Stockholm University

Complex multivalued maps & their invariant sets

Nils Hemmingsson

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Abstract

This thesis consists of five papers, each of which investigates the dynamics of multivalued maps on the Riemann sphere or in the complex plane. The problems studied stem from, and are special cases of, the Pólya–Schur problem.

In Papers I and III, we initiate the study of continuous Hutchinson invariance and characterize the associated invariant sets both topologically and geometrically. In Paper I, we establish necessary and sufficient conditions for the existence of a unique invariant set that is minimal under inclusion, and determine when this set is non-trivial and when it is compact. Paper III focuses on the boundary of this set, providing bounds on its complexity in a specific sense. Moreover, we classify the different types of boundary points.

Papers II and IV examine the dynamics of holomorphic correspondences. In Paper II, we construct explicit differential operators and associated holomorphic correspondences such that, for the differential operator, there exists a unique Hutchinson-invariant set in high degrees that is minimal under inclusion, and we study the equidistribution of the associated holomorphic correspondences. In Paper IV, we explore conformal measures of a class of (anti)holomorphic correspondences, prove their existence, and derive bounds on the Hausdorff dimension of the limit sets.

Paper V focuses on a question more closely related to the Pólya--Schur theory. For a given differential operator T and a degree n, a set $S \subset \mathbb{C}$ is said to be T_n-invariant if T maps any polynomial of degree n with all zeros in S to a polynomial with all zeros in S, or to the zero polynomial. We find conditions on T that guarantee the existence of a unique T_n-invariant set that is minimal under inclusion and establish some of its properties.

Keywords: *Complex dynamics, Multivalued maps.*

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Sammanfattning

Denna avhandling består av fem artiklar vilka samtliga studerar dynamiken hos flervärda avbildningar på Riemannsfären eller i det komplexa planet. De studerade problemen har sitt ursprung i och kan sägas utgöra specialfall av Pólya—Schur-problemet.

I Artikel I och III introducerar vi begreppet kontinuerlig Hutchinson-invarians och beskriver de invarianta mängderna både topologiskt och geometriskt. I Artikel I fastställer vi nödvändiga och tillräckliga villkor för existensen av en unik invariant mängd som är minimal under inklusion, och bestämmer när denna mängd är icke-trivial och när den är kompakt. Artikel III fokuserar på randen av denna mängd och begränsar dess komplexitet i en särskild mening. Vidare klassificerar vi de olika typerna av randpunkter.

Artikel II och IV behandlar dynamiken hos holomorfa korrespondenser. I Artikel II konstruerar vi explicita differentialoperatorer och associerade holomorfa korrespondenser sådana att det för differentialoperatorn finns en unik Hutchinson-invariant mängd i höga grader som minimal under inklusion och vi studerar ekvidistributionen av de associerade holomorfa korrespondenserna. I Artikel IV studerar vi konforma mått av en klass av (anti)holomorfa korrespondenser, bevisar deras existens och härleder begränsningar för Hausdorff-dimensionen av korrespondensernas gränsmängder.

Artikel V fokuserar på en frågeställning som är än närmare relaterad till Pólya–Schur-teorin. Givet en differentialoperator T och en grad n, sägs en mängd $S \subset \mathbb{C}$ vara T_n-invariant om T avbildar varje polynom av grad n med alla nollställen i S på ett polynom med alla nollställen i S, eller på nollpolynomet. Vi hittar villkor på T som garanterar existensen av en unik minimal under inklusion T_n-invariant mängd och fastställer några av dess egenskaper.

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this PhD thesis.

Paper I: **Linear first order differential operators and their Hutchinson Invariant sets**
Journal of Differential Equations, **391** (2024), 265-320, Published by Elsevier
Per Alexandersson, Nils Hemmingsson, Dmitry Novikov, Boris Shapiro and Guillaume Tahar

Paper II: **Equidistribution of iterations of holomorphic correspondences and Hutchinson invariant sets**
Conformal Geometry and Dynamics, **28** (2024), 97-114, Published by the American Mathematical Society
Nils Hemmingsson

Paper III: **On boundary points of minimal continuously Hutchinson invariant sets**
Submitted, arXiv: 2202.10197
Per Alexandersson, Nils Hemmingsson, Dmitry Novikov, Boris Shapiro and Guillaume Tahar

Paper IV: **Conformal measures of (anti)holomorphic correspondences**
Submitted, arXiv: 2409.01361
Nils Hemmingsson, Xiaoran Li and Zhiqiang Li

Paper V: **An inverse problem in Pólya–Schur theory II. Exactly solvable operators and complex dynamics**
Submitted, arXiv: 2412.01643
Per Alexandersson, Nils Hemmingsson and Boris Shapiro

 Preprint versions of Papers I and II appear in the author's licentiate thesis [Hem23]. The arXiv versions of Papers IV and V are identical to those included in this thesis, except for some wording changes. Compared to the arXiv version of Paper III, the version included in this thesis includes an additional short final section.

Author's contribution: The joint papers resulted from collaborative efforts. The author of this thesis took an active role in shaping the research direction, proving results, and writing the papers.

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Contents

1. Introduction

The papers in this thesis concern sets in the complex plane or on the Riemann sphere that are invariant under certain multivalued maps. The problems studied relate to Pólya–Schur theory and the multivalued maps under consideration arise from linear differential operators acting on either complex polynomials or real powers of linear polynomials. Many of the questions studied in this thesis are also related to the dynamics of rational functions.

The purpose of this introduction is to familiarize a mathematician who is not an expert in the subject with important concepts of the thesis, as well as to put the results into context within the general body of research in the area. As such, we will introduce, in order, *the dynamics of rational functions*, *linear differential operators*, *Pólya–Schur theory* and its *inverse*, *Hutchinson invariance*, *continuous Hutchinson invariance*, *(anti)holomorphic correspondences*, *conformal measures* and *rational vector fields and their separatrix graph*, as well as other relevant related concepts.

We assume that the reader has basic knowledge in complex analysis, functional analysis, algebraic geometry, and measure theory.

Versions of Sections 1.5.2 and 1.7 in this introduction originally appeared in the author's licentiate thesis, but they have been revised for this thesis. The first part of Section 1.5 is adapted from Paper IV. A version of the introduction of the problems and definitions related to Pólya–Schur theory in Sections 1.3-1.4 appears in the introduction of Paper II.

1.1 Dynamics of rational functions

A *rational* function (or map) $R(z)$ is a function from the Riemann sphere $\widehat{\mathbb{C}}$ to itself that can be written as

$$R(z) := \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials in $\mathbb{C}[z]$ with no common zeros. For $n \geq 1$, we define iteratively $R^n(z) := R(R^{n-1}(z))$, where

$R^0(z) = z$. When one talks of the *dynamics* of the rational function R, one means, roughly speaking, the behavior of the sequence of iterates $R(z_0), R^2(z_0), R^3(z_0), \ldots$ for different starting points $z_0 \in \widehat{\mathbb{C}}$. If $d := \max(\deg P(z), \deg Q(z)) \leq 1$, the dynamics of R is fairly easy to describe and not what this subsection is directed towards. However, if $d \geq 2$, the dynamics is very rich and even the simplest case of this instance, when $P(z)$ is a degree 2 polynomial and $Q(z)$ a nonzero constant, is not fully understood at the present moment. Much research has been devoted to the case when $d \geq 2$, and we shall here state a few results that will be relevant for this thesis.

The Fatou set $\mathcal{F}(R) \subset \widehat{\mathbb{C}}$ of the rational function R is the maximal open subset of $\widehat{\mathbb{C}}$ on which the family of functions $\{R^n(z)\}_{n\in\mathbb{N}}$ is normal, i.e. $\mathcal{F}(R) \subset \widehat{\mathbb{C}}$ is the maximal open subset such that each infinite sequence of functions from $\{R^n(z)\}_{n\in\mathbb{N}}$ contains a subsequence that converges locally uniformly on $\mathcal{F}(R)$ (see, e.g., [Bea00]). The Julia set $\mathcal{J}(R)$ of R is the complement of $\mathcal{F}(R)$. We say that a point $z_0 \in \widehat{\mathbb{C}}$ is *periodic* if there exists $n \geq 1$ such that $R^n(z_0) = z_0$. A periodic point z_0 is *attracting* (resp. *repelling*) if $|(R^n)'(z_0)| < 1$ (resp. $|(R^n)'(z_0)| > 1$), where $n \geq 1$ is such that $R^n(z_0) = z_0$. If $R^n(z_0) = z_0$ for some $n \geq 1$ and $(R^n)'(z_0)$ is a root of unity, we say that z_0 is a *parabolic* periodic point. Moreover, a set $S \subset \widehat{\mathbb{C}}$ is *forward* (resp. *backward*) *invariant* (under R) if $R(S) \subset S$, (resp. $R^{-1}(S) \subset S$) and *completely invariant* if it is both forward and backward invariant.

The following theorem may be found in e.g. [Bea00].

Theorem 1.1.1. *Let R be a rational function with $d \geq 2$. Then $\mathcal{J}(R)$ is the closure of the set of repelling periodic points of R. It is the minimal under inclusion, closed and completely invariant set containing at least three points.*

The following definition will be relevant for the results stated in later sections.

Definition 1.1.2. The *Mandelbrot set* $\mathcal{M}$ is the set

$$\mathcal{M} := \{c \in \widehat{\mathbb{C}} : \mathcal{J}(z^2 + c) \text{ is connected}\}.$$

1.2 Linear differential operators

Several of the papers in this thesis are either based on, or at least motivated by, the study of linear differential operators from $\mathbb{C}[z]$ to itself. Here, we shall introduce a few relevant concepts. Let $\mathcal{T}$ be the space of all linear operators sending polynomials in $\mathbb{C}[z]$ to polynomials in $\mathbb{C}[z]$. The following fundamental result holds.

Theorem 1.2.1 ([Pee59]). *Let $T \in \mathcal{T}$. Then there are polynomials Q_j, $j = 0, 1, \ldots$ such that*

$$T = \sum_{j=0}^{\infty} Q_j(z) \frac{d^j}{dz^j}.$$

Our study mainly concerns those $T \in \mathcal{T}$ which have finite order $k \in \mathbb{N}$, i.e. those that can be written as

$$T = \sum_{j=0}^{k} Q_j(z) \frac{d^j}{dz^j},$$

with polynomials $Q_j(z)$. The following definitions are important for Papers II and V.

Definition 1.2.2. Consider

$$T = \sum_{j=0}^{k} Q_j(z) \frac{d^j}{dz^j}.$$

The integer $\rho_T = \max\limits_{0 \le j \le k} (\deg Q_j - j)$ is the *Fuchs index* of T.

Definition 1.2.3. If $\rho_T = 0$, we say that T is *exactly solvable.*

Definition 1.2.4. If $\rho_T = \deg Q_k - k$, then T is called *non-degenerate.*

We have the following statement that will be useful in Paper V.

Theorem 1.2.5 ([BR02]). *The following statements hold.*

(i) *For any exactly solvable operator T and each non-negative integer i, one has*

$$T(x^i) = \lambda_i^T x^i + \text{ lower order terms.} \tag{1.1}$$

Additionally, for i large, the numbers λ_i^T have monotone increasing absolute values.

(ii) *For any exactly solvable operator T and any sufficiently large positive integer n, there exists a unique monic eigenpolynomial $p_n^T(x)$ of T of degree n. The eigenvalue of p_n^T equals λ_n^T, where λ_n^T is given by* (1.1).

1.3 Pólya–Schur theory

Pólya–Schur theory, first systematically studied in [PS14], concerns a form of invariance that we shall presently introduce. As mentioned, the papers in this thesis concern variants of the Pólya–Schur problem. The objective of Pólya–Schur theory is to solve the following problem.

Problem 1.3.1 (Pólya–Schur problem). *Given a set $S \subset \mathbb{C}$, find the set of all linear operators $T \in \mathcal{T}$ such that for any $p \in \mathbb{C}[z]$ with all its zeros in S, $T(p)$ has all zeros in S or is the zero polynomial.*

Significant progress has recently been made in this area. In particular, Problem 1.3.1 has been solved when the set S equals the image of the unit disk or the unit circle under a Möbius transformation [BB09], and for strips [BC17]. However, for more general sets S, Problem 1.3.1 remains widely open.

1.4 Inverse Pólya–Schur theory

In [ABS] the "inverse" of Problem 1.3.1 was introduced. It can be stated as follows.

Problem 1.4.1. *For a given linear differential operator T, find the sets S such that T sends each polynomial with all its zeros in S to a polynomial with all its zeros in S, or to the zero polynomial.*

A set S satisfying the property stated in Problem 1.4.1 for a given linear differential operator T is called T-invariant. Among other results, the following theorem is shown in [ABS].

Theorem 1.4.2 ([ABS]). *Let*

$$T = \sum_{j=0}^{k} Q_j(z) \frac{d^j}{dz^j}$$

be a given linear differential operator. Then the following statements hold.

1. *Any T-invariant set is convex.*
2. *If $Q_k(z)$ is non-constant, then any T-invariant set contains the zeros of $Q_k(z)$. In particular, there exists a unique invariant set M_T that is minimal under inclusion.*

Moreover, in [ABS] the set of sets $\mathcal{I}_{\geq n}$ is introduced. For a given linear differential operator T, a set S belongs to $\mathcal{I}_{\geq n}$ if it sends any polynomial of degree at least n with all its zeros in S to a polynomial with all its zeros in S, or to the zero polynomial. The set $\mathrm{M}^T_{\geq n}$ is the unique set in $\mathcal{I}_{\geq n}$ that is minimal under inclusion, if it exists. The following theorem is shown. Here, and throughout this thesis, $Conv(Q_k)$ denotes the convex hull of the set of zeros of $Q_k(z)$.

Theorem 1.4.3 ([ABS])**.** *Suppose that T is a non-degenerate linear differential operator with Q_k non-constant. Then the Hausdorff limit $\lim_{n\to\infty} \mathrm{M}^T_{\geq n}$ exists and equals $Conv(Q_k)$.*

Unless one imposes very strong assumptions on T, a complete characterization of T-invariant sets seems to be very difficult with current methods. Therefore, in [ABS], three other forms of invariance are introduced, which we define here.

Definition 1.4.4. For a given linear differential operator $T : \mathbb{C}[w] \to \mathbb{C}[w]$, a closed set $S \subset \mathbb{C}$ is *Hutchinson invariant in degree n* (or $T_{H,n}$-invariant for short) if for each $z \in S$, the zeros of $T[(w-z)^n]$ belong to S.

For a given T, if there exists a unique Hutchinson invariant in degree n that is minimal under inclusion, it is denoted by $\mathrm{M}^T_{H,n}$,

The next definition, related to the ones above, is the following.

Definition 1.4.5. For a given linear differential operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$, a closed set $S \subset \mathbb{C}$ is *continuously Hutchinson invariant* (or T_{CH}-invariant for short) if for each $u \in S$, the zeros of $T[(z-u)^t]$, for all $t \geq 0$, belong to S.

For a given T, if there exists a unique continuously Hutchinson invariant set that is minimal under inclusion, it is denoted by M^T_{CH}.

Remark 1.4.6. The reason Definition 1.4.4 is stated with the variables w and z and Definition 1.4.5 is stated with the variables u and z, is that this is the format in the corresponding Papers; Paper II and Papers I and III respectively.

Definition 1.4.7. For a given linear differential operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$, a closed set $S \subset \mathbb{C}$ is T_n-invariant if for each polynomial p of degree n with all its zeros in S, $T[p(z)]$ has all zeros in S or equals the zero polynomial.

When it exists, we denote by M_n^T the unique T_n-invariant set that is minimal under inclusion. All of the three variations of the inverse Pólya–Schur problem introduced above are studied in this thesis. As we shall see, sets that are invariant in the sense of Definitions 1.4.4, 1.4.5, and 1.4.7 are generally substantially more complicated than the "nice", convex T-invariant sets.

1.5 Holomorphic correspondences

Let $P(z,w)$ (or $P(\overline{z},w)$) be a polynomial with coefficients in $\mathbb{C}$. The equation $P(z,w)=0$ defines an algebraic curve $\Gamma \subseteq \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$. The algebraic curve Γ, and hence also P, defines an *(anti)holomorphic correspondence* $F\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$F(z) \coloneqq \{w : (z,w) \in \Gamma\}.$$

One often writes $F\colon z \to w$ to indicate the direction of F. For $n \geq 2$, one iteratively defines

$$F^n(z) \coloneqq \big\{w \in F(y) : y \in F^{n-1}(z)\big\}.$$

For a given correspondence $F\colon z \to w$, defined by a polynomial $P(z,w)$ (or $P(\overline{z},w)$), $F^{-1} : w \to z$ is the natural inverse correspondence, i.e., the one defined by the same polynomial. In Paper II, we use the notation $F^\dagger$ instead of F^{-1}. Further, F^0 is the identity map. A set $S \subset \widehat{\mathbb{C}}$ is *forward* (resp. *backward*) *invariant* under F if $F(S) \subset S$ (resp. $F^{-1}(S) \subset S$), and *completely invariant* if it is both forward and backward invariant. When the correspondence F is clear from the situation, we will simply say that a set is *forward invariant*, and omit "under F".

(Anti)holomorphic correspondences may be seen as generalizations of rational maps and finitely generated Kleinian groups as follows. If $R(z) = \dfrac{P(z)}{Q(z)}$ is a rational map, then the correspondence $F_R : z \to w$ defined by the polynomial relation

$$P(z,w) = P(z) - wQ(z) = 0$$

is readily seen to represent the map R in the sense that

$$F_R(z) = \{R(z)\}.$$

Similarly, if G is a Kleinian group generated by the Möbius transformations

$$\gamma_i = \frac{a_i z + b_i}{c_i z + d_i}, \qquad i \in \{1, \ldots, n\},$$

then the correspondence defined by

$$P(z,w)=\prod_{i=1}^{n}(a_iz+b_i-w(c_iz+d_i))=0$$

has the same full orbits as the group G acting on $\widehat{\mathbb{C}}$.

Next, a branch f of a correspondence F defined on a set S is an (anti)holomorphic map $f : S \to \widehat{\mathbb{C}}$ such that $f(z) \in F(z)$ for each $z \in S$. Lastly, we will need the definition of the *forward limit set* of a correspondence F with respect to a starting point $x \in \widehat{\mathbb{C}}$. It is the set

$$\Lambda_+(x)=\bigcap_{n=0}^{+\infty}\overline{\bigcup_{k=n}^{+\infty}F^k(x)}.$$

1.5.1 Holomorphic correspondences and Hutchinson invariance

Take

$$T=\sum_{j=0}^{k}Q_j(z)\frac{d^j}{dz^j}$$

and consider

$$\frac{T[(z-u)^n]}{(z-u)^{n-k}}=0.$$

Without loss of generality, we assume that $n \geq k$. Then

$$\frac{T[(z-u)^n]}{(z-u)^{n-k}}$$

is a polynomial in the variables z and u with coefficients in $\mathbb{C}$. Denote by $T_n : u \to z$ the holomorphic correspondence defined by

$$\frac{T[(z-u)^n]}{(z-u)^{n-k}}=0.$$

By definition, a set $S \subset \mathbb{C}$ is Hutchinson invariant if and only if, for each $u \in S$, $T_n(u) \setminus \{\infty\} \subset S$. However, $T_n(u) \setminus \{\infty\}$ equals the zeros in z of $T[(z-u)^n] = 0$ (possibly after removing the point u from the latter set). As such, the study of Hutchinson invariance boils down to the study of the forward invariant sets under the holomorphic correspondence defined by T_n for different T and n. In Paper V, we show that for every polynomial $P(u,z)$ in two variables of degree k, there is a unique exactly solvable operator T of order at most k such that

$$T[(z-u)^k]=P(u,z).$$

Thereby, the study of Hutchinson invariance is the same as the study of invariant sets under holomorphic correspondences (taking a bit extra care of the point ∞.) This direct connection between holomorphic correspondences and Hutchinson invariance is also used in Paper V to deduce results about T_n-invariant sets.

Remark 1.5.1. Note that for a given linear differential operator T, there is the holomorphic correspondence T_n and the notion of T_n-invariant sets. However, for a given linear differential operator T, forward invariance under the correspondence T_n is *not* equivalent to a set being T_n-invariant (Definition 1.4.7). While this is a bit unfortunate, we maintain the notation in this introduction to remain consistent with the papers included in the thesis.

1.5.2 Correspondences as matings

In order to properly state the results of Paper IV, we shall introduce the Bullett–Penrose and LLMM correspondences.

First, we will need the notion of quasiconformal maps and hybrid equivalences. A *quasiconformal map* from a subset of $\widehat{\mathbb{C}}$ to a subset of $\widehat{\mathbb{C}}$ is, roughly, an orientation preserving homeomorphism that sends infinitesimal circles to infinitesimal ellipses of bounded eccentricity, for the precise definition see e.g. [Bea00].

By a *pinched neighborhood* $U \subseteq \widehat{\mathbb{C}}$ of a set S, *pinched at* a finite set P, we mean the closure of an open set V, such that V has finitely many connected components, V contains $S \setminus P$ but no points in P, and $U = \overline{V}$ contains S. A *pinched neighborhood* of a set S is a pinched neighborhood of S pinched at some finite set P. Denote

$$\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

We say that g is *hybrid equivalent* to a map h on a set S if there is a pinched neighborhood U of S and a quasiconformal map ϕ defined on U, such that $\phi \circ g = h \circ \phi$ and $\bar{\partial}\phi = 0$ almost everywhere on S. The definition of hybrid equivalence introduced here is tailored in order to introduce Definition 1.5.3 and Theorem 1.5.4 below. For the classical definition of hybrid equivalences between maps, see [DH85] and [Lom15].

So far, not very much is known about the dynamics of general (anti)holomorphic correspondences, but recently more attention has been brought to the area. We shall now go through a few recently discovered results. First, let us introduce two families of (anti)holomorphic correspondences that have been shown to be *matings* (explained below) between rational maps and Hecke groups.

The Bullett–Penrose correspondences F_a are defined by the polynomial equation

$$\left(\frac{az+1}{z+1}\right)^2+\left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right)+\left(\frac{aw-1}{w-1}\right)^2-3=0, \qquad (1.2)$$

where a is a complex parameter.

Theorem 1.5.2 ([BL20a, BL20b]). *There is a set $\mathcal{M}_\Gamma$ homeomorphic to the Mandelbrot set, such that for $a\in\mathcal{M}_\Gamma$, $\widehat{\mathbb{C}}$ is partitioned into two disjoint sets Ω and Λ, each completely invariant under F_a such that*

(1) *Ω is open and conformally equivalent to the open upper halfplane,*

(2) *$\Lambda=\Lambda_{a,-}\cup\Lambda_{a,+}$, the set $\Lambda_{a,-}\cap\Lambda_{a,+}$ is a singleton and $\Lambda_{a,+}$ and $\Lambda_{a,-}$ are closed, connected and forward respectively backward invariant. F_a has a branch of carrying $\Lambda_{a,-}$ onto itself of degree 2, F_a^{-1} has a branch carrying $\Lambda_{a,+}$ onto itself of degree 2 and the remaining branch of F_a carries $\Lambda_{a,-}$ to $\Lambda_{a,+}$ homeomorphically.*

We should mention that, since the Mandelbrot set is compact and has a nonempty interior, the set $\mathcal{M}_\Gamma$ in Theorem 1.5.2 is not unique. However, it can be chosen as the *connectedness locus* $\mathcal{C}_\Gamma$ of the Bullett–Penrose correspondences (see [BL20a, BL20b]), which uniquely determines it. In this case, the set $\mathcal{M}_\Gamma$ is called the *modular Mandelbrot set.* We refer the curious reader to the texts [BL20a, BL20b] for further details. From now on, we let $\mathcal{M}_\Gamma$ denote the modular Mandelbrot set.

A correspondence $F:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ is, in colloquial terms, a *mating* between a rational map R and a group G if there exists a partition of $\widehat{\mathbb{C}}$ into disjoint and under F completely invariant sets A and B, such that the following holds. On the set A, the correspondence F is equivalent to the action of G and on certain subsets B_i of B and with suitable forward or backward branches f_i of F, B_i is forward invariant under f_i, and f_i is conjugate to R on B_i. In the case of the Bullett–Penrose correspondence, we mean the following more specifically.

Definition 1.5.3. We say that the correspondence F_a defined by Eq. (1.2) is a *mating* between the quadratic rational map $P_A(z)=z+1/z+A$ and the modular group, if the following conditions hold:

(1) F_a is on Ω conformally conjugate to the pair $\{\alpha,\beta\}$ of the generators $\alpha:z\to z+1,\beta:z\to\dfrac{z}{z+1}$ of the modular group acting on the complex upper half-plane.

(2) the 2-to-1 branch of F_a fixing $\Lambda_{a,-}$ is hybrid equivalent to P_A on $\Lambda_{a,-}$.

The reason for the term "mating" between the modular group and the quadratic map P_A, is, roughly speaking, as follows. The correspondence F defined by (1.2) behaves as the generators of the modular group on Ω, and (branches of) F behaves as $P_A(z)$ on $\Lambda_{a,-}$. On their common boundary, these actions agree, thereby *mating* these two actions.

Bullett and Lomonaco proved the following result.

Theorem 1.5.4 ([BL20a],[BL20b]). *For a in the modular Mandelbrot set M_Γ, the correspondence F_a is a mating between a quadratic rational map $P_A(z) = z + 1/z + A$ and the modular group.*

We now introduce the family of LLMM correspondences. Let f be a rational map of degree $d+1$ that is injective on the closure of the unit disk $\overline{\mathbb{D}}$. Denote by η the reflection in the unit circle, i.e., $\eta(z) := 1/\overline{z}$ for all $z \in \widehat{\mathbb{C}}$. The antiholomorphic LLMM correspondence $F\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ associated to (or defined by) f is given by

$$F(z) := \left\{ w \in \widehat{\mathbb{C}} : \frac{f(w) - f(\eta(z))}{w - \eta(z)} = 0 \right\} \tag{1.3}$$

for all $z \in \widehat{\mathbb{C}}$. A version of these correspondences was introduced in [LLMM21] and mating results for the LLMM correspondences similar to those in Theorem 1.5.4 were proven in [LMM24]. Recently, other strong mating results were found in [BLLM24].

1.5.3 Equidistribution of correspondences

Paper II concerns the study of Hutchinson invariant sets and in it we introduce a family of explicit non-degenerate operators for which there exists a unique Hutchinson invariant set that is minimal under inclusion. We further show that the iterates under T_n equidistribute with respect to a certain measure, whose support equals the aforementioned minimal invariant set. In this section, we shall therefore introduce a few results about the equidistribution of correspondences. First however, let us state the following fundamental result, which is a generalization of Brolin's theorem, see [Bro65].

Theorem 1.5.5 ([Lju83]). *Suppose that $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map of degree $d \geq 2$. Then there is a set $\mathcal{E} \subset \widehat{\mathbb{C}}$, with $|\mathcal{E}| \leq 2$, and a probability measure μ supported on the Julia set of R, such that $R_*\mu = \mu$ and*

$$\frac{1}{d^n} \sum_{R^n(z)=a} \delta_z \xrightarrow{weak*} \mu \quad as\ n \to \infty$$

for all $a \in \widehat{\mathbb{C}} \setminus \mathcal{E}$.

This result was also independently provede in [FLM83].

Let us now give the definition of the pushforward of measures by holomorphic correspondences. Let $\Gamma \subset \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ be an algebraic curve and F be the correspondence induced by it, i.e.

$$F(z) = \{w : (z, w) \in \Gamma\}.$$

We suppose that Γ contains no vertical lines. That is, for each for each $z_0 \in \widehat{\mathbb{C}}$, we have

$$(z_0, \widehat{\mathbb{C}}) \not\subset \Gamma.$$

We may decompose Γ into its irreducible components using the formal sum

$$\Gamma = \sum_{j=1}^{N} m_j \Gamma_j,$$

where Γ_j are the distinct irreducible components of Γ and $m_j \geq 1$ their multiplicity. Recall that $\pi_i : \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$, for $i = 1, 2$, denotes the projection onto the i-th coordinate.

Let $C(\widehat{\mathbb{C}})$ be the set of continuous functions from $\widehat{\mathbb{C}}$ to $\mathbb{C}$. For a given measure μ, we define the functional $F_\mu : C(\widehat{\mathbb{C}}) \to \mathbb{R}$ by,

$$F_\mu(\phi) = \int_{\widehat{\mathbb{C}}} \left(\sum_{j=1}^{N} m_j \sum_{(x,y)\in\Gamma_j} \phi(y) \right) d\mu(x) = \int_{\widehat{\mathbb{C}}} \sum_{y\in F(x)} \phi(y) d\mu(x).$$

Here and elsewhere in this section, the containments $y \in F(x)$, $(x, y) \in \Gamma_j$ and similar expressions are considered with multiplicity. By the Riesz representation theorem (see, e.g., [Fol99, Theorem 7.2]), there is a unique Radon measure, denoted $F_*\mu$ and called the pushforward of μ under F, such that

$$\int_{\widehat{\mathbb{C}}} \phi(x) d(F_*\mu)(x) = \int_{\widehat{\mathbb{C}}} \sum_{y\in F(x)} \phi(y) d\mu(x)$$

for each $\phi \in C(\widehat{\mathbb{C}})$. The measure $F_*\mu$ may also be denoted as $F_*(\mu)$, $(F)_*\mu$ or $(F)_*(\mu)$, depending on the context.

In the special case $\mu = \delta_a$, where δ_a is the Dirac delta measure at a point $a \in \widehat{\mathbb{C}}$, we have

$$\int_{\widehat{\mathbb{C}}} \phi \, d(F_*\delta_a) = \sum_j m_j \sum_{(a,x)\in\Gamma_j} \phi(x) = \sum_{x\in F(a)} \phi(x).$$

For the definition of pushforwards of more general *currents* under holomorphic correspondences see, e.g., [BS16] and [DKW20]. We also define

$F^*\mu = (F^{-1})_*\mu$. Next, a function $\phi : \widehat{\mathbb{C}} \to [-\infty, \infty)$ is *quasi-psh* if it can be locally written as a sum of a smooth and a plurisubharmonic function. A proper subset A of $\widehat{\mathbb{C}}$ is called *pluripolar* if it is contained in $\{z \in \widehat{\mathbb{C}} : \phi(z) = -\infty\}$ for some quasi-psh function $\phi \not\equiv -\infty$.

In the remainder of this section, we assume that $P(z, w)$ has no factors of the form $(z - z_0)$ or $(w - w_0)$, that Γ is defined by $P(z, w) = 0$, and that F is induced by Γ. Let d_1 (resp. d_2) denote the degree of $P(z, w)$ in z (resp. w). With the definitions introduced above, a special case of Theorem 1.2 in [DS06] is the following.

Theorem 1.5.6. *Suppose that F is a holomorphic correspondence induced by Γ, such that $d_2 > d_1 \geq 1$. Then there is a measure μ_F and a pluripolar set $\mathcal{E} \subsetneq \widehat{\mathbb{C}}$ such that $F_*\mu_F = d_2\mu_F$ and if $a \notin \mathcal{E}$, then*

$$\frac{1}{d_2^n}(F^n)_*\delta_a \xrightarrow{weak*} \mu_F \quad \text{as } n \to \infty.$$

The holomorphic correspondence F induced by Γ is called *weakly modular* if there is a positive measure μ on Γ and probability measures μ_1, μ_2 on $\widehat{\mathbb{C}}$ such that $\mu = (\pi_{1|\Gamma})^*\mu_1$ and $\mu = (\pi_{2|\Gamma})^*\mu_2$. A special case of a theorem in [DKW20] is then the following.

Theorem 1.5.7 ([DKW20]). *Suppose that F is a non-weakly modular correspondence induced by Γ and $d_1 = d_2 \geq 2$. Then there are two measures $\mu_\pm$ such that $F^*\mu_+ = d_2\mu_+$ and $F_*\mu_- = d_2\mu_-$.*

If no critical value of F^{-1} is periodic, then there is a constant $0 < \lambda < 1$ such that for any $a \in \widehat{\mathbb{C}}$ and test function ϕ of class C^β with $0 < \beta \leq 1$, we have

$$\left| \left\langle \frac{1}{d_2^n}(F^n)_*\delta_a - \mu_-, \phi \right\rangle \right| \leq A_\beta \|\phi\|_{C^\beta} \lambda^{\beta n}$$

where $A_\beta > 0$ is a constant independent of $n, a,$ and ϕ.

Naturally, if no critical value of F is periodic, then the analogous statement for μ_+ holds. We refer the reader to [DKW20, Section 2] for the definition of critical points of a correspondence, which is used in Theorem 1.5.7. We should point out that their definition is not equivalent to the one we use in Paper IV, but nonetheless closely related. We further want to stress that in their original formulation, the statements Theorem 1.5.6 and Theorem 1.5.7 are more general, and we refer the reader to the texts [DS06, DKW20] for more details. Moreover, the texts also contain statements about the convergence of the normalized iterated pushforwards (and pullbacks in the latter text) of the Fubini–Study form to μ_F, as well as to μ_- and μ_+, respectively.

In Paper II, we show results that are strongly related to Theorems 1.5.5, 1.5.6 and, 1.5.7, but for other, explicit, holomorphic correspondences.

1.6 Conformal measures

In Paper IV, we initiate the study of conformal measures on limit sets of (anti)holomorphic correspondences. These considerations build upon the ideas of [Pat76], in which conformal measures for Fuchsian groups were introduced. In [Sul83], similar ideas were used to construct conformal measures on the Julia sets of rational functions. Here, we introduce the notion of conformal measures for rational functions, as the ideas employed in their construction most closely resembles the methods we utilize in Paper IV.

Definition 1.6.1. Let R be a rational function of degree at least 2. A Borel probability measure μ supported on $\mathcal{J}(R)$ is *δ-conformal* (for R) if for each set A on which R is injective,

$$\mu(R(A)) = \int_A |R'|^\delta d\mu,$$

where $|\cdot|$ denotes the absolute value in the spherical metric.

The following definition is not the classical one, but turns out to be equivalent to it [DU91].

Definition 1.6.2. A rational map R is *expansive* if $\mathcal{J}(R)$ contains no critical points of R.

Let $\mathrm{HD}(S)$ denote the Hausdorff dimension of a set S. The following theorem holds.

Theorem 1.6.3 ([Sul83, DU91, ADU93]). *Suppose that R is expansive. Then there exists a unique $\delta > 0$ such that a non-atomic δ-conformal measure exists, and* $\mathrm{HD}(\mathcal{J}(R)) = \delta$.

The text [McM00] provides an excellent exposition on conformal measures of rational functions and contains many more results beyond the one stated above.

1.7 Rational vector fields and their separatrix graphs

Let $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function. Consider the rational differential equation

$$\frac{dz}{dt} = R(z), \quad t \in \mathbb{R}, \tag{1.4}$$

to which we associate the rational vector field $R(z)\partial_z$ on $\widehat{\mathbb{C}}$. We call solutions to (1.4) *integral curves*. To understand the continuously Hutchinson invariant sets of a first-order linear operator

$$T = Q(z)\frac{d}{dz} + P(z),$$

it turns out to be crucial to get an understanding of the rational vector field $\frac{Q(z)}{P(z)}\partial_z$, which is why this subsection is included in this introduction.

Let us begin with the notions of singular and critical points of a rational vector field. Let

$$R(z) = \frac{Q(z)}{P(z)},$$

where $Q(z)$ and $P(z)$ have no common zeros. If $\deg Q \leq \deg P + 2$ then the singular points are precisely the zeros of P (if any), and their orders equal their multiplicities as zeros of P. If $\deg Q > \deg P + 2$, then besides the zeros of P, ∞ is also a singular point of order $\deg Q - \deg P - 2$. If $\deg Q \geq \deg P + 2$ then the critical points are precisely the zeros of Q (if there are any) and their orders equal their multiplicities as zeros of Q. If $\deg Q < \deg P + 2$, then other than the zeros of Q, ∞ is also a critical point of order $\deg P + 2 - \deg Q$. Points that are not singular or critical are called regular. Note that ∞ is a regular point if and only if $\deg Q = \deg P + 2$.

In our considerations, an important concept is that of the separatrices of a rational vector field.

Definition 1.7.1. Let $z(t)$ be a solution to (1.4) defined on its maximal domain of definition $(t_{\min}, t_{\max}) \subset \mathbb{R}$. If at least one of the limits $\lim_{t \to t_{\min}} z(t)$ and $\lim_{t \to t_{\max}} z(t)$ exists and equals a singular point of $R(z)\partial_z$, then $z(t)$ is a *separatrix*.

Fix an integral curve $z(t)$ with maximal domain of definition $(t_{\min}, t_{\max})$. Denote

$$z_{\min} := \lim_{t \to t_{\min}} z(t), \quad z_{\max} := \lim_{t \to t_{\max}} z(t)$$

If $z_{\min}$ equals a singular point, then $t_{\min} \in \mathbb{R}$, and if $z_{\max}$ equals a singular point, then $t_{\max} \in \mathbb{R}$.

Regardless of whether $z(t)$ is a separatrix, if $z_{\min}$ is not singular, then $t_{\min} = -\infty$, and either $z_{\min}$ is a critical point of $R\partial_z$ or $z(t)$ is periodic (and similarly for $t_{\max}$). Note that this implies that $z_{\min}$ exists if and only if $z_{\max}$ exists. In particular, if $z(t)$ is a separatrix, then both limits exist. If the limits $z_{\min}$ and $z_{\max}$ exist, they are called the *endpoints* of $z(t)$.

Now, one of the endpoints of a separatrix equals singular point and the other endpoint may equal the same singular point, another singular point or a critical point. Therefore, the separatrices split into four distinct categories:

1. *Homoclinic separatrices* are those for which $z_{\min} = z_{\max}$ is the same singular point.

2. *Heteroclinic separatrices* are those for which $z_{\min} \neq z_{\max}$ are distinct singular points.

3. *Outgoing separatrices* are those for which $z_{\min}$ is a singular point and $z_{\max}$ is a critical point.

4. *Ingoing separatrices* are those for which $z_{\min}$ is a critical point and $z_{\max}$ is a singular point.

It should be noted that for each of the above types of separatrices, there exist vector fields that have separatrices of this type, i.e. all types of separatrices actually occur.

Separatrices $z(t)$ of the vector field $-R\partial_z$ such that $z_{\min}$ is a zero of $P(z)$ are of particular importance in Papers I and III. We call them P-starting separatrices (of $-R\partial_z$).

Next, while the global behavior of a given rational vector field may be complicated, its local behavior is easily understood. Let us explain the local portraits of a rational vector field near critical and singular points. Let z_0 be a singular or a critical point. After a change of variables, we can assume that $z_0 = 0$ and that

$$R(z)\partial_z = (z^k + \text{higher order terms in z})\partial_z, \quad k \in \mathbb{Z} \setminus \{0, 1\}$$

or

$$R(z)\partial_z = (\lambda z + \text{higher order terms in } z)\partial_z,$$

close to 0. Let us begin with the former case and assume that $k < 0$, i.e., $z_0 = 0$ is a singular point. Then, there are outgoing separatrices of

$R\partial z$ going from 0 in the directions

$$\frac{2\pi j}{k+1}, \quad j = 0, 1, ..., 2k-1$$

and ingoing separatrices going to 0 in the directions

$$\frac{2\pi j + \pi}{k+1}, \quad j = 0, 1, ..., 2k-1.$$

Next, suppose that $k > 1$, so that 0 is a critical point. In this case, there are integral curves going out from 0 in the directions

$$\frac{2\pi j}{k-1}, \quad j = 0, ..., k-2$$

and integral curves going to 0 in the directions

$$\frac{2\pi j + \pi}{k-1}, \quad j = 0, ..., k-2.$$

Furthermore, there exist outgoing and ingoing separatrices going to and from 0 in these directions respectively (see, e.g., [DG21]). We turn to the remaining case, where

$$R(z)\partial_z = (\lambda z + \text{higher order terms in } z)\partial_z.$$

We have three distinct cases depending on the real part $\Re(\lambda)$ of λ. If $\Re(\lambda) < 0$, then 0 is a sink, and all nearby integral $z(t)$ curves tend to 0 as $t \to \infty$. If $\Re(\lambda) > 0$, then 0 is a source and all nearby integral curves $z(t)$ tend to 0 as $t \to -\infty$. If $\Re(\lambda) = 0$, then 0 is a center and all nearby integral curves are closed curves with the same period τ for some $\tau > 0$. Examples of phase portraits near singular and critical points are shown in Fig. 1.1, which is taken from the appendix of Paper I.

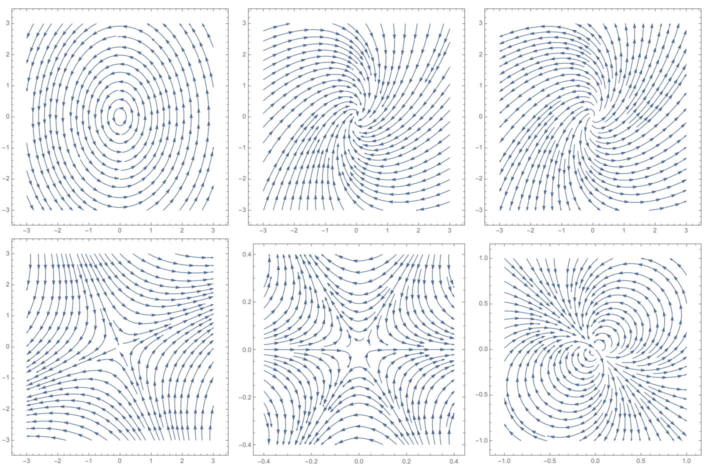

Figure 1.1: Singular points of analytic vector fields (left-to-right; top-to-bottom): center, sink, source, saddle, saddle of order 2, and zero of order 2.

2. Summaries of papers

2.1 Paper I

We refer the reader to the definitions in Section 1.4. In Paper I, we initiate the study of continuously Hutchinson invariant sets of first order linear differential operators

$$T = Q(z)\frac{d}{dz} + P(z). \tag{2.1}$$

This is the first paper that studies this question systematically, and we focus on establishing foundational results. Here, let us assume that neither $Q(z)$ nor $P(z)$ is identically zero. The T_{CH}-invariant sets when either $Q(z)$ or $P(z)$ is identically zero are easy to describe and are fully classified in the paper. In this summary, we shall further assume that $P(z)$ and $Q(z)$ have no common zeros in order to make the statements of the theorems below more straightforward.

We present several key findings, including the following statements.

Proposition 2.1.1 (Proposition 1.8 (ii) of Paper I)**.** *Let T be given as in* (2.1)*. If at least one of P and Q is non-constant, then there exists a unique and nonempty T_{CH}-invariant set that is minimal under inclusion, denoted by M^T_{CH}, equal to the intersection of all the T_{CH}-invariant sets.*

This statement is not difficult to prove, but is an important foundation. We sketch the proof.

Sketch of proof. Take a T_{CH}-invariant set $S \subset \mathbb{C}$, $u \in S$, and consider

$$\frac{T[(z-u)^t]}{(z-u)^{t-1}} = tQ(z) + (z-u)P(z) \tag{2.2}$$

for $t \geq 0$. By definition, the zeros of the expressions in (2.2) belong to S. Sending $t \to 0$, gives that the zeros of $P(z)$ belong to S, and sending $t \to \infty$ gives that the zeros of $Q(z)$ belongs to S. Here, we use the convention that if P or Q is constant, then their set of zeros is empty. By assumption, this implies that every T_{CH}-invariant set has some points in common, namely the zeros of P and Q. Then, noting that the intersection of any family of T_{CH}-invariant sets is T_{CH}-invariant, it follows that there exists a unique and nonempty T_{CH}-invariant set that is minimal under inclusion. $\square$

Now, set

$$R(z) \coloneqq \frac{Q(z)}{P(z)}.$$

We have the following important notion.

Definition 2.1.2. The *associated ray* $r(z)$ of a point $z \in \mathbb{C}$ that is not a zero of P is the set

$$\{z + tR(z) : t \geq 0\}.$$

By dividing the expression in (2.2) by $P(z)$ (assuming z is not a zero of P), and rearranging, we obtain the following useful proposition.

Theorem 2.1.3 (Theorem 3.18 of Paper I)**.** *A closed set $S \subset \mathbb{C}$ is T_{CH}-invariant if and only if the associated ray of each point in the complement S^c is disjoint from S.*

Theorem 2.1.4 (Lemma 5.4 in Paper I)**.** *Let T be given as in* (2.1)*. If at least one of P and Q is non-constant, then each component of every T_{CH}-invariant set is simply connected.*

Proof. As the zeros of Q and P belong to S, it follows that for each $z \in S^c$, the associated ray of z is a half line, and thus each component of S is simply connected. $\square$

Proposition 2.1.5 (Proposition A.2 and A.5 of Paper I)**.** *Let T be given as in* (2.1) *and suppose that S is T_{CH}-invariant. If $z \in S$ and $z(t) \subset \mathbb{C}$ is a solution to the differential equation*

$$\frac{dz}{dt} = -R(z)$$

with $z(0) = z$, then $z(t) \in S$ for each $t \geq 0$. Moreover, if $z(t)$ is a P-starting separatrix of $-R(z)\partial_z$ and $z\left((t_{\min}, t_0)\right) \subset \mathbb{C}$, then $z\left((t_{\min}, t_0)\right) \subset S$.

This theorem provides a direct connection between rational vector fields, introduced in Section 1.7 and continuous Hutchinson invariance. In the text, we provide examples of minimal T_{CH}-invariant sets that have boundaries equal to the closure of the P-starting separatrices of $-R(z)\partial_z$. The proof of the first part of the theorem is based on the backward Euler method, see, e.g., [But03]. For the second part, we use the fact that for each T_{CH}-invariant set S, there exist appropriate points near the zeros of P that belong to S. We then apply the first part of the theorem along with the fact that S, by definition, is closed. For more details, we refer the interested reader to the appendix of Paper I.

Theorem 2.1.6 (Theorem 1.11 of Paper I)**.** *Take an operator T given as in* (2.1) *and set* $\deg Q - \deg P = K$. *Then the only T_{CH}-invariant set is $\mathbb{C}$ if and only if either $|K| \geq 2$ or $K = 1$ and in the expansion of the vector field*

$$R(z)\partial_z = (\lambda z + \textit{lower order terms in } z)\,\partial_z,$$

at ∞, one has $\Re(\lambda) < 0$.

This is one of the main theorems of Paper I and it has a long proof based on many different scenarios. Therefore, we only briefly mention a few ideas we of the proof. The main tools we use are the associated rays and how the vector field $R(z)\partial_z$ "looks" like near infinity, see Section 1.7. A possible proof strategy is as follows. First, we know that the P-starting separatrices of $-R(z)\partial_z$ as in Proposition 2.1.5 are contained in any T_{CH}-invariant set. By assumption on K, if there are no P-starting separatrices of $-R\partial z$ tending to ∞, then for any $u \in \mathbb{C}$, there are solutions to (2.2) tending to ∞, as $t \to 0^+$ or as $t \to \infty$. We may further deduce how the separatrices, or these solutions to (2.2) tend to ∞. We can then use how $R(z)\partial_z$ looks like near infinity and see that "many" points near infinity have associated rays that intersect these separatrices or the mentioned solutions to (2.2). Iterating this argument and using that each component of every T_{CH}-invariant set is simply connected, we obtain the result.

Theorem 2.1.7 (Theorem 1.12 of Paper I)**.** *Take an operator T given as in* (2.1) *and set* $\deg Q - \deg P = K$. *Then there exists a compact T_{CH}-invariant set $\mathbb{C}$ if and only if $K = 1$ and in the expansion of the vector field*

$$R(z)\partial_z = (\lambda z + \textit{lower order terms in } z)\,\partial_z,$$

at ∞, one has $\Re(\lambda) \geq 0$.

In order to prove this statement, one can begin by applying Theorem 2.1.6. Then, in the case $K = 0$ or $K = -1$, one can see that at least one solution in z of Eq. (2.2) for any $u \in \mathbb{C}$ tends to ∞ as $t \to 0$. In the case $K = 1$ and $\Re(\lambda) > 0$, one can use associated rays to see that sufficiently large disks are T_{CH}-invariant. In the case $\Re(\lambda) = 0$, near infinity the integral curves of $R(z)\partial_z$ are closed curves. One can show that sufficiently close to infinity, the closed, bounded and simply connected set whose boundary equals any of these integral curves is convex, and that for each point in the complement of this set, its associated ray lies in the complement. Thereby, the aforementioned set is T_{CH}-invariant.

We also fully describe the existence of irregular T_{CH}-invariant sets, that is, sets which has the property that they are not equal to the closure of their interior. Moreover, we fully classify when there are T_{CH}-invariant sets with empty interior. We refer the interested reader to Paper I for the statements and their proofs.

We further have the following relation between T_{CH}-invariant sets and Julia sets of rational maps. Denote $\mathcal{J}_{\mathbb{C}}(f) = \mathcal{J}(f) \cap \mathbb{C}$, where f is a rational map of degree at least 2. We call $\mathcal{J}_{\mathbb{C}}(f)$ the *plane Julia set* of f.

Proposition 2.1.8 (Proposition 10.1 of Paper I)**.** *Let T be given as in* (2.1)*. Suppose that the map $R_t(z) = z + tR(z)$ has degree $d \geq 2$ for some $t_0 > 0$. Then the plane Julia set $\mathcal{J}_{\mathbb{C}}(R_{t_0})$ is contained in any T_{CH}-invariant set.*

Sketch of proof. One has to be a bit careful with the point ∞, but the general idea is that if u belongs to the T_{CH}-invariant set S, then by definition, $R_t^{-1}(u) \cap \mathbb{C} \subset S$. Now, for all but at most two points $u \in \widehat{\mathbb{C}}$, we have

$$\mathcal{J}(R_t) \subset \overline{\bigcup_{n=0}^{\infty} R_t^{-n}(u)}.$$

Thus, by iteratively using the fact that $R_t^{-1}(u) \cap \mathbb{C} \subset S$ for each $u \in S$, recalling that any T_{CH}-invariant set is closed by definition, and properly accounting for the point ∞, we conclude that $\mathcal{J}_{\mathbb{C}}(R_t) \subset S$. □

We conclude this summary with what we believe to be an interesting phenomenon. Namely, we expect that there are cases when

$$\mathrm{M}_{CH}^{T} = \bigcup_{t>0} \mathcal{J}(R_t),$$

see Fig. 2.1.

2.2 Paper II

In Paper II, we study the equidistribution of holomorphic correspondences. Our first result establishes the equidistribution of a particular

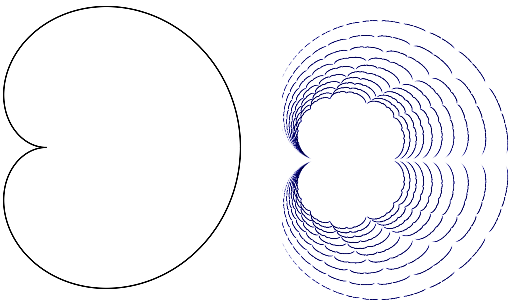

Figure 2.1: The left image shows the boundary of the minimal T_{CH}-invariant set for the operator $T = z^2\frac{d}{dz} + (z-1)$. The right image shows the union of the Julia sets of the maps $z \mapsto z + t\frac{z^2}{z-1}$, for $t \in \{0.2, 0.4, 0.6, \ldots, 1.8, 2.0\}$. This figure is reproduced from the appendix of Paper I.

family of correspondences.

Theorem 2.2.1 (Theorem A of Paper II). *Let $R_0(w)$ be any polynomial of degree d with only simple zeros, $P(z,w)$ any polynomial of (total) degree at most d and let $F_\delta : z \to w$ be the holomorphic correspondence defined by $R_0(w) + \delta P(z,w) = 0$, for $\delta \in \mathbb{C}$. For any $\epsilon > 0$, there exists $\Delta > 0$ such that if $|\delta| < \Delta$, then there exists a probability measure μ_{F_δ} on $\widehat{\mathbb{C}}$ satisfying $(F_\delta)_*(\mu_{F_\delta}) = \mu_{F_\delta} d$ and, for any $a \in \mathbb{C}$,*

$$\frac{1}{d^m}(F_\delta^m)_*(\delta_a) \xrightarrow{weak*} \mu_{F_\delta}, \qquad as\ m \to \infty,$$

and the support $\operatorname{supp}\mu_{F_\delta}$ is contained in the ϵ-neighborhood of the zeros of $R_0(z)$. Moreover, $\operatorname{supp}\mu_{F_\delta}$ is the unique closed and forward invariant set under F_δ that is minimal under inclusion and that contains a point in $\mathbb{C}$.

Remark 2.2.2. We find conditions on P that guarantees that a can be chosen equal to ∞, see Proposition 3.6 in Paper II.

Let now

$$T = \sum_{j=0}^{k} Q_j(w)\frac{d^j}{dw^j} \tag{2.3}$$

be a linear differential operator from $\mathbb{C}[z]$ to itself.

Recalling the definitions from Section 1.4, we relate Theorem 2.2.1 to Hutchinson invariant sets in the following way.

Theorem 2.2.3 (Theorem B of Paper II). *Suppose that T, given as in (2.3), is non-degenerate and all zeros of $Q_k(w)$ are simple. For any $\epsilon > 0$, there exists N such that for each $n \geq N$, the following holds:*

- *T_n has the properties of F_δ with $|\delta| < \Delta$ in Theorem 2.2.1 and $\mathrm{M}^T_{H,n}$ equals the support of the measure μ_{T_n}.*
- *Each point in $\mathrm{M}^T_{H,n}$ is the limit of a sequence of attracting periodic points of T_n and is contained in the ϵ-neighborhood of the zeros of $Q_k(w)$.*
- *If $k \geq 2$ and there is $j < k$ such that $Q_j(w)$ is not identically zero, then $\mathrm{M}^T_{H,n}$ is a Cantor set.*

Remark 2.2.4. For the existence of $\mathrm{M}^T_{H,n}$ it is sufficient that one zero of $Q_k(w)$ is simple.

Sketch of proof of Theorem 2.2.1. First, write

$$P(z, w) = \sum_{j=0}^{d} P_j(w)(w - z)^{d-j},$$

with polynomials $P_j(w)$ such that $\deg P_j(w) \leq j$ and denote

$$R_\beta(z, w) = R_0(w) + \sum_{j=0}^{d} \beta_j P_j(w)(w - z)^{d-j}, \tag{2.4}$$

with $\beta = (\beta_0, \beta_1, \ldots, \beta_d)$. Throughout, when we say that β is small, we mean that each component of β has a sufficiently small absolute value. We may suppose that $\epsilon > 0$ and β are so small that the ϵ-neighborhoods of distinct zeros of $S_\beta(w) = R_0(w) + \beta P_k(w)$ do not overlap. Let the union of the ϵ-neighborhoods of the zeros of $S_\beta(w)$ be denoted by U, and let the component of U containing u_i be denoted by U_i. Next, take $M > 0$ a real number such that the disk $D(0, M)$ of radius M centered at 0 contains U. By investigating the leading terms in Eq. (2.4), one may deduce, possibly by increasing M and decreasing β, that for all $u \geq 2M$, all points $w \in T_n(u)$ satisfy $|w| \leq |u|/2$. Moreover, possibly after decreasing β, we have that for each $u \in D(0, M)$, $F_\beta(u) \subset U$, and for each zero u_i of $S_\beta(w)$, there exists $w \in F_\beta(u)$ of distance no more than ϵ from u_i. Combining these properties, we obtain that for each $v \in \mathbb{C}$, there exists $m \geq 0$ such that $T_n^m(v) \subset U$ and $U_i \cap T_n^m(v) \neq \emptyset$ for each $i = 1, 2, \ldots, d$.

Using (2.4), one may see that the zeros u_i of $S_\beta(w)$ are fixed points of F_β (i.e., $u_i \in F_\beta(u_i)$), and by the assumption that the zeros of $R_0(w)$ are simple, there exists a unique branch f of F_β, defined on U_i, such that $f(u_i) = u_i$, provided that β is sufficiently small. Differentiating implicitly, one finds that for sufficiently small β, $|f'(z)| < 1$ for each $z \in U_i$, which implies that $f^n(z) \to u_i$ as $n \to \infty$ for each $z \in U_i$. This shows that u_i belongs to any set that contains a point in $\mathbb{C}$ and is forward invariant under F_β. One can then use the ideas of [Lju83] to show that the measure μ_{F_β} with the desired properties exists. □

We turn to a sketch of the proof of Theorem 2.2.3

Sketch of proof of Theorem 2.2.3. Consider

$$\frac{T[(w-z)^n]}{(w-z)^{n-k}} = 0,$$

which has the same solutions as

$$\frac{T(w-z)^n}{(n)_k(w-z)^{n-k}} = Q_k(w) + \sum_{j=0}^{k-1} \frac{(n)_j}{(n)_k}(w-z)^{k-j}Q_j(w) = 0,$$

with $(n)_j = n(n-1)\cdots(n-j+1)$ being the Pochhammer symbol. Identifying $d = k$, $Q_j(w) = P_j(w)$ for $j \le k-1$, $P_k(w) \equiv 0$, $Q_k(w) = R_0(w)$ and $\beta_j = \dfrac{(n)_j}{(n)_k}$ it is clear that

$$\frac{T(w-z)^n}{(n)_k(w-z)^{n-k}}$$

has the form of $R_\beta(z,w)$ in (2.4). Then, Theorem 2.2.1 gives the first part of Theorem 2.2.3.

For the second part, one uses that M^T_{CH} is the minimal under inclusion $T_{H,n}$-invariant set together with an argument based on Brouwer's fixed point theorem.

For the third part, one first shows that $\mathrm{M}^T_{H,n}$ is a perfect. This is done by using minimality and, in essence, that the zeros u_i of $S_\beta(w)$ are limit points of $\mathrm{M}^T_{H,n}$ (which may be shown using the analysis done in the proof of Theorem 2.2.1). Then, one can show that M^T_{CH} is a Cantor set by using that the branch f (along with the other locally defined branches fixing the zeros of $S_\beta(w)$) can be chosen to have arbitrarily small derivatives in the set U. □

Errata for Paper II

This section clarifies minor inconsistencies in the version of Paper II included in this thesis. It is important to note that these do not affect the validity of the theorems in the paper.

- There is a typo in the first sentence of the statement of Theorem A. The condition $\delta \geq 0$ is overly restrictive; it should instead state that $\delta \in \mathbb{C}$.

- There is a typo in Remark 1.1 and Proposition 3.6. They should state:

  > [...] *for the minimal* j *such that* $P_j(w) \not\equiv 0$, *we have that* $P_j(w)$ *is not a constant and* $j < d$ [...].

- The second sentence of the proof of Proposition 3.6 should read:

  > *Denote the degree in* w *of the coefficient of the leading term in* z *by* $d_1 > 1$.

- On page 98, we define the *pullback* $F_*(S)$ of a current S by F. However, this should instead be referred to as the *pushforward* of the current S by F, as in the introduction of this thesis.

- The definitions of a quasi-p.s.h. function and pluripolar sets on page 103 contain typos. The correct definitions are as follows:

  - A function $\phi : \widehat{\mathbb{C}} \to [-\infty, \infty)$ is called *quasi-p.s.h.* if it can be expressed locally as the sum of a smooth function and a plurisubharmonic function.
  - A proper subset A of $\widehat{\mathbb{C}}$ is *pluripolar* if it is contained in

    $$\{z \in \widehat{\mathbb{C}} : \phi(z) = -\infty\}$$

    for some quasi-p.s.h. function $\phi \not\equiv -\infty$.

- In (3.1), R_0 should be replaced by $R_0(w)$.

2.3 Paper III

In Paper III, we investigate the boundary of the minimal continuously Hutchinson invariant set for a first order linear differential operator

$$T = Q(z)\frac{d}{dz} + P(z). \tag{2.5}$$

We introduce a compactification of the complex plane that is useful for studying continuously Hutchinson invariant sets. *The extended complex plane* $\mathbb{C} \cup \mathbb{S}^1$ is set-theoretically the disjoint union of $\mathbb{C}$ and $\mathbb{S}^1$ endowed with the topology defined by the following basis of neighborhoods:

- for a point $x \in \mathbb{C}$, we choose the usual open neighborhoods of x in $\mathbb{C}$;
- for a direction $\theta \in \mathbb{S}^1$, we choose open neighborhoods of the form $I \cup C(z, I)$ where I is an open interval of $\mathbb{S}^1$ containing θ and $C(z, I)$ is an open cone with apex $z \in \mathbb{C}$ whose opening (i.e. the interval of directions) is I.

We denote by $\overline{\mathrm{M}_{CH}^T}$ the closure of M_{CH}^T in the extended complex plane. Also, let

$$\mathfrak{I}_R \coloneqq \{z \in \mathbb{C} : \Im(R'(z)) = 0\},$$

where $\Im(c)$ denotes the imaginary part of a complex number c, and let $\mathcal{Z}(f)$ denote the set of zeros of the function f. We call $\mathfrak{I}_R$ the *curve of inflections* of the vector field $R(z)\partial z$.

Definition 2.3.1. The curve of inflections $\mathfrak{I}_R$ of the field $R(z)\partial_z$ decomposes into:

- the *singular locus* $\mathfrak{S}_R$, formed by the points where several branches of $\mathfrak{I}_R$ intersect;
- the *tangency locus* $\mathfrak{T}_R$, consisting of the non-singular points at which $R(z)\partial_z$ is tangent to $\mathfrak{I}_R$;
- *transverse locus* $\mathfrak{I}_R^*$, consisting of the non-singular points of $\mathfrak{I}_R$ at which $R(z)\partial_z$ is transverse to $\mathfrak{I}_R$.

Definition 2.3.2. The *positive trajectory* of the vector field $R(z)\partial_z$ starting at a point $x \in \mathbb{C}$, $x \notin \mathcal{Z}(PQ)$ is the set $\overline{\{z(t) : t \in [0, t_0)\}} \setminus \mathcal{Z}(PQ)$, where $z(t)$ is an integral curve of $R(z)\partial_z$, defined on its maximal domain of definition, such that $z(0) = x$ and t_0 is the maximal extended real number such that $z(t) \in \mathbb{C}$ for all $t \in [0, t_0)$. Here, the closure is taken in $\mathbb{C} \cup \mathbb{S}^1$.

The following correspondences are crucial in our analysis.

Definition 2.3.3. For any $x \in \partial \mathrm{M}^T_{CH} \setminus \mathcal{Z}(PQ)$, we define:

- $\Gamma(x) = \{y \in \gamma^+_x \mid y \neq x\} \cap \overline{\mathrm{M}^T_{CH}}$, where γ^+_x is the positive trajectory of the vector field $R(z)\partial_z$ starting at x;
- $\Delta(x) = \{y \in r(x) \mid y \neq x\} \cap \overline{\mathrm{M}^T_{CH}}$.

Here, $r(x) := \overline{\{x + tR(x) : t \geq 0\}}$ is the closure of associated ray of x in $\mathbb{C} \cup \mathbb{S}^1$.

It can be shown that if $y \in \Gamma(x)$ or $y \in \Delta(x)$, for some $x \in \partial \mathrm{M}^T_{CH} \setminus \mathcal{Z}(PQ)$, then $y \in \partial \mathrm{M}^T_{CH}$, as well.

We introduce the following boundary points of $\partial \mathrm{M}^T_{CH} \setminus (\mathcal{Z}(PQ) \cup \mathfrak{I}_R)$.

Definition 2.3.4. A point of $\partial \mathrm{M}^T_{CH} \setminus (\mathcal{Z}(PQ) \cup \mathfrak{I}_R)$ is a point of:

- *local type* if $\Gamma(z) \neq \emptyset$ and $\Delta(z) = \emptyset$;
- *global type* if $\Gamma(z) = \emptyset$ and $\Delta(z) \neq \emptyset$;
- *extruding type* if $\Gamma(z) \neq \emptyset$ and $\Delta(z) \neq \emptyset$.

We also say that points simply are *local, global and extruding*, respectively.

Definition 2.3.5. A *local arc* of $\partial \mathrm{M}^T_{CH}$ is a maximal open connected subset of an integral curve $z(t) \subset \partial \mathrm{M}^T_{CH}$ of $R(z)\partial_z$ that contains only local points. A *global arc* is a maximal open connected arc of $\partial \mathrm{M}^T_{CH}$ consisting only of points of global type.

In the paper, we show that $\partial \mathrm{M}^T_{CH}$ is locally connected and hence parametrizable at all points except those through which a straight integral curve of $R\partial_z$ passes (these points belong to the tangency locus $\mathfrak{T}_R$). Thus, the notion of global arcs makes sense.

In order to define the other types of boundary points, we need the following definitions.

Definition 2.3.6. For any $z \in \mathfrak{I}_R \setminus \mathcal{Z}(PQ)$, we put $\Delta(z) = \Delta^-(z) \cup \Delta^0(z) \cup \Delta^+(z)$ where $u \in \Delta(z) \cap \mathbb{C}$ belongs to:

- $\Delta^-(z)$ if $R'(z) \leq -\dfrac{R(z)}{u - z}$;
- $\Delta^0(z)$ if $R'(z) = -\dfrac{R(z)}{u - z}$;

- $\Delta^+(z)$ if $R'(z) \geq -\dfrac{R(z)}{u-z}$,

and $u \in \Delta(z) \cap \mathbb{S}^1$ belongs to

- $\Delta^-(z)$ if $R'(z) \leq 0$;
- $\Delta^0(z)$ if $R'(z) = 0$;
- $\Delta^+(z)$ if $R'(z) \geq 0$.

We are now in a position to state the main results of the paper.

Theorem 2.3.7 (Theorem 1.9 of Paper III)**.** *Let T be a differential operator given as in* (2.5)*, and set $d = 3\deg P + \deg Q - 1$. Any point $z \in \partial \mathrm{M}_{CH}^T$ belongs to one of the following types:*

- zeros of the polynomials P *and* Q *(at most* $\deg P + \deg Q$ *of them);*
- singular points of the curve of inflections *(at most* $2d$ *of them);*
- tangency points between the curve of inflections and the field $R(z)\partial_z$*:*
  - *straight segments, half-lines and lines (contained in at most* $\deg P + \deg Q + 1$ *lines);*
  - *at most* $2d^2$ *isolated points;*
- points of the transverse locus $\mathfrak{I}_R^*$ *belonging to one of the four subclasses:*
  - bouncing type*:* $\Delta^+(z) \neq \emptyset$ *and* $\Gamma(z) \cup \Delta^-(z) \neq \emptyset$*;*
  - switch type*:* $\Delta^+(z) \neq \emptyset$ *and* $\Gamma(z) \cup \Delta^-(z) = \emptyset$*;*
  - C^1-inflection type*:* $\Delta^+(z) = \emptyset$*,* $\Delta^-(z) \neq \emptyset$ *and* $\Gamma(z) = \emptyset$*;*
  - C^2-inflection type*:* $\Delta^+(z) = \emptyset$ *and either* $\Delta^-(z) = \emptyset$ *or* $\Gamma(z) \neq \emptyset$*.*
- points not on the curve of inflections *belonging to one of the three subclasses:*
  - local type*:* $\Gamma(z) \neq \emptyset$ *and* $\Delta(z) = \emptyset$*;*
  - global type*:* $\Gamma(z) = \emptyset$ *and* $\Delta(z) \neq \emptyset$*;*
  - extruding type*:* $\Gamma(z) \neq \emptyset$ *and* $\Delta(z) \neq \emptyset$*.*

The proof of Theorem 2.3.7 is done case by case and is too long to describe here. However, we can explain how we show that all points in $\partial \mathrm{M}_{CH}^T \setminus (\mathcal{Z}(PQ) \cup \mathfrak{I}_R)$ are of either local, global or extruding type.

Sketch of proof. If this statement were false, there would be a point $z \in \partial \mathrm{M}_{CH}^T \setminus (\mathcal{Z}(PQ) \cup \mathfrak{I}_R)$ such that $\Gamma(z) = \Delta(z) = \emptyset$. If this is the case, one can find a neighborhood U of z such that all points in U have associated rays belonging to $U \cup (\mathrm{M}_{CH}^T)^c$. But by Theorem 2.1.3 in Section 2.1, this would then show that $(U \cup (\mathrm{M}_{CH}^T)^c)^c \subsetneq \mathrm{M}_{CH}^T$ is T_{CH}-invariant. □

A description of the geometric features of different boundary points is given below, see also Fig. 2.2. Here, we use a natural orientation of local and global arcs. The orientation of local arcs is inherited from the vector field $R(z)\partial_z$. The orientation of global arcs is more intricate, and we refer the reader to the text for details.

- at *points of extruding type*, the boundary of $\partial \mathrm{M}_{CH}^T$ is not locally convex and it switches from a global to a local arc.
- at *points of bouncing type*, $\partial \mathrm{M}_{CH}^T$ intersects the curve of inflections, but does not cross it. Here, $\partial \mathrm{M}_{CH}^T$ is not locally convex.
- at *points of switch type*, $\partial \mathrm{M}_{CH}^T$ is strictly convex, crosses the curve of inflections and the boundary switches from a local to a global arc of opposite orientation.
- at points of C^1-*inflection type*, $\partial \mathrm{M}_{CH}^T$ crosses the curve of inflections and it switches from a global to another global arc having the opposite orientation. At such a point, $\partial \mathrm{M}_{CH}^T$ is C^1 but not C^2.
- at points of C^2-*inflection type*, $\partial \mathrm{M}_{CH}^T$ crosses the curve of inflections and the boundary switches from a global arc to a local arc. Besides, $\partial \mathrm{M}_{CH}^T$ is C^2 at such a point.

A second result of the paper is an upper bound on the number of local arcs of M_{CH}^T, in terms of the degrees of P and Q.

Theorem 2.3.8 (Corollary 1.11 of Paper III)**.** *For any operator T given as in* (2.5)*, the boundary $\partial \mathrm{M}_{CH}^T$ of the minimal set contains at most $d^{16d} + d(2d+1)$ local arcs, where $d = 3 \deg P + \deg Q - 1$.*

We also deduce multiple general topological and geometric properties of M_{CH}^T. For instance, we show the following theorem, stating that the minimal T_{CH}-invariant set is generically connected in $\mathbb{C}$.

Theorem 2.3.9 (Theorem 1.12 of Paper III)**.** *For any linear differential operator T as in* (2.5)*, the minimal continuously Hutchinson invariant set M_{CH}^T is a connected subset of $\mathbb{C}$, with the possible exception of the case where $R(z)$ is of the form $\lambda + \frac{\mu}{z} + o(z^{-1})$ with $\lambda \in \mathbb{C}^*$ and $\mu/\lambda \in \mathbb{R}$. In this case, M_{CH}^T has at most $\frac{1}{2}\deg P + \frac{1}{2}\deg Q$ connected components.*

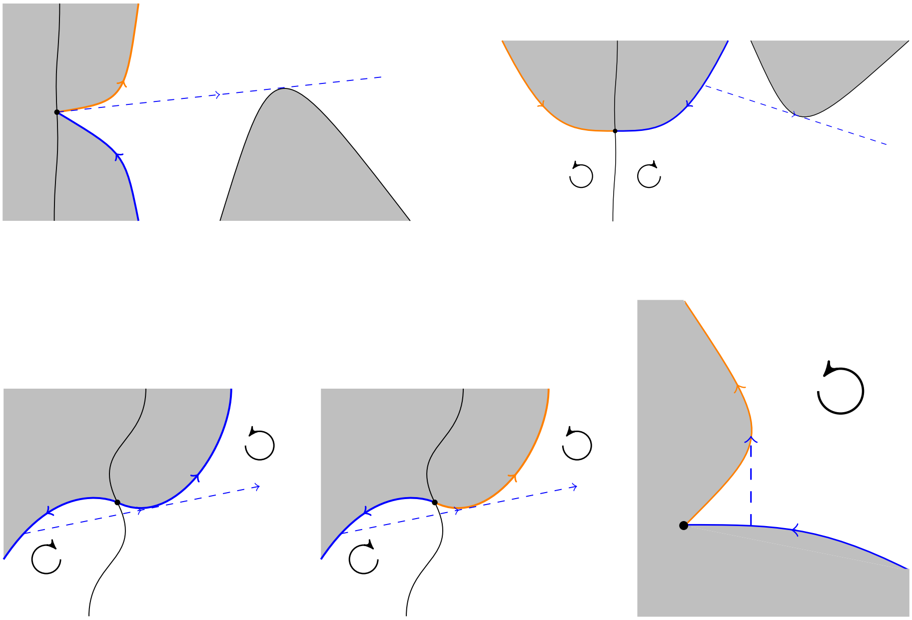

Figure 2.2: Top row from the left: Extruding type, Switch type. Bottom row from the left: C^1-inflection type, C^2-inflection type, Bouncing type. In the pictures, blue arcs are global arcs, red arcs are local arcs and the black arc is a germ of the curve of inflections. The dashed lines are associated rays. The figure also appears in Paper III.

2.4 Paper IV

In Paper IV, we study conformal measures for correspondences. These are probability measures supported on the forward limit set $\Lambda_+(z)$ of an (anti)holomorphic correspondence F, for some suitable z, with the following property. There exists $\delta \in \mathbb{R}$ such that for any Borel set $A \subset \widehat{\mathbb{C}}$ on which a branch f of F is defined and injective, we have

$$\mu(f(A \cap \Lambda_+(x))) = \int_{A \cap \Lambda_+(x)} |f'|^\delta d\mu. \tag{2.6}$$

In the paper, inspired by the ideas in [DU91], we establish sufficient conditions for the existence of non-atomic conformal measures and deduce results on the Hausdorff dimension of the limit sets. We further demonstrate that these conditions are satisfied in two notable cases. For the Bullett–Penrose correspondences, we obtain the following theorem. We ask the reader to recall the necessary definitions from Section 1.5 and remark that if a belongs to the modular Mandelbrot set, then for almost every $z \in \Lambda_{+,a}$, we have that $\Lambda_+(z) = \partial\Lambda_{+,a}$ for the Bullett–Penrose correspondence F_a.

Theorem 2.4.1 (Theorem B of Paper IV)**.** *For $a \in \mathbb{C}$ in the interior of a hyperbolic component of the modular Mandelbrot set, the Bullett–Penrose correspondence F_a admits a δ-conformal measure with $1 \leq \delta < 2$ and $HD(\partial\Lambda_{a,+}) \leq \delta$.*

Analogous results hold for the LLMM correspondences defined by a rational function f injective on the closed unit disk $\overline{\mathbb{D}}$, under some minor extra assumption on the function f; see Theorem A of Paper IV.

Let us explain the ideas we use to obtain Theorem 2.4.1. First, let us introduce the notion of the *Poincaré series*. Suppose that $x \in \widehat{\mathbb{C}}$ is such that there exists a neighborhood U of x on which all branches of F^n for all $n \geq 1$ are defined. For $n \geq 0$, we denote by $M_n := M_n(U)$ the number of branches of F^n defined in U, and denote these branches by $f_{n,j}$, where $j = 1, \ldots, M_n$. For $s > 0$, we define the Poincaré series

$$P_s(x) := \sum_{n=0}^{\infty} \sum_{j=1}^{M_n} |Df_{n,j}(x)|^s. \tag{2.7}$$

Here, $|Dg|$ denotes the absolut value in the spherical metric of the total derivative of the (anti)holomorphic function g.

Definition 2.4.2. If there exist $s > 0$ and $t > 0$ such that $P_s(x)$ converges and $P_t(x)$ diverges, then

$$\delta_{\text{crit}}(x) := \inf\{s > 0 : P_s(x) < \infty\}.$$

If no $s > 0$ such that $P_s(x) < \infty$ exists, then $\delta_{\text{crit}}(x) := \infty$, and if $P_s(x) < \infty$ for each $s > 0$, then $\delta_{\text{crit}}(x) := 0$. The extended real number $\delta_{\text{crit}}(x)$ is called the *critical exponent of F at x.*

As in [Sul83] and using an argument based on the Köbe distortion theorem, one can show that if $x \in (\Lambda_{+,a})^\circ$ is not a periodic point and disjoint from $\mathcal{CV}_{F^{-1}}$, then $\delta_{\text{crit}}(x) \leq 2$. From now on we assume that x has this property, and that $\Lambda_+(x) = \partial\Lambda_{a,+}$. For now, let us further assume that $\delta_{\text{crit}}(x) > 0$ and that $\lim\limits_{s \searrow \delta_{\text{crit}}(x)} P_s(x) = \infty$ and introduce for $s > \delta_{\text{crit}}(x)$ the probability measures

$$\mu_s := \frac{1}{P_s(x)} \sum_{n=0}^{\infty} \sum_{j=1}^{M_n} |Df_{n,j}(x)|^s \delta_{f_{n,j}(x)},$$

where $\delta_{f_{n,j}(x)}$ is the Dirac mass at $f_{n,j}(x)$. Then, sending $s \to \delta_{\text{crit}}(x)$, using the chain rule and the divergence of $\lim_{s \searrow \delta_{\text{crit}}(x)} P_s(x) = \infty$, one can show that any weak-* limit of the measures μ_s as $s \searrow \delta_{\text{crit}}(x)$ is a $\delta_{\text{crit}}(x)$-conformal measure. If $\lim_{s \searrow \delta_{\text{crit}}(x)} P_s(x) \neq \infty$ one can change the sum slightly to still ensure divergence and obtain conformal measures.

A major difference between the setting we are working in and the situation for rational maps is that $Df_{n,j}$ may be 0, while the inverse branch of a rational map never has derivative 0. It turns out that in the case of the Bullett–Penrose correspondence, the point 0 has the property that one of the branches defined in a neighborhood of 0 has a critical point at 0, and the other fixes the point 0 and has derivative equal to 1. Hence, the Dirac measure at 0, δ_0, is a δ-conformal measure for any $\delta > 0$. In order to conclude the existence of non-atomic conformal measures, one needs to show that $\delta_{\text{crit}}(x) > c$, for an explicit number c, where $c \in [1/2, 1)$. We show that under certain assumptions on general holomorphic correspondences F (not merely the Bullett–Penrose correspondences), we have $\delta_{\text{crit}}(x) \geq 1$. This is done as follows. One fixes a simple and closed curve C passing through x disjoint from $\mathcal{CV}_{F^{-1}}$, that has a repelling fixed periodic point (which exists by assumption on a) in one of the two connected components of the complement of C. Then, one considers

$$\bigcup_{n=0}^{\infty} F^n(C)$$

and observes that this set has infinite one-dimensional Lebesgue measure. Using the Köbe distortion theorem, one can show that $\delta_{\text{crit}}(y) = \delta_{\text{crit}}(x)$ for all $x, y \in C$. The definition of the Poincaré series then implies that $P_1(x) = \infty$ for all $x \in C$, and hence $\delta_{\text{crit}}(x) \geq 1$. This is one of the new tools of Paper IV.

We have thus described how we can construct a non-atomic conformal measure supported on $\partial\Lambda_{a,+}$. Then, one uses similar arguments to those found in [DU91, ADU93], to show that $\partial\Lambda_{+,a} \leq \delta_{\text{crit}}(x) < 2$.

We now give the general statements obtained for (anti)holomorphic correspondences, and refer the reader to Paper IV for the necessary definitions.

Theorem 2.4.3 (Theorem C of Paper IV). *Let $x \in \widehat{\mathbb{C}}$ and F be an (anti)holomorphic correspondence that is relatively hyperbolic on $\Lambda_+(x)$ and such that $\Lambda_+(x)$ is minimal. If $\delta_{\text{crit}}(x)$ satisfies*

$$\sup_{\omega \in \Omega_+(x)} p(\omega)/(p(\omega)+1) < \delta_{\text{crit}}(x) < \infty,$$

then there exists a non-atomic $\delta_{\text{crit}}(x)$-conformal measure for F and $\Lambda_+(x)$, and

$$\mathrm{HD}(\Lambda_+(x)) \leq \delta_{\text{crit}}(x) < 2.$$

Theorem 2.4.4 (Theorem D of Paper IV)**.** *Let F be an invariantly inverse-like (anti)holomorphic correspondence on a closed set $S \subsetneq \widehat{\mathbb{C}}$. Suppose that $g_{F,S}$ has an attracting periodic orbit in the interior of S, with immediate basin of attraction $\mathcal{A}$. Suppose further that there exists $x \in \mathcal{A} \setminus \mathcal{PC}_{F^{-1}}$ such that F is relatively hyperbolic on $\Lambda_+(x)$ and that $\Lambda_+(x)$ is minimal. Then $\mathrm{HD}(\Lambda_+(x)) < 2$ and there exists a non-atomic δ-conformal measure for F and $\Lambda_+(x)$, for some $1 \leq \delta < 2$.*

2.5 Paper V

Let

$$T = \sum_{j=0}^{k} Q_j(z) \frac{d^j}{dz^j}$$

be a linear differential operator from $\mathbb{C}[z]$ to itself. In Paper V, we study the T_n-invariant sets of such an operator (recall Definition 1.4.7). Let $\mathcal{I}_n^T$ denote the set of T_n-invariant sets (which is non-empty as it contains $\mathbb{C}$). Our first result is the following. Here, the numbers λ_i correspond to those given in Theorem 1.2.5.

Theorem 2.5.1 (Theorem 3.1 of Paper V)**.** *For an exactly solvable operator T and a non-negative integer n such that*

(i) *among the numbers $\Lambda_n^T = \{\lambda_0^T, \lambda_1^T, \ldots, \lambda_n^T\}$ there exists a unique λ_ℓ^T, $\ell \geq 1$ with the largest absolute value, and*

(ii) *there are no 1-point T_n-invariant sets, i.e., T_n-invariant sets being a single point,*

then there exists unique minimal under inclusion nonempty closed set $\mathrm{M}_n^T \in \mathcal{I}_n^T$.

Sketch of proof. Let S be a T_n-invariant set. After an appropriate linear change of variables, we assume that $\{0, 1\} \subset S$. Now, the polynomials $q_j(z)$, for $j = 0, 1, \ldots, n$, defined by

$$q_j(z) = z^{n-j}(z-1)^j,$$

forms a basis of the space of polynomials of degree at most n. One can then express one of the polynomials q_m as a linear combination of the monic eigenpolynomials provided by Theorem 1.2.5 such that the coefficient of p_ℓ is non-zero. Then, by expanding the expression

$$\frac{T^n(q_m(z))}{\lambda_\ell^n}$$

one sees that a subset of its zeros converges to the zeros of $p_\ell(z)$ as $n \to \infty$. Hence, the zeros of $p_\ell(z)$ belong to any T_n-invariant set. Using that the intersection of two T_n-invariant sets is again T_n-invariant, the statement follows. □

Using similar techniques, we establish the following proposition.

Proposition 2.5.2 (Proposition 3.5 of Paper V). *Let T be an exactly solvable operator with $\lambda_n \neq 0$ and S be a T_n-invariant such that $|S| \geq 2$. Then,*

(i) *if there exists $\ell < n$ such that $|\lambda_\ell| > |\lambda_j|$, $j = 0, 1, \ldots, n$, $j \neq \ell$, then S is unbounded, hence infinite; or*

(ii) *if $\lambda_\ell \neq 0$ for some $\ell < n$ and there exists no integer $m > 0$ such that $\lambda_\ell^m = \lambda_n^m$, then S is infinite.*

We further show the following analog of analog of Theorem 1.4.3. Recall that $Conv(Q_k)$ is the convex hull of the set of zeros of $Q_k(z)$. In this statement, the Hausdorff limits are assumed to be closed.

Theorem 2.5.3 (Theorem 3.7 of Paper V). *For any non-degenerate T where at least one zero of Q_k is simple, there exists an integer $n_0 > 0$ such that for each $n \geq n_0$, M_n^T exists and the sequence $\{\mathrm{M}_n^T\}_{n=n_0}^\infty$ converges in the Hausdorff metric to $Conv(Q_k)$ as $n \to \infty$.*

Sketch of proof. First, by Theorem 2.2.3 and its following remark, $\mathrm{M}_{H,n}^T$ exists, and since $\mathrm{M}_{H,n}^T \subset \mathrm{M}_n^T$ by definition, it follows that M_n^T exists. Furthermore, by following the proof of Theorems 2.2.3 and 2.2.1, one sees that for large n, there exist points near the zeros of $Q_k(z)$ that belong to $\mathrm{M}_{H,n}^T \subset \mathrm{M}_n^T$. Similar arguments to those found in the proof [ABS, Theorem 2.2] may then be used to show that for any $z \in Conv(Q_k)$ and $\epsilon > 0$ there exists $N \geq 0$ such that the ϵ-ball $B(z, \epsilon)$ intersects M_n^T for all $n \geq N$. Moreover, it follows by Theorem 1.4.3, that M_n^T is contained in the ϵ-neighborhood of $Conv(Q_k)$ for all sufficiently large n. Combining the above statements, it follows that $\{\mathrm{M}_n^T\}_{n=n_0}^\infty$ exists and equals $Conv(Q_k)$. □

In the paper, we further show strong relations between T_n-invariant sets and Julia sets of associated rational functions.

Denote $p_{k,n}(x) = (x-1/2)^{n-k}(x+1/2)^k$ for $0 \leq k \leq n$. We conclude this summary with the following proposition and a related conjecture.

Proposition 2.5.4 (Proposition 4.10 of Paper V). *Let $T : \mathbb{C}_n[x] \to \mathbb{C}_n[x]$ be an invertible linear operator that sends all degree n polynomial to degree n polynomials. Suppose further that $S \subset \mathbb{C}$ is a finite T_n-invariant set of cardinality* 2*. After a linear change of variables, we have that either T is the identity (up to multiplication by a non-zero constant) or $T(p_k) = p_{n-k,n}$ for each $0 \leq k \leq n$. Moreover, T is induced by the change of variaibles $z \to -z$ in the sense that after multiplying T by $(-1)^n$, we have*

$$T\left[\prod_{i=1}^{m}(z-a_i)\right] = \prod_{i=1}^{m}(-z-a_i)$$

provided that $0 \leq m \leq n$. In particular, for any $N \in \mathbb{N}$ there are finite T_n-invariant sets of cardinality N.

Conjecture 2.5.5 (Conjecture 8.1 of Paper V). *Let $T : \mathbb{C}_n[x] \to \mathbb{C}_n[x]$ be an invertible linear operator and $S \in \mathcal{I}_n^T$, such that S contains at least* 2 *points and has finite cardinality. Then for all $f \in \mathbb{C}_n[x]$, we have*

$$T[f] = (cx+d)^n f\left(\frac{ax+b}{cx+d}\right),$$

where $\phi(x) := (ax+b)/(cx+d)$ is a Möbius map such that $\phi^k(x) = x$ for some $k \geq 1$.

In the special case that T sends all degree n polynomials to degree n polynomials, then T is induced by a rotation by a rational angle. That is, after a linear change of variables, we have

$$T\left[\prod_{i=1}^{n}(z-a_i)\right] = \prod_{i=1}^{n}(\alpha z-a_i)$$

for some root of unity α.

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