Stockholm University

Operator models for meromorphic functions of bounded type

Christian Emmel

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Abstract

This thesis consists of four papers dealing with generalizations of de Branges`s model theory of (cyclic) self-adjoint operators and applications to the extension theory of symmetric operators.

The main project of this PhD thesis, consisting of three papers, is concerned with constructing operator models for meromorphic functions of bounded type. Specifically, it is shown that these functions can be realized as Q-functions of partially fundamentally reducible relations on Krein spaces in a minimal way. The main result can be found in Paper III, while Papers I and II contain related results of a smaller scope.

The main argument of our construction in Paper 3 can be used to generalize the extension theory for symmetric operators with deficiency index (1,1). Specifically, we characterize all one dimensional extensions with non-empty resolvent set of such an operator via a Krein-type resolvent formula and investigate their spectral properties. This is the content of Paper IV.

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Abstract

This thesis consists of four papers dealing with generalizations of de Branges‘s model theory of (cyclic) self-adjoint operators and applications to the extension theory of symmetric operators.

The main project of this PhD thesis, consisting of three papers, is concerned with constructing operator models for meromorphic functions of bounded type. Specifically, it is shown that these functions can be realized as Q-functions of partially fundamentally reducible relations on Krein spaces in a minimal way. The main result can be found in Paper III, while Papers I and II contain related results of a smaller scope.

The main argument of our construction in Paper 3 can be used to generalize the extension theory for symmetric operators with deficiency index $(1, 1)$. Specifically, we characterize all one dimensional extensions with non-empty resolvent set of such an operator via a Krein-type resolvent formula and investigate their spectral properties. This is the content of Paper IV.

Sammanfattning

Denna avhandling består av fyra artiklar som handlar om generaliseringar av de Branges modellteori för (cykliska) själv-adjungera operatorer och tillämpningar på extensionsteorin för symmetriska operatorer.

Huvuddelen av denna avhandling, som består av tre artiklar, handlar om att konstruera operatormodeller för meromorfa funktioner av bounded typ. Specifikt visas att dessa funktioner kan realiseras som Q-funktioner av delvis fundamentalt reducerbara relationer på Krein rum i ett minimalt sätt. Huvudresultatet finns i Artikle III, medan Artikle I och II innehåller delresultat.

Huvudargumentet för vår konstruktion i Artikle III kan användas för att eralisera extensionsteorin för symmetriska operatorer med index (1,1). Specifikt karaktäriserar vi alla endimensionella extension med en icke-tom resolventmängd för en sådan operator via en Krein-typ resolventformel och undersöker deras spektral egenskaper. Detta är innehållet i Artikle 4.

List of papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

Paper I: **Realizations of meromorphic functions of bounded type**
Christian Emmel and Annemarie Luger, *Operator Theory: Advances and Applications, vol. 291 (06/2023), Pages: 501-522 DOI: 10.1007/978-3-031-31139-0_18*

Paper II: **Minimal Realizations of atomic density functions**
Christian Emmel, *Complex Analysis and Operator Theory, vol 17(5) (07/2023), article 52, DOI: 10.1007/s11785-023-01360-w.*

Paper III: **Operator models for meromorphic functions of bounded type**
Christian Emmel, *accepted for publication in: Transactions of the American Mathematical Society, Preprint:arXiv:2401.05214.*

Paper IV: **A generalization of Krein's extension formalism for symmetric relations with deficiency index (1,1)**
Christian Emmel, *submitted, Preprint: arXiv:2407.07491.*

Reprints were made with permission from the publishers. Paper I and a preprint version of Paper II were already included in the licentiate thesis [6]

Contents

Chapter 1

General introduction

The numerous intersections between complex analysis and operator theory have inspired a lot of research in the last century. One important example of such an interdisciplinary area is de Branges's model theory, which connects reproducing kernel Hilbert spaces, symmetric and self-adjoint operators and Herglotz-Nevanlinna functions. The goal of this thesis is to generalize this model theory. Specifically, we will show that meromorphic functions of bounded type admit models generated by partially fundamentally reducible relations on Krein spaces.

The purpose of this introduction is to discuss the necessary preliminaries and contextualize our main results. We begin with motivating the key concepts of this thesis by considering Q-functions associated to hermitian matrices in this chapter. The second chapter is dedicated to de Branges's model theory for self-adjoint operators on Hilbert spaces. Moving forward, in the subsequent third chapter we discuss reproducing kernel Hilbert and Krein spaces, which provide the natural language for this thesis. The fourth chapter is concerned with the function-theoretic side, namely with meromorphic functions of bounded type. Chapter 5 then deals with the operator-theoretic side. There we will introduce and discuss partially fundamentally relations, which extend the notion of self-adjoint relations. Finally, this introduction is completed by a discussion of our main results in Chapter 6.

We point out that Chapter 2 and Chapter 4 were adapted from the licentiate thesis. This means that these chapters were significantly reworked, but have roughly the same structure as in the licentiate thesis [6]. Furthermore, Papers I and II were also included in the licentiate thesis.

With that being said, let us turn to Q-functions associated to Hermitian matrices next. Let $A \in \mathbb{C}^{n\times n}$ be a hermitian matrix, $\sigma(A)$ its eigenvalues and $u \in \mathbb{C}^n$ an element. Moreover, let $[\cdot,\cdot]_{\mathbb{C}^n}$ denote the usual inner product on $\mathbb{C}^n$. The associated Q-function is then defined as

$$q(\zeta) = \big[(A-\zeta)^{-1}u, u\big]_{\mathbb{C}^n}, \quad \zeta \in \mathbb{C}\setminus\sigma(A) \tag{1.1}$$

The expression (1.1) establishes a connection between two conceptually different objects, namely the function q and the matrix A. We would like to use this connection to investigate the matrix A with the help of the function q and vice versa. However, if $u \in \mathbb{C}^n$ is chosen completely arbitrary then this Q-function is not particularly useful, as the choice $u = 0$ immediately shows. Thus, in order to obtain a rich theory we additionally assume that the element u is cyclic, which means that

$$\operatorname{span}\{u, Au, A^2u, ...\} = \mathbb{C}^n. \tag{1.2}$$

Let us first investigate what type of function q can appear. Since A is a matrix, it follows that q is a rational analytic function. Next, since A is furthermore assumed to be hermitian, it follows that

$$\begin{aligned} q(\zeta) = \big[(A-\zeta)^{-1}u, u\big]_{\mathbb{C}^n} &= \big[u, (A-\overline{\zeta})^{-1}u\big]_{\mathbb{C}^n} \\ &= \overline{\big[(A-\overline{\zeta})^{-1}u, u\big]_{\mathbb{C}^n}} = \overline{q(\overline{\zeta})}. \end{aligned}$$

Moreover, using the resolvent identity

$$(A-w)^{-1} - (A-z)^{-1} = (w-z)(A-w)^{-1}(A-z)^{-1}$$

we compute for an arbitrary element z in the upper-half plane $\mathbb{C}^+$

$$\begin{aligned} \operatorname{Im}(q(z)) &= 2i\cdot(q(z) - \overline{q(z)}) = 2i\cdot(q(z) - q(\overline{z})) \\ &= 2i\cdot\big(\big[(A-z)^{-1}u, u\big]_{\mathbb{C}^n} - \big[(A-\overline{z})^{-1}u, u\big]_{\mathbb{C}^n}\big) \\ &= 2i\cdot(z-\overline{z})\,\big[(A-\overline{z})^{-1}(A-z)^{-1}u, u\big]_{\mathbb{C}^n} \\ &= \operatorname{Im}(z)\cdot\big[(A-z)^{-1}u, (A-z)^{-1}u\big]_{\mathbb{C}^n} \\ &= \operatorname{Im}(z)\cdot\|(A-z)^{-1}u\| > 0. \end{aligned}$$

In summary, we have established that the function q satisfies $q(\overline{\zeta}) = \overline{q(\zeta)}$ and $q(\mathbb{C}^+) \subset \overline{\mathbb{C}^+}$, which means that q is a rational Herglotz-Nevanlinna function. The good news here is that these functions have been around for a century now and are very well understood [24, Chapter 3]. In particular, we know that these functions can only have poles on the real line and that they are at most of order one.

Now let us see how we can use our acquired knowledge to investigate the eigenvalues of A. First, it is clear by construction that if q has a pole at $w \in \mathbb{R}$, then A must have an eigenvalue there. A more involved analysis using our cyclicity assumption shows that the order of the poles of q coincides with the algebraic multiplicities of the eigenvalues of A. This means that the spectrum of A is completely characterized by the function-theoretic properties of q. In particular, since q can only have poles of order one, we see that hermitian matrices having a cyclic element can only have eigenvalues of multiplicity one.

While this theory is already quite interesting in finite dimensions, its true value comes from its infinite dimensional version. This is the subject of the next chapter.

Chapter 2

De Branges's functional model

In this second chapter, we discuss the key concepts of de Brange's functional model for (cyclic) self-adjoint operators. The original source for this theory is the fundamental book [5], while a more modern and complete account of this theory is given in [4, Chapter 4].

Let $A : \mathrm{dom}(A) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let us denote the inner product on $\mathcal{H}$ by $[\cdot,\cdot]_{\mathcal{H}}$. At the center of de Branges's model theory lies the associated Q-function, which we have already encountered in the context of hermitian matrices. However, in infinite dimensions we have to take some extra care in defining it, since self-adjoint operators are usually only defined on a dense subset. More precisely, for a given element $v \in \mathcal{H}$ we define the associated Q-function as

$$q(\zeta) = -i \cdot [v,v]_{\mathcal{H}} + (\zeta - \bar{i}) \left[\left(I + (\zeta - i)(A-\zeta)^{-1} \right) v, v \right]_{\mathcal{H}} \tag{2.1}$$

and we say that v is cyclic if

$$\overline{\mathrm{span}}\{(I + (\zeta - i)(A-\zeta)^{-1})v : \zeta \in \mathbb{C} \setminus \mathbb{R}\} = \mathcal{H}. \tag{2.2}$$

Now if $v \in \mathrm{dom}(A)$, then we can set $u := (A+i)v$, and in this case formula (2.1) simplifies to formula (1.1). Of course, this is always the case in finite dimensions, in which case also condition (2.2) simplifies to condition (1.2).

In the same way as before, it can be shown that q satisfies $q(\overline{\zeta}) = \overline{q(\zeta)}$ and $q(\mathbb{C}^+) \subset \overline{\mathbb{C}^+}$, which means that q is a Herglotz-Nevanlinna function. As already mentioned, these functions are very well understood [24, Chapter 3.4]. While they have many nice properties, the most important one is certainly the following integral representation [24, Chapter 3.4]:

Proposition 2.1. *Let $q : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ be an analytic function satisfying $q(\overline{\zeta}) = \overline{q(\zeta)}$. Then the following are equivalent:*

(A) The function q is a Herglotz-Nevanlinna function, which means that $\operatorname{Im} q(\zeta) \geq 0$ *for* $\operatorname{Im} \zeta > 0$.

(B) There are $a \in \mathbb{R}$, $b \geq 0$ and a positive Borel measure ν satisfying

$$\int_{\mathbb{R}} \frac{\mathrm{d}\nu(t)}{1+t^2} < \infty,$$

such that the following holds:

$$q(\zeta) = a + b\zeta + \int_{\mathbb{R}} \left(\frac{1}{t-\zeta} - \frac{t}{1+t^2} \right) \mathrm{d}\nu(t) \quad \forall \zeta \in \mathbb{C} \setminus \mathbb{R}. \tag{2.3}$$

Finally, our discussion of the first chapter can be generalized in the following way [24, Chapter 3]:

Remark 2.2. *Let A be a self-adjoint operator on $\mathcal{H}$, $\sigma(A)$ its spectrum, and $v \in \mathcal{H}$ be a cyclic vector. Then q is a Herglotz-Nevanlinna function satisfying $b = 0$ in its integral representation. Moreover, the function q encodes the spectral properties of A, for example it holds that*

$$\sigma(A) = \overline{\left\{ x \in \mathbb{R} : 0 < \liminf_{\epsilon \downarrow 0} \operatorname{Im}(q(x + \mathrm{i}\epsilon)) \right\}}.$$

The function q also characterizes the absolutely continuous and the singular spectrum. We refrain from going into details here, and refer to the comprehensive textbook [4] instead.

In summary, we can use Herglotz-Nevanlinna functions to study self-adjoint operators which have a cyclic element. This can be improved even further by showing that such operators can be represented in a canonical way. This is de Branges's functional model, for which the language of reproducing kernels provides a convenient framework [22]:

Definition 2.3. *Let $Z \subset \mathbb{C}$ be an open set and $\mathcal{H}$ a Hilbert space where the elements are complex valued functions defined on Z. Then $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS) if there exists a function*

$$K : Z \times Z \to \mathbb{C}$$

such that for all $w \in Z$ the following holds:

1. *The function $\zeta \mapsto K(\zeta, w)$ belongs to $\mathcal{H}$.*

2. *For any $f \in \mathcal{H}$ it holds that $f(w) = [f(\cdot), K(\cdot, w)]_{\mathcal{H}}$.*

We will discuss reproducing kernel Hilbert spaces in detail in the next chapter. For the moment, we are only interested in the following class of RKHS [5, Chapter 1]:

Proposition 2.4 (de Branges 1968)**.** *Let $q : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ be a Herglotz-Nevanlinna function such that $b = 0$ in its integral representation* (2.3) [1] *. Then there exists a reproducing kernel Hilbert space $\mathcal{L}(q)$ with reproducing kernel*

$$N_q(\zeta, w) := \frac{q(\zeta) - \overline{q(w)}}{\zeta - \overline{w}}.$$

Moreover, there exists a self-adjoint operator A such that $(A - w)^{-1}$ acts as the difference-quotient operator defined as follows:

$$(A - w)^{-1} : \mathcal{L}(q) \to \mathcal{L}(q), \quad (A - w)^{-1}(f)(\zeta) = \frac{f(\zeta) - f(w)}{\zeta - w}.$$

Finally, the kernel function $N_q(\cdot, i)$ is a cyclic element, which means that

$$\overline{\mathrm{span}}\{(A - \zeta)^{-1}(N_q(\cdot, i)) : \zeta \in \varrho(A)\} = \mathcal{L}(q).$$

In summary, we see that every Herglotz-Nevanlinna function q defines a self-adjoint operator with a cyclic element, and this self-adjoint operator can be represented in simple function-theoretic terms. It is easy to check that q is the Q-function associated to A and the element $N_q(\cdot, i)$. Conversely, every self-adjoint operator admitting a cyclic element is of this form [5].

Proposition 2.5 (de Branges 1968)**.** *Let A be a self-adjoint operator on $\mathcal{H}$ and $v \in \mathcal{H}$ be a cyclic vector. Moreover, let q be the associated Q-function, meaning that q is given as*

$$q(\zeta) = -i \cdot [v, v] + (\zeta - \overline{i}) \left[\left(I + (\zeta - i)(A - \zeta)^{-1} \right) v, v \right]_{\mathcal{H}}.$$

Then q is a Herglotz-Nevanlinna function (satisfying $b = 0$ in its integral representation) and there exists an isometric isomorphism

$$F : \mathcal{H} \to \mathcal{L}(q)$$

such that $(A - w)^{-1}$ acts as the difference-quotient operator

$$F(A - w)^{-1} F^* : \mathcal{L}(q) \to \mathcal{L}(q), \quad F(A - w)^{-1} F^*(f)(\zeta) = \frac{f(\zeta) - f(w)}{\zeta - w},$$

and it holds that $F(v) = N_q(\cdot, i)$.

[1] The additional assumption $b = 0$ is not essential here and only added in order to exclude some edge case which would obscure the key ideas.

The explicit description of $(A-w)^{-1}$ as the difference-quotient operator is very helpful in many applications. For example, we will see in Chapter 5 how the quotient rule can be used to characterize all (singular) rank-one perturbation of A. More applications, in particular with respect to differential operators, can be found in [4].

In summary, we have seen that we can establish an intimate connection between self-adjoint operators on Hilbert spaces and Herglotz-Nevanlinna functions. The main result of this thesis is a generalization of this theory, where we will have partially fundamentally reducible operators instead of self-adjoint operators on the one side and meromorphic functions of bounded type instead of Herglotz-Nevanlinna functions on the other. The result will closely resemble Propositions 2.4 and 2.5.

Finally, we address the additional assumption $b = 0$. We have seen that self-adjoint operators correspond to Herglotz-Nevanlinna functions with this additional condition via Propositions 2.4 and 2.5. We can remove this extra assumption on q if we allow A to be a self-adjoint relation instead of merely a self-adjoint operator. However, this should not obscure the key ideas, and we recommend that readers who are not familiar with relations ignore the term "relation" whenever it appears and think of operators instead. For the sake of completeness, we include a formal definition of linear relations which can be thought of as multi-valued operators:

Definition 2.6. *Let $\mathcal{H}$ be a Hilbert space and $A \subset \mathcal{H} \times \mathcal{H}$. Then A is a linear relation if*

$$\text{for all } (x, y), (u, v) \in A, \lambda \in \mathbb{C} \text{ it holds that } (x + \lambda u, y + \lambda v) \in A.$$

Moreover, we define the adjoint relation as

$$A^* := \{(u, v) \in \mathcal{H} \times \mathcal{H} : \forall (x, y) \in A : (u, y)_{\mathcal{H}} = (v, x)_{\mathcal{H}}\}.$$

We call A self-adjoint if $A = A^*$.

Now given an operator $A : \operatorname{dom}(A) \subset \mathcal{H} \to \mathcal{H}$ we can consider its graph

$$\operatorname{GR}(A) := \{(u, A(u)) : u \in \operatorname{dom}(A)\} \subset \mathcal{H} \times \mathcal{H}.$$

The graph is then a linear relation. The resolvent set (and the spectrum) of a linear relation is defined in the same way as for a linear operator, meaning that $w \in \varrho(A)$ if and only if $(A - wI)$ has a continuous inverse.

Chapter 3

Reproducing kernel spaces

We have already seen the definition of RKHSs in the last chapter, and we will flesh out their theory in what follows. In addition, we will also introduce and discuss reproducing kernel Krein spaces, which generalize the notion of RKHSs. These more general objects provide the framework for generalizing de Branges‘s model theory.

3.1 Reproducing kernel Hilbert spaces

The concept of reproducing kernels goes back to the work from Bergmann, Aronszajn and Moore eighty years ago [2]. Nowadays, reproducing kernel spaces have found widespread applications in many areas of pure and applied mathematics besides complex analysis, for example in statistics and machine learning. A comprehensive introduction to their theory is given in [22], where almost all of the following results can be found.

Let us consider a RKHS $\mathcal{H}$ and its reproducing kernel $K : Z \times Z \to \mathbb{C}$. The first important observation is that point evaluations are continuous functionals on $\mathcal{H}$. In fact, using Riesz‘s representation theorem one can even show that a Hilbert space consisting of functions is a RKHS if and only if point evaluations are continuous.

The reproducing kernel function K encodes many properties of the space $\mathcal{H}$. For example, if K is continuous, then all functions in $\mathcal{H}$ are continuous as well. This is due to the fact that the set of kernel functions $K(\cdot, y),\ y \in Z$ forms a dense set:

Proposition 3.1. *Let $\mathcal{H}$ be a RKHS on Z with reproducing kernel K. Then the following holds:*

$$\overline{\operatorname{span}}\{K(\cdot, y) : y \in Z\} = \mathcal{H}.$$

So if we are given a RKHS $\mathcal{H}$ then we have by definition an associated reproducing kernel $K : Z \times Z \to \mathbb{C}$. Conversely, one could ask which functions $K : Z \times Z \to \mathbb{C}$ arise that way, i.e. which functions are reproducing kernels of RKHS‘s. The symmetry of the inner product gives a first necessary condition, since it implies that

$$K(x,y) = \overline{K(y,x)}$$

holds for all $x, y \in Z$. Generalizing this condition in the right way, as it was done by Moore in the 1950‘s, gives an answer to this question [22][Part I, Chapter 2]:

Theorem 3.2 (Moore)**.** *A function $K : X \times X \to \mathbb{C}$ is the reproducing kernel of a RKHS $\mathcal{H}$ if and only if for every $n \in \mathbb{N}$ and every choice of distinct points $\{x_1, ..., x_n\} \subset X$ the matrix*

$$(K(x_i, x_j))_{i,j \in \{1,...,n\}}$$

is non-negative. Here, a matrix $A \in \mathbb{C}^{n \times n}$ is non-negative if

$$[Ax, x]_{\mathbb{C}^n} \geq 0 \quad \forall x \in \mathbb{C}^n.$$

We call such a function K a non-negative kernel function.

The archetypical example of a non-negative kernel function is the Cauchy kernel

$$\mathcal{C}(\zeta, w) = \frac{1}{-i \cdot (\zeta - \overline{w})} \quad \forall \zeta, w \in \mathbb{C}^+.$$

The corresponding reproducing kernel Hilbert space is the Hardy space $H^2(\mathbb{C}^+)$, which will be discussed in detail in the next section. For the moment, we only mention that the point evaluation

$$f(w) = (f, \mathcal{C}(\cdot, w))_{H^2(\mathbb{C}^+)}, \quad f \in H^2(\mathbb{C}^+).$$

will be a generalized version of Cauchy‘s integral formula.

On another note, this theory also allows us to give yet another characterization of Herglotz-Nevanlinna functions [4]:

Proposition 3.3. *Let $q : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ be an analytic function satisfying $q(\overline{\zeta}) = \overline{q(\zeta)}$. Then the following are equivalent:*

(A) The function q is a Herglotz-Nevanlinna function, i.e., $\operatorname{Im} q(\zeta) \geq 0$ *for* $\operatorname{Im} \zeta > 0$.

(B) The function $N_q(\zeta, w) := \frac{q(\zeta)-\overline{q(w)}}{\zeta-\overline{w}}$ *is a non-negative kernel function.*

Now if q is not a Herglotz-Nevanlinna function then N_q can still be defined, but it is not a non-negative kernel function any more. One way of generalizing de Branges‘s model theory is following the ideas of Krein and Langer from the 1980‘s, which lead into the realm of indefinite inner product spaces.

3.2 Reproducing kernel Krein spaces

In this section, we introduce and discuss reproducing kernel Krein spaces. To this end, it is necessary to collect some basic properties of general Krein spaces first.

3.2.1 Krein spaces: Definitions and basic examples

Let us begin with the definition of an indefinite inner product:

Definition 3.4. *Let* $\mathcal{K}$ *be a (complex) vector space. We call*

$$[\cdot,\cdot]_{\mathcal{K}} : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$$

an inner product, if

$$[\alpha v + \beta w, x]_{\mathcal{K}} = \alpha[v,x]_{\mathcal{K}} + \beta[w,x]_{\mathcal{K}} \quad \forall \alpha, \beta \in \mathbb{C},\ v, w, x \in \mathcal{K}. \tag{i}$$

$$[v,x]_{\mathcal{K}} = \overline{[x,v]_{\mathcal{K}}} \quad \forall x, v \in \mathcal{K} \tag{ii}$$

$$[x,y]_{\mathcal{K}} = 0 \quad \forall y \in \mathcal{K} \Rightarrow x = 0. \tag{iii}$$

Condition (iii) means that the inner product is non-degenerate. In the classical definition of an inner product we demand more, namely that the inner product is even positive definite. Krein spaces are then defined as follows:

Definition 3.5. *A pair* $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$ *consisting of a vector space* $\mathcal{K}$ *and an inner product* $[\cdot,\cdot]_{\mathcal{K}}$ *is called Krein space if there is a fundamental decomposition, i.e. a direct orthogonal sum*

$$\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-,$$

where $(\mathcal{K}_+, [\cdot,\cdot]_{\mathcal{K}})$ *and* $(\mathcal{K}_-, -[\cdot,\cdot]_{\mathcal{K}})$ *are Hilbert spaces.* $\mathcal{K}$ *is a Pontryagin space, if either* $\dim(\mathcal{K}_+) < \infty$ *or* $\dim(\mathcal{K}_-) < \infty$.

Note the minus sign in the condition that $(\mathcal{K}_-, -[\cdot,\cdot]_{\mathcal{K}})$ is a Hilbert space. Of course, a fundamental decomposition comes with an natural norm, namely the one of the Hilbert space $\mathcal{K}_+ \oplus (-\mathcal{K}_-)$. Here, $(-\mathcal{K}_-)$ simply means that we flipped the sign of the inner product.

Let $\mathcal{K}$ be a Krein space. Then it has in general many different fundamental decompositions. However, all norms coming from fundamental decompositions are equivalent. Thus notions like density, the convergence of sequences and the boundedness of linear operators can be defined with reference to one fixed fundamental decomposition and the associated norm.

There are various way to construct Krein spaces. In this thesis we will exclusively work with the following obvious construction:

Example 3.6. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. Define the following indefinite inner product space:*

$$\mathcal{K} := \mathcal{H}_1 \oplus \mathcal{H}_2, \quad [(x_1, x_2), (y_1, y_2)]_{\mathcal{K}} := [x_1, y_1]_{\mathcal{H}_1} - [x_2, y_2]_{\mathcal{H}_2}.$$

Then $\mathcal{K}$ is a Krein space.

3.2.2 Reproducing kernel Krein spaces

Reproducing Kernel Krein spaces are defined in the same way as RKHS:

Definition 3.7. *Let $Z \subset \mathbb{C}$ and $\mathcal{K}$ a Krein space consisting of complex valued functions defined on Z. Then $\mathcal{K}$ is a reproducing kernel Krein space (RKKS) if there exists a function*

$$H : Z \times Z \to \mathbb{C}$$

such that for all $w \in Z$:

1. *The function $\zeta \mapsto H(\zeta, w)$ belongs to $\mathcal{K}$.*
2. *For any $f \in \mathcal{K}$ it holds that $f(w) = [f(\cdot), H(\cdot, w)]_{\mathcal{K}}$.*

It is straightforward to verify that the kernel functions $H(\cdot, w), w \in Z$ form a total set.

We had seen in Theorem 3.2 which functions $K : Z \times Z \to \mathbb{C}$ generate reproducing kernels. The analogous question for Krein spaces is answered by the following result [13]:

Theorem 3.8. *Let $K : Z \times Z \to \mathbb{C}$ be a function. Then K is the reproducing kernel of a RKKS if and only if there exists two non-negative kernel functions $H_1 : Z \times Z \to \mathbb{C}$ and $H_2 : Z \times Z \to \mathbb{C}$ such that*

$$K = H_1 - H_2 \tag{3.1}$$

The decomposition in (3.1) is in general not unique.

Now let us recall the Nevanlinna kernel:

$$N_q(\zeta, w) := \frac{q(\zeta) - \overline{q(w)}}{\zeta - \overline{w}}$$

Krein and Langer characterized which functions q give rise to a reproducing kernel Pontryagin space in the 1970‘s [16]. This leads to a generalization of de Branges‘s model theory, which is the subject of the next section.

3.2.3 Cyclic operators on Pontryagin spaces

As in the Hilbert space also indefinite inner products give rise to self-adjoint operators. We state the definition for the sake of completeness in the more general setting of linear relations [1] :

Definition 3.9. *Let A be a linear relation on $\mathcal{K}$. Then we define the adjoint relation as*

$$A^+ := \{(u, v) \in \mathcal{K} \times \mathcal{K} : \forall (x, y) \in A : [u, y]_{\mathcal{K}} = [v, x]_{\mathcal{K}}\}.$$

We call A self adjoint if $A = A^+$.

If A is a densely defined operator, then A^+ is also an operator and it is defined by

$$A^+(u) = v \;\; \Leftrightarrow \forall x \in \operatorname{dom}(A) : [u, A(x)]_{\mathcal{K}} = [v, x]_{\mathcal{K}}.$$

The self-adjointness condition on Krein spaces is not restrictive enough to give rise to a rich theory. For example, the only restriction on the spectrum of such a relation is that it has to be symmetric with respect to $\mathbb{R}$, but apart from that it can be completely arbitrary. However, there are various ways to impose some additional conditions in order to obtain rich theories. The simplest one is to consider self-adjoint relations on Pontryagin spaces. If we additionally assume that they have a cyclic element, then there exists an analogous model theory. The corresponding functions are now the so called generalized Nevanlinna functions:

Definition 3.10. *Let $q : \mathcal{D} \subset \mathbb{C} \setminus \mathbb{R}$ be a meromorphic function on $\mathbb{C} \setminus \mathbb{R}$. Then q is a generalized Nevanlinna function if there exists a Herglotz-Nevanlinna function g and a rational function h satisfying $h(\mathbb{R}) \subset \mathbb{R} \cup \{\infty\}$ such that $q = g \cdot h$.*

[1] Again, it is sufficient for the purpose of this introduction to ignore the term relation whenever it shows up and think of operators instead.

Here, we point out that this is not the original definition, but an important equivalent characterization which was established in [8]. The generalized model theory can then be stated as follows [16] (see also [17]):

Proposition 3.11 (Krein, Langer). *Let $q : \mathcal{D} \subset \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be a generalized Nevanlinna function. Then there exists a reproducing kernel Pontryagin space $\mathcal{L}(q)$ with reproducing kernel*

$$N_q(\zeta, w) := \frac{q(\zeta) - \overline{q(w)}}{\zeta - \overline{w}}.$$

Moreover, there exists a self-adjoint relation A such that $(A - w)^{-1}$ acts as the difference-quotient operator, i.e. it holds that

$$(A - w)^{-1} : \mathcal{L}(q) \to \mathcal{L}(q), \quad (A - w)^{-1}(f)(\zeta) = \frac{f(\zeta) - f(w)}{\zeta - w}.$$

Finally, the kernel function $N_q(\cdot, i)$ is a cyclic element, which means that

$$\overline{\operatorname{span}}\{(I + (\zeta - i)(A - \zeta)^{-1})(N_q(\cdot, i)) : \zeta \in \varrho(A)\} = \mathcal{L}(q).$$

Conversely, every self-adjoint relation A on a Pontryagin space with a cyclic element v is, up to an isometric isomorphim, of this form.

This settles the Pontryagin space case. But what happens if we are considering even more general functions q than generalized Nevanlinna functions? It turns out that for meromorphic functions of bounded type, we can still develop a model theory. These functions are the subject of the next chapter. Before we will discuss them, we collect two technical results in the next section.

3.2.4 Operations on reproducing kernel Krein spaces

There are two operations on reproducing kernel spaces that are relevant for this thesis [22, Part I, Chapter 5].

Proposition 3.12 (Summation). *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two reproducing kernel Krein spaces on the same domain Z and H_1 and H_2 their respective kernels. If it holds that*

$$\mathcal{K}_1 \cap \mathcal{K}_2 = \{0\},$$

then $\mathcal{K}_1 \oplus \mathcal{K}_2$ is a reproducing kernel Krein space with kernel $H_1 + H_2$.

Proposition 3.13 (Products)**.** *Let $\mathcal{K}$ be a reproducing kernel Krein space on Z and kernel H. Moreover, let g be an analytic function on Z which is not identically zero on any component of Z.*

Consider the vector space $g \cdot \mathcal{K} \subset \mathcal{O}(Z)$ and the bijective operator

$$M_g : \mathcal{K} \to g \cdot \mathcal{K}, \quad f \mapsto g \cdot f.$$

Now let us equip $g \cdot \mathcal{K}$ with the inner product that turns M_g into an isomorphism of Krein spaces.

Then $g \cdot \mathcal{K}$ is a reproducing kernel Krein space on Z with kernel

$$H_g(\zeta, w) := g(\zeta) H(\zeta, w) \overline{g(w)}.$$

Chapter 4

Meromorphic functions of bounded type

In this chapter, we discuss the necessary preliminaries from complex analysis to obtain a good understanding of meromorphic functions of bounded type. A full account of their theory is beyond the scope of this introduction, instead we will focus on those concepts that will allow us to construct our operator models. The most important one is Helson‘s representation, which requires us to also discuss the Hardy space $H^2(\mathbb{C}^+)$ and certain subspaces called model spaces.

4.1 The Hardy spaces H^2 and H^∞

The most important function spaces for us are the Hardy spaces $H^2(\mathbb{C}^+)$ and $H^\infty(\mathbb{C}^+)$. They were initially introduced by Frigyes Riesz over a century ago, and have inspired a lot of research ever since. A comprehensive introduction to their theory can be found in the textbook [23], where all the upcoming results can be found.

Let us first discuss their definition. To this end, we denote by $\mathcal{O}(\mathbb{C}^+)$ the space of analytic functions on $\mathbb{C}^+$. The Hardy space $H^\infty(\mathbb{C}^+)$ is then defined as the Banach space of bounded analytic functions with the obvious norm

$$H^\infty(\mathbb{C}^+) := \left\{ f \in \mathcal{O}(\mathbb{C}^+) : \sup_{\zeta\in\mathbb{C}^+} |f(\zeta)| < \infty \right\}, \ \|f\|_\infty := \sup_{\zeta\in\mathbb{C}^+} |f(\zeta)|.$$

There are many different, but equivalent, ways to define the Hardy space $H^2(\mathbb{C}^+)$. For example, we had already encountered $H^2(\mathbb{C}^+)$ as the RKHS generated by the Cauchy-Szegö kernel in the last chapter. However, originally $H^2(\mathbb{C}^+)$ was defined in terms of a certain regularity condition when approaching the boundary:

Definition 4.1. *We define the Hardy space $H^2(\mathbb{C}^+)$ as the space of analytic functions*

$$H^2(\mathbb{C}^+) := \left\{ f \in \mathcal{O}(\mathbb{C}^+) : \sup_{0<y<\infty} \left(\int_{-\infty}^{\infty} |f(x+iy)|^2 \mathrm{d}x \right)^{\frac{1}{2}} < \infty \right\}$$

equipped with the norm

$$\|f\|_2 := \sup_{0<y<\infty} \left(\int_{-\infty}^{\infty} |f(x+iy)|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

At first glance, it is not clear that the norm of $H^2(\mathbb{C}^+)$ is induced by an inner product. In order to make sense of this, we first note that functions in H^2 (or H^∞ for that matter) have well-defined boundary functions [23, Chapter 5]:

Theorem 4.2. *Let $f \in H^p(\mathbb{C}^+)$ with $p \in \{2, \infty\}$. Then*

$$\lim_{y \downarrow 0} f(x+iy)$$

exists for almost every $x \in \mathbb{R}$. We define the boundary function as

$$\tilde{f}(x) := \lim_{y \downarrow 0} f(x+iy), \quad x \in \mathbb{R}.$$

Finally, the operator

$$\mathcal{F} : H^2(\mathbb{C}^+) \to L^2(\mathbb{R}), \quad f \mapsto \tilde{f}.$$

is an isometry.

Therefore, we can consider $H^2(\mathbb{C}^+)$ as a subspace of the Hilbert space $L^2(\mathbb{R})$, and the norm is then given as

$$\|f\|^2_{H^2(\mathbb{C}^+)} = (\tilde{f}, \tilde{f})_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |\tilde{f}(x)|^2 \mathrm{d}x.$$

Thus, we see that $H^2(\mathbb{C}^+)$ is indeed a Hilbert space. Now let us take for granted that the Cauchy-Szegö kernel

$$\mathcal{C}(\zeta, w) = \frac{1}{-i \cdot (\zeta - \overline{w})} \quad \forall \zeta, w \in \mathbb{C}^+.$$

is indeed the reproducing kernel of $H^2(\mathbb{C}^+)$. This allows us to recover the function from its boundary values via a generalized Cauchy transform. More precisely, for $f \in H^2(\mathbb{C}^+)$ it holds that

$$f(w) = (f, \mathcal{C}(\cdot, w))_{H^2(\mathbb{C}^+)} = \left(\tilde{f}, \frac{1}{-i(\cdot - \overline{w})}\right)_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{\tilde{f}(x)}{i \cdot (x - w)} \mathrm{d}x.$$

Here, we have used that the boundary function of $\mathcal{C}(\zeta, w)$ is simply $\frac{1}{i\cdot(x-\overline{w})}$. A more detailed account can be found in [23, Chapter 5.15]

4.2 Inner functions and model spaces

We are actually not so much interested in the Hardy space $H^2(\mathbb{C}^+)$, but rather in certain subspaces called model spaces. These spaces, or more precisely suitable generalizations of them, are at the center of our construction of operator models. In what follows, we discuss their definition and their most important properties. To this end, we first recall that a function $V \in H^\infty(\mathbb{C}^+)$ is called inner, if

$$|\tilde{V}(w)| = 1 \quad \text{for almost every } w \in \mathbb{R}.$$

Note that $\tilde{V}(w)$ was defined as the radial limit. Model spaces are then defined as follows:

Proposition 4.3. *Let V be an inner function. Then we define the model space $H(V) \subset H^2(\mathbb{C}^+)$ associated to V as*

$$H(V) := (V \cdot H^2(\mathbb{C}^+))^\perp.$$

The space $H(V)$ is then a reproducing kernel Hilbert space with kernel

$$s_V(\zeta, w) = \frac{1 - V(\zeta)\overline{V(w)}}{-i \cdot (\zeta - \overline{w})}. \tag{4.1}$$

The theory of model spaces is filling books [11]. Here, we are particularly interested in the following two facts. First, every model space is invariant under the difference quotient operator [11, Chapter 5]:

Proposition 4.4. *Let V be an inner function and $H(V)$ the associated model space. Then, given $w \in \mathbb{C}^+$, the difference-quotient operator*

$$D_w : H(V) \to H(V), \quad D_w(f) = \frac{f(\zeta) - f(w)}{\zeta - w}$$

is well-defined and bounded.

Second, we can define a multiplicative structure on the set of inner functions:

Definition 4.5. *Let V_1 and V_2 be inner functions. Then we say that*

- *the function V_1 divides V_2 if there exists an inner function V_3 such that $V_2 = V_1 \cdot V_3$.*
- *the functions V_1 and V_2 are relatively prime, if there exists no non-constant inner function which divides both V_1 and V_2.*

This structure corresponds to the following ordering result on the level of the model spaces [11, Chapter 5]:

Proposition 4.6. *Let V_1 and V_2 be inner functions. Then the following holds :*

- *The function V_1 divides V_2 if and only if*

$$H(V_1) \subset H(V_2).$$

- *The functions V_1 and V_2 are relatively prime if and only if*

$$H(V_1) \cap H(V_2) = \{0\}.$$

4.3 Meromorphic functions of bounded type

Meromorphic functions of bounded type were originally studied in the context of value distributions [20], but have found their way into many different areas of mathematics since then such as number theory and systems theory. We will explain their special importance in a moment, but we first want to give a formal definition:

Definition 4.7. *Let $f : \mathcal{D} \subset \mathbb{C}^+ \to \mathbb{C}$ be a meromorphic function on $\mathbb{C}^+$. Then f is of bounded type, if there are two functions $f_1 \in H^\infty(\mathbb{C}^+)$ and $f_2 \in H^\infty(\mathbb{C}^+)$ such that f can be represented as follows:*

$$f(\zeta) = \frac{f_1(\zeta)}{f_2(\zeta)} \quad \forall \zeta \in \mathcal{D}.$$

We note that since every such function f can be written as the quotient of two $H^\infty(\mathbb{C}^+)$ functions, the function f has an almost everywhere defined boundary function by Proposition 4.2.

The class of analytic functions of bounded type is, in some sense, "maximal" for the existence of boundary functions (see [21] for details). This explains their special role in complex analysis. In addition, it comes to no surprise that functions in the Hardy space $H^2(\mathbb{C}^+)$ are of bounded type. Yet another class of examples is given by generalized Nevanlinna functions, if we restrict them to the upper half plane.

4.3.1 Real complex functions

An important class of functions of bounded type are the so called real complex functions:

Definition 4.8. *Let f be a meromorphic function of bounded type. Then f is called a real complex function if it has real boundary values almost everywhere.*

These functions played an important role in defining a continuous analog of the Riesz projection on H^p for $0 < p < 1$, and have applications in many other areas. An extensive survey is given in [12].

The fundamental result in this area is Helson‘s representation [14]. Specifically, let f be of bounded type with real boundary values. Then there exist inner function V_1 and V_2 such that

$$f = i \cdot \frac{V_1 - V_2}{V_1 + V_2} \quad \text{and} \quad H(V_1) \cap H(V_2) = \{0\}.$$

Here, $H(V_1)$ and $H(V_2)$ are the associated model spaces. A generalization of this theorem to the class of arbitrary functions of bounded type is one of the main results of this thesis and the key for constructing operator models for meromorphic functions of bounded type.

Chapter 5

Partially fundamentally reducible relations

As mentioned before, the self-adjoint condition on Krein spaces is not very restrictive, and in order to obtain a rich theory we have to make some additional assumption(s). One obvious restriction is to consider self-adjoint operators on Pontryagin spaces, which was done in Chapter 2. Another interesting approach is to demand that the operator behaves well with respect to a fundamental decomposition. For example, one can consider fundamentally reducible operators. These are self-adjoint operators on a Krein space $\mathcal{K}$ for which there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-$ such that A splits into an orthogonal sum of self-adjoint operators $A = A_1 \oplus A_2$ with respect to this decomposition.

Here, we want to relax this condition slightly, which then leads to the notion of partially fundamentally reducible operators. Instead of demanding a clean decomposition into two self-adjoint operators, we allow for a one-dimensional defect. This is best explained with the help of symmetric operators, which are highly relevant for this thesis for multiple reasons. Therefore, we will discuss them in some detail first.

5.1 Symmetric operators

Symmetric operators were initially introduced and studied by von Neumann over a century ago [19], and have attracted substantial interest ever since. A comprehensive account of the current state of research is given in the textbook [4], where all the upcoming results can be found. Let us first state the formal definition:

Definition 5.1. *A linear operator* $S : \mathrm{dom}(S) \subset \mathcal{H} \to \mathcal{H}$ *is called sym-*

metric, if for all $x, y \in \mathrm{dom}(S)$ the following holds:

$$[Sx, y]_{\mathcal{H}} = [x, Sy]_{\mathcal{H}}.$$

Symmetry is a weaker condition than self-adjointness. The main difference is that we allow for S to have some "defect" which is quantified as follows:

Definition 5.2. *Let S be a symmetric operator and $w \in \mathbb{C} \setminus \mathbb{R}$. Then the defect subspace $\eta_w(S)$ is defined as*

$$\eta_w(S) := \mathrm{ran}(S - \overline{w})^{\perp} = \ker(S^* - w).$$

The dimension $\dim(\eta_w(S))$ is constant on $\mathbb{C}^+$ and $\mathbb{C}^-$. Consequently, we define the deficiency index as

$$(n, m) = (\dim(\eta_i(S)), \dim(\eta_{-i}(S))).$$

The defect index measures how close a symmetric operator is to being self-adjoint, as self-adjoint operators are exactly the closed symmetric operators with deficiency index $(0, 0)$.

One of the most important questions about symmetric operators is if they have self-adjoint extensions, and if so, what they look like. The first question is answered in the next theorem [4, Chapter 1.7]:

Theorem 5.3 (von Neumann)**.** *Let S be a symmetric operator on a Hilbert space $\mathcal{H}$. Then S has a self-adjoint extension if and only if $\dim(\eta_{-i}(S)) = \dim(\eta_i(S))$.*

In this introduction, we are mostly interested in symmetric operators which satisfy the following additional condition:

Definition 5.4. *A symmetric operator S is simple, if the following holds:*

$$\overline{\mathrm{span}}\{\eta_\lambda(S) : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \mathcal{H}.$$

This additional assumption resembles the cyclicity condition (1.2) for self-adjoint relations. And in fact, we can model simple symmetric operators with deficiency index $(1, 1)$ in a very similar way [4, Chapter 4]:

Proposition 5.5. *Let q be a Herglotz-Nevanlinna function and $\mathcal{L}(q)$ the RKHS $\mathcal{L}(q)$ with kernel N_q. Then the multiplication operator by the independent variable*

$$S_q : \{f \in \mathcal{L}(q) : \zeta \cdot f(\zeta) \in \mathcal{L}(q)\} \to \mathcal{L}(q), \quad S_q(f)(\zeta) = \zeta \cdot f(\zeta) \tag{5.1}$$

is a simple symmetric operator with deficiency index $(1,1)$.

Conversely, let S *be a simple symmetric operator with deficiency index* $(1,1)$ *on* $\mathcal{H}$. *Then there exists a Herglotz-Nevanlinna function* q *and an isometric isomorphism*

$$U : \mathcal{H} \to \mathcal{L}(q)$$

such that $USU^* = S_q$.

Calculating the function q is a bit more involved, we will not go into details here.

Now we want to clarify the relationship between this model theory and the one for cyclic self-adjoint operators. First, recall that on $\mathcal{L}(q)$ there exists a self-adjoint relation A generating the difference-quotient operator:

$$(A-w)^{-1}(f)(\zeta) = \frac{f(\zeta)-f(w)}{\zeta - w}. \tag{5.2}$$

A simple algebraic calculation allows us to rewrite this relation in the following more explicit way:

$$A := \{(f,g) \in \mathcal{L}(q) \times \mathcal{L}(q) : \exists c \in \mathbb{C} : g(\zeta) - \zeta \cdot f(\zeta) \equiv c\}$$

Thus, A is a self-adjoint extension of S_q.

So we have found *one* self-adjoint extension of S_q. Conveniently, knowing one self-adjoint extension allows us to express all other self-adjoint extensions via Krein's resolvent formula [10]:

Proposition 5.6. *Let* S_q *be the symmetric operator given by equation* (5.1) *acting on* $\mathcal{L}(q)$ *and* A *the self-adjoint extension given by equation* (5.2). *Then all other self-adjoint extensions* A_c *are parameterized by* $c \in \mathbb{R}$ *and given by Krein's resolvent formula:*

$$(A_c - w)^{-1}(f) = (A-w)^{-1}(f) - \frac{f(w)}{q(w)+c} \cdot N_q(\cdot, w) \quad w \in \varrho(A).$$

5.2 Partially fundamentally reducible relations

Now we are able to define partially fundamentally reducible relations:

Definition 5.7. *Let* A *be a self-adjoint relation in a Krein space* $\mathcal{K}$ *satisfying* $\varrho(A) \neq \emptyset$. *Then*

- *the relation A is called fundamentally reducible if there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-$ such that A splits into an orthogonal sum of self-adjoint relations $A = A_1 \oplus A_2$ with respect to this decomposition.*

- *the relation A is called partially fundamentally reducible if there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-$ such that the relations*

  $$S_+ := A \cap (\mathcal{K}_+ \times \mathcal{K}_+) \quad \text{and} \quad S_- := A \cap (\mathcal{K}_- \times \mathcal{K}_-)$$

  are closed, simple symmetric operators on the Hilbert spaces $\mathcal{K}_+$ and $-\mathcal{K}_-$ with deficiency index $(1, 1)$.

A fundamentally reducible relation is relatively easy to analyze, since such a relation is also self-adjoint with respect to the induced Hilbert space inner product. As the name suggests, partially fundamentally reducible relations are relatively close to fundamentally reducible ones. The main difference is that we allow a one dimensional defect in the decomposition. This can also be expressed in a different way, namely a partially fundamentally reducible relation is a suitable one-dimensional perturbation of a fundamentally reducible relation.

Chapter 6

The main results

In this chapter, we discuss the main results of this thesis. We structured them into two parts, the first one is concerned with operator models for meromorphic functions of bounded type and the second one with extensions of symmetric operators. In order to simplify our presentation, we denote for a given function f the difference-quotient at the point w by $D_w(f)$, meaning that

$$D_w(f)(\zeta) = \frac{f(\zeta) - f(w)}{\zeta - w}.$$

6.1 Operator models for meromorphic functions of bounded type

The Papers I, II and III are concerned with operator models for meromorphic functions of bounded type. The main result can be found in Paper III, Papers 1 and 2 contain related results of a smaller scope. Our plan for this section is as follows. First, we will briefly sketch the contents of the first two papers, and after that discuss Paper III in much more detail. This section is then completed by a section in which we discuss important related results.

6.1.1 Paper I

The first step in our search for operator models was to show that every meromorphic function of bounded type can be realized as the Q-function of a self-adjoint operator on a Krein space. More precisely, let f be such a function and extend it to $\mathbb{C}^-$ via $f(\overline{z}) = \overline{f(z)}$. Then there exists a self-adjoint relation A on a Krein space $\mathcal{K}$ and a vector $v \in \mathcal{K}$ such that

$$f(\zeta) = -i \cdot [v, v]_{\mathcal{H}} + (\zeta - \bar{i}) \left[\left(I + (\zeta - i)(A - \zeta)^{-1} \right) v, v \right]_{\mathcal{H}}.$$

The limitation of the construction in Paper I is that the element v is not cyclic.

6.1.2 Paper II

In our second paper we considere certain real complex functions, which we called “atomic”. These are functions of the form

$$\frac{(V+1)^2}{V}, \quad \frac{V}{(V+1)^2}, \quad -\frac{(1-V)^2}{(1+V)^2}$$

where V is an inner function. The reason why these functions are interesting is that any function h, which is of bounded type with positive boundary values, can be decomposed as

$$h = \frac{(V+1)^2}{V} \frac{W}{(W+1)^2} \prod_{n=-\infty}^{\infty} \frac{(1-X_n)^2}{(1+X_n)^2}, \tag{6.1}$$

where V, W and X_n for $n \in \mathbb{Z}$ are inner functions. Here, the right hand side converges locally uniformly.

We show that these functions appear as Q-functions of self-adjoint operators on Krein spaces with a *cyclic* element v. The arguments used in this paper are generalized in Paper III, where we prove our main result.

6.1.3 Paper III

In this paper, we construct operator models for the entire class of meromorphic functions of bounded type. The key result allowing us to do so is a generalization of Helson‘s representation for real complex functions, which was discussed in Chapter 4. We will begin with discussing that result in more detail, which is of interest in its own right.

To this end, let us consider the following function class:

Definition 6.1. *Let h be a meromorphic function on $\mathbb{C} \setminus \mathbb{R}$. Then h is in the class $\mathcal{S}_0$ if*

$$h_{|\mathbb{C}^+} \in \mathcal{H}^\infty(\mathbb{C}^+) \setminus \{0\}, \quad \|h_{|\mathbb{C}^+}\|_{\mathcal{H}^\infty(\mathbb{C}^+)} \leq 1, \quad \text{and } h(\bar{\zeta}) = \frac{1}{\overline{h(\zeta)}}.$$

Note that a function $h \in \mathcal{S}_0$ is determined by its values on $\mathbb{C}^+$, and then simply extended via a symmetry condition. Moreover, the boundary values of the restriction $h_{|\mathbb{C}^+}$ have absolute value at most 1. These functions define the following reproducing kernel Hilbert spaces:

Theorem 6.2 (de Branges)**.** *Let $h \in \mathcal{S}_0$ and $U(h)$ its domain. Then there exists a reproducing kernel Hilbert space $\mathcal{B}(h)$ with kernel*

$$s_h(z,w) : U(h) \times U(h) \to \mathbb{C} \quad s_h(z,w) = \frac{1 - h(z)\overline{h(w)}}{-\mathrm{i}(z - \overline{w})}.$$

Moreover, the difference-quotient operator D_w is well-defined and bounded on $\mathcal{B}(h)$ for every $w \in U(h)$.

These RKHS resemble the model spaces that we encountered in Chapter 4. What‘s more, if $h \in \mathcal{S}_0$ such that $h_{|\mathbb{C}^+}$ is an inner function, then the restriction operator

$$\mathrm{Res} : \mathcal{B}(h) \to H(h_{|\mathbb{C}^+}),\ f \mapsto f_{|\mathbb{C}^+}$$

is an isometric isomorphism. Recall that $H(h_{|\mathbb{C}^+})$ denotes the model space associated to an inner function.

Model spaces appeared in the context of Helson‘s representation theorem for real complex functions in Chapter 4 Section 4.3.1. Using the more general class $\mathcal{S}_0$ we can prove the following generalization which is the first of our main theorems:

Theorem 6.3 (Paper III, Theorem 3.3)**.** *Let f be a non-constant function of bounded type on $\mathbb{C}^+$, which we continue to $\mathbb{C}^-$ via the rule $f(\overline{\zeta}) = \overline{f(\zeta)}$. Then there exists two functions $h_1, h_2 \in \mathcal{S}_0$ such that the following holds:*

$$f = i \cdot \frac{h_2 - h_1}{h_2 + h_1} \quad \textit{and} \quad \mathcal{B}(h_1) \cap \mathcal{B}(h_2) = \{0\}.$$

With that being established, we can turn our attention towards operator models for meromorphic functions of bounded type. This builds on our generalized Helson representation and is our second main result:

Theorem 6.4 (Paper III)**.** *Let $f : \mathcal{D} \subset \mathbb{C}^+ \to \mathbb{C}$ be a meromophic function of bounded type on $\mathbb{C}^+$, which we continue to $\mathbb{C}^-$ via the rule $f(\overline{\zeta}) = \overline{f(\zeta)}$. Then there exists a reproducing kernel Krein space $\mathcal{L}(f)$ with kernel*

$$N_f(\zeta, w) = \frac{f(\zeta) - \overline{f(w)}}{\zeta - \overline{w}}.$$

Moreover, there exists a partially fundamentally reducible relation A on $\mathcal{L}(f)$ such that $(A-w)^{-1}$ acts as the difference-quotient operator D_w. Finally, the kernel function $N_f(\cdot, i)$ is a cyclic element and f is the Q-function associated to A and $N_f(\cdot, i)$.

The Krein space $\mathcal{L}(f)$ is described by the fundamental decomposition

$$\mathcal{L}(f) := \frac{\sqrt{2}}{h_1 + h_2} \cdot (\mathcal{B}(h_1) \oplus (-\mathcal{B}(h_2))) \tag{6.2}$$

where $h_1, h_2 \in \mathcal{S}_0$ are the functions representing f in the generalized Helson representation (Theorem 6.3). This is the decomposition with respect to which A is partially fundamentally reducible.

Conversely, let A be a fundamentally reducible relation on a Krein space $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ and $v \in \mathcal{K}$. Then the associated Q-function f (restricted to $\mathbb{C}^+$) is of bounded type.

This generalizes the functional model developed by Krein and Langer. In particular, we have the following curious corollary:

Corollary 6.5. *Let A be a self-adjoint operator on a Pontryagin space and v a cyclic element. Then A is partially fundamentally reducible.*

We want to sketch the proof of the first part of Theorem 6.4 here. A complete argument with all details can be found in Paper III.

Sketch of the Proof of the first part of Theorem 6.4. In the first step, we verify that we indeed produce a reproducing kernel Krein space with the correct kernel. To this end, we first note that the kernel of the Krein space $\mathcal{L}(f)$ is given as

$$\frac{\sqrt{2}}{h_1(\zeta) + h_2(\zeta)} \cdot (s_{h_1}(\zeta, w) - s_{h_2}(\zeta, w)) \cdot \frac{\sqrt{2}}{\overline{h_1(w)} + \overline{h_2(w)}}$$

A long but elementary calculation shows that this expression coincides with $N_f(\zeta, w)$.

In the second step, we show that $\mathcal{L}(f)$ is invariant under the difference quotient operator D_w. To this end, let $g \in \mathcal{B}(h_1)$ be arbitrary and consider the function

$$\frac{1}{h_1 + h_2} \cdot g \in \mathcal{L}(f).$$

By linearity it is sufficient to show that $D_w(g) \in \mathcal{L}(f)$. First, we calculate using the product rule for difference-quotients

$$D_w\left(\frac{1}{h_1 + h_2} \cdot g\right) = \frac{1}{h_1 + h_2} \cdot D_w(g) + g(w) \cdot D_w\left(\frac{1}{h_1 + h_2}\right).$$

Since $\mathcal{B}(h_1)$ is invariant under D_w, it follows that $\frac{1}{h_1+h_2} \cdot D_w(g) \in \mathcal{L}(f)$. Moreover, we calculate

$$D_w\left(\frac{1}{h_1+h_2}\right) = -\frac{1}{h_1+h_2} \cdot (D_w(h_1) + D_w(h_2)) \cdot \frac{1}{h_1(w)+h_2(w)}$$

Since it is well known that $D_w(h_i) \in \mathcal{B}(h_i)$, it follows that $D_w(g) \in \mathcal{L}(f)$.

Finally, it is straightforward to check that there exists a self-adjoint relation A satisfying $(A-w)^{-1} = D_w$. In the final step, one needs to show that this relation is partially fundamentally reducible with respect to the decomposition given in Equation (6.2). □

6.1.4 Related results

Let us recall that the original result in this research area is de Branges's functional model for cyclic self-adjoint operators on Hilbert spaces [5]. Starting from that, there are roughly two different direction in which one can proceed.

The first one is to allow the inner product to be indefinite, which, for example, enables the construction of Krein and Langer discussed in Chapter 3. Of course, our operator models for meromorphic functions of bounded type also falls into this category.

Yet another result in this direction is concerned with definitizable self-adjoint operators on Krein spaces. Here, an operator A is definitizable, if there exists a polynomial p such that $p(A)$ is non-negative. These operators generate so called definitizable functions as their Q-functions, which are slightly more general than generalized Nevanlinna functions [15].

The second direction is to consider higher-dimensional cyclicity conditions. Specifically, let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $\Gamma_i : \mathbb{C}^n \to \mathcal{H}$ a bounded linear operator. The generalized cyclicity condition is then formulated as

$$\overline{\operatorname{span}}\{(I + (\zeta - i)(A-\zeta)^{-1})\Gamma_i(c) : c \in \mathbb{C}^n,\ \zeta \in \mathbb{C} \setminus \mathbb{R}\} = \mathcal{H}.$$

Note that Γ_i plays the role of v in equation (2.2) here. The counterpart on the level of (higher-dimensional) Q-functions are then matrix-valued Herglotz-Nevanlinna functions [4].

Last but not least, it is of course also possible to proceed in both directions simultaneously, for example an operator model for matrix valued Generalized-Nevanlinna functions was constructed in [17].

So far, we had only discussed global theories, but there also exist local ones. Let $U \subset \mathbb{C}$ be an open set containing i and $f : U \to \mathbb{C}$ an analytic function. Then we can always find

- an open set $D \subset U$ containing i
- a self-adjoint operator A on a Krein space $\mathcal{K}$ which satisfies $D \subset \varrho(A)$
- a cyclic element v

such that

$$f(\zeta) = -i \cdot [v, v]_{\mathcal{H}} + (\zeta - \bar{i}) \left[\left(I + (\zeta - i)(A - \zeta)^{-1} \right) v, v \right]_{\mathcal{H}} \ \forall \zeta \in D$$

The drawback here is that the operator A is just a self-adjoint operator on a Krein space without any further restrictions. As we have already mentioned, self-adjointness on Krein spaces is way too general to be a very useful property. In particular, it is impossible to say more about the resolvent set of A than $D \subset \varrho(A)$. So this should really be regarded as a local theory. This result can be found in [9].

A different local theory was established in [1]. There they considered (operator-valued) functions which are analytic on a disc and continuous on the boundary.

6.2 A generalization of Krein's extension formula

Now we are turning our attention to the extension theory of symmetric operators:

6.2.1 Paper IV

Let S be a symmetric operator on a Hilbert space $\mathcal{H}$ with deficiency index $(1,1)$. The classical extension theory is concerned with self-adjoint extensions, or, slightly more general, extensions satisfying $S \subset A \subset S^*$ and $\varrho(A) \neq \emptyset$ [4]. In contrast, here we are interested in regular extensions, which are defined as follows:

Definition 6.6. *Let S be a symmetric operator on a Hilbert space $\mathcal{H}$ with deficiency index $(1,1)$. A closed linear relation A on $\mathcal{H}$ is a regular extension of S if $S \subset A$ and $\varrho(A) \neq \emptyset$.*

The difference to the classical theory is that we do *not* assume that $A \subset S^*$. In other words, regular extensions can be thought of as one dimensional extensions of S where we do not impose any further symmetry conditions.

In what follows, we will present the main result of Paper IV, which characterizes regular extensions in terms of a generalized Krein-type resolvent formula. First, we will state our result in full generality. In the second part, we will explain what this means for the multiplication operator by the independent variable on $\mathcal{L}(q)$, where q is a Herglotz-Nevanlinna function. Recall that these operators are models for simple symmetric operators with deficiency index $(1,1)$. We recommend that the reader who is not familiar with symmetric operators focus on the second part, because the result can be stated in much simpler terms in that framework.

So let S be a symmetric operator with deficiency index $(1,1)$ on a Hilbert space $\mathcal{H}$ and fix a self-adjoint extension A. Moreover, let us also fix a non-trivial element ϕ in the defect space $\eta_i(S)$ at the point i. Now for a given *target* vector $v \in \mathcal{H}$ we define the "two-sided" Q-function as

$$q(\zeta) = -i \cdot [\phi, v]_{\mathcal{H}} + (\zeta - \bar{i}) \left[\left(I + (\zeta - i)(A - \zeta)^{-1} \right) \phi, v \right]_{\mathcal{H}} .$$

Note that if $\phi = v$, then q is a Herglotz-Nevanlinna function and we are back in the classical framework. A simple polarization arguments shows that q can always be represented as

$$q = (h_1 - h_2) + i \cdot (h_3 - h_4), \tag{6.3}$$

where $h_j,\ j \in \{1,2,3,4\}$ are Herglotz-Nevanlinna functions. This means that q is a Quasi-Herglotz function, which were initially introduced and studied in [18].

Now we are almost ready to state our main theorem. Lastly, in order to simplify the presentation, we define for a given $x \in \mathcal{H}$ the function φ_x as

$$\varphi_x(w) := \big(I + (w-i)(A-w)^{-1}\big)\, x \quad \forall w \in \varrho(A).$$

Theorem 6.7 (Paper IV, Theorem 3.5)**.** *Let S be a symmetric operator with deficiency index $(1,1)$ on a Hilbert space $\mathcal{H}$ and fix a self-adjoint extension A and a non-trivial element ϕ in the defect space $\eta_i(S)$.*

First, let $v \in \mathcal{H}$ be a target element, q the associated "two-sided" Q-function and $c \in \mathbb{C}$ be such that $q + c$ is not identically zero. Then there exists a regular extension $\tilde{A}_{v,c}$ such that

$$\varrho(A) \setminus \{\zeta : q(\zeta) + c \neq 0\} \subset \varrho(\tilde{A}_{v,c})$$

and

$$(\tilde{A}_{v,c} - \zeta)^{-1} = (A-\zeta)^{-1} - \frac{[\,\cdot\,, \varphi_\phi(\overline{\zeta})]_{\mathcal{H}}}{q(\zeta)+c} \cdot \varphi_v(\zeta) \tag{6.4}$$

Conversely, let $\tilde{A}$ be a regular extension of S. Then there exists

- *a target vector $v \in \mathcal{H}$ with associated Q-function q*
- *a constant $c \in \mathbb{C}$ such that $q + c$ is not identically zero*

such that $\tilde{A} = \tilde{A}_{v,c}$.

This means that a regular extension is determined by the pair of parameters (v,c) in terms of a Krein type resolvent formula. We recover the self-adjoint extensions by setting $v := \phi$ and choosing $c \in \mathbb{R}$.

Now let us explain what all this means for the operator

$$S_q : \{f \in \mathcal{L}(q) : \zeta \cdot f(\zeta) \in \mathcal{L}(q)\} \to \mathcal{L}(q), \quad S_q(f)(\zeta) = \zeta \cdot f(\zeta),$$

where q is a Herglotz-Nevanlinna function. To this end, we define the space $\mathcal{M}(q)$ as

$$\mathcal{M}(q) := \{g : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C} \text{ analytic } \mid D_w(g) \in \mathcal{L}(q) \quad \forall w \in \mathbb{C}\setminus\mathbb{R}\}.$$

Note that $\mathcal{L}(q) \subset \mathcal{M}(q)$, because $\mathcal{L}(q)$ is invariant under the difference quotient operator. The functions in $\mathcal{M}(q)$ then completely describe the regular extensions in the following way:

Theorem 6.8 (Paper IV, Theorem 4.2)**.** *Let q be a Herglotz-Nevanlinna function, S_q the multiplication operator by the independent variable on $\mathcal{L}(q)$ and A the self-adjoint extension satisfying $(A-w)^{-1} = D_w$.*

Then every function $g \in \mathcal{M}(q) \setminus \{0\}$ defines a regular extension A_g of S_q via the formula

$$(A_g - w)^{-1}(f) = D_w(f) - \frac{f(w)}{g(w)} \cdot D_w(g), \quad w \in \varrho(A) : g(w) \neq 0. \quad (6.5)$$

And conversely, every regular extension is of this form.

When we compare this to equation (6.4), then $D_w = (A-w)^{-1}$ is the part given by the resolvent of a fixed self-adjoint extension and g is the "two-sided" Q-function. The self-adjoint extensions are given by the choice $g = q + c$, where $c \in \mathbb{R}$. Indeed, in this case, equation (6.5) simplifies to the formula given in Proposition 5.6 [1].

In addition to the generalized Krein extension formula, Paper IV also includes a discussion of the spectrum of these extensions.

6.2.2 Related results

Let S be a symmetric operator on a Hilbert space $\mathcal{H}$. As already mentioned, the classical extension theory is concerned with self-adjoint extensions, or, slightly more general, extensions satisfying $S \subset A \subset S^*$ and $\varrho(A) \neq \emptyset$.

In this introduction, we have discussed the classical theory when S has deficiency indices $(1,1)$. Starting from that, there are again roughly two different ways to proceed. First, one can consider higher deficiency indices, in which case Krein's resolvent formula involves matrix-valued Nevanlinna functions. And second, one can consider symmetric operators and self-adjoint extensions on indefinite inner product spaces. In the case of Pontryagin spaces, Krein's resolvent formula then involves generalized Nevanlinna functions. Of course, one can also proceed in both directions simultaneously.

A complete account of the current state of research concerning the Hilbert space theory can be found in the comprehensive textbook [4]. A good source for the indefinite theory is [7].

When it comes to regular extensions, meaning extensions that merely satisfy $S \subset A$, significantly less is known. The most important result in this area, and also the closest one to ours, is the model theory developed

[1] Note that $N_q(\cdot, \overline{w}) = D_w(q)$.

in [3]. There they consider simple symmetric operators with deficiency index $(1, 1)$, but additionally assume that the self-adjoint extension have no absolutely continuous spectrum. This, and the simplicity assumption, are significant restrictions. In any case, their model also characterizes all regular extensions of these symmetric operators, but in a different way.

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