Stockholm University

Model categories, pro-categories and functors

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Abstract

This thesis consists of five papers. The first three are concerned with various model structures on ind- and pro-categories, while the last two are concerned with the homotopy theory of functors.

In Paper I, a general method for constructing simplicial model structures on ind- and pro-categories is described and its basic properties are studied. This method is particularly useful for constructing "profinite" analogues of known model categories. It recovers various known model structures and also constructs many interesting new model structures.

In Paper II, it is shown that a profinite completion functor for (simplicial or topological) operads with good homotopical properties can be constructed as a left Quillen functor from an appropriate model category of infinity-operads to a certain model category of profinite infinity-operads. The construction of the latter model category is inspired by the method described in Paper I, but there are a few subtle differences that make its construction more involved.

In Paper III, the general method from Paper I is used to give an alternative proof of a result by Arone, Barnea and Schlank. This result states that the stabilization of the category of noncommutative CW-complexes can be modelled as the category of spectral presheaves on a certain category M. The advantage of this alternative proof is that it mainly relies on well-known results on (stable) model categories.

In Paper IV, the question of whether an ordinary functor between enriched categories is equivalent to an enriched functor is addressed. This is done for several types of enrichments: namely when the base of enrichment is (pointed) topological spaces, (pointed) simplicial sets or orthogonal spectra. Simple criteria are obtained under which this question has a positive answer.

In Paper V, the Goodwillie calculus of functors between categories of enriched diagram spaces is described. It is shown that the layers of the Goodwillie tower are classified by certain types of diagrams in spectra, directly generalizing Goodwillie's original classification. Using this classification, an operad structure on the derivatives of the identity functor is constructed that generalizes an operad structure originally constructed by Ching.

Keywords: *Homotopy theory, Quillen model categories, Pro-categories, Enriched categories, Goodwillie calculus.*

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Abstract

This thesis consists of five papers. The first three are concerned with various model structures on ind- and pro-categories, while the last two are concerned with the homotopy theory of functors.

In Paper I, a general method for constructing simplicial model structures on ind- and pro-categories is described and its basic properties are studied. This method is particularly useful for constructing "profinite" analogues of known model categories. It recovers various known model structures and also constructs many interesting new model structures.

In Paper II, it is shown that a profinite completion functor for (simplicial or topological) operads with good homotopical properties can be constructed as a left Quillen functor from an appropriate model category of ∞-operads to a certain model category of profinite ∞-operads. The construction of the latter model category is inspired by the method described in Paper I, but there are a few subtle differences that make its construction more involved.

In Paper III, the general method from Paper I is applied to give an alternative proof of one of the main results of [ABS22]. This result states that the stabilization of the category of noncommutative CW-complexes can be modelled as the category of spectral presheaves on a certain category $\mathcal{M}$. The advantage of this alternative proof is that it mainly relies on well-known results on (stable) model categories.

In Paper IV, the question of whether an ordinary functor between enriched categories is equivalent to an enriched functor is addressed. This is done for several types of enrichments: namely when the base of enrichment is (pointed) topological spaces, (pointed) simplicial sets or orthogonal spectra. Simple criteria are obtained under which this question has a positive answer.

In Paper V, the Goodwillie calculus of functors between categories of enriched diagram spaces is described. It is shown that the layers of the Goodwillie tower are classified by certain types of diagrams in spectra, directly generalizing Goodwillie's original classification. Using this classification, an operad structure on the derivatives of the identity functor is constructed that generalizes Ching's operad structure [Chi05].

Sammanfattning

Denna avhandling består av fem artiklar. De tre första handlar om olika modellstrukturer på ind- och pro-kategorier och de två sista handlar om homotopiteori för funktorer.

I Artikel I beskrivs en allmän metod för att konstruera simpliciella modellstrukturer på ind- och pro-kategorier och deras egenskaper studeras. Denna metod är särskilt användbar för att konstruera "pro-ändliga" analoger av kända modellkategorier. Med metoden återfås olika kända modellstrukturer, men den konstruerar även många intressanta nya modellstrukturer.

I Artikel II visas att en pro-ändlig kompletteringsfunktor för (simpliciella eller topologiska) operader med goda homotopiska egenskaper kan konstrueras som en vänster-Quillen-funktor från en lämplig modellkategori av ∞-operader till en viss modellkategori av pro-ändliga ∞-operader. Konstruktionen av denna modellkategori av pro-ändliga ∞-operader är inspirerad av metoden som beskrivs i Artikel I, men det finns några subtila skillnader som gör dess konstruktion mer komplicerad.

I Artikel III används den allmänna metoden från Artikel I för att ge ett alternativt bevis av ett av huvudresultaten från [ABS22]. Detta resultat säger att stabiliseringen av kategorin av icke-kommutativa CW-komplex kan modelleras som kategorin av spektrala förkärvar på en viss kategori $\mathcal{M}$. Fördelen med detta alternativa bevis är att det huvudsakligen bygger på välkända resultat om (stabila) modellkategorier.

I Artikel IV behandlas frågan om en vanlig funktor mellan berikade kategorier är ekvivalent med en berikad funktor. Detta görs för flera typer av berikningar: nämligen när basen för berikning är topologiska rum, simpliciella mängder, deras punkterade analoger eller ortogonala spektra. Enkla kriterier erhålls under vilka denna fråga har ett positivt svar.

I Artikel V beskrivs Goodwillie-kalkylen för funktorer mellan kategorier av berikade diagramrum. Det visas att lagren i Goodwillie-tornet klassificeras av vissa typer av diagram i spektra, vilket direkt generaliserar Goodwillies ursprungliga klassificering. Med hjälp av denna klassificering konstrueras en operadstruktur på derivatorna av identitetsfunktorn som generaliserar Chings operadstruktur [Chi05].

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I also wish to thank Ieke Moerdijk, who was the supervisor for my master's thesis and who coauthored two of the included papers. I learned a great deal about thinking mathematically and doing research from him.

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The program "Higher algebraic structures in algebra, topology and geometry", hosted at the Mittag-Leffler institute during the spring of 2022, deserves special mention. After having spent most of my time during the Covid-19 pandemic in mathematical isolation, this program was an eye-opener for me: it made me aware of what type of research problems people are currently working on and it exposed me to modern approaches to homotopy theory. I am grateful to the organizers and all the participants of the program for making this possible.

Finally, I would like to thank my family and friends, both in Sweden and in the Netherlands, for their unconditional love and support during the past four years. Especially Rolinde, for always being there for me. Words cannot express how grateful I am to have you in my life.

List of papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

Paper I: **Simplicial model structures on pro-categories**
Thomas Blom, Ieke Moerdijk
Preprint, accepted for publication in *Algebraic & Geometric Topology*.

Paper II: **Profinite ∞-operads**
Thomas Blom, Ieke Moerdijk
Advances in Mathematics 408 (2022), Paper No. 108601.

Paper III: **A note on noncommutative CW-spectra**
Thomas Blom
Preprint

Paper IV: **Replacing functors with enriched ones**
Thomas Blom
Preprint

Paper V: **Goodwillie calculus for diagram categories**
Thomas Blom
Preprint

Earlier versions of Papers I-III have also appeared in the author's licentiate thesis [Blo21].
There are also preprint versions of Papers I-IV available on arXiv. Papers I, III and IV virtually identical to their arXiv preprint versions [BM20; Blo22a; Blo22b], with the exception of the layout and several typographical errors that have been corrected. However, the published version of Paper II included in this thesis is substantially different from the arXiv preprint version [BM21].

Contents

General introduction

This chapter starts with a general introduction to homotopy theory, aimed at a very general mathematical audience, e.g. a second-year student. In the remaining sections, we touch upon a few topics that relate to the papers included in this thesis: enriched category theory, model categories, ind- and pro-categories and Goodwillie calculus. These topics are introduced informally, providing references with more details for those readers that are interested. These sections of the general introduction are intended for a mathematical audience that already has some familiarity with algebraic topology and category theory.

This thesis builds partly upon the author's licentiate thesis [Blo21], which was defended on November 10, 2021. Sections 3 and 4 of this general introduction are slightly updated versions of Sections 1 and 2 of the licentiate thesis. Of the papers included in this thesis, (earlier versions of) Papers I, II and III were part of the licentiate.

1 Homotopy theory

Historically, homotopy theory refers to the study of spaces "up to continuous deformation", also called "homotopy", and it is basically synonymous with algebraic topology. In the second half of the twentieth century, it was observed that the ideas from homotopy theory are also present in other areas of mathematics; for example, maps between chain complexes admit a notion of homotopy that behaves in a similar way as homotopies between maps of spaces. For this reason, the meaning of "homotopy theory" changed over time, now being something along the lines of "the study of a mathematical situation in which there is a notion of deformation between maps". In this introduction to homotopy theory, we tell the story of how, starting from a concept of "space", one may arrive at the definition of a *homotopy type*. After this, we will briefly discuss the more modern meaning of homotopy theory mentioned above.

1.1 From spaces to homotopy types

If one were to make precise the notion of a "space", then a natural starting point would be a subset of $\mathbb{R}^3$. However, from a mathematical perspective this does not offer much flexibility, and one might want to want a more intrinsic definition that does not refer to an ambient object, in this case $\mathbb{R}^3$. Moreover, one could argue that if in a certain situation one can talk about things being close to or far away from each other, then these "things" form a space. This very soon leads to the definition of a *metric space*, which is a set X equipped with a distance function $d\colon X \times X \to X$ satisfying certain natural axioms. An example is the three-dimensional space $\mathbb{R}^3$, where the distance between two points (x_1, x_2, x_3) and (y_1, y_2, y_3) is given by $\sqrt{x^2 + y^2 + z^2}$. The natural functions to consider between metric spaces are *continuous maps*; intuitively, these are the functions $f\colon X \to Y$ with the property that if two points in X are close to each other, then their image in Y is also close. It turns out that the definition of a continuous map does not really depend on the metric: there can be many different metrics on a set such that the continuous maps out of it or into it are the same. In a sense, a metric space contains a lot of redundant information. One can show that the definition of continuity only depends on the collection of *open subsets* of a metric space; these are the subsets $U \subseteq X$ such that for every point $x \in U$, there exists an $\epsilon > 0$ such that for any point $y \in X$, if $d(x, y) < \epsilon$, then $y \in U$. If one is mainly interested in continuous maps and not in metric spaces per se, then this might lead one to axiomatize the properties that this collection of open subsets has. This leads to the definition of a topological space.

Definition 1.1. A topological space consists of a set X together with a collection $\mathcal{T}$ of subsets of X, called *open subsets*, such that

(1) $\varnothing$ and X are in $\mathcal{T}$,

(2) for any collection $\{U_i\}_{i \in I}$ of elements of $\mathcal{T}$, their union $\cup_i U_i$ lies in $\mathcal{T}$, and

(3) for any two elements $U, V \in \mathcal{T}$, their intersection $U \cap V$ lies in $\mathcal{T}$.

In algebraic topology, one studies these topological spaces by assigning invariants to them, which are often objects such as groups or rings. However, it turns out there are many continuous maps of topological spaces that induce isomorphisms on these invariants but that do not have a continuous inverse. Such maps are called *weak homotopy equivalences*. If one is only interested in these invariants, then this means that the notion of a topological space still contains a lot of redundant information that is preventing these weak homotopy equivalences from having inverses. However, it is not so easy to point out what this redundant information is, and how to get rid of it. Instead of forgetting some of the structure of the objects that we study, we will try to change the maps between them.

It turns out that if two maps $f, g\colon X \to Y$ can be "continuously deformed into each other", then they induce the same map on the invariants that algebraic topologists care about. The idea of such a "continuous deformation" is that one has a collection of maps $\{h_t\colon X \to Y\}_{t\in[0,1]}$ such that $h_0 = f$, $h_1 = g$ and the maps h_t depend continuously on t. This is made precise as follows

Definition 1.2. Let $f, g\colon X \to Y$ be continuous maps of topological spaces. A *homotopy* from f to g is a continuous map

$$H\colon X \times [0,1] \to Y$$

such that $H(x,0) = f$ and $H(x,1) = g$ for any $x \in X$. For the definition of the topology on the product $X \times [0,1]$, the reader is referred to [Mun75, §2.4]. Two maps are called *homotopic* if there exists a homotopy between them. Note that this defines an equivalence relation on the set of continuous maps between X and Y. The equivalence classes for this equivalence relation are called *homotopy classes*.

If two maps $f\colon X \to Y$ and $g\colon Y \to X$ are given such that gf and fg are homotopic to the identity maps, then f and g induce inverse maps on the invariants that algebraic topologists care about. In the terminology from above, f and g are weak homotopy equivalences. This suggests that an algebraic topologist might wish to replace continuous maps between topological spaces with homotopy classes of continuous maps, since this would turn f and g into actual inverses of each other.

Strictly speaking, if we change the maps between mathematical objects then we are not changing the objects themselves. However, it makes sense to think of them as different objects: if one takes the view that a mathematical object is determined by how it relates to other mathematical objects, then changing the maps into it and out of it changes the object itself. In this example, since we are identifying certain continuous maps with each other, one could argue that we lost some of the structure that was present on these objects: if two maps are homotopic, then our new objects are not able to distinguish them anymore.

Unfortunately, the objects that we obtain this way still have too much structure: there are still weak homotopy equivalences that are not isomorphisms. However, it turns out that there is a special class of topological spaces, called CW-complexes, such that the weak homotopy equivalences between them admit inverses up to homotopy. In particular, these maps become isomorphisms once we divide out the equivalence relation given by the notion of homotopy. One can furthermore show that for any topological space X there exists a canonical[1] weak homotopy equivalence $Y \to X$ where Y is a CW-complex. In particular, the algebraic topologist might wish to work with CW-complexes and homotopy classes of maps between them. The resulting objects are sometimes called *homotopy types*.

[1]The word canonical should be interpreted loosely; there are many natural choices here, but one can show that up to homotopy they are all the same.

1.2 Abstract homotopy theory

In practice, working with these homotopy types is not ideal: many constructions that one commonly does with mathematical objects can't be performed with homotopy types. It turns out that declaring homotopic maps to be equal is too crude. Ideally, we wish to think of two homotopic maps as equal, but also remember the reasons why they are equal; that is, the homotopies witnessing that they are "equal". Two abstract frameworks for dealing with this issue are model categories and ∞-categories.

An ∞-category is a mathematical structure that consists of a collection of "objects", a collection of "morphisms" between them that can be composed as if they are maps between mathematical objects, a collection of morphisms between these morphisms, morphisms between those morphisms, etc. Moreover, one usually assumes that all morphisms between morphisms are invertible, in analogy with the fact that homotopies can be reversed. While it may seem from the description given here that the notion of an ∞-category would be something unwieldy, there has been a lot of progress in the development of their theory over the past twenty years, most notably through the work of Lurie [Lur09]. They have been fruitfully applied within algebraic topology but also outside of it, since there are many situations where there is a natural notion of "continuous deformation" between maps. These days, for some mathematicians homotopy theory has become synonymous with ∞-category theory.

A different framework for organizing this type of situation is model categories. These were originally invented by Quillen [Qui67] to apply techniques from algebraic topology in other settings. Since all papers in this thesis make extensive use of model categories, we will devote Section 3 to them and refrain from discussing them here.

2 Enriched category theory

Many of the categories that commonly occur in mathematics come with extra structure that is not really captured by the definition of a category: often, there is a natural way of viewing their hom-sets as more sophisticated mathematical objects than just sets. For example, there might be a natural way to view their hom-sets as Abelian groups or as topological spaces. This extra structure can sometimes be exploited to give streamlined constructions or proofs that would be hard or even impossible without it, so ignoring this type of extra structure is probably unwise. The notion of an *enriched category* captures exactly this type of extra structure.

We wish to warn the reader that the subject of enriched category theory can be quite technical and dry if not studied with a collection of examples in mind. The focus of this section is therefore not on precise definitions, for which the reader is referred to [Rie14, §3] and [Kel82]. Instead, we mainly focus on a few natural examples and motivate the topic of Paper IV.

2.1 Enriched categories

If we wish to define categories whose hom-objects are other mathematical objects than just sets, then we should first decide on what we want these objects to be. In the definition of an enriched category, these are defined to be objects in some *symmetric monoidal category*.

Definition 2.1. A *symmetric monoidal category* is a category $\mathcal{V}$ equipped with a functor $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, sometimes called the *tensor product*, and an object $\mathbb{1}$ called the *unit*, such that $\otimes$ is unital, associative and commutative up to natural (coherent) isomorphism. See [Kel82, §§1.1 & 1.4] for a precise definition.

Example 2.2. The category of Abelian groups **Ab** is symmetric monoidal under the tensor product, where $\mathbb{Z}$ is the unit.

Example 2.3. The categories of topological spaces **Top** and of simplicial sets **sSet** are both symmetric monoidal under the Cartesian product, where the terminal object $*$ plays the role of the unit. More generally, for any category that admits finite products there is a canonical way to make it into a symmetric monoidal category where the Cartesian product is the tensor product and the terminal object is the unit.

Example 2.4. The category $\mathbf{Set}_*$ of pointed sets and basepoint-preserving maps forms a symmetric monoidal category under the *smash product* $\wedge$. The smash product of two pointed sets (X, x) and (Y, y) is defined as the quotient

$$\frac{X \times Y}{X \times \{y\} \cup \{x\} \times Y}$$

The categories of pointed topological spaces[2] and pointed simplicial sets can be given a similar symmetric monoidal structure.

We will generally write $\mathcal{V}$ for a symmetric monoidal category, suppressing the tensor product, unit and natural isomorphisms from the notation.

Definition 2.5. Let $\mathcal{V}$ be a symmetric monoidal category. A $\mathcal{V}$-enriched category $\mathcal{C}$ consists of

- a collection $\mathrm{Ob}(\mathcal{C})$, whose elements are called *objects*,
- for each pair $c, d \in \mathrm{Ob}(\mathcal{C})$ a *hom-object* $\mathcal{C}(c, d) \in \mathrm{Ob}(\mathcal{V})$,
- for each $c \in \mathrm{Ob}(\mathcal{C})$ a morphism $\mathrm{id}_c\colon \mathbb{1} \to \mathcal{C}(x, x)$ in $\mathcal{V}$ called the *identity (at c)*, and

[2] This requires one to work with a category of "nice enough" topological spaces, such as the category of compactly generated ones.

- for each triple $c, d, e \in \mathrm{Ob}(\mathcal{C})$ a morphism $c\colon \mathcal{C}(d,e) \times \mathcal{C}(c,d) \to \mathcal{C}(c,e)$ called *composition*,

such that the associativity and unitality diagrams of [Kel82, (1.3)-(1.4)] commute.

Informally, one can think of a $\mathcal{V}$-enriched category as an ordinary category where each hom-set is endowed with extra structure making it an object of $\mathcal{V}$, and where the composition maps respect this extra structure in each variable separately.

Example 2.6. A **Top**-enriched category is an ordinary category where all hom-sets come equipped with a topology and the composition maps

$$\mathrm{Hom}(d,e) \times \mathrm{Hom}(c,d) \to \mathrm{Hom}(c,e)$$

are continuous with respect to the product topology on the domain.

Example 2.7. In an **Ab**-enriched category, all hom-sets come equipped with the structure of an Abelian group and the composition maps are maps

$$\mathrm{Hom}(d,e) \otimes \mathrm{Hom}(c,d) \to \mathrm{Hom}(c,e).$$

Note that by the universal property of the tensor product these correspond to bilinear maps

$$\mathrm{Hom}(d,e) \times \mathrm{Hom}(c,d) \to \mathrm{Hom}(c,e).$$

In particular, an **Ab**-enriched category is simply an ordinary category such that all hom-sets are equipped with an Abelian group structure and the composition maps are bilinear.

Example 2.8. A simplicial category $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}(\mathcal{C})$, simplicial sets $\mathcal{C}(c,d)$ and maps of simplicial sets

$$\mathcal{C}(d,e) \times \mathcal{C}(c,d) \to \mathcal{C}(c,e)$$

satisfying the associativity and unitality axioms. We leave it as an exercise to the reader to verify that this is the same data as a functor $\mathcal{C}_\bullet\colon \Delta^{op} \to \mathbf{Cat}$ with the property that $\mathrm{Ob}(\mathcal{C}_\bullet)$ defines a constant simplicial set.

For **Top**- or **Ab**-enriched categories, there is an obvious way to see them as ordinary categories with extra structure on their hom-sets. However, for the general definition of a $\mathcal{V}$-enriched category this is not so clear. Nevertheless, there exists a construction that associates to any $\mathcal{V}$-enriched category $\mathcal{C}$ an ordinary "underlying" category.

Definition 2.9. Let $\mathcal{V}$ be a symmetric monoidal category with unit $\mathbb{1}$. The *underlying category* $\mathcal{C}_0$ of a $\mathcal{V}$-enriched category $\mathcal{C}$ the ordinary category

- whose objects are the same as those of $\mathcal{C}$,

- in which the hom-sets are defined by $\mathrm{Hom}(c,d) = \mathrm{Hom}(\mathbb{1}, \mathcal{C}(c,d))$, and
- where the composition of two morphisms $f \in \mathrm{Hom}(\mathbb{1}, \mathcal{C}(d,e))$ and $g \in \mathrm{Hom}(\mathbb{1}, \mathcal{C}(c,d))$ is defined as the composite

$$\mathbb{1} \cong \mathbb{1} \otimes \mathbb{1} \xrightarrow{f \otimes g} \mathcal{C}(d,e) \otimes \mathcal{C}(c,d) \xrightarrow{c} \mathcal{C}(c,e)$$

It is not hard to verify that if $\mathcal{V} = \mathbf{Top}, \mathbf{Ab}$, then the underlying category of a $\mathcal{V}$-enriched category is obtained by considering the underlying sets of the hom-objects. In the case of a simplicial category, its underlying category is obtained by considering the sets of 0-simplices of all hom-objects.

2.2 Enriched functors

When working with enriched categories, it is natural to require of functors that they respect the enrichment. This leads to the definition of a $\mathcal{V}$-functor.

Definition 2.10. A $\mathcal{V}$-functor F consists of a map $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ together with morphisms

$$F_{c,d}\colon \mathcal{C}(c,d) \to \mathcal{D}(Fc, Fd)$$

for every pair $c, d \in \mathrm{Ob}(\mathcal{C})$ such that

$$\begin{array}{ccc} \mathcal{C}(d,e) \otimes \mathcal{C}(c,d) & \xrightarrow{c} & \mathcal{C}(c,e) \\ {\scriptstyle F_{d,e} \times F_{c,d}} \downarrow & & \downarrow {\scriptstyle F_{c,e}} \\ \mathcal{D}(Fd, Fe) \otimes \mathcal{D}(Fc, Fd) & \xrightarrow{c} & \mathcal{D}(Fc, Fe) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{1} & \xrightarrow{\mathrm{id}_c} & \mathcal{C}(c,c) \\ & {\scriptstyle \mathrm{id}_{Fc}} \searrow & \downarrow {\scriptstyle F_{c,c}} \\ & & \mathcal{D}(Fc, Fc) \end{array} \qquad (2.1)$$

commute.

One of the main reasons to work with enriched functors as opposed to ordinary functors between the underlying categories is that they preserve more structure.

Example 2.11. In an **Ab**-enriched category, one can show that any finite product is automatically a coproduct and vice versa. For this reason, they are usually called *biproducts*. An **Ab**-enriched category that admits finite biproducts is called an *additive* category. One can show that a functor between additive categories is an **Ab**-functor precisely if it preserves biproducts.

Example 2.12. In any **Top**-enriched category $\mathcal{C}$ there exists a notion of homotopy: a *homotopy* between two maps $f, g\colon c \to d$ in $\mathcal{C}$ is defined as a continuous path $[0,1] \to \mathcal{C}(c,d)$ between f and g. If $F\colon \mathcal{C} \to \mathcal{D}$ is a **Top**-functor, then since $\mathcal{C}(c,d) \to \mathcal{D}(Fc, Fd)$ is continuous, homotopies between f and g are taken to homotopies between Ff and Fg. In particular, any **Top**-functor preserves homotopy equivalences, which turns out to be very useful in practice.

2.3 Property vs. structure

In Paper IV, we study conditions under which certain ordinary functors are naturally equivalent to enriched functors. This can be viewed as (a homotopical version of) the question of whether being an enriched functor is *extra structure* on the underlying functor, or whether it is a *property*. It was already mentioned in Example 2.11 that a functor between additive categories is an **Ab**-functor precisely if it is an ordinary functor that preserves biproducts, hence in this case being an **Ab**-functor is a property of the underlying functor.

The aim of this section is to discuss other examples in enriched category theory that illuminate the difference between something being a property or it being extra structure. This is not just a question one can ask about functors between enriched categories: sometimes the enrichment itself is a property of the category. More precisely, it may be the case that if an ordinary category satisfies certain criteria, then there is a unique way to make it into a $\mathcal{V}$-enriched category.

Example 2.13. Let $\mathcal{C}$ be an ordinary category that admits a *zero object* 0; that is, an object that is both initial and terminal. For any pair of objects $c, d \in \mathrm{Ob}(\mathcal{C})$, the unique maps $c \to 0$ and $0 \to d$ can be composed to obtain a map $0_{c,d} \in \mathrm{Hom}(c,d)$ called the *zero map*. This map does not depend on the choice of zero object, providing all hom-sets of $\mathcal{C}$ with a canonical basepoint. We leave it to the reader to verify that this gives $\mathcal{C}$ the structure of a $\mathbf{Set}_*$-enriched category and moreover that there cannot exist any other $\mathbf{Set}_*$-enriched category whose underlying category is $\mathcal{C}$.

Example 2.14. Suppose that in the previous example, $\mathcal{C}$ admits both binary products and binary coproducts. The zero maps can then be used to define, for any pair $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$, a canonical map

$$c_1 \sqcup c_2 \to c_1 \times c_2$$

such that the composition

$$c_i \to c_1 \sqcup c_2 \to c_1 \times c_2 \to c_j$$

equals id_{c_i} if $i = j$ and $0_{i,j}$ if $i \neq j$. The existence and uniqueness of this map follow from the universal properties of the coproduct and product. We say that an ordinary category $\mathcal{C}$ *admits finite biproducts* if it admits a zero object, finite products and finite coproducts, and if this canonical map is an isomorphism for any pair of objects in $\mathcal{C}$. The hom-sets of such a category $\mathcal{C}$ can be endowed with the structure of a commutative monoid: given a pair of objects $c, d \in \mathrm{Ob}(\mathcal{C})$, one defines a binary operation $+$ on $\mathrm{Hom}(c,d)$ by sending $f, g \in \mathrm{Hom}(c,d)$ to the composite

$$c \xrightarrow{\Delta} c \times c \xrightarrow{f \times g} d \times d \cong d \sqcup d \xrightarrow{\mathrm{fold}} d.$$

We leave it to the reader to verify that this operation makes $\mathcal{C}$ into a category enriched in commutative monoids. Moreover, it can be shown that this is the only category enriched in commutative monoids whose underlying category is $\mathcal{C}$. Note that this defines an enrichment in **Ab** if and only if for any pair $c, d \in \mathrm{Ob}(\mathcal{C})$, the binary operation $+$ on $\mathrm{Hom}(c, d)$ admits inverses.

Example 2.15. Let $\mathcal{C}$ be an ordinary category. Then there are two canonical ways to make $\mathcal{C}$ into a **Top**-enriched category: one can endow all hom-sets with the discrete topology or with the codiscrete topology. This shows that, contrary to the previous example, there exists no property that an ordinary category can have such that it admits a unique upgrade to a **Top**-enriched category. In particular, a **Top**-enrichment is always extra structure on a category and never a property.

For an ordinary functor $F\colon \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-enriched categories (i.e. a functor $F\colon \mathcal{C}_0 \to \mathcal{D}_0$), one may similarly wonder whether it is a property to be $\mathcal{V}$-enriched or whether one has to provide F with extra structure. To lift an ordinary functor $F\colon \mathcal{C} \to \mathcal{D}$ to a $\mathcal{V}$-enriched one, it is necessary that for every pair $c, d \in \mathrm{Ob}(\mathcal{C})$, the map $F_{c,d}\colon \mathcal{C}_0(c, d) \to \mathcal{D}_0(Fc, Fd)$ lies in the image of

$$\begin{aligned}\mathrm{Hom}(\mathcal{C}(c,d), \mathcal{D}(Fc, Fd)) \to \mathrm{Hom}(&\mathrm{Hom}(\mathbb{1}, \mathcal{C}(c,d)), \mathrm{Hom}(\mathbb{1}, \mathcal{D}(Fc, Fd))) \\ &= \mathrm{Hom}(\mathcal{C}_0(c,d), \mathcal{D}_0(Fc, Fd))\end{aligned}$$

Moreover, if this is the case, then one needs to be able to choose lifts $\widetilde{F}_{c,d}$ of the maps $F_{c,d}$ compatibly; i.e. such that the diagrams Equation (2.1) commute. This indicates the following simple criterion on $\mathcal{V}$ ensuring that being $\mathcal{V}$-enriched is a property.

Example 2.16. Let $\mathcal{V}$ be a symmetric monoidal category and suppose that the functor $\mathcal{V} \to$ **Set** corepresented by $\mathbb{1}$ is faithful; that is, for any pair $v, w \in \mathrm{Ob}(\mathcal{V})$ the map $\mathrm{Hom}(v, w) \to \mathrm{Hom}(\mathrm{Hom}(\mathbb{1}, v), \mathrm{Hom}(\mathbb{1}, w))$ is injective. In particular, if given an ordinary functor $F\colon \mathcal{C} \to \mathcal{D}$, the lifts $\widetilde{F}_{c,d}$ described above are unique whenever they exist. Since the commutativity of the diagrams Equation (2.1) is a property of these unique lifts, it follows that for a functor of $\mathcal{V}$-enriched categories, being $\mathcal{V}$-enriched is a property. Examples of symmetric monoidal $\mathcal{V}$-categories where this is the case are $\mathbf{Set}_*$, **Top** and **Ab**.

Conversely, if $\mathrm{Hom}(\mathbb{1}, -)$ is not faithful, then under a mild assumption on $\mathcal{V}$ we can always construct $\mathcal{V}$-categories $\mathcal{C}, \mathcal{D}$ such that being $\mathcal{V}$-enriched is not a property of a functor $\mathcal{C} \to \mathcal{D}$ but extra structure.

Example 2.17. Let $\mathcal{V}$ be a symmetric monoidal category with an initial object $\varnothing$ and suppose that $\varnothing \otimes v \cong \varnothing$ for any $v \in \mathcal{V}$. Given an object $v \in \mathcal{V}$, one can then define the category $\mathcal{C}_v$ with $\mathrm{Ob}(\mathcal{C}_v) = \{0, 1\}$ and

$$\mathcal{C}_v(0,0) = \mathcal{C}_v(1,1) = \mathbb{1}, \quad \mathcal{C}_v(0,1) = v, \quad \mathcal{C}_v(1,0) = \varnothing.$$

Now suppose there exist $v, w \in \mathrm{Ob}(\mathcal{V})$ such that

$$\mathrm{Hom}(v, w) \to \mathrm{Hom}(\mathrm{Hom}(\mathbb{1}, v), \mathrm{Hom}(\mathbb{1}, w))$$

is not injective and let $f, g \colon v \to w$ be distinct morphisms such that $\mathrm{Hom}(\mathbb{1}, f) = \mathrm{Hom}(\mathbb{1}, g)$. A functor $\mathcal{C}_v \to \mathcal{C}_w$ is simply a morphism $v \to w$, while a functor $(\mathcal{C}_v)_0 \to (\mathcal{C}_w)_0$ is a map $\mathrm{Hom}(\mathbb{1}, v) \to \mathrm{Hom}(\mathbb{1}, w)$. In particular, while f and g give different $\mathcal{V}$-enriched functors $\mathcal{C}_v \to \mathcal{C}_w$, the corresponding functors between the underlying categories are equal. An example where this can occur is when $\mathcal{V} = \mathbf{sSet}$.

In Paper IV, we study a homotopical version of the question of whether being $\mathcal{V}$-enriched is a property, and we moreover try to find simple criteria that ensure that this property is satisfied. For example, we show a converse of Example 2.12, namely that an ordinary functor between suitable **Top**-enriched categories is naturally equivalent to a **Top**-functor precisely if it preserves homotopy equivalences.

3 Model categories

This part aims to motivate the definition of a model category and provide a few examples. Motivated by these examples, the related notions of *cofibrantly generated* and *simplicial* model categories are described, which play an important role in the papers contained in this thesis, especially the first three.

There are many excellent resources for those readers who are interested in learning more about model categories, of which we wish to single out the textbooks [Hov99] and [Hir03, Part II].

3.1 What are they good for?

A mathematician might sometimes find themself in the following situation: there is a certain class of mathematical objects they are interested in, but the notion of isomorphism is too fine to suit the mathematician's taste: there exist morphisms that make two objects "equivalent" in some sense, but that are not isomorphisms. The archetypal example is the one considered in Section 1, namely the category of topological spaces. Here one often only cares about spaces up to (weak) homotopy equivalences. In general, it is quite hard to work with a situation where one is given a category together with a class of morphisms that are in some sense "equivalences", but that are not necessarily isomorphisms. One can try to add formal inverses for these morphisms to construct a new category in which all these "equivalences" become isomorphisms (the *homotopy category*), but this process is very hard to control and one often loses valuable information along the way. For these reasons it is often preferable to work in the original category, but this requires tools that help in dealing with weak equivalences.

In the category of topological spaces, fibrations and cofibrations provide such tools. It is exactly this type of structure that the definition of a model category captures: it is a category equipped with three classes of morphisms, called fibrations, cofibrations and weak equivalences, that satisfy properties analogous to those satisfied by the identically named maps between topological spaces (cf. Definition 3.1 below). Many common constructions used in algebraic topology carry over to arbitrary model categories: for example, one can define cylinder objects, path objects and homotopies between maps, but also more sophisticated constructions such as homotopy (co)limits and mapping spaces. Having such constructions available is helpful for showing that objects with certain favourable properties exist and it can aid computations.

3.2 The definition

For completeness' sake, we include the definition of a model category as given in [Hir03, Def. 7.1.3].

Definition 3.1. Let $\mathcal{M}$ be a complete and cocomplete category. A *model structure* on $\mathcal{M}$ consists of three classes of maps, called *weak equivalences* (denoted $\xrightarrow{\sim}$), *cofibrations* (denoted $\rightarrowtail$) and *fibrations* (denoted $\twoheadrightarrow$), such that the following are satisfied:

(1) If f and g are composable maps in $\mathcal{M}$ and two of f, g and gf are weak equivalences, then so is the third.

(2) If f is a map in $\mathcal{M}$ that is a weak equivalence, a fibration or a cofibration, then so is any retract of f. (For the definition of a retract of a map, see e.g. [Hir03, Def. 7.1.1].)

(3) Given a commutative square

$$\begin{array}{ccc} A & \longrightarrow & L \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & K \end{array} \tag{3.1}$$

in $\mathcal{M}$, a lift exists if either

- i is a cofibration and p is a *trivial fibration* (i.e., a fibration that is also a weak equivalence) or
- i is a *trivial cofibration* (i.e., a cofibration that is also a weak equivalence) and p is a fibration.

(4) Every map $g\colon X \to Y$ in $\mathcal{M}$ admits a functorial factorization $X \rightarrowtail Z \overset{\sim}{\twoheadrightarrow} Y$ into a cofibration followed by a trivial fibration and a functorial factorization $X \overset{\sim}{\rightarrowtail} W \twoheadrightarrow Y$ into a trivial cofibration followed by a fibration. (For the definition of a *functorial* factorization, see e.g. [Rie14, Def. 12.1.1]).

A complete and cocomplete category equipped with a model structure is called a *model category*. An object X in a model category is called *fibrant* if the map $X \to *$ is a fibration, and *cofibrant* if the map $\varnothing \to X$ is a cofibration.

The definition of a model category is evidently self-dual; more precisely, if $\mathcal{M}$ is a model category, then the opposite category $\mathcal{M}^{op}$ can be endowed with a model structure with the same weak equivalences, but with the classes of fibration and cofibrations interchanged.

We will say that a map $i\colon A \to B$ has the *left lifting property* with respect to some map $p\colon L \to K$ and that p has the *right lifting property* with respect to i if for any commutative square as in (3.1), a lift exists. In particular, item (3) of the definition of a model category can be rephrased as stating if a map i is a cofibration, then it has the left lifting property with respect to every trivial fibration, and that if a map p is fibration, then it has the right lifting property with respect to every trivial cofibration.

It turns out that the converse holds as well: a map that has the left lifting property with respect to all trivial fibration is a cofibration, and a map that has the right lifting property with respect to all trivial cofibrations is a fibration. This can be proved using the so-called retract argument, see e.g. [Hir03, Prop. 7.2.3]. In particular, if in a model category the class of weak equivalences and either the class of cofibrations or the class of fibrations are known, then the third class of maps is fully determined. In fact, in many examples of model categories, the cofibrations or fibrations are defined as the maps that have the left lifting property with respect to all trivial fibrations or as the maps that have the right lifting property with respect to all trivial cofibrations, respectively. It is worth pointing out that the cofibrations and the fibrations together also determine the weak equivalence in a model category (cf. [Hir03, Prop. 7.2.7]).

Example 3.2. Finally, let us mention a few examples

(i) ([Qui67, §3]) The category of topological spaces admits a model structure in which the fibrations are the Serre fibrations, the weak equivalences are the weak homotopy equivalences and the cofibrations are the maps that have the left lifting property with respect to all trivial fibrations. One can show that the cofibrant objects are exactly the retracts of cell complexes[3], while all objects are fibrant. This model structure is sometimes called the *Quillen model structure*, to distinguish it from other model structures on the category of topological spaces.

(ii) ([Qui67, §3]) The category of simplicial sets admits the *Kan-Quillen model structure* in which the fibrations are the Kan fibrations, the cofibrations the monomorphisms and the weak equivalences the maps of simplicial sets that become a (weak) homotopy equivalence after geometric realization. In particular, every object is cofibrant and the fibrant objects are

[3]These are like CW-complexes, but where one is also allowed to attach a lower-dimensional cells to higher-dimensional ones.

the Kan complexes. This model category can be shown to be equivalent, in an appropriate sense, to the model category of topological spaces mentioned above.

(iii) ([Joy08, §2], [Lur09, §2.2.5]) The category of simplicial sets can also be endowed with the *Joyal model structure*. In this model structure the cofibrations are again the monomorphisms, but the fibrations and weak equivalences are hard to describe explicitly. Its interest lies in the fact that its fibrant objects are exactly the *quasi-categories*, which serve as a model for the theory of $(\infty, 1)$-categories. It can be shown that the weak equivalences between quasi-categories are (an $(\infty, 1)$-categorical generalization of) the essentially surjective and fully faithful functors.

3.3 (Co)fibrant generation

The classes of cofibrations of the model structures described in Example 3.2 all have something in common: they are all (retracts of) maps that are obtained by repeatedly attaching "cells". More precisely, any monomorphism of simplicial sets can be obtained by attaching standard simplices $\Delta[n]$ along their boundaries $\partial\Delta[n]$, while any cofibration of topological spaces is a retract of a map that can be obtained by attaching discs D^n along their boundaries S^{n-1}. The idea that cofibrations are obtained as (retracts of) repeated cell attachments is formalized through the definition of a *cofibrantly generated model category*. Roughly speaking, a model category is cofibrantly generated if there exist both a set of cofibrations $\mathcal{I}$ and a set of trivial cofibrations $\mathcal{J}$ such that a map is a fibration or a trivial fibration if and only if it has the right lifting property with respect to all maps in the set $\mathcal{J}$ or all maps in the set $\mathcal{I}$, respectively, and if the domains of all maps in $\mathcal{I}$ and $\mathcal{J}$ satisfy a technical "smallness" condition (cf. [Hir03, Def. 10.5.15]). The maps in $\mathcal{I}$ can be seen as "cells" and the maps in $\mathcal{J}$ as "trivial cells". A *cell attachment* to an object X is then defined as a pushout of the form $B \cup_A X$, where $A \to B$ is a map in $\mathcal{I}$ and $A \to X$ is an arbitrary map. A *trivial cell attachment* is defined analogously.

When given such sets of maps $\mathcal{I}$ and $\mathcal{J}$, Quillen's small object argument (cf. [Hir03, Prop. 10.5.16]) can be used to factor any map as an (infinite) sequence of cell attachments followed by a trivial fibration, or an (infinite) sequence of trivial cell attachments followed by a fibration. In particular, the small object argument can be used to construct the functorial factorizations required in item (4) of the definition of a model category. Most natural examples of model categories are cofibrantly generated, and a common technique for constructing them is by defining sets of generating (trivial) cofibrations and a class of weak equivalences, and then using Quillen's small object argument to verify the axioms of a model category (cf. [Hir03, Thm. 11.3.1]). Many model categories $\mathcal{M}$ considered in this thesis are constructed by applying this approach either to $\mathcal{M}$ or to $\mathcal{M}^{op}$.

Example 3.3. All model structures from Example 3.2 are cofibrantly generated.

(i) The Quillen model structure on the category of topological spaces admits the following set of generating cofibrations

$$\mathcal{I} = \{\partial D^n \hookrightarrow D^n \mid n \geq 0\}$$

and the following set of generating trivial cofibrations

$$\mathcal{J} = \{D^n \times \{0\} \hookrightarrow D^n \times [0,1] \mid n \geq 0\}.$$

(ii) The set of boundary inclusions

$$\mathcal{I} = \{\partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0\}$$

is a set of generating cofibrations for the Kan-Quillen model structure on simplicial sets, while a set generating trivial cofibrations is given by the horn inclusions

$$\mathcal{J} = \{\Lambda^k[n] \hookrightarrow \Delta[n] \mid 0 \leq k \leq n \text{ and } n \geq 1\}.$$

(iii) Since the cofibrations of the Joyal model structure on simplicial sets agree with those of the Kan-Quillen model structure, one can use the same set of generating cofibrations. A concrete description of a set of generating trivial cofibration is (to the author's knowledge) not known, but such a set can be shown to exist using abstract arguments.

The attentive reader might have noticed that the definition of a cofibrantly generated model category is not self-dual; that is, if $\mathcal{M}$ is a cofibrantly generated model category, then the dual model structure on $\mathcal{M}^{op}$ is generally not cofibrantly generated. We define a *fibrantly generated* model category to be a model category $\mathcal{M}$ with the property that $\mathcal{M}^{op}$ is cofibrantly generated. Even though fibrantly generated model categories are formally dual to cofibrantly generated model categories, they rarely appear in nature. This is due to the fact that categories that have a lot of "cosmall" objects are not very common. However, this thesis is an exception: most categories considered in Papers I and II are pro-categories, in which every object is automatically cosmall.

3.4 Simplicial model categories

All three examples of model categories given above come with a natural simplicial enrichment. For the category of simplicial sets this is simply the Cartesian closed structure, while the category of topological spaces can be made into a simplicially enriched category by first considering it as a category enriched in topological spaces, where the hom-spaces are endowed with the compact-open topology, and then applying the singular complex functor to each of these hom-spaces.[4] Moreover, these simplicial enrichments interact well with the model structures: for the model category of topological spaces and the Kan-Quillen model structure on simplicial sets for example, the 1-simplices of these simplicial hom-sets correspond to homotopies, the 2-simplices to homotopies between homotopies, etc.[5] This is formalized in the definition of a simplicial model category ([Qui67, Def. II.2.2]).

Definition 3.4. Let $\mathcal{M}$ be a **sSet**-enriched category equipped with a model structure on the underlying category, and write $\mathrm{Map}(-,-)$ for the simplicial hom-set. Then $\mathcal{M}$ is called a *simplicial model category* if the following two axioms hold:

(1) $\mathcal{M}$ is tensored and cotensored in simplicial sets; that is, for every simplicial sets S and for every two objects X and Y in $\mathcal{M}$, there exist objects $X \otimes S$ and Y^S together with natural isomorphisms[6]

$$\mathrm{Map}(X \otimes S, Y) \simeq \mathrm{Map}(S, \mathrm{Map}(X, Y)) \simeq \mathrm{Map}(X, Y^S).$$

(2) For any cofibration $i\colon A \rightarrowtail B$ and any fibration $p\colon L \twoheadrightarrow K$, the map

$$\mathrm{Map}(B, L) \xrightarrow{i^* \times p_*} \mathrm{Map}(A, L) \times_{\mathrm{Map}(A,K)} \mathrm{Map}(B, K) \tag{3.2}$$

is a fibration, which is trivial if either i is a trivial cofibration or p is a trivial fibration.

The second axiom can be seen as a version of the homotopy extension lifting property. It ensures that the simplicial enrichment interacts well with the model structure on the underlying category of $\mathcal{M}$. For example, it ensures that for any cofibrant object A, the map $A \sqcup A \cong A \otimes \partial\Delta[1] \to A \otimes \Delta[1]$ coming from the boundary inclusion $\partial\Delta[1] \rightarrowtail \Delta[1]$ is a cofibration and that it can be used to define homotopies between maps.

Remark 3.5. One can generalize this definition to other bases of enrichments: if $\mathcal{V}$ is a symmetric monoidal category with a compatible model structure, then a $\mathcal{V}$-enriched model category is a model category $\mathcal{M}$ equipped with a $\mathcal{V}$-enrichment such that the analogue of Definition 3.4 holds. For a precise definition, the reader is referred to [Hov99, Def. 4.2.18].

Almost all model categories considered in this thesis are simplicial. In fact, in Papers I-III, we often prove that a model structure exists by first proving a version of the homotopy extension lifting property (2) and then using it to

[4] Strictly speaking, we should restrict our attention to a convenient subcategory of topological spaces, such as the category of compactly generated spaces, to obtain a Cartesian closed structure on **Top**.

[5] In the Joyal model structure, the 1-simplices in the simplicial hom-sets correspond to natural transformations of functors instead.

[6] These isomorphisms need to be natural in the enriched sense, cf. [Rie14, Def. 3.5.8].

verify the lifting axiom in the definition of a model category (item (3) of Definition 3.1). This use of the simplicial enrichment turns out to be crucial, and it is one of the features that sets our approach in Papers I-III apart from other common approaches to constructing model structures on ind- and pro-categories.

Strictly speaking, the above definition is not complete since it does not make clear in which model structure the map (3.2) should be a (trivial) fibration. In the literature, this is virtually always the Kan-Quillen model structure, but the definition also makes sense when considering the Joyal model structure on simplicial sets. Paper I features several model structures which are not simplicial in the former sense, but that are simplicial with respect to the Joyal model structure. We will therefore deviate from the standard terminology and call a model category simplicial if it is either simplicial with respect to the Kan-Quillen model structure or with respect to the Joyal model structure on simplicial sets. It will always be clear from the context which of these two holds.[7]

4 Ind- and pro-categories

We will now introduce ind- and pro-categories. The reader who has not heard of ind- and pro-categories before and is afraid that these are very exotic notions, may wish to take a look at Example 4.4 to discover that many common categories are, in fact, ind-categories. Being aware of this hopefully makes it easier to develop some intuition for them.

For further reading about ind- and pro-categories, we refer the reader to [GAV72, Exposé 1], [EH76], [Joh82], [AM69] and [Isa02]

4.1 What are ind- and pro-categories?

The ind-category $\mathrm{Ind}(\mathcal{C})$ of a category $\mathcal{C}$ is the category obtained by "freely adjoining filtered colimits", while the dual pro-category $\mathrm{Pro}(\mathcal{C})$ is obtained by "freely adjoining cofiltered limits". By a (co)limit, we always mean one that is indexed by a small category. Let us first recall what filtered colimits and cofiltered limits are.

Definition 4.1. A category I is called *filtered* if

- it is non-empty,
- for any two objects i and j in I, there exists an object k together with morphisms $i \to k$ and $j \to k$, and
- for any two parallel morphisms $f, g\colon i \rightrightarrows j$, there exists a morphism $h\colon j \to k$ such that $hf = hg$.

[7]Being simplicial with respect to the Joyal model structure is a strictly more general notion than being simplicial with respect to the Kan-Quillen model structure. This follows since the Joyal model structure has the same class of trivial fibrations as the Kan-Quillen model structure, while it has strictly more fibrations.

Dually, a category J is called *cofiltered* if J^{op} is filtered. A *filtered colimit* is a colimit of a diagram indexed by a filtered category, while a *cofiltered limit* is a limit of a diagram indexed by a cofiltered category.

Examples of filtered colimits are colimits indexed by directed sets, such as the set of natural numbers.

Drawing inspiration from the fact that the category of presheaves on a category can be seen as the free cocompletion of this category, we define ind- and pro-categories as follows.

Definition 4.2. Let $\mathcal{C}$ be a category. The *ind-category* $\operatorname{Ind}(\mathcal{C})$ of $\mathcal{C}$ is defined as the full subcategory of the presheaf category $\mathbf{Set}^{\mathcal{C}^{op}}$ spanned by those objects that can be written as filtered colimits of representables. Dually, the *pro-category* $\operatorname{Pro}(\mathcal{C})$ of $\mathcal{C}$ is defined as the full subcategory of $(\mathbf{Set}^{\mathcal{C}})^{op}$ whose objects are those that can be written cofiltered limits of representables (where the limit is computed in $(\mathbf{Set}^{\mathcal{C}})^{op}$).

It follows directly from this definition that $\operatorname{Pro}(\mathcal{C}) = \operatorname{Ind}(\mathcal{C}^{op})^{op}$, hence the theory of pro-categories is formally dual to that of ind-categories.

The Yoneda embedding gives us fully faithful functors $\mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})$ and $\mathcal{C} \hookrightarrow \operatorname{Pro}(\mathcal{C})$. These can be used to express the following universal property, where we use $\operatorname{Fun}^{\mathrm{f}}(-,-)$ to denote the category of filtered colimit preserving functors and $\operatorname{Fun}^{\mathrm{cf}}(-,-)$ to denote the category of cofiltered limit preserving functors.

Theorem 4.3 ([GAV72, Exposé 1, Prop. 8.7.3])**.** *Let $\mathcal{C}$ be any category. The ind-category $\operatorname{Ind}(\mathcal{C})$ admits all filtered colimits, and for any category $\mathcal{E}$ that admits filtered colimits, restricting along the inclusion $\mathcal{C} \hookrightarrow \operatorname{Ind}(\mathcal{C})$ gives an equivalence of categories $\operatorname{Fun}^{\mathrm{f}}(\operatorname{Ind}(\mathcal{C}), \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$. Dually, the category $\operatorname{Pro}(\mathcal{C})$ admits all cofiltered limits, and for any category $\mathcal{E}$ that admits cofiltered limits, the restriction functor $\operatorname{Fun}^{\mathrm{cf}}(\operatorname{Pro}(\mathcal{C}), \mathcal{E}) \to \operatorname{Fun}^{\mathrm{cf}}(\mathcal{C}, \mathcal{E})$ is an equivalence of categories.*

This universal property justifies calling $\operatorname{Ind}(\mathcal{C})$ the category obtained by freely adjoining all filtered colimits, and $\operatorname{Pro}(\mathcal{C})$ the category obtained by freely adjoining all cofiltered limits to $\mathcal{C}$.

There is another common construction of ind- and pro-categories that is also worth mentioning. Namely, one defines the class of objects of the category $\operatorname{Ind}(\mathcal{C})$ as the class of all diagrams $I \to \mathcal{C}$ where I can be any filtered category, and the set of morphisms between two such diagrams $\{C_i\}_{i \in I}$ and $\{D_j\}_{j \in J}$ is defined by the formula

$$\operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(\{C_i\}, \{D_j\}) = \lim_i \operatorname{colim}_j \operatorname{Hom}_{\mathcal{C}}(C_i, D_j).$$

Using that $\operatorname{Pro}(\mathcal{C}) = \operatorname{Ind}(\mathcal{C}^{op})^{op}$, it is easy to give an analogous definition of $\operatorname{Pro}(\mathcal{C})$. It can indeed be shown that these definitions of $\operatorname{Ind}(\mathcal{C})$ and $\operatorname{Pro}(\mathcal{C})$ are equivalent to the ones given above. While this second definition requires a bit more work to set up the theory (for example, it is more work to show that $\mathrm{Ind}(\mathcal{C})$ admits all filtered colimits in this case), its explicit nature is often useful when trying to compute limits and colimits (cf. [Isa02]). Throughout this thesis, both definitions are used interchangeably.

4.2 Ind- and pro-categories in nature

To give some feeling for how ind- and pro-categories behave, and to motivate their definition, we discuss several examples.

Example 4.4. Many common categories are ind-categories. Namely, any category that admits a collection of objects that are "finitely presented" and that generate the whole category under filtered colimits is an ind-category (see Lemma 2.2 of Paper I for a precise statement).

(i) The category **Set** of sets is equivalent to $\mathrm{Ind}(\mathbf{Set}_{\mathrm{fin}})$, the ind-category of the category of finite sets. The equivalence in the direction $\mathrm{Ind}(\mathbf{Set}_{\mathrm{fin}}) \to \mathbf{Set}$ is obtained by applying the universal property of $\mathrm{Ind}(\mathbf{Set}_{\mathrm{fin}})$ to the inclusion $\mathbf{Set}_{\mathrm{fin}} \hookrightarrow \mathbf{Set}$, while the functor in the other direction is obtained by writing a set as the filtered colimit of its finite subset, ordered by inclusion.

(ii) The category of Abelian groups **Ab** is equivalent to the ind-category $\mathrm{Ind}(\mathbf{Ab}_{\mathrm{fg}})$ of the category finitely generated Abelian groups. (Note that Abelian groups are finitely generated if and only if they are finitely presented.) This equivalence is obtained in the same way as the one between $\mathrm{Ind}(\mathbf{Set}_{\mathrm{fin}})$ and **Set**.

(iii) For general groups, not every finitely generated group is finitely presented. The category of groups **Grp** is equivalent to the ind-category $\mathrm{Ind}(\mathbf{Grp}_{\mathrm{fp}})$ of the category of finitely presented groups, but not to the ind-category of the category of finitely generated groups. The equivalence in the direction $\mathrm{Ind}(\mathbf{Grp}_{\mathrm{fp}}) \to \mathbf{Grp}$ again comes from the universal property of ind-categories. The functor in the other direction is slightly more complicated than in the two cases above: to define what it does on a group G, one needs to consider all maps from finitely presented groups to G and not just subgroup inclusions.

(iv) The category of simplicial sets **sSet** is equivalent to $\mathrm{Ind}(\mathbf{sSet}_{\mathrm{fin}})$, where $\mathbf{sSet}_{\mathrm{fin}}$ denotes the category of *finite* simplicial sets; that is, simplicial sets that have finitely many non-degenerate simplices.

This list can be extended much further; we hope that the general idea is clear and the reader is invited to come up with more examples of common categories that are ind-categories.

Pro-categories do not occur as much in nature as ind-categories (except as duals of ind-categories, of course). However, there are a few situations where pro-objects do arise naturally, usually when there is some kind of duality involved.

Example 4.5. (i) The category of *Stone spaces*, i.e. compact Hausdorff totally disconnected spaces, is the pro-category $\text{Pro}(\mathbf{Set}_{\text{fin}})$ of the category of finite sets. In the direction $\text{Pro}(\mathbf{Set}_{\text{fin}}) \to \mathbf{Stone}$, the equivalence is obtained by applying the universal property to $\mathbf{Set}_{\text{fin}} \hookrightarrow \mathbf{Stone}$, where every finite set is viewed as a discrete topological space. One can check by hand that this functor is essentially surjective and fully faithful, hence an equivalence, but there is also a slick proof using Stone duality. Namely, Stone duality asserts that the category of Stone spaces is dual to the category of Boolean algebras. It is easy to verify that the category of Boolean algebras is the ind-category of the category of finite Boolean algebras, which in turn are dual to finite sets.

(ii) The category of profinite groups, i.e. the pro-category of the category of finite groups $\mathbf{Grp}_{\text{fin}}$, is equivalent to the category of topological groups whose underlying space is a Stone space. This may seem like a simple consequence of the fact that the category of profinite sets is equivalent to the category of Stone spaces, but the proof is actually more involved (see for example [Joh82, §VI.2]).

(iii) Call a groupoid finite if it has finitely many arrows. The category of profinite groupoids, i.e. the pro-category of the category of finite groupoids, turns out to be equivalent to a full subcategory of the category of groupoids internal to Stone spaces. However, there exist groupoids internal to Stone spaces that are not an inverse limit of finite groupoids, so these two categories are not equivalent.

(iv) The category of simplicial Stone spaces is equivalent to the pro-category of a certain full subcategory of the category of simplicial sets, namely that of degreewise finite simplicial sets that are furthermore coskeletal (cf. Theorem 2.3 of Paper I). This fact and variations on it play an important role in Papers I and II.

As mentioned, pro-categories often arise in a setting where there is some kind of duality. Given a cofiltered limit in a category where there is some kind of duality, it is natural to consider the dual as a pro-object; that is, as an object in a pro-category. For example, the finite field extensions of the rational numbers form a filtered diagram whose colimit is the algebraic closure of the rationals. By considering the Galois groups of these finite field extensions, we obtain a cofiltered diagram whose limit is the absolute Galois group of the rationals. This shows that there is a natural way to view the absolute Galois group as a profinite group, and one loses information by only considering the underlying group. Another example is that if one wants to extend Spanier-Whitehead duality to arbitrary spectra, then one should work with pro-spectra (cf. [CI04]).

A different situation where pro-categories come up is when working with completion functors, which are functors that take an object in some category to an object in a pro-category that approximates the original object. These completions often carry extra structure not present on the original object that may help in certain computations. For example, Sullivan shows in his proof of the Adams conjecture [Sul74] that the profinite completions of the classifying spaces for topological K-theory (both real and complex) admit an action of the absolute Galois group of the rationals. This action is the crucial ingredient in his proof of the Adams conjecture.

4.3 Why care about model structures on pro-categories?

In general, the reason why one should care about model structures on pro-categories is that there are various situations where pro-objects naturally come up and where there is also a notion of "homotopy", and their homotopy-invariant properties contain interesting information. By way of illustration, let us consider the étale homotopy type of a scheme, originally defined by Artin–Mazur in [AM69]. This is an example of a naturally arising pro-object where one is interested in its homotopy-invariant properties, such as its homotopy groups or its cohomology. Having standard tools from algebraic topology and homotopy theory available can be helpful to study these, hence the existence of a model category of "pro-spaces" in which this étale homotopy type lives is desirable. While Artin and Mazur constructed the étale homotopy type as a pro-object in the homotopy category of spaces, their construction has later been rectified to land in the category of pro-simplicial sets (cf. [Fri82]), and several model categories describing homotopy theories of pro-spaces, profinite spaces or pro-p spaces have been developed (cf. [Isa01; Isa05; Qui08; Mor96]). These model structures have several applications in algebraic geometry as well as algebraic topology.

5 Goodwillie calculus

In the papers [Goo90; Goo92; Goo03], Goodwillie develops a calculus for functors from the category of (pointed) topological spaces to (pointed) topological spaces or spectra. The idea is to approximate a functor by so-called *n-excisive approximations*, similar to how one approximates a real function by its Taylor polynomials. These n-excisive approximations fit together into a tower called the Goodwillie tower. The layers of this tower correspond to the monomials of the Tayler series under the analogy with real analysis, and they admit "coefficients" which behave like the derivatives of a function.

In this section, we will give a basic introduction to Goodwillie calculus. We will focus on functors $\mathbf{Top}_* \to \mathbf{Top}_*$, but the ideas go through in much greater generality. The reader is assumed to be familiar with homotopy (co)limits. For a more in-depth introduction, including discussions of several applications, we refer the reader to [AC20] and the references cited there.

5.1 Excisive approximations

All functors in this section are assumed to preserve weak equivalences. A functor $\mathbf{Top}_* \to \mathbf{Top}_*$ is defined to be *excisive* if takes homotopy pushouts to homotopy pullbacks. The name is due to the resemblance to the excision property in homology. In fact, the prototypical example of an excisive functor is one of the form

$$X \mapsto \Omega^\infty(H \wedge X),$$

where H denotes a fixed spectrum.

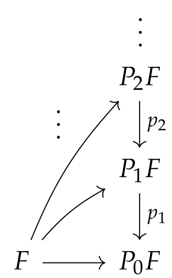
By replacing these pushout and pullback squares with higher-dimensional cubes, one arrives at the definition of an n-excisive functor. Let $\underline{n}$ denote the set $\{1, \ldots, n\}$ of n elements and $\mathcal{P}(\underline{n})$ its collection of subsets, ordered by inclusion. We write $\mathcal{P}_0(\underline{n})$ for the poset of nonempty subsets of $\underline{n}$.

Definition 5.1. An *n-cube* is defined as a diagram indexed by $\mathcal{P}(\underline{n})$. An n-cube $X\colon \mathcal{P}(\underline{n}) \to \mathbf{Top}_*$ will be called *strongly coCartesian* if each of its 2-dimensional faces is a homotopy pushout, and it will be called *Cartesian* if its initial object is the homotopy limit of the restriction of X to $\mathcal{P}_0(\underline{n})$.

Using higher dimensional cubes, one can define a higher version of excision.

Definition 5.2. A functor $F\colon \mathbf{Top}_* \to \mathbf{Top}_*$ is *n-excisive* if it takes strongly coCartesian n-cubes to Cartesian n-cubes.

One can construct, for any functor F, its *n-excisive approximation* P_nF: there exists a universal map $F \to P_nF$ from F into an n-excisive functor. By universal, we mean that any natural map $F \to G$ into an n-excisive functor factors through P_nF up to homotopy, and moreover that this factorization is unique up to homotopy. Since one can show that any n-excisive functor is $(n+1)$-excisive, these approximations fit together into a tower



called the *Goodwillie tower of F*. This tower can be thought of as the analogue of the Taylor series of a real function. The tower is said to *converge* at $X \in \mathbf{Top}_*$ if $F(X) \simeq \lim_n P_nF(X)$. Note that this inverse limit should be interpreted as a homotopy limit.

5.2 Homogeneous layers and derivatives

When trying to understand a tower, it is generally a good idea to study its layers. The *n-th layer* $D_n F$ of the Goodwillie tower is defined as the homotopy fiber of the map $P_n F \to P_{n-1} F$. These layers are examples of *n-homogeneous functors*.

Definition 5.3. A functor $F\colon \mathbf{Top}_* \to \mathbf{Top}_*$ is called *n-reduced* if $P_{n-1}F \simeq *$. It is called *n-homogeneous* if it is both n-excisive and n-reduced. A 1-homogeneous functor is also called *linear*.

It turns out that these homogeneous layers admit a simple description in terms of certain "coefficient spectra", at least when evaluated on finite pointed CW-complexes.

Theorem 5.4 ([Goo03]). *Let $H\colon \mathbf{Top}_* \to \mathbf{Top}_*$ be n-homogeneous. Then there exists an essentially unique spectrum C with an action of the symmetric group Σ_n such that there is a natural isomorphism*

$$H(X) \simeq \Omega^\infty \left(C \wedge X^{\wedge n}\right)_{h\Sigma_n}$$

for any finite pointed CW-complex X.

The coefficient spectrum $\partial_n F$ corresponding to $D_n F$ via this result is called the n-th derivative of F. It turns out that these coefficient spectra are generally easier to compute than the Taylor tower itself; techniques for computing them are discussed in [Goo03, §6]. This suggests an approach to studying a functor $\mathbf{Top}_* \to \mathbf{Top}_*$: if the Taylor tower is known to converge, one could try computing the derivatives of the functor and study how these can be assembled into the Taylor tower of that functor. For a discussion of techniques that can be used for this, we refer the reader to [AC20, §1.4].

In Paper V, we extend Goodwillie's classification of n-homogeneous functors stated in Theorem 5.4 to a more general setting.

Summaries of included papers

Since earlier versions of Papers I, II and III were included in the author's licentiate thesis, the first three summaries are slightly updated versions of the respective summaries in [Blo21].

Paper I

In this paper, we describe a general method for constructing simplicial model structures on ind- and pro-categories. Especially in the case of pro-categories, this method can be used to recover many interesting known model structures, while it can also be applied to produce many new model categories. The main result can be paraphrased as the following theorem:

Theorem. *Let $\mathcal{M}$ be a simplicial model category in which every object is cofibrant and let $\mathcal{C}$ be an essentially small full subcategory of $\mathcal{M}$ closed under finite limits and cotensors by finite simplicial sets. Then for any collection $\mathcal{T}$ of fibrant objects in $\mathcal{C}$, the pro-category* $\mathrm{Pro}(\mathcal{C})$ *carries a fibrantly generated simplicial model structure with the following properties:*

1. *The weak equivalences are the $\mathcal{T}$-local equivalences; that is, a map $f\colon C \to D$ is a weak equivalence if and only if for any $T \in \mathcal{T}$, the map*
$$f^*\colon\ \mathrm{Map}(D, T) \to \mathrm{Map}(C, T)$$
is a weak equivalence.
2. *Every object in* $\mathrm{Pro}(\mathcal{C})$ *is again cofibrant.*
3. *The inclusion $\mathcal{C} \hookrightarrow \mathcal{M}$ induces a simplicial Quillen adjunction $\mathcal{M} \rightleftarrows \mathrm{Pro}(\mathcal{C})$ whose left adjoint can be thought of as a profinite completion functor.*

Examples of known model structures that one can obtain by applying this theorem are Morel's model structure for pro-p spaces [Mor96], Quick's model structure for profinite spaces [Qui08] and Horel's model structure for profinite groupoids [Hor17, §4]. New model structures that one can obtain include profinite versions of the Joyal model structure for quasi-categories and Rezk's model structure for complete Segal spaces, and, for any finite poset P, a model structure describing the homotopy theory of profinite P-stratified spaces (cf. [BGH18, §2.5]). These examples are also studied in some detail.

The construction is based on the notion of a *fibration test category*, which is a small simplicial category $\mathcal{C}$ together with a subset $\mathcal{T}$ of objects and some extra structure that ensures the existence of a model structure on $\mathrm{Pro}(\mathcal{C})$ in which the weak equivalences are the $\mathcal{T}$-local ones. It is then shown that, under the assumptions of the above theorem, the pair $(\mathcal{C}, \mathcal{T})$ can be given the structure of a fibration test category in such a way that the obtained model structure on $\mathrm{Pro}(\mathcal{C})$ satisfies all properties of the above theorem.

It is worth pointing out that in the construction of the model structure on $\mathrm{Pro}(\mathcal{C})$ we work dually; that is, we actually define what a *cofibration test category* is, show that there exists a model structure on the ind-category $\mathrm{Ind}(\mathcal{C}')$ of a cofibration test category $(\mathcal{C}', \mathcal{T}')$, and then dualize this to a result about pro-categories. The reason for this is that the arguments used in constructing the model structure on $\mathrm{Ind}(\mathcal{C}')$ are all very standard, unlike the dual arguments needed in the case of pro-categories, hence easier to follow for the reader.

In the appendix of the paper, we give a precise characterization of the underlying ∞-category of the model structure that we constructed on $\mathrm{Pro}(\mathcal{C})$.

Paper II

The aim of Paper II is to construct a profinite completion functor for ∞-operads.

Recall that dendroidal sets are set-valued presheaves on the category Ω of trees (cf. [HM22]) and that they can be used as models for ∞-operads. The latter is made precise by the existence of a model structure on the category **dSet** of dendroidal sets that is Quillen equivalent to the model category of topological operads and whose fibrant objects are a generalization of Joyal's quasi-categories. In this paper, we construct an analogous model structure on the category of dendroidal Stone spaces, i.e., the category $\mathbf{dStone} = \mathrm{Fun}(\Omega^{op}, \mathbf{Stone})$, that one can view as describing a "homotopy theory of profinite ∞-operads".

The method used for this is similar to that of Paper I. However, the arguments from Paper I cannot be applied directly, as not every object in **dSet** is cofibrant. This somewhat complicates the construction of the desired model structure on **dStone**, making it necessary to study the behaviour of normal monomorphisms between dendroidal Stone spaces in detail.

The methods used are sufficiently general to also allow the construction of model structures on the categories of open and closed dendroidal Stone spaces that describe the homotopy theory of non-unitary and unitary profinite ∞-operads, respectively.

The forgetful functor **dStone** $\to$ **dSet** admits a left adjoint, which is simply the profinite completion functor for sets applied pointwise. Once the model structure (and its variants) described above is established, it is straightforward to show that the profinite completion functor is indeed left Quillen. In particular, this provides a way of describing the profinite completion of an ∞-operad: by using dendroidal sets to model ∞-operads, one can simply apply the profinite completion functor to a dendroidal set and take a fibrant replacement in **dStone** to obtain the profinite completion of an ∞-operad.

A particularly important role in the construction of the model structure on **dStone** (and its variants) is played by the so-called *lean* ∞-operads. For this reason, we also included a precise homotopical characterization of these lean ∞-operads (cf. Section 2.3 of Paper II).

Paper III

In this paper, we present an alternative proof for one of the main results of [ABS22], which states that the stabilization of the category of noncommutative CW-complexes can be modelled by the category of spectral presheaves on a certain spectrum-enriched category $\mathcal{M}_s$. We first show that the methods from Paper I can be used to construct a symmetric monoidal model category describing the homotopy theory of the noncommutative CW-complexes defined in [ABS22, §2]. We then show that this model category can be stabilized by considering functors from the category of finite simplicial sets into this category. This produces a symmetric monoidal stable model category which is enriched over Lydakis's stable model category of simplicial functors [Lyd98]. A slightly modified version of Schwede's and Shipley's theorem [SS03, Theorem 3.9.3.(iii)] is then used to show that this category is Quillen equivalent to the category of enriched presheaves over a certain category $\mathcal{M}_\Delta$ enriched in Lydakis's category of simplicial functors. By changing the base of enrichment, this presheaf category is then shown to be Quillen equivalent to the category of spectral presheaves on $\mathcal{M}_s$, recovering the result of [ABS22].

The appendix to this paper contains a modification of the definition of a cofibration test category from Paper I, which we call a *minimal* cofibration test category. While the data of a minimal cofibration test category is slightly smaller than that of a cofibration test category, its axioms are much simpler and therefore easier to verify.

Paper IV

The problem that Paper IV addresses is that of whether ordinary functors into an enriched model category are weakly equivalent to enriched functors.

In Section 2 of the general introduction, we saw that there are sometimes easy criteria to check whether an ordinary functor can be upgraded to an enriched one. Notable cases are those of **Ab**-enriched functors between additive categories and $\mathbf{Set}_*$-enriched functors between pointed categories. However, for most enrichments one should not expect such simple criteria.

In Paper IV, we consider the homotopical version in this question in the cases where the base of enrichment is either **sSet**, **Top**, $\mathbf{Top}_*$, $\mathbf{sSet}_*$ or the category **Sp** of orthogonal spectra. We show that in these cases, under certain assumptions on the source and target category there exist simple criteria that ensure that a functor is weakly equivalent to an enriched one. Moreover, we show that there is often a Quillen equivalence between the category of functors satisfying these criteria and the category of enriched functors. In particular, if one is given a functor satisfying these criteria, then not only is it equivalent to an enriched one, but moreover the space of enriched functors to which it is equivalent is contractible.

As a corollary of the case of **sSet**-enriched functors, we also obtain a result that is useful for computing Dwyer-Kan localizations of categories (cf. Theorem B of Paper IV).

Paper V

While Goodwillie's calculus of functors from spaces to spaces or spectra is easily generalized to other settings in homotopy theory, this does not hold for his classification of n-homogeneous functors: this result depends on the fact that in $\mathbf{Top}_*$, every object can be obtained as homotopy colimit of copies of S^0. In Paper V, we extend Goodwillie's classification to n-homogeneous functors from $\mathbf{Top}_*^{\mathcal{C}}$ to either $\mathbf{Top}_*^{\mathcal{D}}$ or $\mathbf{Sp}^{\mathcal{D}}$, where $\mathcal{C}$ is a small $\mathbf{Top}_*$-category and $\mathcal{D}$ is a small $\mathbf{Top}_*$- or $\mathbf{Sp}$-category. For brevity, we only state the result in the case that the target is $\mathbf{Top}_*^{\mathcal{D}}$.

Theorem. *Let $\mathcal{C}$ and $\mathcal{D}$ be small $\mathbf{Top}_*$-categories whose hom-spaces are nondegenerately based, and let $F\colon \mathbf{Top}_*^{\mathcal{C}} \to \mathbf{Top}_*^{\mathcal{D}}$ be a functor that preserves weak equivalences and filtered homotopy colimits. Then for every $n \geq 0$, there exists an essentially unique diagram $\partial_n F\colon (\Sigma_n \wr (\mathcal{C}^{op})^{\wedge n}) \wedge \mathcal{D} \to \mathbf{Sp}$ such that there is a natural equivalence*

$$D_nF(X)(d) \simeq \Omega^\infty \left(\int^{\mathcal{C}^{\wedge n}} \partial_n F(c_1, \ldots, c_n, d) \wedge X_1(c_1) \wedge \ldots \wedge X_n(c_n) \right)_{h\Sigma_n}$$

The diagram $\partial_n F\colon (\Sigma_n \wr (\mathcal{C}^{op})^{\wedge n}) \wedge \mathcal{D} \to \mathbf{Sp}$ will be called the *n-th derivative* of F. The category denoted by $\Sigma_n \wr (\mathcal{C}^{op})^{\wedge n}$ is the *wreath product category*, which has the property that functors out of it are n-variable functors out of $\mathcal{C}$ that are symmetric. This theorem is obtained by establishing a Quillen equivalence between the category of functors $(\Sigma_n \wr (\mathcal{C}^{op})^{\wedge n}) \wedge \mathcal{D} \to \mathbf{Sp}$ and a suitable model category of n-homogeneous functors. After establishing this theorem, we also discuss methods for computing these derivatives for a given functor $F\colon \mathbf{Top}_*^{\mathcal{C}} \to \mathbf{Top}_*^{\mathcal{D}}$.

We conclude this paper by studying the derivatives of the identity functor $\mathbf{Top}_*^{\mathcal{C}} \to \mathbf{Top}_*^{\mathcal{C}}$. We show that these can be endowed with an operad structure generalizing that of Ching [Chi05].

Theorem. *The derivatives* $\partial_* \mathrm{Id}_{\mathcal{C}}$ *of the identity functor* $\mathrm{Id}_{\mathcal{C}} \colon \mathbf{Top}_*^{\mathcal{C}} \to \mathbf{Top}_*^{\mathcal{C}}$ *form the (coloured) operad*

$$\partial_* \mathrm{Id}_{\mathcal{C}} = \partial_* \mathrm{Id}_* \otimes_{BV} \mathcal{C},$$

where $\otimes_{BV}$ *denotes the Boardman-Vogt tensor product and* $\partial_* \mathrm{Id}_*$ *denotes the derivatives of the identity* $\mathrm{Id} \colon \mathbf{Top}_* \to \mathbf{Top}_*$ *endowed with Ching's operad structure.*

Note that this statement is somewhat imprecise; for details, the reader is referred to Section 5 of Paper V.

Conclusion

In Paper I, we present a new method for endowing pro-categories with a model structure. While such methods already existed in the literature, our approach applies to a different class of examples. The conditions for our method to work are much easier to verify than in the earlier approaches, while our proofs are surprisingly straightforward. On the one hand, our method recovers model structures that were already constructed before, but by very different methods, shedding new light on them. On the other hand, we also obtain many new model categories.

Besides the examples discussed in Paper I, a notable example of such a new model category is given in Paper III. Here we use the methods from Paper I to construct a model category describing the homotopy theory of noncommutative CW-complexes. This model category is then used to carry out a more direct and straightforward proof of the main result of [ABS22].

In Paper II, we construct a model category of profinite ∞-operads and a profinite completion functor for ∞-operads. Such profinite completions may carry symmetries that do not exist in the original ∞-operad; an example is Horel's construction of an action of the Grothendieck-Teichmüller group on the profinite completion of the little 2-discs operad. We hope that our theory can be used to establish more such results.

The results of Paper IV concern the question of when a functor is equivalent to an enriched one. Such functors generally have a lot of extra structure and properties that can have various applications. An example of why enriched functors are useful is that one can consider enriched Kan extensions of them. A situation where such enriched Kan extensions play an important role is in Paper V. Here we give a classification of n-homogeneous functors, in the sense of Goodwillie calculus, for functors between diagram categories. This classification is defined in terms of enriched Kan extensions and hence requires one to work with enriched functors. This was one of the main motivations for writing Paper IV.

The classification proved in Paper V can be applied to obtain explicit formulas describing the layers of the Goodwillie tower. We apply this to the case of the identity functor between categories of diagrams, where we construct a generalization of Ching's operad structure [Chi05] on the derivatives of the identity $\mathbf{Top}_* \to \mathbf{Top}_*$. This operad structure suggests an extension of the

Arone–Ching chain rule [AC11] for the derivatives of functors $\mathbf{Top}_* \to \mathbf{Top}_*$. In future work, we intend to prove an analogue of the main result of Paper V for functors between compactly generated stable model categories. Moreover, we aim to compute the derivatives of functors between diagram categories in several other examples, such as stable equivariant mapping spaces.

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