Preschool children's collective mathematical reasoning about sharing

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This study focuses on sharing, both in equal parts (groups) or unequal parts. Children at age five attending preschool, are faced with two different tasks where the mathematical properties in their reasoning, or when mathematical reasoning was replaced with an ethical reasoning is analysed. When performing division, different strategies were used, and the norm of fair share was often expressed. It was easier for the children to allocate resources when the dividend was larger than the divisor, and when dealing with a fraction, the cardinality of the number of parts appeared to be a prominent property compared to property 'equal size' of the parts. There were also indications of ethical reasoning where the child used different claims to convince their peer. There was a tension between the norm of equal sharing and the solutions with unequal parts. One implication is that if wanting to challenge children's mathematical reasoning in a division task, it could be fruitful to look at fractions instead of repeating tasks where the dividend is larger than the divisor.

Keywords: Division, ethical reasoning, mathematical reasoning.

Introduction

One of the key concepts in mathematics is division (e.g. Kiselman & Mouwitz, 2008), and although children most often tend to divide into equal parts, when it comes to sharing resources, it is not always as straight forward (Wong & Nunes, 2003). For instance, there are reports that every day experiences can act as an obstacle for understanding of division as equal parts (Smith et al., 2013; Wong & Nunes, 2014), and previous studies have shown that context matters when deciding what is a fair share (Huntsman, 1984; Sigelman & Waitzman, 1991; Wong & Nunes, 2014). As an example, a study on children age five show that they take different aspects into account when deciding how to share resources, so that someone in need might get a larger amount (Enright et al., 1984). However, at the same time, there is a growing body of research indicating that children as young as one can understand sharing into equal parts (e.g. Geraci & Surian, 2011; Sommerville et al., 2013). The norm of fair share appears to be strong already from a young age. This norm has been discussed in studies on ethical reasoning about sharing resources: children express that they know that they should divide resources into equal parts, even when they decide not to do so (Smith et al., 2013). From a mathematical point of view, this is not division (Correa et al., 1998), but it could function as a starting point for ethical reasoning around sharing resources and thereby address issues with respect to sustainability (Hedefalk, 2015). As is stated in the Swedish curriculum for preschool, children should be provided the conditions to develop:

> The ability to discover, reflect on and work out their position on different ethical dilemmas and fundamental questions of life in daily reality (Skolverket, 2019, p.13)

It is therefore relevant not to neglect or disregard children's reasoning where the sharing is done in unequal parts. Instead, it is of interest to understand the arguments supporting the child's reasoning. The aim here is to study preschool children's' collective mathematical reasoning about sharing. The research questions are: (1) What mathematical properties do children use in their reasoning?; and, (2) When is the mathematical reasoning replaced with ethical reasoning?.

Background

Mathematical reasoning can be defined in many different ways (Lithner, 2008; Sumpter, 2016), and here, the choice is to see collective mathematical reasoning as a collective line of arguments that is produced when solving a task. This is seen as a collective effort that aims to create meaning (Eriksson & Sumpter, 2021; Sumpter & Hedefalk, 2018). Reasoning is therefore a social process with the assumption that mathematical reasoning is crucial for the understanding of mathematics (e.g. Herbert & Williams, 2021). Lithner (2008) suggested the following reasoning sequence with four steps as follows: (1) a task situation (TS) is met; (2) a strategy choice (SC) is made where the 'choice' should be interpreted in a wide sense; (3) the strategy is implemented (SI); and, (4) a conclusion (C) is drawn. We then apply Toulmin's (2003) model for each of these steps, which means that the task situation can be supported by identifying arguments (Eriksson & Sumpter, 2021), the strategy choice and implementation can be supported by predictive and verifying arguments (Lithner, 2008), finally, conclusion can be supported by evaluative arguments (Sumpter & Hedefalk, 2018). Each of these steps join in a chain of arguments, an argumentation, that has components described as data, warrant, backing, and conclusion, with the latter step differing from how the conclusion is presented in the reasoning sequence. In this way, based on Sumpter's (2016) integration of mathematical reasoning and argumentation, reasoning is seen as the vertical line of the reasoning sequence (TS – C) whereas argumentation is the horizontal line (i.e. the four different types of arguments). In order to analyse the content of the arguments, Lithner (2008) proposes the notion of 'anchoring' mathematical properties in the components of the arguments. The different mathematical properties are objects (e.g., natural numbers, rational numbers), transformations (e.g., division), and concepts (e.g., the integer concept) that consist of sets of objects and transformations.

Division can be defined as a/b where a is the dividend (numerator) and b is the divisor (denominator), and the result is described as a fraction, quotient, or ratio. Division can be seen as an inverse transformation to multiplication, that $a/b = k$ if and only if $a = bk$ where $b\neq0$ (Kiselman & Mouwitz, 2008). In school mathematics, division is often viewed either as quotition or partition. The common core for either of these is that the shares (i.e. fraction, quotient, or ratio) are of equal size. This is the main difference between division and sharing, where the latter can accept unequal shares (Correa et al., 1998). Studies has shown that when solving mathematical tasks that involve sharing resources, children/ teenagers can use both mathematical properties and ethical properties such as values (Chernyak & Sobel, 2016). For instance, in a study on children five years old and their reasoning, if the child identified a recipient with a greater need, they got a larger share (Enright et al., 1984). Studies has also shown that children as young as two, expect sharing to be in proportion to effort (e.g., Sommerville et al., 2013). Using the same starting point as for mathematical reasoning, we define ethical reasoning as a collective line of arguments that is produced when solving a task, but where the arguments are anchored in values (Sumpter & Hedefalk, forthcoming). This is similar to moral reasoning (Samuelsson & Lindström, 2020). We follow Samuelsson's (2020) criterions for deciding whether an ethical reasoning is sustainable or not by using his SIL methods: (1) coherence (S); (2) information (I); and, (3) vividness (L). This implies that sharing based on ethical reasoning can include division sharing in equal parts as well as sharing in unequal parts. The ethical argument is coherent when it does not contain logical flaws, is based on correct and relevant information and motivations that a listener is willing to accept (Samuelsson & Lindström, 2020). Facts are not enough to make an ethical decision about sharing; the child needs to mentally make the sharing problem vivid to try to understand another person's (or soft toy's) point of view in the sharing experience. If the argumentation consists of all three methods (S, I and L), it is considered that the child has made an ethical argument about sharing (Sumpter & Hedefalk, forthcoming).

Methods

In order to analyse different types of collective reasoning, we used two tasks describing sharing resources to recipients. The tasks were the first two in a set of six (that had been developed and tested earlier) that described different mathematical properties and different ethical issues (Sumpter & Hedefalk, forthcoming). The first task was an open task where the children were asked to divide 12 biscuits (in coloured paper) between three soft toys (a teddy, a dog, and a tiger). If the children decided on a solution that was not equal division, they were asked again as a follow up if they could make the sharing into equal parts. The second task was to divide four biscuits between the three soft toys. Six children worked in pairs together with one of their pre-school teachers. The instruction for the teacher was to ask questions to stimulate arguments such as "What are you thinking?", but not to give any evaluation of the solution (i.e. "This is in/correct"). The children were in the following pairs: (1) Noel (age 5y 8m) and Maya (age 4y and 9m); (2) Nova (5y and 2 m) and Ida (5y 1 m); and, (3) Adam (5y 6m) and Anna (5y 2m). All children are born in Sweden and have another language as a first language, apart from Noel, who arrived in Sweden three months prior to the recordings. Noel speaks almost fluent Swedish.

Their work was videotaped and in a first step of the analysis, these videotapes were transcribed verbatim, including actions according to principles presented by Mergenthaler and Stinson (1992). From these principles follows that an argument could also be a gesture or nonverbal action from the children. The next step of the analyses was to organise the transcripts according to the mathematical reasoning structure, TS, SC, SI, and C (e.g. Lithner, 2008), and arguments for each step were identified. The arguments were then analysed using the notion of anchoring of mathematical properties, for instance the transformation division as a repeated subtraction, thus giving biscuits to each of the three soft toys, one at the time. As a last step of the analyses was to look at the arguments using Samuelsson's (2020) method-based model, originally developed for teaching ethics but here used as an analytical tool (e.g. Sumpter & Hedefalk, forthcoming). Here, we are interested in how the arguments change when children decide to make choices connected to ethical values about sharing. The arguments not based in mathematical properties were analysed using the three SIL criterions: (1) coherence (S); (2) information (I); and, (3) vividness (L). The study follows the ethical principles of the Swedish Research Council. That means, for example, that the parents have signed a letter of consent and that the names of the children are anonymised.

Results

Table 1 presents an overview of the different types of collective reasoning:

Table 1: Mathematical and ethical reasoning

	Pair 1 Noel, Maya	Pair 2 Nova, Ida	Pair 3 Adam, Anna
Task 1	SCI: unequal parts. SIL SC2a: (1+1+1)+(1+1+1)+(1+1+1) 9/3, r=3 SC2b: 3/3 = 1 C: 4 = (1+1+1)+1	SC: unequal parts C={4,3,5}	SC1a: 9/3 3+3+3 reminder 3 SC1b: 3/3 = 1 C: 4 = 3+1
Task 2	SCa: 4/3 = 1 reminder 1 SCb:{1,2,1} - SIL SCc: 1/n (n not determined), shared in 3 SCd: (4/(1/2))/3 Cd: {1.5,1,1.5}	SCa: 4/3 = 1 reminder 1 The surplus biscuit should be eaten up to preserve equal parts. SCb: 1/n (n not determined), shared in 3	SCa: 4/3 = 1 reminder 1 Tension: Anna wants unequal sharing, Adam argues for equal parts. SCb: ½ + ¼ + ¼, cardinal 3 SCc: ¼ + ¼ +¼ + ¼ SCd: (1/4)/3 = 1/12 C: 1 + ¼ +1/12

Starting with the first task, the three pairs used different types of reasoning. The first pair, Noel and Maya, started with an ethical reasoning and there was a tension between them where Noel initially wanted to give more biscuits to two of the soft toys and less to the tiger:

Noel: Me not like tiger, it can eat me!
Teacher: Ok, that is your [way of] thinking. But you like the dog? And that is why it got more of the biscuits?
Noel: Yes, I gave it a lot, a lot, a lot. [stressing the importance]

This part of reasoning is considered an ethical reasoning according to the SIL- method, since the argument for the sharing in unequal parts was justified with the argument from Noel that he does not like the tiger since it is dangerous (it can eat him up), and using opposite argumentation regarding the dog. The motivation was lively (as the child conveys the tigers hunger feelings and can see the consequence if it acts on that), informed (tigers eat humans) and coherent (logical reasoning in his way of thinking). As a second step, when the children were informed to share equally, a conflict arises when Noel wants to give more biscuits to the rabbit who Noel argues is "hungry". Noel’s argument was lively (as the child conveys the rabbits hunger feelings), informed (he is aware of the amounts of biscuits and receivers) and coherent (logical reasoning that can be accepted). The other child, Maya, opposes and justifies that equal also means fair (i.e. the same number of biscuits for all stuffed animals). Maya's argument at that point was informed (i.e. she is aware of the amount of biscuits and receivers), coherent (logical reasoning that can be accepted) but it is not interpreted as lively, according to the SIL-method, since she does not express why and how it will affect the soft toys. This part of the reasoning was considered a mathematical reasoning where Maya's argument of equal parts is accepted by Noel with the transformation 12/3 = 9/3 + 3/3 where both divisions were made as repeated subtraction. The second pair, Nova and Ida, decided to keep the initial decision to share the biscuits in unequal parts although the teacher tried to encourage them to try division. Neither of them gave any arguments to why, and their reasoning was considered neither ethical reasoning nor mathematical founded reasoning. The third pair, Adam and Anna, had no problem to share nine biscuits in groups of three. It took some encouragement from the teacher for them to realise that it was ok to share the remaining three biscuits as well.

Looking at the second task, the first pair again struggled to agree between sharing in unequal parts and division. Maya, again, stated "it has to be fair" whereas Noel argued for a solution where the tiger and the rabbit got one biscuit each whereas the dog got two since "he is really hungry". The reasoning here is considered ethical reasoning according to the SIL-method, whereas when they were encouraged to share in equal parts, they struggled to overcome the impulse to share all biscuits in halves and thereby creating 8/3. The result is sharing in unequal parts. The second pair, Nova and Ida, suggested that the remaining biscuit should be eaten up, without any further arguments. When encouraged to divide the remaining biscuits into smaller parts, they continued to cut the biscuit in smaller and smaller parts and then sharing these to the three recipients without any signal that it should be equal. The third pair experienced the same tension when Adam argued for division and Anna suggested sharing in unequal parts, where it was Anna who took the scissors first. Although they agreed on the strategy choice, to divide the surplus biscuit into pieces, they disagreed on how it should be done, see Table 2:

Table 2: Pair 3 reasoning about 1 divided by 3

Time	Person	Data	Reasoning Structure	Arguments
07:18	Teacher	What can you do then? [putting the scissors in front of the children]	TS initiated	
07:21	Anna	[Bend forward and picks up the scissors and the biscuit] Cut it!	SC:1 divided into parts	Predictive argument: dividing is necessary
07:23	Teacher	Uhm. Do it.	SC confirmed	
07:31	Anna	[Cuts the biscuit into halves]	SI: 1/2	Verifying action
07:36	Adam	Also this one you should cut. [Picks up one of the halves and cuts it into two bits] Everyone should have one bit [Gives one bit to the tiger, here named as lion. At the same time, Anna picks up the other half and the scissors]].. and then	SI: 1 divided into three parts, where the focus is on cardinal value of 3, not equal size.	Verifying arguments: everyone should have one bit. Predictive argument diving is necessary (equal size of the parts)

		[gives one bit to the dog] .. and then [tries to take the bit the girl has in her hand]		Identifying argument: n should be 3
07:48	Anna	No!		
07:48	Adam	It should have it!		Stressing without further arguments.
		[Meanwhile, Girl cuts the half into quarters]	SI: 1/4	
07:50	Adam	Four! [sounds disappointed, open his arms and hands as to stress the conclusion]	C: 1 is divided in 4 parts	Evaluative argument identifying the new problem, once again there are one extra piece: 1 divided by 3.
07:57		The girl shares out the quarters, Adam takes the extra quarters and cuts into three bits which are shared under further discussion between the two.	TS: ¼ should be divided SC: (1/4)/3 =1/12 SI: Straight forward C: 1/12 is added to 1¼	No arguments.

Adam expressed verifying arguments to support the implementation of the strategy, that everyone should have one [piece] of the remaining biscuit, where it appeared not so important that the parts are of equal size. Although the final solution was 1 1/3 biscuits (1 + ¼ + 1/12), the conclusion was not supported by any arguments. Also, given that Adam earlier argued for a solution with unequal sizes of the parts, it is more plausible to assume that the conclusion is a result of random actions more than a result of an informed strategy with an argumentation backed up with claims or warrants.

Discussion

Starting with the mathematical properties in the collective reasoning, the results showed a variation of mathematical components. Looking at the different transformations in how sharing was made, the most common strategy choice was repeated subtraction, one item to each recipient at a time. One pair immediately created the subset ‘9’ of 12, and grouped the nine items into three groups of equal size. Here, we do not have further information on why sharing out the remaining three items was considered a difficulty, which is an interesting topic for further research. Regarding the transformation 1/3 was a challenge for all three groups, including a tension between the idea of equality and other counter arguments. Just as previous studies, this was by no way straight forward (e.g. Wong & Nunes, 2003). One child, Maya, tried several times to convey that when sharing resources, it has to be fair, and here, the norm of equal parts appeared to be strong (e.g. Geraci & Suriam. 2011; Smith et al., 2013; Somerville et al., 2013). However, it is not clear what this norm encompasses. For instance, the suggestion of sharing in equal amount yet unequal sizes had the main argument based on the mathematical property of the object ‘3’, cardinality. No argumentation, at least not explicit in words or actions, was about the size of the parts. This is similar reasoning as seen in Sumpter and Hedefalk (forthcoming). The implication is that if wanting to challenge young children and their reasoning about division, it might not be a question about *a* items shared by *b* recipients as much it is about the sizes of the parts, especially when $a<b$. This means that although it is of importance to understand division both as quotition or partition (e.g. Schmidt & Weiser, 1995), the central mathematical properties of the quotient (ratio) is vital given the difference between division and sharing (Correa et al., 1998). Then, one vital step might be the relationship between division, fraction, and measurement (e.g. Eriksson & Sumpter, 2021), instead of increasing the size of the dividend.

Looking at when mathematical reasoning was replaced with ethical reasoning, there were some instances where the context matter (e.g. Huntsman, 1984; Sigelman & Waitzman, 1991; Sumpter & Hedefalk, forthcoming; Wong & Nunes, 2014): the tiger was scary, the rabbit was hungry, and the dog was more worthy since a child liked it. The context here was mainly emotional, which is one part of ethical reasoning (Samuelsson & Lindström, 2020). The two cases tested here did not invite ethical arguments as such as there being an explicit instruction of one of the recipients having a greater need (e.g. Enright et al., 1984; Sumpter & Hedefalk, forthcoming), and this is something that could be further developed, especially if wanting to use it as a starting point for ethical reasoning, such as sustainability issues (e.g. Hedefalk, 2015; Samuelsson & Lindström, 2020).

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