Stockholm University

Geometry of multi-slit Loewner chains and semigroups of finite shift

Eleftherios Theodosiadis

Doctoral Thesis in Mathematics at Stockholm University, Sweden 2024

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Abstract

This thesis deals with two topics in complex analysis that are related to families of Riemann maps that depend on some parameter function called the driving function of the family. One is Loewner theory and the other is the theory of semigroups of holomorphic self-maps of the unit disc. This thesis consists of four works, three of which lie in the intersection of the two theories and the other one refers solely on semigroups.

In Paper I, we deal with the Loewner equation both in the unit disc (the radial case) and in the upper half-plane (the chordal case). The solutions to these equations, which depend on the space and time variables, are called (radial or chordal) Loewner chains. Its main purpose is to present explicitly solutions to certain choices of driving functions and additionally visualize their geometry as time evolves. In particular, we deal with conformal maps with finitely many slits, for both cases. Thus, the aforementioned evolution involves the growth of multiple curves either in the unit disc or in the upper half-plane. Secondly, we discover the semigroup nature of these families, which we utilize in order to connect the radial with the chordal case through a Möbius transform, although in the general theory this is not always possible.

The second paper is a continuation of Paper I, where we extend the study of the chordal Loewner chains of Paper I to chains with infinitely many slits. Again, we study the geometry of the chains as time evolves and we find the same geometric behaviour as in Paper I. However, this study is more complicated and requires a different approach that involves techniques from classical complex analysis and the use of the harmonic measure.

In Paper III we are concentrated in a specific type of semigroups. We call those semigroups of finite shift. In the general theory of semigroups, several authors have studied the rate of convergence of a semigroup to the Denjoy-Wolff point, in terms of the Euclidean distance. In this direction, we also examine the rate of convergence for this case, in terms of the Euclidean distance, the hyperbolic distance and also in terms of the harmonic measure.

In Paper IV, we present some computational examples of Loewner chains. Some of them are related to those appearing in Papers I and II. We work similarly in the sense that we solve the Loewner equation for some certain driving functions. In addition, we have collected some Loewner chains that do not appear in the literature and we recover their driving functions. Our intention is to visualize these elementary examples in an effort to compare the geometry of the chains with their driving functions.

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In memory of
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Sammanfattning

Denna avhandling behandlar två ämnen i komplex analys som är relaterade till familjer av Riemannavbildningar som beror av en parameterfunktion, Loewnerteori samt teorin för halvgrupper av holomorfa automorfier. Avhandlingen består förutom en inledning av fyra arbeten, varav tre rör sig i gränslandet mellan dessa två ämnen och ett uteslutande handlar om halvgrupper.

I Artikel I betraktar vi Loewnerekvationen i enhetsskivan (det radiella fallet) och i övre halvplanet (det kordala fallet). Lösningar till dessa ekvationer beror av en komplex rumsvariabel och en reell tidsvariabel, och kallas Loewnerkedjor. Huvudsyftet med Artikel I är att härleda explicita lösningar för vissa speciella val av parameterfunktioner samt att beskriva hur lösningarnas geometri varierar med tidsparametern. Mer specifikt behandlar vi konforma avbildningar på komplementet till ändligt många slitsar, både i skivan och i halvplanet. Vi observerar en typ av halvgruppsstruktur hos dessa konforma avbildningar och utnyttjar denna för att uppvisa nära kopplingar mellan de radiella och kordala situationerna.

Artikel II utgör en fortsättning och generalisering av Artikel I där kordala Loewnerkedjor förknippade med oändligt många slitsar betraktas. Vi studerar återigen geometrin hos lösningar till Loewnerekvationen och erhåller liknande beskrivningar som i fallet med ändligt många slitsar. För att bevisa dessa resultat måste mer sofistikerade metoder användas, såsom uppskattningar av harmoniskt mått.

Artikel III handlar om halvgrupper av en särskild typ, så kallade halvgrupper med ändlig förskjutning. Inom den generella teorin för halvgrupper i enhetsskivan har hastigheten vid konvergens mot den entydigt bestämda Denjoy-Wolff-punkten studerats tidigare, uttryckt med hjälp av euklidiskt avstånd. I Artikel III bevisas satser om konvergenshastighet för ovanstående typ av halvgrupper med avseende på både euklidiskt och hyperboliskt avstånd, samt uttryckt med hjälp av harmoniskt mått.

I Artikel IV ges och beskrivs explicita exempel på Loewnerkedjor. Vissa av dessa exempel har nära kopplingar till Artiklar I och II i den mening att de respektive parameterfunktionerna uppvisar strukturella likheter. Vi-

dare bestäms parameterfunktioner på sluten form för vissa Loewnerkedjor som inte återfinns i litteraturen och jämförelser görs mellan analytiska egenskaper hos parameterfunktionerna och geometriska egenskaper hos de konforma avbildningarnas bildmängder.

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this PhD thesis.

Paper I: **Explicit multi-slit Loewner flows and their geometry**
To appear in Computational Methods and Function Theory
arXiv:2210.11396
Eleftherios Theodosiadis

Paper II: **Geometric description of some Loewner chains with infinitely many slits**
Submitted
arXiv:2309.12517
Eleftherios Theodosiadis, Konstantinos Zarvalis

Paper III: **Rates of convergence for holomorphic semigroups of finite shift**
Submitted
arXiv:2403.06883
Maria Kourou, Eleftherios Theodosiadis, Konstantinos Zarvalis

Paper IV: **Miscellaneous examples of Loewner chains**
Preprint
Eleftherios Theodosiadis

Author's contribution. Paper II is a collaborative work between E. K. Theodosiadis and K. Zarvalis and both authors have equally contributed in writing it. Paper III has been a collaborative work of M. Kourou, E. K. Theodosiadis and K. Zarvalis and all authors have equally contributed in writing it.

Contents

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1. Introduction

As the title suggests, this thesis deals with *Loewner's theory*, which describes the evolution of a time-dependent family of analytic and one-to-one functions in a reference domain. Along the introduction, we intend to make the topic accessible to the non-experts and preferably to any interested graduate student.

We therefore make a short introduction to the topic by presenting the material, that is necessary to build the general theory from the bottom up. In other words, this introduction can be considered as a 'what I need to know' material to enter Loewner's theory. Each proposition and theorem is presented without a proof, however, the interested reader can refer to sources as [[4], [7], [9], [19], [20]].

In what follows, we assume that the reader is familiar with elementary complex analysis, such as analyticity, Taylor's expansion and residue calculus, with some basic definitions from measure theory and the theory of Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs).

Note that a list of references is given in the end of the introduction.

Remark. The current thesis builds partly upon the author's Licentiate thesis (defended on November 28, 2022). The Licentiate thesis contained a preprint version of Paper I and an earlier version of Paper IV of this thesis. Also, the general introduction is largely based on the introduction of the Licentiate thesis, though it has been rewritten and more background material related to Paper II and Paper III, has been added.

1.1 Univalent functions in the unit disc

A domain (open and connected) D in the extended complex plane $\hat{\mathbb{C}}$ is called *simply connected* if its complement $\hat{\mathbb{C}}\backslash D$ is connected. We denote by $H(D)$ the set of all complex differentiable (or holomorphic or analytic) functions of D. In $\mathbb{C}$ we have several equivalent definitions of simple connectivity.

Theorem 1. *If D is a domain in $\mathbb{C}$, then the following are equivalent:*

(1) D is simply connected,

(2) for every $f \in H(D)$ nowhere zero, there exists an analytic branch of the logarithm $\log f(z)$ in D,

(3) for every $f \in H(D)$, there exist an antiderivative in D.

We name a *simple curve*, or a *Jordan curve*, a homeomorphic image of the unit circle, in the plane. According to the Jordan curve theorem a simple curve splits $\mathbb{C}$ into two connected components, the interior of the curve and its exterior respectively. The interior of a Jordan curve will be called *Jordan domain* and obviously it is simply connected. Additionally, a *Jordan arc* is a one-to-one correspondence of a circular arc.

According to the Riemann Mapping Theorem, every simply connected, proper subset of $\mathbb{C}$, is conformally equivalent to the unit disk $\mathbb{D}$. It was first formulated by Riemann in 1851, yet it was not fully proved until half a century later with the use of normal families.

Definition 2. *A family $\mathcal{F} = \mathcal{F}(D)$, of analytic functions defined on a domain D, is called normal if for every sequence $(f_n)_{n\geqslant 1} \subset \mathcal{F}$, there exists a locally uniformly convergent subsequence in D.*

Also, a family $\mathcal{F}$ is called locally uniformly bounded, if for every compact $K \subset D$ there exists $M > 0$, so that for every $z \in K$, $|f(z)| \leqslant M$ for all $f \in \mathcal{F}$ and it is called closed if for all $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $f_n \to f$ locally uniformly in D, f belongs to the family $\mathcal{F}$. A closed and normal family of analytic functions, is called compact. One can show that $\mathbf{S}$, the class of analytic and one-to-one functions of the unit disc with $f(0) = 0$ and $f'(0) = 1$, is a compact family.

Montel Theorem. *A family $\mathcal{F} = \mathcal{F}(D)$ of analytic functions in D is normal if and only if it is locally uniformly bounded.*

Vitali Theorem. *Let D be a domain and $(f_n)_{n\geqslant 1}$ be a locally uniformly bounded sequence of $H(D)$. If $(f_n(z))_{n\geqslant 1}$ converges for all $z \in E \subset D$, where E is a set with accumulation points in D, then $(f_n)_{n\geqslant 1}$ converges locally uniformly on D.*

In a contemporary proof of the Riemann mapping theorem (see for example [25]), the Montel Theorem can be applied. Although historically a conformal map is defined as an angle-preserving analytic function, it is common in the literature, nowadays, to identify a conformal map with a *univalent* function, that is an analytic and one-to-one mapping.

Riemann Mapping Theorem. *If $D \subsetneq \mathbb{C}$ is a simply connected domain of $\mathbb{C}$, then there exists a conformal map of $\mathbb{D}$ onto D.*

Moreover, fixing some point $a \in D$ and an angle θ, this map is unique if we require $f(0) = a$ and $\mathrm{Arg} f'(0) = \theta$.

Using the Schwarz lemma, the Riemann map is unique if we demand $f(z_0) = 0$ and $\arg f'(z_0) = \theta_0$, for some fixed $z_0 \in D$ and $\theta_0 \in \mathbb{R}$. Caratheodory proved that the Riemann map can be extended to an homeomorphism of the closure of the unit disk onto the closure of D, whenever the boundary of D is a Jordan curve. See [6], [9] or [19].

Caratheodory's Extension Theorem. *Consider D to be a domain that is enclosed by a Jordan curve C and let f be the Riemann map of $\mathbb{D}$ onto D. Then, f can be extended in an homeomorphism of $\mathbb{D} \cup \partial\mathbb{D}$ onto $D \cup C$.*

In the case where the boundary of D is locally connected, we have that boundary values $f(e^{i\theta}) = \lim_{z \to e^{i\theta}} f(z)$ of the Riemann map, is a continuous function.

If the boundary of D is a continuous path, but it is not a Jordan curve, then the Riemann map is extended continuously, but not injectively. By a continuous path here, we mean a locally connected boundary. We give the definition of local connectedness.

Definition 3. *A compact metric space X is called locally connected if for every $\epsilon > 0$, there exists some $\delta > 0$, such that for all $x, y \in X$ with $d_X(x, y) < \delta$, there is a connected subset A of X containing x,y and satisfying diam$(A) < \epsilon$.*

The boundary behaviour of conformal maps has proven to be a complicated topic in complex analysis. An analytic map with 'nice' behaviour in the interior of a domain can present chaotic behaviour on the boundary. However, in order to visualize a Riemann map, thus find its image, one has to work on the boundary. Most of the work of this text is exactly to obtain a geometric image for the maps under discussion. Especially in the last paper, we try to compare the boundaries of a family of domains, to certain measures that appear in Loewner's theory. For this reason, we consider some basic results from [20], which provides a thorough discussion on the boundary behaviour of conformal maps.

Definition 4. *Let D be a simply connected domain with locally connected boundary and let $f : \mathbb{D} \to D$ be a Riemann map of D. We say that ∂D has a corner of opening $\pi\alpha$, at $f(e^{i\theta_0}) \neq \infty$, if*

$$\arg(f(e^{i\theta}) - f(e^{i\theta_0})) \xrightarrow{\theta \to \theta_0^+} \beta \quad \text{and} \quad arg(f(e^{i\theta}) - f(e^{i\theta_0})) \xrightarrow{\theta \to \theta_0^-} \beta + \pi\alpha.$$

Theorem 5. *Let D be a simply connected domain with locally connected boundary and let $f : \mathbb{D} \to D$ be a Riemann map of D. Consider some $e^{i\theta} \in \partial\mathbb{D}$, so that $f(e^{i\theta}) \neq \infty$. Then, ∂D has a corner of opening $\pi\alpha$, if and only if*

$$\arg\left(\frac{f(z) - f(e^{i\theta})}{(z - e^{i\theta})^{\alpha}}\right) \to \beta - \alpha(\theta + \frac{\pi}{2})$$

as $z \to e^{i\theta}$.

As we discussed, not every conformal map can be extended continuously to the boundary of the unit disc unless the boundary of its image is locally connected. However, every conformal map $f : \mathbb{D} \to \mathbb{C}$ can be extended to almost every point of the unit circle, in the sense of non-tangential limits. Let $\sigma \in \partial\mathbb{D}$. A *Stolz* region with vertex σ and amplitude $R > 1$ is defined as

$$S(\sigma, R) := \{z \in \mathbb{D} : |z - \sigma| < R(1 - |z|)\}.$$

We say that the *non-tangential* limit of f exists at σ, if there exists some $L \in \hat{\mathbb{C}}$ such that

$$\lim_{S(\sigma,R) \ni z \to \sigma} f(z) = L.$$

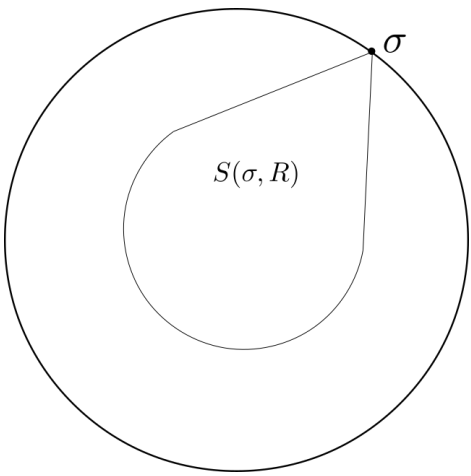

Figure 1.1: A Stoltz angle with vertex σ.

The following theorem states that the non-tangential limit exists, once we know that the limit of f through some continuous curve with one endpoint in the unit circle, exists.

Theorem 6. *Let $f : \mathbb{D} \to \mathbb{C}$ be univalent and let $\gamma : [a, b) \to \mathbb{D}$ be a continuous curve, so that $\lim_{t\to b} \gamma(t) = \sigma \in \partial\mathbb{D}$. If $\lim_{t\to b} f(\gamma(t)) = L \in \hat{\mathbb{C}}$, then $\lim_{z\to\sigma} f(z) = L$, as $z \to \sigma$, non-tangentially.*

According to the next result, the boundary values of a univalent function exist, in the sense of the non-tangential limits, almost everywhere. Given some point $\zeta \in \partial\mathbb{D}$ so that the non-tangential limit exists at ζ and equals L, then we may define $f(\zeta)$ as L.

Theorem 7. *Let $f : \mathbb{D} \to \mathbb{C}$ be univalent. Then, $f(\zeta)$ exists for almost every point $\zeta \in \partial\mathbb{D}$. Moreover, there exists no $L \in \hat{\mathbb{C}}$ and $A \subset \partial\mathbb{D}$ with positive Lebesgue measure, so that $f(\zeta) = L$, for all $\zeta \in A$.*

We conclude this section with a short paragraph on the connection of the Riemann map of a simply connected domain D with the *Green's function* of the domain. Of course, the Green's function can be defined in a wider class of domains. See [23] for an abstract approach. Given a simply connected domain D, the Green's function of D with pole at $w \in D$, denoted by $G_D(z, w)$, is the unique map with the following properties:

1. $G_D(z, w)$ is harmonic on $D\backslash\{w\}$, and bounded outside each neighborhood of w,

2. $G_D(z, w) \to 0$, as $z \to \zeta$ for all $\zeta \in \partial D$,

3. $G_D(w, w) = \infty$ and

$$G_D(z, w) = \begin{cases} -\log|z - w| + O(1), & w \neq \infty \\ \log|z| + O(1), & w = \infty \end{cases}$$

as $z \to w$.

It is also useful to view the Green's function as $G_D(z, w) = h(z, w) - \log|z - w|$ where $h(\cdot, w)$ is the solution to the Dirichlet problem in D, for the boundary value $\zeta \mapsto \log|\zeta - w| \in C(\partial D)$, when D is a Jordan domain (see [9]). We denote by $\psi_w(z)$ the analytic completion of $G_D(z, w) + \log|z - w|$ such that $\psi_w(w) = 0$ (this means that it is unique) and let

$$g(z) = (z - w)e^{-\psi_w(z)}$$

for all $z \in D$. Then, g is analytic, one-to-one and maps D onto the unit disc.

1.2 Sequences of simply connected domains

Now, let us turn to the convergence of a sequence of Riemann maps. Clearly, we are familiar with the notion of convergence of a sequence of functions $(f_n)_{n\geqslant 1}$ defined on some set G. Basically, by convergence we mean the locally uniform convergence, i.e the uniform convergence on compact subsets of G. We will now extend this notion for a sequence such that each f_n is defined on G_n, meaning that we also have a sequence of regions $(G_n)_{n\geqslant 1}$.

It is obvious that we need to pose some restrictions on the behaviour of $(G_n)_{n\geqslant 1}$ (see [6], [7] and [19] for details). For the next, we assume that each G_n is a domain that contains zero and we denote by $D(z,r)$ the disc of center $z\in\mathbb{C}$ and radius $r>0$. We write the set

$$A=\{z\in\mathbb{C}:\ \exists r>0,\ n_0\in\mathbb{N}:\overline{D(z,r)}\subset G_n,\ \forall n\geqslant n_0\}.$$

Definition 8. *If $0\in A$, we define as the kernel of $(G_n)_{n\geqslant 1}$ the maximal open and connected subset of A that contains 0. In this case we say that the kernel of $(G_n)_{n\geqslant 1}$ exists and it by $G=\ker(G_n)$. We simply write $G=\{0\}$ when $0\notin A$, thus, the kernel does not exist.*

Definition 9. *Let $(G_n)_{n\geqslant 1}$ be a sequence of domains that satisfies the previous definition. We say that $(G_n)_{n\geqslant 1}$ converges (in the kernel sense or in the Caratheodory sense) to G and we denote by $G_n\to G$, if G is the kernel of any subsequence $(G_{k_n})_{n\geqslant 1}$.*

Note in the above definition that the kernel must be the same for every subsequence. Even if the kernel exists for the total sequence, it may not coincide with the kernel of any subsequence. Assume for example $(G_n)_{n\geqslant 1}$, with $G_{2n}=\mathbb{D}$ and $G_{2n+1}=D(0,2)$. Then, $\ker(G_n)=\ker(G_{2n})=\mathbb{D}$ and $\ker(G_{2n+1})=D(0,2)$, so $(G_n)_{n\geqslant 1}$ is not convergent.

If $G=\ker(G_n)$, then G is the largest domain containing zero, with the following property:

$$\text{for all compact}\quad K\subset G,\ \ \exists n_0\in\mathbb{N}:\ K\subset G_n,\ \forall n\geqslant n_0.$$

The notion of kernel is necessary to define convergence of a sequence of functions of the form $f_n:G_n\to D_n$.

Definition 10. *Let $(G_n)_{n\geqslant 1}$ be a sequence of domains that contain zero and let also $f_n:G_n\to\mathbb{C}$, $f:G\to\mathbb{C}$, where $G=\ker(G_n)\neq\{0\}$. Then, we say that $(f_n)_{n\geqslant 1}$ converges to f, uniformly on compacta of G, if for every $K\subset G$ and for every $\epsilon>0$, $\exists n_0:\ \forall n\geqslant n_0$,*

$$|f_n(z)-f(z)|<\epsilon,\ \forall z\in K.$$

The following theorem is due to to Caratheodory (see [7, paragraph 3.1]) and connects the locally uniform convergence of a sequence of Riemann maps with the kernel convergence of their images. This is a remarkable theorem that gives a purely geometric interpretation on convergence of sequences of Riemann maps.

Caratheodory's Convergence Theorem. *Let $(D_n)_{n\geqslant 1}$ be a sequence of simply connected domains, with $0 \in D_n \neq \mathbb{C}$ and let $f_n : \mathbb{D} \to D_n$ the unique Riemann maps, with $f_n(0) = 0$ and $f_n'(0) > 0$. Then, $f_n \xrightarrow{u.c} f$ in $\mathbb{D}$ if and only if $D_n \to D$ in the kernel sense.*

Moreover, $D = f(\mathbb{D})$ is simply connected and $f_n^{-1} \to f^{-1}$ uniformly on compacts.

Caratheodory's theorem, often referred to as the Kernel Convergence Theorem, plays important role in our theory because it allows to define a notion of continuity in families of simply connected domains and Loewner's theory deals exactly with such families.

Definition 11. *A family of simply connected domains $(D_t)_{t\geqslant 0}$, such that $0 \in D_t$, for all $t \geqslant 0$, is said to be continuous in the Caratheodory sense, if for every $t_0 \geqslant 0$,*

$$D_{t_n} \to D_{t_0}, \quad n \to +\infty$$

for any sequence $(t_n)_{n\geqslant 1}$ with $t_n \to t_0$. Equivalently, $(D_t)_{t\geqslant 0}$ is continuous if and only if for every $t_0 \geqslant 0$, we have that $f_t \xrightarrow{u.c} f$ in $\mathbb{D}$, as $t \to t_0$, where $f_t : \mathbb{D} \to D_t$, with $f_t(0) = 0$ and $f_t'(0) > 0$.

1.3 Harmonic measure

An important conformal invariant quantity that has a vast field of applications is the harmonic measure, which we will encounter during the course of this thesis. In paper II we make use of it in order to prove some results and in Paper III we study some convergence properties of some families of conformal maps in terms of the harmonic measure.

We shall only present the basic properties we need for this thesis, starting with the unit disc and extending to simply connected domains. Let E be an open arc of the unit circle and let χ_E be the characteristic function of E in $\partial\mathbb{D}$. By definition, the harmonic measure of E is the solution to the generalized *Dirichlet problem* for boundary values χ_E, on the unit disc. Thus it is the function $\omega(z, E, \mathbb{D})$ that satisfies the following conditions:

1. $\omega(\cdot, E, \mathbb{D})$ is harmonic in $\mathbb{D}$,
2. $0 < \omega(z, E, \mathbb{D}) < 1$, for all $z \in \mathbb{D}$,

3. $\omega(z, E, \mathbb{D}) \to 0$, as $z \to E$,

4. $\omega(z, E, \mathbb{D}) \to 1$, as $z \to \partial\mathbb{D}\backslash\overline{E}$.

The preceding function exists and it is unique due to the next result, which is often described as *Lindelöf's maximum principle.*

Theorem 12. *Suppose that u is harmonic and bounded in a domain D, such that $\overline{D} \neq \mathbb{C}$. Let F be a finite subset of ∂D and assume that*

$$\limsup_{z\to\zeta} u(\zeta) \leqslant 0$$

for all $\zeta \in \partial D\backslash F$. Then, $u \leqslant 0$ in D.

It is proved that if E is an open arc with endpoints e^{ia} and e^{ib}, with $0 \leqslant a < b \leqslant 2\pi$, then the harmonic measure is explicitly given as

$$\omega(z, E, \mathbb{D}) = \frac{1}{2\pi}\int_a^b \frac{1-|z|^2}{|z-e^{i\theta}|^2}d\theta = \mathrm{Re}\frac{1}{2\pi}\int_a^b \frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta.$$

We extend this formula to any Borel subset of the unit circle to define the harmonic measure generally.

Definition 13. *Let $E \subset \partial\mathbb{D}$ be a Borel set. Then, the harmonic measure of E at $z \in \mathbb{D}$ is defined as*

$$\omega(z, E, \mathbb{D}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2}\chi_E(e^{i\theta})d\theta.$$

It is immediate by the definition that for each $z \in \mathbb{D}$, the map $E \mapsto \omega(z, E, \mathbb{D})$ defines a probability measure of the unit circle.

We, now, extend the definition to any simply connected domain. Let $D \neq \mathbb{C}$ be simply connected and let $f : \mathbb{D} \to D$ be a Riemann map. Recall that the boundary values $f(\zeta)$ exist for almost every $\zeta \in \partial\mathbb{D}$, non-tangentially. Hence, given some $A \subset \partial D$, we naturally define its preimage as

$$f^{-1}(A) := \{\zeta \in \partial\mathbb{D} : f(\zeta) \in A\}$$

and it is a Borel subset of the unit circle.

Definition 14. *Let $D \neq \mathbb{C}$ be simply connected and let $A \subset \partial D$. Then, the harmonic measure of A at $z \in D$ is defined as*

$$\omega(z, A, D) = \omega(f^{-1}(z), f^{-1}(A), \mathbb{D})$$

for some Riemann map $f : \mathbb{D} \to D$.

It is easy to see that the harmonic measure is Möbius invariant. For this reason, we can directly see that the harmonic measure is independent of the choice of the Riemann map. Also, the simpler (geometrically) the domain is the easier the formlula for the harmonic function to be determined explicitly is, since this is done through the Riemann map of the domain.

Example 15. Consider as a reference domain the upper half-plane $\mathbb{H}$ and let $I = (a, b) \subset \mathbb{R}$, for some $a < b$. Then,

$$\omega(z, I, \mathbb{H}) = \frac{1}{\pi} \arg \frac{z-b}{z-a}$$

for all $z \in \mathbb{H}$. We directly see that if we fix a subset of the boundary then the harmonic measure depends on z.

Example 16. For the angular domain $U_{\alpha,\beta} := \{w \in \mathbb{C} : \alpha < \arg w < \beta\}$, where $0 \leqslant \alpha < \beta < 2\pi$ (or $-\pi \leqslant \alpha < \beta < \pi$), and for $z \in U_{\alpha,\beta}$, we have

$$\omega(z, \{w \in \mathbb{C} : \arg w = \alpha\}, U_{\alpha,\beta}) = \frac{\beta - \arg z}{\beta - \alpha}.$$

Finalizing this section, we note two very basic properties of the harmonic measure, those of monotonicity and conformal invariance.

Proposition 17. *The following statements are true:*

(1) The harmonic measure is conformally invariant in the following sense: if Ω is a Jordan domain $f : \Omega \to \tilde{\Omega}$ is one-to-one, B is a Borel subset of $\partial\tilde{\Omega}$ and

$$f^{-1}(B) := \{\zeta \in \partial\Omega : \text{there exists a continuous curve } \gamma \text{ in } \Omega \text{ ending at } \zeta \text{ such that } \exists \lim_{\gamma \ni z \to \zeta} f(z) \in B \ \}$$

then $f^{-1}(B)$ is a Borel set and $\omega(z, f^{-1}(B), \Omega) = \omega(f(z), B, \tilde{\Omega})$.

(2) Let $D_1, D_2 \neq \mathbb{C}$ be two simply connected domains with $D_1 \subset D_2$ and assume that $A \subset \partial D_1 \cap \partial D_2$ is a Borel set. Then, $\omega(z, A, D_1) \leqslant \omega(z, A, D_2)$ for all $z \in D_1$.

1.4 Classes of analytic functions

The following result is basic for the development of the Loewner-Kufarev theory, as it allows to represent an analytic function with positive real part as a Schwarz integral of a finite Borel measure.

Herglotz Representation Formula. *Let f be analytic in $\mathbb{D}$ with positive real part. Then, there exists a unique bounded, positive Borel measure μ of $\partial\mathbb{D}$, so that*

$$f(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) + i\gamma =: S[d\mu](z) + i\gamma$$

for all $|z| < 1$, where γ is a real constant.

As conformal maps are complicated in general, there are some important subclasses that are well studied, thus they are more convenient to work with. Spirallike functions for instance, form such a subclass. Most of the work in this text makes use of their properties. So, we present, next, a couple of results on spirallike functions, that are necessary for the main results of the thesis. A *logarithmic spiral of angle* $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ in the complex plane is defined as the curve with parameterization $S : w = w_0 \exp(-e^{-i\psi}t), -\infty \leqslant t \leqslant \infty$, for some complex number $w_0 \neq 0$.

Definition 18. *A simply connected domain D, that contains the origin, is said to be ψ-spirallike, if for any point $w_0 \in D$, the logarithmic spiral $S : w = w_0 exp(-e^{i\psi}t), 0 \leqslant t \leqslant \infty$ is contained in D.*

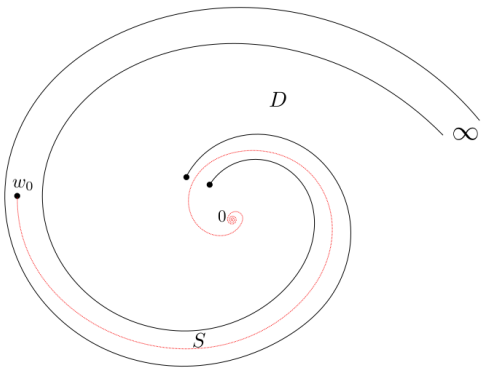

Figure 1.2: The complement of two disjoint logaritmic spirals of the same angle is a spirallike domain.

Definition 19. *A univalent function $f \in H(\mathbb{D})$, with $f(0) = 0$, is said to be ψ- spirallike if it maps the unit disc onto a ψ-spirallike domain D.*

Note that 0-spirals are straight lines emanating from the origin and expanding to infinity. We refer to 0-spirallike domains (resp. functions) as starlike domains (resp. functions). The following theorem gives an analytic characterization of spirallike mappings. A detailed proof is found in [7, paragraph 2.7].

Theorem 20. *Let $f \in H(\mathbb{D})$, with $f'(0) \neq 0$ and $f(z) = 0$ if and only if $z = 0$. Then f is ψ-spirallike, if and only if*

$$Re\left(e^{-i\psi}\frac{zf'(z)}{f(z)}\right) > 0$$

for all $z \in \mathbb{D}$.

For example, the function $f(z) = z(z-i)^{i-1}$ maps the unit disc onto the complement of a logarithmic spiral of angle $\pi/4$, that joins the point $f(1)$ with the point at infinity. Although spirallike functions usually refer to analytic functions of the disc, we can easily transfer the preceding definition on functions of the upper half plane.

Definition 21. *A univalent function $f \in H(\mathbb{D})$, with $f(\beta) = 0$ for some $\beta \in \mathbb{H}$, is said to be ψ-spirallike, if it maps the upper half plane onto a ψ-spirallike domain D.*

Applying the Möbius transform $Tz = \dfrac{z-\beta}{z-\overline{\beta}}$, which maps $\mathbb{H}$ onto the unit disc, using the characterization of spirallike functions of the unit disc, the following proposition follows immediately.

Proposition 22. *Let $f \in H(\mathbb{H})$, with $f'(\beta) \neq 0$ and $f(z) = 0$ if and only if $z = \beta$. Then, f is ψ-spirallike, if and only if*

$$Im\left(e^{-i\psi}\frac{(z-\beta)(z-\bar{\beta})f'(z)}{f(z)}\right) > 0$$

for all $z \in \mathbb{H}$.

Note that by definition, spirallike functions are univalent. However, in the preceding results, we do not assume f to be univalent, but univalence follows from the fact that the real or the imaginary part of the quotients above have constant sign. Thus, these results serve both as conditions for a function to be one-to-one and in addition they describe their geometry.

Another class of analytic functions that appears in the literature and particularly in the theory of semigroups of holomorphic functions in the unit disc, are those that map $\mathbb{D}$ onto some Ω that is *starlike at infinity*.

Definition 23. *A simply connected domain $\Omega \subset \mathbb{C}$ is called starlike at infinity if $\Omega + t \subset \Omega$, for all $t \geqslant 0$.*

A map $h \in H(\mathbb{D})$ is called starlike at infinity with respect to some $\tau \in \partial\mathbb{D}$ if it is univalent, $\limsup_{z\to\tau} \mathrm{Re}\, h(z) = +\infty$ and $h(\mathbb{D})$ is starlike at infinity.

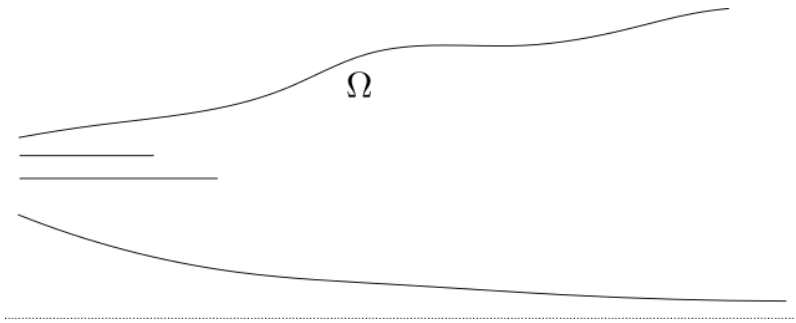

Figure 1.3: A starlike at infinity domain.

Some authors refer to these domain as *convex in the positive direction*. We adopt this terminology in the third paper of this thesis. The following theorem characterizes this class analytically. A detailed proof is found in [4].

Theorem 24. *Let $h : \mathbb{D} \to \mathbb{C}$ be a non-constant analytic function and let $\tau \in \partial\mathbb{D}$. Then, h is starlike at infinity with respect to τ if and only if*

$$\mathrm{Re}(\bar{\tau}(\tau - z)^2 h'(z)) \geqslant 0$$

for all $z \in \mathbb{D}$. Moreover, equality holds at some point if and only if

$$h(z) = ai\frac{\tau + z}{\tau - z} + c$$

where $a \in \mathbb{R}\backslash\{0\}$ and $c \in \mathbb{C}$.

All those classes play substantial role in this thesis as they appear in all papers. In particular, the preceding results are used in order to prove univalence of some functions, arising from the differential equations that we present next, and moreover they describe their geometric behaviour.

1.5 Introduction to Loewner theory

Loewner theory was pioneered by Charles Loewner (1893–1968) in 1923, who studied the properties of continuously (in the Caratheodory sense) evolving families of slit mappings of the unit disc and he discovered that such a family satisfies the so-called Loewner partial differential equation. A few years earlier, in 1916, Ludwig Bieberbach proved that the second coefficient of the Taylor series of a univalent (analytic and one-to-one) mapping of the unit disc, satisfies the inequality $|a_2| \leqslant 2|a_1|$, while he

also conjectured that $|a_n| \leqslant n|a_1|$, for every $n \geqslant 1$. The first theoretical application of Loewner's PDE appeared in the proof of the Bieberbach conjecture for the third coefficient. An affirmative answer to the Bieberbach conjecture was firstly given by Louis de Branges in 1984, while a second and simpler proof by Pommerenke and Fitzgerald in 1985 was possible due to Loewner's theory for slit mappings. See [6] for a detailed proof.

A contemporary and more general version of Loewner's PDE was introduced by Pavel Kufarev (1909–1968) in 1947. According to his study, for any continuous and increasing family of simply connected domains, then their corresponding Riemann maps, satisfy a PDE driven by an analytic function of the unit disc with positive real part and conversely, any such PDE produces a family of solutions, whose images form a continuous and increasing family of simply connected domains. Roughly speaking, the driving functions are replaced by driving measures, generalizing Loewner's theory to non-slit mappings. Besides, one can view the driving functions as point masses moving continuously in time. For this reason, nowadays, we often refer to the Loewner-Kufarev PDE.

An important modern development in Loewner theory is the Schramm-Loewner evolution (or Stochastic-Loewner evolution or SLE), which was introduced in 2000 by Schramm (see [13]). The theory of SLEs gives a probabilistic aspect to Loewner's equation as the driving functions in the classical form, are now replaced by Brownian motions. The curves generated by such a process are random planar curves. It is therefore common in the literature to distinguish Loewner's theory between the deterministic and the stochastic case. In this study, we shall not refer to SLEs, so we return to the deterministic case.

Another version of Loewner's equation that has recently received attention is the Komatu-Loewner equation, firstly introduced by Komatu in 1943. This equation describes the evolution of multiply connected domains. See [1], [2] and [3] for a modern approach and more references.

1.6 The radial Loewner-Kufarev PDE

In this section we will define radial Loewner chains and present the differential equations that govern such chains. The textbooks of Duren [7] and Pommerenke [19] provide an excellent introduction to the topic. An interesting approach using potential-theoretic methods can be found in [24].

Let $(D_t)_{t\geqslant 0}$ be an increasing, i.e $D_t \subset D_s$ for $s < t$, family of simply connected domains in $\hat{\mathbb{C}}$, such that

1. $(D_t)_{t\geqslant 0}$ be an increasing, i.e $D_t \subset D_s$ for $s < t$ and

2. $(D_t)_{t\geqslant 0}$ is continuous in the Caratheodory sense, i.e $D_t \to D_{t_0}$, whenever $t \to t_0$, for all $t \to t_0$.

For each $t \geqslant 0$, consider the unique Riemann map $f_t := f(\cdot, t) : \mathbb{D} \to D_t$, such that $f_t(0) = 0$ and $f_t'(0) > 0$. then f_t expands in Taylor series as

$$f(z,t) = \beta(t)z + \cdots, \quad |z| < 1, \quad t \geqslant 0$$

where β is increasing with time, as follows by the Schwarz Lemma. In the classical notation, $\beta(t)$ is given $\beta(t) = e^t$. This can be considered by reparameterizing time, since β is monotone.

Definition 25. *A function $f : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ with*

$$f(z,t) = e^t z + \cdots \tag{1}$$

is called a Loewner chain if $f(\cdot, t)$ is analytic and univalent in $\mathbb{D}$ for each $t \geqslant 0$ and

$$f(\mathbb{D}, s) \subset f(\mathbb{D}, t), \quad s \leqslant t. \tag{2}$$

We also refer to f as the Loewner flow.

By the second property of the definition, the functions

$$\phi_{s,t}(z) = \phi(z; s, t) := f_t^{-1} \circ f_s(z) : \mathbb{D} \to \mathbb{D}$$

are well-defined in the unit disc for all $s \leqslant t$. We call the family $(\phi_{s,t})_{s\leqslant t}$ the *transition functions* of the chain. The following result follows using a standard argument for the growth inequality on $\boldsymbol{S}$, the space of univalent mappings of the disc. See [7] or [19].

Proposition 26. *Let f be a Loewner chain. Then, for all $0 \leqslant s \leqslant t \leqslant u < +\infty$,*

1. $|f(z,t) - f(z,s)| \leqslant \dfrac{8|z|}{(1-|z|)^4}(e^t - e^s)$

2. $|\phi(z,t,u) - \phi(z,s,u)| \leqslant \dfrac{2}{(1-|z|)^2}(1 - e^{s-t})$ *for all $z \in \mathbb{D}$.*

In particular, for all $z \in \mathbb{D}$, the map $t \mapsto f(z,t)$ is absolutely continuous.

The last statement of the preceding proposition shows that for each $z \in \mathbb{D}$, the time-derivative $\dfrac{\partial f}{\partial t}(z,t)$ exists for almost every $t \geqslant 0$. In fact, using Vitali's theorem, one can prove that the exceptional set where the derivative does not exist is the same for all $z \in \mathbb{D}$. In light of this, the following results are the basis of Loewner's theory.

Theorem 27. *A function $f : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a Loewner chain if and only if there exists some function $p(z,t)$, for $z \in \mathbb{D}$ and $t \geqslant 0$, analytic in z and measurable in t, with $\operatorname{Re} p(z,t) > 0$ and $p(0,t) = 1$ for all $t \geqslant 0$, so that*

$$\frac{\partial f}{\partial t}(z,t) = z f'(z,t) p(z,t) \tag{1.6.1}$$

for all $z \in \mathbb{D}$ and for almost all $t \geqslant 0$.

The function p is called the *driving function* of the Loewner chain. In addition, by Herglotz representation there exists a family of probability measures $(\mu_t)_{t \geqslant 0}$ so that $p(z,t) = S[d\mu_t](z)$. We refer to this family as the *driving measures* of $f(z,t)$. Equation (1.6.1) is called the *Loewner-Kufarev* PDE. As a result, we have that a Loewner chain satisfies equation (1.6.1) and conversely, the solution to (1.6.1) is also a Loewner chain. For the opposite direction we need the following theorem.

Theorem 28. *Let $p(z,t)$ be as above and let $s \geqslant 0$. Then, the initial value problem in $z \in \mathbb{D}$*

$$\frac{dw_s}{dt}(z,t) = -w_s(z,t) p(w_s(z,t), t) \tag{1.6.2}$$

for almost all $t \geqslant s$, with $w_s(z,s) = z$, admits a unique solution $w_s(z,t) = \phi(z;s,t)$ analytic and univalent in $z \in \mathbb{D}$. Furthermore, the limit

$$f(z,s) := \lim_{t \to +\infty} e^t \phi(z;s,t) \tag{1.6.3}$$

exists locally uniformly in $\mathbb{D}$, for all $s \geqslant 0$ and the function f is a Loewner chain.

Conversely, if f is a Loewner chain, then the transition functions satisfy the initial value problem

$$\frac{\partial \phi}{\partial t}(z;s,t) = -\phi(z;s,t) p(\phi(z;s,t), t) \tag{1.6.4}$$

for all $z \in \mathbb{D}$ and for almost all $t \geqslant s$, with $\phi(z;s,s) = z$

We refer to equations (1.6.2) or (1.6.4) as the *Loewner-Kufarev ODE*. The preceding result, answers to the existence question as far as PDE (1.6.1) is concerned. More, it provides a clear connection between Loewner PDE and ODE and how we jump from one to other through the transition functions. Finally, we understand that the solution to equation (1.6.1) is unique if the initial value $f(z,0)$ is given by (1.6.3) for $s = 0$.

As we mentioned in the beginning of this section, we can derive equation (1.6.1), by applying potential theory. Namely, we can relate the Riemann mapping of a simply connected domain with the Green's function of that domain. This was firstly observed by M. Heins (see [11]).

For the sake of completeness, we assume in this case $(D_t)_{t\geqslant 0}$ to be decreasing. Let the corresponding Riemann maps

$$g(z,t) = ze^{-\tilde{G}_{D_t}(z,0)}$$

for all $z \in D_t$, where $\tilde{G}_{D_t}(\cdot,0)$ denotes the unique analytic completion in D_t, of the harmonic function $G_{D_t}(z,0)+\log|z|$, such that $\mathrm{Im}(\tilde{G}_{D_t}(z,0)) = 0$. Now, because $(D_t)_{t\geqslant 0}$ is decreasing, $G_{D_t}(z,0)$ is decreasing with t and therefore it is differentiable with respect to time almost everywhere in $[0,\infty)$ (see [18]). Furthermore, applying the Schwarz integral formula in an arbitrary disc $K \subset D_t$, thus,

$$\tilde{G}_{D_t}(z,0) = \frac{1}{2\pi i}\int_{\partial K}\frac{\zeta+z}{\zeta-z}(G_{D_t}(\zeta,0)+\log|\zeta|)\frac{d\zeta}{\zeta}$$

for all $z \in K$, then we can show that $\tilde{G}_{D_t}(z,0)$ is differentiable with time as well and the derivative is analytic. Consider, therefore, the function $H(z,t) := -\partial_t\tilde{G}_{D_t}(z,0)$ which is well defined in $[0,\infty)\backslash N$ for some null set N, $H(\cdot,t)$ is analytic in D_t for each such $t \notin N$ and it has positive real part due to monotonicity of the Green's function. See [24] for a detailed proof. Finally, a simple differentiation with t, gives us that the Loewner-Kufarev's ODE

$$\frac{\partial g}{\partial t}(z,t) = g(z,t)p(g(z,t),t) \tag{1.6.5}$$

where $p(z,t) := H(f(z,t),t)$, is satisfied for all $z \in D_t$ and almost every $t \geqslant 0$. Finally, letting $f(\cdot,t) = g^{-1}(\cdot,t)$, it is straightforward to see that f satisfies the Loewner-Kufarev PDE

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t). \tag{1.6.6}$$

A natural question in the field is how the geometry of the domains D_t depends on the form of the driving function $p(z,t)$; or equivalently the

form of the driving measures μ_t. For instance, Loewner proved that for slit mappings the driving measures are time-dependent point masses. In fact, Loewner's theory initiated studying evolutions of slit mappings. By definition, a *slit domain* is a simply connected domain of the form $D\backslash\gamma$, for a continuous curve γ in D, that emanates from a boundary point of D. A *slit mapping* of the unit disc is a univalent function so that its image is a slit domain.

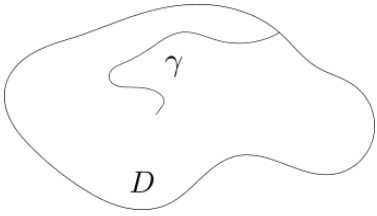

Figure 1.4: A slit domain

Now, given a curve $\hat{\gamma}$, that extends to infinity, with parameterization $\gamma = \gamma(t)$, $t \geqslant 0$, then for the flow $f(\cdot, t) : \mathbb{D} \to \mathbb{C}\backslash\gamma([t, \infty))$, there exists a continuous function $\lambda(t)$, so that Loewner's PDE is written as

$$\frac{\partial f}{\partial t}(z,t) = f'(z,t)z\frac{e^{i\lambda(t)} + z}{e^{i\lambda(t)} - z} \tag{1.6.7}$$

for all $|z| < 1$ and $t \geqslant 0$. In particular, the driving function is given as the preimage of the tip point of the curve, $e^{i\lambda(t)} = f^{-1}(\gamma(t), t)$. In other words, equation (1.6.7) is a special case of (1.6.1) for slit mappings. The opposite direction, on the other hand, is not necessarily true as we point out later on. Thus, given a continuous function λ, the solution to (1.6.7) does not necessarily produce a family of slit domains.

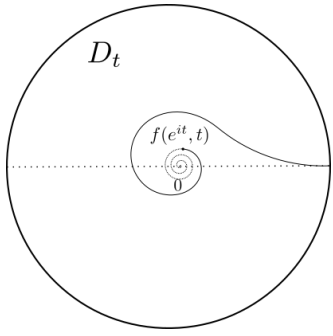

Figure 1.5: The slit domain corresponding to the driving function $\lambda(t) = e^{it}$.

Most part of this thesis is devoted to slit domains both in unit disc and in the upper half-plane. Particularly for the radial case, in Paper I we study the case for $\lambda(t) = e^{it}$, $t \geqslant 0$. Although it seems to be the simplest time-dependent case, solutions to (1.6.7) for this driving function are not found in the literature.

To conclude, we note that we can consider the Loewner-Kufarev equation to the exterior of the unit disk using the inverse transform of f. Our examples in the final paper will be presented either in $\mathbb{D}$ or in Δ. Set

$$\tilde{f}\left(\frac{1}{z}, t\right) = f(z,t)^{-1}, \quad \text{for } |z| < 1,$$

with Laurent expansion $\tilde{f}(z,t) = e^{-t}z + \cdots$, in Δ. Then, the images $\tilde{D}_t = \tilde{f}(D_t)$ also constitute a decreasing family of simply connected domains containing infinity, $\tilde{f}_t$ fix infinity with positive derivative and the PDE becomes

$$\frac{\partial \tilde{f}}{\partial t}(z,t) = z\tilde{f}'(z,t)\int_{\partial\mathbb{D}} \frac{z+\bar{\zeta}}{z-\bar{\zeta}} d\mu_t(\zeta) \tag{1.6.8}$$

for all $z \in \Delta$ and almost all $t \geqslant 0$.

We consider the following time-independent example for the driving measures. Let ζ be some arbitrary point in the unit circle and consider the point mass $\mu_t = \delta_{\{\zeta\}}$, for all $t \geqslant 0$. Then the solution to equation (1.6.8) is given by the function $f(z,t) = \phi^{-1}(e^t\phi(z))$, where

$$\phi(z) = z + 2 + \frac{1}{z}, \quad z \in \Delta.$$

It is direct to see using Theorem 20 that ϕ is starlike, by considering the transformation $z \mapsto \frac{1}{z}$. Then, for each time $t \geqslant 0$, the image $f(\Delta, t)$ is Δ minus an orthogonal line segment emanating from ζ to the tip point $f(\zeta, t)$. Now, if we consider n many point masses, say $\zeta_1, \cdots, \zeta_n$ that are symmetrically placed on the unit circle (the roots of unity for instance) with equal weights, say $\alpha_j = \frac{1}{n}$, then the solution to (1.6.8) is again of the form $f(z,t) = \phi^{-1}(e^t\phi(z))$ for some starlike function ϕ. This formula implies that the image $f(\Delta, t)$ is Δ minus n symmetrically placed orthogonal line segments emanating from the points ζ_j to the tip points $f(\zeta_j, t)$, for $j = 1, \ldots, n$, as in the figure below, and that the trajectories

$$\{f(\zeta_j, t) : t \geqslant 0\}, \quad j = 1, \ldots, n$$

are preimages of straight lines joining the origin and the point at infinity.

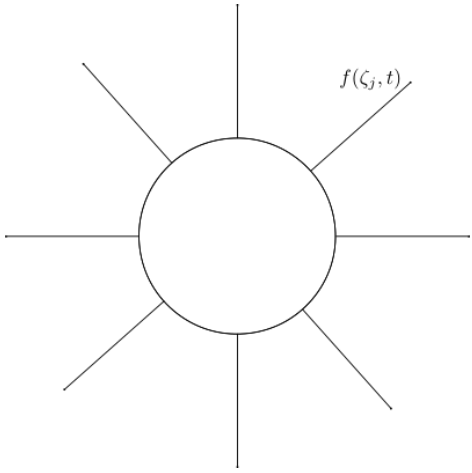

Figure 1.6: Evolution for $\mu_t = \sum_{j=1}^{n} \alpha_j \delta_{\{\zeta_j\}}$, for the roots of unity $\zeta_1, \ldots, \zeta_n$ and equal weights $\alpha_1, \ldots, \alpha_n > 0$.

However, if we choose two points $\zeta_1, \zeta_2 \in \partial\mathbb{D}$, not symmetrically placed, and the driving measures to be $\mu_t = \frac{1}{2}\delta_{\{\zeta_2\}} + \frac{1}{2}\delta_{\{\zeta_2\}}$, the solution to (1.6.8) has the same form as above, but the trajectories of the tip points -which again are inverse images of straight lines connecting the origin with the point at infinity, through a starlike function- are some curves emanating from ζ_1 and ζ_2 towards infinity but are not straight lines.

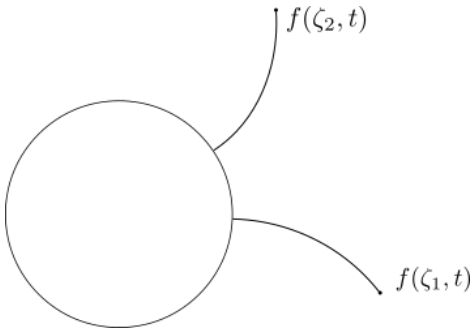

Figure 1.7: The slits repel each other.

In view of the above observations, we see that the solutions to the Loewner equation are not 'linear' in the sense of addition of the driving functions. Indeed, if $p_j(z,t) = S[\frac{1}{2}\delta_{\{\zeta_j\}}] = \frac{1}{2}\frac{z+\bar{\zeta_j}}{z-\bar{\zeta_j}}$ for $j = 1, 2$ and f_j

are the corresponding solutions to (1.6.8), then $f_j(\Delta, t) = \Delta \backslash L_t^j$, where L_t^j is a line segment as in figure 1.6. On the other hand, if f is the solution to the driving function $p_1 + p_2$, then $f(\Delta, t) = \Delta \backslash (L_t^1 \cup L_t^2)$, where L_t^j are not necessarily line segments as in figure 1.7. Thus, we do not just add up the growing slits. Several examples concerning addition of driving measures have been studied in [27].

1.7 The chordal Loewner-Kufarev PDE

The situation in the upper half-plane is somewhat similar to the radial case. Here, we consider a decreasing family of simply connected domains $(D_t)_{t \geqslant 0}$, such that $D_t \subset \mathbb{H}$, $K_t := \mathbb{H} \backslash D_t$ is compact and moreover $\mathbb{H} \backslash K_t$ is also simply connected. To visualize, the last property implies that the hulls K_t are attached to the real line, or as the process evolves, the hulls grow inside $\mathbb{H}$, emanating from its boundary. Such a family of domains is produced by the so-called *chordal* Loewner equation.

Let D be a simply connected domain as above. Then, by Riemann's mapping theorem there exists a conformal map g from D onto $\mathbb{H}$, that fixes infinity, which is a boundary point in this case. Since K is bounded and hence is contained in some disc about zero, then by the Schwarz reflection principle, g extends analytically the complement of the preceding disc onto the complement of a compact interval $I \subset \mathbb{R}$. This means that g admits a Laurent representation with linear analytic part and by normalizing so that the first coefficient is 1 and the constant term is zero, we have that

$$g(z) = z + \frac{2b}{z} + \dots$$

for all $z \in D$. The fact that $g(z) - z \to 0$ as z becomes infinite is referred in the literature as the *hydrodynamic condition* of g. The chordal Loewner flows are always normalized such that each map of the flow satisfies the hydrodynamic condition. The term b is also called the upper half plane capacity of K and is denoted by $\mathrm{hcap}(K)$.

The slit case in the upper half plane is treated as in the radial. A recent, detailed proof is given by A. Monaco and P. Gumenyuk in [17]. See also [28] and [29] for the multiple and infinitely many slits versions respectively. If γ is a Jordan curve emanating from $\mathbb{R}$, with parameterization $\gamma = \gamma(t)$, $0 \leqslant t < \infty$, we then write $K_t := \gamma([0, t])$. Let g_t be the corresponding Riemann maps described above and let f_t be the inverse mappings. It is then proved that $\lambda(t) := g_t(\gamma(t))$ is a continuous real-valued function of t

and for each $T > 0$ we have the chordal Loewner ODE

$$\frac{\partial g}{\partial t}(z,t) = \frac{2}{g(z,t) - \lambda(t)}, \quad g(z,0) = 0 \tag{1.7.1}$$

for all $z \in \mathbb{H}\backslash\gamma([0,T])$ and $0 \leqslant t \leqslant T$. By taking the inverse, the chordal Loewner PDE is written as

$$\frac{\partial f}{\partial t}(z,t) = -f'(z,t)\frac{2}{z - \lambda(t)} \tag{1.7.2}$$

in $\mathbb{H} \times [0,\infty)$. The continuous function λ is called the driving function of the flow $f(z,t)$.

In the opposite direction and more generally, given a family of probability measures $(\mu_t)_{t\geqslant 0}$ of $\mathbb{R}$ with compact support, we consider the ODE in $\mathbb{H} \times [0,\infty)$

$$\frac{\partial w}{\partial t}(z,t) = \int_{\mathbb{R}} \frac{d\mu_t(x)}{w(z,t) - x} \tag{1.7.3}$$

with initial value $w(z,0) = 0$. Let T_z be the supremum of all t such that the solution to equation (1.7.3) is well defined and $g_t(z) \in \mathbb{H}$ for all $t \leqslant T_z$. Then, there exists a unique solution g_t which is conformal in the domain $D_t := \{z : T_z > t\}$, with $\mathrm{hcap}(\mathbb{H}\backslash D_t) = 2t$, satisfying the hydrodynamic condition, mapping D_t onto $\mathbb{H}$. Finally, the Loewner flow $f_t = g_t^{-1}$ satisfies the PDE

$$\frac{\partial f}{\partial t}(z,t) = -f'(z,t)\int_{\mathbb{R}} \frac{d\mu_t(x)}{z - x} \tag{1.7.4}$$

for all $z \in \mathbb{H}$ and $t \geqslant 0$, called the chordal Loewner PDE. The compact sets $K_t := \mathbb{H}\backslash D_t$ are referred to the compact hulls generated by the flow.

Similarly to the radial case, in the chordal case, the point mass at zero produces a slit flow, with a vertical line segment emanating from the origin vertically, inside $\mathbb{H}$ (see [13]). It is straightforward to see that the function $f(z,t) = \sqrt{z^2 - 4t}$ satisfies equation (1.7.2) for $\lambda \equiv 0$. Also in [13], we find that the driving function

$$\lambda(t) = 2\frac{2a-1}{\sqrt{a(1-a)}}\sqrt{t}$$

for $a \in (\frac{1}{2}, 1)$, drives a slit flow with hull a line segment that starts from the origin with angle $(1-a)\pi$. In the last paper, we present a radial Loewner flow for the outside of the unit disc Δ, where the growing hull is also a line segment emanating from the unit circle forming some angle a. However, its driving function is not as trivial as in the chordal case

above. Nevertheless, for t close to zero, the driving function behaves like the square root. Given the opportunity, we note that when comparing the radial and the chordal PDE, although same driving functions may correspond to different hulls, locally at time zero, they have similar behaviour (see [13, paragraph 4.6.3]).

Figure 1.8: Straight slits in $\mathbb{H}$.

Geometrically, the solutions to equation (1.7.2) are not necessarily slit mappings for every continuous functions λ. A sufficient condition for a driving function to produce hulls that are curves are found in [14] and [16]. It is proved that Lip-$\frac{1}{2}$ driving functions, with sufficiently bounded Lip($\frac{1}{2}$)-norm, produce quasi-slit domains. Moreover, the optimal upper bound for the norm is $||\lambda||_{\text{Lip}(1/2)} = 4$ and for every $k \geqslant 4$, there exists some Lip-$\frac{1}{2}$ driving function that does not generate quasi-slit domains. This can be seen explicitly using the driving terms $\lambda_k(t) = k\sqrt{1-t}$. The solutions for this driving functions can be found in [12], [15] and [21], where the corresponding Loewner chains are families of single-slit functions, the geometry of which are either spirals about some point of the upper half-plane, or intersections with the real line, according to the value of k.

Papers I and II are devoted in these driving functions, in a way that we extend the results found in the literature to Loewner chains with multiple or infinitely many slits. As we pointed out in the end of Section 1.6, taking multiple driving functions of the form $\lambda_{k_j}(t) = k_j\sqrt{1-t}$, for $j \geqslant 1$ and $k_j \in \mathbb{R}$, the resulting hulls follow a universal geometric behaviour, rather than 'adding' the slits together with each slit behaving as in the single-slit case.

1.8 Semigroup theory

There is a close connection of Loewner's theory to the semigroup theory for holomorphic self-maps of the unit disc. In fact, due to the Berkson-Porta formula (see [4]), it turns out that a Loewner flow $f(z,t)$ driven by

a time-independent function $p(z,t) = p(z)$, forms a semigroup with fixed point the origin and it is parameterized as

$$f(z,t) = h^{-1}(e^{-t}h(z))$$

for all $z \in \mathbb{D}$ and $t \geqslant 0$, where h is a starlike function with respect to the origin, called the *Koenigs* function (we shall define it shortly). In [27], A. Sola presents explicit examples of Loewner flows, driven by time-independent densities.

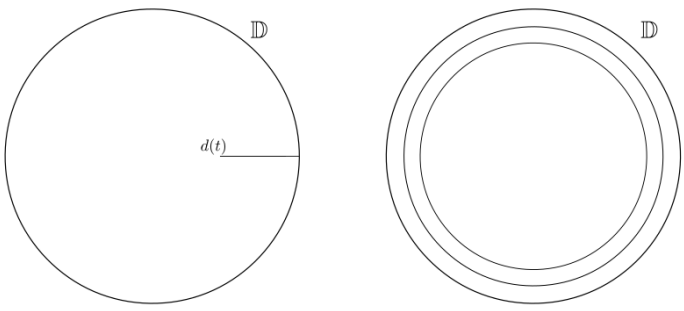

Figure 1.9: The flow $f(\cdot,t) : \mathbb{D} \to \mathbb{D}\backslash[d(t),1]$, corresponding to the point mass at 1, with $d(t) := 2e^t - 1 - 2\sqrt{e^{2t} - e^t}$ on the left and the flow $f(\cdot,t) : \mathbb{D} \to e^{-t}\mathbb{D}$, corresponding to the Lebesgue measure on the right.

To be more precise, a continuous *semigroup* of holomorphic self-maps of the unit disc, is defined as a family $(\phi_t)_{t\geqslant 0} \subset H(\mathbb{D})$ so that

1. $\phi_t : \mathbb{D} \to \mathbb{D}$,
2. $\phi_0 = id_{\mathbb{D}}$
3. $\phi(z, t+s) = \phi(\phi(z,t), s)$, for all $z \in \mathbb{D}$ and $s, t \geqslant 0$,
4. $\mathbb{D} \times [0,\infty) \ni (z,t) \mapsto \phi(z,t)$ is continuous in topology of uniform convergence on compact subsets.

For the rest, we shall only refer to continuous semigroups, thus (4) is always assumed. Also, we only refer to *non-trivial* semigroups, that are by definition $\phi_t = id_{\mathbb{D}}$, for all $t \geqslant 0$. A couple of elementary examples can be easily deduced by hand, if we consider a stationary point mass at 1, thus the constant driving function $\lambda(t) \equiv 0$, or the Lebesgue measure $\frac{1}{2\pi} m|_{[0,2\pi]}$, as in the figure above. In the first case, we get $\phi(z,t) = h^{-1}(e^{-t}h(z))$, where $h(z) = \dfrac{z}{(1+z)^2}$ and $\phi(z,t) = e^{-t}z$, thus $h(z) = z$

for the second one. In the setting of Loewner chains, indeed ϕ corresponds to radial Loewner chains, but as we derive them from time-independent driving functions, in fact, they are also semigroups as defined above.

Back to the semigroup theory, it is proved that for each $t \geqslant 0$, the *iterate* ϕ_t is univalent (one-to-one) in the unit disc. Now, according to the continuous version of the Denjoy-Wolff theorem, for each continuous semigroup $(\phi_t)_{t\geqslant 0}$. there exists a unique point $\tau \in \overline{\mathbb{D}}$, so that for all $z \in \mathbb{D}$ we have that $\lim_{t\to+\infty} \phi_t(z) \to \tau$. It is stated as follows.

Denjoy-Wolff Theorem. *Let $(\phi_t)_{t\geqslant 0}$ be a continuous semigroup. Then, there exists some unique $\tau \in \overline{\mathbb{D}}$, such that $\phi_t(z) \to \tau$, as $t \to +\infty$, for every $z \in \mathbb{D}$. Moreover,*

(1) if $\tau \in \mathbb{D}$, then there exists some $\lambda \in \mathbb{C}\backslash\{0\}$, with $\mathrm{Re}\lambda \geqslant 0$, so that $\phi_t'(\tau) = e^{-\lambda t}$, for all $t \geqslant 0$,

(2) if $\tau \in \partial\mathbb{D}$, then there exists some $\lambda \geqslant 0$, so that the limit $\phi_t'(\tau) := \lim_{z\to\tau} \phi_t'(z)$ exists non-tangentially and equals $\phi_t'(\tau) = e^{-\lambda t}$, for all $t \geqslant 0$.

We call λ the *spectral value* of the semigroup. In abbreviation of the definition, we might refer to $e^{-\lambda t}$ as the spectral value instead. See [4, Proposition 8.1.4] for details on the Denjoy-Wolff theorem and the spectral value. Semigroups are classified in different classes, the first of which are defined according to the Denjoy-Wolff point and the spectral value.

Definition 29. *Let $(\phi_t)_{t\geqslant 0} \subset H(\mathbb{D})$ be a continuous semigroup, with Denjoy-Wolff point $\tau \in \overline{\mathbb{D}}$ and spectral value λ. Then, we say that*

1. *$(\phi_t)_{t\geqslant 0}$ is elliptic if $\tau \in \mathbb{D}$,*
2. *$(\phi_t)_{t\geqslant 0}$ is parabolic if $\tau \in \partial\mathbb{D}$ and $\lambda = 0$*
3. *$(\phi_t)_{t\geqslant 0}$ is hyperbolic if $\tau \in \partial\mathbb{D}$ and $\lambda > 0$.*

We refer to parabolic or hyperbolic semigroups as non-elliptic.

Diving deeper into the theory of semigroups of holomorphic functions, we further classify semigroups in terms of the geometry of the so-called *Koenigs function*. To fully understand the theory of the Koenigs function and reach the aforementioned classification, one needs to go through a technical study that involves Riemann surfaces and *holomorphic models*, that we will define now. For the purposes of this thesis, we only need to know the concluding results of this theory, so we will try to be narrative and describe the general ideas. For details, we refer to [4], chapter 9.

let us mention at this point that it is only natural to define a continuous semigroup $(\psi_t)_{t\geqslant 0}$ in any Riemann surface Ω, endowed with the Euclidean topology, the same way we did before for unit disc. We begin with the definition of a holomorphic model.

Definition 30. *Let $(\phi_t)_{t\geqslant 0}$ be a semigroup in $\mathbb{D}$. A holomorphic model for $(\phi_t)_{t\geqslant 0}$ is a triplet $(\Omega, h, (\psi_t)_{t\geqslant 0})$ where Ω is a Riemann surface, $h : \mathbb{D} \to \Omega$ is univalent and $(\psi_t)_{t\geqslant 0}$ is a continuous semigroup of automorphisms of Ω such that*

1. *$h \circ \phi_t = \psi_t \circ h$ and*
2. *$\bigcup_{t\geqslant 0} \psi_t^{-1}(h(\mathbb{D})) = \Omega$.*

A closer look in the conjugation formula in the definition shows that such a mapping h allows to transform the initial semigroup to another one. Once we manage to prove that ψ_t is simple and Ω is 'nicely' described, this becomes useful. Geometrically speaking, this is interpreted in accordance to the orbits $\gamma_z := \{\phi_t(z) : t \geqslant 0\}$, because as we will see in the coming theorems, $h(\gamma_z)$ are either horizontal half-lines, or logarithmic spirals, for all $z \in \mathbb{D}$. Initially, we have to prove that such a holomorphic model exists and it is unique up to isomorphisms.

Definition 31. *Let $(\Omega, h, (\psi_t)_{t\geqslant 0})$, $(\tilde{\Omega}, \tilde{h}, (\tilde{\psi_t})_{t\geqslant 0})$ be two holomorphic models for a semigroup $(\phi_t)_{t\geqslant 0}$. We say that the models are isomorphic, if there exists a one-to-one, holomorphic map $\eta : \Omega \to \tilde{\Omega}$, so that $\tilde{h} = \eta \circ h$ and $\tilde{\psi}_t \circ \eta = \eta \circ \psi_t$, for all $t \geqslant 0$.*

Theorem 32. *Let $(\phi_t)_{t\geqslant 0}$ be a semigroup in $\mathbb{D}$. Then, there exists a unique up to isomorphisms holomorphic model $(\Omega, h, (\psi_t)_{t\geqslant 0})$ for $(\phi_t)_{t\geqslant 0}$. Moreover, Ω is biholomorphic, either to $\mathbb{D}$ or to $\mathbb{C}$.*

Using the theorem above, we can construct very particular models for $(\phi_t)_{t\geqslant 0}$, called *canonical models* and we classify the semigroups according to the following geometric characterizations. For the next, we assume that $(\phi_t)_{t\geqslant 0}$ is not the trivial semigroup, thus $\phi_t = \mathrm{id}_{\mathbb{D}}$, for all $t \geqslant 0$, nor is it a semigroup of automorphisms of $\mathbb{D}$.

Theorem 33. *Let $(\phi_t)_{t\geqslant 0}$ be a semigroup in $\mathbb{D}$ as above. Then,*

(1) $(\phi_t)_{t\geqslant 0}$ is elliptic with spectral value $\lambda \in \mathbb{C}$, with $\mathrm{Re}\lambda > 0$, if and only if $(\phi_t)_{t\geqslant 0}$ has a holomorphic model $(\mathbb{C}, h, z \mapsto e^{-\lambda t} z)$.

(2) $(\phi_t)_{t\geqslant 0}$ is hyperbolic with spectral value $\lambda > 0$, if and only if $(\phi_t)_{t\geqslant 0}$ has a holomorphic model $(\mathbb{S}_{\frac{\pi}{\lambda}}, h, z \mapsto z + t)$, where we denote by S_y the horizontal strip of width y.

(3) $(\phi_t)_{t\geqslant 0}$ *is parabolic, if and only if* $(\phi_t)_{t\geqslant 0}$
(a) either has a holomorphic model $(\mathbb{H}, h, z \mapsto z + t)$ *or* $(\overline{\mathbb{H}}, h, z \mapsto z + t)$,
(b) or has a holomorphic model $(\mathbb{C}, h, z \mapsto z + t)$

In case (3a), $(\phi_t)_{t\geqslant 0}$ *is called parabolic of positive hyperbolic step and in case (3b),* $(\phi_t)_{t\geqslant 0}$ *is called parabolic of zero hyperbolic step. Furthermore, the above models are called canonical and the map* h *the Koenigs function of the semigroup.*

Geometrically, the preceding classifications constitute for two specific classes of univalent functions that we presented in Section 1.4. We may observe that the Koenigs function of elliptic semigroups are spirallike function, while non-elliptic semigroups have Koenigs functions that are starlike at infinity, with respect to the Denjoy-Wolff point.

Now, we point out that the Koenigs function is related to another characteristic quantity, the *infinitesimal generator* of the semigroup. This in turn, describes a particular differential equation that the semigroup satisfies and conversely the solution of which is a semigroup. Given a semigroup $(\phi_t)_{t\geqslant 0}$, then and only then, there exists a unique vector field $G \in H(\mathbb{D})$, such that the ODE

$$\frac{\partial \phi}{\partial t}(z,t) = G(\phi(z,t)) \tag{1.8.1}$$

is satisfied in $\mathbb{D} \times T$, where T is an interval containing $[0, \infty)$. We call G the *infinitesimal generator* of the semigroup. The Berkson-Porta theorem (or formula) states, that given a non-constant $G \in H(\mathbb{D})$, then G is the infinitesimal generator of a semigroup if and only there exists some $\tau \in \overline{\mathbb{D}}$ and $p \in H(\mathbb{D})$ with $\mathrm{Re}(p(z)) > 0$ in $\mathbb{D}$, so that

$$G(z) = (1 - \bar{\tau} z)(\tau - z)p(z). \tag{1.8.2}$$

In the Loewner language, when $\tau = 0$, then p is the driving function as we previously mentioned. A generalization for the infinitesimal generator towards Loewner's theory is studied by F. Bracci, M.D. Contreras and S. Diaz-Madrigal in [5].

We finalize this section by noting the connection of the Koenigs function with infinitesimal generator. Assume at first that $(\phi_t)_{t\geqslant 0}$ is elliptic with spectral value λ. By Theorem 33 we have that $h(\phi_t(z)) = e^{-\lambda t} h(z)$. Therefore, differentiating with respect to t, by (1.8.1) we see that

$$G(z) = -\lambda \frac{h(z)}{h'(z)} \tag{1.8.3}$$

in $\mathbb{D}$. Observe by Berkson-Porta's formula and the characterization of spirallike functions, that by taking the real part of the preceding equation, then h is an $\mathrm{Arg}(\lambda)$-spirallike function of $\mathbb{D}$. In a similar manner, assume that $(\phi_t)_{t\geqslant 0}$ is non-elliptic. In this case we have that $h(\phi_t(z)) = t + h(z)$, and so we get that

$$G(z) = \frac{1}{h'(z)} \tag{1.8.4}$$

in $\mathbb{D}$. Again by (1.8.2) and Theorem 24, we deduce that h is starlike at infinity with respect to $\tau \in \partial\mathbb{D}$. Many times along this thesis shall we see that we transform the Loewner's ODE to the preceding two equations, since we start p and we wish to describe the corresponding Loewner chains.

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2. Summaries of papers

2.1 Summary of Paper I

Our purpose in this article is to describe some Loewner chains, both in the unit disc (the radial case) and in the upper half-plane (the chordal case), by choosing in advance the corresponding driving functions (equivalently the driving measures). To do so, we first give a technique to solve the PDEs and then, implementing methods from classic complex analysis, we describe geometrically the solutions.

Starting with the radial case as in (1.6.1), we choose the following $2n+1$ parameters. Let $\zeta_1, \ldots, \zeta_n \in \partial\mathbb{D}$ (point masses), $b_1, \ldots, b_n > 0$ (weights) and $a \in \mathbb{R}$ (angle). We, now, consider the time-dependent driving measures $\mu_t := \sum_{j=1}^{n} b_j \delta_{\{\zeta_j e^{iat}\}}$, equivalently, the driving function $p_a(z,t) = \sum_{k=1}^{n} b_k \frac{\zeta_k e^{iat} + z}{\zeta_k e^{iat} - z}$. We then prove that the solutions to this radial Loewner equation are of the form $f_a(z,t) = h_a^{-1}(e^{(ia-1)t} h_a(e^{-iat} z)$, where h_a is an $(\mathrm{Arccot}(a) - \frac{\pi}{2})$-spirallike function of $\mathbb{D}$, in the sense of Definition 19.

Therefore, we know how h_a behaves geometrically, and in particular we can see that it maps the unit circle onto n logarithmic spirals that extend to infinity. This will allow us to visualize the image of $f_a(\cdot, t)$, as $t \to \infty$, that is the unit disc minus n slits emanating from $\zeta_1, \ldots, \zeta_n$ spiralling about the origin. It is worth noting that we can observe that as $a \to \infty$, $f_a(\cdot, t)$ converges locally uniformly to ze^{-t}, for all $t \geqslant 0$. Hence, although the family of the Loewner chain $(f_a)_{a\in\mathbb{R}}$ converges to another Loewner chain, their driving functions $(p_a)_{a\in\mathbb{R}}$ do not converge with respect to a.

We act similarly for the chordal case, which is the second part of the paper. In this setting, we choose the $2n$ parameters $k_1 \ldots, k_n \in \mathbb{R}$ (point masses) and $b_1, \ldots, b_n$ (weights), and we consider the driving measures $\nu_t := \sum_{j=1}^{n} b_j \delta_{\{k_j \sqrt{1-t}\}}$. Equivalently, the driving function is given as

$p(z,t) = \sum_{j=1}^{n} \frac{2b_j}{z - k_j\sqrt{1-t}}$, for all $z \in \mathbb{H}$ and $0 \leqslant t < 1$. As in the radial case, we explicitly solve the corresponding Loewner PDE and we study the geometric behaviour of the Loewner chains as $t \to 1$. Then, depending on the choice of the parameters, the hulls are n disjoint slits that are attracted either to some point $\beta \in \mathbb{H}$ and they spiral about that point, or they are attracted to some point $\rho \in \mathbb{R}$, in such a way that the following possibilities take place: (a) all curves intersect $\mathbb{R}$ non-tangentially, each curve with a different angle, (b) all curves intersect $\mathbb{R}$ tangentially, all of them with same angle 0 or π and (c) if $n \geqslant 2$, all curves intersect $\mathbb{R}$ orthogonally. This result mirrors the case discussed in Section 1.7.

Finally, we point out that when we take spirals in the chordal case, then we can map the chordal case to the radial case, using a Möbius transform and a time translation. This is easily done once we observe their close connection to semigroups of holomorphic self maps.

2.2 Summary of Paper II

In a continuation of the first paper, our aim is to extend those results by deducing Loewner chains with infinitely many slits. In the setting of the chordal Loewner PDE (1.7.4), we now consider a sequence of points $(k_n)_{n\geqslant 1} \subset \mathbb{R}$, such that $d := \inf_{n\neq m} |k_n - k_m| > 0$ and a summable sequence of positive numbers $(b_n)_{n\geqslant 1}$. We then consider the driving function to be written as $p(z,t) = \sum_{n\geqslant 1} \frac{2b_n}{z - k_n\sqrt{1-t}}$ for $z \in \mathbb{H}$ and $0 \leqslant t < 1$. To solve the chordal Loewner PDE in this case, we initially follow the method in the first paper. The reader will be able to see that for this, we need to consider the auxiliary function $P_{\mathbb{H}}(z) := z + \sum_{n\geqslant 1} \frac{4b_n}{z - k_n}$, as we transpose the chordal PDE to the ODE $\frac{dv}{P_{\mathbb{H}}(v)} = \frac{dt}{2(1-t)}$. Therefore, one has to integrate $\frac{1}{P_{\mathbb{H}}}$.

Assuming that $b_n > 0$, for only finitely many $n \geqslant 1$, thus considering the setting of the first paper,we can immediately see that $P_{\mathbb{H}}$ is a rational function with poles the points k_j and therefore, its reciprocal $\frac{1}{P_{\mathbb{H}}}$ is also a rational function with poles the zeroes of $P_{\mathbb{H}}$. But then, it would be easy to integrate it by means of partial fraction decomposition. So, the idea is to appropriately represent $\frac{1}{P_{\mathbb{H}}}$ so that we can easily perform integration.

Obviously, the preceding idea is not applicable in the case of infinitely many positive weights b_n and hence we need other methods. We manage to overcome this using a classical result in complex analysis, the Cauchy fraction expansions.

Then, $\dfrac{1}{P_{\mathbb{H}}}$ is represented as the series of its principal parts, at the poles $\lambda_n \in \mathbb{R}$, where λ_n are the zeroes of $P_{\mathbb{H}}$ and only now are we able to explicitly present the chordal Loewner chain that corresponds to the driving function we chose initially. Namely, we find the same four geometric possibilities as in the previous work, only now these take place for an infinite amount of slits.

In conclusion, the three of the four cases, correspond to intersections of the slits with the real axis. As a new approach, we derive the angles of intersection using the harmonic measure. The interest of this method lies on the fact that we can visualize our claims, by recalling the connection of the harmonic measure to Brownian motion. Then, using Caratheodory's lemma, we deduce that the level sets of the harmonic measure in the unit disc with respect to some prescribed arc $E \subset \partial\mathbb{D}$, are circular arcs intersecting E under some specific angle.

2.3 Summary of Paper III

In this work, we deal with a special class of semigroups of holomorphic maps of the unit disc, called semigroups of *finite shift*. We let $(\phi_t)_{t\geqslant 0}$ be a parabolic semigroup of positive hyperbolic step with Denjoy-Wolff point $\tau \in \mathbb{D}$, according to Section 1.8. By definition, it is of finite shift if for all $z \in \mathbb{D}$, there exists some $R_z > 0$, such that $\phi_t(z) \notin E(\tau, R_z)$, where $E(\tau, R_z)$ is the horodisc $D\left(\dfrac{\tau}{1+R_z}, \dfrac{R_z}{1+R_z}\right)$. Simply, all orbits of the semigroup converge to τ, not only tangentially, but also without intersecting some horodisc.

Because of this geometric behaviour, we examine the rate of convergence of $(\phi_t)_{t\geqslant 0}$ to the Denjoy-Wolff point in terms of the Euclidean distance $|\phi_t(z) - \tau|$ with the help of the hyperbolic distance $d_{\mathbb{D}}(\phi_t(z), z)$ and the harmonic measure $\omega(\phi_t(z), E, \mathbb{D})$, for some open arc $E \subset \partial\mathbb{D}$ that depends only on τ.

As far as the Euclidean distance is concerned, we show for all $z \in \mathbb{D}$, there exist constants $c_j = c_j(z)$, $j = 1, 2$, so that

$$\frac{c_1}{t} \leqslant |\phi_t(z) - \tau| \leqslant \frac{c_2}{t}$$

for all $t \geqslant 0$. The upper bound follows from two known results in the

literature. On the other hand, the rate for the harmonic measure has not been studied before. Studying the geometry of the Koenigs function of the semigroup, we deduce an upper bound for the quantity $t\omega(\phi_t(z), E, \mathbb{D})$, where ω is the harmonic measure defined in Section 1.3, and we prove that this bound is also sharp. On top of that, the aforementioned parameters depend on the geometry of the Koenigs function of the semigroup.

More on the geometry of the Koenigs function, it is well-know in the literature that a parabolic semigroup of positive hyperbolic step, with the property that the Koenigs domain contains a half-plane, is of finite shift. As it turned out, in Papers I and II we explicitly present conformal maps that have this property. In fact, driven by the ideas of Paper II, we construct a counterexample for the converse to the preceding statement. In other words, there exists a parabolic semigroup of positive hyperbolic step, that it is of finite shift, yet its Koenigs domain does not contain a half-plane.

2.4 Summary of Paper IV

Our aim in this final work is to present some elementary Loewner chains, mostly in the radial setting, but we also describe some solutions to the chordal Loewner PDE, that are relevant to the first two articles.

We start off by slightly generalizing the first result of Paper I, thus we assume the driving function to be of the form $p(z,t) = \sum_{k=1}^{n} b_k \dfrac{r_k\zeta_k e^{iat} + z}{r_k\zeta_k e^{iat} - z}$, by choosing $b_k > 0$, $\zeta_k \in \partial\mathbb{D}$, $a \in \mathbb{R}$ and $r_k \geqslant 1$, so that $\min r_k < \max r_k$. Recall that in the first paper we studied the case where $r_1 = \cdots = r_n = 1$. In a similar manner, we describe the chordal Loewner chain driven by $q(z,t) = \sum_{k=1}^{n} \dfrac{2b_k}{z - c_k\sqrt{1-t}}$, for some $b_k > 0$ and some $c_k \notin \mathbb{H}$. This in turn, is also a slight generalization of the chordal setting in the same paper, since we there consider the case where all parameters $c_1, \ldots, c_n$ are real. We see that the results are almost the same in form, and by paying a little attention, we use very similar methods.

In addition to the chordal setting above, we next consider the driving function $q(z,t) = \sum_{n=1}^{+\infty} \dfrac{2b_n}{z - k_n\sqrt{1+t}}$, for some summable sequence of non-negative numbers $(b_n)_{n\geqslant 1}$ and a sequence of real points $(k_n)_{n\geqslant 1} \subset \mathbb{R}$, with the assumption that $d := \inf_{n\neq m} |k_n - k_m|$ is strictly positive. Again in an attempt to compare with the results in Paper II, where we study the $\sqrt{1-t}$ forcing, we firstly use same techniques, while the formulas of the

chains have the same form. However, we find in the case of the $\sqrt{1-t}$ forcing four different geometric descriptions, each one depending on the choice of the parameters. But in the case of $\sqrt{1+t}$ forcing we only have one regardless of the choice of the parameters.

Finally, we wish to construct some elementary Loewner chains driven by some absolutely continuous measures. The way we present this, is by considering some specific families of simply connected domains $(D_t)_{t\geqslant 0}$ and writing down their corresponding Riemann maps, say $f(\cdot,t)$. Moving on, we then recover the corresponding driving measures of the chains, say $d\mu_t = \rho(\theta,t)d\theta$ (equivalently the driving functions $p(z,t)$). It is our intention to visualize possible connections of the driving function of a Loewner chain with the geometry of their corresponding hulls.