Stockholm University

Amenable-like properties of étale groupoids

Gabriel Favre

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Abstract

This thesis consists of three papers related to analytic and representation theoretic properties of étale groupoids.

In the first paper, we characterize algebraically the type I and CCR property for ample groupoids and their non-commutative duals: Boolean inverse semigroups. Our results use and generalize Thoma's work on discrete groups. Algebraic characterizations in the more general context of non-Hausdorff groupoids have been obtained in the author's licentiate thesis. They use a non-Hausdorff version of the Clark-van Wyk topological characterization. We also characterize type I inverse semigroups using the Booleanization of inverse semigroups introduced by Lawson. The inverse semigroups of type I are characterized by excluding specific subquotients of their Booleanization.

In the second paper, we show that any free action of a connected Lie group of polynomial growth on a finite dimensional locally compact space has a finite tubular dimension by constructing a tubular cover of appropriate multiplicity. As a consequence, the C^*-algebras associated to the corresponding transformation groupoids all have finite nuclear dimension. The proof strategy is adapted from the strategy for R-actions of Hirshberg-Wu to the polynomial growth setting. As a corollary, we obtain that the groupoids associated to model sets in connected simply connected nilpotent Lie groups admit a classifiable C^*-algebra.

In the third paper, we study inner amenability for groupoids attached to irregular point sets in general second countable locally compact groups. Upon imposing a regularity condition on the point set–finite local complexity–we are able to show inner amenability of the corresponding ample groupoid. The motivation for this work is the question of Anantharaman-Delaroche asking whether all étale groupoids are inner amenable. As a motivating example, model sets arising from arithmetic lattices give inner amenable groupoids, even in non-amenable groups.

Keywords: *Etale groupoids, Representation theory, amenability, inverse semigroups, C*-algebra.*

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À mes amours.

"Climbing to the top is optional.
Getting down is mandatory"
Ed Viesturs

Abstract

This thesis consists of three papers related to analytic and representation theoretic properties of étale groupoids.

In the first paper, we characterize algebraically the type I and CCR property for ample groupoids and their non-commutative duals: Boolean inverse semigroups. Our results use and generalize Thoma's work on discrete groups. Algebraic characterizations in the more general context of non-Hausdorff groupoids have been obtained in the author's licentiate thesis. They use a non-Hausdorff version of the Clark-van Wyk topological characterization. We also characterize type I inverse semigroups using the Booleanization of inverse semigroups introduced by Lawson. The inverse semigroups of type I are characterized by excluding specific subquotients of their Booleanization.

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Sammanfattning

Denna avhandling består av tre artiklar relaterade till analytiska och representationsteoretiska egenskaper hos étale gruppoider.

I den första artikeln karakteriserar vi algebraiskt typ I- och CCR-egenskapen för ymniga gruppoider och deras icke-kommutativa dual: booleska inversa semigrupper. Våra resultat är förbättringar av Thomas arbete med diskreta grupper. Icke-Hausdorff algebraiska karakteriseringar har erhållits i författarens licentiatavhandling för både ymniga och étale gruppoider som är beroende av en icke-Hausdorff-version av Clark-van Wyk topologiska karakterisering. Vi karakteriserar också typ I inversa semigrupper med Lawsons booleska inverskomplettering. De inversa semigrupperna av typ I kännetecknas av att de exkluderar särskilda delkvoter av deras booleanisering.

I den andra artikeln visar vi att en godtycklig fri verkan av en sammanhängande Lie-grupp med polynomtillväxt på ett ändligtdimensionellt lokalt kompakt topologiskt rum har en ändlig tubulär dimension genom att konstruera ett tubulär övertäckning med lämplig mångfald. Som en konsekvens har C^*-algebrorna associerade med motsvarande transformationsgruppoider alla en ändlig kärndimension. Bevisstrategin är anpassad från strategin för $\mathbb{R}$-verkningar av Hirshberg-Wu till den polynomtillväxtmiljön. Som en följd av detta får vi att gruppoiden som är associerad med modellmängder i sammanhängande enkelt sammanhängande nilpotenta Lie-grupper medger en klassificerbar C^*-algebra.

I den tredje artikeln studerar vi inre foglighet för gruppoider fästa vid oregelbundna punktmängder i allmänna andra uppräkneliga lokalt kompakta grupper. Genom att införa ett regularitetsvillkor på punktmängden - ändlig lokal komplexitet - kan vi visa inre foglighet för motsvarande ymniga gruppoid. Ramen för detta arbete är frågan av Anantharman-Delaroche som frågar om alla étale gruppoider är inre fogliga. Ett motiverande exempel är modellmängder som härrör från aritmetiska gitter, som ger upphov till inre fogliga gruppoider, även i icke-fogliga grupper.

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this PhD thesis.

Paper I **An algebraic characterization of ample type I groupoids**
Semigroup Forum **104**, 58–71 (2022)
Gabriel Favre, Sven Raum

Paper II **Free actions of polynomial growth Lie groups and classifiable C^*-algebras**
Preprint, arXiv:2307.15013
Ulrik Enstad, Gabriel Favre, Sven Raum

Paper III **A note on inner amenability for FLC point sets**
Submitted, arXiv:2307.01880
Gabriel Favre

 Paper I ([FR21]) appeared in the author's Licentiate thesis. Paper II ([EFR23]) and Paper III ([Fav23]) are identical to their first arXiv versions, up to the correction of some typos. Paper III appeared on the arXiv slightly before paper II, but this choice makes it easier for the reader to follow the introduction.

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Contents

1. Introduction

The results of the Papers included in this thesis all relate to *étale groupoids* and related structures. The goal of Chapter 2 is to familiarize the reader with such groupoids and the analytic properties of interest in order to place the PhD thesis in a historical context. The main objective is to situate its relevance within current research from the point of view of a variety of different subjects, while a common point is the relation between groupoids and operator algebras. After these preliminaries, a brief summary of the results and proofs of each paper follows in Chapter 3.

Remark. This thesis builds partly upon the author's licentiate thesis ([Fav21]) defended on January 20th, 2022. The licentiate thesis was a monograph consisting of the results of Paper I of the doctoral thesis and their generalizations to the settings of non-Hausdorff groupoids and étale groupoids. These general statements and their proofs can be found in [Fav21].

2. Mathematical context

We begin the preliminaries by introducing étale groupoids, highlighting the place they occupy within operator algebras in Section 2.1. The key groupoid constructions used in Papers II and III will be introduced therein. These constructions can serve as a red thread throughout the exposition. After this, we introduce the analytic properties of C^*-algebras pivotal to the thesis which are the type I and CCR properties in Section 2.2, finite nuclear dimension in Section 2.3 and inner amenability in Section 2.4.

2.1 Étale groupoids

While a more down to earth approach will follow, a groupoid can be defined from the point of view of category theory as a small category in which every morphism is an isomorphism. Since the focus of the thesis is on topological groupoids, one can instead consider groupoid objects in the category of topological spaces.

A reader unfamiliar with the concept of topological groupoid should think of the following situation. Consider a discrete group Γ acting on a topological space X. Using the multiplication of Γ and the group action, one can define a product and inverse which turn the space $\Gamma \times X$ into a groupoid called *transformation groupoid*. An element (γ, x) of the space $\Gamma \times X$ can be viewed as map sending x to γx via γ. The source and range maps $s, r : \Gamma \times X \to X$ defined by $s(g, x) = x$, $r(g, x) = gx$ are local homeomorphisms. That observation makes $\Gamma \times X$ an *étale* groupoid. This groupoid is usually denoted by $\Gamma \ltimes X$. Let us first give the following informal definition of a topological groupoid.

A *topological groupoid* is a second countable topological space $\mathcal{G}$ together with a Hausdorff locally compact subset $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ of *units*, continuous source and range maps $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$, a continuous multiplication map $m : \{(\gamma, \gamma') \in \mathcal{G} \times \mathcal{G} \mid s(\gamma) = r(\gamma')\} \to \mathcal{G}$ and a continuous inverse map $\iota : \mathcal{G} \to \mathcal{G}$ satisfying certain relations.

A groupoid $\mathcal{G}$ is called *étale*[1] if the source and range maps s and r are local homeomorphisms. In other words, a groupoid $\mathcal{G}$ is étale if the

[1]after the fact that r and s are étale maps between topological spaces.

topology on $\mathcal{G}$ has a basis consisting of open sets $U \subseteq \mathcal{G}$ for which $r|_U$ and $s|_U$ are homeomoprhisms onto their image. Making the last two paragraphs rigorous, we make the following formal definition.

Definition 2.1.1. An *étale groupoid* is a topological space $\mathcal{G}$ together with a distinguished subspace $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a continuous multiplication map $\mathcal{G}^{(2)} \to \mathcal{G}$, $(\alpha, \beta) \mapsto \alpha\beta$, and a continuous inversion map $\mathcal{G} \to \mathcal{G}$, $\alpha \mapsto \alpha^{-1}$, such that the following properties are satisfied.

1. If $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^{(2)}$, then $(\alpha\beta, \gamma), (\alpha, \beta\gamma) \in \mathcal{G}^{(2)}$, and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;

2. $(\alpha, \alpha^{-1}) \in \mathcal{G}^{(2)}$ for all $\alpha \in \mathcal{G}$, and for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$ we have that $\alpha^{-1}(\alpha\beta) = \beta$ and $(\alpha\beta)\beta^{-1} = \alpha$;

3. For all $\alpha \in \mathcal{G}$ we have that $(\alpha^{-1})^{-1} = \alpha$;

4. The maps $\mathrm{s}, \mathrm{r} \colon \mathcal{G} \to \mathcal{G}^{(0)}$ given by $\mathrm{s}(\alpha) = \alpha^{-1}\alpha$ and $\mathrm{r}(\alpha) = \alpha\alpha^{-1}$ for $\alpha \in \mathcal{G}$ are local homeomorphisms, where $\mathcal{G}^{(0)}$ is the subspace of units $\mathcal{G}^{(0)} = \{\alpha^{-1}\alpha \mid \alpha \in \mathcal{G}\} = \{\alpha\alpha^{-1} \mid \alpha \in \mathcal{G}\}$.

The open subsets U of a groupoid $\mathcal{G}$ for which the restrictions $s|_U$ and $r|_U$ of the source and range maps are injective are called open *bisections*. When the topology on $\mathcal{G}$ has a basis consisting of compact open bisections, $\mathcal{G}$ is called *ample*. For an étale groupoid $\mathcal{G}$, this corresponds to the case where the unit space $\mathcal{G}^{(0)}$ is totally disconnected. Open bisections can be multiplied using the multiplication of the groupoid and they collectively form a semigroup $\Gamma(\mathcal{G})$ which is in fact an inverse semigroup, meaning that every open bisection U has a unique generalized inverse V satisfying $UVU = U$ and $VUV = V$.

If further $\mathcal{G}$ is ample, then the semigroup $\Gamma_{\mathrm{CO}}(\mathcal{G})$ of compact open bisections has more structure. In addition to being an inverse semigroup, there is a symmetric difference in $\Gamma_{\mathrm{CO}}(\mathcal{G})$. More intuitively, in the case where $\mathcal{G}^{(0)}$ is compact, the idempotents have complements. The idempotents of $\Gamma_{\mathrm{CO}}(\mathcal{G})$ collectively form a (generalized) Boolean algebra. The semigroup $\Gamma_{\mathrm{CO}}(\mathcal{G})$ is called a Boolean inverse semigroup. Arbitrary inverse semigroups are not automatically Boolean inverse semigroups. However one can obtain a Boolean inverse semigroup from an inverse semigroup in a functorial way via the so-called Booleanization (see [Law20; LV21]). This completion was an important tool in Paper I. The semigroup $\Gamma_{\mathrm{CO}}(\mathcal{G})$ also encodes all the information about an ample groupoid $\mathcal{G}$ via a Stone-type duality and we will come back to this in Section 2.2.

For a groupoid $\mathcal{G}$ and a unit $u \in \mathcal{G}^{(0)}$ the *isotropy group* at u is the group

$$\mathcal{G}_u^u = \{\gamma \in \mathcal{G} \mid s(\gamma) = r(\gamma) = u\}.$$

When the groupoid $\mathcal{G}$ is étale, its isotropy groups are discrete. Some representation theoretic information on the groupoid $\mathcal{G}$ is encoded in the isotropy groups. An occurrence of this phenomenon is a characterization of the ideal intersection property of the C^*-algebra of $\mathcal{G}$ in terms of (absence of certain sections of) the isotropy groups in [Ken+21]. We will see another instance of this theme in Section 2.2.

Apart from transformation groupoids, sources of étale groupoids include the following examples.

Example 2.1.2. A disjoint union of discrete groups $\bigsqcup_{i \in I} \Gamma_i$ is a groupoid called group bundle. The unit space is the set $\{e_{\Gamma_i} \mid i \in I\}$ and the source and range maps are defined by

$$s(g) = r(g) = gg^{-1}.$$

The isotropy group at e_{Γ_i} is identified with Γ_i, so the groupoid $\bigsqcup_{i \in I} \Gamma_i$ is the union of its isotropy groups.

Example 2.1.3. An equivalence relation $\mathcal{R} \subseteq X \times X$ is a groupoid when equipped with the multiplication $m((x,y),(y,z)) = (x,z)$ and inverse $\iota(x,y) = (y,x)$. The unit space of $\mathcal{R}$ is identified with X via its diagonal embedding. The source and range maps are given by projections on the first and second coordinate respectively. This groupoid is étale when X is discrete.

There is a philosophy of viewing every étale groupoid as a generalized transformation groupoid which can be made precise by considering the action of the semigroup of open bisections on the unit space of an étale groupoid and making some identifications. Transformation groupoids also model most of the groupoid terminology, as can be seen from the use of "isotropy", "orbit", etc..

Uniform approximate lattices and their groupoids

In this section we present important examples of transformation groupoids which we study both in Paper II and Paper III. A resource on recent advances in the topic is the survey article [Mac23].

Example 2.1.4. Let Λ be a uniform lattice in a locally compact group G, meaning Λ is a cocompact discrete subgroup of G. The group G acts by left translation on the space G/Λ. The groupoid $G \ltimes G/\Lambda$ is étale precisely when G is discrete.

One can perform a similar construction upon relaxing the subgroup condition on a uniform lattice Λ in a locally compact group G. Recall the following definition of an approximate subgroup which goes back to Tao (see [Gre12]).

Definition 2.1.5. Let Λ be a symmetric discrete subset of a locally compact group G containing the identity. We say that Λ is an *approximate subgroup of* G if there is a finite subset $F \subseteq G$ such that $\Lambda\Lambda \subseteq F\Lambda$.

Since approximate subgroups are much more flexible, pathological behavior around the identity might be observed. Consequently, one needs to impose an extra regularity condition for approximate subgroups to resemble lattices in general locally compact groups. We say that a discrete subset Λ of a locally compact group G is *uniformly discrete* if there exists an identity neighborhood $U \subseteq G$ such that $\Lambda\Lambda^{-1} \cap U = \{e\}$.

Definition 2.1.6. [BH18, Definition 1.2] Let G be a locally compact group. We say that $\Lambda \subseteq G$ is a *uniform approximate lattice* if Λ is a uniformly discrete approximate subgroup of G and there is a compact set $K \subseteq G$ such that $\Lambda K = G$.

The groupoid construction in Example 2.1.4 can be generalized to more general point sets in general second countable locally compact groups, such as uniform approximate lattices.

Example 2.1.7. Let Λ be a uniform approximate lattice in a second countable locally compact group G. Denote by $\mathcal{C}(G)$ the set of all closed subsets of G. Let us endow $\mathcal{C}(G)$ with the Chabauty-Fell topology given by the subbasis consisting of the sets

$$O_K = \{C \in \mathcal{C}(G) \mid C \cap K = \emptyset\},$$
$$O_V = \{C \in \mathcal{C}(G) \mid C \cap V \neq \emptyset\},$$

where K ranges over all compact subsets of G and V ranges over all open subsets of G (see [ER22] Section 3.1). The space $\mathcal{C}(G)$ is compact and the group G acts by left translation on the space $\mathcal{C}(G)$. Consider the orbit of $\{\Lambda\}$ in $\mathcal{C}(G)$ and denote by $\Omega(\Lambda)$ its closure in the Chabauty-Fell topology. The compact G-space $\Omega(\Lambda)$ is called the *hull*. The set Λ can then be studied through the action $G \curvearrowright \Omega(\Lambda)$ and its associated transformation groupoid $G \ltimes \Omega(\Lambda)$.

Remark. The groupoid $G \ltimes \Omega(\Lambda)$ in Example 2.1.7 need not be étale. This can be fixed by restricting the unit space of the groupoid $G \ltimes \Omega(\Lambda)$ to the subsets of all elements of $\Omega(\Lambda)$ containing the identity of G. One obtains an étale groupoid denoted by $\mathcal{G}(\Lambda)$ which has a compact unit space (see ([ER22] Proposition 3.10). The groupoids $G \ltimes \Omega(\Lambda)$ and $\mathcal{G}(\Lambda)$ are Morita equivalent ([EFR23] Lemma 7.3). This equivalence allows to study Λ through the groupoid $\mathcal{G}(\Lambda)$ which is theoretically well-behaved. We employ this construction in Paper II and Paper III.

There are also more general notions of approximate lattices which are more technical to define and will be omitted here (see [BH18], Definitions 1.3 and 1.4). The study of approximate lattices and their connection to model sets (Example 2.1.9) in various non-commutative spaces is a very recent and active research line (see [Mac23, Section 1.1]). For general second countable locally compact groups, several definitions were suggested, notably by Björklund and Hartnick in [BH18]. As a side note, the different definitions have recently been compared in [Mac23, Theorem 1.0.2]. In their paper [BH18], Björklund and Hartnick established some of the foundational theorems concerning lattices in locally compact second countable groups in this extended setting. An important instance of this phenomenon is the following automatic-uniformity theorem which we employ in Paper II.

Theorem 2.1.8. [BH18, Theorem 4.25] Every approximate lattice in a nilpotent Lie group is a uniform approximate lattice.

The study of approximate lattices is being pushed forward in [BHP18; BHP21; BHP22; BHY21; BH18; BH20; Mac20a; Mac20b] and the groupoid construction of Example 2.1.7 has proved to be a very helpful tool. An important source of approximate lattices is provided by model sets which were introduced by Meyer for abelian groups in [Mey72]. These sets are models of mathematical quasicrystal and they have been extensively studied and constitute by now a very classical object in point set analysis, as witnessed by the profusion of references in [BG13]. We study the groupoids attached to model sets in Paper II and Paper III.

Example 2.1.9. Let G and H be locally compact second countable groups and Γ a lattice in $G \times H$ such that the projection $\pi_G : G \times H \to G$ is injective when restricted to Γ and the projection $\pi_H : G \times H \to H$ has dense image when restricted to Γ. One can think of algebraic examples such as the inclusions

$$\begin{aligned} SL_n(\mathbb{Z}(\sqrt{d})) &\subseteq SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \\ (a + b\sqrt{d}) &\mapsto (a + b\sqrt{d}, a - b\sqrt{d}), \end{aligned}$$

for d congruent to 2 or 3 modulo 4.

If $W \subset H$ is a compact neighborhood of the identity (which is regular enough) and the lattice Γ is uniform, the point set

$$\Lambda = \pi_G(\Gamma \cap (G \times W))$$

is a uniform approximate lattice. Such point sets Λ are called *uniform model sets*. When the lattice Γ is not uniform, Λ is simply called a *model set*.

Meyer famously showed in [Mey72] that when G is abelian, every uniform approximate lattice is a (relatively dense) subset of a model set. This result has been generalized to nilpotent Lie groups in [Mac20a] and the relationship between model sets and approximate lattices has been studied for some classes of Lie groups and algebraic groups (see [BHS19; Hru20; Mac22; Mac23]). The case of non-algebraic totally disconnected groups is still very unclear. Other important open problems in the area include the existence of approximate lattices for different classes of locally compact groups (see also [Mac23]).

Étale groupoids and C^*-algebras

Étale groupoids are essential objects in the modern theory of C^*-algebras. They have been brought to the operator algebra community by Renault in his thesis [Ren80] which is still a reference for the general theory as of today. His work built upon previous efforts of Glimm, Westman, Effros-Hahn and Zeller-Meier (see [Gli60; Wes67; EH67; Zel68]) on convolution algebras attached to principal groupoids and transformation groupoids. One of the first historical provenances of C^*-algebras was provided via the group C^*-algebra construction. In turn, the C^*-algebras associated to groupoids also constitute a vast source of examples of C^*-algebras. Provided their unit space is compact, étale groupoids have the strong asset that they give rise to unital C^*-algebras whose construction will be sketched now.

The case of a transformation groupoid is simplest. Given a discrete group Γ acting on a locally compact space X, one can define an action of G on $C_0(X)$, the commutative C^*-algebra of all continuous functions from X to $\mathbb{C}$ which vanish at infinity. One can now form a C^*-algebra from the commutative C^*-algebra $C_0(X)$ and the action of Γ, which is called the *reduced crossed product* and denoted by $\Gamma \ltimes_{\mathrm{red}} C_0(X)$. The idea of the reduced crossed product construction is similar to the semi-direct product construction of groups. The C^*-algebra $\Gamma \ltimes_{\mathrm{red}} C_0(X)$ contains a copy of Γ as unitary elements and a copy of $C_0(X)$ such that the action of Γ on $C_0(X)$ is given by conjugation inside $\Gamma \ltimes_{\mathrm{red}} C_0(X)$.

Example 2.1.10. If Γ is a discrete group acting on a space X, the reduced C^*-algebra of the groupoid $\Gamma \ltimes X$ is the reduced crossed product $\Gamma \ltimes_{\text{red}} C_0(X)$.

We now give the more general construction of maximal and reduced C^*-algebras of an étale groupoid. For simplicity, we assume groupoids to be Hausdorff and second countable.

Definition 2.1.11. Let $\mathcal{G}$ be an étale groupoid. Denote by $C_c(\mathcal{G})$ the subset of $\ell^\infty(\mathcal{G})$ of compactly supported functions on $\mathcal{G}$. The set $C_c(\mathcal{G})$ is a $*$-algebra. The (maximal) C^*-algebra $C^*_{\max}(\mathcal{G})$ of $\mathcal{G}$ is the enveloping C^*-algebra of $C_c(\mathcal{G})$.

Without entering into details, there is a family of convolution representations $(\lambda_x, B(\ell^2(\mathcal{G}_x)))_{x\in\mathcal{G}^{(0)}}$ of a groupoid $\mathcal{G}$. The C^*-completion of $C_c(\mathcal{G})$ under this family of representations gives a C^*-algebra $C^*_{\text{red}}(\mathcal{G})$, which is called the reduced C^*-algebra of $\mathcal{G}$. The reduced and maximal C^*-algebras of a groupoid are very important examples of C^*-algebras.

Several invariants can be attached to a C^*-algebra, for instance its K-theory and its trace space (see Definition 2.3.1). The class of C^*-algebras for which these invariants are complete are called the *classifiable* C^*-algebras. One can alternatively define classifiable C^*-algebras as simple C^*-algebras possessing good regularity properties. This statement will be made more precise in Section 2.3. For a sufficiently regular C^*-algebra, computing its K-theory and its trace spaces allows to determine the C^*-algebra up to isomorphism and describe its structure. The class of classifiable C^*-algebras is important in the modern theory of C^*-algebras and is perhaps the most studied class of C^*-algebras.

Conversely, it is possible to construct a classifiable C^*-algebra with prescribed K-theoretic and tracial data. In fact, Li managed to construct (twisted) étale groupoids whose C^*-algebra have a prescribed K-theoretic and tracial data in his paper [Li19]. Roughly speaking, the adjective *twisted* refers to a suitable version of a cocycle deformation of the C^*-algebra. Using these twists, he thus could obtain the following astonishing result which stresses the importance of étale groupoids for the general theory of C^*-algebras.

Theorem 2.1.12. [Li19, Theorem 1.2, Corollary 1.5] Let A be a classifiable C^*-algebra. There exists a principal second countable locally compact Hausdorff étale groupoid $\mathcal{G}$ and a twist Σ such that $A \cong C^*_r(\mathcal{G}, \Sigma)$

Amenable groupoids

Due to the importance of topological groupoids both for operator algebras and dynamical systems, many group theoretic properties have been generalized to groupoids from the 80's onward. An important historical instance is that of amenability which has many characterizations for discrete groups whose generalizations fail to be equivalent for general étale groupoids.

Proposition 2.1.13. [Pie84, Chapter 2] Let Γ be a discrete group. The following conditions are equivalent.

(i) There is a left-invariant mean on $\ell^\infty(\Gamma)$;

(ii) There is a net $\{\xi_i\}_i$ of unit vectors in $\ell^2(\Gamma)$ such that $\|g\xi_i - \xi_i\|$ converges pointwise to 0;

(iii) The reduced group C^*-algebra $C_{\mathrm{red}}(\Gamma)$ of Γ is nuclear, meaning for every C^*-algebra B, there is a unique C^*-norm on the algebraic tensor product $C^*_{\mathrm{red}}(\Gamma) \otimes_{\mathrm{alg}} B$;

(iv) Γ has the weak containment property, meaning that the natural quotient morphism $C^*(\Gamma) \to C^*_{\mathrm{red}}(\Gamma)$ is an isomorphism of C^*-algebras.

These notions have been generalized to the context of groupoids, and gave rise to topological amenability (ii), and the weak containment property (iv), while notion (iii) exists for general C^*-algebras under the name nuclearity. Understanding the relations between these properties is challenging and it is only in the last decade that Willett gave an example of a non topologically amenable étale groupoid whose reduced and maximal C^*-algebras are the same (see [Wil15]), thus showing that the weak containment property is strictly weaker than topological amenability in general. This result exemplifies the difficulty to generalize analytic properties in the groupoid setting. In Section 2.4, we will introduce a notion of amenability which is weaker than topological amenability called inner amenability. It is also the property of interest in Paper III. That property has the important feature of relating the topological amenability of an étale groupoid to the nuclearity of its C^*-algebra. In Sections 2.2 and 2.3, we will study two strong structural properties which imply amenability. These properties are motivated by the classification of von Neumann algebras and C^*-algebras respectively.

2.2 Operator algebras and inverse semigroups of type I

This section contains background which is specifically related to Paper I. We will discuss the type I property for von Neumann algebras, C^*-algebras, groups and groupoids and the connection with Boolean inverse semigroups. Although Paper I also treats the CCR property, we will only briefly mention it here. Both properties are closely related and often studied together, especially for groupoids.

Operator algebras of type I

The type I property takes its roots in the foundational papers of Murray and von Neumann [MN36; MN37; Neu40] on rings of operators in the thirties. In their seminal series of papers, they introduce and begin a classification of von Neumann algebras which are divided into type I, II and III. The different types are defined in terms of existence (or non-existence for type III) of specific projections in the algebra. For a von Neumann algebra M, there exists unique pairwise orthogonal central projections p_I, p_{II}, p_{III} in M summing up to 1 such that the compressions $p_I M$, $p_{II} M$, $p_{III} M$ are of type I, II and III respectively, whence the decomposition of M as a direct sum of different types. The type I von Neumann algebras are the simplest of the three building blocks of the theory to understand.

Definition 2.2.1. A projection $p \in M \subseteq B(H)$ is *abelian* if the von Neumann algebra pMp is abelian. Moreover, a von Neumann algebra M is type I if every projection p has a non-zero abelian subprojection q.

Observe immediately from the definition that any abelian von Neumann algebra is type I. Note also that for any Hilbert space H, the space of bounded linear operators $B(H)$ on H is type I, because a non-zero projection p has abelian (since minimal) subprojections of the form $\xi \otimes \overline{\xi}$ for any unit vector $\xi \in pH$. In fact, when the von Neumann algebra M has a trivial center (M is called a *factor*), then M is of type I precisely when $M = B(H)$ and H is a κ-dimensional Hilbert space for some cardinal κ. When $\kappa = n$ is an integer, it means that $M = M_n(\mathbb{C})$.

When the von Neumann algebra M is not a factor, the type I property takes the following form.

Theorem 2.2.2. A von Neumann algebra M is type I if and only if there exists a cardinal number κ and abelian von Neumann algebras $Z_{\kappa'}$ for each $\kappa' < \kappa$ such that

$$M \cong \bigoplus_{\kappa' < \kappa} Z_{\kappa'} \otimes B(\ell^2(\kappa')).$$

For separable von Neumann algebras, the structural consequences of the type I property can be specified the following way.

Theorem 2.2.3. A separable von Neumann algebra M is type I if and only if there exists abelian von Neumann algebras Z_n for each $n \in \mathbb{N} \cup \{\infty\}$ such that

$$M \cong Z_\infty \otimes B(\ell^2(\mathbb{N})) \oplus \bigoplus_n Z_n \otimes M_n(\mathbb{C}).$$

In the context of C^*-algebras, a C^*-algebra A is type I if every von Neumann algebra generated by a representation of A is a type I von Neumann algebra. This property has been shown to be equivalent to other operator algebraic properties in the 1950's and 1960's, such as being *GCR* or *postliminal*. The equivalence between all the different definitions is now classical (see [Dix83]). The adjective postliminal qualifies C^*-algebras for which every unitary representation contains the compact operators. This allows to leverage the elementary structure of the latter algebra. Postliminal C^*-algebras are not simple, unless they are equal to the compact operators. An important contributor to the proof of these equivalences is Glimm ([Gli61]) who also showed that C^*-algebras which are not of type I possess both representations which generate a type II and a type III von Neumann algebra. As a consequence, the type distinction for C^*-algebras does not nearly make as much sense as for von Neumann algebras. The classification of certain C^*-algebras that we will see in Section 2.3 is of a very different nature.

In parallel with the type I property, another structural property for C^*-algebras has been developped.

Definition 2.2.4. Let A be a C^*-algebra. We say that A is CCR if for every irreducible representation (π, H_π) of A, the set $\pi(A)$ is equal to the set of all compact operators on H.

Though it is not clear from this definition, the CCR property is stronger than the type I property. Additionally, the spectrum of a C^*-algebra which is CCR is a T_1 topological space, meaning that for any pair of points, there is an open set containing the first one but not the second one.

Type I groups

Around the same time in the 60's, the type I property was naturally transferred to topological groups via the maximal group C^*-algebra construction giving the following definition.

Definition 2.2.5. A locally compact group is type I if its maximal group C^*-algebra is type I.

This property admits intrinsic characterizations of different natures. As a sample, a group is type I if every unitary representation can be decomposed as a direct integral of irreducible representation in an essentially unique way. This is a statement about uniqueness of measure on the unitary dual and not about the existence of such a decomposition (see [Dix83], Theorem 8.5.2). In that sense, type I groups generalize compact groups by (a version of) the Peter-Weyl theorem stating that representations of a compact group can be uniquely decomposed as a direct sum of irreducible ones.

Alternatively, type I groups are the most general class of groups for which a sensible version of the Plancherel theorem holds, which makes harmonic analysis on such groups particularly interesting (see [Dix83] Chapter 18). It makes it possible to extend several results from locally compact abelian groups to type I groups due to their representation theoretic behavior. These facts motivated the community to study the class of locally compact type I groups. Outside of compact and abelian groups, the first classes of groups shown to be type I were algebraic groups. Building on Harish-Chandra's work in [Har53], Dixmier showed that all real linear groups are type I in [Dix57]. Dixmier also showed that nilpotent groups are type I in [Dix59]. Bernstein then gave a proof that linear groups over the p-adics are type I (see [Ber74]). More recently, Bekka and Echterhoff showed in [BE20] that finite extensions of reductive groups over non archimedean fields are type I. The case of unipotent groups over local fields in positive characteristic is still an open problem. In the discrete world, the characterization of the type I property is due to Thoma in [Tho64].

Theorem 2.2.6. [Tho64] A discrete group Γ is of type I if and only if it has an abelian subgroup of finite index.

By taking inverse limits, this result has been generalized to locally compact groups which have a conjugation invariant neighborhood basis of the identity in [Kan71]. For totally disconnected groups acting on trees, the type I property is not well understood, as tools from algebraic groups or Lie theory are no longer available. Among the few instruments available to show the type I property is a combinatorial independence property (Tits' independence property). Groups that act transitively on the boundary ([Ols80; Ama03; Cio15]) were also shown to be type I. More recently, Semal has shown in [Sem22] that Radu groups are type

I (and CCR in fact). Inspired by a conjecture of Nebbia in [Neb99], Raum-Houdayer suggest that a group acting minimally on a tree should be type I if and only if it acts transitively on the boundary of the tree (see [HR19]). This conjecture is backed up by their characterization of type I Burger-Mozes groups as the Burger-Mozes groups which are boundary transitive. Recently, Caprace, Kalantar and Monod have shown in [CKM23] that if a group acting minimally on a tree is type I, then the action is transitive on the boundary of the tree. The main tool they use in order to prove this result is the following result which shoud remind of Thoma's theorem (Theorem 2.2.6).

Theorem 2.2.7. [CKM23, Theorem B] Let G be a hyperbolic locally compact group admitting a uniform lattice.

If G is type I, then G has a cocompact amenable subgroup

Based on this result, they conjecture that every second countable locally compact group of type I admits a cocompact amenable subgroup. This structural consequence of being type I calls for a better algebraic understanding of the type I property (see also [CW21]).

Type I groupoids

In the context of groupoids, the type I property can be defined by prescribing that the maximal C^*-algebra of the groupoid is type I. This notion was introduced and characterized topologically for transformation groupoids by Gootman and Williams in [Goo73; Wil81], then for principal groupoids by Clark in [Cla07b] and later for general (Hausdorff) groupoids by Van Wyk in [Wyk18; Wyk19]. The understanding of the type I property culminated in the following characterization in [Wyk19].

Theorem 2.2.8. A locally compact Hausdorff groupoid is type I if and only if its orbit space is a T_0 topological space and its isotropy groups are type I.

Recall that a topological space is called T_0 if for any pair of points, there is an open set containing one but not the other.

Since the proof of Theorem 2.2.8 is rather involved, we will instead explain the characterization of the CCR property which we enunciate now.

Theorem 2.2.9. The maximal C^*-algebra of a locally compact Hausdorff groupoid is CCR in the sense of Definition 2.2.4 if and only if its orbit space is a T_1 topological space and its isotropy groups are CCR.

Recall that a topological space is called T_1 if for any pair of points, there is an open set containing the first one but not the second one.

At first glance, the condition on the orbit space might look odd. The requirement on the orbit space essentially allows to compare representations of the isotropy groups with representations of the groupoid. Indeed, one can factor representations of $\mathcal{G}$ through representations of $\mathcal{G}|_{\overline{\mathcal{G}x}}$, where $\mathcal{G}x = \{y \in \mathcal{G}^{(0)} \mid \exists \gamma \in \mathcal{G},\ s(\gamma) = x \text{ and } r(\gamma) = y\}$ is the orbit of $x \in \mathcal{G}^{(0)}$. Since the orbit space of $\mathcal{G}$ is T_1, the points of the orbit space are closed. Hence $\mathcal{G}|_{\overline{\mathcal{G}x}} = \mathcal{G}|_{\mathcal{G}x}$. This process allows to pass from representations of $\mathcal{G}$ to representations of $\mathcal{G}|_{\mathcal{G}x}$ which is Morita equivalent to $\mathcal{G}_x^x$, the isotropy group of $\mathcal{G}$ at x. The proof of this Morita equivalence also uses that the orbit space is T_1, see [MRW87, Theorem 3.1] (in fact T_0 is enough).

Furthermore, if the orbit space of $\mathcal{G}$ is not T_1, Clark showed in [Cla07a] that there exists a continuous map from the spectrum of $C^*(\mathcal{G})$ to the orbit space of $\mathcal{G}$. Clark then used this map to prove that the spectrum of $C^*(\mathcal{G})$ is not T_1, hence $C^*(\mathcal{G})$ is not CCR.

Hoping to shed some light on the structure of type I groupoids, we give in Paper I a completely algebraic characterization of the type I property for ample groupoids (see Theorem B of Paper I). In order to get such a statement, one has to interpret differently the condition on the orbit space of Theorem 2.2.8. This can be done by carefully analyzing the semigroup of compact open bisections of the groupoid. A similar characterization can be obtained for étale groupoids as well which takes the form of forbidden subquotients for the semigroup of open bisections of the groupoid. These results can be transferred to the realm of inverse semigroups by the means of a so-called noncommutative Stone duality. The question of characterizing somewhat intrinsically which inverse semigroups are type I was asked at the "equilibrium states in semigroup theory, K-theory and number theory" workshop in Oslo, 2019. It was in fact the main motivation for Paper I.

Type I inverse semigroups

The classical Stone duality connects totally disconnected topological spaces with generalized Boolean algebras. Indeed, the set of all compact open subsets of a totally disconnected space forms a generalized Boolean algebra. On the other hand, the set of all the non-zero Boolean algebra morphism to $\{0, 1\}$ of a generalized Boolean algebra forms a totally disconnected space. Ample groupoids have a totally disconnected unit space and the classical Stone duality has been extended to the ample groupoid setting by the works of Lawson and Lenz in [Law10; Law12; LL13].

Given an ample groupoid $\mathcal{G}$, the set of all compact open bisections $\Gamma_{\mathrm{CO}}(\mathcal{G})$ forms a Boolean inverse semigroup, meaning it is an inverse semigroup whose set of idempotents has a generalized Boolean algebra structure. Conversely, given a Boolean inverse semigroup S, consider the space K of all non-zero semigroup homomorphisms from the set of all idempotents of S to $\{0, 1\}$. The natural (partial) action of S on K yields an étale transformation groupoid $\mathcal{G}(S) = S \ltimes K$ which is ample since K is totally disconnected. There are several equivalent ways to build an ample groupoid from S which lead to topologically equivalent groupoids (see [Arm+22]).

Noncommutative Stone duality establishes that ample groupoids and Boolean inverse semigroups are dual to each other. The morphisms involved in this duality are not the continuous groupoid homomorphisms which is a subtle point of the duality. To each object of the duality S, $\mathcal{G}(S)$, $\Gamma_{\mathrm{CO}}(\mathcal{G}(S))$ one can associate a C^*-algebra. These C^*-algebras turn out to be isomorphic and these isomorphisms have been exploited in Paper I to deduce an algebraic characterization of the type I property for Boolean inverse semigroups, using the characterization we had for ample groupoids. The step of investigating general inverse semigroups has been made possible with further work via the Boolean inverse completion introduced by Lawson in [Law20] which allowed to answer the original question: which inverse semigroups are type I. The result we obtained is not completely intrinsic, as it involves the algebraic structure of the Boolean inverse completion.

This approach using noncommutative Stone duality allows to study an inverse semigroup via its associated ample groupoid, which we employ in Paper I. A dictionary between the two frameworks has been built over the last decade. Properties of an inverse semigroup such as being fundamental, 0-additively simple, having a simple Steinberg algebra and so on have all been characterized at a groupoid level and successfully exploited (see [SS21; LV21; LMS13]). Paper I also naturally fits within this picture and complements this dictionary.

2.3 Classification of C^*-algebras

We saw in Section 2.2 that the purpose of the type I property was initially to classify von Neumann algebras. In order to classify C^*-algebras, many regularity properties have been developed over the last two decades. Among them, finite nuclear dimension has emerged as a key property for the classification of C^*-algebras. In this Section, we will first explain the origin of the notion of nuclear dimension which take its roots in the classification program. Then we will define the nuclear dimension and give a context for Paper II. We will keep the technical level at the bare minimum. For details, useful references on the topic include [Str20; RS02; Cas+20; WZ10; TWW17].

It has to be said that the regularity property for C^*-algebras known as $\mathcal{Z}$-stability appeared first. Both $\mathcal{Z}$-stability and finite nuclear dimension stem from the classification program. The relationship between the two notions for C^*-algebras will be explained.

The Elliott invariant

In his foundational paper ([Ell76]), Elliott showed that it was possible to classify approximately finite dimensional C^*-algebras by their K_0-group. Elliott showed that two unital approximately finite-dimensional (AF) algebras A and B are isomorphic if and only if there is an isomorphism of dimension groups

$$(K_0(A), K_0(A)^+, [1_A]) \cong (K_0(B), K_0(B)^+, [1_B]),$$

where $K_0(A)^+$ is the positive cone of $K_0(A)$ and $[1_A]$ is the K_0-class of the unit of A. His result can be viewed as a generalization of an earlier result of Glimm which classifies UHF algebras via supernatural numbers in [Gli60]. This led Elliott to conjecture that all separable simple nuclear C^*-algebras should be classified by such an invariant.

However, in order to classify C^*-algebras of increasing generalities, the invariant had to be modified and more information had to be included. Among others, the works of Elliott, Dadarlat, Gong and Lin were pivotal to obtain more and more general classification results ([Ell93; DG97; EGL07]). The invariant which emanated from all of their work justifies the following definition.

Definition 2.3.1. To every C^*-algebra A, one can associate an invariant called the *Elliott invariant* of A given by

$$Ell(A) = (K_0(A), K_0(A)^+, [1_A], K_1(A), Tr(A), \rho_A),$$

where $K_1(A)$ is the K_1 group of A, $Tr(A)$ is the trace space of A and ρ_A is the natural pairing between $Tr(A)$ and $K_0(A)$. The pair $(Tr(A), \rho_A)$ is called the *tracial data* of A.

The original conjecture of Elliott that all separable nuclear C^*-algebras should be classified by their Elliott invariant has been shown to be false in that generality. The following theorem is an optimal classification result for C^*-algebras in the literature, in the sense that one cannot have a better result without changing the invariant. It has many contributors.

Theorem 2.3.2. [TWW17, Corollary D] Let A and B be two simple, separable, nuclear, $\mathcal{Z}$-stable, C^*-algebras satisfying the UCT. Then A and B are isomorphic if and only if they have the same Elliott invariant.

Compared to Elliott's original expectations, there are two extra conditions in Theorem 2.3.2. The first one is a technical condition called the *UCT* (the universal coefficient theorem) which holds for large classes of nuclear C^*-algebras among which amenable groupoid C^*-algebras (see [Tu99]). The UCT condition is believed to hold for all nuclear C^*-algebras. This is in fact an important open problem within operator algebras. Without entering into too much details, the UCT hypothesis allows in certain cases to construct a quasi-morphism between C^*-algebras from a morphism of the K-groups via KK-theory. This strategy has been used by Kirchberg and Phillips in order to classify Kirchberg algebras in [Phi00] (see also [RS02, Chapter 8]). More recently, the UCT was used by Tikuisis, White and Winter to show that faithful traces on separable nuclear C^*-algebras satisfying the UCT are quasidiagonal. This was a key step to prove their classification result in [TWW17].

The second novel condition appearing in Theorem 2.3.2 to be able to classify C^*-algebras is $\mathcal{Z}$-*stability*. The letter $\mathcal{Z}$ denotes the Jiang-Su algebra which has a special importance, namely it is a unital separable nuclear strongly self-absorbing C^*-algebras. Concretely speaking, this algebra has the property that $Ell(A) = Ell(A \otimes \mathcal{Z})$ for any separable unital C^*-algebra A, forcing any potentially classifiable C^*-algebras to satisfy $A \otimes \mathcal{Z} = A$. The Jiang-Su algebra $\mathcal{Z}$ is not an easy object to understand and has various definitions by now (see [JS99; RW10; Mas17]). Understanding $\mathcal{Z}$ and studying $\mathcal{Z}$-stability are very different tasks. Over the last decades, $\mathcal{Z}$-stability has received a lot of attention due to its necessity towards a classification. The more recent notion of nuclear dimension which we will introduce in the next Section has been shown to be equivalent to $\mathcal{Z}$-stability for all C^*-algebras in the scope of Theorem 2.3.2.

Nuclear dimension of C^*-algebras

Winter and Zacharias introduced in [WZ10] a non-commutative covering dimension called *nuclear dimension*. This notion is modeled over the decomposition rank for C^*-algebras and can be outlined as the existence of a non-commutative flexible partition of unity, allowing to approximate elements by finite dimensional data. The different generalizations of topological covering dimension for C^*-algebras known as stable rank, real rank and decomposition rank introduced in [BP91; Rie83; KW04] are pivotal to understanding non-commutative spaces and their applications (see [EGL07; Tom08; Win07; Win14; Lin14; Yu98; Con94]). Although the existing classification results for C^*-algebras did not explicitly appeal to these properties, they all seemed to play an important role. Classification of C^*-algebras together with giving an appropriate rank notion were the main motivations to introduce nuclear dimension.

Definition 2.3.3. [WZ10, Definition 2.1] A C^*-algebra A has nuclear dimension at most n if there exists a net $(F_\lambda, \psi_\lambda, \phi_\lambda)_{\lambda\in\Lambda}$ such that the F_λ are finite-dimensional C^*-algebras, and such that $\psi_\lambda : A \to F$ and $\phi_\lambda : F_\lambda \to A$ are completely positive maps satisfying

(i) $\psi_\lambda \circ \phi_\lambda(a) \to a$ uniformly on finite subsets of A;

(ii) $\|\psi_\lambda\| = 1$;

(iii) for each λ, F_λ decomposes into $n+1$ ideals $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that each $\phi_\lambda|_{F_\lambda^{(i)}}$ is a completely positive contractive order 0 map for $i = 0, \ldots, n$.

This definition is hard to check and very scarcely used. In practice, one uses existing nuclear dimension bounds and results comparing different bounds in order to produce a new nuclear dimension estimate instead. Paper II employs this technique as we will see in Section 3.2.

The study of nuclear dimension starts in [WZ10], where explicit upper bounds on the nuclear dimension of important C^*-algebras such as Cuntz algebras, Kirchberg algebras and Roe algebras are given. Briefly after, Strung-Winter and Toms-Winter give direct applications of finite nuclear dimension to classification of C^*-algebras associated with dynamical systems in [SW11; TW09; TW13]. Toms and Winter conjectured that finite nuclear dimension should be equivalent to $\mathcal{Z}$-stability for some classes of C^*-algebra, which goes by the name of the Toms-Winter conjecture. They also conjecture these properties to be equivalent to strict comparison of projections. The equivalence between $\mathcal{Z}$-stability and finite nuclear dimension was recently proven for simple separable amenable unital non-elementary C^*-algebras in [Cas+20], while showing the equivalence with strict comparison of projections is still an open problem. In particular, the following result can be deduced from Theorem 2.3.2.

Theorem 2.3.4. Let A and B be two infinite dimensional simple, separable, nuclear C^*-algebras satisfying the UCT of *finite nuclear dimension*. Then A and B are isomorphic if and only if they have the same Elliott invariant.

The classification result in Theorem 2.3.4 constitutes an important step forward to the understanding of classifiable C^*-algebras, as showing finite nuclear dimension for a given C^*-algebra is often more practical than proving $\mathcal{Z}$-stability.

Remark. As a disclaimer to the last statement, $\mathcal{Z}$-stability has been shown in different settings where no direct proof of finite nuclear dimension is available. See for instance [Con+18; EN14; Ker20; KS20]

A major task in current research is to determine which C^*-algebras have finite nuclear dimension. In the foundational paper [WZ10], the following question is posed.

Question 1. [WZ10, Problem 9.4] Let G be a topological group acting on a C^*-algebra A. Under which conditions on A, G and the action, has the crossed product $A \rtimes G$ finite nuclear dimension?

One strategy to approach this problem is to place conditions on the action. In [HWZ15], Hirshberg, Winter and Zacharias developed the theory of Rokhlin dimension for finite groups and integers actions on unital separable C^*-algebras. They show that the nuclear dimension of the crossed product $A \rtimes G$ is bounded by a polynomial which only depends on the nuclear dimension of A and the Rokhlin dimension of the action, when G is a finite group or $\mathbb{Z}$. The concept of Rokhlin dimension has been extended to various settings and results ensuring finite nuclear dimension have flourished. A non-exhaustive list of papers in this direction is [Gar18; Sza13; Hir+16].

A result of particular importance to this thesis was obtained by Hirshberg and Wu in [HW17], where they give a concrete polynomial bound on the nuclear dimension of algebras of the form $\mathbb{R} \ltimes C_0(X)$ depending only on the dimension of X without imposing any condition on the action. They solve the non-free case of Question 1 for $\mathbb{R}$-actions on commutative C^*-algebras. Their methods are based on the previous work of Hirshberg, Szabó, Winter and Wu in [Hir+16].

In Paper II, we adapt their methods to show that the C^*-algebra associated to a free action of a connected Lie group G of polynomial growth on a locally compact space X has finite nuclear dimension. The proof is sketched in Section 3.2.

We also consider in Paper II groupoids of the form $G \ltimes \Omega^{\times}(\Lambda)$, where Λ is a uniform approximate lattice in a connected Lie group G of polynomial growth, and their restrictions $\mathcal{G}(\Lambda)$ (see Examples 2.1.4 and 2.1.7). Using freeness of the G-action on $\Omega^{\times}(\Lambda)$, we prove finite nuclear dimension of $C^*(G \ltimes \Omega^{\times}(\Lambda))$ and of $C^*(\mathcal{G}(\Lambda))$. In fact this algebra is classifiable when the action $G \curvearrowright \Omega^{\times}(\Lambda)$ is minimal. This can be viewed as a generalization of a result of Ito, Whittaker and Zacharias for tilings in $\mathbb{R}^n$ (see [IWZ19]).

2.4 Inner amenability for étale groupoids

We conclude the chapter on preliminaries by introducing the weakening of amenability known as inner amenability. This property is a bridge between topological amenability of an object and amenability of its C^*-algebra (and von Neumann algebra). First appearing for locally compact groups in the work of Losert and Rindler [LR87], inner amenability was explicitly defined later by Paterson in [Pat88] as the existence of a Haar continuous conjugation invariant mean on the bounded function of the group. Losert and Rindler investigated the class of inner amenable groups and showed in [LR87] that it contains [IN]-groups and that inner amenability is in fact equivalent to having a conjugation invariant neighborhood of the identity for connected groups. More generally, Lau and Paterson have shown the following fundamental result.

Theorem 2.4.1. [LP91] A locally compact group G is amenable if and only if it is inner amenable and its C^*-algebra is nuclear.

Since type I groups have a nuclear C^*-algebra, inner amenable type I groups are amenable. Furthermore, Connes proved in [Con76] that almost connected groups also have a nuclear C^*-algebra. In particular, inner amenable almost connected groups are amenable. These two connecting results motivated the community to further study the class of inner amenable groups.

Remark. A different notion of inner amenability has been introduced by Effros in [Eff75] requiring atomlessness of means. Effros proved that property Γ of the group von Neumann algebra implies inner amenability in his sense. Vaes showed in [Vae12] that the converse does not hold.

In this thesis, we do not require atomlessness of means, which makes discrete groups automatically inner amenable, since the Dirac measure at the identity is a conjugation invariant Haar continuous mean in that case.

For groupoids, inner amenability has been defined by Anantharaman-Delaroche for transformation groupoids first in [Ana00], and more generally for locally compact groupoids in [Ana16].

Definition 2.4.2. A locally compact groupoid $\mathcal{G}$ is inner amenable if for every compact $K \subseteq \mathcal{G}$, $\epsilon > 0$, there exist a properly supported, continuous, positive type function ϕ on $\mathcal{G} \times \mathcal{G}$ which is ϵ-close to 1 on $K \times K$.

Here, a function f is said to be *properly supported* if for every compact subset $K \subseteq \mathcal{G}$, the sets $supp(f) \cap (\mathcal{G} \times K)$ and $supp(f) \cap (K \times \mathcal{G})$ are compact.

This notion is compatible with the one for groups. When G is an inner amenable group, then for every compact set $K \subseteq G$ and positive ϵ, there is a unit vector $\xi \in L^2(G)$ such that $\|\lambda(g)\rho(g)\xi - \xi\|_{L^2(G)} < \epsilon$ for any $g \in K$, where λ and ρ denote the left and right regular representations of G respectively. The matrix coefficient

$$\begin{aligned} \phi : G \times G &\to \mathbb{C} \\ (g,h) &\mapsto \langle \lambda(g)\rho(h)\xi, \xi\rangle_{L^2(G)} \end{aligned}$$

witnesses the inner amenability of G as a groupoid. Crann-Tanko have shown the converse in [CT17], namely that if a group G is inner amenable as a groupoid, then it has a conjugation invariant invariant mean. Therefore, the notion of inner amenability of a locally compact groupoid introduced by Anantharaman-Delaroche generalizes the classical inner amenability for locally compact groups.

The study of this notion has been started by Anantharaman-Delaroche and she showed the following proposition.

Proposition 2.4.3. Let $\rho : \mathcal{G} \to \mathcal{H}$ be a continuous morphism of locally compact groupoids. Assume that $\mathcal{H}$ is inner amenable. Assume also that the map ρ is locally proper, meaning that the map

$$\begin{aligned} \rho \times s \times r : \mathcal{G} &\to \mathcal{H} \times \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} \\ \gamma &\mapsto (\rho(\gamma), s(\gamma), r(\gamma)) \end{aligned}$$

is proper. Then $\mathcal{G}$ is inner amenable.

An example of locally proper map is an inclusion of the form $\mathcal{H}|_X \to \mathcal{H}$, for $X \subseteq \mathcal{H}^{(0)}$ (when the range map of $\mathcal{H}|_X$ is open) or the inclusion maps of closed subgroupoids. For transformation groupoids, if a group G acts on a space X, the homomorphism

$$G \ltimes X \to G$$

is locally proper. Since discrete groups are inner amenable, one gets the following examples of inner amenable étale groupoids.

Example 2.4.4. Let Γ be a discrete group acting on a space X. The transformation groupoid $\Gamma \ltimes X$ is inner amenable.

In light of this fact, Anantharaman-Delaroche asks the following natural question.

Question. [Ana16, Question 11.1 (3)] Is there a non inner amenable étale groupoid?

Very little progress had been made on this question since it was posed. Besides amenable groupoids and transformation groupoids coming from discrete group actions, no major class of étale groupoids is known to be inner amenable. Investigating this class was the main motivation for Paper III, where the main result states that the étale groupoid associated with a point set (see example 2.1.7) of finite local complexity in a locally compact second countable group is inner amenable. The finite local complexity condition is a regularity condition which is satisfied by all approximate lattices. This result provides a class of example of inner amenable étale groupoids, which are not constructed as transformation groupoids.

As for the locally compact group case, a strong connection between topological amenability of a groupoid and amenability of its operator algebras holds for locally compact groupoids provided they are inner amenable. Anantharaman-Delaroche showed the following groupoid counterpart of Theorem 2.4.1 in [Ana16].

Theorem 2.4.5. A transformation groupoid $\mathcal{G}$ is amenable if and only if it is inner amenable and its C^*-algebra is nuclear.

She also showed that this statement holds in the context of étale groupoids. Another major aspect of inner amenability is the fact that for inner amenable étale groupoids, the notions of topological amenability, amenability at infinity and nuclearity of the reduced C^*-algebra are all equivalent. There are still lots of open problems around comparing these notions for more general groupoids, already for locally compact groups (see [Ana16, Section 11.2]). All the mentioned results further motivate the study of the class of inner amenable groupoids.

3. Summaries of papers

We now have enough background to summarize the three papers of the thesis. In this chapter, we describe their results, briefly discuss the ideas used to prove them and sketch potential lines of research suggested by these results.

3.1 Summary of Paper I

In this paper, co-authored with Sven Raum, we investigate the type I and CCR properties for locally compact Hausdorff ample groupoids and give an algebraic characterization of both properties. The proof takes the topological characterization of Clark and van Wyk's (Theorem 2.2.8) and translates it into our algebraic characterization. Given a locally compact Hausdorff ample groupoid $\mathcal{G}$, we show on the one hand that the isotropy groups of $\mathcal{G}$ can be viewed as subsemigroups of the compact open bisections $\Gamma_{\mathrm{co}}(\mathcal{G})$. On the other hand, the T_0 condition on the orbit space of $\mathcal{G}$ can be translated as a forbidden subquotients condition for $\Gamma_{\mathrm{co}}(\mathcal{G})$. Combining both results yields a characterization of the type I property (similarly for the CCR property) for $\mathcal{G}$ in terms of the algebraic structure of $\Gamma_{\mathrm{co}}(\mathcal{G})$.

Using the non-commutative Stone duality, we were able to obtain an algebraic characterization of the type I property for Boolean inverse semigroups in terms of forbidden subquotients (see Theorem 2 of Paper I). This characterization cannot be automatically derived from the Stone duality and the ample groupoid statement due to the poor behavior of the duality at the morphism level. As one of the ingredients of the proof, we show that restrictions of unit spaces do induce Boolean inverse semigroup morphisms.

Using the Boolean inverse completion of an inverse semigroup, we were able to give an algebraic characterization of the type I property for inverse semigroups, in the spirit of the result of Thoma (see Theorem 4 of Paper I). A natural next step would be to investigate left cancellative semigroups which simultaneously generalize inverse semigroups and Ore semigroups, both of which possess a type I characterization. Left cancellative semigoups are connected to inverse semigroups via the inverse hull construction (see [Li11; Nor14]).

Going beyond the case of ample Hausdorff groupoids, characterizations of the type I and CCR properties can be obtained in the étale non-Hausdorff setting by first adapting the proof of Clark-Van Wyk to obtain topological characterizations for the type I and CCR properties for étale non-Hausdorff groupoids. Exploiting the duality between étale groupoids and pseudogroups of Lawson, Lenz and Kudryavtseva in [Law12; LL13; Kud12; Kud13], one should obtain algebraic characterizations for the type I and CCR properties in the non-Hausdorff étale setting following the proofs of Theorem 1 and Theorem 2 of Paper I.

3.2 Summary of Paper II

In this paper, co-authored with Ulrik Enstad and Sven Raum, we show that free actions of connected Lie groups of polynomial growth on finite dimensional locally compact spaces have finite tube dimension. As a consequence, we get explicit nuclear dimension bounds for transformation groupoids arising from such free actions. The resulting C^*-algebras are amenable, hence automatically satisfy the UCT by a result of Tu. As a corollary and motivating example, we show that the cut-and-project sets constructed from irreducible lattices in products of connected nilpotent Lie groups give rise to groupoids with a classifiable C^*-algebra. Since these C^*-algebras have a unique trace, the remaining data to calculate the Elliott invariant is the ordered K_0-group, that is $(K_0, K_0^+, [1])$, the pairing of K_0 with the unique trace and K_1. Our results invite for concrete computations of the Elliott invariant.

We adapt the strategy of proof of Hirshberg-Wu in [HW17] for $\mathbb{R}$-actions in order to prove our main result which is the following theorem.

Theorem 3.2.1. Let $G \curvearrowright X$ be a free action of a connected Lie group of polynomial growth on a second-countable, locally compact Hausdorff space of finite covering dimension. Then the nuclear dimension of $C_0(X) \rtimes G$ is finite.

Our result gives a concrete bound depending on the dimension of X and the growth constant of G.

Roughly speaking, a C^*-algebra B has finite nuclear dimension if one can approximate B on finite subsets by C^*-algebras of controlled dimension in a sufficiently regular way. This is the content of the following proposition.

Proposition 3.2.2. [HW17, Lemma 1.2] Let B be a separable, nuclear C^*-algebra and let B_0 be a dense subset of the unit ball of B. Then $\dim_{\mathrm{nuc}}(B) \leq d$ if and only if for every finite subset $F \subseteq B_0$ and every $\epsilon > 0$ there exist a C^*-algebra $A_\epsilon = \bigoplus_{l=0}^m A_\epsilon^{(l)}$ and completely positive maps $\phi = \bigoplus_{l=0}^m \phi^{(l)} \colon B \to A_\epsilon$, $\psi = \bigoplus_{l=0}^m \psi^{(l)} \colon A_\epsilon \to B$ such that

1. ϕ is contractive;
2. $\psi^{(l)} = \sum_{k=0}^{d^{(l)}} \psi^{(l,k)}$ where each $\psi^{(l,k)}$ is an order zero contraction;
3. $\|\psi(\phi(x)) - x\| < \epsilon$ for all $x \in F$;
4. $\sum_{l=0}^m (\dim_{\mathrm{nuc}}(A_\epsilon^{(l)}) + 1)(d^{(l)} + 1) \leq d + 1$.

We will now outline the proof of Theorem 3.2.1, trying to keep the technical level at a minimum. Fix a free action $G \curvearrowright X$ of a connected Lie group of polynomial growth on a second-countable, locally compact Hausdorff space. The proof's main ingredient is the existence of open covers of X which locally trivialize the action and which have a controlled multiplicity. Given a compact subset $K \subseteq G$ and a set $Y \subseteq X$, one can construct an open cover $\mathcal{U}$ of X with the following properties.

- For each element of $y \in Y$, the K-fattening Ky is a subset of some element of the cover;
- Each element U of the cover $\mathcal{U}$ trivializes the action, meaning it is contained in a set of the form $U = K \times S = KS \subseteq X$. The sets are called tubes;
- The multiplicity of the cover is uniformly bounded.

Here, the multiplicity d of a cover $\mathcal{U}$ is the least integer such that any intersection of $d + 1$ distinct elements of $\mathcal{U}$ is empty.

The existence of the tube coverings together with a partition of unity argument is enough to approximate finite subsets of $C^*(G \ltimes X)$ by considering (compressions of) the C^*-algebras associated with restrictions of $G \ltimes C_0(X)$ to the tubes.

Compared to the proof for $\mathbb{R}$-action in [HW17], there were essential difficulties to be overcome. The first one was to prove the existence of tubes for free actions of $\mathbb{R}$-linear groups, which locally trivialize the action. This part of the proof relies upon earlier work of Palais in [Pal61]. Once these tubes were made available, we had to show the existence of a tubular covering of specific topological dimension in the non-abelian setting. Here we adapted to connected Lie groups of polynomial growth the proof of Kasprowski-Rüping in [Kas17] which treated the case of $\mathbb{R}$-actions. In contrast to [HW17], the partitions of unity we obtained from the tubular covering were not Lipschitz and the argument giving finite nuclear dimension (point 3. of Proposition 3.2.2 above) had to be slightly adjusted there as well.

Along the way, we clarify some details in the argument of Kasprowski-Rüping in [Kas17] which we could not reproduce. Our arguments lead to slightly different covering bounds. This also affects the nuclear dimension bounds in [HW17]. It must be remarked that none of the obtained bounds is sharp and there is no ambition to obtain sharp bounds or any potential application of such.

Without freeness of the action, it is not clear how to adapt the methods of Kaspowski-Rüping, even for $\mathbb{R}^n$-actions. Solving this problem would be very satisfying, as one would obtain as a particular case that the C^*-algebra of the groupoid of any uniformly discrete point set in a connected Lie group of polynomial growth has finite nuclear dimension.

3.3 Summary of Paper III

In Paper III, we study the étale groupoid associated to approximate lattices in locally compact second countable groups (see Example 2.1.7), and more generally groupoids attached to point sets of finite local complexity (FLC). The main result states that such groupoids are inner amenable, providing new examples of inner amenable étale groupoids. The main ingredient of the proof is the use of discreteness of the space $X = \bigcup_{P \in \Omega_0(\Lambda)} P$ for FLC point sets Λ. This discreteness is used in order to construct continuous, properly supported positive type functions on $\mathcal{G}(\Lambda) \times \mathcal{G}(\Lambda)$ which converge uniformly to 1 on the diagonal. These functions witness the inner amenability of $\mathcal{G}(\Lambda)$. Specializing to the case where Λ is a model set in an algebraic group, we get interesting new examples of inner amenable, non-amenable étale groupoids.

By perturbing randomly the points of a lattice, one obtains a uniformly discrete point set Λ. For point sets in $\mathbb{R}^n$, the resulting dynamical system appears frequently in probability theory and applications. Although not of finite local complexity, Λ still has some regularity properties. Its associated groupoid $\mathcal{G}(\Lambda)$ seem to be ample and similar techniques could be used to show its inner amenability.

Bibliography

[Ama03] Oliver Éric Amann. Groups of tree-automorphisms and their unitary representations. PhD thesis. ETH Zürich, 2003.

[Ana00] Claire Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C*-algebras. *Transactions of the American Mathematical Society* 354 (2000), pp. 4153–4178.

[Ana16] Claire Anantharaman-Delaroche. Exact groupoids. *Institut Denis Poisson* (2016). arXiv:1605.05117.

[Arm+22] Becky Armstrong, Lisa Orloff Clark, Astrid an Huef, Malcolm Jones, and Ying-Fen Lin. Filtering germs: Groupoids associated to inverse semigroups. *Expositiones Mathematicae* 40.1 (2022), pp. 127–139.

[BG13] Michael Baake and Uwe Grimm. *Aperiodic Order: Volume 1, A Mathematical Invitation*. Cambridge University Press, 2013.

[BE20] Bachir Bekka and Siegfried Echterhoff. *On unitary representations of algebraic groups over local fields*. Preprint, arXiv:2003.02297. 2020.

[Ber74] Joseph Bernstein. All reductive p-adic groups are tame. English translation. *Functional Analysis and Its Applications* 8.2 (1974), pp. 91–93.

[BH18] Michael Björklund and Tobias Hartnick. Approximate lattices. *Duke Mathematical Journal* 167.15 (2018), pp. 2903–2964.

[BH20] Michael Björklund and Tobias Hartnick. Analytic properties of approximate lattices. *Annales de l'Institut Fourier* 70.5 (2020), pp. 1903–1950.

[BHP18] Michael Björklund, Tobias Hartnick, and Felix Pogorzelski. Aperiodic order and spherical diffraction, I: auto-correlation of regular model sets. *Proceedings of the London Mathematical Society (3)* 116.4 (2018), pp. 957–996.

[BHP21] Michael Björklund, Tobias Hartnick, and Felix Pogorzelski. Aperiodic order and spherical diffraction, III: the shadow transform and the diffraction formula. *Journal of Functional Analysis* 281.12 (2021), Paper No. 109265, 59.

[BHP22] Michael Björklund, Tobias Hartnick, and Felix Pogorzelski. Aperiodic order and spherical diffraction, II: translation bounded measures on homogeneous spaces. *Mathematische Zeitschrift* 300.2 (2022), pp. 1157–1201.

[BHS19] Michael Björklund, Tobias Hartnick, and Thierry Stulemeijer. Borel density for approximate lattices. *Forum of Mathematics: Sigma* 7 (2019).

[BHY21] Michael Björklund, Tobias Hartnick, and Karasik Yakov. Intersection spaces and multiple transverse recurrence. Preprint, arXiv:2108.09064. 2021.

[BP91] Lawrence G. Brown and Gert K. Pedersen. C*-algebras of real rank zero. *Journal of Functional Analysis* 99.1 (1991), pp. 131–149.

[CKM23] Pierre-Emmanuel Caprace, Mehrdad Kalantar, and Nicolas Monod. A type I conjecture and boundary representations of hyperbolic groups. *Proceedings of the London Mathematical Society* 127.2 (2023), pp. 447–486.

[CW21] Pierre-Emmanuel Caprace and George A. Willis. *A totally disconnected invitation to locally compact groups*. Preprint, arXiv:2110.05991. 2021.

[Cas+20] Jorge Castillejos, Samuel Evington, Aaron Tikuisis, Stuart White, and Wilhelm Winter. Nuclear dimension of simple C*-algebras. *Inventiones Mathematicae* 224.1 (2020), pp. 245–290.

[Cio15] Corina Ciobotaru. *A note on type I groups acting on d-regular trees.* Preprint, arXiv:1506.02950. 2015.

[Cla07a] Lisa Orloff Clark. CCR and GCR Groupoid C*-algebras. *Indiana University Mathematics Journal* 56.5 (2007), pp. 2087–2110.

[Cla07b] Lisa Orloff Clark. Classifying the types of principal groupoid C^*-algebras. *Journal of Operator Theory* 57.2 (2007), pp. 251–266.

[Con+18] Clinton Conley, Steve Jackson, David Kerr, Andrew Marks, Brandon Seward, and Robin Tucker-Drob. Folner tilings for actions of amenable groups. *Mathematische Annalen* 371 (2018), pp. 663–683.

[Con76] A. Connes. Classification of Injective Factors Cases I_1, II_∞, III_λ, $\lambda \neq 1$. *Annals of Mathematics* 104.1 (1976), pp. 73–115.

[Con94] Alain Connes. *Noncommutative geometry.* San Diego, CA: Academic Press, 1994.

[CT17] Jason Crann and Zsolt Tanko. On the operator homology of the Fourier algebra and its cb-multiplier completion. *Journal of Functional Analysis* 273.7 (2017), pp. 2521–2545.

[DG97] Marius Dadarlat and Guihua Gong. A Classification Result for Approximately Homogeneous C*-algebras of Real Rank Zero. *Geometric and Functional Analysis* 7 (1997), pp. 646–711.

[Dix57] Jacques Dixmier. Sur les représentations unitaires des groupes de Lie algébriques. *Annales de l'institut Fourier* 7 (1957), pp. 315–328.

[Dix59] Jacques Dixmier. Sur les représentations unitaires des groupes de Lie nilpotents. V. *Bulletin de la Société Mathématique de France* 87 (1959), pp. 65–79.

[Dix83] Jacques Dixmier. *C*-algebras.* Second edition. Amsterdam: North-Holland Publishing Company, 1983.

[Eff75] Edward G. Effros. Property Γ and inner amenability. *Proceedings of the American Mathematical Society* 47.2 (1975), pp. 483–486.

[EH67] Edward G. Effros and Frank Hahn. Locally compact transformation groups and C^*-algebras. *Bulletin of the American Mathematical Society* 73.2 (1967), pp. 222–226.

[EN14] George Elliott and Zhuang Niu. The C*-algebra of a minimal homeomorphism of zero mean dimension. *Duke Mathematical Journal* 166 (2014), pp. 3569–3594.

[Ell76] George A Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *Journal of Algebra* 38.1 (1976), pp. 29–44.

[Ell93] George A. Elliott. On the classification of C*-algebras of real rank zero. *Journal für die reine und angewandte Mathematik* 443 (1993), pp. 179–219.

[EGL07] George A. Elliott, Guihua Gong, and Liangqing Li. On the classification of simple inductive limit C*-algebras, II: The isomorphism theorem. *Inventiones Mathematicae* 168.2 (2007), pp. 249–320.

[EFR23] Ulrik Enstad, Gabriel Favre, and Sven Raum. *Free actions of polynomial growth Lie groups and classifiable C*-algebras*. Preprint, arXiv:2307.15013. 2023.

[ER22] Ulrik Enstad and Sven Raum. *A dynamical approach to sampling and interpolation in unimodular groups*. Preprint, arXiv:2207.05125. 2022.

[Fav21] Gabriel Favre. The type I and CCR properties for groupoids and inverse semigroups. Licentiate Thesis. Stockholm University, 2021.

[Fav23] Gabriel Favre. *A note on inner amenability for FLC point sets*. Preprint, arXiv:2307.01880. 2023.

[FR21] Gabriel Favre and Sven Raum. An algebraic characterisation of ample type I groupoids. *Semigroup Forum* 104 (2021), pp. 58–71.

[Gar18] Eusebio Gardella. Compact group actions with the Rokhlin property. *Transactions of the American Mathematical Society* 371 (2018), pp. 2837–2874.

[Gli60] James Glimm. On a certain class of operator algebras. *Transactions of the American Mathematical Society* 95.2 (1960), pp. 318–340.

[Gli61] James Glimm. Locally compact transformation groups. *Transactions of the American Mathematical Society* 101.1 (1961), pp. 124–138.

[Goo73] Elliot Gootman. The type of some C* and W*-algebras associated with transformation groups. *Pacific Journal of Mathematics* 48 (1973), pp. 93–106.

[Gre12] Ben Green. WHAT IS...an Approximate Group? *Notices of the American Mathematical Society* 59.05 (2012), pp. 655–656.

[Har53] Harish-Chandra. Representations of a semisimple Lie group on a Banach space. I. *Transactions of the American Mathematical Society* 75 (1953), pp. 185–243.

[Hir+16] Ilan Hirshberg, Gábor Szabó, Wilhelm Winter, and Jianchao Wu. Rokhlin Dimension for Flows. *Communications in Mathematical Physics* 353.1 (2016), pp. 253–316.

[HWZ15] Ilan Hirshberg, Wilhelm Winter, and Joachim Zacharias. Rokhlin Dimension and C*-Dynamics. *Communications in Mathematical Physics* 335.2 (2015), pp. 637–670.

[HW17] Ilan Hirshberg and Jianchao Wu. The nuclear dimension of C*-algebras associated to homeomorphisms. *Advances in Mathematics* 304 (2017), pp. 56–89.

[HR19] Cyril Houdayer and Sven Raum. Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras. *Commentarii Mathematici Helvetici* 94 (2019), pp. 185–219.

[Hru20] Ehud Hrushovski. *Beyond the Lascar Group*. arXiv:2011.12009. 2020.

[IWZ19] Luke J. Ito, Michael F. Whittaker, and Joachim Zacharias. *Classification of tiling C^*-algebras*. Preprint, arXiv:1908.00770. 2019.

[JS99] Xinhui Jiang and Hongbing Su. On a simple unital projectionless C*-algebra. *American Journal of Mathematics* 121 (1999), pp. 359–413.

[Kan71] Eberhard Kaniuth. Die Struktur der regulären Darstellung lokalkompakter Gruppen mit invarianter Umgebungsbasis der Eins. *Mathematische Annalen* 194.3 (1971), pp. 225–248.

[Kas17] Daniel Kasprowski. Long and thin covers for flow spaces. *Groups, Geometry, and Dynamics* 11.4 (2017), pp. 1201–1229.

[Ken+21] Matthew Kennedy, Se-Jin Kim, Xin Li, Sven Raum, and Dan Ursu. *The ideal intersection property for essential groupoid C*-algebras*. Preprint, arXiv:2107.03980. 2021.

[Ker20] David Kerr. Dimension, comparison, and almost finiteness. *Journal of the European Mathematical Society* 22.11 (2020), pp. 3697–3745.

[KS20] David Kerr and Gabor Szabo. Almost finiteness and the small boundary property. *Communications in Mathematical Physics* 374 (2020), pp. 1–31.

[KW04] Eberhard Kirchberg and Wilhelm Winter. Covering dimension and quasidiagonality. *International Journal of Mathematics* 15.01 (2004), pp. 63–85.

[Kud12] Ganna Kudryavtseva. A refinement of Stone duality to skew Boolean algebras. *Algebra universalis* 67.4 (2012), pp. 397–416.

[Kud13] Ganna Kudryavtseva. A dualizing object approach to noncommutative Stone duality. *Journal of the Australian Mathematical Society* 95.3 (2013), pp. 383–403.

[LP91] Anthony To Ming Lau and Alan L. T. Paterson. Inner amenable locally compact groups. *Transactions of the American Mathematical Society* 325.1 (1991), pp. 155–169.

[Law10] Mark V. Lawson. A noncommutative generalization of Stone duality. *Journal of the Australian Mathematical Society* 88.3 (2010), pp. 385–404.

[Law12] Mark V. Lawson. Non-commutative Stone duality: inverse semigroups, topological groupoids and C*-algebras. *International Journal of Algebra and Computation* 22.6 (2012), p. 1250058.

[Law20] Mark V. Lawson. The booleanization of an inverse semigroup. *Semigroup Forum* 100 (2020), pp. 283–314.

[LL13] Mark V. Lawson and Daniel H. Lenz. Pseudogroups and their étale groupoids. *Advances in Mathematics* 244 (2013), pp. 117–170.

[LMS13] Mark V. Lawson, Stuart W. Margolis, and Benjamin Steinberg. The étale groupoid of an inverse semigroup as a groupoid of filters. *Journal of the Australian Mathematical Society* 94 (2013), pp. 234–256.

[LV21] Mark V. Lawson and Alina Vdovina. The universal Boolean inverse semigroup presented by the abstract Cuntz–Krieger relations. *Journal of Noncommutative Geometry* 15.1 (2021), pp. 279–304.

[Li11] Xin Li. Semigroup C*-algebras and amenability of semigroups. *Journal of Functional Analysis* 262 (2011), pp. 4302–4340.

[Li19] Xin Li. Every classifiable simple C*-algebra has a Cartan subalgebra. *Inventiones mathematicae* 219.2 (2019), pp. 653–699.

[Lin14] Huaxin Lin. Localizing the Elliott conjecture at strongly self-absorbing C*-algebras, II. *Journal für die reine und angewandte Mathematik* 2014.692 (2014), pp. 233–243.

[LR87] Viktor Losert and Harald Rindler. Conjugation-invariant means. *Colloquium Mathematicae* 51.1 (1987), pp. 221–225.

[Mac20a] Simon Machado. Approximate Lattices and Meyer Sets in Nilpotent Lie Groups. *Discrete Analysis* 1 (2020).

[Mac20b] Simon Machado. *Approximate Lattices in Higher-Rank Semi-Simple Groups.* Preprint, arXiv:2011.01835. 2020.

[Mac22] Simon Machado. Discrete approximate subgroups of Lie groups. PhD thesis. University of Cambridge, 2022.

[Mac23] Simon Machado. *Approximate lattices: structure in linear groups, definition(s) and beyond.* Preprint, arXiv:2306.09899. 2023.

[Mas17] Shuhei Masumoto. The Jiang–Su algebra as a fraïssé limit. *The Journal of Symbolic Logic* 82.4 (2017), pp. 1541–1559.

[Mey72] Yves Meyer. *Algebraic numbers and harmonic analysis.* Vol. 2. North-Holland mathematical library. Elsevier, 1972.

[MRW87] Paul Muhly, Jean Renault, and Dana Williams. Equivalence and isomorphism for groupoid C* - algebras. *Journal of Operator Theory* 17 (1987), pp. 3–22.

[MN36] Francis J. Murray and John von Neumann. On rings of operators. *Annals of Mathematics (2)* 37.1 (1936), pp. 116–229.

[MN37] Francis J. Murray and John von Neumann. On rings of operators. II. *Transactions of the American Mathematical Society* 41 (1937), pp. 208–248.

[Neb99] Claudio Nebbia. Groups of isometries of a tree and the CCR property. *Rocky Mountain Mathematics Consortium* 29.1 (1999), pp. 311–316.

[Neu40] John. von Neumann. On rings of operators. III. *Ann. Math. (2)* 41 (1940), pp. 94–167.

[Nor14] Magnus Dahler Norling. Inverse Semigroup C*-Algebras Associated with Left Cancellative Semigroups. *Proceedings of the Edinburgh Mathematical Society* 57.2 (2014), pp. 533–564.

[Ols80] Grigorii Iosifovich Olshanskii. New “large” groups of type I. Russian. *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii* 16 (1980), pp. 31–52.

[Pal61] Richard S. Palais. On the Existence of Slices for Actions of Non-Compact Lie Groups. *Annals of Mathematics* 73.2 (1961), pp. 295–323.

[Pat88] Alan Paterson. *Amenability.* Vol. 29. Mathematical Surveys and Monographs. American Mathematical Society, 1988.

[Phi00] N. Christopher Phillips. A classification theorem for nuclear purely infinite simple C*-algebras. *Documenta Mathematica* 5 (2000), pp. 49–114.

[Pie84] Jean Paul Pier. *Amenable locally compact groups.* Pure and Applied Mathematics. John Wiley and Sons, 1984.

[Ren80] Jean Renault. *A groupoid approach to C*-algebras.* Vol. 793. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 1980.

[Rie83] Marc A. Rieffel. Dimension and stable rank in the K-theory of C*-algebras. *Proceedings of the London Mathematical Society* 46.2 (1983), pp. 301–333.

[RS02] Mikael Rørdam and Erling Størmer. *Classification of Nuclear C*-Algebras. Entropy in Operator Algebras.* Springer Berlin Heidelberg, 2002.

[RW10] Mikael Rørdam and Wilhelm Winter. The Jiang–Su algebra revisited. *Journal für die reine und angewandte Mathematik* 2010.642 (2010), pp. 129–155.

[Sem22] Lancelot Semal. *Radu groups acting on trees are CCR.* Preprint, arXiv:2203.04189. 2022.

[SS21] Benjamin Steinberg and Nóra Szakács. Simplicity of inverse semigroup and étale groupoid algebras. *Advances in Mathematics* 380 (2021), pp. 107611–107653.

[Str20] Karen R. Strung. *An introduction to C*-algebras and the classification program.* Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser, 2020.

[SW11] Karen R. Strung and Wilhelm Winter. Minimal dynamics and $\mathcal{Z}$-stable classification. *International Journal of Mathematics* 22.01 (2011), pp. 1–23.

[Sza13] Gabor Szabo. The Rokhlin dimension of topological $\mathbb{Z}^m$-actions. *Proceedings of the London Mathematical Society* 110 (2013), pp. 673–694.

[Tho64] Elmar Thoma. Über unitäre Darstellungen abzählbarer, diskreter Gruppen. *Mathematische Annalen* 153 (1964), pp. 111–138.

[TWW17] Aaron Tikuisis, Stuart White, and Wilhelm Winter. Quasidiagonality of nuclear C*-algebras. *Annals of Mathematics (2)* 185 (2017), pp. 229–284.

[Tom08] Andrew Toms. On the classification problem for nuclear C*-algebras. *Annals of Mathematics* 167.3 (2008), pp. 1029–1044.

[TW09] Andrew S. Toms and Wilhelm Winter. Minimal dynamics and the classification of C*-algebras. *Proceedings of the National Academy of Sciences* 106.40 (2009), pp. 16942–16943.

[TW13] Andrew S. Toms and Wilhelm Winter. Minimal Dynamics and K-Theoretic Rigidity: Elliott's Conjecture. *Geometric and Functional Analysis* 23.1 (2013), pp. 467–481.

[Tu99] Jean-Louis Tu. La conjecture de Baum–Connes pour les feuilletages moyennables. *K-Theory* 17.3 (1999), pp. 215–264.

[Vae12] Stefaan Vaes. An inner amenable group whose von Neumann algebra does not have property Gamma. *Acta Mathematica* 208.2 (2012), pp. 389–394.

[Wes67] Joel J. Westman. *Nontransitive groupoid algebras.* Lecture Notes. University of California at Irvine, 1967.

[Wil15] Rufus Willett. A non-amenable groupoid whose maximal and reduced C*-algebras are the same. *Münster Journal of Mathematics* 8 (2015), pp. 241–252.

[Wil81] Dana Williams. The topology on the primitive ideal space of transformation group C*-algebras and CCR transformation group C*-algebras. *Transactions of the American Mathematical Society* 266 (1981), pp. 335–359.

[Win07] Wilhelm Winter. Simple C*-algebras with locally finite decomposition rank. *Journal of Functional Analysis* 243.2 (2007), pp. 394–425.

[Win14] Wilhelm Winter. Localizing the Elliott conjecture at strongly self-absorbing C*-algebras. *Journal für die reine und angewandte Mathematik* 2014.692 (2014), pp. 193–231.

[WZ10] Wilhelm Winter and Joachim Zacharias. The nuclear dimension of C*-algebras. *Advances in Mathematics* 224.2 (2010), pp. 461–498.

[Wyk18] Daniel W. van Wyk. The orbit spaces of groupoids whose C*-algebras are GCR. *Journal of Operator Theory* 80.1 (2018), pp. 167–185.

[Wyk19] Daniel W. van Wyk. The orbit spaces of groupoids whose C*-algebras are CCR. *Journal of Mathematical Analysis and Applications* 478.1 (2019), pp. 304–319.

[Yu98] Guoliang Yu. The Novikov Conjecture for Groups with Finite Asymptotic Dimension. *Annals of Mathematics* 147.2 (1998), pp. 325–355.

[Zel68] G. Zeller-Meier. *Produits croisés d'une C*-algèbre par un groupe d'automorphismes.* Gauthier-Villars, 1968.