[ s_h \frac{\text{d} \varepsilon_{L_k,h}}{\text{d} z_{ij}} \left( \sum_{q=1}^{\ell} \frac{\phi_{C_q}}{\alpha_{C_q}} \Pi_{L_q}^{0} \varepsilon_{L_q,h} \right) - s_h \varepsilon_{L_k,h} \sum_{q=1}^{\ell} \frac{\phi_{C_q}}{\alpha_{C_q}} \Pi_{L_q}^{0} \frac{\text{d} \varepsilon_{L_q,h}}{\text{d} z_{ij}} \right]^{-2} \left( \sum_{q=1}^{\ell} \frac{\phi_{C_q}}{\alpha_{C_q}} \Pi_{L_q}^{0} \varepsilon_{L_q,h} \right) \right], \tag{2.25}
\end{equation}

The proof begins with the first statement in Proposition 2.2, which claims that an increase in the attention that a lobby receives from any member of the public increases, ceteris paribus, the public influence of that lobby.

The attention that agent \( j \in L_k \) receives from agent \( i \in P \) is captured by element \( z_{ij} \) of submatrix \( P_L \) of the adjacency matrix \( Z \). Differentiating thus \( \hat{w}_{P,L_k}^{(\infty)}(s) \) with respect to \( z_{ij} \) gives

\begin{equation}
\frac{\text{d} \hat{w}_{P,L_k}^{(\infty)}(s)}{\text{d} z_{ij}} = \frac{\phi_{L_k}}{\alpha_{L_k} \Pi_{L_k}^{0}} \sum_{h \in P} \left[ s_h \frac{\text{d} \varepsilon_{L_k,h}}{\text{d} z_{ij}} \left( \sum_{q=1}^{\ell} \frac{\phi_{C_q}}{\alpha_{C_q}} \Pi_{L_q}^{0} \varepsilon_{L_q,h} \right) - s_h \varepsilon_{L_k,h} \sum_{q=1}^{\ell} \frac{\phi_{C_q}}{\alpha_{C_q}} \Pi_{L_q}^{0} \frac{\text{d} \varepsilon_{L_q,h}}{\text{d} z_{ij}} \right]^{-2} \left( \sum_{q=1}^{\ell} \frac{\phi_{C_q}}{\alpha_{C_q}} \Pi_{L_q}^{0} \varepsilon_{L_q,h} \right) \right],
\end{equation}

where the second equality holds true because \( \frac{\text{d} \varepsilon_{L_q,h}}{\text{d} z_{ij}} = 0 \) since \( j \notin L_q \) for \( q \neq k \).

In order to compute \( \frac{\text{d} \varepsilon_{L_k,h}}{\text{d} z_{ij}} \) for \( i,h \in P, j \in L_k \), and \( q \in \{1, \ldots, \ell\} \) we need to calculate first

\begin{equation}
\frac{\text{d} \varepsilon_{L_k}(s)}{\text{d} z_{ij}} = \left( I_{n(P)} - \beta P P \right)^{-1} \frac{\text{d} P_{L_k}}{\text{d} z_{ij}},
\end{equation}
\[
(\mathbf{I}_{n(P)} - \beta \mathbf{P} \mathbf{P})^{-1} = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial z_{ij}}{\partial z_{ij}} \\
\ddots \\
0
\end{bmatrix}
\begin{bmatrix}
r_{L_k,1} \\
\vdots \\
r_{L_k,n(P)}
\end{bmatrix}
\geq 0_{n(P)}
\]

where \(a_{hi}\) denotes the \((h,i)\)-th element of matrix \((\mathbf{I}_{n(P)} - \beta \mathbf{P} \mathbf{P})^{-1}\). Nonnegativity of the vector above follows from nonnegativity of matrix \((\mathbf{I}_{n(P)} - \beta \mathbf{P} \mathbf{P})^{-1}\) (which is the Neumann series of non-negative matrices \(\beta \mathbf{P} \mathbf{P}\)). In fact, strict inequality must hold for at least one element \(a_{hi} r_{L_k,j}\). This is true because agent \(i \in \mathcal{P}\) must receive positive attention from at least one agent (recall that the network is connected). It follows hence that

\[
\frac{\partial \varepsilon_{L_k,h}}{\partial z_{ij}} = a_{hi} r_{L_k,j} \geq 0
\]

for all \(h \in \mathcal{P}\), with strict inequality for at least one \(h \in \mathcal{P}\). Substituting now this into expression (2.26) gives

\[
\frac{\partial w^{(\infty)}_{P,L_k}(s)}{\partial z_{ij}} = \frac{\phi_{L_k} \pi_{L_k}^0 a_{hi} r_{L_k,j}}{a_{L_k}} \sum_{h \in \mathcal{P}} \left[ \sum_{q=1}^{\ell} \frac{\phi_{L_q} \pi_{L_q}^0 \varepsilon_{L_q,h}}{\alpha_{L_q} \sum_{q=1}^{\ell} \pi_{L_q}^0 \varepsilon_{L_q,h}} \right] \cdot (2.27)
\]

which is a positive number, since all parameters in this expression are non-negative, and at least one of them is positive.

The second part of the statement asserts that the gains are larger for the lobby if the additional attention is directed towards lobbyists with high relative
openness \( r_{L_k,j} \). Observe that expression (2.27) is larger for higher values of \( r_{L_k,j} \), all other things being equal.

It is straightforward to prove the second statement by calculating the derivative of the public influence of lobby \( L_k \), as given by \( \tilde{w}^{(\infty)}_{P,L_k} \) in (2.26):

\[
\frac{d\tilde{w}^{(\infty)}_{P,L_k}(s)}{d\pi^0_{L_k}} = \frac{\phi_{L_k}}{\alpha_{L_k}} \sum_{h \in P} \left[ \frac{s_h \varepsilon_{L_k,h}}{\sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \pi^0_{L_q} \varepsilon_{L_q,h}} \right] - \frac{\phi_{L_k} \pi^0_{L_k}}{\alpha_{L_k}} \times \\
\times \sum_{h \in P} \left[ \frac{s_h \varepsilon_{L_k,h}}{\left( \sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \pi^0_{L_q} \varepsilon_{L_q,h} \right)^2} \frac{\phi_{L_k} \varepsilon_{L_k,h}}{\alpha_{L_k}} \right] \\
= \frac{\phi_{L_k}}{\alpha_{L_k}} \sum_{h \in P} \left[ \frac{s_h \varepsilon_{L_k,h}}{\sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \pi^0_{L_q} \varepsilon_{L_q,h}} \left( \sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \pi^0_{L_q} \varepsilon_{L_q,h} \right) \right] - \frac{\phi_{L_k}}{\alpha_{L_k}} \times \\
\times \sum_{h \in P} \left[ \frac{s_h \varepsilon_{L_k,h}}{\left( \sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \pi^0_{L_q} \varepsilon_{L_q,h} \right)^2} \left( \sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \pi^0_{L_q} \varepsilon_{L_q,h} \right) \right] \\
= \frac{\phi_{L_k}}{\alpha_{L_k}} \sum_{h \in P} \left[ \frac{s_h \varepsilon_{L_k,h}}{\sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \pi^0_{L_q} \varepsilon_{L_q,h}} \left( \sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \pi^0_{L_q} \varepsilon_{L_q,h} \right) \right] \geq 0
\]

with strict inequality if \( s_h \varepsilon_{L_k,h} > 0 \) for at least one \( h \in P \), since all other terms in the last expression are non-negative. Equality will hold only if every member of the public \( h \) who pays attention to \( L_k \) (implying that \( \varepsilon_{L_k,h} > 0 \)) is assigned zero significance \( (s_h = 0) \).

To prove the third statement, recall that the correlation between centrality and expertise in a lobby \( L_k \) is captured by the centrality–expertise dispersion parameter \( \alpha_{L_k} \). Higher values of \( \alpha_{L_k} \) imply that expertise is, on average,
“dispersed” towards more peripheral agents, while smaller values indicate that expertise is concentrated on the central agents. As above, differentiating the expression for public influence with respect to that parameter yields

$$
\frac{d\hat{w}_{P,L_k}^{(\infty)}(s)}{d\alpha_{L_k}} = -\frac{\phi_{L_k}}{\alpha_{L_k}^2} \sum_{h \in P} s_h e_{L_k,h} \left[ \frac{s_h e_{L_k,h}}{\alpha_{L_k}} \right] + 
$$

$$
+ \frac{\phi_{L_k}}{\alpha_{L_k}^2} \sum_{h \in P} \sum_{q=1}^{\ell} \left( \frac{\phi_{L_q}}{\alpha_{L_q}} \right)^2 \sum_{h \in P} \left[ \frac{s_h e_{L_k,h}}{\alpha_{L_k}} \right] + 
$$

$$
\sum_{q=1}^{\ell} \left( \frac{\phi_{L_q}}{\alpha_{L_q}} \right)^2 \sum_{h \in P} \left[ \frac{s_h e_{L_k,h}}{\alpha_{L_k}} \right] 
$$

$$
= -\frac{\phi_{L_k}}{\alpha_{L_k}^2} \sum_{h \in P} s_h e_{L_k,h} \left( \sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \sum_{h \in P} e_{L_q,h} \right) + 
$$

$$
+ \frac{\phi_{L_k}}{\alpha_{L_k}^2} \sum_{h \in P} \sum_{q=1}^{\ell} \left( \frac{\phi_{L_q}}{\alpha_{L_q}} \right)^2 \sum_{h \in P} \left[ \frac{s_h e_{L_k,h}}{\alpha_{L_k}} \right] 
$$

$$
= -\frac{\phi_{L_k}}{\alpha_{L_k}^2} \sum_{h \in P} s_h e_{L_k,h} \left( \sum_{q=1}^{\ell} \frac{\phi_{L_q}}{\alpha_{L_q}} \sum_{h \in P} e_{L_q,h} \right) \leq 0,
$$

with the equality being strict if $s_h e_{L_k,h} > 0$ for at some $h \in P$, as discussed above.

Recall that the centrality–openness dispersion parameter $\phi_{L_k}$ captures the correlation between centrality and relative openness in a lobby $L_k$. It follows
from expression (2.25) that

\[
\frac{\partial w^{(\infty)}_{\mathcal{P}, \mathcal{L}_k}(s)}{\partial \phi_{\mathcal{L}_k}} \bigg|_{\mathcal{E}_{\mathcal{L}_k}} = \frac{\pi_{\mathcal{L}_k}^o}{\alpha_{\mathcal{L}_k}} \sum_{h \in \mathcal{P}} \frac{s_h \epsilon_{\mathcal{L}_k, h}}{\left( \sum_{q=1}^{\ell} \left( \frac{\phi_{\mathcal{L}_q}}{\alpha_{\mathcal{L}_q}} \pi_{\mathcal{L}_q}^o \epsilon_{\mathcal{L}_q, h} \right) \right)^2} = \frac{\pi_{\mathcal{L}_k}^o}{\alpha_{\mathcal{L}_k}} \sum_{h \in \mathcal{P}} \frac{s_h \epsilon_{\mathcal{L}_k, h}}{\left( \sum_{q=1}^{\ell} \left( \frac{\phi_{\mathcal{L}_q}}{\alpha_{\mathcal{L}_q}} \pi_{\mathcal{L}_q}^o \epsilon_{\mathcal{L}_q, h} \right) \right)^2} \geq 0,
\]

establishing thus the fourth statement.

The first part in the fifth statement, namely that larger lobbies are more influential than smaller ones, ceteris paribus, follows directly from the first statement in Proposition 2.2. As long as there is at least one member of the public \( i \) with positive significance \( (s_i > 0) \) who assigns a non-zero weight to the additional members in the larger lobby, that lobby receives more attention from the public than the smaller lobby, all other things being equal. Subsequently its public influence must be larger, as shown above.

According to the second part of this statement, however, an increase in the size of a lobby has an ambiguous effect on its public influence. The source of
this ambiguity lies in the characteristics of the agent who is added to lobby. Let us analyse the effects triggered by this addition.

First, for the reasons discussed just above, enlargement on a first order cannot decrease the influence of a lobby, and may well increase it (cf. first statement). Furthermore, unless the new member’s expertise is zero, it will add to the lobby’s aggregate expertise, and hence increase its influence, as stipulated in the second statement. Third, enlargement may increase or decrease the correlation between centrality and expertise in the lobby, depending on the expertise of the new agent, and the position he or she will assume in the network; this in turn will increase or decrease the public influence of the lobby according to the third statement in this proposition. Similarly, the effect through its impact on the relationship between centrality and openness is also unclear. Hence, the overall effect of the increase of the lobby’s size can potentially increase its public influence, reduce it, or even leave it unchanged. This completes the proof of Proposition 2.2.
Chapter 3
As a large body of literature in sociology and economics has shown, social interaction induces conformism, and as it has been observed, behaviours deviating from the social norm tend to be punished. Although conformism has been studied in a network setup before, this is one of the first papers to examine conformism under incomplete information, and the first to provide and discuss thoroughly a comprehensive theoretical framework. Social interaction is modelled as a Bayesian network game, which is the natural setup for analysing decisions whose potential returns or costs are *ex ante* uncertain (e.g. education, crime). We establish existence and uniqueness of the equilibrium, characterise the optimal decisions, and examine conditions under which policy interventions can be welfare-improving.

*This essay is co-authored with Marc Sommer (University of Zurich and Stockholm University) and Yves Zenou (Stockholm University and Monash University).*
3.1 Introduction

Social and professional networks are omnipresent in people’s everyday lives, and directly or indirectly influence their choices and their behaviour. The decision of an individual to pursue college education, consume a product, work hard or shirk, and even engage into criminal activities, has been shown to be affected by their social environment.

Games on networks can be used as a tool to analyse a wide variety of situations where agents’ payoffs, and therefore their behavior, depend on the actions of their peers.\(^1\) An important class of games are the so-called games of strategic complements. They are used to model settings in which an individual’s additional benefit from taking a given action increases as their peers increase the intensity which they engage into this action. According to the standard cost–benefit analysis, undertaking an action yields some benefit to the actor, but at the same time also imposes a cost on them: this trade-off is captured by the private component of the agents’ utility function. Conformism enters the model through the social component of the utility function. The mechanisms through which these complementarities operate are captured by the social component of the utility functions, which can be defined by two distinct models, known in the literature respectively as the local-aggregate and the local-average models.\(^2\)

The local-aggregate model is used to model situations in which an individual’s returns from engaging into an activity depend on the total prevalence of this activity in their neighbourhood, irrespectively of how this is distributed among their peers. The local-average model, on the other hand, builds on the premise that it is the average, rather than the aggregate level of activity that matters. In other words, the latter, formalises of the idea that, due to social pressure, individuals who deviate from the social norm suffer some

\(^1\) For a review of the literature on games on networks, see Jackson and Zenou (2015).

\(^2\) For a thorough and up-to-date review of the theoretical and empirical literature on this area, see Topa and Zenou (2015).
punishment in the form of reputational damage, loss of social status, or even social exclusion.

The theoretical local-aggregate model, first proposed by Calvó-Armengol and Zenou (2004), and Ballester, Calvó-Armengol, and Zenou (2006), has been used to model peer effects in crime (Ballester, Calvó-Armengol, and Zenou, 2010; Lindquist and Zenou, 2014), and in R&D partnerships (König, Liu, and Zenou, 2014).

The local-average model has received significant empirical attention ever since Bramoullé, Djebarri, and Fortin (2009) proposed an instrumental variables approach that enables identification of peer effects in such type of models. In particular, it has been used to estimate such effects in a large spectrum of setups where social norms may play an important role, for example crime (Liu, Patacchini, Zenou, and Lee, 2012; Lindquist and Zenou, 2014), education (Calvó-Armengol, Patacchini, and Zenou, 2009; De Giorgi, Pellizzari, and Redaelli, 2010; Liu and Lee, 2010), and consumption (De Giorgi, Frederiksen, and Pistaferri, 2016).

Despite the significant and fertile literature in this area, there is still a lot of work that needs to be done in order to gain a deeper understanding of the impact of peer effects on individual decisions. First, the majority of the empirical studies pursue the estimation of the significance and the magnitude of peer effects in some setting of interest, assuming the model they use (most commonly, the local-average) is the relevant one. Although the choice made is rigorously motivated, and quite often, intuitively plausible, in many cases the use of the alternative model could be also theoretically justifiable. In fact, it could be argued that in some settings, a hybrid model should be used. The only studies that we are aware of that test the two benchmark models against each other are those by Liu, Patacchini, and Zenou (2014). They find that, even within the same network, some choices (e.g. study effort) are driven by social norms, and hence are better described by the local-average model, while others (e.g. delinquent behavior) are better accounted for by the amount of exposure to peers’ activity in total, pointing
thus towards the local-aggregate model.

Second, an important assumption implicitly underlying the existing literature has been that the model’s parameters are known, if not by the social planner, at least by the decision makers. In many of the aforementioned applications, however, this may be quite a strong assumption. When deciding whether to pursue further education, high school graduates may not know their financial and non-pecuniary benefits of college education. Similarly, potential criminals do not know with certainty what the probability of getting caught is. Uncertainty, though, may apply to the social component of the utility function as well. Individuals in a society, for example, may not be fully cognisant of the strength and the enforceability of the social norm, that is, how tolerant society is towards non-conformist behaviors, and how harsh social punishment is expected to be.

To the best of our knowledge, the only other paper to examine social interactions in the local-average framework with incomplete information is the one by Blume, Brock, Durlauf, and Jayaraman (2015). Their focus though lies more on providing the framework and the tools that will allow identification and estimation of the local-average model under incomplete information, rather than studying the theoretical model per se. Our paper, instead, provides a closed-form solution the the agents’ optimisation problem, and studies the implications that the introduction of uncertainty has for equilibrium behaviour and social welfare; it can be therefore seen as complementary to their work. Understanding the mechanics and the channels through which peer effects operate in this setup is of primary importance, since the results of the baseline model do not in general carry over to the incomplete information case. The present work aims moreover to close a gap that exists in the literature, since the local-aggregate model has already been studied in an incomplete information setup by de Martí and Zenou (2015). This paper hence intends, together with the econometric model introduced by Blume et al. (2015), to give the researcher the necessary tools to study peer effects in settings where the local-average model has been found to be more
relevant, as discussed above.

We model the agents' decision problem as a Bayesian network game. More specifically, we allow uncertainty to creep in through any one of the three main parameters of the model: private benefit of an action, private cost, or taste for conformity, the latter being a parameter that captures the network-induced cost due to deviation from the endogenously determined social norm. These parameters are potentially heterogeneous across agents, and consist of two components: a generic one that is common to all agents and represents the objective payoffs or costs of an action, and an idiosyncratic one, that is known to the agents and captures their individual characteristics.

In our model, agents receive some signals about the state of the world and optimally decide their actions. Even though individuals are affected directly only by their direct links in the network and need to infer their actions through the signals received, in equilibrium, they also need to infer the actions of agents located more than one-link away from them (higher order beliefs). We establish the existence and uniqueness of the Bayesian Nash equilibrium (BNE), characterise the optimal decisions, and examine conditions under which policy interventions can be welfare-improving. We also perform different comparative statics and show, in particular, how social environment, and players' characteristics affect their actions and welfare.

The rest of the paper is structured as follows. Section 3.2 introduces formally the Bayesian network game described above. Section 3.3 studies the complete information case, which will serve as our benchmark. Our main theorems and results are presented in Section 3.4. Section 3.5 concludes. All proofs have been deferred to the Appendix of this Chapter, where we also provide a list of the most frequently used symbols and their definition.
3.2 The incomplete information network game

The Bayesian network game to be described hereinafter is a static, non-cooperative game with incomplete information, where the players take their decisions simultaneously and independently of one another. The players are assumed to be rational in the sense that they seek to maximise their welfare.

There are \( n > 1 \) players participating in the game. In what follows, each mathematical object associated with a particular player will be indexed by an element of the set \( \mathcal{I} := \{1, \ldots, n\} \). Even a player will be abstractly represented by an element of \( \mathcal{I} \), so that \( \mathcal{I} \) corresponds with the set of players of the game.

Let \((\Omega, \mathcal{S})\) be a measurable space with \( \Omega \neq \emptyset \). The set \( \Omega \) is called the state space; it represents all the possible states of the world that are relevant for the game. We assume that the players have a common prior (that is, probability measure) \( P \) on \((\Omega, \mathcal{S})\). Thus, the probabilistic nature of the game is represented by the probability space \((\Omega, \mathcal{S}, P)\).

For all \( i \in \mathcal{I} \), let \( \alpha_i : \Omega \to \mathbb{R}_{++}, \beta_i : \Omega \to \mathbb{R}_{++}, \) and \( \gamma_i : \Omega \to \mathbb{R}_+ \) be integrable random variables defined on \((\Omega, \mathcal{S}, P)\). The triple of functions \((\alpha_i, \beta_i, \gamma_i)\) is referred to as player \( i \)'s payoff parameters (see also utility function (3.1) below).

For all \( i \in \mathcal{I} \), let \( S_i : \Omega \to \mathbb{R} \) be a simple random variable\(^3\) defined on \((\Omega, \mathcal{S}, P)\). The function \( S_i \) is called player \( i \)'s private signal, which may convey some information about his payoff parameters. A player's signal is said to convey no information about a payoff parameter if the signal and the parameter are (stochastically) independent. We assume that the players' signals have a common support, denoted by \( \Theta \), which is also referred to as the players' common type space. The signals that players receive may be correlated with each other. Note that \( 0 < |\Theta| < \infty \) because \( \Omega \neq \emptyset \) and a simple

\(^3\) A random variable is called simple if it can assume only a finite number of values. In this case, this implies that the range of \( S_i, \mathcal{R}(S_i) = \{S_i(\omega) | \omega \in \Omega\} \) is a finite set.
random variable has a finite support. Hereinafter, we shall write $\Theta = \{\theta_t \mid t \in \{1, \ldots, T\}\}$, where $T := |\Theta| \geq 1$. For all $i \in I$, player $i$’s type is given by an element $\theta_i$ of $\Theta$. A type profile is an $n$-tuple $(\theta_1, \ldots, \theta_n) \in \Theta^n$.

The set of actions available to player $i \in I$ is equal to $\mathbb{R}_+$. An action of player $i \in I$ is denoted by $x_i$. The set of all possible action profiles $x = (x_1, \ldots, x_n)$ is equal to $\mathbb{R}_+^n$. A pure strategy of player $i \in I$ is a map $x_i : \Theta \to \mathbb{R}_+$. A strategy is therefore a rule that assigns an action to each possible type. The set of all pure strategies of player $i \in I$ is denoted by $\mathbb{R}_+^{\Theta}$. The set of all possible strategy profiles $x = (x_1, \ldots, x_n)$ is equal to $\prod_{i \in I} \mathbb{R}_+^{\Theta}$.

As it is characteristic for a game, a player’s well-being depends not only on his or her action but may also depend on the actions of other players. This dependence is made explicit by means of a network through which the players are connected. We assume that this network is fixed and common knowledge and that it can be represented by a directed graph $G$ on $I$. The graph $G$ encodes the information about the identities of all the players who directly affect a player’s well-being through their actions. For a particular player, the set of all the players who directly affect his well-being is called his or her out-neighbourhood. For all $i \in I$, player $i$’s out-neighbourhood (in $G$) is denoted by $D^+_G(i)$, and its cardinality, the so-called out-degree of $i$ (in $G$), by $\deg^+_G(i)$. It is assumed that $i \in D^+_G(i)$, implying that $G$ has no self-loops. The assumption of a directed graph implies that a player is not necessarily an out-neighbour of his or her out-neighbours, that is, $j \in D^+_G(i)$ does not necessarily imply that $i \in D^+_G(j)$. In short, the dependence of a player’s well-being on the actions of his or her out-neighbours is potentially unidirectional.

In many applications, including games on networks, it is more convenient to represent a network using a matrix $G := [g_{i,j}]_{(i,j) \in I^2} \in \{0,1\}^{n \times n}$, where $g_{i,j} := 1$ if player $i$ is an out-neighbour of player $j$, and $g_{i,j} := 0$ otherwise. In network theory terminology, matrix $G$ is referred to as the adjacency matrix of network $G$. 
We assume that every player has at least one out-neighbour, as it is stated in the following assumption.

Assumption 3.1. For every player \( i \in I \) it holds that \( D_G^+(i) \neq \emptyset \).

Players' welfare can be represented by a family of utility functions \( \{ u_i : \Omega \times \mathbb{R}_+^n \rightarrow \mathbb{R} \}_{i \in I} \) given by

\[
 u_i(\omega, (x_1, \ldots, x_n)) = \alpha_i(\omega)x_i - \frac{\beta_i(\omega)}{2}x_i^2 - \frac{\gamma_i(\omega)}{2}\left(x_i - \frac{\sum_{j \in D_G^+(i)}x_j}{\deg_G^+(i)}\right)^2,
\]

where \( \alpha_i(\omega) > 0, \beta_i(\omega) > 0, \) and \( \gamma_i(\omega) \geq 0 \), for all \( i \in I \).

The common functional form of the players' utility functions is known to all players. Incomplete information may, however, arise by the players' ignorance about the value of some of the payoff parameters. A particular payoff parameter can hereby give rise to incomplete information only if it is not a constant function.

Some comments on utility function (3.1) are in order. To this end, let \( \omega \in \Omega \), and let \( (x_1, \ldots, x_n) \in \mathbb{R}_+^n \) be an action profile.

Player \( i \)'s utility function is symmetric in his or her out-neighbours’ actions. It exhibits local strategic complements if \( \gamma_i > 0 \) (that is, \( \gamma_i \) is a positive function) because for all \( j \in I \),

\[
 \frac{\partial^2 u_i(\omega, (x_1, \ldots, x_n))}{\partial x_i \partial x_j} = \begin{cases} \frac{\gamma_i(\omega)}{\deg_G^+(i)} & \text{if } j \in D_G^+(i), \\ 0 & \text{if } j \notin D_G^+(i). \end{cases}
\]

It does, however, not exhibit positive or negative local externalities.\(^4\)

The utility that every player \( i \in I \) ascribes to the action profile \( (x_1, \ldots, x_n) \), \( u_i(\omega, (x_1, \ldots, x_n)) \), consists of two components: the private component and

\(^4\) In accordance with the terminology introduced by Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010, pp. 226–27), player \( i \)'s utility function is said to exhibit negative (respectively, positive) local externalities if for all \( \omega \in \Omega \), for all \( (x_1, \ldots, x_n) \in \mathbb{R}_+^n \), and for all \( (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}_+^n \) with \( \tilde{x}_i = x_i \) and \( \{ \tilde{x}_j - x_j \mid j \in D_G^+(i) \} \subset \mathbb{R}_+ \), \( u_i(\omega, (x_1, \ldots, x_n)) \leq u_i(\omega, (\tilde{x}_1, \ldots, \tilde{x}_n)) \) (respectively, \( u_i(\omega, (x_1, \ldots, x_n)) \geq u_i(\omega, (\tilde{x}_1, \ldots, \tilde{x}_n)) \)).
the social component.

The private component is defined as $\alpha_i(\omega)x_i - (\beta_i(\omega)/2)x_i^2$. Since $\alpha_i > 0$ and $\beta_i > 0$, the point $\alpha_i(\omega)/\beta_i(\omega)$ is defined and lies in the interior of $\mathbb{R}_+$, from which it follows that the private component function $x_i \mapsto \alpha_i(\omega)x_i - (\beta_i(\omega)/2)x_i^2$ is strictly increasing on $(0, \alpha_i(\omega)/\beta_i(\omega))$ and strictly decreasing on $(\alpha_i(\omega)/\beta_i(\omega), +\infty)$, with a global maximum point at $\alpha_i(\omega)/\beta_i(\omega)$. The private component can in turn be decomposed into two parts: the private benefit and the private cost. The private benefit is defined as $\alpha_i(\omega)x_i$ and the private cost is defined as $-(\beta_i(\omega)/2)x_i^2$.

The social component is defined as

$$-\frac{\gamma_i(\omega)}{2} \left( x_i - \frac{\sum_{j \in D_G^+(i)} x_j}{\deg_G^+(i)} \right)^2.$$

The social component represents player $i$’s social cost (if any; $\gamma_i = 0$ is possible) from deviating from a social norm that is given by the arithmetic mean of his or her out-neighbours’ actions. The coefficient $\gamma_i(\omega)$, referred to as the social conformism parameter or the strength of the social norm, captures the magnitude of the social cost for player $i$, relatively to the private payoff parameters of that player ($\alpha_i, \beta_i$). The distance between player $i$’s action and his or her social norm is referred to as the social distance between player $i$ and his or her out-neighbours (see also Akerlof, 1997). It is important to note that the players’ social norms are endogenous and potentially heterogeneous (in equilibrium) because the players may vary in their out-neighbourhoods, and they may choose different actions (in equilibrium).

The following definition is useful in order to give a compact representation of the players’ utility functions.
**Definition 3.1: Row-Normalised Matrix**

The row-normalised adjacency matrix of $\mathcal{G}$ with respect to the canonical enumeration of $\mathcal{I}$, $\text{id}_\mathcal{I}$, is denoted by $\bar{\mathcal{G}}$. The component in row $i$ and column $j$ of $\bar{\mathcal{G}}$ is denoted by $\bar{g}_{i,j}$, that is

$$\bar{g}_{i,j} = \frac{g_{i,j}}{\deg_C(i)} \quad \forall (i,j) \in \mathcal{I}^2.$$

Using Definition 3.1, for all $i \in \mathcal{I}$, player $i$’s utility function can be written as follows:

$$u_i(\omega, (x_1, \ldots, x_n)) = \alpha_i(\omega)x_i - \frac{\beta_i(\omega)}{2}x_i^2 - \frac{\gamma_i(\omega)}{2}\left(x_i - \sum_{j \in \mathcal{I}} \bar{g}_{i,j}x_j\right)^2.$$

The timing of the Bayesian network game is as follows:

**Step 1: Nature moves:** A state $\omega \in \Omega$ is realised (but not observed).

**Step 2: Players receive information:** Each player $i \in \mathcal{I}$ observes $S_i(\omega)$, the value of his or her private signal $S_i$ at the state $\omega$, which determines his or her type $\vartheta_i \in \Theta$.

**Step 3: Players move:** Each player $i \in \mathcal{I}$ chooses an action $x_i(\vartheta_i) \in \mathbb{R}_+$ conditional on his or her type $\vartheta_i$.

**Step 4: Payoffs are realised:** Each player $i \in \mathcal{I}$ receives the payoff that corresponds with the realised state (and, hence, the values of the players’ signals and their types) and the chosen strategy profile, $u_i(\omega, (x_1(\vartheta_1), \ldots, x_n(\vartheta_n)))$.

### 3.3 The case of complete information

This section discusses the case in which players have a complete knowledge of all the parameters of the game. It is assumed therefore that players know
with certainty not only their payoff parameters, but potentially those of all
the other players in the network too. In some cases, this may not be a bad
approximation, since individuals may be aware of the preferences or the
characteristics of the others, or at least those of their friends. Even if this
assumption may not be as plausible in many setups, it is a case that is still
worthwhile examining. Apart from serving as a benchmark, it also provides
some basic intuition on how the various forces in the model interact to give
rise to an equilibrium, before this is perplexed further by the introduction of
uncertainty.

Using the formal framework introduced in the previous section, complete
information corresponds to the case of $|\Omega| = 1$. It follows that for all players
$i \in I$, the payoff parameters $\alpha_i, \beta_i$, and $\gamma_i$ are known constants, that is, there
exists a triple $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i) \in \mathbb{R}^2_+ \times \mathbb{R}_+$ such that $\alpha_i(\Omega) = \{\bar{\alpha}_i\}$, $\beta_i(\Omega) = \{\bar{\beta}_i\}$, and
$\gamma_i(\Omega) = \{\bar{\gamma}_i\}$. It follows also that the players’ signals are constant and identi-
cal, that is, $T = 1$ and $\Theta = \{\theta_1\}$. Let $\Gamma := (I, G, (\Omega, S, \mathbb{P}), \{(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i)\}_{i \in I}, \{\theta_1\})$
 denote the Bayesian network game with complete information. Let $\bar{\alpha} :=
(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)^T$, $\bar{\beta} := (\bar{\beta}_1, \ldots, \bar{\beta}_n)^T$, and $\bar{\gamma} := (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)^T$. In addition, let $D_{\bar{\beta} + \bar{\gamma}}$ and $D_{\bar{\gamma}}$ be the diagonal matrices of order $n$ with the components in row
$i$ and column $i$ equal to $\bar{\beta}_i + \bar{\gamma}_i$ and $\bar{\gamma}_i$, respectively. Then the following
statement holds true.

**Theorem 3.1: BNE Under Complete Information**

Assume that the payoff parameters of all players are constant and common
knowledge, that is $(\alpha(\Omega), \beta(\Omega), \gamma(\Omega)) = (\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$. Then the
game $\Gamma$ has a unique and interior (Bayesian) Nash equilibrium $x^*$, which
is given by

$$x^*(\theta_1) = \left(D_{\bar{\beta} + \bar{\gamma}} - D_{\bar{\gamma}} G\right)^{-1} \bar{\alpha}.$$

Notice that game $\Gamma$ is strategically equivalent to the Bayesian network game
$(I, G, (\Omega, S, \mathbb{P}), \{\bar{\alpha}_i/\bar{\beta}_i, 1, \bar{\gamma}_i/\bar{\beta}_i\}_{i \in I}, \{\theta_1\})$. 
The equilibrium strategy of each agent can be also written as a scalar.

**Corollary 3.1.** Player $i$’s strategy in the (Bayesian) Nash equilibrium is given by

$$x^*_i(\theta_1) = \frac{\bar{\alpha}_i}{\bar{\beta}_i + \bar{\gamma}_i} + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} \bar{g}_i \left( D \bar{\beta} + D \bar{\gamma} \bar{G} \right)^{-1} \bar{\alpha}$$

where $\bar{g}_i := (\bar{g}_{i,1}, \ldots, \bar{g}_{i,j}, \ldots, \bar{g}_{i,n})$ is $i$-th row of the row-normalised adjacency matrix $\bar{G}$.

It would be now interesting to study how this equilibrium is affected as the parameters of the model change. In particular, we consider two types of changes: individual and global. In order to facilitate our analysis, we can decompose the agents’ payoff parameters as follows:

$$\bar{\alpha}_i = \bar{\alpha}^G + \bar{\alpha}^I_i, \quad \bar{\beta}_i = \bar{\beta}^G + \bar{\beta}^I_i, \quad \bar{\gamma}_i = \bar{\gamma}^G + \bar{\gamma}^I_i,$$

where $\bar{\alpha}^G$ is the global component of the marginal benefit parameter, common to all players $i \in I$ (for example, $\bar{\alpha}^G := \frac{1}{n} \sum_{j=1}^{n} \bar{\alpha}_j$), and $\bar{\alpha}^I_i := \bar{\alpha}_i - \bar{\alpha}^G_i$ is the idiosyncratic component, which may vary across players. Parameters $\bar{\beta}^G$, $\bar{\beta}^I_i$, $\bar{\gamma}^G$, and $\bar{\gamma}^I_i$ are defined in a similar way.

The global part of each parameter characterises some attribute of the activity in question that does not directly depend on the specific characteristics of an individual. Consider, for example, the problem of optimal investment in education. In that case, $\bar{\alpha}^G$ can be interpreted as the expected marginal increase in earnings from an additional year of college attendance, while $\bar{\beta}^G$ captures the additional (pecuniary or not) cost incurred by the average individual (e.g. tuition fees, average income foregone in the duration of studies). Yet some students may possess skills that enable them to benefit more than the average, while the opposite may be true for lower-skill students. Costs may also vary across agents as well. Students of higher ability are more likely to receive a scholarship compared to low-ability students, and their disutility from studying may be lower as well. Alternatively, the opportunity cost of obtaining a postgraduate degree may be higher for an individual who is already employed compared to an individual who is unemployed or has just
finished college. Similarly, $\gamma_G$ represents the prevailing strength of social norm in the society, while some individuals may feel more ($\gamma_i^I > 0$) or less compelled ($\gamma_i^I > 0$) to adhere to that norm.

**Proposition 3.1: The Effect of Shifts in Idiosyncratic Parameters**

- An increase in the idiosyncratic benefit parameter of player $i$ ($\alpha_i^I$) strictly increases his or her equilibrium action, and weakly increases the equilibrium action of all agents $j \in I$.
- An increase in the idiosyncratic cost parameter of agent $i$ ($\beta_i^I$) strictly decreases his or her equilibrium action, and weakly decreases the equilibrium action of all agents $j \in I$.
- An increase in the idiosyncratic strength of social norm of agent $i$ ($\gamma_i^I$) weakly increases the equilibrium action of all agents $j \in I$ if agent $i$’s equilibrium action is lower than his or her social norm, and weakly decreases the equilibrium effort of all agents $j \in I$ if agent $i$’s equilibrium action is higher than his or her social norm.

The first statement is quite intuitive: if exerting effort becomes more beneficial for player $i$, that agent will respond by increasing his or her effort. At a second order, this will raise the social norm of player $i$’s in-neighbours, inducing them to increase their effort as well. By the same token, the latter will induce their own in-neighbours to to increase their level of activity, propagating this effect via the network. In many networks, this effect may end up raising the social norm in the out- neighbourhood of player $i$, leading to a further increase in his or her effort. This chain effect will go on dwindling, until the new equilibrium is reached. At the same time though, this increase will propagate through the network, tending to increase the effort of the neighbours of neighbours of agent $i$, and subsequently their neighbours, may have a second-order effect, and so on. A similar analysis, albeit towards the opposite direction, applies to an increase in the idiosyncratic
cost parameter, $\bar{\beta}_i$.

The intuition behind the third statement is similar, although the mechanics are slightly different. An increase in the idiosyncratic strength of the social norm of agent $i$ causes deviations to be more costly for that agent. If therefore agent $i$’s effort is lower than his or her social norm, this will cause his or her effort to increase, giving rise to the chain process described above. If agent $i$’s effort is higher than his or her social norm, this effect will work towards the opposite direction.

The following proposition stipulates how the above findings change in the case of a shift in the global payoff parameters.

**Proposition 3.2: The Effect of Shifts in Global Parameters**

- An increase in the global benefit parameter ($\bar{\alpha}^G$) strictly increases the equilibrium effort of all agents in the network.
- An increase in the global cost parameter ($\bar{\beta}^G$) strictly decreases the equilibrium effort of all agents in the network.
- The effect of an increase in the global strength of the social norm ($\bar{\gamma}^G$) is *ex ante* unclear, and may be different for different agents in the network.

The first two statements are quite intuitive, and the consequences of an increase in a global parameter are realised in a way similar to the one analysed above. The third statement though presents more interest; in this case, unlike the preceding ones, the findings of Proposition 3.1 on idiosyncratic parameter shifts do not carry over to global shifts. The reason is that an increase in the strength of the social norm will affect agents asymmetrically, and even the aggregate effect may be unclear. At a first order, agents will intensify their effort if it is relatively low, and reduce it if it is relatively high compared to their social norm. Yet the higher-order effects are indeterminate. Consider, for example, an agent who reduces her effort following an
increase in the strength of the social norm. It may well be the case that her out-neighbours reduce their effort, reducing thus her social norm. This will offset the first-order increase, changing again the social norm of that agent’s in-neighbours. At the same time, both her in- and out-neighbours will be affected by other agents; the effort thus of each agent in the new equilibrium will depend of the particular values of the parameters, and the structure of the network.

### 3.4 The case of incomplete information

The previous section studied how a social network influences individuals’ choices, and how changes in the environment, or even in the characteristics of a single player, can affect the equilibrium outcomes of all individuals. It can be, however, argued that in more cases than not, individuals have to make their choices in an uncertain environment, without knowing *ex ante* the exact returns or costs of their actions. This paper focuses on three cases in particular: incomplete information about the benefit deriving from increasing the level of their activity (exerting more effort), the cost of that additional effort, and the strength of the social norm among one’s peers. Using the terminology introduced in Section 3.2, there will be in general at least two different types $\theta \in \Theta$ of players in the game, so that $|\Theta| = T > 1$. In the present section, we use this to model uncertainty about each of the three aforementioned parameters.

#### 3.4.1 Incomplete information about the marginal benefit parameters $\alpha$

We begin with the case in which the return of their actions is unknown to the players. A college student, for example, has to decide on the amount of effort to put down in studying, without knowing how this will reflect on her performance in the exams, or even how obtaining a degree will affect her future labour market outcomes. Similarly, an employee at a firm may be uncertain on whether additional effort and longer working hours will
translate into higher output, and potentially a higher salary or a promotion. To formally model this, let \( \alpha := (\alpha_1, \ldots, \alpha_n) : \Omega \to \mathbb{R}_{++}^n \) be a non-degenerate random \( n \)-vector of the players’ marginal private benefit parameters. Furthermore, assume that for all \( i \in I \), the payoff parameters \( \beta_i \) and \( \gamma_i \) are constant, that is, there exists a pair \( (\tilde{\beta}_i, \tilde{\gamma}_i) \in \mathbb{R}_{++} \times \mathbb{R}_{+} \) such that \( \beta_i(\Omega) = \{\tilde{\beta}_i\} \) and \( \gamma_i(\Omega) = \{\tilde{\gamma}_i\} \). Let \( \Gamma_\alpha := (I, G, (\Omega, S, \mathbb{P}), ((\alpha_i, \tilde{\beta}_i, \tilde{\gamma}_i))_{i \in I}, \Theta) \) denote the Bayesian network game with incomplete information about the value of the players’ \( \alpha \)'s. Let \( \bar{\beta}, \bar{\gamma}, \mathbf{D}_{\bar{\beta}+\bar{\gamma}} \), and \( \mathbf{D}_\gamma \) be defined as in Section 3.3, and let \( \Theta := (\theta_1, \ldots, \theta_T)^T \). For every \( i \in I \), let

\[ \alpha_i^\Theta(\Theta) := (E[\alpha_i | S_i = \theta_1], \ldots, E[\alpha_i | S_i = \theta_T])^T \]

be the vector of the posterior expectations of player \( i \) for all different values of signal \( S_i \). For all strategy profiles \( x = (x_1, \ldots, x_n)^T \), and for all players \( i \in I \), define the strategy vector of player \( i \) as

\[ x_i(\Theta) := (x_i(\theta_1), \ldots, x_i(\theta_T))^T. \quad (3.2) \]

Observe that (3.2) expresses the strategy of player \( i \) as a vector, whose \( r \)-th element gives that player action for signal \( S_i = \theta_r \). Let

\[ \alpha^\Theta[\Theta] := \begin{pmatrix} \alpha_1^\Theta(\Theta) \\ \vdots \\ \alpha_n^\Theta(\Theta) \end{pmatrix} \quad \text{and} \quad x[\Theta] := \begin{pmatrix} x_1(\Theta) \\ \vdots \\ x_n(\Theta) \end{pmatrix} \]

denote the concatenations of vectors \( \alpha_i^\Theta(\Theta) \) and \( x_i(\Theta) \) of all players \( i \in I \). Moreover, for all \( (j, i) \in \{1, \ldots, n\}^2 \), let \( \Pi_{j,i} \) be the square matrix of order \( T \) with the component in row \( q \) and column \( r \) equal to \( \mathbb{P}(S_j = \theta_q | S_i = \theta_r) \):

\[
\Pi_{j,i} := \begin{pmatrix}
\mathbb{P}(S_j = \theta_1 | S_i = \theta_1) & \ldots & \mathbb{P}(S_j = \theta_1 | S_i = \theta_T) \\
\vdots & \ddots & \vdots \\
\mathbb{P}(S_j = \theta_q | S_i = \theta_1) & \ldots & \mathbb{P}(S_j = \theta_q | S_i = \theta_T) \\
\vdots & \ddots & \vdots \\
\mathbb{P}(S_j = \theta_T | S_i = \theta_1) & \ldots & \mathbb{P}(S_j = \theta_T | S_i = \theta_T)
\end{pmatrix}
\]

\[ \text{Recall that } \Theta = \{\theta_t | t \in \{1, \ldots, T\} \}. \]
That is, the $r$-th column of matrix $\Pi_{j,i}$, with $r \in \{1,\ldots,T\}$, is essentially the conditional probability mass function of player $j$’s signal $S_j$, given that player $i$ observes signal $S_i = \theta_r$. It follows thus that matrix $\Pi_{j,i}$ is column-stochastic for all $(j, i) \in I^2$, and in particular, $\Pi_{i,i} = I_T$. Let $\Pi$ be the square matrix of order $Tn$ defined by

$$
\Pi := \begin{pmatrix}
\Pi_{1,1} & \ldots & \Pi_{1,i} & \ldots & \Pi_{1,n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\Pi_{j,1} & \ldots & \Pi_{j,i} & \ldots & \Pi_{j,n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\Pi_{n,1} & \ldots & \Pi_{n,i} & \ldots & \Pi_{n,n}
\end{pmatrix}
$$

With the preliminaries in order, we can now proceed to state the main result of this section.

**Theorem 3.2: BNE with Unknown Benefit**

The game $I_\alpha$ has a unique and interior Bayesian Nash equilibrium $x^*$, which is given by

$$
x^*[\theta] = \left( D_{\tilde{\beta}+\tilde{\gamma}} \otimes I_T - \left( (D_{\tilde{y}} \hat{G}) \otimes J_T \right) \circ \Pi^T \right)^{-1} \alpha^*[\theta].
$$

where $J_T := 1_T \otimes 1_T$ is a $T \times T$ matrix of ones.

A first observation is that the above formula is very similar to the one derived in Theorem 3.1 for the complete information case. There are three main differences. The first one is rather mechanical: Players’ strategies are no longer degenerate functions of their signal, and in general they assume different values for different signals. Strategy vectors $x_i(\theta)$ will thus be proper vectors, not scalars, as under complete information. In fact, this increases the dimension of the equilibrium by the number $T$ of different signals (the cardinality of $\Theta$), and hence the use of Kronecker products.

The second difference is that the equilibrium strategies are, unsurprisingly,
functions of the expected marginal benefit parameters, $\alpha^e[\theta]$. This is rather intuitive, since the exact values of these parameters depend on the state of the world, which is no longer known. Notice that based on their private signal, players are required to form a posterior expectation not only for their marginal benefit parameter, $E[\alpha_i | S_i]$, but also for those of the other players in the network, $E[\alpha_j | S_i]$.

This remark gives rise to yet another difference compared to the previous case. Forming expectations on other players’ $\alpha$’s is not enough. Players need to form expectations about their neighbours’ expectations, since the strategies of the latter will be based on their expectations. The same argument goes for the expectations of the expectations of their neighbours, and so on, ad infinitum. This interdependence structure is formally introduced in the model through block matrix $\Pi$, which is interacted with the adjacency matrix $\bar{G}$ to channel this effect only through the links stipulated by the network.

In order to see more clearly that each player’s optimal action in equilibrium depends on the entire network structure, and not just that player’s immediate out-neighbours, we can write his or her optimal strategy as a scalar, real-valued function of signal value $\theta_r$.

**Corollary 3.2.** Player $i$’s strategy in the Bayesian Nash equilibrium is given by

$$x_i^*(\theta_r) = \frac{E[\alpha_i | S_i = \theta_r]}{\bar{\beta}_i + \bar{\gamma}_i} + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} c_i(\theta_r) \left( D_{\bar{\beta}_i + \bar{\gamma}_i} \otimes I_T - \left( D_{\bar{\gamma}_i} \bar{G} \otimes J_T \right) \circ \Pi^T \right)^{-1} \alpha^e[\theta]$$

where $c_i(\theta_r)$ is a $Tn$-dimensional block vector defined as

$$c_i(\theta_r) := \left( \bar{g}_{i,1} \, \pi_{1,i}(r) \quad \cdots \quad \bar{g}_{i,j} \, \pi_{j,i}(r) \quad \cdots \quad \bar{g}_{i,n} \, \pi_{n,i}(r) \right)$$

and $\pi_{j,i}(r)$ is the $r$-th row of matrix $\Pi^T_{j,i}$,

$$\pi_{j,i}(r) := \left( P(S_j = \theta_1 | S_i = \theta_r) \quad \cdots \quad P(S_j = \theta_q | S_i = \theta_r) \quad \cdots \quad P(S_j = \theta_T | S_i = \theta_r) \right).$$
3.4.2 Incomplete information about the private cost parameters $\beta$

A different potential source of uncertainty for the individuals lies in the cost entailed by undertaking some action. One such example is the decision of an individual to smoke. Smoking is an activity whose intensity is largely affected by social networks, especially among adolescents (see, for example, Bisin, Moro, and Topa, 2011). The impact of smoking on health, however, is \textit{ex ante} unknown to the smoker. Similarly, individuals engaging in wide range of delinquent or illegal activities, such as bullying, petty crime (Patacchini and Zenou, 2012), hooliganism, or tax evasion do not know with certainty the probability of getting caught or the exact punishment they would incur in such a case.

In terms of technical analysis, this case is very similar to one with incomplete information about $\alpha$ studied above. The unknown cost parameters of the players are represented by an $n$-dimensional random vector $\beta := (\beta_1, \ldots, \beta_n): \Omega \to \mathbb{R}^n_{++}$. The remaining payoff parameters, $\alpha$ and $\gamma$ are assumed to be constant, that is, there exists a pair of vectors $(\bar{\alpha}, \bar{\gamma}) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_+$ such that for each of their elements it holds $\alpha_i(\Omega) = \{\bar{\alpha}_i\}$ and $\gamma_i(\Omega) = \{\bar{\gamma}_i\}$, for all $i \in I$. Let $I^\beta := (I, G, (\Omega, S, P), \{(\bar{\alpha}_i, \beta_i, \bar{\gamma}_i)\}_{i \in I}, \Theta)$ denote the Bayesian network game with incomplete information about the value of the players’ cost parameters $\beta$’s. Moreover, similarly to Section 3.4.1, define player $i$’s vector of posterior expected values for $\beta$ as

$$\beta^e_i(\theta) := \left( E[\beta_i | S_i = \theta_1], \ldots, E[\beta_i | S_i = \theta_T] \right)^T,$$

and its concatenation for all players $i \in I$ as

$$\beta^e[\theta] := \left( \beta^e_1(\theta)^T, \ldots, \beta^e_n(\theta)^T \right)^T.$$

The equilibrium of this game is given below.
Theorem 3.3: BNE with Unknown Private Cost

The game \( \Gamma_{\beta} := (I, G, (\Omega, S, \mathbb{P}), \{(\tilde{\alpha}_i, \beta_i, \tilde{\gamma}_i)\}_{i \in I}, \Theta) \) has a unique and interior Bayesian Nash equilibrium \( x^* \) given by

\[
x^*[\theta] = (D_{[\beta'[\theta]+\tilde{\gamma}_T]} + (D_{\tilde{\gamma} \beta'} G_{J_T}) \circ \Pi_T)^{-1} D_{\beta'[\theta]+\tilde{\gamma}_T} (\tilde{\alpha} \otimes 1_T).
\]

The main intuition behind the above formula is similar to the one for Theorem 3.2 presented in Section 3.4.1. Uncertainty in this case though concerns the cost rather than the benefit parameters. The result is presented below in scalar form.

Corollary 3.3. Player \( i \)'s strategy in the Bayesian Nash equilibrium is given by

\[
x_i^* (\theta_r) = \frac{\tilde{\alpha}_i}{E[\beta_i | S_i = \theta_r] + \tilde{\gamma}_i} + \frac{\tilde{\gamma}_i}{E[\beta_{i+} | S_i = \theta_r] + \tilde{\gamma}_i} c_i (\theta_r) \times
\]

\[
\times (D_{[\beta'[\theta]+\tilde{\gamma}_T]} + (D_{\tilde{\gamma} \beta'} G_{J_T}) \circ \Pi_T)^{-1} D_{\beta'[\theta]+\tilde{\gamma}_T} (\tilde{\alpha} \otimes 1_T),
\]

where \( c_i (\theta_r) \) and \( \pi_{j,i} (r) \) are defined as in Corollary 3.2.

3.4.3 Incomplete information about the social conformism parameters \( \gamma \)

Finally, uncertainty in the present model may stem from ignorance concerning the social, rather than the private costs of the activity in question. In many cases it may be more reasonable to treat the strength of social norm as unknown to the individuals. Consider, for example, a group of freshmen in a college, families moving to a new neighbourhood, or immigrants settling down to a country with different culture. Individuals in the aforementioned environments most likely understand that behaving very disparately from their peers may entail consequences, ranging from failure to create a positive image to as far as social marginalisation. It may be less clear, however,
how important is conformist behaviour regarded in each situation, or which aspects of everyday life it applies to.

Although the formal setup is similar to the cases discussed above, the mechanism through which uncertainty affects individuals’ behaviour is slightly different, as it will be seen. Let \( \gamma := (\gamma_1, \ldots, \gamma_n): \Omega \rightarrow \mathbb{R}_+^n \) be a random \( n \)-dimensional vector, representing the social conformism parameters of the players. Private marginal benefit and cost parameters, \( \alpha_i \) and \( \beta_i \) are assumed to be constant (for all \( i \in I, (\bar{\alpha}_i, \bar{\beta}_i) \in \mathbb{R}_+^2 \)) and common knowledge, with \( \alpha_i(\Omega) = \{\bar{\alpha}_i\} \) and \( \beta_i(\Omega) = \{\bar{\beta}_i\} \). The Bayesian game with incomplete information about players’ social conformism parameter \( \gamma \) is denoted with \( \Gamma_\alpha := (I, G, (\Omega, \mathcal{S}, \mathbb{P}), \{(\alpha_i, \bar{\beta}_i, \bar{\gamma}_i)\}_{i \in I}, \Theta) \). Apart, however, from the standard definitions,

\[
\gamma^e_i(\theta) := \left( E[\gamma_i | S_i = \theta_1], \ldots, E[\gamma_i | S_i = \theta_T] \right)^T
\]

and

\[
\gamma^e[\theta] := \left( \gamma^e_1(\theta)^T, \ldots, \gamma^e_n(\theta)^T \right)^T,
\]

that are similar to the ones used above, we furthermore need to define the \( Tn \)-dimensional square block matrix

\[
\Gamma := \begin{pmatrix}
0_T & \cdots & \Gamma_{1,i} & \cdots & \Gamma_{1,n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\Gamma_{i,1} & \cdots & 0_T & \cdots & \Gamma_{i,n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\Gamma_{n,1} & \cdots & \Gamma_{n,i} & \cdots & 0_T
\end{pmatrix},
\]

where each block \( \Gamma_{i,j} \), for \( (i, j) \in I^2 \) with \( i \neq j \), is given by

\[
\Gamma_{i,j} := \begin{pmatrix}
E[\gamma_i | S_i = \theta_1, S_j = \theta_1] & \cdots & E[\gamma_i | S_i = \theta_1, S_j = \theta_T] \\
\vdots & \ddots & \vdots \\
E[\gamma_i | S_i = \theta_r, S_j = \theta_1] & \cdots & E[\gamma_i | S_i = \theta_r, S_j = \theta_T] \\
\vdots & \ddots & \vdots \\
E[\gamma_i | S_i = \theta_T, S_j = \theta_1] & \cdots & E[\gamma_i | S_i = \theta_T, S_j = \theta_T]
\end{pmatrix}
\]
\[ \Gamma_{i,j} := \begin{pmatrix} E[\gamma_i | S_i = \theta_1, S_j = \theta_1] & \cdots & E[\gamma_i | S_i = \theta_1, S_j = \theta_q] & \cdots & E[\gamma_i | S_i = \theta_1, S_j = \theta_T] \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ E[\gamma_i | S_i = \theta_r, S_j = \theta_1] & \cdots & E[\gamma_i | S_i = \theta_r, S_j = \theta_q] & \cdots & E[\gamma_i | S_i = \theta_r, S_j = \theta_T] \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ E[\gamma_i | S_i = \theta_T, S_j = \theta_1] & \cdots & E[\gamma_i | S_i = \theta_T, S_j = \theta_q] & \cdots & E[\gamma_i | S_i = \theta_T, S_j = \theta_T] \end{pmatrix} \]

The equilibrium is then described in the following theorem.

**Theorem 3.4: BNE with Unknown Social Cost**

The game \( \Gamma_{\gamma} \) has a unique and interior Bayesian Nash equilibrium \( x^\star \), which is given by

\[
x^\star[\theta] = \left( D(\bar{\beta} \otimes 1_T + \gamma[\theta]) - \left( D\gamma[\theta] \bar{G} \right) \otimes J_T \right) \circ \Pi^T \circ \Gamma^{-1} \left( \bar{\alpha} \otimes 1_T \right).
\]

As expected, uncertainty in this case has been shifted to the social conformism parameter \( \gamma \). The above formula, however, includes an additional term compared to the cases with incomplete information about the private payoff parameters that were analysed. Matrix \( \Gamma \) implies that players need to form their expectation about the unknown parameter based not only on their signal, but also considering every possible signal of their peers.

As discussed following the previous results, uncertainty about the private payoff parameters (say, for example, \( \beta_i \)) infiltrates the individuals decision problem through two channels. Firstly, because this parameter is unknown, there is uncertainty concerning the net private returns of the increasing the level of activity; this is true irrespectively of any potential network effects. At the same time though, uncertainty about the value of that parameter gives rise to uncertainty concerning the behaviour of others, and hence the social norm. Notice that these two effects are independent of each other: If the player knew his or her private parameter, the former source of uncertainty would disappear; the latter source of uncertainty too if the social norm that the player faces was known.
This observation, however, does not carry over to the present case. Even if the players' social conformism parameter was known, there would still be uncertainty about the social norm they face. Conversely, even with a known social norm, the social component of the utility function is not known because the social conformism parameter is unknown. Hence, in order to resolve this, players need to form expectation about the that parameter while treating the actions of the others as fixed to a specific values and do this for all possible values that the latter can assume.

3.5 Conclusions

This paper provides a comprehensive theoretical framework for studying the effects of social conformism when some parameters of the model are unknown to the individuals. More specifically, uncertainty enters our analysis through three parameters in the agents’ decision problem: private benefit of an action, private costs, or costs due to socially divergent behaviour. There is a wide range of potential applications of our model, including the study of decisions on education, work effort, the practice of religion, socially delinquent behaviours, and crime. Although the role of social networks in influencing behaviour in these areas has been studied in the recent literature, it would be interesting to revisit their under the light of our findings in this paper, since a game of incomplete information appears to constitute a more natural approach to these decision problems.

We also find that an increase in social pressure can have ambiguous effects on the level of activity in a society, which is in line with findings from the empirical string of the networks literature (Bisin et al., 2011). A reason behind this indeterminacy is that the direction of these effects is contingent on the structure of the network. Based on our analysis, we conjecture that in networks with strongly prevalent homophily, an increase in the social conformism parameter reinforces the existing differences in behaviour among different groups, while in networks with milder homophily it may actually
dampen them. We believe, moreover, that the magnitude of these effects is more likely to be larger under higher uncertainty. Corroborating though these conjectures requires further research.
APPENDIX

3.A Proofs

3.A.1 Some results on matrix analysis

The following statements will be useful in proving some of the theorems and propositions presented in the main part of this paper.

**Lemma 3.1.** Let $A \in \mathbb{R}^{n \times n}$ be a non-negative square matrix, and consider a scalar $c \in \mathbb{R}_+$. Then the matrix $I_n - cA$ is nonsingular with a nonnegative inverse if and only if $c\rho(A) < 1$.

**Proof.** The statement is trivial if $c = 0$. Assume that $c > 0$ in what follows. First, note that matrix $I_n - cA$ is nonsingular with a nonnegative inverse if and only if $(1/c)I_n - A$ is nonsingular with a nonnegative inverse because $c > 0$ and $I_n - cA = c((1/c)I_n - A)$. Second, note that $(1/c)I_n - A$ is an M-matrix if and only if $(1/c) \geq \rho(A)$ (for the definition of M-matrices see, for example, Berman and Plemmons, 1994, Definition 1.2 on p. 133). Third, note that $(1/c)I_n - A$ is singular if $1/c = \rho(A)$; indeed, if $1/c = \rho(A)$, then $(1/c) \in \sigma(A)$ (Proposition B.1). Thus, $(1/c)I_n - A$ is a nonsingular M-matrix if and only if $(1/c) > \rho(A)$. Fourth, note that $(1/c)I_n - A$ is nonsingular with a nonnegative inverse if and only if $(1/c)I_n - A$ is a nonsingular M-matrix (see, for example, Berman and Plemmons, 1994, Theorem 2.3 on pp. 134–138, especially Condition N38 on p. 137). It follows from the foregoing results that $I_n - cA$ is nonsingular with a nonnegative inverse if and only if $1/c > \rho(A)$, or equivalently $c\rho(A) < 1$. ■

**Lemma 3.2.** Matrix $D_{\beta+\gamma} - D_{\gamma}\tilde{G}$ is invertible, with $(D_{\beta+\gamma} - D_{\gamma}\tilde{G})^{-1} \geq O_n$. Moreover, for any vector $y \in \mathbb{R}^{n}_+$, it holds that $(D_{\beta+\gamma} - D_{\gamma}\tilde{G})^{-1}y > 0_n$. 
Proof. The above lemma maintains that matrix $D_{\hat{\beta}+\hat{\gamma}} - D_{\gamma} \bar{G}$ has an inverse that is nonnegative, and has at least one positive element in each row. First, rewrite this matrix as

$$D_{\hat{\beta}+\hat{\gamma}} - D_{\gamma} \bar{G} = \left( \bar{G} - D_{\hat{\beta}+\hat{\gamma}}^{-1} D_{\gamma} \bar{G} \right)^{-1}.$$ 

and define

$$D_{\hat{\beta}+\hat{\gamma}} - D_{\gamma} \bar{G} =: A = \left( a_{i,j} \right)_{(i,j) \in \mathbb{I}^2}$$

Observe now that since the row-normalised adjacency matrix $\bar{G}$ is, by its definition, a stochastic matrix, $D_{\hat{\beta}+\hat{\gamma}}^{-1} D_{\gamma} \bar{G}$ will be sub-stochastic. This is true since $D_{\hat{\beta}+\hat{\gamma}}^{-1} D_{\gamma} \bar{G}$ multiplies all elements in each row $i$ of matrix $\bar{G}$ by $\bar{g}_i / (\hat{\beta}_i + \hat{\gamma}_i) < 1$ (see Section B.A of the Appendix). It will therefore hold that $\sum_{j=1}^{n} a_{i,j} < 1$, which in turn implies that $\rho(A) < 1$ (well-known result in matrix algebra; see, for example, Meyer, 2001, p.498, expression 7.1.13). Then invertibility of matrix $A$, and nonnegativity of it inverse follows from Lemma 3.1.

To prove that $\left( D_{\hat{\beta}+\hat{\gamma}} - D_{\gamma} \bar{G} \right)^{-1} y > 0_n$ for $y > 0_n$, denote

$$\left( D_{\hat{\beta}+\hat{\gamma}} - D_{\gamma} \bar{G} \right)^{-1} y =: C = \left( c_{i,j} \right)_{(i,j) \in \mathbb{I}^2}. \tag{3.3}$$

Observe that since matrix $C$ is invertible, it cannot contain any rows consisting exclusively of zeros. Hence it must be the case that for every $i \in \{1, \ldots, n\}$, there must exist at least one $j \in \{1, \ldots, n\}$ such that $c_{i,j} > 0$. It follows therefore that $\sum_{j=1}^{n} c_{i,j} y_j > 0$ for all $i \in \{1, \ldots, n\}$. This proves the second statement of this lemma. ■

3.A.2 Proof of Theorem 3.1

In the case of complete information, the utility function (3.1) of player $i$ can be written as

$$u_i(x) = \bar{a}_i x_i - \frac{\bar{\beta}_i}{2} x_i^2 - \frac{\bar{\gamma}_i}{2} \left( x_i - \sum_{j=1}^{n} \bar{g}_{i,j} x_j \right)^2.$$
The first-order condition of player \( i \) is

\[
\bar{\alpha}_i - (\bar{\beta}_i + \bar{\gamma}_i) x_i^*(\theta_1) + \bar{\gamma}_i \sum_{j=1}^{n} \bar{g}_{i,j} x_j^*(\theta_1) = 0. \tag{3.4}
\]

It can be easily seen that the second-order condition is satisfied. The first-order conditions of all players can be written in matrix format as follows

\[
\bar{\alpha} - D_{\bar{\beta}+\bar{\gamma}} x^* + D_{\bar{\gamma}} \bar{G} x^*(\theta_1) = 0_n
\]

\[
(D_{\bar{\beta}+\bar{\gamma}} - D_{\bar{\gamma}} \bar{G}) x^*(\theta_1) = \bar{\alpha}
\]

\[
x^*(\theta_1) = (D_{\bar{\beta}+\bar{\gamma}} - D_{\bar{\gamma}} \bar{G})^{-1} \bar{\alpha}, \tag{3.5}
\]

where existence of \((D_{\bar{\beta}+\bar{\gamma}} - D_{\bar{\gamma}} \bar{G})^{-1}\) is guaranteed from Lemma 3.2.

Interiorness of the equilibrium (that is, \( x^* > 0_n \)) follows directly from expression (3.5) and Lemma 3.2.

It remains now to show that this equilibrium is unique. First, notice that the first-order condition (3.4) is necessary for an interior solution, and thus there can be no other interior solutions apart from the one given by expression (3.5). Assume now towards a contradiction that there is a boundary Bayesian Nash equilibrium, \( \bar{x}^*(\theta_1) \geq 0_n \). Without loss of generality, let \( \bar{x}_i^*(\theta_1) = 0 \), for some \( i \in \mathcal{I} \). It follows then from expression (3.4) that the marginal utility of player \( i \) in equilibrium will be \( \mu_i(\bar{x}^*(\theta_1)) = \alpha_i + \gamma_i \sum_{j=1}^{n} \bar{g}_{i,j} \bar{x}_j^*(\theta_1) \). Notice though that \( \mu_i(\bar{x}^*(\theta_1)) > 0 \), since \( \alpha_i > 0 \) and \( \gamma_i \geq 0 \) by definition, and \( \bar{x}_j^*(\theta_1) \geq 0 \) for all \( j \in \mathcal{I} \). This readily suggests that player \( i \) can become better off by increasing his or her action \( x_i \), which contradicts that \( \bar{x}^*(\theta_1) \) constitutes an equilibrium. This concludes the proof of Theorem 3.1.

### 3.A.3 Proof of Corollary 3.1

First-order condition (3.4) can be written as

\[
x_i^*(\theta_1) = \frac{\bar{\alpha}_i}{\bar{\beta}_i + \bar{\gamma}_i} + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} \sum_{j=1}^{n} \bar{g}_{i,j} x_j^*(\theta_1) \]
\[ = \frac{\bar{\alpha}_i}{\bar{\beta}_i + \bar{\gamma}_i} + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} \bar{g}_i x^*(\theta_1). \]

Then, substituting expression (3.5) for \( x^*(\theta_1) \) in the above formula yields

\[ x^*_i(\theta_1) = \frac{\bar{\alpha}_i}{\bar{\beta}_i + \bar{\gamma}_i} + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} \bar{g}_i \left( D_{\bar{\beta}} + \bar{\gamma}^2 G \right)^{-1} \bar{\alpha} \]

demonstrating thus Corollary 3.1.

### 3.A.4 Proof of Proposition 3.1

This proposition is proved with the use of matrix calculus. Recall that from Theorem 3.1, the equilibrium is given by

\[ x^*(\theta_1) = \left( D_{\bar{\beta}} + \bar{\gamma} G \right)^{-1} \bar{\alpha}. \]

To prove the first statement, let matrix \( C \) be defined as in expression (3.3) as in Section 3.A.2, and consider the derivative

\[
\frac{\partial x^*(\theta_1)}{\partial \bar{\alpha}_i} = \left( D_{\bar{\beta}} + \bar{\gamma} G \right)^{-1} \frac{\partial \bar{\alpha}}{\partial \bar{\alpha}_i}
\]

\[
= \left( D_{\bar{\beta}} + \bar{\gamma} G \right)^{-1} \left( \frac{\partial (\bar{\alpha} + \bar{\alpha}_i^T)}{\partial \bar{\alpha}_i} \right)^T
\]

\[
= \left( \begin{array}{ccc}
  c_{1,1} & \cdots & c_{1,n} \\
  \vdots & \ddots & \vdots \\
  c_{i,1} & \cdots & c_{i,n} \\
  \vdots & \ddots & \vdots \\
  c_{n,1} & \cdots & c_{n,n}
\end{array} \right) \left( \begin{array}{c}
  0 \\
  \vdots \\
  1 \\
  \vdots \\
  0
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
  c_{1,i} \\
  \vdots \\
  c_{i,i} \\
  \vdots \\
  c_{n,i}
\end{array} \right) \geq 0_n.
\]
This proves the first statement. The second statement can be proved in similar way:

\[
\frac{\partial x^*(\theta_1)}{\partial \beta_i^l} = \frac{\partial}{\partial \beta_i^l} (D_{\beta+\bar{\gamma}} - D_{\gamma \bar{G}})^{-1} \bar{\alpha}
\]

\[
= - (D_{\beta+\bar{\gamma}} - D_{\gamma \bar{G}})^{-1} \left[ \frac{\partial}{\partial \beta_i^l} (D_{\beta+\bar{\gamma}} - D_{\gamma \bar{G}}) \right] \left( D_{\beta+\bar{\gamma}} - D_{\gamma \bar{G}} \right)^{-1} \bar{\alpha}
\]

\[
= - \left( \begin{array}{ccc}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{array} \right)
\left( \begin{array}{c}
x_1^*(\theta_1) \\
\vdots \\
x_n^*(\theta_1)
\end{array} \right)
\]

Finally, we prove the third statement.

\[
\frac{\partial x^*(\theta_1)}{\partial \gamma_i^l} = \frac{\partial}{\partial \gamma_i^l} (D_{\beta+\bar{\gamma}} - D_{\gamma \bar{G}})^{-1} \bar{\alpha}
\]

\[
= -(D_{\beta+\bar{\gamma}} - D_{\gamma \bar{G}})^{-1} \left[ \frac{\partial}{\partial \gamma_i^l} (D_{\beta+\bar{\gamma}} - D_{\gamma \bar{G}}) \right] \left( D_{\beta+\bar{\gamma}} - D_{\gamma \bar{G}} \right)^{-1} \bar{\alpha}
\]

with the inequality being strict for all agents \( j \in I \) who are direct or indirect neighbours of agent \( i \).
\[
\begin{pmatrix}
  c_{1,1} & \ldots & c_{1,j} & \ldots & c_{1,n} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  c_{i,1} & \ldots & c_{i,j} & \ldots & c_{i,n} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  c_{n,1} & \ldots & c_{n,j} & \ldots & c_{n,n}
\end{pmatrix}
\begin{pmatrix}
  0 & \ldots & 0 & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  -\bar{g}_{i,1} & \ldots & 1 & -\bar{g}_{i,n} \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \ldots & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x^*_1(\theta_1) \\
\vdots \\
x^*_i(\theta_1) \\
\vdots \\
x^*_n(\theta_1)
\end{pmatrix}
= \begin{pmatrix}
  c_{i,1} & \tilde{g}_{i,1} & \ldots & -c_{1,i} & \ldots & c_{1,i} & \tilde{g}_{i,n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_{i,i} & \tilde{g}_{i,1} & \ldots & -c_{i,i} & \ldots & c_{i,i} & \tilde{g}_{i,n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_{n,i} & \tilde{g}_{i,1} & \ldots & -c_{n,i} & \ldots & c_{n,i} & \tilde{g}_{i,n}
\end{pmatrix}
\begin{pmatrix}
x^*_1(\theta_1) \\
\vdots \\
x^*_i(\theta_1) \\
\vdots \\
x^*_n(\theta_1)
\end{pmatrix}
= \begin{pmatrix}
  c_{1,i} \left( \sum_{h\neq i} \tilde{g}_{i,h} x^*_j(\theta_1) - x^*_i(\theta_1) \right) \\
  \vdots \\
  c_{i,i} \left( \sum_{h\neq i} \tilde{g}_{i,h} x^*_j(\theta_1) - x^*_i(\theta_1) \right) \\
  \vdots \\
  c_{n,i} \left( \sum_{h\neq i} \tilde{g}_{i,h} x^*_j(\theta_1) - x^*_i(\theta_1) \right)
\end{pmatrix}
\]

Notice that unless \( c_{j,i} = 0 \), every element \( j \in \mathcal{I} \) of the above vector will be nonnegative (positive) if and only if

\[
    c_{j,i} \left( \sum_{h\neq i} \tilde{g}_{i,h} x^*_j(\theta_1) - x^*_i(\theta_1) \right) \geq 0
\]

\[
    \sum_{h\neq i} \tilde{g}_{i,h} x^*_j(\theta_1) \geq x^*_i(\theta_1)
\]

that is, if and only if agent \( i \)'s effort does not exceed (is lower than) the average of his or her out-neighbours' effort.
3.A.5 Proof of Proposition 3.2

Maintaining the notation of the preceding proof (Section 3.A.4), this proof starts with the first statement in Proposition 3.2.

\[
\frac{\partial x^*(\theta_1)}{\partial \tilde{\alpha}} = \left(D_{\tilde{\beta}+\tilde{\gamma}} - D_{\tilde{\gamma}} \tilde{G} \right)^{-1} \frac{\partial \tilde{\alpha}}{\partial \tilde{\alpha}^G} \\
= \left(D_{\tilde{\beta}+\tilde{\gamma}} - D_{\tilde{\gamma}} \tilde{G} \right)^{-1} \begin{pmatrix}
\frac{\partial (\tilde{\alpha}^G + \tilde{\alpha}_1)}{\partial \tilde{\alpha}^G} & \cdots & \frac{\partial (\tilde{\alpha}^G + \tilde{\alpha}_n)}{\partial \tilde{\alpha}^G}
\end{pmatrix}^\top
\]

\[
= \begin{pmatrix}
\sum_{j\neq i} c_{1,j} \\
\vdots \\
\sum_{j\neq i} c_{i,j} \\
\vdots \\
\sum_{j\neq i} c_{n,j}
\end{pmatrix} > 0_n.
\]

Similarly, for changes in the global cost parameter it holds that

\[
\frac{\partial x^*(\theta_1)}{\partial \tilde{\beta}^G} = \frac{\partial}{\partial \tilde{\beta}^G} \left(D_{\tilde{\beta}+\tilde{\gamma}} - D_{\tilde{\gamma}} \tilde{G} \right)^{-1} \tilde{\alpha}
\]

\[
= -\left(D_{\tilde{\beta}+\tilde{\gamma}} - D_{\tilde{\gamma}} \tilde{G} \right)^{-1} \left[\frac{\partial}{\partial \tilde{\beta}^G} \left(D_{\tilde{\beta}+\tilde{\gamma}} - D_{\tilde{\gamma}} \tilde{G} \right) \right] \left(D_{\tilde{\beta}+\tilde{\gamma}} - D_{\tilde{\gamma}} \tilde{G} \right)^{-1} \tilde{\alpha}
\]

\[
= \begin{pmatrix}
\sum_{j\neq i} c_{1,j} \\
\vdots \\
\sum_{j\neq i} c_{i,j} \\
\vdots \\
\sum_{j\neq i} c_{n,j}
\end{pmatrix} > 0_n.
\]
\[
\begin{align*}
&\begin{pmatrix}
\sum_{j \neq i} c_{1,j} x_j^*(\theta_1) \\
&\vdots
\sum_{j \neq i} c_{i,j} x_j^*(\theta_1) \\
&\vdots
\sum_{j \neq i} c_{n,j} x_j^*(\theta_1)
\end{pmatrix} < 0_n,
\end{align*}
\]
proving thus the second statement.

3.6 Proof of Theorem 3.2

Player \( i \in I \) chooses strategy \( x_i \) in order to maximise his or her expected utility, given the realisation \( \theta_r \) of his or her signal \( S_i \):

\[
E[u_i(x_i(\theta_r)) \mid S_i = \theta_r] = E[\alpha_i \mid S_i = \theta_r] x_i(\theta_r) - \frac{\beta_i}{2} x_i(\theta_r)^2
\]

\[-\frac{\gamma_i}{2} E \left[ \left( x_i(\theta_r) - \sum_{j=1}^n \bar{g}_{i,j} x_j^* \right)^2 \mid S_i = \theta_r \right] \]

The first-order condition of player \( i \) is

\[
E[\alpha_i \mid S_i = \theta_r] x_i^*(\theta_r) - \beta_i x_i^*(\theta_r) - \gamma_i E \left[ x_i(\theta_r) - \sum_{j=1}^n \bar{g}_{i,j} x_j^* \mid S_i = \theta_r \right] = 0. \tag{3.6}
\]

Player \( i \)'s optimal strategy, given signal \( S_i = \theta_r \) is therefore given by

\[
x_i^*(\theta_r) = \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} E \left[ \sum_{j=1}^n \bar{g}_{i,j} x_j^* \mid S_i = \theta_r \right] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]

\[
= \frac{1}{\beta_i + \gamma_i} E[\alpha_i \mid S_i = \theta_r] + \frac{\gamma_i}{\beta_i + \gamma_i} \sum_{q=1}^T \left( \sum_{j=1}^n \bar{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q) \right)
\]
Hence player $i$’s equilibrium strategy can be written as

$$x^*_i(\theta) = \frac{1}{\bar{\beta}_i + \bar{\gamma}_i} \alpha^*_i(\theta) + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} \sum_{j=1}^{n} g_{i,j}(\theta) x^*_j(\theta).$$

or

$$x^*_i(\theta) = \frac{1}{\bar{\beta}_i + \bar{\gamma}_i} \alpha^*_i(\theta) + \frac{\bar{\gamma}_i}{\bar{\beta}_i + \bar{\gamma}_i} \left( g_{i,1} \Pi_{1,i}^T e \ldots g_{i,j} \Pi_{j,i}^T e \ldots g_{i,n} \Pi_{n,i}^T e \right) \begin{pmatrix} x^*_1(\theta) \\ \vdots \\ x^*_j(\theta) \\ \vdots \\ x^*_n(\theta) \end{pmatrix}.$$

Stacking the equilibrium strategies $x^*_i(\theta)$ of all players $i \in \mathcal{I}$ into vector $x^*[\theta]$ gives

$$x^*[\theta] = \left( D_{\bar{\beta} + \bar{\gamma}}^{-1} \otimes I_T \right) \alpha^*[\theta] + \left( D_{\bar{\beta} + \bar{\gamma}}^{-1} \otimes I_T \right) \left( (\mathcal{G} \otimes J_T) \circ \Pi^T \right) x^*[\theta]$$

$$x^*[\theta] = \left( I_T - \left( D_{\bar{\beta} + \bar{\gamma}}^{-1} \otimes I_T \right) \left( (\mathcal{G} \otimes J_T) \circ \Pi^T \right) \right) x^*[\theta] = \left( D_{\bar{\beta} + \bar{\gamma}}^{-1} \otimes I_T \right) \alpha^*[\theta]$$

$$x^*[\theta] = \left( D_{\bar{\beta} + \bar{\gamma}} \otimes I_T - \left( D_{\bar{\beta} + \bar{\gamma}} \otimes I_T \right) \left( (D_{\bar{\beta} + \bar{\gamma}}^{-1} \otimes I_T) (\mathcal{G} \otimes J_T) \circ \Pi^T \right) \right) x^*[\theta]$$

$$x^*[\theta] = \left( D_{\bar{\beta} + \bar{\gamma}} \otimes I_T - \left( D_{\bar{\beta} + \bar{\gamma}} \otimes I_T \right) \left( \mathcal{G} \otimes J_T \circ \Pi^T \right) \right) x^*[\theta]$$

$$x^*[\theta] = \left( D_{\bar{\beta} + \bar{\gamma}} \otimes I_T - \left( (D_{\bar{\beta} + \bar{\gamma}} \otimes I_T) \right) \left( \mathcal{G} \otimes J_T \circ \Pi^T \right) \right) x^*[\theta]$$

where $[\Pi_{j,i}^T x^*_j(\theta)]_r$ is the $r$-th element of the vector resulting from the matrix multiplication of $\Pi_{j,i}^T$ with $x^*_j(\theta)$.
To establish invertibility of matrix
\[ \tilde{A} := \left( I_T - \left( \left( D_{\beta}^{-1} D_{\gamma} \right) \otimes I_T \right) \left( (\hat{G} \otimes J_T) \circ \Pi^T \right) \right) \]

in the third step above, we will show first that matrix \((\hat{G} \otimes J_T) \circ \Pi^T\) is stochastic. Indeed, notice that the \([(i-1)T + r]-th row of this matrix, with \(i \in \{1, \ldots, n\}\) and \(r \in \{1, \ldots, T\}\), has the form
\[
\left( \hat{g}_{i,1} \pi_{1,i}(r) \quad \hat{g}_{i,j} \pi_{j,i}(r) \quad \ldots \quad \hat{g}_{i,n} \pi_{n,i}(r) \right) \in [0, 1]^T,
\]
where
\[
\pi_{j,i}(r) := \left( \mathbb{P}(S_j = \theta_1 | S_i = \theta_r) \cdots \mathbb{P}(S_j = \theta_q | S_i = \theta_r) \cdots \mathbb{P}(S_j = \theta_T | S_i = \theta_r) \right)
\]
is a stochastic vector of order \(T\). Then the sum of the elements of each row \([(i-1)T + r] \in \{1, \ldots, Tn\}\) can be shown to equal 1:
\[
\sum_{j=1}^{n} \hat{g}_{i,j} \sum_{q=1}^{T} \mathbb{P}(S_j = \theta_q | S_i = \theta_r) = \sum_{j=1}^{n} \hat{g}_{i,j} = 1.
\]
Hence invertibility of \(\tilde{A}\) and interiority of the Bayesian Nash equilibrium can be proved by the use of Lemmas 3.1 and 3.2, similarly to invertibility of \(D_{\beta}^{-1} D_{\gamma} - D_{\gamma} \hat{G}\) and interiority of \(x^{\ast}(\theta_1)\) in Section 3.A.2.

Likewise, to establish uniqueness, observe that there can be no other interior solutions. Furthermore, \(x^{\ast}_i(\theta_r) = 0\) cannot be part of any equilibrium strategy of any player, since expression (3.6) implies that the corresponding conditional expected marginal utility would be positive:
\[
\mathbb{E}[\alpha_i | S_i = \theta_r] x_i^{\ast}(\theta_r) + \hat{\gamma}_i \sum_{j=1}^{n} \hat{g}_{i,j} \mathbb{E}[x_j^{\ast} | S_i = \theta_r] > 0.
\]
This completes the proof of Theorem 3.2.
3.A.7 Proof of Corollary 3.2

Notice that
\[ \Pi_{j,i}^\top x_j^*(\theta) = c_i(\theta_r) x^*[\theta]. \]
Then Corollary 3.2 follows directly from expression (3.9) and Theorem 3.2.

3.A.8 Proof of Theorem 3.3

The proof that follows is similar to that of Theorem 3.2. Given the realisation \( \theta_r \) of his or her signal \( S_i \), player \( i \in I \) chooses strategy \( x_i \) in order to maximise his or her expected utility, which is now given by
\[
E\left[ u_i(x_i(\theta_r)) \mid S_i = \theta_r \right] = \bar{\alpha}_i x_i(\theta_r) - \frac{\tilde{\bar{\beta}}_i}{2} x_i(\theta_r)^2 - \frac{\tilde{\gamma}_i}{2} E\left[ \left( x_i(\theta_r) - \sum_{j=1}^n \tilde{g}_{i,j} x_j \right)^2 \mid S_i = \theta_r \right].
\]
The first-order condition of player \( i \) is
\[
\alpha_i x_i^*(\theta_r) - E[ \tilde{\bar{\beta}}_i \mid S_i = \theta_r ] x_i^*(\theta_r) - \tilde{\gamma}_i E\left[ \sum_{j=1}^n \tilde{g}_{i,j} x_j^* \mid S_i = \theta_r \right] = 0.
\]
Player \( i \)'s optimal strategy, given signal \( S_i = \theta_r \) is therefore given by
\[
x_i^*(\theta_r) = \frac{\alpha_i}{E[\tilde{\bar{\beta}}_i \mid S_i = \theta_r] + \tilde{\gamma}_i} + \frac{\tilde{\gamma}_i}{E[\tilde{\bar{\beta}}_i \mid S_i = \theta_r] + \gamma_i} \sum_{j=1}^n \tilde{g}_{i,j} \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x_j^*(\theta_q)
\]
\[
\alpha_i \beta_i = \frac{\beta_i}{E[\beta_i \mid S_i = \theta_r] + \gamma_i} + \gamma_i \sum_{q=1}^{T} \bar{g}_{i,j} \left[ \Pi_{j,i}^T x_j^*(\theta) \right]_r, \quad (3.10)
\]

where \( \left[ \Pi_{j,i}^T x_j^*(\theta) \right]_r \) is the \( r \)-th element of the vector resulting from the matrix multiplication of \( \Pi_{j,i}^T \) with \( x_j^*(\theta) \).

Notice now that since

\[
\Pi \ \text{multiplication of} \ \theta = \frac{1}{E[\beta_i \mid S_i = \theta_1] + \gamma_i} \begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{pmatrix}
\]

player \( i \)'s optimal action for each value of the signal \( S_i \) can be written in vector form as

\[
x_i^*(\theta) = D_{(\beta_i^* + \gamma_i^* \mathbf{1}_T)}^{-1} \alpha_i \mathbf{1}_T + D_{(\beta_i^* + \gamma_i^* \mathbf{1}_T)}^{-1} \gamma_i \sum_{j=1}^{n} \bar{g}_{i,j} \Pi_{j,i}^T x_j^*(\theta)
\]

or

\[
x_i^*(\theta) = \alpha_i D_{(\beta_i^* + \gamma_i^* \mathbf{1}_T)}^{-1} \mathbf{1}_T + \gamma_i D_{(\beta_i^* + \gamma_i^* \mathbf{1}_T)}^{-1} \left( \bar{g}_{i,1} \Pi_{1,i}^T \ldots \bar{g}_{i,i} \Pi_{i,i}^T \ldots \bar{g}_{i,n} \Pi_{n,i}^T \right) x_j^*(\theta).
\]

Stacking the equilibrium strategies \( x_i^*(\theta) \) of all players \( i \in \mathcal{I} \) into vector \( x^*[\theta] \) gives

\[
x^*[\theta] = D_{(\beta^*[\theta] + \bar{\gamma} \otimes \mathbf{1}_T)}^{-1} (\bar{\alpha} \otimes \mathbf{1}_T) + D_{(\beta^*[\theta] + \bar{\gamma} \otimes \mathbf{1}_T)}^{-1} D_{(\bar{\gamma} \otimes \mathbf{1}_n)} \left( (\bar{G} \otimes J_T) \circ \Pi^T \right) x^*[\theta]
\]

\[
= \left[ \mathbf{I}_T - D_{(\beta^*[\theta] + \bar{\gamma} \otimes \mathbf{1}_T)}^{-1} D_{(\bar{\gamma} \otimes \mathbf{1}_n)} \left( (\bar{G} \otimes J_T) \circ \Pi^T \right) \right]^{-1} D_{(\beta^*[\theta] + \bar{\gamma} \otimes \mathbf{1}_T)}^{-1} (\bar{\alpha} \otimes \mathbf{1}_T)
\]

\[
= \left[ D_{(\beta^*[\theta] + \bar{\gamma} \otimes \mathbf{1}_T)} - D_{(\bar{\gamma} \otimes \mathbf{1}_n)} \left( (\bar{G} \otimes J_T) \circ \Pi^T \right) \right]^{-1} (\bar{\alpha} \otimes \mathbf{1}_T)
\]
\[ x^*[\theta] = \left[ D_{\beta'[\theta]} + \tilde{\gamma} \otimes 1_T \right] - \left( D_{\tilde{\gamma}} \otimes I_n \left[ \left( \bar{G} \otimes J_T \right) \circ \Pi^T \right] \right)^{-1} (\bar{\alpha} \otimes 1_T) \]

\[ x^*[\theta] = \left[ D_{\beta'[\theta]} + \tilde{\gamma} \otimes 1_T \right] - \left[ \left( D_{\tilde{\gamma}} \bar{G} \otimes J_T \right) \circ \Pi^T \right]^{-1} (\bar{\alpha} \otimes 1_T). \]

Invertibility of \( I_T - D_{\beta'[\theta]} + \tilde{\gamma} \otimes 1_T \), as well as uniqueness and interiority of the Bayesian Nash equilibrium can be established similarly to the corresponding statements in Theorems 3.1 and 3.2 (see Sections 3.A.2 and 3.A.6), and thus the complete proof need not be reiterated.

3.A.9 Proof of Corollary 3.3

As above, we can use the observation that

\[ \left[ \Pi^{T}_{j,i} x^*_j(\theta) \right]_{r} = c_i(\theta_r) x^*[\theta]. \]

Then Corollary 3.3 follows directly from expression (3.10) and Theorem 3.3.

3.A.10 Proof of Theorem 3.4

Agents observe the realisation \( \theta_r \) of their signal \( S_i \). Assuming that \( \bar{\alpha} \) and \( \bar{\beta} \) are known, player \( i \)'s expected utility is given by

\[ E\left[ u_i(x_i(\theta_r)) \mid S_i = \theta_r \right] = \bar{\alpha}_i x_i(\theta_r) - \frac{\bar{\beta}_i}{2} x_i(\theta_r)^2 \]

\[ - \frac{1}{2} E\left[ y_i \left( x_i(\theta_r) - \sum_{j=1}^{n} \bar{g}_{i,j} x_j(S_j) \right)^2 \mid S_i = \theta_r \right]. \]

Each player \( i \in I \) maximises his or her strategy \( x_i \) to maximise the above function. The corresponding first-order condition is

\[ \bar{\alpha}_i - \bar{\beta}_i x_i(\theta_r) - E\left[ y_i \left( x_i(\theta_r) - \sum_{j=1}^{n} \bar{g}_{i,j} x_j(S_j) \right) \mid S_i = \theta_r \right] = 0 \]

or

\[ \bar{\alpha}_i - \bar{\beta}_i x^*_i(\theta_r) - E[y_i \mid S_i = \theta_r] x^*_i(\theta_r) + E\left[ \sum_{j=1}^{n} \bar{g}_{i,j} y_i x'_i(S_j) \mid S_i = \theta_r \right] = 0. \]
Given signal $S_i = \theta_r$, player $i$’s optimal strategy is given by

$$x^*_i(\theta_r) = \frac{\alpha_i}{\beta_i + \mathbb{E}[\gamma_i | S_i = \theta_r]} + \frac{1}{\beta_i + \mathbb{E}[\gamma_i | S_i = \theta_r]} \sum_{j=1}^n \tilde{g}_{i,j} \mathbb{E} \left[ \gamma_j x^*_j(S_j) \mid S_i = \theta_r \right].$$

(3.11)

Note that social conformity parameter $\gamma_i$ will be, in general, correlated with $x^*_i(S_j)$, so $\mathbb{E}[\gamma_i]$ cannot be factored out of the last term in the above expression. Recall that $\gamma_i : \Omega \rightarrow \mathbb{R}_+$ is an integrable random variable, and $S_i : \Omega \rightarrow \mathbb{R}$ on the same probability space $(\Omega, \mathcal{S}, \mathbb{P})$. Moreover, denote the pre-image of $\theta_q$ on $\Omega$ under random variable $S_j$ with $S_j^{-1}(\theta_q)$. Then, for any $j \neq i$, the last term can be written as

$$\mathbb{E} \left[ \gamma_j x^*_j(S_j) \mid S_i = \theta_r \right] = \sum_{q=1}^T \int_{\omega \in S_j^{-1}(\theta_q)} \gamma_i(\omega) x^*_j(S_j(\omega)) \mathbb{P}(d\gamma_i(\omega) \mid S_i = \theta_r)$$

$$= \sum_{q=1}^T \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) \times$$

$$\int_{\omega \in S_j^{-1}(\theta_q)} x^*_j(\theta_q) \gamma_i(\omega) \frac{\mathbb{P}(d\gamma_i(\omega) \mid S_i = \theta_r)}{\mathbb{P}(S_j = \theta_q \mid S_i = \theta_r)}$$

$$= \sum_{q=1}^T \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) x^*_j(\theta_q) \mathbb{E} \left[ \gamma_i \mid S_i = \theta_r, S_j = \theta_q \right].$$

Substituting this into expression (3.11) gives

$$x^*_i(\theta_r) = \frac{\alpha_i}{\beta_i + \mathbb{E}[\gamma_i | S_i = \theta_r]} + \frac{1}{\beta_i + \mathbb{E}[\gamma_i | S_i = \theta_r]} \times$$

$$\sum_{j=1}^n \tilde{g}_{i,j} \left( \sum_{q=1}^T \mathbb{P}(S_j = \theta_q \mid S_i = \theta_r) \mathbb{E}[\gamma_j \mid S_i = \theta_r, S_j = \theta_q] x^*_j(\theta_q) \right)$$

$$= \frac{\alpha_i}{\beta_i + \mathbb{E}[\gamma_i | S_i = \theta_r]} + \frac{\bar{\gamma}_i}{\beta_i + \mathbb{E}[\gamma_i | S_i = \theta_r]} \sum_{j=1}^n \tilde{g}_{i,j} \left[ (\Pi^T_{j,i} \circ \Gamma_{i,j}) x^*_j(\theta) \right],$$

6 This will not be the case if signals $S$ are completely non-informative (i.e. they are orthogonal to $\gamma$) or if they are uncorrelated with each other (i.e. $\text{Cov}[S_i, S_j] = 0$).
where \( \left( \Pi_{j,i}^T \circ \Gamma_{i,j} \right) \) is the \( r \)-th element of the vector resulting from the matrix product of \( \Pi_{j,i}^T \circ \Gamma_{i,j} \) and \( x^*_j(\theta) \).

Similarly to the proof of Theorem 3.3 in Section 3.A.8, the range of player \( i \)'s equilibrium strategy can be written as a vector

\[
x^*_i(\theta) = D_{\bar{\beta}^T + \gamma^T(\theta)}^{-1} \bar{\alpha}^T I + D_{\bar{\beta}^T + \gamma^T(\theta)}^{-1} \bar{\gamma}^T \sum_{j=1}^{n} \bar{g}_{ij} \left( \Pi_{j,i}^T \circ \Gamma_{i,j} \right) x^*_j(\theta).
\]

Concatenating vectors \( x^*_i(\theta) \) for all players \( i \in I \) into a block vector, we get

\[
x^*[\theta] = D_{\bar{\beta}^T + \gamma^T(\theta)}^{-1} \bar{\alpha}^T I + D_{\bar{\beta}^T + \gamma^T(\theta)}^{-1} \bar{\gamma}^T \left[ \left( \bar{G} \otimes J^T \right) \circ \Pi^T \circ \Gamma \right] x^*[\theta].
\]

Working now as in Section 3.A.8, we can solve for \( x^*[\theta] \) getting

\[
x^*[\theta] = \left[ D_{\bar{\beta}^T + \gamma^T(\theta)} - \left( D_{\gamma^T(\theta)} \bar{G} \right) \right] \left( \bar{\alpha}^T I \right)^{-1} \left[ \left( \bar{G} \otimes J^T \right) \circ \Pi^T \circ \Gamma \right]^{-1} x^*[\theta].
\]

Finally, following the methodology that we used in the proofs of Theorems 3.1 and 3.2 (Sections 3.A.2 and 3.A.6 of the Appendix) we can prove that this equilibrium is unique and interior.
3.B Notation

This is a list of some frequently used symbols and their definitions.

**Network game**

$I$ set of all players

$\alpha_i$ player $i$'s marginal benefit parameter

$\beta_i$ player $i$'s private cost parameter

$\gamma_i$ player $i$'s social conformism parameter (strength of social norm)

$S_i$ player $i$'s signal

$\Theta$ the set of all types $\{\theta_1, \ldots, \theta_T\}$

$\theta$ the vector of all types $(\theta_1, \ldots, \theta_T)$

$\vartheta$ a type profile $(\vartheta_1, \ldots, \vartheta_n)$

**Graph theory**

$G$ the network describing players' interactions

$D_G^+(i)$ out-neighborhood of player $i$ in network $G$

$\text{deg}_G^+(i)$ out-degree of player $i$ in network $G$

$G$ adjacency matrix of network $G$

**Game parameter vectors**

$\alpha^c_i(\theta)$ vector $\left( E[\alpha_i | S_i = \theta_1], \ldots, E[\alpha_i | S_i = \theta_T] \right)^\top$

$\beta^c_i(\theta)$ vector $\left( E[\beta_i | S_i = \theta_1], \ldots, E[\beta_i | S_i = \theta_T] \right)^\top$

$\gamma^c_i(\theta)$ vector $\left( E[\gamma_i | S_i = \theta_1], \ldots, E[\gamma_i | S_i = \theta_T] \right)^\top$

$\alpha^c[\theta]$ block vector $\left( \alpha^c_1(\theta)^\top, \ldots, \alpha^c_n(\theta)^\top \right)^\top$

$\beta^c[\theta]$ block vector $\left( \beta^c_1(\theta)^\top, \ldots, \beta^c_n(\theta)^\top \right)^\top$

$\gamma^c[\theta]$ block vector $\left( \gamma^c_1(\theta)^\top, \ldots, \gamma^c_n(\theta)^\top \right)^\top$

**Matrix algebra and special matrices**

$0_n$ zero (column) vector in $\mathbb{R}^n$

$I_n$ identity matrix of order $n$

$O_n$ $n$-dimensional square matrix of zeros
$1_n$  (column) vector of ones in $\mathbb{R}^n$

$J_n$  $n$-dimensional square matrix of ones

$\rho(A)$  the spectral radius of square matrix $A$

$\sigma(A)$  the spectrum of square matrix $A$
Appendices
Networks appendix: basic concepts and theory

A.A Basic terms

This section defines formally some basic terminology from network theory that are commonly used in this thesis. More specialised terms are introduced in the relevant chapters.

**Definition A.1: Basic Networks Terminology**

- A *directed walk* in a network $G$ is a sequence of (potentially repeated) nodes that are sequentially connected via directed links. Formally, it is a sequence of nodes $\langle i_1, i_2, \ldots, i_{H-1}, i_H \rangle$ in $G$ such that $g_{i_h i_{h+1}} = 1$ for all $h \in \{1, 2, \ldots, H-1\}$. 
• A directed path from node $i_1 \in \mathcal{N}$ to another node $i_H \in \mathcal{N}$ in a network $\mathcal{G}$ is a directed walk consisting of distinct nodes. Formally, it is a sequence of nodes $(i_1, i_2, \ldots, i_{H-1}, i_H)$ in $\mathcal{G}$, with $i_k \neq i_l$ for $k \neq l$, $k, l \in \{1, 2, \ldots, H\}$, such that $g_{i_h, i_{h+1}} = 1$ for all $h \in \{1, 2, \ldots, H-1\}$.

• A simple cycle of length $H$ in a network $\mathcal{G}$ is a closed walk consisting of $H$ distinct nodes. Formally, it is a sequence of nodes $(i_1, i_2, \ldots, i_{H-1}, i_H)$ in $\mathcal{G}$ such that $g_{i_h, i_{h+1}} = 1$ for $h \in \{1, 2, \ldots, H\}$, with $i_1 = i_H$ and $i_k \neq i_l$ for all other $k, l \in \{1, 2, \ldots, H\}$ with $k \neq l$.

• A network $\mathcal{G}$ is said to be strongly connected if there exists a directed path from any node to any other node in $\mathcal{G}$.

• The period of a strongly connected network $\mathcal{G}$ is defined as the greatest common divisor of the lengths of its simple cycles.

• A strongly connected network is called aperiodic if its period is equal to 1, otherwise it is called periodic.

### A.B Non-negative matrices and networks

The following lemma summarises some well-known results establishing a relationship between the properties of an adjacency matrix and the structure of the network it represents. A rigorous proof is provided for the first statement.

**Lemma A.1.** The following statements are true for any network $\mathcal{G}$.

• A network $\mathcal{G}$ is strongly connected if and only if its adjacency matrix $\mathbf{G}$ is irreducible.

• A strongly connected network $\mathcal{G}$ is aperiodic if and only if its adjacency matrix $\mathbf{G}$ is primitive.
**Proof.** To prove the first statement, it suffices to show that its contrapositive holds true. In other words, it is equivalent to proving the following:

**Statement [CP].** A network \( G = (\mathcal{N}, \mathcal{E}) \) is not strongly connected if and only if its adjacency matrix \( G \) is reducible.

First, notice that pre- and post-multiplying a square matrix respectively by a permutation matrix \( P \) and its transpose \( P^\top \), reorders the rows and the columns of matrix in the same way, preserving thus the structure of the network described by \( G \). Hence the transformation

\[
\tilde{G} := P^\top G P
\]  
(A.1)

simply relabels the agents in \( \mathcal{N} \) without essentially changing the structure of the network; matrix \( \tilde{G} \) represents the same network as \( G \), but with the agents relabelled. More formally, transformation (A.1) can be seen as applying a bijection \( f_N : \mathcal{N} \rightarrow \mathcal{N} \) from the set of nodes to itself, and a corresponding bijection \( f_E : \mathcal{N}^2 \rightarrow \mathcal{N}^2 \) with \( f_E(i,j) := (f_N(i), f_N(j)) \) from the set of edges of \( G^c \) to itself. Except for the node labels, network \( \tilde{G}^G = (f_N(\mathcal{N}), f_E(\mathcal{E}^c)) \), represented by adjacency matrix \( \tilde{G} \), will be identical to \( G^G \). An equivalent, therefore, to Statement [CP] is that network \( \tilde{G}^G \) is not strongly connected if and only if its adjacency matrix \( G \) is reducible. For convenience, denote the new labels of the nodes of the transformed network with \( l_i \), so that \( f_N(\mathcal{N}) = \{l_1, l_2, \ldots, l_n\} \).

Notice that if \( G \) is reducible, then by (B.3) matrix \( P \) can chosen so that

\[
\tilde{G} = \begin{bmatrix}
V_{k \times k} & X_{k \times (n-k)} \\
O_{(n-k) \times k} & Y_{(n-k) \times (n-k)}
\end{bmatrix}
\]

It can now easily be seen from \( \tilde{G} \) that no directed path exists from any of the nodes in \( \mathcal{N}_{\text{isol}} := \{l_{k+1}, l_{k+2}, \ldots, l_n\} \) to any of the nodes in \( \mathcal{N}_{\text{main}} := \{l_1, l_2, \ldots, l_k\} \), since the former group pay attention only to agents within \( \mathcal{N}_{\text{isol}} \). This suggests that network \( \tilde{G}^G \), and hence network \( G^G \), are not strongly connected. This proves the *if* part of Statement [CP].
To prove the only if part of Statement [CP], assume that network $G^G$ is not strongly connected. Then there must exist (at least) two nodes $i, j \in \mathcal{N}$ such that there are no directed paths from node $i$ to node $j$. Consider the set of nodes $\mathcal{M}(j) \subset \mathcal{N}$ consisting of those and only those nodes $h \in \mathcal{N}$ such that there is a directed path from node $h$ to node $j$, and denote its cardinality with $m$, where $1 \leq m \leq n-1$. Since there is no directed path from node $i$ to node $j$, it must be that $i \in \mathcal{N} \setminus \mathcal{M}$. Now consider a transformation similar to (A.1) that assigns labels from $l_1$ to $l_m$ to the nodes in $\mathcal{M}(j)$. This can be implemented through a bijection $f_N$ such as the one described above, with $f_N(\mathcal{M}) = \{l_1, \ldots, l_m\}$ and $f_N(\mathcal{N} \setminus \mathcal{M}) = \{l_{m+1}, \ldots, l_n\}$, together with the corresponding bijection $f_E$ for the edges. Then the transformed matrix can be written as

$$\tilde{G} = P^T G P = \begin{bmatrix}
\tilde{V}_{m \times m} & \tilde{X}_{m \times (n-m)} \\
\tilde{W}_{(n-m) \times m} & \tilde{Y}_{(n-m) \times (n-m)}
\end{bmatrix}$$

where block $\tilde{V}$ captures the edges among the nodes in $\mathcal{M}$, block $\tilde{Y}$ the edges among the nodes in $\mathcal{N} \setminus \mathcal{M}$, block $\tilde{X}$ the edges from nodes in $\mathcal{M}$ to nodes in $\mathcal{N} \setminus \mathcal{M}$, and block $\tilde{W}$ the edges from nodes in $\mathcal{N} \setminus \mathcal{M}$ to nodes in $\mathcal{M}$. Observe, however, that there should not exist any edges from nodes outside $\mathcal{M}$ towards nodes in $\mathcal{M}$. Suppose towards a contradiction that there exists such an edge, emanating without loss of generality from node $q \in \mathcal{N} \setminus \mathcal{M}$. This would imply that there is a directed path from node $q$ to node $j$, and hence $q \in \mathcal{M}$ by the definition of $\mathcal{M}$. Since no such edges exists, it must be that $\tilde{W} = \mathbf{O}_{(n-m) \times m}$, suggesting that $G$ is a reducible matrix. This completes the proof of the first statement in Lemma A.1.

The second statement follows directly from Proposition 1.2 in Neufeld (1996) and the definition of an aperiodic network (see Definition A.1). The statement is presented in the same form as in Lemma 2 in Golub and Jackson (2010). ■

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1 Notice that $\mathcal{M}(j)$ will be non-empty since $j \in \mathcal{M}(j)$. 
B

Mathematical appendix

B.A The Hadamard product

This section discusses shortly the Hadamard product matrix operation and some basic results related to it that are used in the present analysis.

**Definition B.1: The Hadamard Product**

Consider matrices $A = [a_{ij}] \in \mathbb{C}^{m \times n}$ and $B = [b_{ij}] \in \mathbb{C}^{m \times n}$ with $m, n \in \mathbb{N}$. The *Hadamard product*\(^1\) of $A$ and $B$, denoted with $A \circ B$, is defined as the matrix of the scalar products of their corresponding elements

$$A \circ B := [a_{ij} b_{ij}]_{(i,j) \in \mathcal{M} \times \mathcal{N}} \in \mathbb{C}^{m \times n}$$

where $\mathcal{M} := \{1, \ldots, m\}$ and $\mathcal{N} := \{1, \ldots, n\}$.
Let \( A, B, C \in \mathbb{C}^{m \times n} \), and consider a conformable matrix of ones, \( J_{m \times n} := 1_m 1_n^T \), and a scalar \( \kappa \in \mathbb{C} \). The Hadamard product possesses the following properties:

\[
\begin{align*}
[B.H.1] \quad \text{Commutativity:} & \quad A \circ B = B \circ A \\
[B.H.2] \quad \text{Associativity:} & \quad A \circ (B \circ C) = (A \circ B) \circ C \\
[B.H.3] \quad \text{Distributivity:} & \quad A \circ (B + C) = (A \circ B) + (A \circ C) \\
[B.H.4] \quad \text{Identity element} J: & \quad A \circ J_{m \times n} = A \\
[B.H.5] \quad \text{Distributive transposition:} & \quad (A \circ B)^T = A^T \circ B^T \\
[B.H.6] \quad \text{Compatibility with scalar multiplication:} & \quad \kappa (A \circ B) = (\kappa A) \circ B = A \circ (\kappa B)
\end{align*}
\]

In addition, consider vectors \( x \in \mathbb{C}^m, y \in \mathbb{C}^n \), and define the diagonal matrices \( D_x := \text{diag}(x_1, \ldots, x_m) \) and \( D_y := \text{diag}(y_1, \ldots, y_n) \). The following observations will be particularly useful in our analysis:

- **Pre-multiplying** matrix \( A \) by a conformable diagonal matrix \( D_x \) multiplies every element of each row \( i \) of \( A \) by the corresponding element \( x_i \) of vector \( x \), that is
  \[
  D_x A = [a_i x_i]_{i \in M, j \in N} \in \mathbb{C}^{m \times n}.
  \]  
  \[
  \text{(B.1)}
  \]

- **Post-multiplying** matrix \( A \) by a conformable diagonal matrix \( D_y \) multiplies every element of each column \( j \) of \( A \) by the corresponding element \( y_j \) of vector \( y \), that is
  \[
  A D_y = [a_{ij} y_j]_{i \in M, j \in N'} \in \mathbb{C}^{m \times n}.
  \]  
  \[
  \text{(B.2)}
  \]

The above observations give rise to the following properties:

\[
[B.H.7] \quad [\text{Multiply row } i \text{ by } x_i] \quad D_x A = A \circ (x1_m^T) = \left[ \left( x1_m^T \right) \circ I_m \right] A
\]

\[\text{1 Named after French mathematician Jacques Salomon Hadamard (1865-1963). The terms elementwise, entrywise, or Schur product are also encountered in the literature.}\]
Finally, the results below are used in the proofs in the Appendices of Chapters 1, 2, and 3.

\[ D_x(A \circ B)D_y = (D_x A) \circ (B D_y) = (A D_y) \circ (D_x B) = A \circ (D_x B D_y) \]

A proof of the last two statements can be found in Horn and Johnson (1991). More specifically, property [B.H.9] is Lemma 5.1.2 in Chapter 5, while property [B.H.10] follows immediately from Lemma 5.1.3 and the definition of Hadamard product.

**B.B Theory of non-negative matrices**

For the sake of convenience, some basic elements of linear algebra theory are presented below.

**Definition B.2: Irreducible and Primitive Matrices**

- A matrix \( P \in \{0,1\}^{n \times n} \) is called a permutation matrix if in each row and in each column there exists exactly one entry equal to 1, with all other entries being equal to 0.
- A square matrix \( A \in \mathbb{C}^{n \times n} \) is said to be a reducible matrix if there exists a permutation matrix \( P \) such that
  \[
P^T A P = \begin{bmatrix}
  V_{k \times k} & X_{k \times (n-k)} \\
  O_{(n-k) \times k} & Y_{(n-k) \times (n-k)}
  \end{bmatrix}
  \]  
  \[
  (B.3)
  \]
  where \( V, Y \) are square matrices, and \( O \) is a matrix of zeros.
- A square matrix is called irreducible if it is not reducible.
- A non-negative irreducible matrix is said to be a primitive matrix if only one of its eigenvalues lies on its spectral circle.
Note that sometimes, especially in older literature, the terms *indecomposable* and *regular* are encountered respectively instead of irreducible and primitive.

**B.B.1 On irreducible matrices**

The following statement of the Perron–Frobenius theorem\(^2\) is based on Meyer (2001), Chapter 8.3, and has been adapted to the context of the present paper.

**Proposition B.1: The Perron–Frobenius Theorem**

Let \( G \in \mathbb{R}^{n \times n}_+ \) be a non-negative, irreducible square matrix, and denote its spectral radius with \( \rho_G \). The following statements hold true.

[B.PF.1] There exists a simple eigenvalue of \( G \) equal to \( \rho_G \).

[B.PF.2] There exists a positive stochastic eigenvector corresponding to \( \rho_G \), that is, a vector \( p > 0_n \) such that \( Gp = \rho_Gp \) and \( \|p\|_1 = 1 \). This is called the Perron vector of \( G \).

[B.PF.3] The Perron vector is the only non-negative eigenvector of \( G \) up to a positive multiple.

The following formula that appears in Meyer (2001) is used in some of the proofs in main part of the text. We also provide a proof.

**Lemma B.1.** Let \( G \) be a strongly connected, aperiodic network with adjacency matrix \( G \). Denote its spectral radius by \( \rho_G \), and its Perron vector by \( p \). Then

\[
\lim_{t \to +\infty} \left( \frac{G}{\rho_G} \right)^t = \frac{pc^T}{c^Tp}. \tag{B.4}
\]

\(^2\) A first version of the theorem, applying to positive matrices, was proved by German mathematician Oskar Perron in 1907. Five years later his colleague Ferdinand Georg Frobenius showed that most of Perron’s results carry over to non-negative matrices, provided that they are irreducible.
Proof. We know that a network $G$ is aperiodic, and hence strongly connected, if and only if its adjacency matrix $G$ is primitive, and hence irreducible (Lemma A.1). Recall that eigenvector centrality is defined as the left eigenvector of $G$ (Definition 1.1). Then (B.4) follows directly from the theorem on primitive matrices and expression (8.3.10) in Meyer (2001).

B.B.2 On reducible matrices

In many cases, it is easier to work with the canonical form of an adjacency matrix of a network instead of the adjacency matrix per se.

**Definition B.3: Matrix Canonical Form**

Let $A \in \mathbb{C}^{n \times n}$ be a reducible matrix, and let $\mathcal{P}(n)$ denote the set of all permutation matrices of order $n$. Consider a series of successive symmetric permutations $P_k A P_k^T$ of matrix $A$, with $P_k \in \mathcal{P}(n)$, such that

$$A^c := (P_k \cdots P_\ell) A (P_k \cdots P_\ell)^T$$

where every diagonal block $V_{hh}$, for $h \in \{1, \ldots, v\}$, is either irreducible or a zero scalar, and every $Y_q$, for $q \in \{1, \ldots, \chi\}$, is irreducible. Then matrix $A^c$ shall be called the *canonical form* of matrix $A$.

A useful observation is that all adjacency matrices that share a common
canonical form represent essentially the same network (with the labels of their nodes and edges permuted according to the respective symmetric permutation used to obtain the canonical form).

B.C The Silverman–Toeplitz theorem

The following version of the Silverman–Toeplitz theorem is based on Theorems 1 and 2 in Ruder (1966), adapted to the context of the present paper.

Proposition B.2: The Silverman–Toeplitz Theorem

Consider the double sequence \( \{ \tilde{v}_T(t) \}_{T \in \mathbb{N}, t \in \mathbb{N}} \), and the complex-valued sequence \( \{ w(t) \}_{t \in \mathbb{N}} \). The sequence

\[
\left\{ \sum_{t=1}^{T} \tilde{v}_T(t) w(t) \right\}_{T \in \mathbb{N}}
\]

converges whenever \( \{ \tilde{v}_T(t) \}_{T \in \mathbb{N}, t \in \mathbb{N}} \) converges if and only if

[\text{B.ST.1}] \quad \sum_{t=1}^{+\infty} |\tilde{v}_T(t)| \leq \bar{b} \text{ for some } \bar{b} \in \mathbb{R}_+, \text{ for all } T \in \mathbb{N},

[\text{B.ST.2}] \quad \lim_{T \to +\infty} \tilde{v}_T(t) = v_t^{(\infty)} \in \mathbb{C} \text{ for all } t \in \mathbb{N},

[\text{B.ST.3}] \quad \lim_{T \to +\infty} \sum_{t=1}^{T} \tilde{v}_T(t) = s_v \in \mathbb{C}.

Moreover,

\[
\lim_{T \to +\infty} \sum_{t=1}^{T} \tilde{v}_T(t) w(t) = \lim_{t \to +\infty} w(t)
\]

if and only if \( s_v = 1 \) and \( v_t^{(\infty)} = 0 \) for all \( t \in \mathbb{N} \).
Bibliography


