Stockholm University

Intersection Theory on Zip Period Maps

Simon Cooper

Doctoral Thesis in Mathematics at Stockholm University, Sweden 2024

Intersection Theory on Zip Period Maps

Simon Cooper

Academic dissertation for the Degree of Doctor of Philosophy in Mathematics at Stockholm University to be publicly defended on Thursday 23 May 2024 at 09.00 in lärosal 7, hus 1, Albano, Albanovägen 28.

Abstract

This thesis consists of four papers, all motivated by questions about intersection theory on Shimura varieties in positive characteristic. The connection with intersection theory of flag varieties, made using the stack of G-Zips of type μ, is explored throughout. More generally, we work in the setting of intersection theory on spaces X admitting morphisms to the stack of G-Zips of type μ. These morphisms are termed 'zip period maps' in Paper III. The fundamental example of such an X is the special fibre of an integral canonical model of a Shimura variety of Hodge-type. Moreover, there is a notion of 'tautological ring' for any (smooth) zip period map which gives the usual tautological ring in the case of Shimura varieties.

In Paper I the tautological ring of a Hilbert modular variety at an unramified prime is computed. The method generalises van der Geer's approach from the Siegel case and makes use of the properness of the non-maximal Ekedahl-Oort strata closures in this setting.

The pushforward map in the Chow ring between Siegel flag varieties is computed in Paper II. Siegel flag varieties are projective varieties which are quotients of the symplectic group. They appear as the compact dual of the Siegel upper half plane. A conjecture exploring the connection between classes in Chow rings of flag varieties and classes in tautological rings of Shimura varieties is presented. The computation contained in this paper can be viewed as very basic evidence for this conjecture.

In Paper III we develop various conjectures related to positivity in the tautological ring of a zip period map. The notion of strata-effective classes is introduced. Several conjectures are presented regarding classes which we expect to be strata-effective. These are proved in many cases, including for Hilbert modular varieties, which are more accessible for various group-theoretic reasons. A connection between strata-effectivity and the Cone Conjecture of Goldring-Koskivirta is developed and provides examples of tautological and effective classes which nevertheless fail to be strata-effective.

In Paper IV we compute the Grothendieck group of the stack of G-Zips of type μ (as a ring) in the case where the derived group of G is simply connected.

Keywords: *G-Zips, Intersection theory, Tautological rings, Shimura varieties.*

Stockholm 2024
http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-227945

ISBN 978-91-8014-753-8
ISBN 978-91-8014-754-5

Department of Mathematics

Stockholm University, 106 91 Stockholm

INTERSECTION THEORY ON ZIP PERIOD MAPS

Simon Cooper

Stockholm
University

Intersection Theory on Zip Period Maps

Simon Cooper

ISBN print 978-91-8014-753-8
ISBN PDF 978-91-8014-754-5

Printed in Sweden by Universitetsservice US-AB, Stockholm 2024

Abstract

This thesis consists of four papers, all motivated by questions about intersection theory on Shimura varieties in positive characteristic. The connection between intersection theory on Shimura varieties in positive characteristic and intersection theory of flag varieties, made using the algebraic stack G-$\mathtt{Zip}^{\mu}$, is explored throughout. More generally, we work in the setting of intersection theory on spaces X admitting morphisms to G-$\mathtt{Zip}^{\mu}$. These morphisms are termed *zip period maps* in Paper III. The fundamental example of such an X is the special fibre of an integral canonical model of a Shimura variety of Hodge-type. Moreover, there is a notion of *tautological ring* for any (smooth) zip period map which gives the usual tautological ring in the case of Shimura varieties.

In Paper I the tautological ring of a Hilbert modular variety at an unramified prime is computed. The method generalises that of van der Geer from the Siegel case and makes use of properness of the non-maximal Ekedahl-Oort strata closures in this setting.

The pushforward map in the Chow ring between Siegel flag varieties is computed in Paper II. Siegel flag varieties are projective varieties which are quotients of the symplectic group. They appear as the compact dual of the Siegel upper half-plane. A conjecture exploring the connection between classes in Chow rings of flag varieties and classes in tautological rings of Shimura varieties is presented. The computation contained in this paper can be viewed as very basic evidence for this conjecture.

In Paper III we develop various conjectures related to positivity in the tautological ring of a zip period map. The notion of strata-effective classes is introduced. Several conjectures are presented regarding classes which we expect to be strata-effective. These are proved in many cases, including for Hilbert modular varieties, which are more accessible for various group-theoretic reasons. A connection between strata-effectivity and the Cone Conjecture of Goldring-Koskivirta is developed and provides examples of tautological and effective classes which nevertheless fail to be strata-effective.

In Paper IV we compute the K_0-ring $K_0(G\text{-}\mathtt{Zip}^{\mu})$ of the stack G-$\mathtt{Zip}^{\mu}$ in the case where the derived group of G is simply connected. Since G-$\mathtt{Zip}^{\mu}$ is a quotient stack, this computation can be done within the framework of equivariant algebraic K-theory.

Sammanfattning

Denna avhandling består av fyra artiklar, alla motiverade av frågor gällande intersektionsteorin för Shimura-varieteter i positiv karakteristik. Genomgående i avhandlingen så studeras kopplingen mellan intersektionsteorin för Shimura-varieteter i positiv karakteristik och intersektionsteorin för flaggvarieteter, med hjälp av den algebraiska stacken G-$\mathtt{Zip}^{\mu}$. Artikel I och III studerar den tautologiska delringen av Chow-ringen för Shimura-varieteter i positiv karakteristik. Artikel II beräknar framskjutningen av Siegel-flaggvarieteter i Chow-ringen och artikel IV beräknar den algebraiska K-teorin för stacken G-$\mathtt{Zip}^{\mu}$ under det ytterligare antagandet att kommutatorgruppen av G är enkelt sammanhängade.

I artikel I beräknas den tautologiska ringen av en Hilbert-modulär varietet vid ett oförgrenat primtal. Metoden generaliserar van der Geers metod för Siegel-fallet och använder sig av kompakthet hos de icke-maximala Ekedahl-Oort-stratahöljena.

Framskjutnings-avbildningen i Chow-ringerna till Siegel-flaggvarieteter beräknas i artikel II. Siegel flaggvarieteter är projektiva varieteter som är kvoter av den symplektiska gruppen. De antingen uppstår som den kompakta dualen av Siegels övre halvplan. En förmodan som utforskar sambandet mellan klasser i Chow-ringar av flaggvarieteter och klasser i tautologiska ringar av Shimura-varieteter presenteras. Beräkningen i denna artikel kan ses som mycket grundläggande evidens för denna förmodan.

I artikel III utvecklar vi olika förmodanden relaterade till positivitet i den tautologiska ringen. Vi arbetar under antagandet att vi har en generell Zip-periodkarta (ett rum med en avbildning till stacken av G-Zips av typen μ). Begreppet strata-effektiva klasser introduceras. Flera förmodanden presenteras gällande klasser som vi förväntar oss är strata-effektiva. Dessa är bevisade i många fall, inklusive för Hilbert-modulära varieteter vilka är mer tillgängliga av olika gruppteoretiska skäl. En koppling mellan strata-effektivitet och Cone Conjecture av Goldring-Koskivirta har utvecklats och ger exempel på tautologiska och effektiva klasser som ändå inte är strata-effektiva.

I artikel IV beräknas den algebraiska K-teorin $K_0(G\text{-}\mathtt{Zip}^{\mu})$ till stacken G-$\mathtt{Zip}^{\mu}$ i fallet då den kommutatorgruppen av G är enkelt sammanhängade.

Acknowledgements

I would like to thank my supervisor Wushi Goldring for his support and guidance over the last 4 years. His dedication and generosity are greatly appreciated and it has been a pleasure to work with him as a co-author (see Paper III).

Thanks to my colleagues at Stockholm University (and KTH) for providing a nice working environment and for their patience in answering my questions.

Finally, I would like to thank my family and friends for their support throughout the last 4 years.

List of papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

Paper I: **Tautological rings of Hilbert modular varieties**
Simon Cooper
Preprint

Paper II: **Pushforward of Siegel flag varieties in the Chow ring**
Simon Cooper
Preprint

Paper III: **Hodge-Chern classes and strata-effectivity in the tautological ring**
Simon Cooper, Wushi Goldring
Preprint

Paper IV: **Integral K-theory of the stack G-$\mathtt{Zip}^{\mu}$**
Simon Cooper
Preprint

Earlier versions of Papers I-II have also appeared in the author's licentiate thesis, defended November 2022.

Contents

General introduction

Preface

This introductory chapter will be split into three parts, of increasing mathematical technicality. The first part provides an outline of the basic questions studied in the thesis and the contribution of Papers I-IV towards answering them. The idea is that this can be read by anyone with a broad mathematical background, with the strong caveat that many mathematical objects will appear as black-boxes with a reference to the second part, which contains technical background and context. This second part aims to provide background at a level of a mathematician working in this or related areas. The third part provides a presentation of the results of the thesis at the appropriate level of technicality. This is aimed at experts who want to see clearly what is contained in the thesis.

Relation to the author's licentiate thesis:

While similar ideas and themes were present in the introduction to the author's licentiate thesis, this introduction is new.

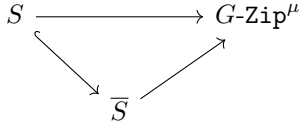
1 Outline

This thesis concerns the topic of intersection theory of Shimura varieties in positive characteristic. Shimura varieties are highly symmetric spaces which are defined from group-theoretic data. They are important objects of interest in number theory, especially in the Langlands program, as they contain lots of arithmetic information. In this thesis we are interested in their *geometry*: how they look as geometric objects. Strictly speaking, one has a tower of Shimura varieties indexed by extra data called the level. Many of the questions we are interested in are compatible with change of level or independent of change of level and so we ignore this subtlety for now. From now on we focus on Shimura varieties of Hodge type, which are those which are moduli spaces of abelian varieties with additional motivic information, very roughly speaking (see 2.1, 2.2). These spaces admit integral models so can be considered in both characteristic 0 and characteristic $p > 0$. One interesting feature of the geometry of Shimura

varieties is that they exhibit different geometric properties in characteristic 0 and characteristic $p > 0$. There is more known structure in characteristic p where, for example, Shimura varieties of Hodge type admit a *stratification* (the Ekedahl-Oort stratification). A stratification of a space S is a collection of pairwise disjoint, locally closed subspaces S_i which cover $S = \sqcup S_i$ such that for every stratum S_i, its boundary $\overline{S}_i \setminus S_i$ is a union of strata. This is a strong property and this stratification will play a large role in our attempts at understanding the geometry of Shimura varieties. Another interesting discrepancy between characterstic 0 and characteristic p is the size of the largest compact subvariety. In the case of the Siegel modular variety the dimension of the largest compact subvariety is greater in characteristic p than in characteristic 0 [KS03]. These are substantial geometric differences.

Much of this extra structure (e.g. the Ekedahl-Oort stratification) in characteristic p can be characterised using an elusive geometric object defined by groups: the algebraic stack $G\text{-}\mathtt{Zip}^{\mu}$. This is the moduli space of G-zips of type μ introduced in a series of papers by Moonen-Wedhorn [MW04] and Pink-Wedhorn-Ziegler [PWZ15]. Its underlying space is a finite set (a set of coset representatives in the Weyl group) and it admits a stratification by substacks with closure relations determined by the combinatorics of the Weyl group. If S is a Shimura variety of Hodge type in characteristic p then there is a canonical morphism $\zeta\colon S \to G\text{-}\mathtt{Zip}^{\mu}$ which is smooth and surjective [Zha18]. The Ekedahl-Oort stratification on S can be recovered by pulling back the stratification from $G\text{-}\mathtt{Zip}^{\mu}$.

It should be noted that there is some geometry that seems impervious to $G\text{-}\mathtt{Zip}^{\mu}$. For example, it is hard to see how one could distinguish the largest compact subvariety of S using $G\text{-}\mathtt{Zip}^{\mu}$. One possible reason for this is that the morphism $S \to G\text{-}\mathtt{Zip}^{\mu}$ factors through a *compactification* $\overline{S}$, that is a projective variety which contains S as a dense open subvariety.



The problem of discerning between S and its compactifications $\overline{S}$ will appear later. It is interesting to think about how much the stack $G\text{-}\mathtt{Zip}^{\mu}$ controls the geometry of the Shimura variety. One can phrase this as the following question initially raised by Goldring-Koskivirta in [GK18, Question A]. This can be seen as a guiding question throughout the thesis, even when the answer is 'not too much'.

Question 1.1. What can the geometry of $G\text{-}\mathtt{Zip}^{\mu}$ tell us about the geometry of S?

The rest of the introduction will place the work contained in the thesis within the context of this question. For us 'geometry' will refer solely to intersection

theory (2.5). In particular, we will be concerned only with two invariants: the Chow ring with rational coefficients $A^\bullet(-)$ and the K_0-ring $K_0(-)$, which take a scheme (or a stack) and give us a ring (2.5). The idea behind any invariant is that while our original space might be very complicated, the invariant should be a simpler and more down-to-earth object. In practice, these invariants $A^\bullet(-)$ and $K_0(-)$ can still prove very difficult to work with. One can instead consider the *tautological ring*, which for us is defined as the subring of the Chow ring generated by classes pulled back from $G\text{-}\mathtt{Zip}^\mu$ (2.6). This is much more accessible given that we can use information about the Chow ring of $G\text{-}\mathtt{Zip}^\mu$, which is fairly well understood.

Remark 1.2. In principle we would like to work in the framework of a general space X admitting a morphism $\zeta : X \to G\text{-}\mathtt{Zip}^\mu$, termed a *zip period map*. In this framework one can ask the more general version of Question 1.1: what can the geometry of $G\text{-}\mathtt{Zip}^\mu$, together with regularity properties of ζ (e.g. smoothness) tell us about the geometry of X? In practice, we predominantly focus on the zip period map with $X = S$ a Shimura variety of Hodge-type in characteristic p, since one obtains some compelling positive answers to Question 1.1 with clear applications to the intersection theory of Shimura varieties.

Remark 1.3. The object $G\text{-}\mathtt{Zip}^\mu$ exists only in the characteristic p setting. Moonen-Wedhorn [MW04] suggest that it should be seen as the characteristic p analogue of the period domain in Hodge theory.

2 Background

We discuss the basic geometric objects in 2.1-2.3 and describe the main tools for studying their geometry in 2.5-2.6. In 2.4 the construction of the canonical morphism $S \to G\text{-}\mathtt{Zip}^\mu$ is recalled.

2.1 Shimura varieties of Hodge type

Shimura varieties are rich arithmetico-geometric objects of intense interest in arithmetic geometry and number theory. They are defined by a *Shimura datum*: that is a pair $(\mathbf{G}, \mathbf{X})$ where $\mathbf{G}$ is a connected reductive group over $\mathbb{Q}$ and $\mathbf{X}$ is a conjugacy class of maps $\mathbb{S} \to \mathbf{G}_\mathbb{R}$ from the Deligne torus $\mathbb{S}$ satisfying a list of properties due to Deligne [Del79, 2.1.1]. Given a compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$ one obtains a complex manifold as a double coset space

$$\mathrm{Sh}_K(\mathbf{G}, \mathbf{X}) \coloneqq \mathbf{G}(\mathbb{Q})\backslash \mathbf{X} \times \mathbf{G}(\mathbb{A}_f)/K$$

It turns out this has the structure of an algebraic variety $\mathbf{S}_K$ over a number field $E = E(G, X)$ (called the reflex field).

Example 2.1 (Siegel modular datum)**.** Let GSp_{2g} be the general symplectic group. There is a symmetric space $\mathbf{X}_g$ called the double Siegel half space such

that $(\mathsf{GSp}_{2g}, \mathbf{X}_g)$ is a Shimura datum [Del79, 1.3.1]. The resulting Shimura variety is called the Siegel modular variety and denoted $\mathbf{S}_{g,K}$. Its reflex field is $\mathbb{Q}$.

The Siegel modular variety $\mathbf{S}_{g,K}$ parametrises abelian varieties with additional structure (principal polarisation and level). This description as a moduli space is vital in many constructions and leads us to focus on the class of Shimura varieties admitting similar descriptions. A Shimura datum $(\mathbf{G}, \mathbf{X})$ is defined to be *of Hodge type* if there is an embedding of Shimura data $(\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathsf{GSp}_{2g}, \mathbf{X}_g)$. This is the class of Shimura varieties we work with in this thesis. The simplest example of a Shimura variety of Hodge type is the Hilbert modular variety.

Example 2.2 (Hilbert modular datum, see Paper I Section 2)**.** Let $F/\mathbb{Q}$ be a totally real field extension of degree d. Define $\mathbf{G}$ to be the fibre product:

$$\begin{array}{ccc} \mathbf{G} & \longrightarrow & Res_{F/\mathbb{Q}}\mathsf{GL}_{2,F} \\ \downarrow & \square & \downarrow \\ \mathbf{G}_{m,\mathbb{Q}} & \longrightarrow & Res_{F/\mathbb{Q}}\mathbf{G}_{m,F} \end{array}$$

Note that $\mathbf{G}_F \subset \prod_{i=1}^{d} \mathsf{GL}_{2,F}$ have the same adjoint group. There is a Shimura datum $(\mathbf{G}, \mathbf{X})$, where $\mathbf{X}$ is the symmetric space $\mathbf{X} = \mathbf{X}_1^d$ consisting of d copies of the upper half plane. This is called the Hilbert modular datum. There is an embedding of Shimura data $(\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathsf{GSp}_{2d}, \mathbf{X}_d)$.

2.2 Integral models

In order to think of Shimura varieties in characteristic p we need to introduce *integral models*, constructed by Kisin [Kis10] and Vasiu [Vas99]. This is somewhat technical but the basic idea is to produce schemes over an integral domain whose base change to E recovers the Shimura variety. The special fibre of such a model is a variety over a field of characteristic $p > 0$.

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum of Hodge type and $p > 0$ a prime such that $\mathbf{G}$ is unramified at p (that is, there exists a smooth reductive group scheme $\mathcal{G}$ over $\mathbb{Z}_p$ whose generic fibre is $\mathbf{G}_{\mathbb{Q}_p}$). Consider, following Kisin [Kis10], the tower of Shimura varieties $(\mathrm{Sh}_{K_pK^p}(\mathbf{G}, \mathbf{X}))_{K^p}$ at *hyperspecial* level $K = K_pK^p \subset \mathbf{G}(\mathbb{A}_f)$. This means we fix $K_p = \mathcal{G}(\mathbb{Z}_p) \subset \mathbf{G}(\mathbb{Q}_p)$ for some smooth reductive model $\mathcal{G}$ over $\mathbb{Z}_p$ and vary $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$. Kisin proves that given a prime $v|p$ in the ring of integers $\mathcal{O} \subset E$, there is a tower of smooth $\mathcal{O}_{(v)}$-schemes $(\mathcal{S}_{K_pK^p})_{K^p}$ such that $\mathcal{S}_{K_pK^p} \otimes_{\mathcal{O}_{(v)}} E \simeq \mathrm{Sh}_{K_pK^p}(\mathbf{G}, \mathbf{X})$ [Kis10, 2.3.8]. Each $\mathcal{S}_{K_pK^p}$ is smooth and quasi-projective and the transition maps are finite etale. Moreover, this tower satisfies a certain extension property that characterises it uniquely, due to Milne [Mil90]. This tower is called the integral canonical model. We define the variety $S_{K_pK^p}$ over $\kappa_v \coloneqq \mathcal{O}_{(v)}/\mathfrak{m}$ to be the special fibre of $\mathcal{S}_{K_pK^p} \to \mathcal{O}_{(v)}$. Given the uniqueness of the integral canonical model it makes sense to call this the

Shimura variety mod p or *Shimura variety in characteristic p* or *special fibre of the Shimura variety* (at level $K = K_pK^p$). We often suppress the level K from the notation and write S for S_K or even the whole tower $(S_K)_K$.

2.3 G-Zips

Much of the structure peculiar to Shimura varieties mod p can be encapsulated in the stack G-$\mathtt{Zip}^\mu$. The definition is fairly technical but this object is of central importance to the thesis so it is worth recalling.

The input for the construction is a *cocharacter datum*: defined to be a pair (G, μ) where G is a connected reductive group over $\mathbb{F}_p$ and $\mu\colon \mathbb{G}_m \to G_k$ is a cocharacter over an algebraic closure k of $\mathbb{F}_p$. This datum determines opposite parabolic subgroups $P^-, P^+ \subset G_k$, where the Lie algebra of the parabolic P^- (resp. P^+) is the sum of the non-positive (resp. non-negative) weight spaces for the action μ on the Lie algebra of G_k. Let $P \coloneqq P^-$, $Q \coloneqq (P^+)^{(p)}$ the Frobenius twist of P^+ and $L \coloneqq P^- \cap P^+$ and $M \coloneqq L^{(p)}$ their Levi factors.

Definition 2.3. Let (G, μ) be a cocharacter datum. A *G-Zip of type μ over a base k-scheme S* is a tuple $\underline{I} = (I, I_P, I_Q, \iota)$ where I is a G_k-torsor over S, $I_p \subset I$ is a P-torsor over S, $I_Q \subset I$ is a Q-torsor over S and

$$\iota\colon (I_P^{(p)})/\mathcal{R}_u(P)^{(p)} \xrightarrow{\simeq} I_Q/\mathcal{R}_u(Q)$$

is an isomorphism of M-torsors over S. Here, $\mathcal{R}_u(-)$ denotes the unipotent radical of an algebraic group.

The category of G-zips of type μ over k-schemes forms an algebraic stack G-$\mathtt{Zip}^\mu$ [PWZ15, 3.2].

2.3.1 Properties of G-$\mathtt{Zip}^\mu$

G-$\mathtt{Zip}^\mu$ admits a description as a quotient stack G-$\mathtt{Zip}^\mu \simeq [E\backslash G_k]$. The E-orbits in G_k form a stratification of G_k by locally closed subsets. They are indexed by a subset ${}^IW \subset W$ of minimal length coset representatives of $W_I\backslash W$, where $I \subset \Delta$ is the type of P and $W_I \subset W$ is the subgroup generated by simple reflections s_α for $\alpha \in I$. This induces a stratification G-$\mathtt{Zip}^\mu = \sqcup_{w \in {}^IW} Z_w$ of G-$\mathtt{Zip}^\mu$ by locally closed substacks Z_w.

2.4 The universal G-Zip for Shimura varieties of Hodge type

For smooth families of schemes in positive characteristic satisfying a certain degeneration condition (see Moonen-Wedhorn [MW04]), the de Rham cohomology admits special structure. This leads to the construction of a G-Zip on the special fibre of a Shimura variety of Hodge type. In particular, essential use is

made of the Hodge and conjugate filtrations and Cartier isomorphisms between them. This section mostly follows the exposition of [GK19, 4.2].

Let $(\mathbf{G}, \mathbf{X})$ be a Shimura datum of Hodge type and $f\colon (\mathbf{G}, \mathbf{X}) \hookrightarrow (\mathbf{GSp}_{2g}, \mathbf{X}_g)$ a Siegel embedding. Let $p > 0$ be a prime such that $\mathbf{G}$ is unramified at p, $v|p$ a place of $\mathcal{O}_E$ above p and denote k an algebraic closure of the residue field $\mathcal{O}_v/\mathfrak{m}_v$. By [Kis10, 2.3.1] there is an $\mathcal{O}_{(v)}$-lattice Λ and embeddings $\mathcal{G} \hookrightarrow GSp(\Lambda) \hookrightarrow GL(\Lambda)$ which lift $\mathbf{G} \hookrightarrow \mathbf{GSp}_{2g} \hookrightarrow \mathbf{GL}_{2g}$ to $\mathcal{O}_{(v)}$. This induces an embedding of integral models $f\colon \mathcal{S}_K \hookrightarrow \mathcal{S}_{g,K'}$, where the level K' is chosen appropriately (see [Kis10] for details).

The integral model of the Siegel modular variety $\mathcal{S}_{g,K'}$ parametrises abelian schemes over $\mathcal{O}_{(v)}$ with principal polarisation and level structure [Kis10, 2.3.3]. Denote $\mathcal{A}/\mathcal{S}_{g,K'}$ the universal abelian scheme over $\mathcal{S}_{g,K'}$ and $A/S_{g,K'}$ its special fibre. This family $A/S_{g,K'}$ plays a crucial role in the construction as it satisfies the degeneration condition of Moonen-Wedhorn [MW04, 7.2].

2.4.1 de Rham cohomology

The de Rham cohomology

$$H^1_{dR}(A/S_{g,K'}) \coloneqq R^1\Gamma(\Omega^\bullet_{A/S_{g,K'}})$$

is a locally free sheaf of rank $2g$ over $S_{g,K'}$. The pullback of the de Rham cohomology $f^*H^1_{dR}(A/S_{g,K'})$ is a locally free sheaf of rank $2g$ on S_K and so its (etale) sheaf of trivialisations[1]

$$\mathcal{I}som_{\mathcal{O}_S}(f^*H^1_{dR}(A/S_{g,K'}), \Lambda_k \otimes \mathcal{O}_S)$$

is a $GL(\Lambda_k)$-torsor on S_K, where Λ_k is the k-vector space $\Lambda \otimes_{\mathcal{O}_{(v)}} k$. Note that $\Lambda_k \otimes \mathcal{O}_S \cong \mathcal{O}_S^{\oplus 2g}$ is free of rank $2g$. A matrix $g \in GL(\Lambda_k)$ acts on an isomorphism $f^*H^1_{dR}(A/S_{g,K'}) \xrightarrow{\sim} \Lambda_k \otimes \mathcal{O}_S$ by post-composition with the isomorphism $g\colon \Lambda_k \otimes \mathcal{O}_S \xrightarrow{\sim} \Lambda_k \otimes \mathcal{O}_S$.

2.4.2 Constructing a G-torsor on S_K

One can realise the subgroup $\mathcal{G}$ as the stabiliser in $GL(\Lambda)$ of a finite collection of Hodge tensors $(s_\alpha) \subset \Lambda^\otimes$ [Kis10, 1.3.2]. This realises $G \hookrightarrow GL(\Lambda_k)$ as the stabiliser of reductions $(\overline{s}_\alpha)$ of (s_α) to $\Lambda_k^\otimes$. This can be used to construct a $\mathcal{G}$-torsor as follows. In [Kis10, 2.3.9] Kisin associates to (s_α) a collection of sections

$$(s_{\alpha,dR}) \in H^0(\mathcal{S}_K, f^*H^1_{dR}(\mathcal{A}/\mathcal{S}_{g,K}))$$

Denote $\overline{s}_{\alpha,dR} \in H^0(S_K, f^*H^1_{dR}(A/S_{g,K}))$ the reduction of $s_{\alpha,dR}$ to the special fibre S_K. The sheaf of trivialisations compatible with these Hodge tensors

$$I \coloneqq \mathcal{I}som_{\mathcal{O}_S}((f^*H^1_{dR}(A/S_{g,K}), (\overline{s}_{\alpha,dR})), (\Lambda_k, (\overline{s}_\alpha)) \otimes \mathcal{O}_S)$$

[1] Given $T \to S$ and locally free $\mathcal{O}_S$-modules $\mathcal{F}, \mathcal{G}$, an element in $\mathcal{I}som_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G})(T)$ is a isomorphism of sheaves $\phi_T\colon \mathcal{F}_T \xrightarrow{\sim} \mathcal{G}_T$.

is a G-torsor on S_K.

2.4.3 Hodge and conjugate filtrations, Cartier isomorphisms

In order to construct subschemes of I which are P and Q-torsors we consider isomorphisms which are compatible with the Hodge and conjugate filtrations.

The existence of Hodge-de Rham and conjugate spectral sequences converging to de Rham cohomology means there is a 2-step Hodge filtration $0 \subset \Omega \subset f^*H^1_{dR}(A/S_{g,K'})$ and 2-step conjugate filtration $0 \subset \Omega_{\text{conj}} \subset f^*H^1_{dR}(A/S_{g,K'})$ [MW04]. On the special fibre S_K there are Cartier isomorphisms

$$\iota_0 \colon \Omega^{(p)} \simeq f^*H^1_{dR}(A/S_{g,K'})/\Omega_{\text{conj}}$$

and

$$\iota_1 \colon (f^*H^1_{dR}(A/S_{g,K'})/\Omega)^{(p)} \simeq \Omega_{\text{conj}}$$

We can similarly define filtrations of the vector space Λ_k. Let μ be, roughly speaking, the reduction mod p of a representative of $\mathbf{X}$ (see [GK19, 4.1.5] for details). The cocharacter $f \circ \mu$ defines filtrations $0 \subset \Lambda_P \subset \Lambda_k$ and $0 \subset \Lambda_Q \subset \Lambda_k$.

The sheaf of trivialisations compatible with Hodge tensors and stabilising the Hodge filtration

$$I_P \coloneqq \mathcal{I}som_{\mathcal{O}_S}((f^*H^1_{dR}(A/S_{g,K}), (\overline{s}_{\alpha,dR}), \Omega), (\Lambda_k, (\overline{s}_\alpha), \Lambda_P) \otimes \mathcal{O}_S)$$

is a P-torsor on S_K. Likewise, the sheaf of trivialisations compatible with Hodge tensors and stabilising the conjugate filtration

$$I_Q \coloneqq \mathcal{I}som_{\mathcal{O}_S}((f^*H^1_{dR}(A/S_{g,K}), (\overline{s}_{\alpha,dR}), \Omega_{\text{conj}}), (\Lambda_k, (\overline{s}_\alpha), \Lambda_Q) \otimes \mathcal{O}_S)$$

is a Q-torsor on S_K.

2.4.4 Universal G-Zip of type μ

Consider the tuple (I, I_P, I_Q, ι), where

$$\iota \colon (I_P^{(p)})/\mathcal{R}_u(P)^{(p)} \xrightarrow{\simeq} I_Q/\mathcal{R}_u(Q)$$

is given by the Cartier isomorphisms (ι_0, ι_1). This forms a G-zip of type μ on S_K and so defines a morphism

$$S_K \to G\text{-}\mathtt{Zip}^\mu$$

This is the main object of interest for us. It forms the fundamental example of a zip period map, which is defined below.

2.4.5 Zip period maps

Definition 2.4. A zip period map is a morphism $\zeta \colon X \to G\text{-}\mathtt{Zip}^\mu$ from a stack X to the stack $G\text{-}\mathtt{Zip}^\mu$, for some cocharacter datum (G, μ). Note that this includes the case $X = G\text{-}\mathtt{Zip}^\mu$, $\zeta = \mathrm{Id}$.

2.5 Intersection theory

The idea behind intersection theory is a simple and ubiquitous one: to study a class of complicated objects by introducing an invariant which is easier to understand.

2.5.1 Chow ring

For the most part, our 'geometry' will be embodied in the *Chow ring* (with rational coefficients), which is an invariant associated with the space X. It is a vector space over $\mathbb{Q}$ with a $\mathbb{Z}$-grading and a ring structure (when X is smooth). For example, when $X = \mathrm{pt}$ is a single point the Chow ring is $A^\bullet(\mathrm{pt}) = \mathbb{Q}$. Given a scheme X we define the group of cycles to be the free abelian group on the set of integral subschemes $Z \subset X$. We impose a coarse equivalence relation on them, called *rational equivalence*. This forms an abelian group, which has a grading by codimension and when X is smooth a ring structure given, more or less, by intersection of subschemes. This is by definition the Chow ring and taking the tensor product with $\mathbb{Q}$ over $\mathbb{Z}$ we obtain the *Chow ring with rational coefficients* which we denote $A^\bullet(X)$. This definition can be extended to quotient stacks using the equivariant Chow ring of Edidin-Graham [EG98].

2.5.2 K-theory

Another related invariant we study is the K_0-ring (the Grothendieck group). Given a scheme X we denote $\mathrm{Vect}_\simeq(X)$ the set of vector bundles on X, up to isomorphism. Then $K_0(X) \coloneqq \mathbb{Z}\{[E] \mid E \in \mathrm{Vect}_\simeq(X)\}/\sim$ where $[E] \sim [E'] + [E'']$ if and only if there exists a short exact sequence of vector bundles:

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

There is a ring structure on $K_0(X)$ given by tensor product $[E] \cdot [F] \coloneqq [E \otimes F]$. The relation between Chow ring and Grothendieck group is encapsulated for a smooth scheme X in the chern character map which is a ring homomorphism $\mathrm{ch}\colon K_0(X) \to A^\bullet(X)$. In this case the extension to rational coefficients $\mathrm{ch}_\mathbb{Q}\colon K_0(X) \otimes \mathbb{Q} \to A^\bullet(X)$ is an isomorphism. The definition of K_0 can be extended to quotient stacks, though now $\mathrm{ch}_\mathbb{Q}$ is only an isomorphism after completion [EG00].

2.6 Tautological rings

The concept of tautological ring appears in different areas of mathematics and originated in the work of Mumford. The notable example outside our zone of interest is the case of the moduli space of curves, where it has been intensely studied. The main idea is that we have some space, usually a moduli space, and we want to understand some invariant on the space, usually the Chow ring. However, the whole Chow ring of the space is often not accessible. Instead

one studies a subring of the Chow ring generated by some 'nice' classes, which are called tautological classes. The resulting subring is called the *tautological ring*. Naturally, the tautological ring depends on which classes are chosen to be tautological. In the case of Shimura varieties, the tautological ring is defined to be the subring of the Chow ring generated by the chern classes of automorphic vector bundles [Gee99][WZ23]. Note that this definition makes sense in both characteristic 0 and characteristic p. There are very natural classes which are not included in this ring, namely certain special cycles, see (2.11). Nevertheless we restrict our attention to this smaller subring, which is somewhat amenable to existing techniques. This definition of tautological ring can be extended to the case of any smooth zip period map $\zeta\colon X \to G\text{-}\mathtt{Zip}^{\mu}$, by defining the tautological ring of X as[2] $T^{\bullet}(X) \coloneqq \zeta^{*}(A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}))$. In the case where $X = S$ is the special fibre of a Shimura variety of Hodge type this recovers the original definition [WZ23, 6.6].

2.7 Compactifications

A common feature of intersection theory is that it is very difficult to work with varieties which are not compact (in the language of algebraic geometry: proper). One reason for this is that many positivity results which hold for proper schemes do not hold for non-proper schemes. For instance, it can happen that the class of a proper subvariety of a quasi-projective variety is zero in the rational Chow ring, something that never happens for a projective variety (e.g. the p-rank 0 locus in the Siegel modular variety). Given that most Shimura varieties are not proper, this leads to the study of their compactifications. Let $\mathbf{S}$ be a Shimura variety (of Hodge type, though this is not always necessary). The *minimal compactification* $\mathbf{S}^{\min}$ is a projective algebraic variety over E containing $\mathbf{S}$ as a dense open subscheme. The main issue is that the boundary $\mathbf{S}^{\min} \setminus \mathbf{S}$ is highly singular. The search for desingularisations leads to the *toroidal compactifications* of $\mathbf{S}$ due to Ash-Mumford-Rapaport-Tai [Ash+10], which depend on some additional combinatorial data Σ called a cone decomposition. Given such a datum Σ one obtains an algebraic variety $\mathbf{S}^{\Sigma}$ containing $\mathbf{S}$ as a dense open. In general, $\mathbf{S}^{\Sigma}$ need not be proper or smooth, but we can choose Σ systematically (i.e. at every stage of our constructions) such that $\mathbf{S}^{\Sigma}$ is smooth and projective and $\mathbf{S}^{\Sigma} \setminus \mathbf{S}$ is a union of smooth normal crossing divisors. Often we implicitly choose such a Σ and simply write $\mathbf{S}^{\mathrm{tor}} = \mathbf{S}^{\Sigma}$. These compactifications are related by a proper map $\mathbf{S}^{\mathrm{tor}} \to \mathbf{S}^{\min}$ which is an isomorphism on the dense open $\mathbf{S}$.

2.7.1 Compactifications of integral models

Let $\mathcal{S}$ be the integral model of a Shimura variety of Hodge type at a prime of good reduction as in (2.2). There is a minimal compactification $\mathcal{S}^{\min}$ and for each cone decomposition Σ a toroidal compactification $\mathcal{S}^{\Sigma}$ (which can be chosen

[2] This depends a priori on ζ, but this is suppressed in the notation.

smooth and proper) of $\mathcal{S}$ [Mad19]. This gives minimal and toroidal compactifications of the special fibre S of the integral model.

2.7.2 Universal G-Zip on compactifications

Let S be a Shimura variety of Hodge type in characteristic $p > 0$ and S^{tor} a smooth proper toroidal compactification of S. Fix a level K. The universal G-Zip of type μ over S_K (2.4.4) can be extended to S_K^{tor} [GK19, 6.2.1]. This gives a morphism $\zeta^{\mathrm{tor}} \colon S_K^{\mathrm{tor}} \to G\text{-}\mathtt{Zip}^{\mu}$ extending $\zeta \colon S_K \to G\text{-}\mathtt{Zip}^{\mu}$.

2.8 Previous results: Chow ring of G-$\mathtt{Zip}^{\mu}$

Given a zip period map $X \to G\text{-}\mathtt{Zip}^{\mu}$, we are interested in what the geometry of $G\text{-}\mathtt{Zip}^{\mu}$ can tell us about the geometry of X. This relies on first understanding the geometry of $G\text{-}\mathtt{Zip}^{\mu}$. The main instance of this is the computation of $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$ for a general cocharacter datum (G, μ) by Brokemper [Bro18]. If (G, μ) is a cocharacter datum associated with the special fibre of a Shimura variety of Hodge type then $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}) \cong A^{\bullet}(G/P) \cong H^{2\bullet}_{\mathrm{sing}}(\mathbf{X}^{\vee}(\mathbb{C}), \mathbb{Q})$ where $\mathbf{X}^{\vee} \cong \mathbf{G}/\mathbf{P}$ is the compact dual of the symmetric space $\mathbf{X}$ [WZ23, 4.12]. This is rather nice as intersection theory of flag varieties (e.g. of G/P) is fairly well understood. For example, the Chow ring of a flag variety is isomorphic to its cohomology ring and there is a famous presentation of this ring due to Borel [Bor53].

2.9 Previous results: Tautological rings of Shimura varieties

One way of phrasing our main question in the context of Chow rings is the following.

Question 2.5. Suppose that X is a scheme admitting a smooth zip period map $\zeta \colon X \to G\text{-}\mathtt{Zip}^{\mu}$. What is the tautological ring $T^{\bullet}(X) \coloneqq \zeta^*(A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}))$?

Wedhorn-Ziegler answer this question positively in the case when $X = S_K^{\mathrm{tor}}$ is a toroidal compactification of a Shimura variety of Hodge type in characteristic $p > 0$ at arbitrary hyperspecial level K. Their result is a springboard for our investigation and of central importance.

Theorem 2.6 ([WZ23] 7.12)**.** *Let S_K^{tor} be a smooth proper toroidal compactification of a Shimura variety of Hodge type in characteristic $p > 0$ at hyperspecial level K.*

$$T^{\bullet}(S_K^{\mathrm{tor}}) \cong A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$$

In this case the map ζ^* is injective and so $A^{\bullet}(S_K^{\mathrm{tor}})$ contains an isomorphic copy of $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$. This seems a very positive answer to 1.1 in this setting. Note also that it shows the tautological ring is independent of the choice of level K.

Remark 2.7. This result is a characteristic p version of a general folk-lore conjecture about tautological rings of toroidal compactifications of Shimura varieties (see 2.9).

Going beyond the compact case things become much more complicated. The map ζ^* is no longer injective [Gee99] and so information is *lost*; this provides a sort of negative answer to Question 1.1. In light of the result of Wedhorn-Ziegler, this prompts the following question. Can one compute $T^\bullet(S_K)$ with the help of some other geometric information about S_K (not coming from G-$\mathtt{Zip}^\mu$)? This was already achieved by van der Geer in the case of the Siegel modular variety. The additional geometric information needed is the existence of a large proper subvariety (the p-rank 0 locus). This allows one to use techniques from Chow rings of proper schemes, namely certain *positivity* properties, to complete the computation (see Paper I for more details).

Remark 2.8. One interesting detail of van der Geer's method is that he is also able to deduce the tautological ring in characteristic 0 by using the properness of the p-rank 0 locus, even though this only exists in characteristic $p > 0$.

Another result contained in [Gee99] is that the chern classes of the Hodge vector bundle are effective. That is, they are non-negative linear combinations of cycle classes of subvarieties. In this case they are represented by p-rank loci. This is generalised in Paper III (see 3.2).

2.10 Context: Tautological rings in characteristic 0

Let $\mathbf{S} = \mathrm{Sh}(\mathbf{G}, \mathbf{X})$ be a Shimura variety (not necessarily of Hodge type). Consider, for a given level K, the subring $T^\bullet(\mathbf{S}_K^{\mathrm{tor}}) \subset A^\bullet(\mathbf{S}_K^{\mathrm{tor}})$ generated by chern classes of canonical extensions of automorphic vector bundles.

Conjecture 2.9 (Folk-lore conjecture)**.** $T^\bullet(\mathbf{S}_K^{\mathrm{tor}}) \cong A^\bullet(\mathbf{X}^\vee)$.

We explain why this is reasonable to expect and relate it with another conjecture regarding vanishing of chern classes. There is a cartersian diagram due to Milne [Mil90] (explained in [Lan12, 5.2]) of the form

$$\begin{array}{ccc} \mathbf{Y}_K & \longrightarrow & \mathbf{X}^\vee \\ \downarrow & \square & \downarrow \pi \\ \mathbf{S}_K^{\mathrm{tor}} & \xrightarrow{\sigma} & [\mathbf{X}^\vee/\mathbf{G}^c] \end{array}$$

where $\mathbf{Y}_K \to \mathbf{S}_K^{\mathrm{tor}}$ is a $\mathbf{G}^c$-torsor[3] and $\mathbf{Y}_K \to \mathbf{X}^\vee$ is $\mathbf{G}$-equivariant. Canonical extensions of automorphic vector bundles on $\mathbf{S}_K^{\mathrm{tor}}$ are *by definition* vector bundles

[3] $\mathbf{G}^c$ is a central quotient of $\mathbf{G}$ with equality $\mathbf{G} = \mathbf{G}^c$ in the Hodge-type case.

of the form $\sigma^*\mathcal{V}$ where $\mathcal{V}$ is a vector bundle on $[\mathbf{X}^\vee/\mathbf{G}^c]$. This gives a diagram

$$\begin{array}{ccccc} A^\bullet([*/\mathbf{G}^c]) & \longrightarrow & A^\bullet([\mathbf{X}^\vee/\mathbf{G}^c]) & \xrightarrow{\sigma^*} & T^\bullet(\mathbf{S}_K^{\mathrm{tor}}) \\ & & \downarrow{\scriptstyle \pi^*} & \nearrow & \\ & & A^\bullet(\mathbf{X}^\vee) & & \end{array} \tag{2.1}$$

If $\sigma^*(\ker(\pi^*)) = 0$ then σ^* factors through π^* and Hirzebruch-Mumford proportionality [Mum77] implies that the resulting map $A^\bullet(\mathbf{X}^\vee) \to T^\bullet(\mathbf{S}_K^{\mathrm{tor}})$ of graded rings is injective [WZ23, 7.11], giving the desired result. It is known that the kernel of π^* is the image of $A^{\bullet>0}([*/\mathbf{G}^c])$. Its image $\sigma^*(A^{\bullet>0}([*/\mathbf{G}^c]))$ is the subring of $A^\bullet(\mathbf{S}_K^{\mathrm{tor}})$ generated by chern classes of *flat* automorphic vector bundles i.e. those coming from a representation of $\mathbf{G}^c$. This leads to the following conjecture.

Conjecture 2.10 (Attributed to Esnault, see discussion in [EH17])**.** Chern classes of flat automorphic vector bundles vanish in the Chow ring.

This is known only in the Siegel case [EV02] and appears inaccessible beyond that with the techniques currently available. Our discussion above amounts to the following.

Proposition 2.11. *Conjecture 2.10 implies Conjecture 2.9.*

The converse is also true: assuming that an isomorphism fits in the commutative diagram (2.1), Conjecture 2.9 implies Conjecture 2.10.

2.11 Special cycles

In general, there are important and natural classes in the Chow ring $A^\bullet(S_K)$ of a Shimura variety which are not tautological. A major example is special cycles which are the cycle classes of special subvarieties. Special subvarieties are, roughly, Shimura sub-varieties and their translates. One has examples of special cycles which are not tautological already for the Siegel modular threefold (at least in characteristic 0). Here the rational Picard group is generated by special cycles [HH12] but the codimension 1 part of the tautological ring is 1-dimensional, generated by λ_1 [Gee99]. Given that the rational Picard group of the Siegel modular threefold has rank greater than 1 [Wei92] we deduce that there are special divisors which are not tautological. Another example of a natural class which is not tautological is the product of sub[4] Siegel modular varieties $[\mathbf{S}_1 \times \mathbf{S}_5] \in A^\bullet(\mathbf{S}_6)$. It has recently been shown that this class is not tautological. The result is announced in [Can+24] and a proof is to appear in [COP24].

Remark 2.12. It might be interesting to investigate a bigger ring given by taking cycle classes of subvarieties $X_K \subset S_K$ at every level K, which are compatible with the $G(\mathbb{A}_f)$-action on the tower. The tautological ring should be a subring of this and the special cycles considered above should also be contained in it.

[4] we suppress here the choice of compatible level K

3 Presentation of results contained in this thesis

This thesis consists of four papers, whose results can be seen as concerning different interpretations of Question 1.1. For the most part, excluding Paper IV, Question 1.1 is studied through the lens of the Chow ring. In particular, we restrict our attention to the tautological ring.

3.1 Tautological ring

As above, for smooth $\zeta\colon X \to G\text{-}\mathtt{Zip}^{\mu}$ the tautological ring is defined to be $T^{\bullet}(X) \coloneqq \zeta^{*}(A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}))$. In the Chow ring context Question 1.1 can be interpreted as:

Question 3.1. Let X be a scheme admitting a smooth map $\zeta\colon X \to G\text{-}\mathtt{Zip}^{\mu}$. What is the tautological subring $T^{\bullet}(X) \subset A^{\bullet}(X)$?

This is quite a concrete question in that we are asking for an explicit description of this ring. For example, if $X = S_1^{\mathrm{tor}}$ is the compactified Siegel modular curve in characteristic $p > 0$ then $T^{\bullet}(S_1^{\mathrm{tor}}) = \mathbb{Q}[\lambda_1]/(\lambda_1^2)$ is a vector space of dimension 2 with basis $\{1, \lambda_1\}$ [Gee99]. For reasons described in (2.7) it is much easier to work with schemes X which are proper. We have seen that ζ^{*} is injective for the toroidal compactification S^{tor} of a Shimura variety of Hodge type in characteristic $p > 0$ and so $T^{\bullet}(S^{\mathrm{tor}}) \cong A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$ [WZ23, 7.12]. Emboldened by van der Geer's computation for the (non-proper) Siegel modular variety we can ask the following question.

Question 3.2. Let S be a Shimura variety of Hodge type in characteristic $p > 0$. What is the tautological ring $T^{\bullet}(S)$?

In Paper I we generalise van der Geer's method to tackle the next simplest Shimura variety of Hodge type and compute $T^{\bullet}(X)$ for X the Hilbert modular variety of dimension d at an unramified prime. The geometric input needed here is more complicated than in the Siegel case and consists of d proper subvarieties of codimension 1 in X (these are Zariski closures of Ekedahl-Oort strata), which we prove have linearly independent cycle classes in the Chow ring. In Paper I we also explain why this method can only work for very specific examples such as the two above, essentially due to issues of dimension. In general, new techniques are required to compute the tautological ring for Shimura varieties of Hodge type in characteristic $p > 0$.

The main tool for accessing the Chow groups $A_i(X)$ of a quasi-projective variety X is the localisation exact sequence

$$A_i(\overline{X} \setminus X) \longrightarrow A_i(\overline{X}) \longrightarrow A_i(X) \longrightarrow 0$$

where $\overline{X}$ is a projective variety containing X as a dense open subscheme. It allows one to compute the Chow ring of the space X using the Chow ring of a

compactification $\overline{X}$ and the classes in $A^\bullet(\overline{X})$ which are supported on the boundary $\overline{X} \setminus X$. For S a Shimura variety of Hodge type in characteristic $p > 0$ this gives the following diagram of graded $\mathbb{Q}$-vector spaces:

$$\begin{array}{ccccccc} A^{\bullet-1}(S^{\mathrm{tor}} \setminus S) & \longrightarrow & A^\bullet(S^{\mathrm{tor}}) & \longrightarrow & A^\bullet(S) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \zeta^{\mathrm{tor},*}} & \nearrow{\scriptstyle \zeta^*} & & & \\ & & A^\bullet(G\text{-}\mathtt{Zip}^\mu) & & & & \end{array}$$

One very general question regarding the isomorphism $T^\bullet(S^{\mathrm{tor}}) \cong A^\bullet(G/P)$ is:

Question 3.3. Given a class in $T^\bullet(S^{\mathrm{tor}})$, can one describe the corresponding class in $A^\bullet(G/P)$?

More specifically, can we describe the classes in $A^\bullet(G/P)$ which correspond to the tautological classes supported on the boundary? In Paper II we conjecture an answer to this. The boundary $\partial \coloneqq S^{\mathrm{tor}} \setminus S$ is stratified into boundary strata ∂_J indexed by standard parabolic subgroups $P_J \subset G$ (these are determined by a subset of the simple roots $J \subset \Delta$).

Conjecture 3.4. The tautological classes supported on the boundary correspond to the subspace generated by $[L_J/L_J \cap P] \in A^\bullet(G/P)$ ranging over $J = \Delta \setminus \{\alpha\}$ for simple roots $\alpha \in \Delta$.

In light of the localisation sequence, this would provide a description of $T^\bullet(S)$ for the non-proper Shimura varieties S of Hodge type in characteristic $p > 0$ and thus an answer to Question 3.1. In Paper II we are able to show that Conjecture 3.4 is consistent with the two known cases of the tautological ring just described: the Siegel modular variety and the Hilbert modular variety. A much more modest conjecture, which follows from Conjecture 3.4 is the following:

Conjecture 3.5. Suppose that $\alpha \in T^\bullet(S^{\mathrm{tor}})$. If $\alpha|_S = 0$ (equivalently, α is supported on the boundary) then $\alpha \in T_{\bullet \leq e}(S^{\mathrm{tor}})$ where $e = \dim(S^{\mathrm{min}} \setminus S)$ is the dimension of the boundary of the minimal compactification.

This follows from (3.4) since the minimal boundary $S^{\mathrm{min}} \setminus S$ is stratified by smaller Shimura varieties S_J for $J \subset \Delta$. The largest of these occur when J is of the form $\Delta \setminus \{\alpha\}$. In this case $T^\bullet(S_J^{\mathrm{tor}}) \cong A^\bullet(L_J/L_J \cap P)$ and $\dim(L_J/L_J \cap P) = \dim(S_J^{\mathrm{tor}}) \leq \dim(S^{\mathrm{min}} \setminus S)$.

Remark 3.6. On **underlying topological spaces** the morphism $\zeta^{\mathrm{tor}} \colon S_K^{\mathrm{tor}} \to G\text{-}\mathtt{Zip}^\mu$ factors through $\pi \colon S_K^{\mathrm{tor}} \to S_K^{\mathrm{min}}$ [GK19, 6.2.3] i.e. set-theoretically. If this factored as a morphism of stacks then Conjecture 3.5 would immediately follow. However, this seems unlikely as it would imply that all automorphic vector bundles extend to the minimal compactification as locally free sheaves. This definitely does not occur in characteristic 0 (e.g. the Hodge vector bundle on the Siegel modular variety doesn't extend to the minimal compactification). There should be an obstruction to the Hodge vector bundle extending to the minimal compactification in characteristic $p > 0$, which doesn't vanish at some level K, but this remains to be worked out in detail.

3.2 Strata effectivity

Another approach to our guiding question involves investigating certain properties within the tautological ring. For example, one could hope to generalise van der Geer's result that the chern classes of the Hodge vector bundle are effective from the Siegel case to other Shimura varieties of Hodge type. In Paper III we tackle this and make a conjecture generalising van der Geer's result in several different directions. In order to state this we introduce the notion of strata effectivity.

Definition 3.7. Let X be a stack admitting a stratification $\bigsqcup_{w\in I} X_w$ indexed by a finite set I. A class $\alpha \in A^\bullet(X)$ is strata-effective if $\alpha = \sum_{w\in I} a_w[\overline{X}_w]$ for some $a_w \geq 0$.

Recall that there is a stratification $G\text{-}\mathtt{Zip}^\mu = \sqcup Z_w$ by locally closed substacks. Let $\zeta\colon X \to G\text{-}\mathtt{Zip}^\mu$ be a morphism satisfying certain regularity properties (see Paper III). The locally closed subschemes $X_w \coloneqq \zeta^{-1}(Z_w)$ form a stratification of X. We consider generalisations of the Hodge bundle called Griffiths-Hodge bundles which take as input data $((G,\mu),r)$ where (G,μ) is a cocharacter datum, $r\colon G_k \to \mathsf{GL}(V)$ is a k-representation and give a vector bundle $\mathrm{Grif}(G\text{-}\mathtt{Zip}^\mu, r)$ on $G\text{-}\mathtt{Zip}^\mu$. In Paper III we conjecture the following.

Conjecture 3.8. The chern classes of $\mathrm{Grif}(G\text{-}\mathtt{Zip}^\mu, r)$ are strata-effective.

We prove several instances of this conjecture, including the case where (G,μ) is associated with the Hilbert modular variety and $\mathrm{Grif}(G\text{-}\mathtt{Zip}^\mu, r)$ are pieces of the Hodge bundle. A connection is developed between strata-effectivity in the tautological ring and the Cone Conjectures of Goldring-Koskivirta [GK18], which compare global sections of automorphic vector bundles on $G\text{-}\mathtt{Zip}^\mu$ and X. In particular, this provides an example of a tautological class which is effective but not strata-effective (on the Hilbert modular threefold at an inert prime).

3.2.1 The flag space

Given a cocharacter datum (G,μ) one also has a flag bundle $\pi\colon G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$ introduced in [GK19]. Given a morphism $X \to G\text{-}\mathtt{Zip}^\mu$, this defines a flag space Y/X by pullback

$$\begin{array}{ccc} Y & \longrightarrow & G\text{-}\mathtt{ZipFlag}^\mu \\ \downarrow & \square & \downarrow \\ X & \longrightarrow & G\text{-}\mathtt{Zip}^\mu \end{array}$$

The stack $G\text{-}\mathtt{ZipFlag}^\mu$ also admits a stratification, indexed by W, which pulls back to a stratification $Y = \sqcup_{w\in W} Y_w$ of the flag space. This gives a notion of strata-effectivity for classes on Y. There are two reasons why one might be interested in studying strata-effectivity on Y. Firstly, the space Y is closely

related to X so one could hope to deduce results about X. For example, if $\alpha \in A^\bullet(Y)$ is strata-effective then $\pi_*(\alpha)$ is strata-effective. Secondly, in many ways $G\text{-}\mathtt{ZipFlag}^\mu$ is better understood than $G\text{-}\mathtt{Zip}^\mu$. A concrete example is that every automorphic vector bundle on $G\text{-}\mathtt{Zip}^\mu$ is the pushforward of some 'automorphic' line bundle on $G\text{-}\mathtt{ZipFlag}^\mu$. This is used in [GK18] to study sections of automorphic vector bundles. Indeed in Paper III we show that, given (G, μ) a cocharacter datum, all tautological effective divisors for flag spaces are strata effective if and only if the partial Hasse cone conjecture holds for all spaces $X \to G\text{-}\mathtt{Zip}^\mu$ (satisfying certain mild conditions).

3.2.2 Generically ordinary curves

The previous result pertained to classes in codimension 1. We also study classes on Y in dimension 1: in several cases it is shown that all effective tautological curves which are generically ordinary (i.e. have open dense intersection with the maximal stratum) are strata-effective. A sufficient condition, phrased solely in terms of the group-theoretic data, is presented.

3.3 K-theory

As previously mentioned, one central aspect to answering Question 1.1 lies in understanding the geometry of $G\text{-}\mathtt{Zip}^\mu$. In the case of the Chow ring with rational coefficients this was done by Brokemper [Bro18]. Moving beyond the rational Chow ring, one can try to understand what other invariants look like for $G\text{-}\mathtt{Zip}^\mu$. For instance, it is natural to consider the Grothendieck group K_0. In Paper IV we study $K_0(G\text{-}\mathtt{Zip}^\mu)$. Under the additional assumption that G^{der} is simply connected we compute the ring $K_0(G\text{-}\mathtt{Zip}^\mu)$. The proof strategy is based on Brokemper's computation [Bro18] in the Chow case but there are numerous difficulties that occur in equivariant K-theory which must be dealt with and also account for the additional assumption.

Summaries of included papers

Paper I

The tautological subring of the Chow ring is computed in the case of the Hilbert modular variety. The method develops and generalises van der Geer's method from the Siegel modular variety (which was the only known tautological ring of a non-proper Shimura variety) to the Hilbert modular variety. This makes key use of the geometric fact that closures of non-maximal Ekedahl-Oort strata in the Hilbert modular variety are proper.

There is an arXiv version available at arXiv 2105.12642.

Paper II

The result of Paper II concerns computing the pushforward map in the Chow ring between Siegel flag varieties. A conjecture is presented relating such pushforward classes with classes in the tautological ring which are supported on the boundary. We show that this is consistent with the case of the Siegel modular variety and the Hilbert modular variety.

Paper III

In this paper we present several conjectures regarding positivity in the tautological ring. In the presence of a stratification we introduce the notion of a cycle class being strata-effective. A strata-effective class is effective. We conjecture that the chern classes of Griffiths-Hodge bundles (generalisations of the Hodge vector bundle) are strata-effective and prove this in several cases, including Hilbert modular varieties and Shimura varieties with a linear stratification. We also investigate which effective classes in the tautological ring are strata-effective. Using a connection with the Cone Conjectures of Goldring-Koskivirta, we prove the existence of effective tautological classes which are not strata-effective.

Equal contribution from both authors.

Paper IV

The introduction of the stack of G-Zips of type μ has enriched the study of moduli spaces in characteristic $p > 0$. In this paper we compute its K_0-ring $K_0(G\text{-}\mathtt{Zip}^{\mu})$ under the additional assumption that the derived group of G is simply connected. This is an invariant capturing information about vector bundles over the space and is closely related to the Chow ring, which was previously computed for the stack of G-Zips of type μ by Brokemper. Our computation uses Thomason's equivariant algebraic K-theory.

Bibliography

[Ash+10] A. Ash et al. *Smooth compactifications of locally symmetric varieties*. Second. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010, pp. x+230. ISBN: 978-0-521-73955-9. DOI: 10.1017/CBO9780511674693. With the collaboration of Peter Scholze.

[Bor53] A. Borel. "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts". *Ann. of Math. (2)* 57 (1953), pp. 115–207. DOI: 10.2307/1969728.

[Bro18] D. Brokemper. "On the Chow ring of the stack of truncated Barsotti-Tate groups". *Pacific J. Math.* 296.2 (2018), pp. 271–303. DOI: 10.2140/pjm.2018.296.271.

[COP24] S. Canning, D. Oprea, and R. Pandharipande. *Cycles on the moduli space of principally polarized abelian varieties*. 2024. In preparation.

[Can+24] S. Canning et al. *Tautological projection for cycles on the moduli space of abelian varieties*. 2024. arXiv: 2401.15768 [math.AG].

[Del79] P. Deligne. "Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques". In: *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*. Vol. XXXIII. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1979, pp. 247–289. ISBN: 0-8218-1437-0.

[EG98] D. Edidin and W. Graham. "Equivariant intersection theory". *Invent. Math.* 131.3 (1998), pp. 595–634. DOI: 10.1007/s002220050214.

[EG00] D. Edidin and W. Graham. "Riemann-Roch for equivariant Chow groups". *Duke Math. J.* 102.3 (2000), pp. 567–594. DOI: 10.1215/S0012-7094-00-10239-6.

[EH17] H. Esnault and M. Harris. "Chern classes of automorphic vector bundles". *Pure Appl. Math. Q.* 13.2 (2017), pp. 193–213. DOI: 10.4310/PAMQ.2017.v13.n2.a1.

[EV02] H. Esnault and E. Viehweg. "Chern classes of Gauss-Manin bundles of weight 1 vanish". *K-Theory* 26.3 (2002), pp. 287–305. DOI: 10.1023/A:1020619014609.

[Gee99] G. van der Geer. "Cycles on the moduli space of abelian varieties". In: *Moduli of curves and abelian varieties*. Friedr. Vieweg, Braunschweig, 1999, pp. 65–89. DOI: 10.1007/978-3-322-90172-9_4.

[GK18] W. Goldring and J.-S. Koskivirta. "Automorphic vector bundles with global sections on G-$\mathtt{Zip}^{\mathcal{Z}}$-schemes". *Compos. Math.* 154.12 (2018), pp. 2586–2605. DOI: 10.1112/S0010437X18007467.

[GK19] W. Goldring and J.-S. Koskivirta. "Strata Hasse invariants, Hecke algebras and Galois representations". *Invent. Math.* 217.3 (2019), pp. 887–984. DOI: 10.1007/s00222-019-00882-5.

[HH12] H. He and J. W. Hoffman. "Picard groups of Siegel modular 3-folds and θ-liftings". *J. Lie Theory* 22.3 (2012), pp. 769–801.

[KS03] S. Keel and L. Sadun. "Oort's conjecture for $A_g \otimes \mathbb{C}$". *J. Amer. Math. Soc.* 16.4 (2003), pp. 887–900. DOI: 10.1090/S0894-0347-03-00431-4.

[Kis10] M. Kisin. "Integral models for Shimura varieties of abelian type". *J. Amer. Math. Soc.* 23.4 (2010). DOI: 10.1090/S0894-0347-10-00667-3.

[Lan12] K.-W. Lan. "Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties". *J. Reine Angew. Math.* 664 (2012), pp. 163–228. DOI: 10.1515/crelle.2011.099.

[Mad19] K. Madapusi Pera. "Toroidal compactifications of integral models of Shimura varieties of Hodge type". *Ann. Sci. Éc. Norm. Supér. (4)* 52.2 (2019). DOI: 10.24033/asens.2391.

[Mil90] J. S. Milne. "Canonical models of (mixed) Shimura varieties and automorphic vector bundles". In: *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*. Vol. 10. Perspect. Math. Academic Press, Boston, MA, 1990, pp. 283–414. ISBN: 0-12-176651-9.

[MW04] B. Moonen and T. Wedhorn. "Discrete invariants of varieties in positive characteristic". *Int. Math. Res. Not.* 72 (2004), pp. 3855–3903. DOI: 10.1155/S1073792804141263.

[Mum77] D. Mumford. "Hirzebruch's proportionality theorem in the noncompact case". *Invent. Math.* 42 (1977), pp. 239–272. DOI: 10.1007/BF01389790.

[PWZ15] R. Pink, T. Wedhorn, and P. Ziegler. "F-zips with additional structure". *Pacific J. Math.* 274.1 (2015), pp. 183–236. DOI: 10.2140/pjm.2015.274.183.

[Vas99] A. Vasiu. "Integral canonical models of Shimura varieties of preabelian type". *Asian J. Math.* 3.2 (1999), pp. 401–518. DOI: 10.4310/AJM.1999.v3.n2.a8.

[WZ23] T. Wedhorn and P. Ziegler. "Tautological rings of Shimura varieties and cycle classes of Ekedahl-Oort strata". *Algebra Number Theory* 17.4 (2023), pp. 923–980. DOI: 10.2140/ant.2023.17.923.

[Wei92] R. Weissauer. "The Picard group of Siegel modular threefolds". *J. Reine Angew. Math.* 430 (1992), pp. 179–211. DOI: 10.1515/crll.1992.430.179. With an erratum: "Differential forms attached to subgroups of the Siegel modular group of degree two" [J. Reine Angew. Math. **391** (1988), 100–156; MR0961166 (89i:32074)] by the author.

[Zha18] C. Zhang. "Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type". *Canad. J. Math.* 70.2 (2018), pp. 451–480. DOI: 10.4153/CJM-2017-020-5.