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Cooking up model structures on ind- and pro-categories

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Abstract

This licentiate thesis consists of three papers related to model structures on ind- and pro-categories.

In Paper I, a general method for constructing simplicial model structures on ind- and pro-categories is described. This method is particularly useful for constructing "profinite" analogues of known model categories. The construction quickly recovers Morel's model structure for pro-p spaces and Quick's model structure for profinite spaces, but it can also be applied to construct many interesting new model structures. In addition, some general properties of this method are studied, such as its functorial behaviour and its relation to Bousfield localization. The construction is compared to the ∞-categorical approach to ind- and pro-categories in an appendix.

In Paper II, it is shown that a profinite completion functor for (simplicial or topological) operads with good homotopical properties can be constructed as a left Quillen functor from an appropriate model category of ∞-operads to a certain model category of profinite ∞-operads. The method for constructing this model category of profinite ∞-operads and the profinite completion functor is similar to the method described in Paper I, but there are a few subtle differences that make this construction more involved. In understanding the model structure for profinite ∞-operads, an important role is played by the so-called lean ∞-operads. It is shown that these lean ∞-operads can, up to homotopy, be characterized by certain homotopical finiteness properties. Several variants of the construction are also discussed, such as the cases of unital (or closed) ∞-operads and of ∞-categories.

In Paper III, the general method from Paper I is used to give an alternative proof of one of the main results of [ABS21]. This result states that the stabilization of the category of noncommutative CW-complexes can be modelled as the category of spectral presheaves on a certain category

$\mathcal{M}$. The advantage of this alternative proof is that it mainly relies on well-known results on (stable) model categories.

Sammanfattning

Denna licentiatavhandling består av tre artiklar, som är relaterade till modellstrukturer på ind- och pro-kategorier.

I Artikel I beskrivs en allmän metod för att konstruera simpliciella modellstrukturer på ind- och pro-kategorier. Denna metod är särskilt användbar för att konstruera "pro-ändliga" analoger av kända modellkategorier. Med konstruktionen återfås snabbt Morels modellstruktur för pro-p rum och Quicks modellstruktur för pro-ändliga rum, men den kan också tillämpas för att konstruera många intressanta nya modellstrukturer. Dessutom studeras några allmänna egenskaper hos denna metod, såsom kompatibilitet med Quillen-funktorer och Bousfield-lokalisering. Konstruktionen jämförs med det ∞-kategoriska tillvägagångssättet för ind- och pro-kategorier i en appendix.

I Artikel II visas att en pro-ändlig kompletteringsfunktor för (simpliciella eller topologiska) operader med goda homotopiska egenskaper kan konstrueras som en vänster-Quillen-funktor från en lämplig modellkategori av ∞-operader till en viss modellkategori av pro-ändliga ∞-operader. Metoden för att konstruera denna modellkategori av pro-ändliga ∞-operader och den pro-ändliga kompletteringsfunktorn liknar metoden som beskrivs i Artikel I, men det finns några subtila skillnader som gör denna konstruktion mer komplicerad. För att förstå modellstrukturen för pro-ändliga ∞-operader spelar de så kallades "magra" ∞-operaderna en viktig roll. Det visas att dessa magra ∞-operader, upp till homotopi, kan karaktäriseras av vissa homotopiska ändlighetsegenskaper. Flera varianter av konstruktionen diskuteras också, såsom fall av stängda ∞-operader och ∞-kategorier.

I Artikel III används den allmänna metoden från Artikel I för att ge ett alternativt bevis av ett av huvudresultaten från [ABS21]. Detta resultat säger att stabiliseringen av kategorin av icke-kommutativa CW-komplex kan modelleras som kategorin av spektrala förkärvar på en viss kategori

$\mathcal{M}$. Fördelen med detta alternativa bevis är att det huvudsakligen bygger på välkända resultat om (stabila) modellkategorier.

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Furthemore, I wish to thank everyone who has been part of the topology group in Stockholm during the past two and a half years. I have learned a lot from all the seminars, reading groups and discussions, even during the pandemic when contact was a bit more scarce.

Finally, I would like to express my gratitude to Rolinde for always supporting me and for letting me work from her place, where most of this thesis was written. Even during the past one and a half years when times were particularly rough, she was always there for me.

List of papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

Paper I: **Simplicial model structures on pro-categories**
Thomas Blom, Ieke Moerdijk, *Submitted*.

Paper II: **Profinite ∞-operads**
Thomas Blom, Ieke Moerdijk

Paper III: **A note on noncommutative CW-spectra**
Thomas Blom

Contents

General introduction

This thesis consists of three papers concerned with constructing and studying model structures on ind- and pro-categories. This means that, by the very nature of this subject, these papers are on the somewhat technical side. In particular, the readers of these papers are assumed to at least have a good working knowledge of model categories and of ind- and pro-categories. The goal of this general introduction is twofold: first, to informally introduce model categories and ind- and pro-categories, so that the reader should be able to read the summaries of the three included papers and knows where to find literature containing the required background to understand the actual papers, and, second, to motivate why one should be interested model structures on ind- and pro-categories in the first place. The more advanced reader who is already familiar with model categories and ind- and pro-categories may wish to skip this general introduction or only glimpse over it, as it most probably does not contain anything that is new to them.

It is assumed that the reader of this general introduction has a basic background in algebraic topology and category theory; for example, basic knowledge of concepts such as CW-complexes, (weak) homotopy equivalences, simplicial sets, categorical (co)limits and enriched categories is assumed.

1 Model categories

The aim of this part is to motivate the definition of a model category and provide a few examples. Motivated by these examples, the related notions of "cofibrantly generated" and "simplicial" model categories are described, which play an important role in all three papers contained in this thesis.

There are many excellent resources for those readers who are interested in learning more on model categories, of which we wish to single out the textbooks [Hov99] and [Hir03, Part II].

1.1 What are they good for?

A situation that an algebraic topologist commonly finds himself in is the following: One is interested in a certain class of mathematical objects which, together with a notion of “morphism” between them, form a category satisfying many convenient properties. However, the notion of “equivalence” between objects induced by the isomorphisms is too fine to suit the algebraic topologist’s taste: there exist morphisms between objects that make these objects “equivalent” in some sense, but that are not isomorphisms. The archetypal example is the category of topological spaces, where one only cares spaces up to (weak) homotopy equivalences. In general, it is quite hard to work with the situation where one is given a category together with a class of morphisms that are in some sense “equivalences”, but that are not necessarily isomorphisms. One can try to add formal inverses for these morphisms to construct a new category in which all these “equivalences” become isomorphisms (the *homotopy category*), but this process is very hard to control and one often loses valuable information along the way. For these reasons it is often preferable to work in the original category, but this requires tools that help in dealing with weak equivalences. In the category of topological spaces, fibrations and cofibrations provide such tools. It is exactly this type of structure that the definition of a model category captures: it is a category equipped with three classes of morphisms, called fibrations, cofibrations and weak equivalences, that satisfy properties analogous to those satisfied by the identically named maps between topological spaces (cf. Definition 1.1 below). Many common constructions used in algebraic topology carry over to arbitrary model categories: for example, one can define cylinder objects, path objects and homotopies between maps, but also more sophisticated constructions such as homotopy (co)limits and mapping spaces. Having such constructions available is helpful for showing that objects with certain favourable properties exist and it can aid computations.

1.2 The definition

For completeness’ sake, we include the definition of a model category as given in [Hir03, Definition 7.1.3].

Definition 1.1. Let $\mathcal{M}$ be a complete and cocomplete category. A *model structure* on $\mathcal{M}$ consists of three classes of maps, called *weak equivalences* (denoted $\xrightarrow{\sim}$), *cofibrations* (denoted $\rightarrowtail$) and *fibrations* (denoted $\twoheadrightarrow$), such that the following are satisfied:

(1) If f and g are composable maps in $\mathcal{M}$ and two of f, g and gf are weak equivalences, then so is the third.

(2) If f is a map in $\mathcal{M}$ that is a weak equivalences, a fibration or a cofibration, then so is any retract of f. (For the definition of a retract of a map, see e.g. [Hir03, Definition 7.1.1].)

(3) Given a commutative square

$$\begin{array}{ccc} A & \longrightarrow & L \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & K \end{array} \tag{1.1}$$

in $\mathcal{M}$, a lift exists if either

- i is a cofibration and p is a *trivial fibration* (i.e., a fibration that is also a weak equivalence) or
- i is a *trivial cofibration* (i.e., a cofibration that is also a weak equivalence) and p is a fibration.

(4) Every map $g\colon X \to Y$ in $\mathcal{M}$ admits a functorial factorization $X \rightarrowtail Z \overset{\sim}{\twoheadrightarrow} Y$ into a cofibration followed by a trivial fibration and a functorial factorization $X \overset{\sim}{\rightarrowtail} W \twoheadrightarrow Y$ into a trivial cofibration followed by a fibration. (For the definition of a *functorial* factorization, see e.g. [Rie14, Definition 12.1.1]).

A complete and cocomplete category equipped with a model structure is called a *model category*. An object X in a model category is called *fibrant* if the map $X \to *$ is a fibration, and *cofibrant* if the map $\varnothing \to X$ is a cofibration.

The definition of a model category is evidently self dual; more precisely, if $\mathcal{M}$ is a model category, then the opposite category $\mathcal{M}^{op}$ can be endowed with a model structure with the same weak equivalences, but with the classes of fibration and cofibrations interchanged.

We will say that a map $i\colon A \to B$ has the *left lifting property* with respect to some map $p\colon L \to K$ and that p has the *right lifting property* with respect to i if for any commutative square as in (1.1), a lift exists. In particular, item (3) of the definition of a model category can be rephrased as stating if a map i is a cofibration, then it has the left lifting property with respect to every trivial fibration, and that if a map p is fibration, then it has the right lifting property with respect to every trivial cofibration.

It turns out that the converse holds as well: a map that has the left lifting property with respect to all trivial fibration is a cofibration, and a map that has the right lifting property with respect to all trivial cofibrations is a fibration. This can be proved using the so-called retract argument, see e.g. [Hir03, Proposition 7.2.3]. In particular, if in a model category the class of weak equivalences and either the class of cofibrations or the class of fibrations are known, then the third class of maps is fully determined. In fact, in many examples of model categories, the cofibrations or fibrations are actually defined as the maps that have the left lifting property with respect to all trivial fibrations or as the maps that have the right lifting property with respect to all trivial cofibrations, respectively. It is worth pointing out that the cofibrations and the fibrations together also determine the weak equivalence in a model category (cf. [Hir03, Proposition 7.2.7]).

Example 1.2. Finally, let us mention a few examples

(i) ([Qui67, §3]) The category of topological spaces admits a model structure in which the fibrations are the Serre fibrations, the weak equivalences are the weak homotopy equivalences and the cofibrations are the maps that have the left lifting property with respect to all trivial fibrations. One can show that the cofibrant objects are exactly the retracts of CW-complexes, while all objects are fibrant. This model structure is sometimes called the *Quillen model structure*, to distinguish it from other model structures on the category of topological spaces.

(ii) ([Qui67, §3]) The category of simplicial sets admits the *Kan-Quillen model structure* in which the fibrations are the Kan fibrations, the cofibrations the monomorphisms and the weak equivalences the maps of simplicial sets that become a (weak) homotopy equivalence after geometric realization. In particular, every object is cofibrant and the fibrant objects are the Kan complexes. This model category can be shown to be equivalent, in an appropriate sense, to the model category of topological spaces mentioned above.

(iii) ([Joy08, §2], [Lur09, §2.2.5]) The category of simplicial sets can also be endowed with the *Joyal model structure*. In this model structure the cofibrations are again the monomorphisms, but the fibrations and weak equivalences are hard to describe explicitly. Its interest lies in the fact that its fibrant objects are exactly the *quasi-categories*, which serve as a model the theory of $(\infty, 1)$-categories. It can be shown that the weak equivalences between quasi-categories are (an $(\infty, 1)$-categorical generalization of) the essentially surjective and fully faithful functors.

1.3 (Co)fibrant generation

The classes of cofibrations of the model structures described in Example 1.2 all have something in common: they are all (retracts of) maps that are obtained by repeatedly attaching "cells". More precisely, any monomorphism of simplicial sets can be obtained by attaching standard simplices $\Delta[n]$ along their boundary $\partial\Delta[n]$, while any cofibration of topological spaces is a retract of a map that can be obtained by attaching discs D^n along their boundary S^{n-1}. The idea that cofibrations are obtained as (retracts of) repeated cell attachments in formalized in the definition of a *cofibrantly generated model category*. Roughly speaking, a model category is cofibrantly generated if there exist a set of cofibrations $\mathcal{I}$ and a set of trivial cofibrations $\mathcal{J}$ such that a map is a fibration or a trivial fibration if and only if it has the right lifting property with respect to all maps in the set $\mathcal{J}$ or all maps in the set $\mathcal{I}$, respectively, and if the domains of all maps in $\mathcal{I}$ and $\mathcal{J}$ satisfy a technical "smallness" condition (cf. [Hir03, Definition 10.5.15]). The maps in $\mathcal{I}$ can be seen as "cells" and the maps in $\mathcal{J}$ as "trivial cells". A *cell attachment* to an object X is then defined as a pushout of the form $B \cup_A X$, where $A \to B$ is a map in $\mathcal{I}$ and $A \to X$ an arbitrary map. A *trivial cell attachment* is defined analogously.

When given such sets of maps $\mathcal{I}$ and $\mathcal{J}$, Quillen's small object argument (cf. [Hir03, Proposition 10.5.16]) can be used to factor any map as an (infinite) sequence of cell attachments followed by a trivial fibration, or an (infinite) sequence of trivial cell attachments followed by a fibration. In particular, the small object argument can be used construct the functorial factorizations required in item (4) of the definition of a model category. Most natural examples of model categories are cofibrantly generated, and a common technique for constructing them is by defining sets of generating (trivial) cofibrations and the class of weak equivalences, and then using Quillen's small object argument to verify the axioms of a model category (cf. [Hir03, Theorem 11.3.1]). In fact, every model category $\mathcal{M}$ constructed in this thesis (with the exception of a few model structures in Paper III) is constructed by applying this approach either to $\mathcal{M}$ or to $\mathcal{M}^{op}$.

Example 1.3. All model structures from Example 1.2 are cofibrantly generated.

(i) The Quillen model structure on the category of topological spaces admits the following set of generating cofibrations

$$\mathcal{I} = \{\partial D^n \hookrightarrow D^n \mid n \geq 0\}$$

and the following set of generating trivial cofibrations

$$\mathcal{J} = \{D^n \times \{0\} \hookrightarrow D^n \times [0,1] \mid n \geq 0\}.$$

(ii) The set of boundary inclusions

$$\mathcal{I} = \{\partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0\}$$

is a set of generating cofibrations for the Kan-Quillen model structure on simplicial sets, while a set generating trivial cofibrations is given by the horn inclusions

$$\mathcal{J} = \{\Lambda^k[n] \hookrightarrow \Delta[n] \mid 0 \leq k \leq n \text{ and } n \geq 1\}.$$

(iii) Since the cofibrations of the Joyal model structure on simplicial sets agree with those of the Kan-Quillen model structure, one can use the same set of generating cofibrations. A concrete description of a set of generating trivial cofibration is (to the author's knowledge) not known, but can be shown to exist using abstract arguments.

The attentive reader might have noticed that the definition of a cofibrantly generated model category is not self-dual; that is, if $\mathcal{M}$ is a cofibrantly generated model category, then the dual model structure on $\mathcal{M}^{op}$ is generally not cofibrantly generated. We define a *fibrantly generated* model category to be a model category $\mathcal{M}$ with the property that $\mathcal{M}^{op}$ is cofibrantly generated. Even though fibrantly generated model categories are formally dual to cofibrantly generated model categories, they rarely appear in nature. This is due to the fact that categories that have a lot of "cosmall" objects are not very common. However, this thesis is an exception: many categories considered in this thesis are pro-categories, in which every object is automatically cosmall.

1.4 Simplicial model categories

All three examples of model categories given above come with a natural simplicial enrichment. For the category of simplicial sets this is simply the cartesian closed structure, while the category of topological spaces can be made into a simplicially enriched category by first considering it as a category enriched in topological spaces, where the hom-spaces are endowed with the compact-open topology, and then applying the singular complex functor to each of these hom-spaces.[1] Moreover, these simplicial enrichments interact well with the model structures: for the model category of topological spaces and the Kan-Quillen model structure on simplicial sets for example, the 1-simplices of these simplicial hom-sets correspond to homotopies, the 2-simplices to homotopies between homotopies, etc.[2] This is formalized in the definition of a simplicial model category ([Qui67, Definition II.2.2]).

Definition 1.4. Let $\mathcal{M}$ be a category enriched in simplicial sets equipped with a model structure on the underlying category, and write $\mathrm{Map}(-,-)$ for the simplicial hom-set. Then $\mathcal{M}$ is called a *simplicial model category* if the following two axioms hold:

(1) $\mathcal{M}$ is tensored and cotensored in simplicial sets; that is, for every simplicial sets S and for every two objects X and Y in $\mathcal{M}$, there exist objects $X \otimes S$ and Y^S together with natural isomorphisms

$$\mathrm{Map}(X \otimes S, Y) \simeq \mathrm{Map}(S, \mathrm{Map}(X,Y)) \simeq \mathrm{Map}(X, Y^S).$$

(2) For any cofibration $i\colon A \rightarrowtail B$ and any fibration $p\colon L \twoheadrightarrow K$, the map

$$\mathrm{Map}(B,L) \xrightarrow{i^* \times p_*} \mathrm{Map}(A,L) \times_{\mathrm{Map}(A,K)} \mathrm{Map}(B,K) \qquad (1.2)$$

[1]Strictly speaking, we should restrict our attention to a convenient subcategory of topological spaces, such as the category of compactly generated spaces, to obtain a cartesian closed structure on **Top**.

[2]In the Joyal model structure, the 1-simplices in the simplicial hom-sets correspond to natural transformations of functors instead.

is a fibration, which is trivial if either i is a trivial cofibration or p is a trivial fibration.

The second axiom can be seen as a version of the homotopy extension lifting property. It ensures that the simplicial enrichment interacts well with the model structure on the underlying category of $\mathcal{M}$. For example, it ensures that for any cofibrant object A, the map $A \sqcup A \cong A \otimes \partial\Delta[1] \to A \otimes \Delta[1]$ coming from the boundary inclusion $\partial\Delta[1] \rightarrowtail \Delta[1]$ is a cofibration and that it can be used to define homotopies between maps.

All model categories considered in this thesis are simplicial.[3] In fact, we will usually construct these model categories by first proving a version of the homotopy extension lifting property (2) and then using it to verify the lifting axiom in the definition of a model category (item (3) of Definition 1.1). This use of the simplicial enrichment turns out to be crucial, and it is one of the features that sets our approach apart from other common approaches to constructing model structures on ind- and pro-categories.

Strictly speaking, the above definition is not complete since it does not make clear in which model structure the map (1.2) should be a (trivial) fibration. In the literature, this is virtually always the Kan-Quillen model structure, but the definition also makes sense when considering the Joyal model structure on simplicial sets. This thesis features several model structures which are not simplicial in the former sense, but that are simplicial with respect to the Joyal model structure. We will therefore deviate from the standard terminology and call a model category simplicial if it is either simplicial with respect to the Kan-Quillen model structure or with respect to the Joyal model structure on simplicial sets. It will always be clear from the context which of these two holds. [4]

[3]Technically speaking, the model category of dendroidal sets considered in Paper II is not simplicial since the enrichment is only associative "up to homotopy". However, it does satisfy both the axioms of a simplicial model category, so in practice one can simply work with it as if it were a simplicial model category.

[4]Being simplicial with respect to the Joyal model structure is a strictly more general notion than being simplicial with respect to the Kan-Quillen model structure. This follows since the Joyal model structure has the same class of trivial fibrations as the Kan-Quillen model structure, while it has strictly more fibrations.

2 Ind- and pro-categories

We will now introduce ind- and pro-categories. The reader who has not heard of ind- and pro-categories before and is afraid that these are very exotic notions, may wish to take a look at Example 2.4 to discover that many common categories are, in fact, ind-categories. Being aware of this hopefully makes it easier to develop some intuition for them.

To learn more about ind- and pro-categories, we refer the reader to [GAV72, Exposé 1], [EH76], [Joh82], [AM86] and [Isa02]

2.1 What are ind- and pro-categories?

The ind-category $\mathrm{Ind}(\mathcal{C})$ of a category $\mathcal{C}$ is the category obtained by "freely adjoining filtered colimits", while the dual pro-category $\mathrm{Pro}(\mathcal{C})$ is obtained by "freely adjoining cofiltered limits". By a (co)limit, we always mean one that is indexed by a small category. Let us first recall what filtered colimits and cofiltered limits are.

Definition 2.1. A category I is called *filtered* if

(i) it is non-empty,

(ii) for any two objects i and j in I, there exists an object k together with morphisms $i \to k$ and $j \to k$, and

(iii) for any two parallel morphisms $f, g\colon i \rightrightarrows j$, there exists a morphism $h\colon j \to k$ such that $hf = hg$.

Dually, a category J is called *cofiltered* if J^{op} is filtered.

A *filtered colimit* is a colimit of a diagram indexed by a filtered category, while a *cofiltered limit* is a limit of a diagram indexed by a cofiltered category.

Examples of filtered colimits are colimits indexed by directed sets, such as the set of natural numbers.

Drawing inspiration from the fact that the category of presheaves on a category can be seen as the free cocompletion of this category, we define ind- and pro-categories as follows.

Definition 2.2. Let $\mathcal{C}$ be a category. The *ind-category* $\mathrm{Ind}(\mathcal{C})$ of $\mathcal{C}$ is defined as the full subcategory of the presheaf category $\mathbf{Set}^{\mathcal{C}^{op}}$ spanned by those presheaves that can be written as filtered colimits of representables. Dually, the *pro-category* $\mathrm{Pro}(\mathcal{C})$ of $\mathcal{C}$ is defined as the full subcategory of $(\mathbf{Set}^{\mathcal{C}})^{op}$ whose objects are are those that can be written cofiltered limits of representables (where the limit is computed in $(\mathbf{Set}^{\mathcal{C}})^{op}$).

It follows directly from this definition that $\mathrm{Pro}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C}^{op})^{op}$, hence the theory of pro-categories is formally dual to that of ind-categories.

The Yoneda embedding gives us fully faithful functors $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ and $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$. These can be used to express the following universal property, where we use $\mathrm{Fun}^{\mathrm{f}}(-,-)$ to denote the category of filtered colimit preserving functors and $\mathrm{Fun}^{\mathrm{cf}}(-,-)$ to denote the category of cofiltered limit preserving functors.

Theorem 2.3 ([GAV72, Exposé 1, Proposition 8.7.3])**.** *Let* $\mathcal{C}$ *be any category. The ind-category* $\mathrm{Ind}(\mathcal{C})$ *admits all filtered colimits, and for any category* $\mathcal{E}$ *that admits filtered colimits, restricting along the inclusion* $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ *gives an equivalence of categories* $\mathrm{Fun}^{\mathrm{f}}(\mathrm{Ind}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$. *Dually, the category* $\mathrm{Pro}(\mathcal{C})$ *admits all cofiltered limits, and for any category* $\mathcal{E}$ *that admits cofiltered limits, the restriction functor* $\mathrm{Fun}^{\mathrm{cf}}(\mathrm{Pro}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cf}}(\mathcal{C}, \mathcal{E})$ *is an equivalence of categories.*

This universal property justifies calling $\mathrm{Ind}(\mathcal{C})$ the category obtained by freely adjoining all filtered colimits, and $\mathrm{Pro}(\mathcal{C})$ the category obtained by freely adjoining all cofiltered limits to $\mathcal{C}$.

There is another common construction of ind- and pro-categories that is also worth mentioning. Namely, one defines the objects of the category $\mathrm{Ind}(\mathcal{C})$ as the class of all diagrams $I \to \mathcal{C}$ where I can be any filtered category, and the set of morphisms between two such diagrams $\{C_i\}_{i \in I}$ and $\{D_j\}_{j \in J}$ is defined by the formula

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\{C_i\}, \{D_j\}) = \lim_i \operatorname{colim}_j \mathrm{Hom}_{\mathcal{C}}(C_i, D_j).$$

Using that $\mathrm{Pro}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C}^{op})^{op}$, it is easy to give an analogous definition of $\mathrm{Pro}(\mathcal{C})$. It can indeed be shown that these definitions of $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ are equivalent to the one given above. While this second definition requires a bit more work to set up the theory (for example, it is more work to show that $\mathrm{Ind}(\mathcal{C})$ admits all filtered colimits in this case), its explicit nature is often useful when trying to compute limits and colimits (cf. [Isa02]). Throughout this thesis, both definitions are used interchangeably.

2.2 Ind- and pro-categories in nature

To give some feeling for how ind- and pro-categories behave, and to motivate their definition, we discuss several examples.

Example 2.4. Many common categories are ind-categories. Namely, any category admits a collection of objects that are "finitely presented" and that generate the whole category under filtered colimits is an ind-category (see Lemma 2.2 of Paper I for a precise statement).

(i) The category of sets **Set** is equivalent to $\mathrm{Ind}(\mathbf{Set}_{\mathrm{fin}})$, the ind-category of the category of finite sets. The equivalence in the direction $\mathrm{Ind}(\mathbf{Set}_{\mathrm{fin}}) \to \mathbf{Set}$ is obtained by applying the universal property of $\mathrm{Ind}(\mathbf{Set}_{\mathrm{fin}})$ to the inclusion $\mathbf{Set}_{\mathrm{fin}} \hookrightarrow \mathbf{Set}$, while the functor in the other direction is obtained by writing a set as the filtered colimit of its finite subset, ordered by inclusion.

(ii) The category of abelian groups **Ab** is equivalent to the ind-category $\mathrm{Ind}(\mathbf{Ab}_{\mathrm{fg}})$ of the category finitely generated abelian groups. (Note that abelian groups are finitely generated if and only if they are finitely presented.) This equivalence is obtained in the same way as the one between $\mathrm{Ind}(\mathbf{Set}_{\mathrm{fin}})$ and **Set**.

(iii) For general groups, not every finitely generated group is finitely presented. The category of groups **Grp** is equivalent to the ind-category $\mathrm{Ind}(\mathbf{Grp}_{\mathrm{fp}})$ of the category of finitely presented groups, but not to the ind-category of the category of finitely generated groups. The equivalence in the direction $\mathrm{Ind}(\mathbf{Grp}_{\mathrm{fp}}) \to \mathbf{Grp}$ again comes from the universal property of ind-categories. The functor in the other direction is slightly more complicated than in the two cases above: to define what it does on a group G, one needs to consider all maps from finitely presented groups towards G and not just subgroup inclusions.

(iv) The category of simplicial sets **sSet** is equivalent to $\mathrm{Ind}(\mathbf{sSet}_{\mathrm{fin}})$, where $\mathbf{sSet}_{\mathrm{fin}}$ denotes the category of *finite* simplicial sets; that is, simplicial sets that have finitely many non-degenerate simplices.

This list can be extended much further; we hope that the general idea is clear and the reader is invited to come up with more examples of common categories that are ind-categories.

Pro-categories do not occur as much in nature as ind-categories (except as duals of ind-categories, of course). However, there are various situations where pro-objects arise naturally, usually when there is some kind of duality involved.

Example 2.5. (i) The category of *Stone spaces*, i.e., compact Hausdorff totally disconnected spaces, is the pro-category $\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$ of the category of finite sets. In the direction $\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}) \to \mathbf{Stone}$, the equivalence is obtained by applying the universal property to $\mathbf{Set}_{\mathrm{fin}} \hookrightarrow \mathbf{Stone}$, where every finite set is viewed as a discrete topological space. One can check by hand that this functor is essentially surjective and fully faithful, hence an equivalence, but there is also a slick proof using Stone duality. Namely, Stone duality asserts that the category of Stone spaces is dual to the category of Boolean algebras. It is easy to verify that the category of Boolean algebras is the ind-category of the category of finite Boolean algebras, which in turn are dual to finite sets.

(ii) The category of profinite groups, i.e., the pro-category of the category of finite groups $\mathbf{Grp}_{\mathrm{fin}}$, is equivalent to the category of topological groups whose underlying space is a Stone space. This may seem like a simple consequence of the fact that the category of profinite sets is equivalent to the category of Stone spaces, but the proof is actually more involved (see for example [Joh82, §VI.2]).

(iii) Call a groupoid finite if it has finitely many arrows. The category of profinite groupoids, i.e., the pro-category of the category of finite groupoids, turns out to be equivalent to a full subcategory of the category of groupoids internal to Stone spaces. However, there exist groupoids internal to Stone spaces that are not an inverse limit of finite groupoids, so these two categories are not equivalent.

(iv) The category of simplicial Stone spaces is equivalent to the pro-category of a certain full subcategory of the category of simplicial sets, namely that of degreewise finite sets that are furthermore coskeletal (cf. Theorem 2.3 of Paper I). This fact and variations on it play an important role in Paper I and Paper II.

As mentioned, pro-categories often arise in a setting where there is some kind of duality. Given a cofiltered limit in a category where there is some kind of duality, it is natural to consider the dual as a pro-object; that is, as an object in a pro-category. For example, the finite field extensions of the rational numbers form a filtered diagram, whose colimit is the algebraic closure of the rationals. By considering the Galois groups of these finite field extensions, we obtain a cofiltered diagram whose limit is the absolute Galois group of the rationals. This shows that there is a natural way to consider the absolute Galois group as a profinite group, and one loses information by only considering the underlying group. Another example is that if one wants to extend Spanier-Whitehead duality to arbitrary spectra, then one should work with pro-spectra (cf. [CI04]).

Another situations where pro-categories come up is when working with completion functors, which are functors that take an object in some category to an object in a pro-category that approximates the original object. These completions often carry extra structure not present on the original object that may help in certain computations. For example, Sullivan shows in his proof of the Adams conjecture [Sul74] that the profinite completions of the classifying spaces for topological K-theory (both real and complex) admit an action of the absolute Galois group of the rationals. This action is the crucial ingredient in his proof of the Adams conjecture.

2.3 Why care about model structures on pro-categories?

In general, the reason why one should care about model structures on pro-categories is that there are various situations where pro-objects naturally come up and where there is also a notion of "homotopy", and their homotopy-invariant properties contain interesting information. By way of illustration, let us consider the étale homotopy type of a scheme, originally defined by Artin and Mazur in [AM86]. This is an example of a naturally arising pro-object where one is interested in its homotopy-invariant properties, such as its homotopy groups or its cohomology. Having standard tools from algebraic topology and homotopy theory available can be helpful to study these, hence the existence of a model category of "pro-spaces" that this étale homotopy type lives in is desirable. While Artin and Mazur constructed the étale homotopy type as a pro-object in the homotopy category of spaces, their construction has later been strictified to land in the category of pro-simplicial sets (cf. [Fri82]), and several model categories describing homotopy theories of pro-spaces, profinite spaces or pro-p spaces have been developed (cf. [Isa01; Isa05; Qui08; Mor96]). These model structures have several applications in algebraic geometry and in algebraic topology as well.

Summaries of included papers

Paper I

In this paper, we describe a general method for constructing simplicial model structures on ind- and pro-categories. Especially in the case of pro-categories, this method can be used to recover many interesting known model structures, while it can also be applied to produce many new model categories. The main result can be paraphrased as the following theorem:

Theorem. *Let $\mathcal{M}$ be a simplicial model category in which every object is cofibrant and let $\mathbf{C}$ be an essentially small full subcategory of $\mathcal{M}$ closed under finite limits and cotensors by finite simplicial sets. Then for any collection $\mathbf{T}$ of fibrant objects in $\mathbf{C}$, the pro-category $\mathrm{Pro}(\mathbf{C})$ carries a fibrantly generated simplicial model structure with the following properties:*

1. *The weak equivalences are the $\mathbf{T}$-local equivalences; that is, a map $f\colon C \to D$ is a weak equivalence if and only if for any $t \in \mathbf{T}$, the map*

$$f^*\colon \mathrm{Map}(D,t) \to \mathrm{Map}(C,t)$$

   is a weak equivalence.

2. *Every object in $\mathrm{Pro}(\mathbf{C})$ is again cofibrant.*

3. *The inclusion $\mathbf{C} \hookrightarrow \mathcal{M}$ induces a simplicial Quillen adjunction $\mathcal{M} \rightleftarrows \mathrm{Pro}(\mathbf{C})$ whose left adjoint is the profinite completion functor.*

Examples of known model structures that one can obtain by applying this theorem are Morel's model structure for pro-p spaces [Mor96], Quick's model structure for profinite spaces [Qui08] and Horel's model structure for profinite groupoids [Hor17, §4]. New model structures that one can obtain include profinite versions of the Joyal model structure for quasi-categories and Rezk's model structure for complete Segal spaces, and, for any finite poset P, a model structure describing the homotopy theory of profinite P-stratified spaces (cf. [BGH20, §2.5]). These examples are also studied in some detail.

The construction is based on the notion of a *fibration test category*, which is a small simplicial category $\mathbf{C}$ together with a subset $\mathbf{T}$ of objects and some extra structure that ensures the existence of a model structure on $\mathrm{Pro}(\mathbf{C})$ in which the weak equivalences are the $\mathbf{T}$-local ones. It is then shown that, under the assumptions of the above theorem, the pair $(\mathbf{C}, \mathbf{T})$ can be given the structure of a fibration test category in such a way that the obtained model structure on $\mathrm{Pro}(\mathbf{C})$ satisfies all properties of the above theorem.

It is worth pointing out that in the construction of the model structure on $\mathrm{Pro}(\mathbf{C})$ we work dually; that is, we actually define what a *cofibration test category* is, show that there exists a model structure on the ind-category $\mathrm{Ind}(\mathbf{C}')$ of a cofibration test category $(\mathbf{C}', \mathbf{T}')$, and then dualize this to a result about pro-categories. The reason for this is that the arguments used in constructing the model structure on $\mathrm{Ind}(\mathbf{C}')$ are all very standard, unlike the dual arguments needed in the case of pro-categories, hence easier to follow for the reader.

In the appendix of the paper, we give a precise characterization of the underlying ∞-category of the model structure that we constructed on $\mathrm{Pro}(\mathbf{C})$.

Paper II

The aim of Paper II is to describe what the profinite completion of an ∞-operad could be.

Recall that a dendroidal set is a set-valued presheaf on the category Ω of trees (cf. [HM]) and that they can be used as a model for ∞-operads. The latter is made precise by the existence of a model structure on the category **dSet** of dendroidal sets that is Quillen equivalent to the model category of topological operads and whose fibrant objects are a generalization of Joyal's quasi-categories. In this paper, we construct an analogous model structure on the category of dendroidal Stone spaces, i.e., the category $\mathbf{dStone} = \mathrm{Fun}(\Omega^{op}, \mathbf{Stone})$, that one can view as describing a "homotopy theory of profinite ∞-operads".

The method used for this is similar to that of Paper I. However, the arguments from Paper I cannot be applied directly, as not every object in **dSet** is cofibrant. This somewhat complicates the construction of the desired model structure on **dStone**, making it necessary to study the behaviour of normal monomorphisms between dendroidal Stone spaces in detail.

The methods used are sufficiently general to also allow the construction of model structures on the categories of open and closed dendroidal Stone spaces that describe the homotopy theory of non-unitary and unitary profinite ∞-operads, respectively.

The forgetful functor **dStone** $\to$ **dSet** admits a left adjoint, which is simply the profinite completion functor for sets applied pointwise. Once the model structure (and its variants) described above is established, it is straightforward to show that the profinite completion functor is indeed left Quillen. In particular, this provides a way of describing the profinite completion of an ∞-operad: by using dendroidal sets to model ∞-operads, one can simply apply the profinite completion functor to a dendroidal set and take a fibrant replacement in **dStone** to obtain the profinite completion of an ∞-operad.

A particularly important role in the construction of the model structure on **dStone** (and its variants) is played by the so-called *lean* ∞-operads. For this reason, we also included a precise characterization of which ∞-operads are weakly equivalent to such lean ∞-operads (cf. Section 2.3 of Paper II).

Paper III

In this paper, we present an alternative proof for one of the main results of [ABS21], which states that the stabilization of the category of noncommutative CW-complexes can be modelled by the category of spectral presheaves on a certain spectrum-enriched category $\mathcal{M}_s$. We first show that the methods from Paper I can be used to construct a symmetric monoidal model category describing the homotopy theory of the noncommutative CW-complexes defined in [ABS21, §2]. We then show that this model category can be stabilized by considering functors from the category of finite simplicial sets into this category. This produces a symmetric monoidal stable model category which is enriched over Lydakis's stable model category of simplicial functors [Lyd98]. A slightly modified version of Schwede's and Shipley's theorem [SS03, Theorem

3.9.3.(iii)] is then used to show that this category is Quillen equivalent to the category of enriched presheaves over a certain category $\mathcal{M}_\Delta$ enriched in Lydakis's category of simplicial functors. By changing the base of enrichment, this presheaf category is then showed to be Quillen equivalent to the category of spectral presheaves on $\mathcal{M}_s$, recovering the result of [ABS21].

The appendix to this paper contains a modification of the definition of a cofibration test category from Paper I, which we call a *minimal* cofibration test category. While the notion of a minimal cofibration test category is slightly less general than that of a cofibration test category, its axioms are much simpler and they apply to any example of interest.

The proofs in this paper are mostly formal, and they can be shown to work for any (minimal) cofibration test category. In particular, for any (minimal) cofibration test category $(\mathbf{C}, \mathbf{T})$, one can show that the stabilization of $\mathrm{Ind}(\mathbf{C})$ is equivalent to the category of spectral presheaves on a certain spectrum-enriched category $\mathcal{T}$, whose objects can be thought of as the suspension spectra of the objects in $\mathbf{T}$.

Bibliography

[ABS21] G. Arone, I. Barnea, and T. M. Schlank. *Noncommutative CW-spectra as enriched presheaves on matrix algebras*. Jan. 2021. arXiv: 2101.09775.

[AM86] M. Artin and B. Mazur. *Etale homotopy*. Vol. 100. Lecture Notes in Mathematics. Reprint of the 1969 original. Springer-Verlag, Berlin, 1986, pp. iv+169.

[BGH20] C. Barwick, S. Glasman, and P. Haine. *Exodromy*. Aug. 2020. arXiv: 1807.03281.

[CI04] J. D. Christensen and D. C. Isaksen. "Duality and pro-spectra". *Algebr. Geom. Topol.* 4 (2004), pp. 781–812.

[EH76] D. A. Edwards and H. M. Hastings. *Čech and Steenrod homotopy theories with applications to geometric topology*. Lecture Notes in Mathematics, Vol. 542. Springer-Verlag, Berlin-New York, 1976, pp. vii+296.

[Fri82] E. M. Friedlander. *Étale homotopy of simplicial schemes*. Annals of Mathematics Studies, No. 104. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982, pp. vii+190.

[GAV72] A. Grothendieck, M. Artin, and J.-L. Verdier. *Théorie des topos et cohomologie étale des schémas (SGA4)*. Vol. 269. Lecture Notes in Mathematics. Springer-Verlag, 1972.

[HM] G. Heuts and I. Moerdijk. *Trees in algebra and topology*. Book draft, available on https://sites.google.com/site/gijsheuts/research. (Visited on 06/14/2021).

[Hir03] P. S. Hirschhorn. *Model categories and their localizations*. Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457.

[Hor17] G. Horel. “Profinite completion of operads and the Grothendieck-Teichmüller group”. *Adv. Math.* 321 (2017), pp. 326–390.

[Hov99] M. Hovey. *Model categories*. Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999, pp. xii+209.

[Isa01] D. C. Isaksen. “A model structure on the category of pro-simplicial sets”. *Trans. Amer. Math. Soc.* 353.7 (2001), pp. 2805–2841.

[Isa02] D. C. Isaksen. “Calculating limits and colimits in pro-categories”. *Fund. Math.* 175.2 (2002), pp. 175–194.

[Isa05] D. C. Isaksen. “Completions of pro-spaces”. *Math. Z.* 250.1 (2005), pp. 113–143.

[Joh82] P. T. Johnstone. *Stone spaces*. Vol. 3. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1982, pp. xxi+370.

[Joy08] A. Joyal. *The theory of quasi-categories and its applications*. Notes for course at CRM, Barcelona. 2008. URL: `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf`.

[Lur09] J. Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925.

[Lyd98] M. Lydakis. *Simplicial functors and stable homotopy theory*. Unpublished preprint, available at `http://users.math.uoc.gr/~mlydakis/papers/sf.pdf`. May 6, 1998.

[Mor96] F. Morel. “Ensembles profinis simpliciaux et interprétation géométrique du foncteur T”. *Bull. Soc. Math. France* 124.2 (1996), pp. 347–373.

[Qui08] G. Quick. “Profinite homotopy theory”. *Doc. Math.* 13 (2008), pp. 585–612.

[Qui67] D. G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967, iv+156 pp. (not consecutively paged).

[Rie14] E. Riehl. *Categorical homotopy theory*. Vol. 24. New Mathematical Monographs. Cambridge University Press, Cambridge, 2014, pp. xviii+352.

[SS03] S. Schwede and B. Shipley. "Stable model categories are categories of modules". *Topology* 42.1 (2003), pp. 103–153.

[Sul74] D. Sullivan. "Genetics of homotopy theory and the Adams conjecture". *Ann. of Math. (2)* 100 (1974), pp. 1–79.

Paper I

Simplicial model structures on pro-categories

Based on: Thomas Blom, Ieke Moerdijk, *Simplicial model structures on pro-categories*. Submitted.

Paper II

Profinite ∞-operads

Based on: Thomas Blom, Ieke Moerdijk, *Profinite ∞-operads*.

Paper III

A note on noncommutative CW-spectra

Based on: Thomas Blom, *A note on noncommutative CW-spectra*.