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Relative self-equivalences and graph complexes

Robin Stoll

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Relative self-equivalences and graph complexes

Robin Stoll

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Abstract

This thesis consists of three papers.

In Paper I, we identify the cohomology of the stable classifying space of homotopy automorphisms (relative to an embedded disk) of connected sums of $S^k \times S^l$, where $3 \leq k < l \leq 2k - 2$. We express the result in terms of Lie graph complex homology.

In Paper II, we construct a rational model for the classifying space $\mathrm{Baut}_A(X)$ of homotopy automorphisms of a simply connected finite CW-complex X relative to a simply connected subcomplex A. Using this model, we provide a purely algebraic description of the cohomology of this classifying space. This constitutes an important input for the results of Paper I.

In Paper III, we show that modular operads are equivalent to modules over a certain simple properad which we call the Brauer properad. Furthermore we show that the Feynman transform corresponds to the cobar construction for modules of this kind. To make this precise, we extend the machinery of the bar and cobar constructions relative to a twisting morphism to modules over a general properad. As an application, we provide the foundations of a Koszul duality theory for modular operads.

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Contents

Abstract iii

Sammanfattning v

Acknowledgments vii

List of papers ix

General introduction xi

1. Prelude . . . xi
2. Summary of Paper I . . . xv
3. Interlude . . . xvi
4. Summary of Paper II . . . xvii
5. Outlook . . . xviii
6. Interlude . . . xix
7. Summary of Paper III . . . xxiii

Bibliography xxv

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Sammanfattning

Denna avhandling består av tre artiklar.

I **Artikel I** identifierar vi kohomologin av det stabila klassificerande rummet för homotopiautomorfier (relativt till en inbäddad disk) av sammanhängande summor av $\mathrm{S}^k \times \mathrm{S}^l$, där $3 \leq k < l \leq 2k - 2$. Vi uttrycker resultatet i termer av homologi av Lie grafkomplex.

I **Artikel II** konstruerar vi en rationell modell för klassificerande rummet $\mathrm{Baut}_A(X)$ för homotopiautomorfier av ett enkelt sammanhängande ändligt CW-komplex X relativt ett enkelt sammanhängande delkomplex A. Med hjälp av denna modell ger vi en rent algebraisk beskrivning av kohomologin av detta klassificerande rum. Detta utgör en viktig ingrediens som behövs i Artikel I.

I **Artikel III** visar vi att modulära operader är ekvivalenta med moduler över en viss enkel properad som vi kallar Brauerproperaden. Dessutom visar vi att Feynmantransformen kan uttryckas i termer av kobarkonstruktionen för moduler av detta slag. För att åstadkomma detta, utvidgar vi maskineriet för bar- och kobarkonstruktioner relativt en vridande morfi till moduler över en allmän properad. Som en tillämpning lägger vi grunden för en Koszuldualitetsteori för modulära operader.

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First and foremost, I would like to thank my PhD advisor, Alexander Berglund, for guiding me through the process of writing this thesis and the papers contained within; without him, and the many discussion we had, this thesis would not exist.

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List of papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

Paper I: **The stable cohomology of self-equivalences of connected sums of products of spheres**
Robin Stoll. Forum of Mathematics Sigma 12, 2024, Article e1. DOI: 10.1017/fms.2023.113.

Paper II: **Equivariant algebraic models for relative self-equivalences**
Alexander Berglund and Robin Stoll.

Paper III: **Modular operads as modules over the Brauer properad**
Robin Stoll. Theory and Applications of Categories 38.40, 2022, pp. 1538–1607.

Earlier versions of Papers I and III were contained in my licentiate thesis [Sto22]. Paper II is a collaborative project joint with Alexander Berglund; the initial broad strategy was due to him, while, in the part included here, I worked out most of the details and wrote everything up. Particularly Sections 4.3 to 4.5 are due to myself.

General introduction

In this introduction, we explain the background of the papers contained in this thesis and provide short summaries of their content. The order of the papers was chosen to fit the narrative of this introduction, even though it does not reflect the order in which they were completed. This introduction reuses, in a slightly modified form, much of the text of the general introduction of the author's licentiate thesis [Sto22].

1. Prelude

The first paper contained in this thesis is concerned with describing the stable rational cohomology of the classifying spaces

$$\mathrm{Baut}_{\partial}(M)$$

of fibrations with fiber certain high-dimensional manifolds M with boundary a sphere. We begin by explaining what this object is and why one might be interested in it.

The study of different kinds of *bundles* (to which fibrations are closely related) is fundamental to many different fields of mathematics, topology foremost among them. Given a (topological) space F, a bundle with fiber F consists of a map of (topological) spaces $p\colon E \to B$ such that every point of B has a *trivializing neighborhood*, i.e. a neighborhood U such that there is a homeomorphism $p^{-1}(U) \cong U \times F$ that is compatible with the projections to U.

One important example is the tangent bundle $\tau\colon \mathrm{T}M \to M$ of a d-dimensional manifold M: its fiber over a point $m \in M$ is the tangent space $\mathrm{T}_m M$ of M at m. The tangent bundle is a d-dimensional *(real) vector bundle*, which means that it is a bundle with fiber $\mathbb{R}^d$ and, for two trivializing neighborhoods U and V (potentially of different points) and $x \in U \cap V$, the two identifications of $\tau^{-1}(x)$ with $\mathbb{R}^d$ differ by a linear map — we say its *structure group* is $\mathrm{GL}_d(\mathbb{R})$.

Invariants that can distinguish bundles are a powerful tool. For instance, when applied to tangent bundles, one might hope to be able to answer intrinsic question about manifolds. This approach has seen much success over the past hundred years. The main families of invariants used are *characteristic classes.* A characteristic class λ assigns to each bundle $p\colon E \to B$ of a certain type a cohomology class $\lambda(p) \in \mathrm{H}^*(B)$. In most cases, one can think of this cohomology class as simply a number; this allows to distinguish between two bundles p and p' by proving that the two numbers $\lambda(p)$ and $\lambda(p')$ are different, which is often much more tractable. Characteristic classes are moreover required to be natural in maps of the base space: when $f\colon B' \to B$ is a continuous map, we ask that $\lambda(f^*p) = f^*(\lambda(p))$. Here f^*p denotes the pullback of p along f; it is the bundle over B' whose fiber over a point b' is given by $p^{-1}(f(b'))$. Important examples of characteristic classes for vector bundles are Stiefel–Whitney, Pontryagin, and Euler classes (for real vector bundles) as well as the Chern classes (for complex vector bundles). They have, for example, been used to study the existence of embeddings and immersions of manifolds into Euclidean space (see e.g. [MS74]) or the existence of complex structures on manifolds (see e.g. [BS53]).

Closely related to the study of bundles and characteristic classes are so-called "classifying spaces". It is possible to construct a *universal bundle*

$$u\colon \mathrm{EGL}_d \longrightarrow \mathrm{BGL}_d$$

such that every d-dimensional vector bundle p over a (paracompact) space B arises (up to isomorphism) as a pullback of u along some map $f_p\colon B \to \mathrm{BGL}_d$ that is unique up to homotopy. The space BGL_d is the *classifying space* for d-dimensional vector bundles. The group GL_d appears because it is, as mentioned above, the structure group of this kind of bundle. A similar construction can be carried out for any topological group.

A characteristic class λ in particular yields an element $\lambda(u) \in \mathrm{H}^*(\mathrm{BGL}_d)$. Conversely, given a cohomology class $a \in \mathrm{H}^*(\mathrm{BGL}_d)$, we can define a characteristic class λ_a by setting $\lambda_a(p) := f_p^*(a)$. It is easy to see that this is a bijection. Thus understanding characteristic classes is equivalent to understanding the cohomology of the classifying space. For both real and complex vector bundles, this cohomology has been computed completely (see e.g. [Bro82]).

Let us now leave the realm of vector bundles and turn to bundles whose fibers are smooth manifolds and whose structure group is the diffeomorphism group of the manifold. Such bundles already arise in the study of vector bundles when one considers the unit sphere in each fiber. Another class of examples are *surface bundles*, whose fibers are (oriented) surfaces. These occur naturally in the study of 3- and 4-manifolds (see e.g. [Thu86; Ago13]). A short survey can be found in [ST20].

We will now focus on the latter example. The study of characteristic classes of (oriented) surface bundles, and hence of the classifying spaces

$\mathcal{M}_g \coloneqq \mathrm{BDiff}^+(\Sigma_g)$ of the groups of orientation preserving diffeomorphisms of an oriented surface of genus g (which are also sometimes called the *moduli spaces* of Riemann surfaces), has been of considerable interest for a long time. However, much of their cohomology still remains mysterious; for example, Harer–Zagier [HZ86] computed their Euler characteristics and showed that they grow super-exponentially in g, but no family of classes is known that would explain this behavior.

On the other hand, significant progress has been made in cohomological degrees that are small compared to g. One important result is the *stability theorem* asserting that the *stabilization maps*

$$\begin{aligned} \mathrm{H}^k\big(\mathrm{BDiff}_\partial(\Sigma_{g+1,1})\big) &\longrightarrow \mathrm{H}^k\big(\mathrm{BDiff}_\partial(\Sigma_{g,1})\big) \\ \mathrm{H}^k\big(\mathrm{BDiff}^+(\Sigma_g)\big) &\longrightarrow \mathrm{H}^k\big(\mathrm{BDiff}_\partial(\Sigma_{g,1})\big) \end{aligned}$$

are isomorphisms for k smaller than roughly $\frac{2}{3}g$, the *stable range.* Here $\Sigma_{g,r}$ denotes a genus g surface with r boundary components, and the subscript ∂ indicates that the diffeomorphisms fix a neighborhood of the boundary pointwise (which implies that they are orientation preserving). The maps are induced by gluing on a copy of $\Sigma_{1,2}$ respectively $\Sigma_{0,1} \cong \mathrm{D}^2$. This theorem was first proven by Harer [Har85] with a lower stable range; for a modern treatment, including the improved version we stated, see [Wah13].

The stability theorem implies that to understand the cohomology of $\mathcal{M}_g$ in the stable range, it is enough to understand the *stable cohomology*, i.e. the limit

$$\lim_{g\in\mathbb{N}} \mathrm{H}^*\big(\mathrm{BDiff}_\partial(\Sigma_{g,1})\big) \cong \mathrm{H}^*\big(\operatorname*{colim}_{g\in\mathbb{N}} \mathrm{BDiff}_\partial(\Sigma_{g,1})\big)$$

of the stabilization maps. With rational coefficients this was conjectured by Mumford [Mum83] to be isomorphic to

$$\mathbb{Q}[\kappa_i \mid i \geq 1]$$

where κ_i is the *Miller–Morita–Mumford class* of degree $2i$, which admits an explicit description in terms of the Euler class of the tangent bundle of the surface. This conjecture was proven in celebrated work of Madsen–Weiss [MW07] (their proof actually even yields an explicit description of the integral cohomology, which is more complicated).

Considering these result, one natural question is whether any of this generalizes to higher dimensions. To this end, let $n \geq 3$ and write

$$M^{2n}_{g,1} \coloneqq \mathop{\#}_{g}(\mathrm{S}^n \times \mathrm{S}^n) \setminus \mathring{\mathrm{D}}^{2n}$$

for the g-fold connected sum of products of the n-sphere, with one open disk removed. As before, we can define stabilization maps

$$\mathrm{H}^k\big(\mathrm{BDiff}_\partial(M^{2n}_{g+1,1})\big) \longrightarrow \mathrm{H}^k\big(\mathrm{BDiff}_\partial(M^{2n}_{g,1})\big)$$

which also have been shown to induce isomorphisms in a certain stable range by Galatius–Randal-Williams [GR17]. Moreover, the same authors [GR14] proved an analogue of the Madsen–Weiss theorem for these manifolds, again yielding an explicit description of the stable rational cohomology of the classifying space as a polynomial ring on certain "generalized Miller–Morita–Mumford" classes.

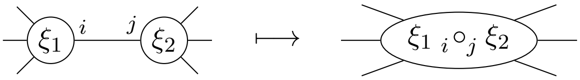
In a similar realm falls recent work of Berglund–Madsen [BM20]. They consider the same manifolds $M^{2n}_{g,1}$, but consider *homotopy automorphisms* instead of diffeomorphisms; that is, continuous maps which have an inverse up to homotopy. The corresponding classifying space $\mathrm{Baut}_\partial(M^{2n}_{g,1})$ classifies certain fibrations with fiber $M^{2n}_{g,1}$ (this can be thought of as being similar to a bundle, except that they are locally only "homotopically trivial"), see [HL15, Appendix B]. For these spaces, Berglund–Madsen prove a rational stability result analogous to the ones above and identify the rational stable cohomology (though using very different methods than Galatius–Randal-Williams).

Their identification takes the form

$$\lim_{g\in\mathbb{N}} \mathrm{H}^*\big(\mathrm{Baut}_\partial(M^{2n}_{g,1});\mathbb{Q}\big) \cong \mathrm{H}^*(\Gamma_\infty;\mathbb{Q}) \otimes \mathrm{H}^*\big(\mathfrak{UG}^{2n}(\mathcal{L}\mathrm{ie})^\vee\big)$$

where Γ_∞ is the infinite orthogonal group $\operatorname{colim}_{g\in\mathbb{N}} \mathrm{O}_{g,g}(\mathbb{Z})$ if n is even, and the infinite symplectic group $\operatorname{colim}_{g\in\mathbb{N}} \mathrm{Sp}_{2g}(\mathbb{Z})$ if n is odd. Their rational cohomologies have been identified explicitly by Borel [Bor74] as certain polynomial algebras. The other tensor factor is the cohomology of the *Lie graph complex* $\mathfrak{UG}^{2n}(\mathcal{L}\mathrm{ie})$. This is an object that was first described by Kontsevich [Kon93; Kon94] and has since then been studied by many different authors, see e.g. [GK98; CV03; LV08].

The Lie graph complex $\mathfrak{UG}^m(\mathcal{L}\mathrm{ie})$ is generated as a graded rational vector space by isomorphism classes of graphs (potentially with loops and multiple edges) whose vertices have valence at least three and which are labeled by elements of the (cyclic) Lie operad. This is quotiented by the action of the automorphism groups of the graphs. To specify this action (and the grading) precisely, we should think of a vertex as having homological degree $1-m$ and of an edge as having degree m; additionally we obtain a sign $(-1)^{m+1}$ whenever the orientation of an edge is flipped. The differential of the graph complex is given by the sum over all ways to contract a non-loop edge; the resulting new vertex is labeled by the (cyclic) operadic composition of the labels of the two old vertices, as in the following figure.



The homology of this graph complex is an object of significant interest, not the least because of its connection to the rational homology of the groups $\mathrm{Out}(\mathrm{F}_g)$ of outer automorphisms of the free group on g generators; this was observed by Kontsevich [Kon93] and worked out in detail by Conant–Vogtmann [CV03] in the case that m is even, and by Lazarev–Voronov [LV08] in the case that m is odd. However, knowledge about these homology groups is limited; see [CHKV16; BW24] for a summary of what is known.

Let us now turn away from $M^{2n}_{g,1}$ and consider the more general manifolds

$$M^{k,l}_{g,1} := \mathop{\#}_{g}(\mathrm{S}^k \times \mathrm{S}^l) \setminus \mathring{\mathrm{D}}^{k+l}$$

which in particular also exist in odd dimensions. In the cases $4 \leq k < l \leq 2k-3$, a stability result for the integral cohomology of $\mathrm{BDiff}_\partial(M^{k,l}_{g,1})$ was proven by Perlmutter [Per15]. However, when $k+l$ is odd, the stable cohomology remains mysterious even rationally. This includes in particular the case $l = k+1$ which has been of particular interest since the results of Galatius–Randal-Williams in the $l = k$ case. In this situation important steps of the stable identification have been carried out by Hebestreit–Perlmutter [HP19]; however for one of the steps no analogue exists yet, even though a lot of work in recent years has been put towards resolving this. In fact, no generally accepted conjecture what the result should be appears to exist; the most straightforward generalization of the even-dimensional case has been shown to be false by Ebert [Ebe13]. The only result known to the author is a recent computation of the stable rational cohomology in degrees up to about k, due to Ebert–Reinhold [ER24]. See further below for an outlook on future work, building on the results of this thesis, that significantly extends this computation; in fact, this was a major motivation for carrying out the work described here.

Turning again to homotopy automorphisms, rational stability for the cohomology of $\mathrm{Baut}_\partial(M^{k,l}_{g,1})$ has been shown by Grey [Gre19] in the cases $3 \leq k < l \leq 2k-2$. Identifying the stable rational cohomology is the content of Paper I.

2. Summary of Paper I

We prove the following main theorem (as well as a version for cohomology with certain local coefficients).

Theorem. *There is an isomorphism of graded algebras*

$$\lim_{g\in\mathbb{N}} \mathrm{H}^*\big(\mathrm{Baut}_\partial(M_{g,1}^{k,l});\mathbb{Q}\big) \cong \mathrm{H}^*\big(\mathrm{GL}(\mathbb{Z});\mathbb{Q}\big) \otimes \mathrm{H}^*\big(\mathfrak{UG}^{k+l-2}(\mathcal{L}\mathrm{ie})^\vee\big)$$

where $\mathrm{GL}(\mathbb{Z}) \coloneqq \mathrm{colim}_{g\in\mathbb{N}} \mathrm{GL}_g(\mathbb{Z})$.

Noting that the cohomology of $\mathrm{GL}(\mathbb{Z})$ has an easy explicit description as a polynomial algebra by work of Borel [Bor74], this yields a description of the cohomology of $\mathrm{Baut}_\partial(M_{g,1}^{k,l})$ in terms of the cohomology of the Lie graph complex.

While the statement is analogous to the theorem of Berglund–Madsen mentioned above (for the case $k = l$), the proof requires an extra step in addition to adapting their methods to a different situation. Using results of Grey [Gre19] and Li–Sun [LS19], as well as Paper II below[1], we reduce the problem to computing the cohomology of the $\mathrm{GL}_g(\mathbb{Q})$-invariants of the Chevalley–Eilenberg chains of a certain graded Lie algebra. However, in contrast to the work of Berglund–Madsen, the result is most naturally expressed in terms of a graph complex based on *directed* graphs. To arrive at the description we give above, we prove that the homology of this directed graph complex is isomorphic to the homology of $\mathfrak{UG}^{k+l-2}(\mathcal{L}\mathrm{ie})$ (the argument we use for this is similar to one sketched by Willwacher [Wil14]).

3. Interlude

In Paper II, summarized below, we provide, in particular, an important input for the proof of the main result of Paper I. There, we require an isomorphism of graded algebras, compatible with the stabilization maps,

$$\mathrm{H}^*\big(\mathrm{Baut}_\partial(M_{g,1}^{k,l});\mathbb{Q}\big) \cong \mathrm{H}^*\big(\mathrm{GL}_g(\mathbb{Z});\mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_g)\big) \tag{1}$$

where $\mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_g)$ denotes the Chevalley–Eilenberg cohomology of a certain graded Lie algebra $\mathfrak{g}_g$. More precisely, it is (a truncation of) the graded Lie algebra $\mathrm{Der}(\mathbb{L}V_g \parallel \omega)$ of derivations, annihilating a certain element ω, of the free graded Lie algebra on a shift $V_g \coloneqq \mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(M_{g,1}^{k,l};\mathbb{Q})$ of the reduced homology of the manifold.

---

[1] At the time of publication of Paper I, Paper II had not yet appeared (in the beginning, it was planned to be an appendix of the former, but it became too long). The results of Paper II are cited as upcoming work in Proposition 5.3; in that proposition the condition that the Γ_g-module P is finite-dimensional was erroneously omitted. This is of no consequence for the rest of Paper I.

Isomorphisms of a form similar to this have recently been obtained by Berglund–Zeman [BZ22]. Given a simply connected finite CW-complex X, they prove that

$$\mathrm{H}^*\big(\mathrm{Baut}(X);\mathbb{Q}\big) \cong \mathrm{H}^*\big(\mathcal{R}(X);\mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}_X)\big)$$

where $\mathfrak{g}_X$ is a certain dg Lie algebra of (curved) derivations of a minimal dg Lie model L_X of X, and $\mathcal{R}(X)$ is a quotient of the group $\mathcal{E}(X) \coloneqq \pi_0(\mathrm{aut}(X))$ of homotopy classes of homotopy automorphisms of X, which acts on $\mathfrak{g}_X$ by conjugation via a non-canonically chosen map $\mathcal{R}(X) \to \mathrm{Aut}(L_X)$. Underlying this isomorphism is a result stating that $\mathfrak{g}_X$ is an equivariant rational model for the covering space of $\mathrm{Baut}(X)$ associated to an explicit normal subgroup $U \subseteq \mathcal{E}(X)$, equipped with the action of the deck transformation group $\mathcal{R}(X) \coloneqq \mathcal{E}(X)/U$.

Results of this form for the universal covering of $\mathrm{Baut}(X)$ go back to Sullivan [Sul77] and Tanré [Tan83]. More recently, Félix–Fuentes–Murillo [FFM22] generalized this to a larger class of coverings. However, neither of these incorporates the deck transformation action; for the universal covering this was done by Lazarev [Laz14] on the level of homotopy groups. That, for a certain covering, this is possible on the space level was the main insight of Berglund–Zeman (rediscovering and making precise an old idea of Sullivan [Sul77, p. 314]).

In the relative situation, a model for the universal covering of $\mathrm{Baut}_A(X)$ was obtained by Berglund–Saleh [BS20], and Félix–Fuentes–Murillo [FFM23] again generalized this to a larger class of coverings. As above, neither of these incorporates the deck transformation action and thus neither yields a model for the classifying space itself.

4. Summary of Paper II

This paper is joint work with Alexander Berglund. We develop an analogue of his work with Zeman [BZ22] in the relative situation, i.e. for the classifying space $\mathrm{Baut}_A(X)$ of the space of homotopy automorphisms of a simply connected finite CW-complex X that fix a simply connected subcomplex A pointwise. We provide an explicit normal subgroup $U \subseteq \mathcal{E}_A(X) \coloneqq \pi_0(\mathrm{aut}_A(X))$ and show that the covering space of $\mathrm{Baut}(X)$ associated to U together with the action of the deck transformation group $\mathcal{R}_A(X) \coloneqq \mathcal{E}_A(X)/U$ is modeled by an explicit dg Lie subalgebra $\mathrm{Der}^{\mathrm{u}}(L_X \parallel L_A)$ of $\mathrm{Der}(L_X \parallel L_A)$ together with the conjugation action via a non-canonical map $\mathcal{R}_A(X) \to \mathrm{Aut}_{L_A}(L_X)$. Here $L_A \to L_X$ is a minimal relative dg Lie model (in the sense introduced by Espic–Saleh [ES20]) for the inclusion $A \to X$, and $\mathrm{Der}(L_X \parallel L_A)$ denotes the dg Lie algebra of derivations of

L_X that vanish on L_A. From this we deduce, among other results, the following.

Theorem. *There is an isomorphism*

$$\mathrm{H}^*\big(\mathrm{Baut}_A(X);\mathbb{Q}\big) \cong \mathrm{H}^*\big(\mathcal{R}_A(X);\mathrm{H}^*_{\mathrm{CE}}(\mathrm{Der}^{\mathrm{u}}(L_X \parallel L_A))\big)$$

of graded algebras.

In the case that X is an oriented simply connected n-dimensional manifold (fulfilling some extra conditions) with boundary $A := \partial M$ homeomorphic to S^{n-1}, we furthermore relate these constructions to the dg Lie algebra $\mathrm{Der}(\mathbb{L}V \parallel \omega)$ mentioned in Section 3 above, where $V := \mathrm{s}^{-1}\widetilde{\mathrm{H}}_*(X;\mathbb{Q})$. This yields the isomorphism (1) as a special case. We also prove that these isomorphisms are compatible with taking boundary connected sums, implying in particular a compatibility with the stabilization maps $\mathrm{aut}_\partial(M^{k,l}_{g,1}) \to \mathrm{aut}_\partial(M^{k,l}_{g+1,1})$.

To do this, we show that, under certain conditions, the more general equivariant rational models we produce are compatible with two types of constructions: the map

$$\mathrm{aut}_B(X) \times \mathrm{aut}_C(Y) \longrightarrow \mathrm{aut}_{B \amalg_A C}(X \amalg_A Y)$$

given by taking pushouts under a common subspace A of B and C, and the forgetful map

$$\mathrm{aut}_B(X) \longrightarrow \mathrm{aut}_A(X)$$

restricting the fixed subspace to a smaller one. Much of the paper is devoted to making this precise. As these constructions only make sense in the relative situation, they have no analogue in the work of Berglund–Zeman [BZ22].

Apart from their work, this paper also heavily relies on results of Espic–Saleh [ES20], Berglund–Saleh [BS20], as well as Lindell–Saleh [LS21].

5. Outlook

As explained at the end of Section 1, computing the stable rational cohomology of the classifying spaces $\mathrm{BDiff}_\partial(U^k_{g,1})$ of the odd-dimensional manifolds $U^k_{g,1} := M^{k,k+1}_{g,1}$ is of significant interest, and so far the only known result is the computation of Ebert–Reinhold [ER24] in degrees up to about k. Their approach is to first compute the stable cohomology of the related *block classifying spaces* $\mathrm{B}\widetilde{\mathrm{Diff}}_\partial(U^k_{g,1})$ in the same range. These spaces sit between $\mathrm{BDiff}_\partial(U^k_{g,1})$ and $\mathrm{Baut}_\partial(U^k_{g,1})$, and were invented to approximate the former. In particular, using classical methods as well as a recent result of Krannich [Kra22], knowledge of its cohomology in a range allows to compute the cohomology of $\mathrm{BDiff}_\partial(U^k_{g,1})$ in the same range, up to a maximum of roughly $2k$.

An ongoing project of the author, building on Paper I and work of Berglund–Madsen [BM20], is concerned with computing the full stable cohomology of these block classifying spaces, again in terms of a graph complex. This also relies on an upcoming generalization (again joint with Berglund) of Paper II to classifying spaces of homotopy automorphisms of certain types of bundles ξ over a CW-complex X; by a result of Berglund–Madsen, this recovers the block classifying space by taking ξ to be the stable tangent bundle of the manifold. As explained above, this computation then enables one to obtain the cohomology of $\mathrm{BDiff}_\partial(U^k_{g,1})$ up to degree about $2k$. One particularly interesting part of this work is to see how the known relations in this cohomology, found by Ebert [Ebe13] and Ebert–Reinhold [ER24], appear from the point of view of graph complexes. In fact, these applications were one of the main motivations for Papers I and II to begin with.

6. Interlude

In Paper I, graph complexes (more precisely the Lie graph complex) played an important role. Since their introduction by Kontsevich [Kon93; Kon94], these have been studied intensively. One important systematic treatment is the work of Getzler–Kapranov [GK98] on *modular operads*. In the last paper contained in this thesis, we give a new definition of modular operads and the *Feynman transform*; the latter constructs a modular operad from another and generalizes the graph complexes of Kontsevich. Let us begin by explaining what a modular operad is.

First recall that a $\mathbf{\Sigma}$-module A is a sequence of rational chain complexes $(A(n))_{n\in\mathbb{N}_0}$ such that A_n is equipped with an action of the symmetric group Σ_n. Classically, modular operads arise from a certain functor $\mathbb{M}$ from the category of $\mathbf{\Sigma}$-modules to itself: the underlying chain complex of $\mathbb{M}(A)(n)$ is spanned by isomorphism classes of connected graphs (potentially with multiple edges and loops) with n hairs (i.e. distinguished vertices of valence 1) whose non-hair vertices v are decorated by elements of $A(|v|)$, where $|v|$ denotes the valence of v. The hairs are labeled from 1 to n and Σ_n acts by permuting the labels. The result is quotiented by the action of those automorphisms of the graphs that fix the hairs. There is a canonical map $\mathbb{M}(\mathbb{M}(A)) \to \mathbb{M}(A)$ given by "flattening" (or "grafting") an element of $\mathbb{M}(\mathbb{M}(A))$, which is represented by a graph whose vertices are labeled by graphs, to a single graph. This natural transformation equips $\mathbb{M}$ with the structure of a monad. Algebras over this monad are modular operads. (Actually, modular operads are equipped with an extra “genus grading” which keeps track of the genus of the graphs; we omit this for the sake of exposition.) Unwinding this definition, we see that a modular operad is a $\mathbf{\Sigma}$-module M together with a “composition operation” that takes a connected graph that is labeled by M and has n hairs, and returns an element of $M(n)$. This composition operation is then further required to fulfill a certain associativity property.

One example of a modular operad is the collection

$$\Big(\bigoplus_{g} \mathrm{H}^*\big(\mathrm{BDiff}_{\partial}(\Sigma_{g,r})\big)\Big)_{r\in\mathbb{N}_0}$$

where the composition is given by gluing the surfaces along their boundary components (here the extra genus grading is given by the actual genus g). This is sometimes called the “hypercommutative” modular operad. (This example was the motivation for the name “modular” operad, since it is constructed from moduli spaces.)

We also note that every cyclic operad can be considered to be a modular operad by letting all compositions along non-tree graphs be trivial (this could also be used as the definition of a cyclic operad). The construction of the Lie graph complex $\mathfrak{UG}^m(\mathcal{L}\mathrm{ie})$ above can be generalized to any cyclic operad instead of $\mathcal{L}\mathrm{ie}$, and in fact, keeping the preceding observation in mind, to any modular operad. To make this precise, we need to introduce a certain “twisted” analogue of $\mathbb{M}$.

To this end, let $\mathfrak{D}$ be a functor from the category of connected graphs with hairs to chain complexes (together with some extra structure we will not dwell on); this is called a *hyperoperad*. Then we can define $\mathbb{M}_{\mathfrak{D}}$ to be a “twisted” version of $\mathbb{M}$ where a labeled graph G is additionally decorated by an element of $\mathfrak{D}(G)$ (and $\mathrm{Aut}(G)$ acts both on G and on $\mathfrak{D}(G)$). Setting

$$\mathfrak{C}_m(G) := \bigotimes_{e\in\mathrm{Edge}(G)} \mathrm{s}^m \,\mathrm{Or}(e)^{\otimes m+1}$$

where s denotes a degree shift and $\mathrm{Or}(e)$ is the one-dimensional vector space of orientations of e, we see that the underlying graded vector space of the subcomplex $\mathfrak{UG}^m_{\mathrm{conn}}(\mathcal{L}\mathrm{ie}) \subset \mathfrak{UG}^m(\mathcal{L}\mathrm{ie})$, spanned by the connected graphs, is isomorphic to $\mathbb{M}_{\mathfrak{C}_m}(\mathrm{s}^{1-m}\,\mathcal{L}\mathrm{ie})(0)$. For any modular operad M, it is possible to equip $\mathbb{M}_{\mathfrak{C}_m}(\mathrm{s}^{1-m}M)$ with a differential in the same way as for the graph complex: it is defined as a sum over all ways to contract an edge where the new vertex is labeled by the composition of the two old vertices. In the case $M = \mathcal{L}\mathrm{ie}$ this recovers the differential of $\mathfrak{UG}^m(\mathcal{L}\mathrm{ie})$.

This fits into a more general framework. To explain this, let

$$\mathfrak{K}(G) \coloneqq \bigotimes_{e \in \mathrm{Edge}(G)} \mathrm{s}^{-1}\mathbb{Q}$$

and $\mathfrak{D}$ some hyperoperad such that $\mathfrak{D}(G)$ is one-dimensional for all G (this is called a *cocycle*). Then one can define a functor

$$\begin{aligned} \mathsf{F}_{\mathfrak{D}} \colon \mathbf{ModOp}_{\mathfrak{D}} &\longrightarrow \mathbf{ModOp}_{\mathfrak{K}\otimes\mathfrak{D}^\vee} \\ M &\longmapsto \mathbb{M}_{\mathfrak{K}\otimes\mathfrak{D}^\vee}(M^\vee) \end{aligned}$$

where $\mathbf{ModOp}_{\mathfrak{D}}$ denotes the category of *modular* $\mathfrak{D}$*-operads*, i.e. algebras over the monad $\mathbb{M}_{\mathfrak{D}}$. The differential on $\mathbb{M}_{\mathfrak{K}\otimes\mathfrak{D}^\vee}(M^\vee)$ is defined dually to the one mentioned above. In particular we have

$$\mathfrak{UG}^m_{\mathrm{conn}}(\mathcal{L}\mathrm{ie})^\vee \cong \mathsf{F}_{\mathfrak{K}\otimes\mathfrak{C}_m}(\mathrm{s}^{1-m}\,\mathcal{L}\mathrm{ie})$$

(using implicitly that $\mathrm{s}^{1-m}\mathcal{C}$ is a modular $(\mathfrak{K}\otimes\mathfrak{C}_m)$-operad for any cyclic operad $\mathcal{C}$). The functor $\mathsf{F}_{\mathfrak{D}}$ is called the *Feynman transform*. It has some nice properties. First of all $\mathsf{F}_{\mathfrak{D}}$ preserves quasi-isomorphisms. Secondly, for any modular $\mathfrak{D}$-operad M, there is a natural quasi-isomorphism $\mathsf{F}_{\mathfrak{K}\otimes\mathfrak{D}^\vee}(\mathsf{F}_{\mathfrak{D}}(M)) \to M$. In particular the Feynman transform is an equivalence of categories up to quasi-isomorphism.

This is reminiscent of the classical bar and cobar constructions of associative algebras (or, more generally, algebras over an operad). Let us recall this now (omitting some technicalities) following the account of Loday–Vallette [LV12]. There is an adjoint pair of functors

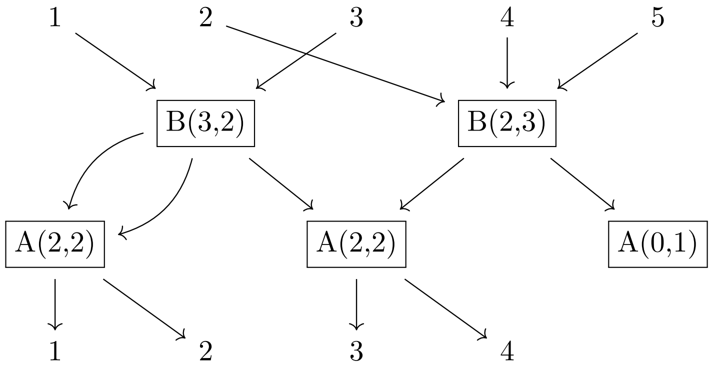
$$\begin{tikzcd} \mathbf{Alg} \arrow[r, bend left, "\mathrm{B}"] \arrow[r, phantom, "\top"] & \mathbf{CoAlg} \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

where **Alg** and **CoAlg** denote the categories of differential graded (co)associative (co)algebras. The bar construction $\mathrm{B}(A)$ of an algebra A is defined by taking the cofree coassociative coalgebra $\mathrm{T}^{\mathrm{c}}(\mathrm{s}A) = \bigoplus_n (\mathrm{s}A)^{\otimes n}$ and equipping it with the differential defined by taking the sum over all ways of multiplying two adjacent elements. The cobar construction Ω is defined dually. They again enjoy some nice properties (under some weak assumptions). First of all, both B and Ω preserve quasi-isomorphisms. Secondly, both the unit $C \to \mathrm{B}(\Omega(C))$ and the counit $\Omega(\mathrm{B}(A)) \to A$ are quasi-isomorphisms. In particular this yields an, often useful, way of constructing *quasi-free* resolutions (i.e. a quasi-isomorphic algebra whose underlying graded algebra, without the differential, is free).

In Paper III of this thesis, we prove that the similarity of the properties of the Feynman transform and the (co)bar construction of algebras over an operad is not just a coincidence by providing a common generalization. This also yield a new, relatively simple, definition of modular operads.

Various other treatments of modular operads and the Feynman transform exist in the literature. For example they appear as a special case of the theory of Feynman categories due to Kaufmann–Ward [KW17], of the theory of groupoid-colored operads by work of Ward [War21] (see also Dotsenko–Shadrin–Vaintrob–Vallette [DSVV24]), and of the theory of operadic categories due to Batanin–Markl [BM15; BM23b; BM23a]. A definition of modular operads (though not the Feynman transform) as presheaves on a category of graphs that fulfill a strict Segal condition has been given by Hackney–Robertson–Yau [HRY20a; HRY20b], and as certain strong symmetric monoidal functors by Costello [Cos04]. The approaches of Ward and Batanin–Markl are similar to the one we employ in the sense that, there too, modular operads appear as algebras over an operad-like object. Our definition is arguably simpler, however; the reason for this is that by virtue of our setup no relations need to be encoded in the governing object.

Our approach uses the theory of *properads* developed by Vallette [Val07] and Merkulov–Vallette [MV09]. A properad is a generalization of an operad for modeling algebraic structures which have operations with both multiple inputs and multiple outputs (such as a bialgebra). To be able to summarize the results of Paper III, we now sketch the definition of a properad. To this end, we first define a $\mathbf{\Sigma}$-bimodule to be a family $(A(n,m))_{n,m\in\mathbb{N}_0}$ such that $A_{n,m}$ is equipped with an action of $\Sigma_n \times (\Sigma_m)^{\mathrm{op}}$. We think of $A(n,m)$ as a collection of operations with m inputs and n outputs. There is a monoidal structure $\boxtimes$ on the category of $\mathbf{\Sigma}$-bimodules such that $A \boxtimes B$ is spanned by pictures of the following form



i.e. it is spanned by “connected, directed, 2-level graphs with sources and sinks” with vertices of the lower level labeled by elements of A and vertices of the upper level labeled by B. A properad is a monoid in this monoidal category. Unwinding the definition, this means that the elements of a properad can be composed along connected directed graphs without directed loops.

7. Summary of Paper III

Let $\mathfrak{t}$ be a Σ_2-module. We define the $\mathfrak{t}$-twisted *Brauer properad* $\mathcal{B}_\mathfrak{t}$ to be the free properad generated by $\mathfrak{t}$ in biarity $(0,2)$. We prove the following main theorem.

Theorem. *There is an equivalence of categories*

$$\Psi_\mathfrak{t} \colon \left\{ \begin{matrix} \textit{stable weight graded purely outgoing} \\ \textit{left modules over the Brauer properad } \mathcal{B}_\mathfrak{t} \end{matrix} \right\} \xrightarrow{\simeq} \{\textit{modular } \mathrm{h}(\mathfrak{t})\textit{-operads}\}$$

for a certain hyperoperad $\mathrm{h}(\mathfrak{t})$.

Moreover, under this equivalence, the Feynman transform corresponds to the bar construction of left modules over $\mathcal{B}_\mathfrak{t}$.

By a *left module* we mean here a left module in the monoidal category of $\boldsymbol{\Sigma}$-bimodules (equipped with the tensor product $\boxtimes$) over the monoid $\mathcal{B}_\mathfrak{t}$. Such a module M is *purely outgoing* if $M(m,n) \cong 0$ for $n > 0$. The condition of being “stable weight graded” corresponds to the genus grading on a modular operad we briefly mentioned above.

To make the second part of the theorem above precise, we extend the theory of the bar and cobar constructions to modules over a properad. We do this relative to a *twisting morphism*, which is a certain kind of map α from a coproperad $\mathcal{C}$ to a properad $\mathcal{P}$ (introduced by Merkulov–Vallette [MV09]). The statement we prove is the following (omitting some technical conditions).

Theorem. *There is an adjunction*

$$\begin{tikzcd}
\{\textit{modules over } \mathcal{P}\} \arrow[r, bend left, "\mathrm{B}_\alpha"] \arrow[r, phantom, "\top"] & \{\textit{comodules over } \mathcal{C}\} \arrow[l, bend left, "\Omega_\alpha"]
\end{tikzcd}$$

such that both B_α *and* Ω_α *preserve quasi-isomorphisms.*

Moreover $\mathrm{B}_\alpha\mathcal{P}$ *is acyclic if and only if* $\Omega_\alpha\mathcal{C}$ *is acyclic. In this situation both the unit* $K \to \mathrm{B}_\alpha\Omega_\alpha K$ *and the counit* $\Omega_\alpha\mathrm{B}_\alpha M \longrightarrow M$ *of the adjunction are quasi-isomorphisms for all* $\mathcal{C}$*-comodules* K *and* $\mathcal{P}$*-modules* M.

This generalizes the case of algebras over an operad due to Getzler–Jones [GJ94]. It also produces various constructions of Vallette [Val07] as special cases.

Bibliography

[Ago13] Ian Agol. “The virtual Haken conjecture”. In: *Documenta Mathematica* 18 (2013), pp. 1045–1087. DOI: `10.2307/2044298`.

[BM15] Michael Batanin and Martin Markl. “Operadic categories and duoidal Deligne’s conjecture”. In: *Advances in Mathematics* 285 (2015), pp. 1630–1687. DOI: `10.1016/j.aim.2015.07.008`.

[BM23a] Michael Batanin and Martin Markl. *Koszul duality for operadic categories.* 2023. DOI: `10.32408/compositionality-5-4`.

[BM23b] Michael Batanin and Martin Markl. “Operadic categories as a natural environment for Koszul duality”. In: *Compositionality* 5.3 (2023). DOI: `10.32408/compositionality-5-3`.

[BM20] Alexander Berglund and Ib Madsen. “Rational homotopy theory of automorphisms of manifolds”. In: *Acta Mathematica* 224.1 (2020), pp. 67–185. DOI: `10.4310/ACTA.2020.v224.n1.a2`.

[BS20] Alexander Berglund and Bashar Saleh. “A dg Lie model for relative homotopy automorphisms”. In: *Homology, Homotopy and Applications* 22.2 (2020), pp. 105–121. DOI: `10.4310/hha.2020.v22.n2.a6`.

[BZ22] Alexander Berglund and Tomáš Zeman. *Algebraic models for classifying spaces of fibrations.* Preprint. 2022. arXiv: `2203.02462v3`.

[BS53] A. Borel and J.-P. Serre. “Groupes de Lie et Puissances Reduites de Steenrod”. In: *American Journal of Mathematics* 75.3 (1953), pp. 409–448. DOI: `10.2307/2372495`.

[Bor74] Armand Borel. “Stable real cohomology of arithmetic groups”. In: *Annales scientifiques de l’École Normale Supèrieure* Ser. 4, 7.2 (1974), pp. 235–272. DOI: `10.24033/asens.1269`.

[Bro82] Edgar H. Brown. “The Cohomology of BSO_n and BO_n with Integer Coefficients”. In: *Proceedings of the American Mathematical Society* 85.2 (1982), pp. 283–288. DOI: `10.2307/2044298`.

[BW24] Simon Brun and Thomas Willwacher. “Graph homology computations”. In: *New York Journal of Mathematics* 30 (2024), pp. 58–92.

[CHKV16] James Conant, Allen Hatcher, Martin Kassabov, and Karen Vogtmann. “Assembling homology classes in automorphism groups of free groups”. In: *Commentarii Mathematici Helvetici* 91.4 (2016), pp. 751–806. DOI: `10.4171/cmh/402`.

[CV03] James Conant and Karen Vogtmann. “On a theorem of Kontsevich”. In: *Algebraic & Geometric Topology* 3.2 (2003), pp. 1167–1224. DOI: `10.2140/agt.2003.3.1167`.

[Cos04] Kevin Costello. *The A-infinity operad and the moduli space of curves*. Unpublished manuscript. 2004. arXiv: `math/0402015v2`.

[DSVV24] Vladimir Dotsenko, Sergey Shadrin, Arkady Vaintrob, and Bruno Vallette. “Deformation theory of cohomological field theories”. In: *Journal für die reine und angewandte Mathematik* (2024). DOI: `10.1515/crelle-2023-0098`.

[Ebe13] Johannes Ebert. “A vanishing theorem for characteristic classes of odd-dimensional manifold bundles”. In: *Journal für die reine und angewandte Mathematik* 2013.684 (2013), pp. 1–29. DOI: `10.1515/crelle-2012-0012`.

[ER24] Johannes Ebert and Jens Reinhold. “Some rational homology computations for diffeomorphisms of odd-dimensional manifolds”. In: *Journal of Topology* 17.1, e12324 (2024). DOI: `10.1112/topo.12324`.

[ES20] Hadrien Espic and Bashar Saleh. *On the group of homotopy classes of relative homotopy automorphisms*. Preprint. 2020. arXiv: `2002.12083v4`.

[FFM22] Yves Félix, Mario Fuentes, and Aniceto Murillo. “Lie models of homotopy automorphism monoids and classifying fibrations”. In: *Advances in Mathematics* 402, 108359 (2022). DOI: `10.1016/j.aim.2022.108359`.

[FFM23] Yves Félix, Mario Fuentes, and Aniceto Murillo. *Rational homotopy type of relative universal fibrations*. Preprint. 2023. arXiv: `2311.14132v1`.

[GR14] Søren Galatius and Oscar Randal-Williams. “Stable moduli spaces of high-dimensional manifolds”. In: *Acta Mathematica* 212.2 (2014), pp. 257–377. DOI: `10.1007/s11511-014-0112-7`.

[GR17] Søren Galatius and Oscar Randal-Williams. “Homological stability for moduli spaces of high dimensional manifolds. I”. In: *Journal of the American Mathematical Society* 31.1 (2017), pp. 215–264. DOI: `10.1090/jams/884`.

[GJ94] E. Getzler and J. D. S. Jones. *Operads, homotopy algebra, and iterated integrals for double loop spaces.* Unpublished manuscript. 1994. arXiv: `hep-th/9403055v1`.

[GK98] E. Getzler and M. M. Kapranov. “Modular operads”. In: *Compositio Mathematica* 110.1 (1998), pp. 65–125. DOI: `10.1023/A:1000245600345`.

[Gre19] Matthias Grey. “On rational homological stability for block automorphisms of connected sums of products of spheres”. In: *Algebraic & Geometric Topology* 19.7 (2019), pp. 3359–3407. DOI: `10.2140/agt.2019.19.3359`.

[HRY20a] Philip Hackney, Marcy Robertson, and Donald Yau. “A graphical category for higher modular operads”. In: *Advances in Mathematics* 365 (2020). DOI: `10.1016/j.aim.2020.107044`.

[HRY20b] Philip Hackney, Marcy Robertson, and Donald Yau. “Modular operads and the nerve theorem”. In: *Advances in Mathematics* 370 (2020). DOI: `10.1016/j.aim.2020.107206`.

[HZ86] J. Harer and D. Zagier. “The Euler characteristic of the moduli space of curves”. In: *Inventiones Mathematicae* 85.3 (1986), pp. 457–485. DOI: `10.1007/bf01390325`.

[Har85] John L. Harer. “Stability of the Homology of the Mapping Class Groups of Orientable Surfaces”. In: *Annals of Mathematics* 121.2 (1985), pp. 215–249. DOI: `10.2307/1971172`.

[HP19] Fabian Hebestreit and Nathan Perlmutter. “Cobordism categories and moduli spaces of odd dimensional manifolds”. In: *Advances in Mathematics* 353 (2019), pp. 526–590. DOI: `10.1016/j.aim.2019.07.002`.

[HL15] Richard Hepworth and Anssi Lahtinen. “On string topology of classifying spaces”. In: *Advances in Mathematics* 281 (2015), pp. 394–507. DOI: `10.1016/j.aim.2015.03.022`.

[KW17] Ralph M. Kaufmann and Benjamin C. Ward. “Feynman Categories”. In: *Astérisque* 387 (2017). DOI: `10.24033/ast.1015`.

[Kon93] Maxim Kontsevich. “Formal (Non)-Commutative Symplectic Geometry”. In: *The Gelfand Mathematical Seminars, 1990–1992.* Ed. by L. Corwin, I. Gelfand, and J. Lepowsky. Birkhäuser, 1993, pp. 173–187. DOI: `10.1007/978-1-4612-0345-2_11`.

[Kon94] Maxim Kontsevich. "Feynman Diagrams and Low-Dimensional Topology". In: *First European Congress of Mathematics.* Ed. by A. Joseph et al. Vol. II. Progress in Mathematics 120. Birkhäuser, 1994, pp. 97–121. DOI: `10.1007/978-3-0348-9112-7_5`.

[Kra22] Manuel Krannich. "A homological approach to pseudoisotopy theory. I". In: *Inventiones mathematicae* 227 (2022), pp. 1093–1167. DOI: `10.1007/s00222-021-01077-7`.

[Laz14] A. Lazarev. "Models for classifying spaces and derived deformation theory". In: *Proceedings of the London Mathematical Society* 109.1 (2014), pp. 40–64. DOI: `10.1112/plms/pdt069`.

[LV08] A. Lazarev and A. A. Voronov. "Graph homology: Koszul and Verdier duality". In: *Advances in Mathematics* 218.6 (2008), pp. 1878–1894. DOI: `10.1016/j.aim.2008.03.022`.

[LS19] Jian-Shu Li and Binyong Sun. "Low degree cohomologies of congruence groups". In: *Science China Mathematics* 62.11 (2019), pp. 2287–2308. DOI: `10.1007/s11425-018-9476-8`.

[LS21] Erik Lindell and Bashar Saleh. *Representation stability for homotopy automorphisms.* Preprint. To appear in Algebraic & Geometric Topology. 2021. arXiv: `2105.11325v3`.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads.* Grundlehren der mathematischen Wissenschaften 346. Springer-Verlag, 2012. DOI: `10.1007/978-3-642-30362-3`.

[MW07] Ib Madsen and Michael Weiss. "The stable moduli space of Riemann surfaces: Mumford's conjecture". In: *Annals of Mathematics* 165.3 (2007), pp. 843–941. DOI: `10.4007/annals.2007.165.843`.

[MV09] Sergei Merkulov and Bruno Vallette. "Deformation theory of representations of prop(erad)s I". In: *Journal für die reine und angewandte Mathematik* 2009.634 (2009), pp. 51–106. DOI: `10.1515/CRELLE.2009.069`.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic Classes.* Annals of Mathematics Studies 76. Princeton University Press, 1974. DOI: `10.1515/9781400881826`.

[Mum83] David Mumford. "Towards an Enumerative Geometry of the Moduli Space of Curves". In: *Arithmetic and Geometry.* Ed. by Michael Artin and John Tate. Progress in Mathematics 36. Birkhäuser, 1983, pp. 271–328. DOI: `10.1007/978-1-4757-9286-7_12`.

[Per15] Nathan Perlmutter. "Homological stability for the moduli spaces of products of spheres". In: *Transactions of the American Mathematical Society* 368.7 (2015), pp. 5197–5228. DOI: `10.1090/tran/6564`.

[ST20] Nick Salter and Bena Tshishiku. "Surface bundles in topology, algebraic geometry, and group theory". In: *Notices of the American Mathematical Society* 67.2 (2020). DOI: `10.1090/noti2016`.

[Sto22] Robin Stoll. "Homotopy automorphisms, graph complexes, and modular operads". Licentiate thesis. Stockholm University, 2022.

[Sul77] Dennis Sullivan. "Infinitesimal computations in topology". In: *Publications mathématiques de l'IHÉS* 47.1 (1977), pp. 269–331. DOI: `10.1007/bf02684341`.

[Tan83] Daniel Tanré. *Homotopie Rationnelle. Modèles de Chen, Quillen, Sullivan.* Lecture Notes in Mathematics 1025. Springer-Verlag, 1983. DOI: `10.1007/bfb0071482`.

[Thu86] William P Thurston. "A norm for the homology of 3-manifolds". In: *Memoirs of the American Mathematical Society* 339 (1986), pp. 99–130.

[Val07] Bruno Vallette. "A Koszul duality for props". In: *Transactions of the American Mathematical Society* 359.10 (2007), pp. 4865–4944. DOI: `10.1090/s0002-9947-07-04182-7`.

[Wah13] Nathalie Wahl. "Homological stability for mapping class groups of surfaces". In: *Handbook of Moduli.* Ed. by Gavril Farkas and Ian Morrison. Vol. 3. Advanced Lectures in Mathematics 26. International Press, 2013. arXiv: `1006.4476v2`.

[War21] Benjamin C. Ward. "Massey Products for Graph Homology". In: *International Mathematics Research Notices* 2022.11 (2021), pp. 8086–8161. DOI: `10.1093/imrn/rnaa346`.

[Wil14] Thomas Willwacher. "M. Kontsevich's graph complex and the Grothendieck–Teichmüller Lie algebra". In: *Inventiones mathematicae* 200.3 (2014), pp. 671–760. DOI: `10.1007/s00222-014-0528-x`.