Endpoint estimates for bilinear operators

Sergi Arias

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Abstract

The present thesis is based on the material presented in three research papers, whose main goal is to obtain endpoint estimates for bilinear pseudodifferential operators. In particular, the study is focused on obtaining several estimates involving the endpoint space of functions with local bounded mean oscillation, denoted by bmo(R^n).

In Paper I we establish boundedness properties for bilinear Coifman-Meyer multipliers in the product spaces H^1(R^n) x bmo(R^n) and L^p(R^n) x bmo(R^n), with 1<p<infty. As a consequence, we are able to study the pointwise product of a function in bmo(R^n) with functions in the Hardy space H^1(R^n), in the local Hardy space h^1(R^n) and in L^p(R^n), with 1<p<infty.

Paper II is devoted to the study of endpoint estimates for bilinear pseudodifferential operators with symbol in the bilinear Hörmander class BS^m_{1,1}, involving bmo(R^n). In combination with the estimates in Paper I, we obtain fractional Leibniz rules for the product of a function in bmo(R^n) and a function in the Hardy space h^p(R^n), with 0<p<=infty.

In Paper III we continue our study on boundedness properties for bilinear pseudodifferential operators with symbol in BS^m_{1,1}. This time, we study the action of those operators on functions in Triebel-Lizorkin spaces of the type F^{n/p}_{p,q}(R^n). In particular, we obtain some estimates for the pointwise product of two functions in F^{n/p}_{p,q}(R^n) with 1<p<\infty, where the spaces involved fail to be multiplicative algebras.

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Abstract

The present thesis is based on the material presented in three research papers, whose main goal is to obtain endpoint estimates for bilinear pseudodifferential operators. In particular, the study is focused on obtaining several estimates involving the endpoint space of functions with local bounded mean oscillation, denoted by $\mathrm{bmo}(\mathbb{R}^n)$.

In Paper I we establish boundedness properties for bilinear Coifman-Meyer multipliers in the product spaces $H^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$, with $1 < p < \infty$. As a consequence, we are able to study the pointwise product of a function in $\mathrm{bmo}(\mathbb{R}^n)$ with functions in the Hardy space $H^1(\mathbb{R}^n)$, in the local Hardy space $h^1(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$, with $1 < p < \infty$.

Paper II is devoted to the study of endpoint estimates for bilinear pseudodifferential operators with symbol in the bilinear Hörmander class $BS^m_{1,1}$, involving $\mathrm{bmo}(\mathbb{R}^n)$. In combination with the estimates in Paper I, we obtain fractional Leibniz rules for the product of a function in $\mathrm{bmo}(\mathbb{R}^n)$ and a function in the Hardy space $h^p(\mathbb{R}^n)$, with $0 < p \leq \infty$.

In Paper III we continue our study on boundedness properties for bilinear pseudodifferential operators with symbol in $BS^m_{1,1}$. This time, we study the action of those operators on functions in Triebel-Lizorkin spaces of the type $F^{n/p}_{p,q}(\mathbb{R}^n)$. In particular, we obtain some estimates for the pointwise product of two functions in $F^{n/p}_{p,q}(\mathbb{R}^n)$ with $1 < p < \infty$, where the spaces involved fail to be multiplicative algebras.

Sammanfattning

Den här avhandlingen är baserad på materialet som presenteras i tre forskningsartiklar, vars huvudmål är att bevisa ändpunktsuppskattningar för bilineära pseudodifferentialoperatorer. Särskilt fokuserar studien på att erhålla flera uppskattningar som involverar ändpunktsfunktionella rum för funktioner med lokal begränsad medeloscillation, betecknad som $\mathrm{bmo}(\mathbb{R}^n)$.

I Artikel I etablerar vi begränsningsegenskaper för bilinjära Coifman-Meyer-multiplikatorer i produktrummet $H^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$ och $L^p(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$, med $1 < p < \infty$. Som en konsekvens av detta kan vi studera den punktvisa produkten av en funktion i $\mathrm{bmo}(\mathbb{R}^n)$ med funktioner i Hardyrummet $H^1(\mathbb{R}^n)$, i det lokala Hardyrummet $h^1(\mathbb{R}^n)$ och i $L^p(\mathbb{R}^n)$, med $1 < p < \infty$.

Artikel II ägnas åt studien av ändpunktsuppskattningar för bilinjära pseudodifferentialoperatorer med symboler i den bilinjära Hörmanderklassen $BS^m_{1,1}$, som involverar $\mathrm{bmo}(\mathbb{R}^n)$. I kombination med uppskattningarna i Artikel I får vi fraktionella Leibnizregler för produkten av en funktion i $\mathrm{bmo}(\mathbb{R}^n)$ och en funktion i Hardyrummet $h^p(\mathbb{R}^n)$, med $0 < p \leq \infty$.

I Artikel III fortsätter vi vår studie av begränsningsegenskaper för bilinjära pseudodifferentialoperatorer med symboler i $BS^m_{1,1}$. Den här gången studerar vi hur dessa operatorer verkar på funktioner i Triebel-Lizorkin-rum av typen $F^{n/p}_{p,q}(\mathbb{R}^n)$. Särskilt erhåller vi några uppskattningar för den punktvisa produkten av två funktioner i $F^{n/p}_{p,q}(\mathbb{R}^n)$ med $1 < p < \infty$, där de involverade rummen inte är multiplikativa algebror.

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this PhD thesis.

Paper I: **Some endpoint estimates for bilinear Coifman-Meyer multipliers**
Journal of Mathematical Analysis and Applications , **498** (2021), no. 2, Paper No. 124972, 27 pp.
Sergi Arias, Salvador Rodríguez-López

Paper II: **Endpoint estimates for bilinear pseudodifferential operators with symbol in $BS^m_{1,1}$**
Journal of Mathematical Analysis and Applications , **515** (2022), no. 1, Paper No. 126453, 23 pp.
Sergi Arias, Salvador Rodríguez-López

Paper III: **Bilinear pseudodifferential operators with symbol in $BS^m_{1,1}$ on Triebel-Lizorkin spaces with critical Sobolev index**
Collect. Math. (2023),
https://doi.org/10.1007/s13348-023-00400-0
Sergi Arias, Salvador Rodríguez-López

Author's contribution

In paper I, S.A. led the resolution of the proposed problem, and took main responsibility on its submission process and diffusion. In papers II and III, S.A. fundamentally contributed to set the research problem, its resolution, and took main responsibility on its submission processes and diffusion.

All authors were involved in discussing the results of all the papers and contributed to the final manuscript.

Papers not included in the thesis The following paper is also co-authored with S. Rodriguez-Lopez, but it is not included in the thesis:

A weighted generalisation of Carleman's inequality. *Math. Inequal. Appl.* 27 (2024), no. 4, 887–907, dx.doi.org/10.7153/mia-2024-27-60

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Contents

1. Introduction

The present thesis is based on the material contained in the three research papers presented below. The main purpose of this thesis is the study of boundedness properties for certain bilinear operators, of which we are interested in finding endpoint estimates involving the space of functions with local bounded mean oscillation, denoted by $\mathrm{bmo}(\mathbb{R}^n)$. We will refer to these papers as Paper I, Paper II and Paper III, following a chronological order of publication.

The aim of this introduction is to present the necessary foundational knowledge for understanding the contributions of Papers I, II, and III. We will provide an overview of key theories, concepts, and prior research papers relevant to our study. Additionally, we will introduce and describe the function spaces and bilinear operators that play a crucial role in our analysis.

1.1 Preliminaries

The Fourier transform is a powerful tool in mathematical analysis and it can be applied in several fields, such as harmonic analysis, partial differential equations or probability theory, among others. One instance of its usefulness is the fact that, due to its fundamental properties, it can turn convolution or partial differential equations into algebraic equations.

The main idea behind the Fourier transform is similar to that of the Fourier series. Given a periodic function (or signal) on some real interval, the role of the Fourier series is, roughly speaking, to decompose the function into a linear combination of sine and cosine functions (waveforms) with different natural frequencies. The amplitude associated to each frequency is given by the Fourier coefficients, which are obtained by distilling the function into each frequency component. Geometrically, these are obtained as the orthogonal projection of the signal on the space generated by each waveform, measuring the amount of the signal that aligns with each frequency.

In the non-periodic setting, where functions are defined across the entire Euclidean space, the Fourier transform aims to decompose functions into a combination of elementary waveforms with real frequencies. This is achieved by replacing series with integrals. Instead of discrete Fourier coefficients (which form a sequence of complex numbers), we now use a function defined across the entire space, known as the *Fourier transform*.

Let us recall that, for an integrable function f on $\mathbb{R}^n$, we define its Fourier transform by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \mathrm{d}x, \quad \xi \in \mathbb{R}^n,$$

where we simply write $x\xi$ to denote the inner product $x_1\xi_1 + ... + x_n\xi_n$. In this definition, the Fourier transform acts as a tool to analyse how different frequencies, represented by ξ, contribute to the original function f. From now on, we shall refer to x as the variable in the *spatial domain* and ξ as the variable in the *frequency domain*.

As in the discrete setting, where certain periodic functions can be recovered from their Fourier coefficients, we can also express a function f in terms of its Fourier transform, through the synthesis relation given by

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\xi}\, đ\xi, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $đ\xi = (2\pi)^{-n}\mathrm{d}\xi$ denotes the normalised Lebesgue measure. The relation in (1.1) is known as the *Fourier inversion formula*.

In general, not all integrable functions can be recovered from $\widehat{f}$ via (1.1). For the sake of avoiding technicalities in our discussion, every function considered in this introduction will have this property, since we will assume that all functions in this exposition belong to the *Schwartz class*, denoted by $\mathcal{S}(\mathbb{R}^n)$. Formally, those are smooth functions that, jointly with their derivatives, decay faster than the reciprocal of any polynomial.

Before giving a more rigorous definition of $\mathcal{S}(\mathbb{R}^n)$, let us introduce some notation that will be useful later. We recall that any element $\alpha = (\alpha_1, ..., \alpha_n)$ that belongs to $\mathbb{N}^n$ is called a *multi-index*, whose *length* is defined by

$$|\alpha| = \alpha_1 + ... + \alpha_n.$$

We also follow the standard notation that, given $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and a multi-index α, we write

$$x^\alpha := x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n},$$

while given a function f defined in $\mathbb{R}^n$, we use the notation

$$\partial^\alpha f(x_1, ..., x_n) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} ... \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f(x_1, ..., x_n).$$

The space of Schwartz functions, $\mathcal{S}(\mathbb{R}^n)$, is defined as the set of all smooth functions f satisfying

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta f(x) \right| < \infty, \tag{1.2}$$

for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$.

An essential property of the Fourier transform is that derivatives in the spatial domain become polynomials in the frequency domain. More specifically, for all $\alpha \in \mathbb{N}^n$ and all $\xi \in \mathbb{R}^n$, it holds that

$$\widehat{\partial^\alpha f}(\xi) = (i\xi)^\alpha \widehat{f}(\xi). \tag{1.3}$$

As a consequence of this behaviour, the Fourier transform becomes a very useful tool in the study of partial differential equations (see, for instance, [13, Sections 7.1 and 7.5] or [16, Section 4.3.1.b)]). Indeed, one of the most characteristic virtues of this property is that it can turn certain partial differential equations into algebraic equations or ordinary differential equations.

To illustrate the previous idea with an example (see for instance [53, Section 2.1] or [31, Section 1.5])), let us consider the *linear Schrödinger equation*

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = 0, & x \in \mathbb{R}^n,\, t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

where Δ denotes the Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

We shall also use this example to motivate some of the contributions of the present thesis and introduce the main elements involved.

We notice that, due to the property in (1.3), Δ satisfies the identity

$$\widehat{\Delta f}(\xi) = -|\xi|^2 \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \tag{1.5}$$

where $|\xi| = (\xi_1^2 + \ldots + \xi_n^2)^{1/2}$. In particular, the Fourier inversion formula (1.1) provides an alternative expression for Δf given by

$$\Delta f(x) = -\int_{\mathbb{R}^n} |\xi|^2 \widehat{f}(\xi) e^{ix\xi}\, đ\xi. \tag{1.6}$$

We shall notice that there is nothing special in the structure of Δ in order to obtain an expression as (1.6) since, in general, every differential operator with constant coefficients can be expressed in such a way. Indeed, let us use the notation $D := -i\partial$ and consider a generic *linear differential operator with constant coefficients* of order $N \in \mathbb{N}$, given by

$$p(D) := \sum_{|\alpha| \le N} c_\alpha D^\alpha, \quad \{c_\alpha : |\alpha| \le N\} \subseteq \mathbb{C}. \tag{1.7}$$

Then, applying (1.3), $p(D)f$ can be expressed as in (1.6) replacing the factor $-|\xi|^2$ by the polynomial

$$p(\xi) := \sum_{|\alpha|\leq N} c_\alpha \xi^\alpha.$$

More generally, for suitable functions m defined on $\mathbb{R}^n$ (not necessarily of polynomial type), one defines a broader class of operators, known as *Fourier multipliers*, associated to m by

$$m(D)f(x) := \int_{\mathbb{R}^n} m(\xi)\widehat{f}(\xi)e^{ix\xi}\, đ\xi, \quad x\in\mathbb{R}^n.$$

To fix ideas, in the setting of this introduction, it is enough to assume that m is smooth and has, at most, polynomial growth.

As we can see, a Fourier multiplier is a combination of three transformations: we filter the function f in the frequency ξ by applying the Fourier transform, we modify it with the factor $m(\xi)$ and, finally, we add the different frequencies by integrating over $\mathbb{R}^n$. The function m is usually referred to as the *symbol* or simply *(Fourier) multiplier*.

In a more general context, generalising the structure of differential operators with constant coefficients and that of Fourier multipliers, we can consider the wider class of *pseudodifferential operators*. These operators are defined by

$$T_\sigma f(x) := \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\xi}\, đ\xi, \quad x\in\mathbb{R}^n.$$

where $\sigma(x,\xi)$ is a certain function which is also called the *symbol* of T_σ. We shall observe that, in comparison to Fourier multipliers, pseudodifferential operators may have a symbol that not only depends on the variable ξ, but also on the variable in the spatial domain.

The symbols of these operators are assumed to satisfy certain growth conditions, that naturally appear imposed by the problem at hand. A classical example of symbols are those satisfying the so-called *Hörmander-Mihlin condition* (see for instance [18, Section 6.2.3]): for a given $\alpha\in\mathbb{N}^n$ there exists a constant $C_\alpha > 0$ such that for all $\xi\neq 0$

$$\left|\partial_\xi^\alpha \sigma(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|}. \tag{1.8}$$

Another relevant family of symbols are those belonging to the so-called *Hörmander classes* $S^m_{\rho,\delta}$, with $m\in\mathbb{R}$ and $0\leq\rho,\delta\leq 1$, which are characterised by satisfying that, given $\alpha,\beta\in\mathbb{N}^n$, there exists $C_{\alpha,\beta}>0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}. \tag{1.9}$$

The class $S^m_{1,0}$ recovers the class of symbols introduced by J.J Khon and L. Nirenberg (see for instance [30], [26]).

Going back to solving the Schrödinger equation presented in (1.4), we will apply the Fourier transform in the x-variable and use (1.5) to reduce ourselves to the study of the initial value problem

$$\begin{cases} i\partial_t \widehat{u}(\xi,t) - |\xi|^2\,\widehat{u}(\xi,t) = 0, & \xi \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ \widehat{u}(\xi,0) = \widehat{u_0}(\xi), & \xi \in \mathbb{R}^n. \end{cases}$$

This equation is an ordinary differential equation whose solution is given by

$$\widehat{u}(\xi,t) = e^{-it|\xi|^2}\widehat{u_0}(\xi), \quad \xi \in \mathbb{R}^n. \tag{1.10}$$

Hence, the Fourier inversion formula in (1.1) yields an explicit solution for (1.4), given by

$$u(x,t) = \int_{\mathbb{R}^n} e^{-it|\xi|^2}\widehat{u_0}(\xi)e^{ix\xi}\,đ\xi.$$

As a conclusion, we see that u is expressed as a pseudodifferential operator (or more precisely, a Fourier multiplier) applied to the initial data.

The theory of pseudodifferential operators started its development in the decades of 1950s and 1960s, and it was formalised in the works of J. J. Khon and L. Nirenberg [30], L. Hörmander [25] and A. Unterberger and J. Bokobza [55].

In the present thesis, one of the main objects of study is the bilinear counterpart of pseudodifferential operators. In the same manner, bilinear pseudodifferential operators arise naturally when studying certain partial differential equations. To explore further this concept, we introduce next a *quadratic nonlinear Schrödinger equation* and see how it is related to the study of some type of bilinear operators.

We are going to consider the initial value problem

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = u(x,t)^2, & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.11}$$

In order to find a solution, we proceed using the same approach detailed earlier for the linear case (see, for instance, the exposition in [31, Section 5.1]). To this end, we apply the Fourier transform in the x-variable to get

$$i\partial_t \widehat{u}(\xi,t) - |\xi|^2\,\widehat{u}(\xi,t) = \widehat{u(\cdot,t)^2}(\xi), \quad \xi \in \mathbb{R}^n, \tag{1.12}$$

which yields an equation with a more complex structure than the one obtained before when solving the linear Schrödinger equation. In addition, one can rewrite (1.12) as

$$\partial_t\left(e^{it|\xi|^2}\widehat{u}(\xi,t)\right) = -ie^{it|\xi|^2}\widehat{u(\cdot,t)^2}(\xi), \quad \xi \in \mathbb{R}^n. \tag{1.13}$$

Before continuing, we shall introduce the notation

$$[e^{it\Delta}f](x) := \int_{\mathbb{R}^n} e^{-it|\xi|^2}\widehat{f}(\xi)e^{ix\xi}\,đ\xi,$$

so that the solution to the linear equation in (1.4) can be simply denoted by $u(x,t) = [e^{it\Delta}u_0](x)$. Having this in mind, we integrate (1.13) in time between 0 and t, and we use the Fourier inversion formula to get that

$$u(x,t) = [e^{it\Delta}u_0](x) - i\int_0^t [e^{i(t-\nu)\Delta}u(\cdot,\nu)^2](x)\mathrm{d}\nu =: \Gamma_{u_0}(u). \tag{1.14}$$

The previous expression is a particular instance of *Duhamel's formula* (see [16, Section 2.3.1.c)] for its more general description).

As a consequence of the identity in (1.14), solutions to (1.11) can be seen as fixed points of the operator Γ_{u_0}. This fact suggests that a possible way to find a solution to our original quadratic non-linear equation is to apply a fixed point type theorem to the operator Γ_{u_0}.

In order to find existence of fixed points for Γ_{u_0} and capture features of the solution to our original equation, we need to consider a particular function space. In harmonic analysis, the most common function spaces vary between Lebesgue spaces, Sobolev-type spaces, Hardy spaces, spaces of functions with bounded mean oscillation or spaces of Besov and Triebel-Lizorkin type. We refer to Section 1.2 for the definitions and detailed information on those spaces.

The standard way of showing the existence of fixed points for Γ_{u_0} is as follows. To start with, we assume that the initial data u_0 belongs to a Banach function space, that we will denote by $(X, \|\cdot\|_X)$. Then, one shall prove the existence of a certain time $T = T(u_0) > 0$, which may depend on the data function u_0, such that Γ_{u_0} is a contraction on a ball centered at the origin, lying in the space $\mathcal{C}([-T,T],X)$. Here $\mathcal{C}([-T,T],X)$ denotes the space of functions which are continuous from $([-T,T],|\cdot|)$, being $|\cdot|$ the absolute value in $\mathbb{R}$, to $(X, \|\cdot\|_X)$, endowed with the norm

$$\|f\|_{\mathcal{C}([-T,T],X)} := \sup_{t\in[-T,T]} \|f(t)\|_X.$$

In order to find the time $T = T(u_0)$, we shall look for a positive real number $R = R(u_0)$, so that the closed ball centred at zero with radius R in $\mathcal{C}([-T,T],X)$ is mapped to itself after the action of Γ_{u_0}. Throughout this process, we shall need to estimate the $\|\cdot\|_{\mathcal{C}([-T,T],X)}$-norm of $\Gamma_{u_0}(u)$. One first step would be to use Minkowski's integral inequality to obtain

$$\begin{aligned}\|\Gamma_{u_0}(u)\|_{\mathcal{C}([-T,T],X)} \leq & \sup_{t\in[-T,T]} \left\|e^{it\Delta}u_0\right\|_X \\ & + \int_0^t \sup_{\nu\in[-T,T]} \left\|e^{i(t-\nu)\Delta}[u(\cdot,\nu)^2]\right\|_X \mathrm{d}\nu, \end{aligned} \tag{1.15}$$

in such a way that it becomes crucial to obtain estimates on both terms

$$\left\|e^{it\Delta}u_0\right\|_X \quad \text{and} \quad \left\|e^{i(t-\nu)\Delta}[u(\cdot,\nu)^2]\right\|_X. \tag{1.16}$$

The choice of X depends on which features of the solution (such as amplitude or regularity) one aims to study. In order to avoid technical discussions, a natural choice could be to consider X as a subspace of the Lebesgue space $L^2(\mathbb{R}^n)$, defined as the set of functions for which

$$\|f\|_{L^2(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}x\right)^{1/2} < \infty.$$

This choice is convenient due to the fact that the operator $e^{it\Delta}$ is unitary in $L^2(\mathbb{R}^n)$, meaning that $\left\|e^{it\Delta}f\right\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$. This property is a consequence of *Plancherel's Theorem*, which asserts that an $L^2(\mathbb{R}^n)$-function and its Fourier transform have the same norm, that is,

$$\|f\|_{L^2(\mathbb{R}^n)} = \left\|\widehat{f}\right\|_{L^2(\mathbb{R}^n)}.$$

More specifically, we consider X to be the subspace of $L^2(\mathbb{R}^n)$-functions whose first N derivatives also lie in $L^2(\mathbb{R}^n)$, denoted by $L^2_N(\mathbb{R}^n)$ and endowed with the norm

$$\|f\|_{L^2_N(\mathbb{R}^n)} := \left(\sum_{|\alpha|\leq N} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}.$$

The space $L^2_N(\mathbb{R}^n)$ is refered to as the *Sobolev space* of order N.

Before continuing, let us introduce some common notation in our field of study. From now on, we will write $A \lesssim B$ to indicate the existence of a constant $C > 0$ such that $A \leq CB$. Similarly, we will write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$ are satisfied simultaneously. Moreover, we shall use the common notation

$$\langle \xi \rangle := (1 + |\xi|^2)^{1/2}.$$

One can express the norm of spaces like $L^2_N(\mathbb{R}^n)$ in terms of the boundedness properties of a Fourier multiplier. More specifically, we observe that applying Plancherel's theorem and (1.3), the Sobolev norm can be rewritten as

$$\begin{aligned}\left(\sum_{|\alpha|\leq N} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2} &\approx \left(\int_{\mathbb{R}^n} \left(\sum_{|\alpha|\leq N} |\xi|^{2|\alpha|}\right) \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \\ &\approx \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^N \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \\ &= \left\|\langle D\rangle^N f\right\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{1.17}$$

This description allows to give a meaning to the notion of fractional regularity. More precisely, given $s \in \mathbb{R}$, one defines the *Sobolev space of order* s, denoted by $L^2_s(\mathbb{R}^n)$, as the set of functions f (or, more rigorously, tempered distributions, which are those linear functionals lying in the dual space of the Schwartz class) for which

$$\|f\|_{L^2_s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \langle \xi \rangle^s \widehat{f}(\xi) e^{ix\xi} \, đ\xi \right|^2 \mathrm{d}x \right)^{1/2} = \| \langle D \rangle^s f \|_{L^2(\mathbb{R}^n)} < \infty. \tag{1.18}$$

Sobolev spaces of fractional order, and many other function spaces in harmonic analysis, are part of a more general scale of spaces, known as *spaces of generalised smoothness*, which we introduce in Section 1.2.1.

As an alternative expression, in the case $s < 0$, the operator $\langle D \rangle^s$ admits the representation

$$\langle D \rangle^s f(x) = \int_{\mathbb{R}^n} K_s(x-y) f(y) \mathrm{d}y, \tag{1.19}$$

where the kernel K_s equals the inverse Fourier transform of the function $\langle \cdot \rangle^s$. We shall point out that the inverse Fourier transform of a Schwartz function is defined by

$$f \mapsto \int_{\mathbb{R}^n} f(\xi) e^{ix\xi} \, đ\xi.$$

Operators of the type in (1.19) are usually referred to as *potential type operators* and the function K_s is called the *potential function* (in the precise case of the operator $\langle D \rangle^s$ we call K_s the *Bessel potential* of order s). In this way, function spaces defined through a condition like (1.18) inherit the terminology *potential type spaces* (see [19, Sections 1.2 and 1.3] for further details and the precise description of the Bessel potential). Some additional information on those type of function spaces is given in Section 1.2.2.

Let us go back to estimate the quantities in (1.16). If we consider the choice $X = L^2_s(\mathbb{R}^n)$ and we take into account that the operator $e^{it\Delta}$ is unitary in $L^2(\mathbb{R}^n)$ (and hence in $L^2_s(\mathbb{R}^n)$), then finding estimates for the quantities in (1.16) reduce the matter to bound the norm $\left\| u(\cdot, \nu)^2 \right\|_{L^2_\nu(\mathbb{R}^n)}$. More precisely, we are interested in having estimates of the type

$$\left\| u(\cdot, \nu)^2 \right\|_{L^2_s(\mathbb{R}^n)} \leq C \left\| u(\cdot, \nu) \right\|_{L^2_s(\mathbb{R}^n)} \left\| u(\cdot, \nu) \right\|_{L^2_s(\mathbb{R}^n)}, \tag{1.20}$$

which, in particular, is known to hold for $s > n/2$. Indeed, the spaces $L^2_s(\mathbb{R}^n)$ are multiplicative algebras for $s > n/2$, which can be deduced from boundedness properties of bilinear operators and Sobolev-type embeddings, as we will discuss later in this introduction (see also [51] for a direct proof).

In order to conclude the argument for showing the existence of a fixed point for Γ_{u_0}, we combine (1.15) with (1.20) to obtain that

$$\|\Gamma_{u_0}(u)\|_{\mathcal{C}([-T,T],X)} \leq C\left(\|u_0\|_X + T\,\|u\|^2_{\mathcal{C}([-T,T],X)}\right),$$

for a positive constant C. Therefore, considering functions u in the closed ball of $\mathcal{C}([-T,T],X)$ centred at the origin with radius $R = 2C\,\|u_0\|_X$, yields

$$\|\Gamma_{u_0}(u)\|_{\mathcal{C}([-T,T],X)} \leq \frac{R}{2} + CTR^2 = R\left(\frac{1}{2} + CTR\right),$$

where the right hand side can be made smaller or equal than R if we choose $T \leq 1/2CR$. The bilinearity of the product allows to write

$$\Gamma_{u_0}(u) - \Gamma_{u_0}(v) = (u-v)u + v(u-v),$$

from which, by choosing R small enough, one obtains as a consequence, that Γ_{u_0} is a contraction on a closed ball in $\mathcal{C}([-T,T],X)$, from where the existence of a fixed point for Γ_{u_0} is deduced and, consequently, the existence of solutions for the quadratic non-linear equation described in (1.11).

Let us highlight at this point that the following two issues play a fundamental role in the previous discussion:

- **The choice of the function space** X**.** Qualitative features of solutions in terms of that of the initial data, is reflected in the choice of the function space. For instance, the Sobolev space $L^2_s(\mathbb{R}^n)$ chosen in the previous discussion, encodes information about size and regularity of its elements. More generally, the scale of Triebel-Lizorkin and Besov spaces generalise many of the classical function spaces, such as Lebesgue and Sobolev spaces, and give refined information of the size and regularity properties of their elements.

- **Bilinear estimates.** The bilinear estimates of the term u^2 on the chosen function space, as those in (1.20), are instrumental to show that the operator Γ_{u_0} has a fixed point.

We shall notice that the bilinearity of the product was also essential in the argument above. Actually, there are several type of non-linearities that may be treated as bilinear terms. For instance, let us consider a non-linearity of the type $F(u(x,t))$ with $F \in \mathcal{C}^\infty(\mathbb{R})$ such that $F(0) = 0$. In this case, we shall mention that if $u(x,t) \in L^2_s(\mathbb{R}^n)$ for $s > n/2$, then it is possible to write

$$F(u(x,t)) = \Pi(F'(u(x,t)), u(x,t)) + \text{error term},$$

where Π is a bilinear operator called *Bony's paraproduct*, which we define later in (1.43), and the error term is a function in a Sobolev space with regularity larger than s (see [9] or [14, Theorem 37]). As a consequence, having estimates of the type

$$\|\Pi(f,g)\|_{L^2_s(\mathbb{R}^n)} \leq C\,\|f\|_{L^2_s(\mathbb{R}^n)}\,\|g\|_{L^2_s(\mathbb{R}^n)}$$

for bilinear paraproducts would help us to study the well-posedness of the equation in (1.11) with $u(x,t)^2$ replaced by $F(u(x,t))$. In both Papers I and II we provide new estimates on bilinear paraproducts, which we describe in Section 1.3

In addition, we shall notice that by using twice the Fourier inversion formula, one can write

$$u(x,t)^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{u(\cdot,t)}(\xi)\widehat{u(\cdot,t)}(\eta)e^{ix(\xi+\eta)}\,đ\xi\,đ\eta.$$

In a similar fashion, the bilinear paraproduct Π admits the integral representation

$$\Pi(f,g) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\,đ\xi\,đ\eta, \tag{1.21}$$

for a certain smooth function σ. To be more precise, the function σ is supported on a set of the type $\{|\xi| > C|\eta|\}$, for a particular constant $C > 0$, where one of the frequencies dominates the other one. In addition, the function σ satisfies the condition of R. Coifman and Y. Meyer, which requires the function and its derivatives to satisfy a certain growth behaviour (see Section 1.3.1 for the precise details).

In this way, the bilinear paraproduct above belongs to the so called class of *bilinear Coifman-Meyer multipliers*. This kind of operators are studied in Paper I, where we obtain new boundedness properties for them, which are described later in Section 1.3.1.

More generally, the expression on the right hand side of (1.21), under some essential requirements for σ, not necessarily the ones of R. Coifman and Y. Meyer, belongs to the class of *bilinear multiplier* operators.

Analogously to the pseudodifferential operators exposed earlier, if the function σ in the right hand side of (1.21) depends also on the spatial variable x, we obtain a new class of operators. Indeed, we define *bilinear pseudodifferential operators* by

$$T_\sigma(f,g)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\,đ\xi\,đ\eta, \tag{1.22}$$

where the function σ, called the *symbol* of the operator, is required to satisfy certain growth conditions.

Starting with the seminal work of R. R. Coifman and Y. Meyer [14], there is a wide variety of authors whose work allowed to develop the theory of bilinear pseudodifferential operators, such as L. Grafakos and R. Torres [22], Á. Bényi and R. Torres [5; 6], Á. Bényi, D. Maldonado, V. Naibo and R. Torres [4], Á. Bényi, F. Bernicot, D. Maldonado and V. Naibo [3], A. Miyachi and N. Tomita [37–39], N. Michalowski, D. Rule and W. Staubach [36], V. Naibo [41; 42], K. Koezuka and N. Tomita [29] or V. Naibo and A. Thomson [44], among others. We refer to [43] for a recent survey on the topic.

As a summary of the discussion above, we conclude that having estimates of the type

$$\|T_\sigma(f,g)\|_Z \leq C\,\|f\|_X\,\|g\|_Y \tag{1.23}$$

for bilinear pseudodifferential operators is useful, for instance, to obtain well-posedness results for some non-linear partial differential equations. In the three papers Paper I, Paper II and Paper III we obtain new estimates of the type in (1.23) for bilinear operators, that we will expose later in Section 1.3.

As an example, we show in Corollary 4.7 in Paper III that for a small enough time $T>0$, and given a data function $u_0 \in L^2_{n/2}(\mathbb{R}^n)$, the initial value problem

$$\begin{cases} i\partial_t u + |D|^s u = m(D)(T_\sigma(u,u)) \\ u(t,0) = u_0 \end{cases} \tag{1.24}$$

has a unique solution in $\mathcal{C}([0,T], L^2_{n/2}(\mathbb{R}^n))$. Here m is a linear Fourier multiplier which must satisfy a certain growing condition and T_σ is a bilinear pseudodifferential operator whose symbol σ lies in one of the bilinear Hörmander classes introduced in Section 1.3.2. This well-posedness result follows from an estimate of the type in (1.23) for a certain scale of function spaces that include the Sobolev space $L^2_{n/2}(\mathbb{R}^n)$ (see Theorem 2.3.3 below for the precise statement).

In addition, the equation in (1.24) recovers some well-known equations in mathematical physics for different choices of $s>0$. For instance, in the case $s=2$, it becomes the *Schrödinger equation* and for $s=4$, it becomes the *biharmonic Schrödinger equation*. Moreover, for $s=1$, the equation is a *half-wave equation*, while given $0<s<2$ with $s\neq 1$, we recover the so-called *fractional Schrödinger equation*.

To conclude this section, let us introduce a type of estimates that can be derived from the study of bilinear pseudodifferential operators. To fix ideas, let us first consider two differentiable functions defined in $\mathbb{R}$. The product of those is differentiable and its derivative is given by the Leibniz rule

$$(fg)' = f'g + fg'.$$

If we would like to estimate the size of the derivative $(fg)'$, a good way to do so would be to consider its norm in some Lebesgue space $L^p(\mathbb{R})$ with $0 < p \leq \infty$ (see Section 1.2 for the precise defintion). To this end we can make use of Hölder's inequality (detailed in (1.41) below) to get that for any $1/2 \leq p \leq \infty$ and $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ such that $1/p = 1/p_1 + 1/q_1 = 1/p_1 + 1/q_2$, it holds that

$$\left\|(fg)'\right\|_{L^p(\mathbb{R})} \lesssim \left\|f'\right\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{q_1}(\mathbb{R})} + \|f\|_{L^{p_2}(\mathbb{R})} \left\|g'\right\|_{L^{q_2}(\mathbb{R})}.$$

The last inequality has a similar counterpart for derivatives of higher (and fractional) order of functions in $\mathbb{R}^n$. More precisely, for $s > 0$, we have the estimate

$$\|\langle D\rangle^s (fg)\|_{L^p(\mathbb{R}^n)} \lesssim \|\langle D\rangle^s f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{q_1}(\mathbb{R}^n)} + \|f\|_{L^{p_2}(\mathbb{R}^n)} \|\langle D\rangle^s g\|_{L^{q_2}(\mathbb{R}^n)}, \tag{1.25}$$

for those indexes for which $1 < p < \infty$, $1 < p_1, p_2, q_1, q_2 \leq \infty$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ (further details are given in Section 1.4). This kind of estimates are known as *Kato-Ponce inequalities*, which take their name after the seminal work of T. Kato and G. Ponce [28], who introduced those in their study of local existence results for the nonstationary Navier-Stokes and Euler equations. Several authors studied these kind of inequalities since its appearance in the late 80s and, as a main reference for this thesis, we used the works of L. Grafakos and S. Oh [21], L. Grafakos, D. Maldonado and V. Naibo [20], V. Naibo and A. Thomson [45], J. Brummer and V. Naibo [10; 11] and K. Koezuka and N. Tomita [29].

The relation of those type of inequalities with bilinear operators lies in the fact that, roughly speaking, the bilinear operator $\langle D\rangle^s (fg)$ can be decomposed as a sum of bilinear Coifman-Meyer multipliers. Therefore, having estimates on these yields inequalities of Kato-Ponce type.

In this thesis we are interested in inequalities as (1.25) where some indexes lie at the endpoint of their range, which we refer to as *endpoint estimates*. We started an investigation on Kato-Ponce inequalities in Paper I which we completed in Paper II, as we describe with more detail in Section 1.4 later. Those kind of estimates play a role in establishing some algebraic properties of function spaces. To illustrate this, consider the particular case where $q_1 = p_2 = \infty$ and $p = p_1 = q_2 = 2$, for which (1.25) yields

$$\|fg\|_{L^2_s(\mathbb{R}^n)} \lesssim \|f\|_{L^2_s(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^2_s(\mathbb{R}^n)}. \tag{1.26}$$

Under the condition $s > n/2$, the *classical Sobolev embedding theorem* asserts that distributions with enough regularity are indeed continuous and bounded functions. More specifically, it holds that

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{L^2_s(\mathbb{R}^n)}, \quad \text{if } s > n/2.$$

Combining the previous embedding with the estimate in (1.26), we obtain that $L^2_s(\mathbb{R}^n)$ has the structure of a multiplicative algebra under the condition $s > n/2$, since it holds that

$$\|fg\|_{L^2_s(\mathbb{R}^n)} \lesssim \|f\|_{L^2_s(\mathbb{R}^n)} \|g\|_{L^2_s(\mathbb{R}^n)} .$$

Furthermore, if $s \leq n/2$ then the embedding does not hold and neither is $L^2_s(\mathbb{R}^n)$ an algebra under that range of s (see for instance [51]). However, a natural question is whether we can quantify how far is the space $L^2_s(\mathbb{R}^n)$ from being an algebra.

In that direction, it is known that the product of two functions in $L^2_{n/2}(\mathbb{R}^n)$ can take place in any Sobolev space with less regularity than $n/2$ (see for instance [2, Proposition 2.3]). More specifically, for any $\varepsilon > 0$, we have the inequality

$$\|fg\|_{L^2_{n/2-\varepsilon}(\mathbb{R}^n)} \lesssim \|f\|_{L^2_{n/2}(\mathbb{R}^n)} \|g\|_{L^2_{n/2}(\mathbb{R}^n)} . \tag{1.27}$$

In Paper III we show that the estimate

$$\left\| \frac{1}{\mathrm{w}}(D)(fg) \right\|_{L^2_{n/2}(\mathbb{R}^n)} \lesssim \|f\|_{L^2_{n/2}(\mathbb{R}^n)} \|g\|_{L^2_{n/2}(\mathbb{R}^n)} \tag{1.28}$$

holds for a smooth function w that behaves pointwise like the logarithmic function $w(t) = (1 + \log_+ t)^{1/2}$. The estimate in (1.28) recovers that in (1.27) since the left hand side in (1.28) can be shown to be an upper bound for $\|fg\|_{L^2_{n/2-\varepsilon}(\mathbb{R}^n)}$. In addition, we also show that (1.28) is sharp in the sense that the exponent $1/2$ of the logarithmic weight w cannot be made smaller.

Now that the main objects of study in this thesis have been put in some context and, hopefully, their usefulness has been clarified throughout the above exposition, we proceed to present in the upcoming sections more detailed and precise information on our contributions.

1.2 Function spaces of generalised smoothness

In harmonic analysis we find a vast amount of function spaces which measure different properties of functions, such as their size or regularity. For instance, a basic example that could come to mind would be the classical *Lebesgue spaces*. We denote them by $L^p(\mathbb{R}^n)$, with $0 < p \leq \infty$, and they allow us to measure the size of a function, which is said to belong to $L^p(\mathbb{R}^n)$ if

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \right)^{1/p} < \infty,$$

when $0 < p < \infty$, and

$$\|f\|_{L^\infty(\mathbb{R}^n)} := \inf\{C \geq 0 : |f(x)| \leq C \quad \text{for} \quad \text{a.e} \quad x \in \mathbb{R}^n\}.$$

In a different direction, we may also think of function spaces which encode information on the regularity of a function. One example could be the class $\mathcal{C}_b^m(\mathbb{R}^n)$, $m \in \mathbb{N}$, consisting of those functions f for which $\partial^\alpha f$ is bounded and uniformly continuous on $\mathbb{R}^n$, for all $|\alpha| \leq m$. More generally, one may also consider the spaces of *Hölder type* $C^s(\mathbb{R}^n)$, $s > 0$. In that case, if $[s]$ and $\{s\}$ respectively denote the integer and fractionary part of s, a function f belongs to $C^s(\mathbb{R}^n)$ if $f \in \mathcal{C}_b^{[s]}(\mathbb{R}^n)$ and if there exists a constant $C > 0$ such that

$$|\partial^\alpha f(x) - \partial^\alpha f(y)| \leq C|x-y|^{\{s\}}$$

for any multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = [s]$ and all $x, y \in \mathbb{R}^n$, with $x \neq y$.

A more sophisticated example would include the scale of *Sobolev spaces*, which measure the regularity of a function, as well as its size and that of its derivatives. Those spaces, that we will denote by $L_m^p(\mathbb{R}^n)$, with $1 < p < \infty$ and $m \in \mathbb{N}$, consist of those functions satisfying

$$\|f\|_{L_m^p(\mathbb{R}^n)} := \left(\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p} < \infty.$$

1.2.1 The scale of Besov and Triebel-Lizorkin spaces

All the spaces mentioned before, each of them being very useful to analyse different properties of functions, can be unified in a single scale: the class of *Besov and Triebel-Lizorkin spaces*. The underlying feature of these spaces relies on taking advantage of the so-called *Littlewood–Paley theory*, which encodes properties of functions through frequency localisation.

Indeed, Fourier Analysis lies in the core of the theory initiated by J. E. Littlewood and R. E. A. C. Paley [32–34]. The underlying idea is to decompose the frequency space in dyadic annulus associated to a dyadic partition of unity, and thus expressing functions as a sum of a countable number of analytic functions, whose Fourier transform is supported in those annulus. More specifically, given a positive and radially decreasing Schwartz function φ_0 , supported in the ball $\{|\xi| \leq 3/2\}$, which is identically one on $\{|\xi| \leq 1\}$, we define for $\xi \in \mathbb{R}^n$

$$\varphi(\xi) := \varphi_0(\xi) - \varphi_0(2\xi), \qquad \text{and} \qquad \varphi_j(\xi) := \varphi(2^{-j}\xi), \quad j \geq 1.$$

We notice that each φ_j is supported in the annulus $\{2^{j-1} \leq |\xi| \leq 3 \cdot 2^{j-1}\}$ and it holds that

$$1 = \sum_{j \geq 0} \varphi_j(\xi), \quad \text{for all } \xi \in \mathbb{R}^n. \tag{1.29}$$

For the sake of simplicity, we restrict the next discussion to functions in the Schwartz class, but the reader will have no trouble extending it to the space of tempered distributions. In this way, for a Schwartz function f, the identity in (1.29) yields

$$f(x) = \int 1\widehat{f}(\xi)e^{ix\xi}\, đ\xi = \sum_{j\geq 0} \varphi_j(D)f(x),$$

with

$$\varphi_j(D)f(x) = \int \varphi_j(\xi)\widehat{f}(\xi)e^{ix\xi}\, đ\xi.$$

We may see each of the terms $\varphi_j(D)f$ as a filter for the frequencies of order 2^j of f.

Due to the support properties of the partition of unity $\{\varphi_j\}_{j\geq 0}$, and using Plancherel's identity, one can show the existence of constants $C, D > 0$ such that

$$C\sum_{j\geq 0} \left\|\varphi_j(D)f\right\|_{L^2}^2 \leq \|f\|_{L^2}^2 \leq D\sum_{j\geq 0} \left\|\varphi_j(D)f\right\|_{L^2}^2. \tag{1.30}$$

In this way, one can identify $L^2(\mathbb{R}^n)$ with the space of functions for which

$$\left\{\left\|\varphi_j(D)f\right\|_{L^2}\right\}_{j\geq 0} \in \ell^2(\mathbb{N}_0),$$

where $\ell^2(\mathbb{N}_0)$ denotes the L^2-Lebesgue sequence space, defined by

$$\ell^2(\mathbb{N}_0) = \left\{ (a_n)_{j\in\mathbb{N}_0} : \left(\sum_{j\in\mathbb{N}_0} \left|a_j\right|^2\right)^{1/2} < \infty \right\}.$$

As mentioned in Section 1.1, for a given multi-index $\alpha \in \mathbb{N}^n$, the operator ∂^α can be expressed as

$$\partial^\alpha f(x) = \int_{\mathbb{R}^n} (i\xi)^\alpha \widehat{f}(\xi)e^{ix\xi}\, đ\xi. \tag{1.31}$$

Using this fact together with Plancherel's formula, and taking into account the support properties of φ_j, we have the equivalence

$$\left\|\varphi_j(D)(\partial^\alpha f)\right\|_{L^2(\mathbb{R}^n)} \approx 2^{j|\alpha|} \left\|\varphi_j(D)f\right\|_{L^2(\mathbb{R}^n)}, \tag{1.32}$$

for any multi-index $\alpha \in \mathbb{N}^n$ and all $j \geq 1$. The previous equivalence and (1.30) allow us to identify the Sobolev space $L^2_m(\mathbb{R}^n)$ with the space of functions satisfying

$$\sum_{j\geq 0} 2^{j2m} \left\|\varphi_j(D)f\right\|_{L^2}^2 = \left\| \sum_{j\geq 0} 2^{j2m} \left|\varphi_j(D)f\right|^2 \right\|_{L^2}^2 < +\infty$$

or, equivalently,

$$\left\{2^{jm}\left\|\varphi_j(D)f\right\|_{L^2}\right\}_{j\geq 0}\in \ell^2(\mathbb{N}_0).$$

We shall observe that in the previous discussion there were three parameters involved: the regularity m, the exponent of the Lebesgue norm and that of the discrete norm. Bearing this example in mind, one defines the Besov and Triebel Lizorkin spaces, expressed in terms of Littlewood-Paley decompositionsas follows.

Definition 1.2.1. Let $s\in\mathbb{R}$, $0<p\leq\infty$ and $0<q\leq\infty$.

- We define the *Besov space*, $B^s_{p,q}(\mathbb{R}^n)$, to be the set of all tempered distributions f for which

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq}\left\|\varphi_j(D)f\right\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} < \infty, \tag{1.33}$$

  when $q<\infty$, and

$$\|f\|_{B^s_{p,\infty}(\mathbb{R}^n)} := \sup_{j\geq 0}\left(2^{js}\left\|\varphi_j(D)f\right\|_{L^\infty(\mathbb{R}^n)}\right) < \infty. \tag{1.34}$$

- If $0<p<\infty$, we define the *Triebel-Lizorkin space*, $F^s_{p,q}(\mathbb{R}^n)$, to be the set of all tempered distributions f for which

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} := \left\|\left(\sum_{j=0}^{\infty} 2^{jsq}\left|\varphi_j(D)f\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} < \infty, \tag{1.35}$$

  when $q<\infty$, and

$$\left\|\sup_{j\geq 0}\left(2^{js}\left|\varphi_j(D)f\right|\right)\right\|_{L^p(\mathbb{R}^n)} < \infty. \tag{1.36}$$

From now on we will use $A^s_{p,q}(\mathbb{R}^n)$ to denote either $B^s_{p,q}(\mathbb{R}^n)$ or $F^s_{p,q}(\mathbb{R}^n)$ when we refer to both spaces at the same time.

Remark 1.2.2. As mentioned earlier, many classical spaces can be expressed in the scale $A^s_{p,q}(\mathbb{R}^n)$. For instance, we have the following identifications between function spaces, in the sense of equivalent norms, given by

$$L^p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n) \quad \text{for} \quad 1<p<\infty,$$

$$C^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n) \quad \text{for} \quad s>0,$$

and

$$L^p_m(\mathbb{R}^n) = F^m_{p,2}(\mathbb{R}^n) \quad \text{for} \quad 1 < p < \infty, \quad m \in \mathbb{N}.$$

Similarly, many other spaces in harmonic analysis are encompassed within that scale, such as local Hardy spaces, inhomogeneous Sobolev spaces or the space of functions with local bounded mean oscillation (see [54, Section 2.3.5] for more details).

The indexes p and q in the spaces $A^s_{p,q}(\mathbb{R}^n)$ measure the size of the functions while the index s encodes information on their regularity, as suggested by (1.32). Furthermore, the differential operator $(I-\Delta)^m$, with $m \in \mathbb{N}$, maps isomorphically $A^s_{p,q}(\mathbb{R}^n)$ onto $A^{s-m}_{p,q}(\mathbb{R}^n)$ (see [54, Section 2.3.8]). In other words, we get the equivalence

$$\|(I-\Delta)^m f\|_{A^0_{p,q}(\mathbb{R}^n)} \approx \|f\|_{A^{2m}_{p,q}(\mathbb{R}^n)} \tag{1.37}$$

for any $m \in \mathbb{N}$, so that the order of the differential operator is transferred to the s-index of the space.

In the way of extending the scope of $A^s_{p,q}(\mathbb{R}^n)$, the so-called *spaces of generalised smoothness* started to appear in the decade of 1980s. Some authors generalised the spaces of Besov and Triebel-Lizorkin type using different approaches: see, for instance, the works of C. Merucci [35] or G. A. Kalyabin and P. I. Lizorkin [27].

The more relevant version of those generalisations for the present thesis are the Besov and Triebel-Lizorkin spaces of generalised smoothness introduced by D. Edmunds and H. Triebel [15] and S. Moura [40]. In their study, the authors define the spaces $B^{s,w}_{p,q}(\mathbb{R}^n)$ and $F^{s,w}_{p,q}(\mathbb{R}^n)$ in terms of an admissible weight w. The quasinorms defining this broader scale of spaces is almost identical to the classical ones. More precisely, we define

$$\|f\|_{B^{s,w}_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq} w(2^{-j})^q \left\| \varphi_j(D) f \right\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q},$$

for $s \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q < \infty$, in the case of $B^{s,w}_{p,q}(\mathbb{R}^n)$. Regarding $F^{s,w}_{p,q}(\mathbb{R}^n)$, the norm is defined by

$$\|f\|_{F^{s,w}_{p,q}(\mathbb{R}^n)} := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} w(2^{-j})^q \left| \varphi_j(D) f \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q < \infty$.

If $q = \infty$, the norm of both $B^{s,w}_{p,\infty}(\mathbb{R}^n)$ and $F^{s,w}_{p,\infty}(\mathbb{R}^n)$ is defined as in (1.34) and (1.36), respectively, replacing the term 2^{js} by $2^{js} w(2^{-j})$.

In our work, slightly modifying the original definition by D. Edmunds and H. Triebel [15], we consider the following class of weight functions to be admissible.

Definition 1.2.3. Let $w:(0,1]\to(0,\infty)$ be a monotonic function and extend it to $w:(0,\infty)\to(0,\infty)$ by defining $w(t)=w(1)$ for all $t\geq 1$. We say that w is an *admissible* weight if there exist $c,d>0$ such that

$$cw(2^{-j})\leq w(2^{-2j})\leq dw(2^{-j}),\quad j\geq 0. \tag{1.38}$$

Example 1.2.4. The kind of admissible weights that the reader may bear in mind are functions of the form

$$w(t):=(1+\log_+ 1/t)^{\lambda}\left(1+\log(1+\log_+ 1/t)\right)^{\mu},$$

with λ, $\mu\in\mathbb{R}$ and $\lambda\cdot\mu\geq 0$.

We shall point out that the index s in $A^{s,w}_{p,q}(\mathbb{R}^n)$ still encodes the essential information on the regularity, since w just entails a slight modification. Indeed, the contribution of admissible weights is basically of logarithmic type. To be more precise, for any admissible weight it is possible to find $c_1,c_2>0$ and $b\geq 0$ such that

$$c_1\frac{1}{(1+|\log t|)^b}\leq w(t)\leq c_2(1+|\log t|)^b,\quad t>0. \tag{1.39}$$

To conclude this section, we shall observe that in Definition 1.2.1 the case $p=\infty$ was excluded when defining Triebel-Lizorkin spaces. Indeed, one of the essential properties satisfied by the quasinorms described in (1.33) and (1.35) is that they do not depend on the underlying resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ (see [54, Proposition 1]). However, in the case where $p=\infty$ and $0<q<\infty$ the quantity in (1.35) may depend on the choice of $\{\varphi_j\}_{j=0}^{\infty}$ (see [54, Remark 2.3.1/4]). To complete the picture, the norms in (1.34) and (1.36) are equivalent when $p=q=\infty$, so that the space $F^s_{\infty,\infty}(\mathbb{R}^n)$ can be defined by using (1.36) and it coincides with $B^s_{\infty,\infty}(\mathbb{R}^n)$.

Due to the mentioned issues, an alternative description to (1.35) shall be given for $p=\infty$ and $0<q<\infty$, so that the spaces $F^s_{\infty,q}(\mathbb{R}^n)$ can be defined in a coherent way. We used in this thesis the following definition, introduced by M. Frazier and B. Jawerth in [17, Section 12].

Definition 1.2.5. Let $\mathcal{D}$ be the set of all dyadic cubes in $\mathbb{R}^n$ and $0<q<\infty$. We define $F^{s,w}_{\infty,q}(\mathbb{R}^n)$ to be the set of all tempered distributions f for which

$$\|f\|_{F^s_{\infty,q}(\mathbb{R}^n)}:=\|\varphi_0(D)f\|_\infty + \sup_{\substack{Q\in\mathcal{D}\\ \ell(Q)\leq 1}}\left(\frac{1}{|Q|}\int_Q\sum_{j=-\log_2\ell(Q)}^{\infty}2^{sjq}\left|\varphi_j(D)f(x)\right|^q\mathrm{d}x\right)^{1/q}$$

is finite, where $\{\varphi_j\}_{j\geq 0}$ is a resolution of unity as in (1.29). Here $\ell(Q)$ denotes the side lenght of the cube Q.

However, we could not find in the literature a version of the spaces $F^s_{\infty,q}(\mathbb{R}^n)$ with a smoothness of generalised type. Therefore, one of the contributions of the present thesis is to define such spaces (Definition 2.7 in Paper II), which are denoted by $F^{s,w}_{\infty,q}(\mathbb{R}^n)$, where w is an admissible weight. We also provide later in Paper III a detailed proof on the fact that these spaces do not depend on the underlying resolution of unity and, additionally, we also give some proofs in Paper IIIfor certain lifting properties satisfied by those spaces (see Propositions 2.10, 2.15 and 2.17 in Paper III).

The definition that we introduced for those spaces is analogous to the one given for $F^{s,w}_{p,q}(\mathbb{R}^n)$ when $p \neq \infty$, where the dyadic term 2^{js} is replaced by $2^{js}w(2^{-j})$. More precisely, we define the norm in $F^{s,w}_{\infty,q}(\mathbb{R}^n)$ by

$$\|f\|_{F^{s,w}_{\infty,q}(\mathbb{R}^n)} := \|\varphi_0(D)f\|_\infty + \sup_{\substack{Q\in\mathcal{D}\\ \ell(Q)\leq 1}} \left(\frac{1}{|Q|}\int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} 2^{sjq} w(2^{-j})^q \left|\varphi_j(D)f(x)\right|^q \mathrm{d}x\right)^{1/q}.$$

1.2.2 Potential-type spaces

As we saw earlier in Section 1.1, we may extend the notion of Sobolev spaces from having a natural order to having a fractionary order $s \in \mathbb{R}$, in terms of a Fourier multiplier, through the condition

$$\|f\|_{L^2_s(\mathbb{R}^n)} = \|\langle D\rangle^s f\|_{L^2(\mathbb{R}^n)} < \infty.$$

Those spaces can be extended in a natural manner to the larger scale of *inhomogeneous Sobolev spaces*, by considering $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ (and not only $p = 2$). In that case, for $s \in \mathbb{R}$ fixed, we define $L^p_s(\mathbb{R}^n)$ as the set of tempered distributions for which

$$\|f\|_{L^p_s(\mathbb{R}^n)} := \|\langle D\rangle^s f\|_{L^p(\mathbb{R}^n)} < \infty.$$

When $s < 0$, the term $\langle D\rangle^s$ can be written as a convolution operator, as described in (1.19), whose kernel is given by the *Bessel potential*. In that way, the spaces $L^p_s(\mathbb{R}^n)$ are sometimes refered to as *spaces of potential type*. Moreover, it is common to denote $J^s = \langle D\rangle^s$ and write $L^p_s(\mathbb{R}^n)$ as $J^{-s}(L^p(\mathbb{R}^n))$, that is, the image of $L^p(\mathbb{R}^n)$ under the action of J^{-s}.

Another instance of potential type spaces, related to the ones above, are the BMO-Sobolev spaces (see for instance [52] or [46]). Those were developed as a replacement of $L^p_s(\mathbb{R}^n)$ at the endpoint $p = \infty$, where one considers the

Sobolev space to be based on $\mathrm{BMO}(\mathbb{R}^n)$ instead of $L^\infty(\mathbb{R}^n)$. More specifically, the BMO-*Sobolev space* is defined as the set of tempered distributions f for which $\|\langle D\rangle^s f\|_{\mathrm{BMO}(\mathbb{R}^n)} < \infty$.

Extending the notion present in the examples above, given a function space X, and a symbol (potential) σ, one may refer to the space of distributions f for which $\|\sigma(D)f\|_X < \infty$ as a potential type space. In Paper I, we needed to introduce spaces with that structure, where the potential operator involved is defined in terms of an admissible weight w. We introduce them next.

To start with, S. Rodríguez-López and W. Staubach [48] defined a potential-type space involving $\mathrm{BMO}(\mathbb{R}^n)$ when studying endpoint estimates for bilinear Coifman-Meyer multipliers in $\mathrm{bmo}(\mathbb{R}^n)$.

Let us recall that the space of functions with bounded mean oscillation, denoted by $\mathrm{BMO}(\mathbb{R}^n)$, is defined as the set of all those locally integrable functions f in $\mathbb{R}^n$ for which

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|\mathrm{d}x < \infty.$$

Here Q denotes a cube in $\mathbb{R}^n$ whose sides are parallel to the axis, $|Q|$ stands for its Lebesgue measure and f_Q is the average of f over Q, that is, $f_Q = \frac{1}{|Q|}\int_Q f(x)\mathrm{d}x$.

Its local version, denoted by $\mathrm{bmo}(\mathbb{R}^n)$, is called the space of *functions with local bounded mean oscillation*. It is defined as the set of all those locally integrable functions f in $\mathbb{R}^n$ for which

$$\|f\|_{\mathrm{bmo}(\mathbb{R}^n)} := \sup_{\ell(Q)<1} \frac{1}{|Q|}\int_Q |f(x) - f_Q|\mathrm{d}x + \sup_{\ell(Q)\geq 1} \frac{1}{|Q|}\int_Q |f(x)|\mathrm{d}x < \infty,$$

where $\ell(Q)$ denotes the side lenght of the cube Q.

The authors define in [48] the operator $J_w f := w(D)f$, associated to an admissible weight w (see Definition 1.2.3 above), and they introduce the space $J_w(\mathrm{BMO}(\mathbb{R}^n))$ of tempered distributions of the type $f = J_w g$ with $g \in \mathrm{BMO}(\mathbb{R}^n)$, endowed with the norm

$$\|f\|_{J_w(\mathrm{BMO}(\mathbb{R}^n))} := \left\|\frac{1}{w}(D)f\right\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Remark 1.2.6. We shall observe that the Bessel potential $J^s = \langle D\rangle^s$ can be seen as a differential operator of order s (at least, this is clear when $s \in 2\mathbb{N}$). Following this idea, if $f \in L^p_s(\mathbb{R}^n)$, we can think that f has the property that it is the derivative of order $(-s)$ of a funtion in $L^p(\mathbb{R}^n)$.

In this way, since admissible weights are basically behaving as logarithmic functions, as suggested by (1.39), we may think of $J_w(\mathrm{BMO}(\mathbb{R}^n))$ as the set of those tempered distributions which are derivatives of *logarithmic order* of functions in $\mathrm{BMO}(\mathbb{R}^n)$.

The space $J_w(\mathrm{BMO}(\mathbb{R}^n))$ appears naturally in [48] as the range of bilinear Coifman-Meyer multipliers on the product space $\mathrm{bmo}(\mathbb{R}^n)\times\mathrm{bmo}(\mathbb{R}^n)$ (further details are given in Section 1.3.1). In Paper I, which is a continuation of the investigation initiated in [48], we study boundedness properties of those operators in the product spaces $\mathrm{bmo}(\mathbb{R}^n)\times L^p(\mathbb{R}^n)$, with $1<p<\infty$, and $\mathrm{bmo}(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$, where $H^1(\mathbb{R}^n)$ denotes the (real) Hardy space. Therefore, we need to define the corresponding version of the space $J_w(\mathrm{BMO}(\mathbb{R}^n))$ for the $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$.

Let us recall that the *Hardy space*, denoted by $H^1(\mathbb{R}^n)$, is the space of tempered distributions f for which the non-tangential maximal function

$$x\mapsto \sup_{t>0}\sup_{|x-y|<t}|(\Phi_t * f)(y)| \tag{1.40}$$

belongs to $L^1(\mathbb{R}^n)$, endowed with the norm

$$\|f\|_{H^1(\mathbb{R}^n)}:=\left\|\sup_{t>0}\sup_{|x-y|<t}|(\Phi_t * f)(y)|\right\|_{L^1(\mathbb{R}^n)}.$$

Here Φ is a Schwartz function with $\int\Phi=1$ and the $\Phi_t(x)=t^{-n}\Phi(x/t)$, with $t>0$ and $x\in\mathbb{R}^n$.

Definition 1.2.7. (Definition 2.20 in Paper I) Let w be an admissible weight and consider the operator $J_w f := w(D)f$.

- The space $J_w(L^p(\mathbb{R}^n))$ is defined as the set of tempered distributions for which
$$\|f\|_{J_w(L^p(\mathbb{R}^n))}:=\left\|\frac{1}{w}(D)f\right\|_{L^p(\mathbb{R}^n)}<\infty.$$
- The space $J_w(H^1(\mathbb{R}^n))$ is defined as the set of tempered distributions for which
$$\|f\|_{J_w(H^1(\mathbb{R}^n))}:=\left\|\frac{1}{w}(D)f\right\|_{H^1(\mathbb{R}^n)}<\infty.$$

In Paper I we show some basic properties of the spaces defined above, such as the fact that they are Banach spaces or the computation of their dual space (see Proposition 2.23 in Paper I). In addition, to avoid technical discussions in our exposition, we avoided the fact both $J_w(L^p(\mathbb{R}^n))$ and $J_w(H^1(\mathbb{R}^n))$ are defined in terms of a regularised version of the admissible weight w instead of the function w itself. The precise definition of that regularisation can be found in Definition 2.14 in Paper I.

Remark 1.2.8. We shall observe that, for $1 < p < \infty$, the potential type spaces $J_w(L^p(\mathbb{R}^n))$ can be written as Triebel-Lizorkin spaces of generalised smoothness. More precisely, we have the equivalences

$$\|f\|_{J_w(L^p(\mathbb{R}^n))} := \left\|\frac{1}{w}(D)f\right\|_{L^p(\mathbb{R}^n)} \approx \left\|\frac{1}{w}(D)f\right\|_{F^0_{p,2}(\mathbb{R}^n)} \approx \|f\|_{F^{0,1/w}_{p,2}(\mathbb{R}^n)},$$

in such a way that $J_w(L^p(\mathbb{R}^n))$ can be identified with the space $F^{0,1/w}_{p,2}(\mathbb{R}^n)$ (see Proposition 2.26 in Paper I for more details).

In a different direction, when $p = 2$, for suitable w, one can show that the space $J_{\mathrm{w}}(L^2(\mathbb{R}^n))$ lies within the *refined Sobolev scale* introduced by L. Hörmander in [24] (see Proposition 2.29 in Paper I).

1.3 Bilinear pseudodifferential operators

One of the main tasks of our research is finding endpoint estimates for bilinear pseudodifferential operators. In this section we introduce those objects and provide the necessary background to understand the contributions of Papers I, II and III.

The most natural example of a bilinear operation is the product of two functions. Such a basic operator is, of course, well understood, and its boundedness properties on some classical function spaces are well known. For instance, functions in Lebesgue spaces satisfy *Hölder's inequality*, which states that

$$\|fg\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \tag{1.41}$$

for all $1 \leq p, q \leq \infty$ and $1/2 \leq r \leq \infty$ such that $1/r = 1/p + 1/q$.

Beyond the setting of L^p-spaces, and more relevant to the present thesis, some authors have studied the product of a function in $\mathrm{BMO}(\mathbb{R}^n)$ and a function in the Hardy space $H^1(\mathbb{R}^n)$, as in the works of A. Bonami, T. Iwaniec, P. Jones and M. Zinsmeister in [8] and A. Bonami, S. Grellier and L. D. Ky in [7]. Later, J. Cao, L. D. Ky and D. Yang [12] investigated the local counterpart problem by studying the product of a function in the *local Hardy space*, $h^p(\mathbb{R}^n)$, for $0 < p \leq 1$, and a function in $\mathrm{bmo}(\mathbb{R}^n)$.

The local Hardy space $h^p(\mathbb{R}^n)$ coincides with the Lebesgue space $L^p(\mathbb{R}^n)$ when $1 < p < \infty$. In that sense, we extend in Paper I the aforementioned results by investigating the product of a function in $L^p(\mathbb{R}^n)$ and a function in $\mathrm{bmo}(\mathbb{R}^n)$ when $1 < p < \infty$, which we show to lie in $J_w(L^p(\mathbb{R}^n))$ for a certain choice of the weight w (see Corollary 2.1.3 below for the precise statement).

For the particular case $p = 1$, the *local Hardy space* $h^1(\mathbb{R}^n)$, which is the local version of $H^1(\mathbb{R}^n)$, is defined as the space of tempered distributions f for which the truncated non-tangential maximal function

$$x \mapsto \sup_{0<t<\frac{1}{2}} \sup_{|x-y|<t} |(\Phi_t * f)(y)|$$

belongs to $L^1(\mathbb{R}^n)$, where Φ is as in the definition of $H^1(\mathbb{R}^n)$. The space is endowed with the norm

$$\|f\|_{h^1(\mathbb{R}^n)} := \left\| \sup_{0<t<\frac{1}{2}} \sup_{|x-y|<t} |(\Phi_t * f)(y)| \right\|_{L^1(\mathbb{R}^n)}.$$

In Paper I we also study the product of a function in $h^1(\mathbb{R}^n)$ and a function in $\mathrm{bmo}(\mathbb{R}^n)$, obtaining estimates for that product in both spaces of potential type and spaces of Musielak-Orlicz-Hardy type (see Corollaries 2.1.4 and 2.1.6 below, respectively, for the precise statements).

In order to study the product of two functions and, more generally, that of bilinear pseudodifferential operators, we decompose these objects on the frequency side by using the Fourier transform. Indeed, let us consider a radial Schwartz function ψ whose Fourier transform $\widehat{\psi}$ is supported in a ring centred at zero, and satisfies that

$$\int_0^\infty \widehat{\psi}(s\xi) \frac{\mathrm{d}s}{s} = 1, \quad \xi \neq 0.$$

Let us define

$$\widehat{\phi}(\xi) := \int_1^\infty \widehat{\psi}(s\xi) \frac{\mathrm{d}s}{s}, \quad \xi \in \mathbb{R}^n.$$

The function ϕ is in the Schwartz class and its Fourier transform is supported in a ball centred at the origin, and the unity can be decomposed as

$$1 = \int_0^\infty \int_0^\infty \widehat{\psi}(t\xi)\widehat{\psi}(s\eta) \frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s} = \int_0^\infty \widehat{\phi}(t\xi)\widehat{\psi}(t\eta)\frac{\mathrm{d}t}{t} + \int_0^\infty \widehat{\phi}(s\xi)\widehat{\psi}(s\eta)\frac{\mathrm{d}s}{s}.$$

Combining the previous decomposition with the synthesis relation in (1.1), we have that

$$\begin{aligned} fg(x) &= \int\int 1 \cdot \widehat{f}(\xi)\widehat{g}(\eta) e^{ix(\xi+\eta)}\, \text{đ}\xi\, \text{đ}\eta \\ &= \int_0^\infty [\widehat{\psi}(tD)f](x)[\widehat{\phi}(tD)g](x)\frac{\mathrm{d}t}{t} + \int_0^\infty [\widehat{\phi}(tD)f](x)[\widehat{\psi}(tD)g](x)\frac{\mathrm{d}t}{t}. \end{aligned} \tag{1.42}$$

The bilinear operator

$$\Pi(f,g) := \int_0^\infty [\widehat{\psi}(tD)f](x)[\widehat{\phi}(tD)g](x)\frac{\mathrm{d}t}{t} \tag{1.43}$$

is called a *paraproduct* and, according to (1.42), it is one of the two elements that decomposes the product of two functions into the sum

$$fg = \Pi(f,g) + \Pi(g,f). \tag{1.44}$$

We shall observe that the action of the paraproduct Π on the pair (f,g) is not symmetric, since it is acting in a different way on each argument, due to the support conditions of $\widehat{\psi}$ and $\widehat{\phi}$. More precisely, we shall observe that, for a fixed $t > 0$, the operator $\widehat{\psi}(tD)f$ localises $\widehat{f}(\xi)$ in a ring of the form $\{C_1/t \leq |\xi| \leq C_2/t\}$, for two positive constants $C_1 < C_2$ inherent to the support of $\widehat{\psi}$. On the contrary, $\widehat{\phi}(tD)g$ is localising $\widehat{g}(\xi)$ on a ball of the form $\{|\xi| \leq C\}$ for some constant C inherent to the support of $\widehat{\phi}$.

By implementing some simple manipulations, the paraproduct Π can be rewritten as

$$\Pi(f,g)(x) = \iint \left(\int_0^\infty \widehat{\psi}(t\xi)\widehat{\phi}(t\eta)\frac{\mathrm{d}t}{t} \right) \widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\, đ\xi\, đ\eta, \tag{1.45}$$

which recovers the structure of the Bony paraproduct introduced in (1.21), with

$$\sigma(\xi,\eta) = \int_0^\infty \widehat{\psi}(t\xi)\widehat{\phi}(t\eta)\frac{\mathrm{d}t}{t}.$$

One of the main advantages of decomposing fg as in (1.44) is that having boundedness properties for the bilinear paraproduct Π will produce estimates for the product of two functions.

Both the product of two functions and the bilinear paraproducts described above are just two particular instances of the broader class of bilinear pseudodifferential operators. Those objects have been studied by several authors, since the seminal work of R. Coifman and Y. Meyer [14], and they differ from each other depending on the properties that the symbol is required to satisfy. We refer to [43] for a recent survey on the topic.

In the present thesis we will only focus on two different classes of them. We studied bilinear Coifman-Meyer multipliers in Paper I, while bilinear pseudodifferential operators with symbol in the bilinear Hörmander class $BS^m_{1,1}$ were treated in Papers II and III.

1.3.1 Bilinear Coifman-Meyer multipliers

Before introducing the precise definition of bilinear Coifman-Meyer multipliers, let us consider the symbol of the bilinear paraproduct in (1.45), which is given by

$$\sigma(\xi,\eta) = \int_0^\infty \widehat{\psi}(t\xi)\widehat{\phi}(t\eta)\frac{\mathrm{d}t}{t}.$$

If we differentiate σ with respect to ξ and η, we can observe that it is possible to find, given $\alpha,\beta \in \mathbb{N}^n$, a positive constant $C_{\alpha,\beta}$ such that

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)\right| \leq C_{\alpha,\beta} \frac{1}{(|\xi|+|\eta|)^{|\alpha|+|\beta|}} \tag{1.46}$$

for all $(\xi,\eta) \neq (0,0)$. This growth property, which we just saw to be satisfied by bilinear paraproducts, is known as the *Coifman-Meyer condition*. We shall notice that (1.46) is analogous to the Hörmander-Mihlin condition in (1.8) for the bilinear setting.

A bilinear multiplier of the form

$$T_\sigma f(x) := \iint_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix(\xi+\eta)}\, đ\xi\, đ\eta, \quad x \in \mathbb{R}^n,$$

for which σ satisfies (1.46) is called a *bilinear Coifman-Meyer multiplier*. In the present thesis, the study of those operators was useful to obtain estimates for the product of functions. In addition, these bilinear multipliers become very useful to obtain fractional Leibniz rules, since the Bessel potential $\langle D\rangle^s$ can be decomposed as a finite sum of bilinear Coifman-Meyer multipliers (more details on that direction are given in Section 1.4).

These kind of multipliers appeared first in 1978 with the influential work of R. Coifman and Y. Meyer [14], where they use them to obtain boundedness properties for multilinear pseudodifferential operators. More precisely, they show that bilinear Coifman-Meyer multipliers map $L^\infty(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ continuously into $L^2(\mathbb{R}^n)$.

After the work of L. Grafakos and R. Torres [22], who developed the multilinear Calderón-Zygmund theory, bilinear Coifman-Meyer multipliers became particular instances of bilinear Calderón-Zygmund operators. As such, they satisfy

$$L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n) \tag{1.47}$$

for all $1 < p,q \leq \infty$ and $1/2 < r < \infty$ under the relation $1/r = 1/p + 1/q$.

Those estimates fail to be true at the endpoint cases where $p = q = \infty$ and $p = 1$ or $q = 1$. For those choices of indexes, some estimates can still be obtained if one replaces $L^p(\mathbb{R}^n)$ by other types of function spaces, like $H^1(\mathbb{R}^n)$ or $\mathrm{BMO}(\mathbb{R}^n)$. It follows from the general theory of multilinear Calderón-Zygmund operators [22], that bilinear Coifman-Meyer multipliers satisfy the following boundedness properties:

(i). $H^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$;

(ii). $L^p(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, for $1 < p < \infty$;

(iii). $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n)$.

The symmetric counterpart is also satisfied, having $L^\infty(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)\times L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$, for $1<p<\infty$.

We shall notice that in (i) and (iii), both $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ have been replaced in (1.47) by $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$, respectively. In addition, $H^1(\mathbb{R}^n)$ is continuously embedded in $L^1(\mathbb{R}^n)$, while $L^\infty(\mathbb{R}^n)$ is continuously embedded in $\mathrm{BMO}(\mathbb{R}^n)$. In this way, the spaces $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ appear to be natural substitutes for $L^p(\mathbb{R}^n)$ at the endpoint cases $p=1$ and $p=\infty$, where it is common to use the terminology *endpoint spaces* for them.

An alternative endpoint space for the case $p=\infty$ is $\mathrm{bmo}(\mathbb{R}^n)$, which lies in between $L^\infty(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$. S. Rodríguez-López and W. Staubach [48] studied the action of bilinear Coifman-Meyer multipliers on that space, getting an estimate analogous to (iii) with $\mathrm{bmo}(\mathbb{R}^n)$ in place of $L^\infty(\mathbb{R}^n)$. More precisely, they show that bilinear Coifman-Meyer multipliers satisfy the boundedness property

$$\mathrm{bmo}(\mathbb{R}^n)\times\mathrm{bmo}(\mathbb{R}^n)\to J_w(\mathrm{BMO}(\mathbb{R}^n)) \tag{1.48}$$

for the admissible weight $w(t)=1+\log_+ 1/t$.

One of the main goals of Paper I was to continue the investigation developed in [48]. Motivated by (1.48), the purpose was to obtain estimates for bilinear Coifman-Meyer multipliers using $\mathrm{bmo}(\mathbb{R}^n)$ instead of $L^\infty(\mathbb{R}^n)$, completing hence the remaining cases (i) and (ii) (see Theorem 2.1.1 below).

1.3.2 Bilinear pseudodifferential operators with symbol in $BS^m_{1,1}$

As mentioned in Section 1.1, a linear differential operator with constant coefficients as in (1.7) can be written as a Fourier multiplier, whose symbol is given by a polynomial function. In the bilinear setting, the same phenomena occurs for polynomials with two variables. Indeed, let us consider a *bilinear differential operator* of the form

$$T_{N,M}(f,g)(x)=\sum_{\substack{|\delta|\le N\\|\nu|\le M}} c_{\delta,\nu}(x)D^\delta f(x)D^\nu g(x),\quad x\in\mathbb{R}^n, \tag{1.49}$$

for some $N,M\in\mathbb{N}$, where $c_{\delta,\nu}$ are smooth functions in $\mathbb{R}^n$ with compact support. This operator can be written as a bilinear pseudodifferential operator whose symbol is the polynomial in ξ and η given by

$$P_{N,M}(x,\xi,\eta):=\sum_{\substack{|\delta|\le N\\|\nu|\le M}} c_{\delta,\nu}(x)\xi^\delta\eta^\nu.$$

Since the coefficient functions $c_{\delta,\nu}(x)$ are infinitely differentiable and all their derivatives are bounded, given any three multi-index $\alpha,\beta,\gamma\in\mathbb{N}^n$, there is a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma P_{N,M}(x,\xi,\eta)\right| \leq C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{N+M-(|\beta|+|\gamma|)}. \tag{1.50}$$

This property is an example of the type of conditions associated to the *bilinear Hörmander* class for symbols of bilinear pseudodifferential operators, and is the bilinear counterpart of (1.9). More precisely, given three parameters $m \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$, the class of symbols $BS^m_{\rho,\delta}$ is the set of those smooth functions $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ for which, given any three multi-index $\alpha, \beta, \gamma \in \mathbb{N}^n$, there is a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)\right| \leq C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}.$$

As a particular case, in view of (1.50), we see that differential operators of the type in (1.49) are bilinear pseudodifferential operators with symbol in $BS^{N+M}_{1,0}$.

In the present thesis we study endpoint estimates for bilinear pseudodifferential operators with symbol in $BS^m_{1,1}$, for which the condition becomes asking the symbol σ to satisfy the estimate

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)\right| \leq C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m+|\alpha|-|\beta|-|\gamma|} \tag{1.51}$$

for all $(x,\xi,\eta) \in \mathbb{R}^{3n}$, all multi-index $\alpha, \beta, \gamma \in \mathbb{N}^n$ and some $C_{\alpha,\beta,\gamma} > 0$. Here the index m is called the *order* of the symbol.

We shall notice that the function $\sigma \equiv 1$ satisfies the condition in (1.51) and, thus, the operator $T_\sigma(f,g) = fg$ lies within the class that we are considering. In view of Hölder's inequality in (1.41), satisfied by the product fg, one may expect that bilinear pseudodifferential operators with symbol in $BS^0_{1,1}$ satisfy the same type of estimate in the setting of Lebesgue spaces. However, it is actually not possible to obtain such inequality.

In replacement, Á. Bényi and R.H. Torres [5] provided some estimates for those kind of operators when considering inhomogeneous Sobolev spaces $L^p_s(\mathbb{R}^n)$. More precisely, given $s > 0$ and $1 < p, q, r < \infty$ such that $1/p + 1/q = 1/r$, they show that a bilinear pseudodifferential operator with symbol in $BS^0_{1,1}$ (that is, with $m = 0$) satisfies

$$\|T_\sigma(f,g)\|_{L^r_s(\mathbb{R}^n)} \lesssim \left(\|f\|_{L^p_s(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q_s(\mathbb{R}^n)}\right). \tag{1.52}$$

The result was generalised later by K. Koezuka and N. Tomita [29], who considered symbols of general order $m \in \mathbb{R}$, and used the scale of Triebel-Lizorkin and local Hardy spaces. To be more precise, they show that

$$\|T_\sigma(f,g)\|_{F^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{s+m}_{p_1,q}(\mathbb{R}^n)}\|g\|_{h^{p_2}(\mathbb{R}^n)} + \|f\|_{h^{\tilde{p}_1}(\mathbb{R}^n)}\|g\|_{F^{s+m}_{\tilde{p}_2,q}(\mathbb{R}^n)} \tag{1.53}$$

holds for all $f,g \in \mathcal{S}(\mathbb{R}^n)$, where $0 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 < \infty$, $0 < p < \infty$, $1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2 = 1/p$, $0 < q \le \infty$ and

$$s > \begin{cases} \tau_{p,q} & \text{if} \quad 0 < q < \infty, \\ \tau_{p,q} + n & \text{if} \quad q = \infty, 0 < p < 1, \\ n/p & \text{if} \quad q = \infty, 1 \le p \le \infty, \end{cases}$$

with

$$\tau_{p,q} := n \left(\frac{1}{\min(1,p,q)} - 1 \right).$$

Moreover, they also show that the inequality holds for the endpoint cases $p_2 = \infty$ or $\tilde{p}_1 = \infty$, with $L^\infty(\mathbb{R}^n)$ in replacement of the local Hardy space.

Remark 1.3.1. We shall point out that, since we have the identities $F^s_{p,2}(\mathbb{R}^n) = L^p_s(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$, it is the case that (1.53) recovers the estimate in (1.52).

In a recent work, B. J. Park [47] improved (1.53) by allowing the weaker regularity condition $s > \tau_{p,q}$. Under that restriction on s, one of the main contributions in Paper II was to obtain an inequality analogous to (1.53) at the endpoint case $p_2 = \tilde{p}_1 = \infty$, replacing $L^\infty(\mathbb{R}^n)$ by the larger endpoint space $\mathrm{bmo}(\mathbb{R}^n)$.

Throughout the study of the product of functions in the Triebel-Lizorkin spaces of the type $F^{n/p}_{p,q}(\mathbb{R}^n)$, we also obtain in Paper III an estimate on bilinear pseudodifferential operators with symbol in $BS^m_{1,1}$, for functions that lie in the intersection of Triebel-Lizorkin spaces, one of which of the type $F^{n/p}_{p,q}(\mathbb{R}^n)$ (see Theorem 2.3.3 below).

Remark 1.3.2. We shall mention that the boundedness properties on both bilinear Coifman-Meyer multipliers and bilinear pseudodifferential operators with symbol in $BS^m_{1,1}$, are obtained in our research for a larger scale of spaces, rather than only for $\mathrm{bmo}(\mathbb{R}^n)$. More precisely, the estimates are shown for a class of functions denoted by $X_w(\mathbb{R}^n)$, associated to a non-increasing admissible weight w (see Definition 1.2.3 bove), for which $\mathrm{bmo}(\mathbb{R}^n)$ is a particular instance. Those spaces were initially defined in [48].

Given a Schwartz function ϕ, whose frequency support is a ball centred at the origin, and a non-increasing admissible weight w, the space $X_w(\mathbb{R}^n)$ is defined as the set of functions for which

$$\|f\|_{X_w(\mathbb{R}^n)} := \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + \sup_{t>0} \frac{\left\| \widehat{\phi}(tD) f \right\|_{L^\infty(\mathbb{R}^n)}}{w(t)} < \infty. \tag{1.54}$$

We shall mention that the inclusions $L^\infty(\mathbb{R}^n) \subseteq X_w(\mathbb{R}^n) \subseteq \mathrm{bmo}(\mathbb{R}^n)$ hold for any non-increasing admissible weight w, in such a way that the scale of spaces $X_w(\mathbb{R}^n)$ lies in between $L^\infty(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$. At the same time, those 'endpoints' are also part of that scale, since $X_w(\mathbb{R}^n)$ coincides with $L^\infty(\mathbb{R}^n)$ for the choice $w \approx 1$ while $X_w(\mathbb{R}^n)$ coincides with $\mathrm{bmo}(\mathbb{R}^n)$ for the choice $w(t) = 1 + \log_+(1/t)$.

1.4 Fractional Leibniz rules

One of the main applications that can be derived from the boundedness properties obtained for the bilinear pseudodifferential operators in Section 1.3, is to show inequalities for differential operators of fractional order. In this thesis we will refer to a *fractional Leibniz rule* as a bilinear estimate of the type

$$\|J^s(fg)\|_Z \lesssim \|J^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|J^s g\|_{Y_2}, \tag{1.55}$$

for some appropriate normed spaces X_1, X_2, Y_1, Y_2 and Z, where $J^s = \langle D \rangle^s$. Those kind of estimates are also known as *Kato-Ponce inequalities*, due to the seminal work of T. Kato and G. Ponce [28], as mentioned earlier in Section I.

The Bessel potential J^s is considered to be a fractional differentiation operator when $s > 0$. More concretely, in the case where $s = 2m$ for some positive integer m, J^s becomes an ordinary differentiation operator, due to the identity $J^{2m} f = (I - \Delta)^m f$. Thus, the action of J^s can be considered as taking derivatives up to order s. As a consequence, we see that estimates of the type in (1.55) are built in an analogous way to the classical Leibniz rule for the derivative of the product of two functions on $\mathbb{R}$, given by

$$(fg)' = f'g + fg'.$$

The structure of the operator J^s allows to obtain fractional Leibniz rules by means of biliner Coifman-Meyer multipliers. Indeed, the action of J^s on the product of two functions f, g can be decomposed as

$$J^s(fg) = B_1^s(J^s f, g) + B_2^s(f, J^s g) + B_3^s(f, J^s g),$$

where the B_j^s, $j = 1, 2, 3$, are bilinear operators. Both B_1^s and B_2^s can be shown to satisfy the Coifman-Meyer condition in (1.46), while B_3^s only satisfies (1.46) for those multi-index such that $|\alpha| + |\beta| \leq s$ (see [21] for more details). Therefore, boundedness properties of bilinear Coifman-Meyer multipliers may be applied to obtain fractional Leibniz rules.

After the works of A. Gulisashvili and M. A. Kon [23, Theorem 1.4] and L. Grafakos and S.Oh [21, Theorem 1], some estimtes are known in the setting of

Lebesgue spaces. More precisely, let $1 < p_1, p_2, q_1, q_2 \leq \infty$ and $1/2 < p < \infty$ be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Then, if $s > \max\{0, n/p - n\}$ or if s is a positive even integer, it holds that

$$\|J^s(fg)\|_{L^p(\mathbb{R}^n)} \lesssim \|J^s f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{q_1}(\mathbb{R}^n)} + \|f\|_{L^{p_2}(\mathbb{R}^n)} \|J^s g\|_{L^{q_2}(\mathbb{R}^n)}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

In particular, at the endpoint case $q_1 = q_2 = \infty$, the last estimate asserts that for $1 < p < \infty$ and $s > 0$, the inequality

$$\|J^s(fg)\|_{L^p(\mathbb{R}^n)} \lesssim \|J^s f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \|J^s g\|_{L^\infty(\mathbb{R}^n)} \tag{1.56}$$

holds for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

As an extension, S. Rodríguez-López and W. Staubach [48] extended (1.56) by using $\mathrm{bmo}(\mathbb{R}^n)$ instead of $L^\infty(\mathbb{R}^n)$ as an endpoint space. More precisely, their work allows to prove that, under the stronger restriction $s > 4n+1$, it holds that

$$\|J^s(fg)\|_{J_w(\mathrm{BMO}(\mathbb{R}^n))} \lesssim \|J^s f\|_{\mathrm{bmo}(\mathbb{R}^n)} \|g\|_{\mathrm{bmo}(\mathbb{R}^n)} + \|f\|_{\mathrm{bmo}(\mathbb{R}^n)} \|J^s g\|_{\mathrm{bmo}(\mathbb{R}^n)} \tag{1.57}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, with $w(t) = 1 + \log_+ 1/t$.

As a continuation of the investigation conducted to obtain (1.57), one of the contributions of the present thesis was to replace $L^\infty(\mathbb{R}^n)$ by $\mathrm{bmo}(\mathbb{R}^n)$ in (1.56) for the remaining cases $1 < p < \infty$. We obtained such estimates in Paper I and II, where we also included the endpoint case $p = 1$ using the Hardy space $H^1(\mathbb{R}^n)$ in place of $L^1(\mathbb{R}^n)$.

1.5 Some Sobolev embeddings

The classical Sobolev embedding theorem states that if f lies in $L^2_s(\mathbb{R}^n)$ and $s > n/2$, then f is a continuous function vanishing at infinity which satisfies

$$\|f\|_\infty \lesssim \|\langle D \rangle^s f\|_{L^2(\mathbb{R}^n)},$$

or equivalently, the embedding $L^2_s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ holds for all $s > n/2$. In addition, the previous property fails to be true for $s = n/2$ and thus we refer to the index $n/2$ as the *critical Sobolev index*.

As mentioned in Section I, combining the previous embedding with a convenient bilinear estimate (such as (1.26) above) yields algebraic properties for the space $L^2_s(\mathbb{R}^n)$.

Regarding the critical case $s = n/2$, we do have an embedding for $L^2_{n/2}(\mathbb{R}^n)$ if we replace $L^\infty(\mathbb{R}^n)$ by the larger space $\mathrm{bmo}(\mathbb{R}^n)$. Namely, it is possible to obtain that

$$\|f\|_{\mathrm{bmo}(\mathbb{R}^n)} \lesssim \left\| \langle D \rangle^{n/2} f \right\|_{L^2(\mathbb{R}^n)}. \tag{1.58}$$

In a more general setting, we can find similar embeddings in the scale of Triebel-Lizorkin spaces. For instance (see [49, Theorem 2.2.4/1]), if we have $0 < q \leq \infty$, and either $s > n/p$ if $1 < p < \infty$ or $s = n/p$ if $0 < p \leq 1$, it holds that

$$F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n),$$

which recovers the classical Sobolev embedding theorem when $p = q = 2$ (since we have the identification $F^s_{2,2}(\mathbb{R}^n) = L^2_s(\mathbb{R}^n)$).

In addition, regarding the critical case $s = n/p$, we have (see [49, Theorem 2.2.2 and Remark 2.2.3/3]) the embedding

$$F^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{bmo}(\mathbb{R}^n) \tag{1.59}$$

for all $0 < q \leq \infty$ and $1 < p < \infty$, which recovers (1.58) in the particular case $p = q = 2$.

We obtain in Paper III a refined version of the embedding in (1.59) by considering the spaces $X_w(\mathbb{R}^n)$ (see Remark 1.3.2 above). More specifically we obtain the embedding

$$F^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow X_{w_p}(\mathbb{R}^n) \hookrightarrow \mathrm{bmo}(\mathbb{R}^n)$$

for some logarithmic type weight w_p(depending on p). This becomes useful in order to study the product of functions in Triebel-Lizorkin spaces with critical Sobolev index, $F^{n/p}_{p,q}(\mathbb{R}^n)$.

2. Summaries of Papers

2.1 Summary of Paper I

The main goal of this paper is to obtain endpoint estimates for bilinear Coifman-Meyer multipliers, involving the endpoint space $\mathrm{bmo}(\mathbb{R}^n)$.

As a continuation of the investigation initiated in [48], where the authors study bilinear Coifman-Meyer multipliers acting on $\mathrm{bmo}(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$, the purpose is to study boundedness properties for those operators on the product spaces $\mathrm{bmo}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$, with $1 < p < \infty$, and $\mathrm{bmo}(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

The wished estimates are obtained as a particular case of the following result (we refer the reader to Definition 1.2.3 for the definition of admissible weight, Remark 1.3.2 for the that of $X_w(\mathbb{R}^n)$ and to Definition 1.2.7 for that of the spaces $J_w(L^p(\mathbb{R}^n))$ and $J_w(H^1(\mathbb{R}^n))$).

Theorem 2.1.1. *(Theorem 3.2 in Paper I)Let T_σ be a bilinear Coifman-Meyer multiplier and let w be an admissible weight satisfying* $\inf_{t>0} w(t) > 0$.

(i). Given $1 < p < \infty$, there exists a constant $C > 0$ such that

$$\|T_\sigma(f,g)\|_{J_w(L^p(\mathbb{R}^n))} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{X_w(\mathbb{R}^n)}$$

holds for every $f \in L^p(\mathbb{R}^n)$ and $g \in X_w(\mathbb{R}^n)$.

(ii). The symbol σ can be decomposed as the sum of two symbols $\sigma = \sigma_g + \sigma_b$, such that we can find constants $C', C'' > 0$ for which

$$\left\|T_{\sigma_g}(f,g)\right\|_{L^1(\mathbb{R}^n)} \leq C' \|f\|_{H^1(\mathbb{R}^n)} \|g\|_{X_w(\mathbb{R}^n)}$$

and

$$\|T_{\sigma_b}(f,g)\|_{J_w(H^1(\mathbb{R}^n))} \leq C'' \|f\|_{H^1(\mathbb{R}^n)} \|g\|_{X_w(\mathbb{R}^n)}$$

hold for every $f \in H^1(\mathbb{R}^n)$ and $g \in X_w(\mathbb{R}^n)$.

For the particular choice $w \approx 1$ we have that

$$X_w(\mathbb{R}^n) = L^\infty(\mathbb{R}^n), \quad J_w(L^p(\mathbb{R}^n)) = L^p(\mathbb{R}^n) \quad \text{and} \quad J_w(H^1(\mathbb{R}^n)) = H^1(\mathbb{R}^n),$$

from where the theorem recovers the known boundeness results $L^p(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ mentioned in Section 1.3.1.

Moreover, for $w(t) = 1 + \log_+(1/t)$, we have $X_w(\mathbb{R}^n) = \mathrm{bmo}(\mathbb{R}^n)$ and the boundedness properties for bilinear Coifman-Meyer operators in the product spaces $\mathrm{bmo}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$, with $1 < p < \infty$, and $\mathrm{bmo}(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, are obtained.

Similarly to what happens in [48], potential-type spaces appear as the target space of the bilinear multiplier, this time involving $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. Both $J_w(L^p(\mathbb{R}^n))$ and $J_w(H^1(\mathbb{R}^n))$ are defined in this paper, where we also study some of their basic properties, such as that of completeness and the determination of their dual spaces, among others.

The building blocks to study bilinear Coifman-Meyer multipliers (and many other bilinear operators) are bilinear paraproducts. In order to obtain the estimates in Theorem 2.1.1 we first decompose the bilinear operator into paraproducts, following the ideas of R. Coifman and Y. Meyer [14]. The problem reduces to obtain the corresponding estimates for bilinear paraproducts of the type in (1.43), both on the product space $\mathrm{bmo}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$, with $1 < p < \infty$, and $\mathrm{bmo}(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

As for the applications of our results, one of the main consequences that we obtain from Theorem 2.1.1 are some new inequalities of Kato-Ponce type. In this way, we extend the fractional Leibniz rule in (1.56) under the regularity restriction $s > 4n + 1$.

Corollary 2.1.2. *(Corollary 6.1 in Paper I) Let $s > 4n + 1$ and $1 < p < \infty$. Consider the admissible weight $w(t) = 1 + \log_+ 1/t$. The estimate*

$$\|J^s(fg)\|_{J_w(L^p(\mathbb{R}^n))} \lesssim \|J^s f\|_{L^p(\mathbb{R}^n)} \|g\|_{\mathrm{bmo}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \|J^s g\|_{\mathrm{bmo}(\mathbb{R}^n)}$$

is satisfied for all $f, g \in \mathcal{S}(\mathbb{R}^n)$. Moreover, we have that

$$\|J^s(fg)\|_{L^1(\mathbb{R}^n)+J_w(H^1(\mathbb{R}^n))} \lesssim \|J^s f\|_{H^1(\mathbb{R}^n)} \|g\|_{\mathrm{bmo}(\mathbb{R}^n)} + \|f\|_{H^1(\mathbb{R}^n)} \|J^s g\|_{\mathrm{bmo}(\mathbb{R}^n)}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

The requirement $s > 4n + 1$ in Theorem 2.1.2 is due to the method used in the proof, where the symbol of the bilinear multiplier has to satisfy the condition in (1.46) for, at least, all those multi-index α, β for which $|\alpha| + |\beta| \leq 4n + 1$. We are able to remove this technical restriction on s later in Paper II.

As an additional application of Theorem 2.1.1, we also show some inequalities for the product of two functions. For instance, taking $\sigma \equiv 1$ in Theorem 2.1.1 gives an estimate for the product of a function in $\mathrm{bmo}(\mathbb{R}^n)$ and a function in $L^p(\mathbb{R}^n)$.

Corollary 2.1.3. *(Corollary 6.2 in Paper I) Let $1 < p < \infty$ and $w(t) = 1 + \log_+ 1/t$. Then, it holds that*

$$\|fg\|_{J_w(L^p(\mathbb{R}^n))} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{\mathrm{bmo}(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$ *and* $g \in \mathrm{bmo}(\mathbb{R}^n)$.

A more elaborated consequence, which does not follow directly by applying Theorem 2.1.1, is the following result, which gives an estimate for the product of a function in $\mathrm{bmo}(\mathbb{R}^n)$ and a function in the local Hardy space $h^1(\mathbb{R}^n)$.

Corollary 2.1.4. *(Corollary 6.3 in Paper I) Consider the admissible weight* $w(t) = 1 + \log_+ 1/t$. *There exist two continuous bilinear operators on the product space* $h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$, *respectively*

$$B_1 : h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

and

$$B_2 : h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n) \to J_w(H^1(\mathbb{R}^n)),$$

such that

$$fg = B_1(f,g) + B_2(f,g)$$

for all $f \in h^1(\mathbb{R}^n)$ *and* $g \in \mathrm{bmo}(\mathbb{R}^n)$.

Remark 2.1.5. We shall notice that a direct application of Theorem 2.1.1, taking the symbol $\sigma \equiv 1$, gives the previous corollary with $H^1(\mathbb{R}^n)$ in place of $h^1(\mathbb{R}^n)$. However, Corollary 2.1.4 is a stronger result since $H^1(\mathbb{R}^n)$ is continuously embedded in $h^1(\mathbb{R}^n)$.

The argument that we use to show Corollary 2.1.4, in combination with the estimates obtained by A. Bonami, S.Grellier and L. D. Ky in [7, Theorem 1.1], allow us to get an alternative inequality for the product of two functions in $h^1(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$. Instead of a potential-type space, the product is realised in the scale of Musielak-Orlicz-Hardy spaces. More precisely, the target space is the Musielak-Orlicz-Hardy space $H^{\log}(\mathbb{R}^n)$, which is associated to the Musielak-Orlicz function

$$\varphi(x,t) := \frac{t}{\log(e+|x|) + \log(e+t)}, \quad x \in \mathbb{R}^n, t > 0. \tag{2.1}$$

Corollary 2.1.6. *(Corollary 6.4 in Paper I) There exist two continuous bilinear operators on the product space* $h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$, *respectively* $B_1 : h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ *and* $B_2 : h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n) \to H^{\log}(\mathbb{R}^n)$, *such that*

$$fg = B_1(f,g) + B_2(f,g).$$

J. Cao, L.D. Ky and D. Tang obtained in [12, Theorem 1.1 and Remark 3.1] a different decomposition for the product of functions in $h^1(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$ into two bilinear operators. Similarly to Corollary 2.1.6, one of them is mapping continuously $h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, while the other maps $h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$ continuously into $h^{\log}(\mathbb{R}^n)$, where $h^{\log}(\mathbb{R}^n)$ is a local Musielak-Orlicz-Hardy space related to the growth function defined in (2.1). In addition, due to the fact that $H^{\log}(\mathbb{R}^n) \subseteq h^{\log}(\mathbb{R}^n)$, Corollary 2.1.6 above slightly improves [12, Theorem 1.1 and Remark 3.1].

2.2 Summary of Paper II

One of the main goals of Paper II is to extend the estimate in (1.53) at the end-point cases $p_2 = \infty$ or $\tilde{p}_1 = \infty$, replacing $L^\infty(\mathbb{R}^n)$ by $\mathrm{bmo}(\mathbb{R}^n)$. More specifically, we obtain the following estimate for bilinear pseudodifferential operators with symbol in the class $BS^m_{1,1}$.

Corollary 2.2.1. *(Corollary 3.4 in Paper II) Let* $0 < p \leq \infty$, $0 < q \leq \infty$, $m \in \mathbb{R}$, $\sigma \in BS^m_{1,1}$ *and*

$$s > \tau_{p,q} := n\left(\frac{1}{\min(1,p,q)} - 1\right).$$

Then it holds that

$$\|T_\sigma(f,g)\|_{F^{s,1/(1+\log_+ 1/t)}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{s+m}_{p,q}(\mathbb{R}^n)} \|g\|_{\mathrm{bmo}(\mathbb{R}^n)} + \|f\|_{\mathrm{bmo}(\mathbb{R}^n)} \|g\|_{F^{s+m}_{p,q}(\mathbb{R}^n)}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

The first step to show the previous result, following the idea of A. Bényi and R. Torres [5], is to decompose the symbol σ into elementary symbols of the form

$$\tilde{\sigma}(x,\xi,\eta) = \sum_{j=0}^{\infty} \mathtt{m}_j(x)\psi_j(\xi)\phi_j(\eta), \tag{2.2}$$

where $\psi_j(\xi) = \psi(2^{-j}\xi)$ and $\phi_j(\xi) = \phi(2^{-j}\xi)$ with ψ and ϕ satisfying similar properties to the ones defining the bilinear paraproduct in (1.43). In this sense, we could see the expression in (2.2) as the *discrete* analogue of the paraproduct in (1.43), with the exception of the terms $\{\mathtt{m}_j\}_j$, which are smooth functions satisfying a certain growth condition (the precise description of the functions ψ, ϕ and $\mathtt{m}$ can be found in the Introduction in Paper II).

More precisely, Theorem 2.2.1 is obtained as a corollary of the following result, where we show an estimate for bilinear operators with an elementary symbol as described above.

Theorem 2.2.2. *(Theorem 3.1 I) in Paper II) Let us consider the operator $T_{\tilde{\sigma}}$ with $\tilde{\sigma}$ as in* (2.2) *and let w be a non-increasing admissible weight. If $0 < p \leq \infty$, $0 < q \leq \infty$, $m \in \mathbb{R}$ and $s > \tau_{p,q}$, then we can find $C > 0$ such that*

$$\|T_{\tilde{\sigma}}(f,g)\|_{F^{s,1/w}_{p,q}(\mathbb{R}^n)} \leq C\,\|f\|_{F^{s+m}_{p,q}(\mathbb{R}^n)}\,\|g\|_{X_w(\mathbb{R}^n)}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Summarising, the way of showing Theorem 2.2.1 is to decompose the symbol σ in the class $BS^m_{1,1}$ as a sum of elementary symbols $\tilde{\sigma}$ as in (2.2) and apply Theorem 2.2.2 with $w(t) = 1 + \log_+(1/t)$, where $X_w(\mathbb{R}^n) = \mathrm{bmo}(\mathbb{R}^n)$.

We shall notice that due to the structure of the symbol $\tilde{\sigma}$ in Theorem 2.2.1, the operator $T_{\tilde{\sigma}}$ is not symmetric, since the ψ_j's are frequency localising functions into rings, while the ϕ's localise into balls. Hence, we also include in Paper II the following theorem, where it is now the fist argument of the operator $T_{\tilde{\sigma}}$ which is considered to lie in the scale $X_w(\mathbb{R}^n)$ (actually, in a larger scale of Besov spaces).

Theorem 2.2.3. *(Theorem 3.1 II) in Paper II) Let us consider the operator $T_{\tilde{\sigma}}$ with $\tilde{\sigma}$ as in* (2.2) *and let w be a non-increasing admissible weight. Let $0 < p \leq \infty$, $1 < q < \infty$, $m \in \mathbb{R}$, $s > \tau_{p,q}$ and take an admissible weight v such that*

$$\sup_{\ell \geq 0} \left(\sum_{k=\ell}^{\infty} \frac{1}{w(2^{-k})^q} \right)^{1/q} \left(\sum_{k=0}^{\ell} \frac{1}{v(2^{-k})^{q'}} \right)^{1/q'} < \infty. \tag{2.3}$$

Then we can find $C > 0$ such that

$$\|T_{\tilde{\sigma}}(f,g)\|_{F^{s,1/w}_{p,q}(\mathbb{R}^n)} \leq C\,\|f\|_{B^{s+m}_{\infty,\infty}(\mathbb{R}^n)}\,\|g\|_{F^{0,v}_{p,q}(\mathbb{R}^n)}$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Remark 2.2.4. The technical condition (2.3) is used to apply an inequality of Hardy's type. In connection to these type of inequalities, and as a digression of this thesis, we obtain in [1] a weighted generalisation of the classical Carleman's inequality. Since neither the topic nor the obtained results are directly related to those in the present thesis, we have decided not to include them here, and we refer the interested reader to the paper and the references therein.

Following the approach used in Theorem 2.2.1, one can decompose a symbol in the class $BS^m_{1,1}$ as a sum of elementary symbols of the type in (2.2) but, this time, alternate the use of Theorems 2.2.2 and 2.2.3 to bound the corresponding bilinear operators. As a consequence, the following estimate is obtained.

Corollary 2.2.5. *(Corollary 3.3 in Paper II) Let* $0 < p \leq \infty$*,* $1 < q < \infty$*,* $s > \tau_{p,q}$*,* $m \in \mathbb{R}$ *and* $\sigma \in BS^m_{1,1}$*. Set* w*,* v *admissible weights such that* w *is decreasing and* (2.3) *is satisfied. Then it holds that*

$$\|T_\sigma(f,g)\|_{F^{s,1/w}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{s+m}_{p,q}(\mathbb{R}^n)} \|g\|_{X_w(\mathbb{R}^n)} + \|f\|_{F^{0,v}_{p,q}(\mathbb{R}^n)} \|g\|_{B^{s+m}_{\infty,\infty}(\mathbb{R}^n)}$$

for all $f,g \in \mathcal{S}(\mathbb{R}^n)$*.*

One of the main implications of the previous corollary is the following fractional Leibniz rule, which can be obtained by taking the particular choices $\sigma \equiv 1$, $m = 0$, $w(t) = 1 + \log_+(1/t)$ and $v \equiv 1$.

Corollary 2.2.6. *(Corollary 3.5 in Paper II) Let* $0 < p \leq \infty$*,* $1 < q < \infty$ *and* $s > \tau_{p,q}$*. Then*

$$\|J^s(fg)\|_{F^{0,1/(1+\log_+(1/t))}_{p,q}(\mathbb{R}^n)} \lesssim \|J^s f\|_{F^0_{p,q}(\mathbb{R}^n)} \|g\|_{\mathrm{bmo}(\mathbb{R}^n)} + \|f\|_{F^0_{p,q}(\mathbb{R}^n)} \|J^s g\|_{\mathrm{bmo}(\mathbb{R}^n)}$$

for all $f,g \in \mathcal{S}(\mathbb{R}^n)$*.*

We shall notice that $F^0_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\tau_{p,2} = 0$ when $1 < p < \infty$. Therefore, the last corollary is an improvement of Corollary 2.1.2 for the case $1 < p < \infty$ and $q = 2$. Indeed, the restriction $s > 4n+1$ appearing in Corollary 2.1.2 has been now replaced by the weaker requirement $s > 0$ in Corollary 2.2.6.

Furthermore, Corollary 2.2.6 also extends Corollary 2.1.2 in a different direction, since the range of p has been increased from the interval $(1,\infty)$ to $(0,\infty)$, replacing $L^p(\mathbb{R}^n)$ by the local Hardy space $h^p(\mathbb{R}^n)$ in the case where $0 < p \leq 1$ (recall that $F^0_{p,2}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$).

Finally, using the identification $F^0_{\infty,2}(\mathbb{R}^n) = \mathrm{bmo}(\mathbb{R}^n)$, we notice that Corollary 2.2.6 with $p = \infty$ and $q = 2$ yields an improvement of the Kato-Ponce estimate (1.57) obtained in [48], by allowing $s > 0$.

2.3 Summary of Paper III

One of the main contributions of this paper is the following refinement of the Sobolev embedding in (1.59), where we consider the scale of spaces $X_w(\mathbb{R}^n)$.

Theorem 2.3.1. *(Theorem 3.1 in Paper III) Let either* $0 < p < \infty$ *and* $0 < q \leq \infty$*, or* $p = \infty$ *and* $0 < q \leq 2$*. If we set* $r := \max\{1,p\}$ *then the embedding*

$$F^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow X_w(\mathbb{R}^n)$$

holds with $w(t) = (1 + \log_+ 1/t)^{1/r'}$*.*

We shall point out that, if $r = 1$, the weight w is constant and hence $X_w(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. As a consequence, the previous theorem states that the embedding

$$F^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$$

holds for all $0 < q \leq \infty$ and $0 < p \leq 1$, hence recovering partially the stronger result of W. Sickel and H. Triebel [50, Theorem 3.3.1], which asserts that $F^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ holds if, and only if, $0 < q \leq \infty$ and $0 < p \leq 1$.

Following the approach used in Paper II to show Theorems 2.2.2 and 2.2.3 we obtain the following estimate.

Proposition 2.3.2. *(Proposition 4.2 in Paper III) Let $0 < p,q \leq \infty$, $s > \tau_{p,q}$, $m \in \mathbb{R}$ and $\sigma \in BS^m_{1,1}$. Consider two admissible weights u, v with $u \leq v$. Then we have that*

$$\|T_\sigma(f,g)\|_{F^{s,1/v}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{s+m}_{p,q}(\mathbb{R}^n)} \|g\|_{X_u(\mathbb{R}^n)} + \|f\|_{X_v(\mathbb{R}^n)} \|g\|_{F^{s+m}_{p,q}(\mathbb{R}^n)}$$

for all $f,g \in \mathcal{S}(\mathbb{R}^n)$.

As a consequence of the refined Sobolev embedding in Theorem 2.3.1 and the previous proposition, we obtain the following estimate for bilinear pseudodifferential operators with symbol in the class $BS^m_{1,1}$, when both arguments belong to the intersection of Triebel-Lizorkin spaces, one of which has a critical Sobolev exponent.

Theorem 2.3.3. *(Theorem 4.3 in Paper III) Let $0 < p,q \leq \infty$, $s > \tau_{p,q}$, $m \in \mathbb{R}$ and $\sigma \in BS^m_{1,1}$. Set either $0 < \tilde{p}_i < \infty$ and $0 < \tilde{q}_i \leq \infty$ or $\tilde{p}_i = \infty$ and $0 < \tilde{q}_i \leq 2$, with $i \in \{1,2\}$. Consider the function $w(t) = (1 + \log_+ 1/t)^{1/r'}$ with $r = \max\{1, \tilde{p}_1, \tilde{p}_2\}$, where r' denotes the conjugate Hölder exponent. Then it holds that*

$$\|T_\sigma(f,g)\|_{F^{s,1/w}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{s+m}_{p,q}(\mathbb{R}^n)} \|g\|_{F^{n/\tilde{p}_1}_{\tilde{p}_1,\tilde{q}_1}(\mathbb{R}^n)} + \|f\|_{F^{n/\tilde{p}_2}_{\tilde{p}_2,\tilde{q}_2}(\mathbb{R}^n)} \|g\|_{F^{s+m}_{p,q}(\mathbb{R}^n)}$$

for all $f,g \in \mathcal{S}(\mathbb{R}^n)$.

We shall mention that, in the generality it is stated, Theorem 2.3.3 is sharp. More precisely, the exponent in the logarithmic weight cannot be improved, since it is possible to find a counterexample for the product of two functions in the L^2-case with the Sobolev critical index $n/2$ (see Proposition 4.6 in Paper III).

In the particular case where $\sigma \equiv 1$, we obtain as a corollary of Theorem 2.3.3 the following estimate on the product of two functions in the Triebel-Lizorkin space $F^{n/p}_{p,q}(\mathbb{R}^n)$.

Corollary 2.3.4. *(Corollary 4.4 in Paper III) Let* $0 < p < \infty$ *and* $0 < q \leq \infty$ *such that*

$$\min(1, p, q) > \frac{p}{p+1}.$$

If $r = \max\{1, p\}$ *then the inequality*

$$\|fg\|_{F^{n/p,1/w}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{n/p}_{p,q}(\mathbb{R}^n)} \|g\|_{F^{n/p}_{p,q}(\mathbb{R}^n)}$$

holds with $w(t) = (1 + \log_+ 1/t)^{1/r'}$.

It is well known that, given $0 < q \leq \infty$, the spaces $F^{n/p}_{p,q}(\mathbb{R}^n)$ are multiplication algebras if, and only if, $0 < p \leq 1$ (see, for instance, [49, Theorem 4.6.4/1]). In that case, when $0 < p \leq 1$, we observe that $r' = \infty$ and $w \equiv 1$, so that the space $F^{n/p,1/w}_{p,q}(\mathbb{R}^n)$ coincides with the classical version $F^{n/p}_{p,q}(\mathbb{R}^n)$. As a conclusion, Theorem 2.3.4 recovers the fact that $F^{n/p}_{p,q}(\mathbb{R}^n)$ is a multiplication algebra when $0 < p \leq 1$.

Regarding the case $1 < p < \infty$, we shall mention that some estimates for the product of two functions in $F^{n/p}_{p,q}(\mathbb{R}^n)$ were already known. More precisely, it holds (see [2, Proposition 2.3]) that

$$\|fg\|_{F^{n/p-\varepsilon}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{n/p}_{p,q}(\mathbb{R}^n)} \|g\|_{F^{n/p}_{p,q}(\mathbb{R}^n)} \tag{2.4}$$

for all $\varepsilon > 0$. However, the previous estimate can be derived from the result in Corollary 2.3.4 (see Remark 4.5 in Paper III for further details).

Finally, we include in Paper III some detailed proofs of certain properties satisfied by the spaces $F^{s,w}_{\infty,q}(\mathbb{R}^n)$, such as their independence on the chosen resolution of unity defining its norm (see Proposition 2.10 in Paper III), or some lifting properties (see Propositions 2.15 and 2.17 in Paper III).

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