Stockholm University

Stochastic optimal control and stopping, games and time inconsistency

Andi Bodnariu

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Abstract

This thesis contributes to the field of stochastic optimisation considering a version of a dividend problem, a stochastic differential game with incomplete information, as well as game-theoretic approaches to time-inconsistent stopping and control. The content of this thesis is based upon four papers.

Paper I considers a game theoretic approach to a time-inconsistent stopping problem, where the time-inconsistency is due to non-exponential discounting. We introduce a novel class of mixed stopping strategies and provide a verification theorem. Furthermore, we consider an example, where there is no equilibrium when using only pure stopping times. In this case we are able to construct an equilibrium in the class of mixed stopping times.

Paper II considers a continuous time stochastic controller stopper game with incomplete information. The stopper can be seen as owner of an asset and a controller as the manager who is either effective or non-effective. The manager earns a salary paid by the owner. An effective manager can choose to exert effort at a cost in order to increase the drift of the asset while a non-effective manager cannot act. The owner can choose to stop the game at any time based on observations of the movement of the asset. The owner is not able to observe directly whether the manager is effective or non-effective, making this a game of incomplete information. We derive a Nash equilibrium to this game, given as a threshold solution depending on the conditional probability that the manager is effective.

Paper III considers a time-inconsistent singular stochastic control problem, where the time-inconsistency is due to non-exponential discounting. We introduce a new class of "mild" threshold controls, which are given by an exploding rate that generates an inaccessible boundary for the underlying diffusion. These "mild" controls stand in contrast to the "strong" threshold controls that have been considered previously and amount to a Skorokhod reflection at an upper boundary. We provide an appropriate equilibrium condition for these controls as well as a verification theorem. Furthermore we provide an example, where no equilibrium exist if we only consider "strong" threshold strategies. We are, however able to find an equilibrium when considering "mild" threshold control strategies.

Paper IV considers a dividend problem with ruin at zero surplus or if the surplus spends too long time below a certain threshold of distress. We completely solve the problem considering three different cases. If the distress threshold is small or large the optimal control results in paying out dividends above a certain threshold. If the distress threshold takes intermediate values, the optimal control results in paying out dividends in two separated regions.

Collectively, these results advance the theory for optimal stochastic control and stopping, by enriching the literature with new problems, as well as presenting solution structures that have not been considered previously.

Keywords: *Stochastic optimal control, Stochastic optimal stopping, Stochastic differential games, Time-inconsistency.*

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This thesis is based on four papers, numbered I–IV, which are included in the following order.

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II. Bodnariu, Andi and Lindensjö, Kristoffer (2024). A controller-stopper-game with hidden controller type. *Stochastic Processes and their Applications* 173, p. 104361

III. Bodnariu, Andi, Lindensjö, Kristoffer and Rodosthenous, Neofytos (2025). Time-inconsistent singular control problems: Reflection and Absolutely continuous controls with exploding rates. arXiv: `2507.04836`. Submitted.

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Authors' contributions:

In Paper I, based on an idea of K. Lindensjö and S. Christensen, A. Bodnariu was involved equally in the devolopment of the theoretical results as well as writing the paper together with the other co-authors.

In Paper II, A. Bodnariu was involved equally in discussions that lead to the formalization of the problem. Furthermore A. Bodnariu was involved equally in the derivation of the theoretical results as well as writing the paper with K. Lindensjö.

In Paper III, A. Bodnariu was involved equally in discussions that lead to the formalization of the problem together with the co-authors. Furthermore A. Bodnariu was involved equally in the derivation of the theoretical results as well as writing the paper with K. Lindensjö and N. Rodosthenous.

In Paper IV, based on an initial research idea by N. Rodosthenous, A. Bodnariu was involved equally in deriving the theoretical results as well as writing the paper with the other co-authors.

General comments:

Earlier versions of Paper I and Paper II were included in the licentiate thesis of A. Bodnariu (Bodnariu (2024)). A. Bodnariu has asked for permission to reprint the published papers for this thesis, in case this was required. Parts of the introduction of this thesis has been taken from the licentiate thesis with some additional sections to cover the new papers.

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Part I

Introduction

Chapter 1

Motivation

Stochastic optimisation, which is optimisation and decision making under uncertainty is an important problem in finance and other applications. This is due to the fact that decisions have to be made even though most of the time the exact future development is unknown. Typically under uncertainty, any decision can lead to an unfavourable outcome, given an unexpected negative event or disaster. Thus the decision maker is usually satisfied with picking a decision that is optimal in expectation leading to an optimal outcome over a large sample size.

A prime example of stochastic optimisation is the optimal stopping problem. In this problem, the decision maker observes a stochastic process and wishes to choose a time at which the process is stopped, in order to maximize (or minimize) an expected reward (or cost). An application of this is the pricing of a financial contract such as a call option, where the fair price of the option equals the (potentially discounted) expected reward of the option if it is stopped at the optimal time.

Another example is the problem of optimal stochastic control. In this case the decision maker has a direct influence on the process. This can be done by either influencing the dynamics or directly controlling the value of the process. Again the goal is to find an optimal control in order to maximize (or minimize) an expected utility (or cost). An example of this would be an optimal investment problem, in which an investor has the option to invest funds in a risk free and a risky asset. At each point in time the investor can control the stochastic process (funds) by deciding how much of the funds should be allocated in each of the assets. In this thesis, we will consider a dividend problem, which is a stochastic control problem where a company seeks to determine an optimal strategy for paying dividends to its shareholders while balancing the risk of ruin when the firms surplus reaches zero. The main novelty of the version studied in this thesis is that we consider an additional ruin possibility, corresponding to a random time, whose intensity is active when the surplus of the firm is below a critical threshold. This extra form of ruin is considered in order to provide a more realistic model where extended liquidity shortages can trigger operational breakdowns or regulatory action.

In the real world there are typically more than one person acting and affecting a stochastic process. Thus, when facing multiple decision makers or players, there typically no longer exists a strategy that optimises the expected reward for all players. Hence, one typically searches for strategies that form a Nash equilibrium, which means that if all the other players keep the equilibrium strategy, each decision maker is not able to improve his expected reward by deviating from equilibrium. In this thesis, we will consider a specific controller–stopper game that can be interpreted as a game between an owner and a manager of an asset. The owner may decide to stop and terminate the contract if the manager is believed to be non-effective. The manager, in turn, may choose to exert effort to demonstrate effectiveness and thereby retain the position and continue earning a salary.

Additionally, a decision maker might consider a utility function that makes his optimisation problem time-inconsistent. A prime example of this is the so called mean variance optimisation in which an investor wishes to maximize his expected return, while also penalizing the variance or uncertainty. For this problem time-inconsistency arises since the variance is a non-linear function of an expectation. In a time inconsistent problem a decision rule that is optimal today might no longer be optimal when reassessing the optimisation problem at a later time when the value of the asset has changed. In this case game-theoretic approaches can be used to provide a new notion of optimality. This thesis will consider two works in the field of time-inconsistency: one control problem and one stopping problem. In both cases, time-inconsistency arises due to non-exponential discounting. To find an equilibrium within the game-theoretic approach, we will introduce a new class of strategies that does not appear in the time-consistent counterpart but proves to be necessary for establishing a game-theoretic equilibrium

In conclusion there exists a plethora of different stochastic optimisation problems, with some of them mentioned and solved in this thesis. Applications of stochastic optimisation often appear in real-world applications, particularly in finance. Problems of optimal stopping, optimal control as well as further developments including time inconsistency and stochastic games, can be used to tackle these problems, by modelling the scenarios mathematically and then finding optimal solutions.

Chapter 2

Stochastic calculus

This chapter provides a summary of stochastic calculus that will be used to model the underlying stochastic processes as well as solutions appearing in the stochastic optimisation problems presented in this thesis. It follows, unless stated otherwise, Karatzas and Shreve (1991) and Øksendal (2003).

2.1 Preliminaries

In this thesis, a continuous time stochastic process $X = \{X_t : 0 \leq t < \infty\}$ is a collection of $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ valued random variables on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We equip this probability space with a filtration $\{\mathcal{F}_t\} = \{\mathcal{F}_t : 0 \leq t < \infty\}$, which is a non-decreasing collection of sub-σ-algebras of $\mathcal{F}$ i.e. we have $\mathcal{F}_s \subseteq \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s < t < \infty$. Furthermore, we set $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$. A natural choice of filtration is the one generated by the process X itself. It is denoted by $\mathcal{F}_t^X = \sigma(\{X_s : 0 \leq s \leq t\})$. We will refer to the the process X being adapted (to the filtration $\{\mathcal{F}_t\}$), if we have that X_t is $\mathcal{F}_t$ measurable.

For the purpose of this thesis, we will assume the stochastic process X to be adapted and have RCLL paths i.e right continuous paths with left hand limits. Particularly, this implies that the process is progressively-measurable, i.e., that the path mapping $(t, \omega) \to X_t(\omega) : ([0, T] \times \Omega, \mathcal{B}[0, T] \otimes \mathcal{F}_T) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable for each $T \geq 0$. This ensures that functionals (such as stochastic integrals) of a path of the stochastic process i.e. functionals of the form $f(X_{\leq t})$, are $\{\mathcal{F}_t\}$ adapted.

The filtration $\{\mathcal{F}_t\}$ is used to model the flow of information about the process as time progresses. This allows us to mathematically model questions about the future behavior of the processes, given our current (present) information. A prime example of this is the conditional probability $\mathbb{P}(X_t \in \cdot | \mathcal{F}_s)$ for $0 \leq s < t < \infty$. This leads us to the definition of a Markov process (see Definition 1 below). Intuitively, a Markov process is memoryless such that future behavior only depends the information available at the present time.

Definition 1. *We say that an adapted stochastic process X is a Markov process if it satisfies the Markov property:*

$$\mathbb{P}(X_t \in B|\mathcal{F}_s) = \mathbb{P}(X_t \in B|X_s) \quad a.s. \tag{2.1}$$

for all $B \in \mathcal{B}(\mathbb{R})$ and $0 \leq s < t < \infty$.

We continue introducing the concept of a martingale. Intuitively, a martingale can be interpreted as a fair game i.e., a game, where the player's expected profit/loss is zero.

Definition 2. *A stochastic process X with $E[|X_t|] < \infty$, $\forall t \geq 0$, is said to be submartingale if we have $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ a.s. for $0 \leq s < t < \infty$. We refer to a process as supermartingale if the opposite inequality (i.e., $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$) holds. We refer to the process X being a martingale if it is both a submartingale and a supermartingale (i.e., we have $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$).*

Throughout this thesis we will make use of stopping times (see Definition 3 below), which are random times at which the process X satisfies a certain property. Particularly, we will make frequent use of first hitting times

$$\tau_B = \inf\{t \geq 0 : X_t \in B\} \tag{2.2}$$

for some Borel set $B \subset \mathbb{R}$.

Definition 3. *Let $(\Omega, \mathcal{F})$ be a measurable space with filtration $\{\mathcal{F}_t\}$. We say that τ is a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. A random time τ is an optional time if we have that $\{\tau < t\} \in \mathcal{F}_t$ for all $t \geq 0$.*

Let us motivate the above definition heuristically. Stopping/optional times can be interpreted as times when a (random) event happens. The key difference lies with the information that is available at time t represented by the filtration $\mathcal{F}_t$. If τ is a stopping time we know at time t whether the event has occurred or not i.e., we have the information about $\{\tau \leq t\}$. For an optional time we only have the information about $\{\tau < t\}$ meaning that we do not know if event has occurred exactly at time t. This could for example appear if a stochastic process jumps at time t but the new position is still unknown given the information up to t encoded by $\mathcal{F}_t$.

This thesis considers stochastic processes, which satisfy the so called strong Markov property (also referred to as strong Markov processes). A process satisfies the strong Markov property if it satisfies Markov property (2.1) for optional times i.e., for any optional time σ, $t \geq 0$ and $B \in \mathcal{B}(R)$ we have

$$\mathbb{P}(X_{\sigma+t} \in B|\mathcal{F}_{\sigma+}) = \mathbb{P}(X_{\sigma+t} \in B|X_\sigma) \quad \text{a.s. on } \{\sigma < \infty\} \tag{2.3}$$

where $\mathcal{F}_{\sigma+} := \bigcap_{\varepsilon>0} \mathcal{F}_{\sigma+\varepsilon}$ is the σ-algebra of events immediately after σ. Similarly we can define $\mathcal{F}_{t-} := \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right)$, for $t > 0$ (and $\mathcal{F}_{0-} := \mathcal{F}_0$) as the events prior to t. We say that a filtration is right (resp. left) continuos if it satisfies $\mathcal{F}_t = \mathcal{F}_{t+}$ (resp. $\mathcal{F}_t = \mathcal{F}_{t-}$). A filtration is continuos if it is both right and left continuos.

Throughout this thesis we will make the standing assumption that the underlying filtration satisfies the so called usual conditions, which means that it is right continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$ null sets of $\mathcal{F}$ (i.e., for all $N \in \mathcal{F}$ with $\mathbb{P}(N) = 0$ we have $N \in \mathcal{F}_0$). Under this assumption the concept of optional time and stopping time coincide, since by knowing $\mathcal{F}_t$, we already have the information about all the events happening immediately after t given by $\mathcal{F}_{t+}$.

We remark that we can construct a continuos filtration satisfying the usual conditions by augmenting the filtration generated by a left continuos strong Markov process with the collection of null sets. In particular, this can be done for the Brownian motion which is a continuos strong Markov process introduced in the next chapter. We refer to Karatzas and Shreve (1991, Chapter 2.7.) for details.

2.2 The Brownian motion

A standard example of a strong Markov process in continuous time is the Brownian motion W_t on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. It can be seen as a continuous time analog of the symmetric random walk and is characterized by four main properties:

(i) $W_0 = 0$ a.s.

(ii) W is an a.s. continuous process adapted to the filtration $\{\mathcal{F}_t\}$.

(iii) For $0 \leq s < t < \infty$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$.

(iv) For $0 \leq s < t < \infty$, we have that $W_t - W_s \sim \mathcal{N}(0, t-s)$.

(Here $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2). It can be shown that a process satisfying the above properties (i)-(iv) exists, we refer to Karatzas and Shreve (1991) for the proof. The Brownian motion has many useful properties, particularly it is a square-integrable (i.e., $\mathbb{E}[W_t^2] < \infty$, $\forall t \geq 0$) martingale.

2.3 The Itô integral and Itô's formula

Generally, paths of stochastic processes in continuous time (including the Brownian motion) are rough in a sense that they do not admit a derivative in the classical sense. Hence, stochastic calculus is typically formalized in a integral sense. There are two main alternatives for defining a stochastic integral: the Itô integral and the Stratonovich integral. In this thesis we will only consider the Itô integral, for more details regarding the Stratonovich integral see Karatzas and Shreve (1991, Chapter 3.2).

2.3.1 The Itô integral

Roughly speaking, given an adapted stochastic processes X the Itô integral (w.r.t a Brownian motion W) is defined as the limit of a Lebesgue-Stieltjes sum i.e.

$$\int_0^t X_s dW_s = \lim_{n\to\infty} \sum_{i=1}^{n} X_{t_{i-1}^{(n)}} (W_{t_i^{(n)}} - W_{t_{i-1}^{(n)}}), \tag{2.4}$$

for a sequence of partitions $(t_i^{(n)})_{0\leq i\leq n}$ of $[0,t]$ with mesh size going to zero. A main difficulty when defining the Itô integral as a limit of a Lebesgue-Stieltjes type sum is however that the paths of the Brownian have a.s. unbounded variation. Therefore, the Itô integral is typically not introduced path-wise (meaning for each ω separately) but rather as a certain $L^2(\mathbb{P})$ limit in the Hilbert space of square-integrable random variables. Particularly, given the square integrability assumption

$$\mathbb{E}\left[\int_0^T X_t^2 dt\right] < \infty, \tag{2.5}$$

it is possible to show that the left hand side of (2.4) has a unique $L^2(\mathbb{P})$ limit for each fixed $T \geq 0$. By varying T, this limit can be used to define a unique a.s. continuous stochastic process $I_t(X)$, which defines our stochastic integral and we write

$$I_t(X) = \int_0^t X_s dW_s.$$

The Itô integral satisfies the following basic properties for all $t \geq 0$:

(i) $\int_0^t X_s dW_s$ is an adapted a.s. continuous process

(ii) $\int_0^t X_s dW_s$ is a martingale with $\mathbb{E}\left[\int_0^t X_s dW_s\right] = 0$ for all $t \geq 0$.

(iii) $\mathbb{E}\left[\left(\int_0^t X_s dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t X_s^2 ds\right]$.

Here (iii) is referred to as the Itô isometry. It is possible to relax the assumption made in (2.5) and still prove existence of the Itô integral. We refer to Karatzas and Shreve (1991) for details.

2.3.2 Itô's formula

A main tool used in computations throughout this thesis is Itô's formula (see Theorem 4 below). It establishes an identity between a sufficiently "smooth" function of an Itô process and its derivatives. An Itô process X is a process of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \tag{2.6}$$

where μ_s, σ_s are progressively measurable stochastic processes such that corresponding integrals are well defined (see Øksendal (2003, Definition 4.1.1) for details). In this thesis, we will often write equation (2.6) in the commonly used differential notation:

$$dX_t = \mu_t dt + \sigma_t dW_t. \tag{2.7}$$

Theorem 4. *Let $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^{1,2}([0,\infty), \mathbb{R})$ function. Let X be a Itô process given by* (2.6) *and $X_0 \in \mathbb{R}$. Then we have $\mathbb{P}$ a.s. that*

$$\begin{aligned} &f(t, X_t) - f(0, X_0) \\ &= \int_0^t \left(\frac{\partial f}{\partial t}(s, X_s) + \mu_s \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right) ds \\ &+ \int_0^t \sigma_s \frac{\partial f}{\partial x}(s, X_s) dW_s, \end{aligned} \tag{2.8}$$

or written in differential notation

$$\begin{aligned} &df(t, X_t) \\ &= \left(\frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right) dt + \sigma_t \frac{\partial f}{\partial x}(t, X_t) dW_t. \end{aligned} \tag{2.9}$$

Note that there are multiple extensions to the formula (2.8). Notable extensions include multi-dimensional processes in Øksendal (2003, Chapter 4.2), or less "smooth" functions f in Peskir and Shiryaev (2006, Chapter 3.3).

2.4 Stochastic differential equations

A (one-dimensional, time-independent) stochastic differential equation is given by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \tag{2.10}$$

where $x \in \mathbb{R}$, W is the Brownian motion and μ, σ are real valued functions. The goal of this chapter is to examine existence of a stochastic process X that solves Equation (2.10) in a suitable sense. Two main solution concepts arise when dealing with SDE's: X can either be a strong or a weak solution.

A strong solution (see Definition 5 below) satisfies the principle of causality of a dynamical system. This means that the solution X_t at time t of Equation (2.10) does only depend on the parameters chosen as input for the system: The initial coniditon $X_0 = x$ and the path of the specific Brownian motion up to time t (i.e. $\{W_s : 0 \leq s \leq t\}$. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then in order to formally define strong solutions to (2.10), we need to specify the filtration which the solution will be adapted to. We consider the so called augmented filtration generated by the Brownian motion

$$\mathcal{F}_t = \sigma\left(\mathcal{F}_t^W \cup \mathcal{N}\right), \tag{2.11}$$

where $\mathcal{N}$ is the collection of null sets i.e., it is given by

$$\mathcal{N} := \{N \subset \Omega : \exists F \in \mathcal{F}_\infty^W \text{ with } N \subset F \text{ and } \mathbb{P}(F) = 0\}. \tag{2.12}$$

In particular, we get with help of this augmentation that $\{\mathcal{F}_t\}$ is continuos, which was a necessary assumption for the theory presented above, and that W is remains a Brownian motion under this filtration (cf. Karatzas and Shreve (1991, Section 2.7. A)).

Definition 5. *Let W be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\mathcal{F}_t$ is given by* (2.11). *We say that a stochastic process X is a strong solution to* (2.10) *if it has continuous sample paths and satisfies:*

(i) X is adapted to $\{\mathcal{F}_t\}$,

(ii) $\int_0^t |\mu(X_s) + \sigma^2(X_s)| ds < \infty$ a.s. for every $0 \leq t < \infty$,

(iii) X satisfies (2.10).

Given that the coefficients μ, σ satisfy the Lipschitz condition:

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|, \tag{2.13}$$

for $x, y \in \mathbb{R}$ and $K > 0$, it can be shown that (2.10) has a pathwise unique strong solution. Particularly, this solution can also be constructed in a pathwise manner, see Karandikar (1995) for more details.

A weak solution (see Definition below) is a pair (X, W) on a filtered probability space such that W is a Brownian motion and (X, W) solves (2.10). Hence, when considering weak solutions the Brownian motion is part of the solution, which stands in contrast to the strong solution where the Brownian motion was given.

Definition 6. *A weak solution to* (2.10) *is a triplet $((X, W), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\})$, such that*

(i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a filtered probability space.

(ii) X is a continuous adapted real valued stochastic process and W is a Brownian motion (w.r.t. the filtration $\{\mathcal{F}_t\}$).

(iii) The equation (2.10) *is satisfied by* (X, W) *and (ii) in Definition 5 holds.*

It is easily seen that strong solvability implies weak solvability. The converse however does not hold, for example it can be shown that the SDE

$$dX_t = \operatorname{sgn}(X_t)dW_t, \tag{2.14}$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x \leq 0, \end{cases} \tag{2.15}$$

has a weak but not a strong solution. The proof of this is rather technical and can be found in Karatzas and Shreve (1991, Chapter 5.3.). Roughly speaking the proof relies on the fact that if there was a strong solution to the above equation this would imply by definition that we could construct the process X for fixed Brownian motion W. Furthermore it is possible to show that X is again a Brownian motion. However it can also be shown that it is possible to construct W given the process $|X|$ (i.e., the absolute value of X). This would imply that it is possible to construct the process X given $|X|$ which is a contradiction. For more details on weak solutions, particularly regarding questions of existence and uniqueness, we refer to Karatzas and Shreve (1991, Chapter 5).

2.5 Local time

This section introduces the local time of a stochastic process X given by a SDE (2.10). In order to do so we first construct the local time of the Brownian motion. Roughly speaking, the local time l_t^x of a stochastic process quantifies the time spent in the "vicinity" of a specific point $x \in \mathbb{R}$. In order to motivate the local time, let us first consider the level set $\mathcal{L}(x) = \{0 \leq t < \infty : W_t = x\}$. Then it is possible to show that the level set has a.s. measure zero, i.e. we have

$$\lambda(\mathcal{L}(x)) = 0 \quad \text{a.s.}, \tag{2.16}$$

where λ denotes the one-dimensional Lebesgue measure. Thus, the level set does not provide any information of the time the Brownian motion spent in a point. Therefore we instead consider "infinitesimal vicinity" of a point and define the two parameter collection of random variables as the a.s. limit

$$l_t^x := \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t I_{\{|W_s - x| \leq \varepsilon\}} ds. \tag{2.17}$$

It can then be shown that:

(i) The limit on the right hand side of (2.17) a.s. exists and is finite, but not identical to zero.

(ii) The collection of the random variables l_t^x can be chosen to be jointly continuous in t, x.

We then refer to the solution l_t^x satisfying (i)-(ii) as the local time of the Brownian motion x. Another way to define the local time, is via the Tanaka formula. It characterizes the local time as the unique stochastic process satisfying

$$l_t^{x_i} := |W_t - x| - |W_0 - x| - \int_0^t \operatorname{sgn}(W_s - x) dW_s, \tag{2.18}$$

where the sgn is given by (2.15). Particularly, it can be shown that a process given by (2.18) coincides with the collection of random variables l_t^x introduced above. The local time has following basic properties:

(i) For a fixed x the local time l_t^x is a continuous increasing stochastic process.

(ii) For a fixed x, the path of l_t^x is a.s. flat outside of the level set $\mathcal{L}(x)$ i.e. we have

$$\int_0^\infty I_{\{W_t \neq x\}} dl_t^x = 0. \tag{2.19}$$

(iii) For a fixed x, the path of l_t^x strictly increases (singularly) on elements of the level set $\mathcal{L}(x)$.

It is possible to use an analogues construction to define the local time for a wider range of stochastic processes X_t (particularly to those processes given by solutions to SDE's). However, since in this case the distribution of the process is no longer symmetric, multiple versions of local time appear, referred to as the symmetric or one-sided local time. In this thesis, we will only consider the symmetric local time. Following the derivations in Revuz and Yor (2013, Chapter VI) the symmetric local time $l_t^x(X)$ of the process X given by the SDE (2.10) can be defined by

$$l_t^x(X) := |X_t - x| - |X_0 - x| - \int_0^t \tilde{\operatorname{sgn}}(X_s - x) dX_s, \tag{2.20}$$

where $\tilde{\operatorname{sgn}}$ is the symmetric sign function defined as

$$\tilde{\operatorname{sgn}}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \tag{2.21}$$

It can then be shown that $l_t^x(X)$ can be written as the a.s. limit

$$l_t^x(X) = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t \sigma^2(X_s) I_{\{|X_s - x| \leq \varepsilon\}} ds. \tag{2.22}$$

For more detail regarding the local time we refer to Karatzas and Shreve (1991, Chapter 3.6) and Revuz and Yor (2013, Chapter VI).

2.6 Reflected stochastic differential equations

This section introduces stochastic differential equations with reflection following the derivations in Pilipenko (2014). In particular we wish to construct a stochastic differential equation, such that resulting diffusion takes values in the positive half lime $[0, \infty)$, with reflection at 0. Furthermore the dynamics of the SDE should be unaffected by the reflection (i.e. given by (2.10)), when inside the interval $(0, \infty)$.

In order to construct such a process, it is helpful to consider the deterministic problem first (i.e. reflection of a continuos function $f(\cdot)$ at 0). This is the so called (deterministic) Skorokhod problem, defined as follows:

Definition 7. *Let $f \in \mathcal{C}([0,T])$ for some $T > 0$, with $f(0) \geq 0$. A pair (g, l) of continuos functions is a solution to the Skorokhod problem for the function f if*

(i) $g(t) \geq 0, t \in [0,T]$,

(ii) $g(t) = f(t) + l(t), t \in [0,T]$,

(iii) $l(0) = 0$, *l is non decreasing,*

(iv) $\int_0^T I_{g(s)>0} dl(s) = 0$.

The interpretation of the above definition is as follows: Given a function f, there exists a function g that follows the same incremental dynamics as f on $(0, \infty)$. This is due to the fact that while $g(t) > 0$, $l(t)$ remains constant (cf. (iv)). However, when $g(t) = 0$ the incremental dynamics of f that would result in g being negative are compensated by a function l. It can be shown that the solution to this problem exists uniquely and can be characterized by

$$l(t) = - \min_{s \in [0,t]} (f(s) \wedge 0), \tag{2.23}$$

$$g(t) = \Gamma(f)(t) := f(t) - \min_{s \in [0,t]} (f(s) \wedge 0) = f(t) + l(t), \tag{2.24}$$

where Γ is the so called Skorokhod map. Let us remark that the above problem has a solution for an arbitrary continuos function. Since sample paths of solutions to SDEs as in (2.10) are almost surely continuos, it is thus natural to expand the above problem for stochastic processes resulting in the definition of a stochastic differential equation with reflection.

Definition 8. *A solution to the stochastic differential equation*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + L_t, \quad t \geq 0, \quad X_0 = x \geq 0, \tag{2.25}$$

reflected at 0 is a pair (X, L) of continuos $\{\mathcal{F}_t\}$ adapted processes satisfying

(i) $X_t \geq 0$, $t \geq 0$,

(ii) L_t *non decreasing with* $L_0 = 0$*,*

(iii) $\int_0^t I_{\{X_s>0\}} dL_s = 0,\ t \geq 0$*,*

(iv) $X_t = x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s + L_t, \quad t \geq 0,\ a.s,$

such that all these integrals are well defined (cf. Section 2.3.1 and the references therein).

It can be shown that given the standard Lipschitz continuity assumptions on μ, σ the above reflected SDE has a unique solution. In particular it can be constructed using the process Y given as the unique solution to the SDE

$$Y_t = x + \int_0^t \mu(\Gamma Y(s))ds + \int_0^t \sigma(\Gamma Y(s))dW_s \tag{2.26}$$

where Γ is the Skorokhod map defined in (2.24). Then a solution to the reflected SDE (2.25) satisfying the conditions of Definition 8 is given by

$$X_t = \Gamma Y(t), \quad L_t = X_t - Y_t. \tag{2.27}$$

Furthermore it can be shown that the process L_t coincides with the (symmetric) local time of X at 0 i.e. we have

$$L_t = l_t^0(X),$$

where $l_t^0(X)$ is defined in (2.20).

We remark that it is possible to expand the above theory for multiple dimensions, we refer to Pilipenko (2014) for details.

2.7 Stochastic filtering

This section illustrates the fundamentals of non-linear stochastic filtering using a simple example. The example considered in this section is a simple case of the general theory presented in Liptser and Shiryaev (2013).

Consider the following setting: Let θ be a Bernoulli random variable with $p = \mathbb{P}(\theta = 1)$ and X be a stochastic process given by

$$dX_t = \theta\mu t + W_t, \tag{2.28}$$

where W_t is a Brownian motion and $\mu \neq 0$ is a constant. We are now interested in computing the conditional probability $\mathbb{P}(\theta = 1|\mathcal{F}_t^X) = \mathbb{E}\left[\theta|\mathcal{F}_t^X\right]$ based on the information $\mathcal{F}_t^X$ given by the stochastic process X up to time t and our initial guess $p = \mathbb{P}(\theta = 1) = \mathbb{P}(\theta = 1|\mathcal{F}_0^X)$. Naturally, we can see that our estimate $P_t := \mathbb{E}_x\left[\theta|\mathcal{F}_t^X\right]$ evolves randomly over time as we observe X and it thus defines a stochastic process. We are interested in characterizing the process

P as the solution to a stochastic differential equation. Following Theorem 8.1 in Liptser and Shiryaev (2013) we have that

$$dP_t = \mu P_t(1 - P_t)d\hat{W}_t, \quad P_0 = p, \tag{2.29}$$

where the so called innovations process $\hat{W}_t$ is a Brownian motion w.r.t. $\{\mathcal{F}_t^X\}$, given by

$$\hat{W}_t = X_t - \mu \int_0^t P_s ds. \tag{2.30}$$

Combining (2.29) and (2.30) we can express the dynamics of P directly with respect to X. We get

$$dP_t = -\mu^2 P_t^2(1 - P_t)dt + \mu P_t(1 - P_t)dX_t. \tag{2.31}$$

We remark that (2.31) is a stochastic differential equation not with respect to a Brownian motion, but with respect to X which is a so called semi martingale. It is however possible to expand the theory presented in Section 2.4 for this case. Particularly, by Karandikar (1995) it is also possible to compute a solution to (2.31) in a pathwise manner. Thus, given a fixed observation $X(\omega)$, we can deduce the path of the process $P(\omega)$ which represents our conditional probability $\mathbb{P}(\theta = 1|\mathcal{F}_t^X)$. This can be seen as a solution to a Bayesian approach to estimate θ based on our initial guess $p = \mathbb{P}(\theta = 1)$ and the information given by the process X.

Chapter 3

Stochastic optimisation

This chapter provides a summary of classical stochastic optimisation problems as well as time-inconsistency and stochastic games. This serves as an introduction into the field, providing a starting point as well as motivation for the problems considered in the papers of this thesis.

3.1 Optimal stopping problem

In this section we will introduce an optimal stopping problem following the derivations in Peskir and Shiryaev (2006) and Øksendal (2003). We start by developing optimal stopping theory for general strong Markov processes, then in Section 3.1.1 we solve an explicit example in where the underling process is given by a stochastic differential equation. Let X be an continuos adapted strong Markov process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and let G be a continuous real valued function satisfying

$$\mathbb{E}_x \left[\sup_{0 \leq t \leq \infty} |G(X_t)| \right] < \infty, \tag{3.1}$$

where we use the convention that $G(X_t) = 0$ if $t = \infty$ (i.e. we implicitly consider $G(X_t) I_{\{t<\infty\}}$). An optimal stopping problem is then given by

$$V(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[G(X_\tau) \right], \tag{3.2}$$

where $\mathcal{T}$ denotes the set of all stopping times, and we have used the notation $\mathbb{E}_x[\cdot] := \mathbb{E}[\cdot | X_0 = x]$. We will refer to V as the value function and to G as the gain function of the optimal stopping problem. Solving an optimal stopping problem consists of two things: (i) finding an optimal stopping time τ^* such that the supremum in (3.2) is obtained (i.e. $\mathbb{E}_x \left[G(X_{\tau^*}) \right] =$

$\sup_{\tau \in \mathcal{T}} E_x [G(X_\tau)]$), (ii) computing the value function V as close as possible. Regarding (i), we introduce the continuation region

$$C = \{x \in \mathbb{R} : V(x) > G(x)\}, \tag{3.3}$$

the stopping region

$$D = \{x \in \mathbb{R} : V(x) = G(x)\} \tag{3.4}$$

and the stopping time τ_D, which is the first passage time to D (cf. (2.2)). Assuming that V is continuous, we have that D is a closed set. Furthermore, it is possible to show that then the stopping time τ_D is the minimal optimal stopping time (see Peskir and Shiryaev (2006, Theorem 2.4)). Hence, finding the optimal stopping time, typically results in computing the value function.

A key definition used in optimal stopping is the one of a superharmonic function.

Definition 9. *We say that a real valued function F is superharmonic if we have that*

$$\mathbb{E}_x [F(X_\sigma)] \leq F(x), \tag{3.5}$$

for all stopping times σ and $x \in \mathbb{R}$.

Here we assume implicitly that the left hand side of (3.5) is well defined and finite for all σ, x. Let us motivate Definition 9. Assume that F is superharmonic and of the form $F(x) = \mathbb{E}_x[G(X_\tau)]$ for some stopping time τ. Then by the strong Markov property, we have for any stopping time σ that

$$F(x) \geq \mathbb{E}_x[F(X_\sigma)] = \mathbb{E}_x [\mathbb{E}_{X_\sigma} [G(X_\tau)]] = \mathbb{E}_x[G(X_{\sigma + \tau \circ \theta_\sigma})], \tag{3.6}$$

where θ is the shift operator. Omitting a formal definition of the shift operator, we can heuristically describe the stopping time $\sigma + \tau \circ \theta_\sigma$ as follows: First wait until you reach the stopping time σ and then stop the process X starting at X_σ according to the stopping time τ. For a formal definition of the shift operator we refer to Peskir and Shiryaev (2006, Chapter 4). Thus heuristically, (3.6) implies that waiting some time before applying the "stopping rule" implied by τ gives us a suboptimal reward. This is a property we would expect from an optimal stopping time and hence serves as as a motivation for Theorem 10 below, which characterises the value function as the smallest superharmonic function that dominates the gain function.

Theorem 10. *Assume that there exists a optimal stopping time τ^* i.e. we have*

$$V(x) = \mathbb{E}[G(X_{\tau^*})]. \tag{3.7}$$

Then we have that V is the smallest superharmonic function with $V \geq G$ and assuming that V is continuous we have that τ_D satisfies $\tau_D \leq \tau^*$ a.s. and is an optimal stopping time for (3.2).

The converse also holds. Let us assume that we have found the smallest superharmonic function that dominates the gain function, then by Theorem 11 below, we have found the value function.

Theorem 11. *Assume that there exists a continous function $\hat{V}$, which is the smallest superharmonic function that dominates G. Set $\hat{D} = \{x : \hat{V}(x) = G(x)\}$ and assume that $\tau_{\hat{D}}$ is a.s. finite. Then we have that $\hat{V}$ is the value function to the optimal stopping problem* (3.2) *(i.e $\hat{V} = V$ and $\tau_{\hat{D}} = \tau_D$ is the optimal stopping time).*

Hence, solving an optimal stopping problem, typically results in computing the smallest superharmonic function that dominates the gain. Traditionally there are two ways of doing so, either (i) an iterative procedure or (ii) as a solution to a free boundary problem. Both methods have their advantages and disadvantages. An iterative approximation scheme is typically constructive but non-explicit. Solving a free boundary problem results in solving the problem directly. This result can result in an explicit or non-explicit solution. In this thesis we are mostly interested in explicit solutions.

We remark that the theory presented above can be generalized for more general type of processes. Particularly, it is possible to consider general Markov processes, or multi-dimensional processes. We refer to Peskir and Shiryaev (2006) for more details.

3.1.1 Optimal stopping for diffusions: A simple example with exponential discounting

In this chapter we will consider an example of an optimal stopping problem, where the underlying process is given by a solution to a stochastic differential equation. This example is inspired by the derivations in Øksendal (2003, Chapter 10) and the goal of this chapter will be to illustrate common proof techniques when solving such a problem. Let X be given by the SDE (2.10) for some σ, μ. Suppose we are interested in solving the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-r\tau} g(X_\tau)] \tag{3.8}$$

for $r > 0$, where $\mathcal{T}$ is the set of a.s. finite stopping times and g is some bounded increasing gain function with $g \in \mathcal{C}^2(\mathbb{R})$ and $\lim_{x \to -\infty} g(x) = 0$. We remark that problem (3.8) is slightly more general than the problem considered in (3.2) because the gain function is exponentially discounted. However, due to the simple form of time-dependency this does not cause us any additional complications. We remark that in general, time-dependency in the gain function can be removed, by formulating the problem into a time-independent problem in two dimensions (see Øksendal (2003, Chapter 10.2)).

We will solve (3.8) by transforming it into a free boundary problem. Let us assume for simplicity of this motivation, that an optimal stopping time exists, is a.s. finite and is given by

$$\tau^* = \inf\{t : X_t \geq b^*\}, \tag{3.9}$$

for some $b^* \in \mathbb{R}$. Furthermore let us assume that $V \in \mathcal{C}^2(\mathbb{R} \setminus \{b^*\}) \cap \mathcal{C}^1(\mathbb{R})$. Note that by the regularity assumption this implies that

$$V'(b^*) = g'(b^*), \tag{3.10}$$

(since $V(x) = g(x), \ \ x \geq b^*$) which is the so called smooth fit principle. Let us heuristically derive further necessary conditions for the value function using Itô's formula. We will do the derivation for $V \in \mathcal{C}^2(\mathbb{R})$. The case $\hat{V} \in \mathcal{C}^2((-\infty, b^*) \cup (b^*, \infty)) \cap \mathcal{C}^1(\mathbb{R})$ can be done analogously using a more general version of Itô's formula the so called Itô-Tanka-Meyer formula (see Peskir and Shiryaev (2006)). We obtain for

$$\tau_n = n \wedge \inf\{t \geq 0; X_t \notin [-n, n]\} \tag{3.11}$$

and any stopping time τ that

$$\begin{aligned} &e^{-r(\tau_n \wedge \tau)} V(X_{\tau_n \wedge \tau}) - V(x) = \\ &\qquad \int_0^{\tau_n \wedge \tau} e^{-rt} (\mathcal{A} - r) V(X_s) ds + \int_0^{\tau_n \wedge \tau} e^{-rt} \sigma(X_s) V''(X_s) dW_s, \end{aligned} \tag{3.12}$$

where $\mathcal{A} : \mathcal{C}^2(\mathbb{R}) \to C(\mathbb{R})$ is the characteristic operator (of X) given by

$$(\mathcal{A}f)(x) = \frac{\sigma^2(x)}{2} f''(x) + \mu(x) f'(x). \tag{3.13}$$

Taking expectations on both sides, we get that the stochastic integral vanishes since it is a martingale (cf. Chapter 1.2.1). We get

$$\mathbb{E}_x \left[e^{-r(\tau_n \wedge \tau)} V(X_{\tau_n \wedge \tau}) - V(x) \right] = \mathbb{E}_x \left[\int_0^{\tau_n \wedge \tau} e^{-rt} (\mathcal{A} - r) V(X_s) ds \right]. \tag{3.14}$$

Since we have that value function is superharmonic (here we slightly alter Definition 9 to allow for time dependency), we see that the left hand side of (3.14) is negative. Thus by taking $n \to 0$ we (heuristically) get the necessary condition

$$(\mathcal{A} - r) V(x) \leq 0, \quad \forall x \in \mathbb{R}. \tag{3.15}$$

Additionally, we can see that if $x < b^*$ (i.e. $\tau^* > 0$) then we have by the strong Markov property that for any stopping time τ with $\tau < \tau^*$ a.s. we obtain

$$V(x) = \mathbb{E}\left[e^{-r\tau} V(X_\tau) \right], \tag{3.16}$$

which (heuristically) implies that,

$$(\mathcal{A} - r) V(x) ds = 0, \quad \forall x < b^*. \tag{3.17}$$

The above derivations give us necessary conditions for the value function (assuming regularity). Thus, in order to find the value function, we can apply a guess and verify approach. First we find a pair $(\hat{V}, b^*)$ satisfying (3.10), (3.15), (3.17) as well as $\hat{V} \geq g$, and then later we verify that $\hat{V}$ is equals the value function i.e. $(\hat{V} = V)$. Theorem 12 is an example of such a verification theorem, illustrating classical proof techniques.

Theorem 12. *Let $\hat{V} \in \mathcal{C}^2(-\infty, b^*) \cap \mathcal{C}^1(\mathbb{R})$ be a solution to the free boundary value problem*

$$\begin{cases} (\mathcal{A} - r)\hat{V}(x) & = 0, \quad x \in (-\infty, b^*), \\ \hat{V}(x) & = g(x), \quad x \in [b^*, \infty), \\ \lim_{x \to -\infty} \hat{V}(x) & = 0. \end{cases} \tag{3.18}$$

Assume furthermore that

$$\begin{cases} \hat{V}(x) & \geq g(x), \quad x \in (-\infty, b^*), \\ (\mathcal{A} - r)g(x) & \leq 0, \quad x \in [b^*, \infty,), \end{cases} \tag{3.19}$$

and the stopping time

$$\tau^* = \inf\{t \geq 0 : X_t \geq b^*\} \tag{3.20}$$

is a.s. finite. Then we have that $\hat{V} = V$ and τ^* is the optimal stopping time.

Proof. We will only prove the statement for the case $\hat{V} \in \mathcal{C}^2(\mathbb{R})$. Let

$$\tau_n = n \wedge \inf\{t \geq 0; X_t \notin [-n, n]\} \tag{3.21}$$

and let τ be an arbitrary finite stopping time. Since $V(x)$ is bounded, we can use Itô's formula and have that

$$\begin{aligned} &e^{-r(\tau_n \wedge \tau)}\hat{V}(X_{\tau_n \wedge \tau}) - \hat{V}(x) = \\ &\qquad \int_0^{\tau_n \wedge \tau} (\mathcal{A} - r)\hat{V}(X_s)ds + \int_0^{\tau_n \wedge \tau} \sigma(X_s)\hat{V}''(X_s)dW_s. \end{aligned} \tag{3.22}$$

Taking expectations on both sides, we get that the stochastic integral vanishes (since it is a martingale). Using (3.18) and (3.19) we get

$$\hat{V}(x) = \mathbb{E}_x\left[e^{-r(\tau_n \wedge \tau)}\hat{V}(X_{\tau_n \wedge \tau}) - \int_0^{\tau_n \wedge \tau} (\mathcal{A} - r)\hat{V}(X_s)ds\right] \tag{3.23}$$

$$\geq \mathbb{E}_x\left[e^{-r(\tau_n \wedge \tau)}(\hat{V}(X_{\tau_n \wedge \tau}) - g(X_{\tau_n \wedge \tau})) + e^{-r(\tau_n \wedge \tau)}g(X_{\tau_n \wedge \tau})\right] \tag{3.24}$$

$$\geq \mathbb{E}_x\left[e^{-r(\tau_n \wedge \tau)}g(X_{\tau_n \wedge \tau})\right]. \tag{3.25}$$

Thus, taking $n \to \infty$ we get by dominated convergence that

$$\hat{V}(x) \geq \mathbb{E}_x\left[e^{-r\tau}g(X_\tau)\right], \tag{3.26}$$

which gives us $\hat{V}(x) \geq V(x)$. Repeating similar arguments with τ^* we have that

$$\hat{V}(x) = \mathbb{E}_x \left[e^{-r(\tau_n \wedge \tau^*)} \hat{V}(X_{\tau_n \wedge \tau}) - \int_0^{\tau_n \wedge \tau^*} (\mathcal{A} - r)\hat{V}(X_s) \right] \tag{3.27}$$

$$= \mathbb{E}_x \left[e^{-r(\tau_n \wedge \tau^*)} \hat{V}(X_{\tau_n \wedge \tau}) \right], \tag{3.28}$$

where we have used (3.18) and $\tau^* = 0$ for $x \geq b^*$. Using dominated convergence and $\tau^* < \infty$ a.s. we get

$$\hat{V}(x) = \mathbb{E}_x \left[e^{-r\tau^*} \hat{V}(X_{\tau^*}) \right] = \mathbb{E}_x \left[e^{-r\tau^*} g(X_{\tau^*}) \right], \tag{3.29}$$

which gives us $\hat{V}(x) \leq V(x)$. The statement follows. □

3.2 Optimal control problem

This section introduces stochastic optimal control problems, following the derivations in Touzi (2002) and Øksendal (2003). Before we develop a general theory, let us first present a simple example. Consider an investor who has the option to invest between a risk free asset with return $\mu_0 > 0$ and a risky asset with return $\mu_1 > \mu_0$ and volatility $\sigma > 0$. Let $v_t \in (0, 1)$ denote the fraction of the wealth invested in the risky asset. Then the wealth Z_t of the investor can be modelled to follow the SDE

$$dZ_t^v = Z_t^v(\mu_1 v_t + \mu_0(1 - v_t))dt + \sigma v_t Z_t^v dW_t. \tag{3.30}$$

Let us assume that the investor wishes to find v_t in order to maximize the logarithmic expected utility given by

$$\mathbb{E}\left[\log(Z_T^v)\right],$$

at some terminal time $T > 0$. If we make the additional assumption that

$$\frac{\mu_1 - \mu_0}{\sigma^2} \in (0, 1) \tag{3.31}$$

it can be shown that the optimal control v_t^* is given by

$$v_t^* = \frac{\mu_1 - \mu_0}{\sigma^2},$$

(cf. Øksendal (2003, Example 11.2.5.)). Please note that v_t^* is constant and that if (3.31) is violated we should allow for short selling or borrowing (i.e., allowing $v_t \in \mathbb{R}$) in order for v_t^* to be an optimal control.

Let us continue by developing a general theory of control problems. For this purpose, consider a given subset $U \in \mathbb{R}$, we then define the set of controls

$\mathcal{U}_0 = \{v_t : t \in [0, \infty)\}$ as the set of all progressively measurable processes with values in U. For each control process $v \in \mathcal{U}_0$ we define the controlled process X^v to be given by the controlled SDE

$$dX_t = \mu(t, X_t, v_t)dt + \sigma(t, X_t, v_t)dW_t, \tag{3.32}$$

where $\mu, \sigma : [0, \infty) \times \mathbb{R} \times U \to \mathbb{R}$ are continuous functions satisfying a Lipschitz condition, such that the above equation has a unique strong solution (see Theorem 1.1 in Touzi (2002) for details). We introduce the subset of admissible controls $\mathcal{U} \subset \mathcal{U}_0$, which satisfy the additional integrability assumption

$$\mathbb{E}_x \left[\int_0^T (|\mu(t, x, v_t)| + \sigma(t, x, v_t)|^2)dt \right] < \infty, \tag{3.33}$$

for some fixed given time horizon $T > 0$ (which can be infinite). When dealing with a stochastic control problem, we typically want to maximize (or minimize) a reward (or cost) function. Both problems are equivalent and in line with the introduction in Touzi (2002), we will focus on a minimization problem. Let $f, g : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ be cost functionals satisfying the quadratic growth condition

$$|f(x, t, u)| + |g(x)| \le C(1 + x^2), \tag{3.34}$$

for some $C > 0$. Fix a positive discount rate $r > 0$. We then introduce the cost function $J : [0, T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ given by

$$J(x, t, v) : \mathbb{E}_{t,x} \left[\int_t^T e^{-r(s-t)} f(s, X_s, v_s)ds + e^{-r(T-s)} g(X_T) \right], \tag{3.35}$$

where $E_{t,x} = \mathbb{E}\left[\cdot | X_t = x\right]$. We remark that we set $g = 0$ in the case $T = \infty$. Our goal is study the minimization problem

$$V(t, x) := \inf_{v \in \mathcal{U}} J(t, x, v), \tag{3.36}$$

where we are interested in two parts: (i) computing the value function $V(t, x)$, (ii) existence and characterization of the optimal control v^*.

A main tool when solving an optimal control problem is the dynamic programming principle: For any stopping time τ with values in $[t, T]$ we have

$$V(t, x) = \inf_{v \in \mathcal{U}} \mathbb{E}_{t,x} \left[e^{-r(\tau - t)} V(\tau, X_\tau) + \int_t^\tau e^{-r(s-t)} f(s, X_s, v_s)ds \right]. \tag{3.37}$$

Let us motivate (3.37). It states that the global minimization on the whole time interval $[t, T]$ can be split up into a sequence of local minimizations. Since τ was arbitrary, it is reasonable to assume that the optimal control v^* should really only depend on our current value and the infinitesimal dynamics of the process. Similarly to the optimal stopping problem these properties are "encoded" into the characteristic operator. Given a constant control value $u \in U$, we introduce the linear differential operator associated to the process X^u to be given by:

$$(\mathcal{A}^u \varphi)(t,x) = \mu(t,x,u)\varphi_x(t,x) + \frac{\sigma^2(x,t,u)}{2}\varphi_{xx}(t,x), \tag{3.38}$$

where $\varphi \in \mathcal{C}^{1,2}([0,T),\mathbb{R})$ is a real valued function and φ_x,φ_{xx} denote its partial derivatives.

Motivated by the dynamic programming principle, we can characterize the value function as a solution to a local optimisation problem. To this end, define the function $H : [0,T] \times \mathbb{R}^4$ by

$$H(t,x,v,d,s) = \inf_{u \in U}\{\mu(t,x,u)d + \frac{\sigma^2(t,x,u)}{2}s - rv + f(x,t,u)\}. \tag{3.39}$$

We remark that H is closely related to the operator $\mathcal{A}^u$. Particularly,

$$H(t,x,\varphi(t,x),\varphi_x(t,x),\varphi_{xx}(t,x)) = \inf_{u \in U}\{(\mathcal{A}^u\varphi)(t,x) - r\varphi(t,x) + f(t,x,u)\}. \tag{3.40}$$

Given regularity assumptions on the value function, we can now characterize the value function as a solution to a real-valued optimisation problem.

Theorem 13. *Assume that the value function satisfies $V \in \mathcal{C}^{1,2}([0,T),\mathbb{R})$, and that the function H is continuous. Assume furthermore that the cost functional $f(\cdot,\cdot,u)$ is continuous in (t,x) for all fixed $u \in U$. Then we have that V satisfies the Hamilton-Jacobi-Bellman (HJB) equation*

$$V_t(t,x) + H(t,x,V(t,x),V_x(t,x),+V_{xx}(t,x)) = 0 \tag{3.41}$$

for each $(t,x) \in [0,T) \times \mathbb{R}$.

Theorem 13 provides a necessary condition for the value function. Suppose now that we have found a smooth solution $\hat{V}$ solving the HJB equation (3.41). Naturally, we are then interested in sufficient conditions which imply that we have actually found the value function (i.e. we have $\hat{V} = V$). This result is given by a so called verification theorem:

Theorem 14. *Let $\hat{V} \in \mathcal{C}^{1,2}([0,\infty),\mathbb{R})$. Assume that $\hat{V}, f$ are of quadratic growth, meaning there exists a constant $C > 0$ such that*

$$|f(t,x,u)| + |\hat{V}(t,x)| \leq C(1+x^2), \tag{3.42}$$

for all $(x,t,u) \in [0,T) \times \mathbb{R} \times U$. Assume furthermore that $\hat{V}(T,\cdot) = g$ and $\hat{V}$ satisfies the HJB equation (3.41) for some minimizer $\hat{u}(t,x)$, i.e. we have

$$0 = \hat{V}_t(t,x) + H(t,x,\hat{V}(t,x),\hat{V}_x(t,x),\hat{V}_{xx}(t,x)) \tag{3.43}$$

$$= \hat{V}_t(t,x) + (\mathcal{A}^{\hat{u}(t,x)} - r)\hat{V}(t,x) + f(t,x,\hat{u}(t,x)). \tag{3.44}$$

Then, given that the stochastic differential equation

$$dX_s = \mu(s, X_s, \hat{u}(s, X_s))ds + \sigma(s, X_s, \hat{u}(s, X_s))dW_s \tag{3.45}$$

has a unique strong solution for each starting point $X_t = x$ and the control process $v_s^* := \hat{u}(s, X_s)$ is well-defined in $\mathcal{U}$, we have that $\hat{V} = V$ and v^* is an optimal (Markovian) control process.

The proof is basically a result of Itô's formula and is carried out using similar techniques as in the optimal stopping case (Theorem 12). We remark that a key assumption of the verification theorem is that the solution to the HJB equation is smooth. Typically, this is not the case and solutions to (3.41) are instead given in a viscosity sense. For more details regarding stochastic control problems, and particularity viscosity solutions, we refer to Touzi (2002).

3.2.1 Example of a singular stochastic control problem: The dividend problem

In this chapter we wish to study an example of a singular stochastic control problem following the results given in Shreve et al. (1984). Consider a company whose uncontrolled surplus is given by the SDE

$$dX_t^0 = \mu(X_t^0)dt + \sigma(X_t^0)dt, \quad t \geq 0, \quad X_{0-}^0 = x, \tag{3.46}$$

where μ, σ, with $\sigma(\cdot) \geq c > 0$, are Lipschitz continuos with Lipschitz continuos derivatives. Let us consider a control process D that represents the flow of dividend which is dynamically subtracted form the surplus. The controlled dynamics then result in

$$dX_t^D = \mu(X_t^D)dt + \sigma(X_t^D)dt - dD_t, \quad t \geq 0, \quad X_{0-}^D = x. \tag{3.47}$$

Let us remark that this setup is significantly different from the one discussed in the previous chapter, since the control process no longer affects the drift or diffusion term, but rather amounts to a direct subtraction of an arbitrary extraction process. Formally, we are interested in studying the optimisation problem

$$V(x) = \sup_{D \in \mathcal{E}} J(x; D) = \sup_{D \in \mathcal{E}} \mathbb{E}_x \left[\int_0^{\tau_0^D} e^{-rt} dD_t \right], \tag{3.48}$$

for some $r > 0$, where

$$\tau_0^D = \inf\{t \geq 0 : X_t^D \leq 0\} \tag{3.49}$$

and

$$\begin{aligned} \mathcal{E} := \{&(D_t(\omega))_{t \geq 0} \text{ is non-decreasing, right-continuous, } \{\mathcal{F}_t\}\text{-adapted,} \\ &\text{with } D_{0-} = 0, \text{ such that } D_t - D_{t-} \leq X_{t-}^D, \forall\, t \geq 0,\ \mathbb{P}\text{-a.s.}\}. \end{aligned} \tag{3.50}$$

Let us comment on the above problem and conditions. We are interested in a process D that maximizes the discounted dividend stream until default at time 0. By imposing that D is right continuos and adapted we ensure that the controlled SDE (3.47) has a unique solution for $0 \leq t \leq \tau_0$ (cf. Shreve et al. (1984, Page 59.)). Furthermore by imposing the last condition $D_t - D_{t-} \leq X^D_{t-}$, we ensure that the process is not allowed to have jumps larger than the current surplus, leading to a negative surplus level.

Let us continue by characterizing the optimal dividend policy as well as the value function. For this purpose, let us first present this auxiliary result presenting an upper bound of the value function (cf. Shreve et al. (1984, Lemma 3.1)).

Lemma 15. *Let $F : [0, \infty) \to \mathbb{R}$, satisfy*

$$\begin{aligned} F &\in \mathcal{C}^2(0,\infty) \cap \mathcal{C}[0, \infty) \\ F(0) &\geq 0, \\ F'(x) &\geq 1, \quad x \geq 0, \\ F''(x) &\leq 0, \quad x \geq 0, \\ \frac{1}{2}\sigma^2(x)F''(x) + \mu(x)F'(x) - rF(x) &\leq 0, \quad x \geq 0. \end{aligned} \tag{3.51}$$

Then we have that $F(x) \geq V(x)$ for all $x \geq 0$. If we have that $V = F$ (i.e. V satisfies the above properties for F) and $V'(x) > 1$ for all $x \geq 0$, then no optimal dividend policy exists.

Let us explain the consequences of the above Lemma. Firstly, since F is an upper bound of V it gives us a verification approach. Namely, if we would find a policy $D \in \mathcal{E}$ such that the expected reward $J(x; D^*)$ would satisfy the conditions in Lemma 15, then we would have that D^* is optimal and $J(x; D^*) = V(x)$. Secondly if $V'(x) > 1$ for all $x \geq 0$ (and V satisfies the conditions of Lemma 15) then no optimal policy exists, i.e. the supremum is not obtained by a dividend policy.

Since an optimal policy is not guaranteed to exists, we will make the following set of (standard) assumptions on μ, σ, used in a variety of papers e.g. Shreve et al. (1984), Bai et al. (2012) and Christensen and Lindensjö (2022), that will ensure the existence of an optimal policy.

Assumption 1.

- $\mu(0) > 0$
- $\sigma^2(x) > 0, \quad x \geq 0,$
- $\mu'(x) < r$ *and there exists an* $\varepsilon > 0$ *and an* $x_0 > 0$ *such that* $\mu'(x) < r - \varepsilon$ *for all* $x \geq x_0$. (3.52)

Under these assumptions, it can be shown that the value function admits a unique inflection point i.e. a unique point b^* satisfying $V''(b^*) = 0$ (cf. Bai et al. (2012, Theorem 2.3, Proposition 2.5)). Then the optimal policy is to pay out dividends when the process is above the threshold b^* (cf. Shreve et al. (1984, Theorem 4.3). Mathematically, the pair (X^{D^*}, D^*), is then determined by solving (3.32) in form of a Skorokhod reflection at the upper boundary b^* (cf. Section 2.6) and an immediate jump to b^* in case the process starts at $x > b^*$. By the local time characterization, the optimal dividend policy $D^* = D^{b^*}$ can be written as

$$D_t^{b^*} = l_t^{b^*}(X^{D^{b^*}}) + (x - b^*)I_{\{x>b^*\}} \quad x \geq 0. \tag{3.53}$$

Furthermore by following the derivations in Shreve et al. (1984, Chapter 4.), we can determine the value function (as well as the optimal boundary) by solving the free boundary value problem

$$\tfrac{1}{2}\sigma^2(x)V''(x) + \mu(x)V'(x) - rV(x) = 0, \quad x \in (0, b^*), \tag{3.54}$$

$$V(x) = V(b^*) + x - b^*, \qquad x \in [b^*, \infty), \tag{3.55}$$

$$V(0) = 0, \tag{3.56}$$

$$V \in \mathcal{C}^2(0, \infty), \tag{3.57}$$

where we thus have that b^* is determined by "smooth fit" i.e., $V''(b^*) = 0$.

3.3 Stochastic differential games

This section introduces stochastic games and presents the basics of the theory such as the Nash equilibrium and the best response mapping. Stochastic differential games have been studied in a variety of papers in a mostly example based fashion, with the results being dependent on the exact structure of the game at hand. Hence, it is suitable to restrict this section to controller stopper games for a continuous time diffusion. Such games have been studied extensively in the literature, such as in Karatzas and Sudderth (2001), Hernandez-Hernandez et al. (2015) and Hernández-Hernández and Yamazaki (2015) . Recent developments include the addition of incomplete information such as Ekström et al. (2022) or the second paper of this thesis.

A controller stopper game is a game between two players: A controller and a stopper. Given a set $\mathcal{U}$ of admissible controls and a set $\mathcal{T}$ of admissible stopping times, the controller choose their control $v \in \mathcal{U}$ and the stopper chooses their stopping time $\tau \in \mathcal{T}$) in order to maximize (or minimize) their corresponding value function. Since both players act on the same process they are typically in competition, and their expected payoffs will depend on the strategy pair $(v, \tau) \in \mathcal{U} \times \mathcal{T}$. For example, consider a controller that wishes to maximize the value function

$$\mathcal{J}^1(\tau, v, x) = \mathbb{E}_x \left[\int_0^\tau e^{-rt} f_1(X_t^v, v_t) dt + e^{-r\tau} g_1(X_\tau^v) \right] \tag{3.58}$$

and a stopper that wishes to minimize the value function

$$\mathcal{J}^2(\tau, v, x) = \mathbb{E}_x \left[\int_0^\tau e^{-rt} f_2(X_t^v, v_t) dt + e^{-r\tau} g_2(X_\tau^v) \right], \tag{3.59}$$

for some $r > 0$, some functions f_1, f_2, g_1, g_2 and a controlled process X^v (which can for example given by (3.32)). If we have $\mathcal{J}^1 = \mathcal{J}^2$, the game is said to be zero-sum.

Since both players are competing against each other, it is generally not possible to find a strategy pair that would optimise the value function for both players simultaneously i.e. a pair (τ^*, v^*) such that

$$\begin{cases} \mathcal{J}^1(\tau^*, v^*, x) \geq \mathcal{J}^1(\tau, v, x), \\ \mathcal{J}^2(\tau^*, v^*, x) \leq \mathcal{J}^2(\tau, v, x), \end{cases} \tag{3.60}$$

for all $x \in \mathbb{R}$ and all $(\tau, v) \in \mathcal{U} \times \mathcal{T}$. Hence, solutions to stochastic differential games are typically given in form of a Nash equilibrium (see Definition 16 below). A Nash equilibrium is a strategy pair (τ^*, v^*) such that neither player would gain by deviating, meaning that if the controller sees the strategy τ^* of the stopper as fixed, the control v^* is a maximizer of the value function $\mathcal{J}^1$ of the controller (and vice versa for the stopper).

Definition 16 (Nash equilibrium)**.** *A pair of admissible strategies* $(\tau^*, v^*) \in \mathcal{T} \times \mathcal{U}$ *is a said to be a Nash equilibrium if the corresponding rewards,* (3.58)–(3.59)*, satisfy*

$$\begin{cases} \mathcal{J}^1\left(\tau^*, v^*, x\right) \geq \mathcal{J}^1\left(\tau^*, v, x\right), \\ \mathcal{J}^2\left(\tau^*, v^*, x\right) \leq \mathcal{J}^2\left(\tau, v^*, x\right), \end{cases} \tag{3.61}$$

for any pair of deviation strategies $(\tau, v) \in \mathcal{T} \times \mathcal{U}$ *and each starting value* x*.*

For the zero-sum case (i.e the case $\mathcal{J}_1 = \mathcal{J}_2$) the problem simplifies and showing that (τ^*, v^*) is a Nash equilibrium results in checking

$$\mathcal{J}^1(\tau^*, v^*, x) = \inf_{\tau \in \mathcal{T}} \sup_{v \in \mathcal{U}} \mathcal{J}^1(\tau, v, x) = \sup_{v \in \mathcal{U}} \inf_{\tau \in \mathcal{T}} \mathcal{J}^1(\tau, v, x). \tag{3.62}$$

The existence of a Nash equilibrium is typically proven using a fixed point approach. A useful tool when considering such an approach is the so called best response mapping (see Definition 17 below). Given a fixed stopping strategy τ the best response mapping returns the optimal strategy $v(\tau)$ of the controller (and analogously for the stopper). Naturally, the Nash equilibrium can then be seen as a fixed point in such a best response mapping. By proving continuity properties of such a mapping combined with properties of the admissible strategies (such as convexity of $\mathcal{T}, \mathcal{U}$, in a suitable space) it is then possible to prove that such a fixed point i.e. a Nash equilibrium exists using results from fixed point theory.

Definition 17. *Given an arbitrary pair* $(\tilde{\tau}, \tilde{v}) \in \mathcal{T} \times \mathcal{U}$*, we define the best response mapping as the point to set mapping*

$$(\tilde{\tau}, \tilde{v}) \to \left(\arg\min_{\tau \in \mathcal{T}} \mathcal{J}_2(\tau, \tilde{v}, x), \arg\max_{v \in \mathcal{U}} \mathcal{J}_1(\tilde{\tau}, v, x) \right). \tag{3.63}$$

3.4 Introduction to time-inconsistency

In this section we will introduce time-inconsistent control and stopping problems. For this purpose, we will first present the main concepts of time-inconsistency for the stopping problem and then later expand this notion to the control case. This section follows the derivations in Björk et al. (2021).

3.4.1 Time-inconsistent stopping

Roughly speaking, time-inconsistency in optimal stopping problems means that the maximizing/minimizing stopping rule depends on the starting point of the process, i.e. different starting values have different stopping times maximizing the expected reward. This typically means that a stopping time τ^* maximising an expected reward J in the classical sense as

$$\tau^* = \arg\max_{\tau} J_\tau(x),$$

is of the form

$$\tau^* = \tau_{D^*(x)} := \inf\{t \geq 0 : X_t \in D^*(x)\},$$

for some set $D^*(x) \subset \mathbb{R}$ explicitly depending on the starting value x. This appears in a variety of applications such as:

- Non exponential discounting:

  $$J_\tau(x) = \mathbb{E}_x \left[\delta(\tau) G(X_\tau)\right], \tag{3.64}$$

  where δ is a non-exponential discount function i.e., we have $\delta : [0, \infty) \to (0, 1]$ decreasing and not of the type $\delta(t) = c^{-t}$ for some $c > 0$.

- Mean variance utility:

  $$J_\tau(x) = \mathbb{E}_x \left[X_\tau\right] - \frac{\gamma}{2} \mathrm{Var}_x \left[X_\tau\right] \tag{3.65}$$

  for some $\gamma > 0$.

Let us illustrate why different maximizing/minimizing stopping times for different starting points causes the problem to be time-inconsistent. Consider an agent that wishes to maximize a time-inconsistent expected reward i.e. solve a optimal stopping problem where the maximizing stopping rule depends on the starting value (we denote the maximizing stopping rule for x by $D^*(x)$). Let $X_0 = x_0$, then initially (at $t = 0$), $\tau_{D^*(x_0)}$ maximizes the expected reward. However, at each later time point $t > 0$ (assuming the process has not been stopped before), the agent might reconsider his stopping rule to be given by $D^*(X_t)$ which results in the stopping time $\tau_{D^*(X_t)}$ that maximizes the expected reward at the current time-point t instead. Hence, the optimal stopping rule can change over time given the behaviour of the process. This is called time-inconsistency and makes the notion of what is considered an "optimal" stopping time unclear. There are three main ways of dealing with time-inconsistent problems:

- The agent fixes the starting point x_0 and chooses $\tau_{D^*(x_0)}$ to be his stopping time. In this case the agent disregards the fact that his stopping rule is no longer optimal at later time points. This is referred to as pre-commitment.
- For each time t, the agent continuously updates his stopping time to be given by $\tau_{D^*(X_t)}$ (with initial choice $\tau_{D^*(x_0)}$). In the literature, an agent using this strategy is referred to as a naive agent.
- The agent considers a game theoretic perspective, playing an intra-personal game between himself and his future selves, searching for a Nash equilibrium stopping strategy $\hat{\tau}$.

In this thesis we will focus on the third (game-theoretic approach). For the naive approach we refer to Huang et al. (2020) and the references therein.

3.4.1.1 Nash equilibrium formulation

This chapter provides a game theoretic interpretation of a time-inconsistent stopping problem, and gives a suitable intra-personal Nash equilibrium definition. The arguments presented in this chapter are inspired by Christensen and Lindensjö (2018) and Björk et al. (2021).

Consider the non-exponential discounting problem (3.64), driven by a stochastic process X with state space $E \subset \mathbb{R}$ whose dynamics are given by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \tag{3.66}$$

We view this time-inconsistent problem as a stopping problem where the preferences of the stopping agent change over time. More specifically, the agent's preferences change according to the current state of the process. In the game theoretic approach, we interpret this as an intra-personal game between different instances of the agent that depend on the current state of the process. By the Markov property, this is equivalent to considering a sequential game between so called x-agents that represent the agents preference given each different starting point x. Roughly speaking each x-agent has the following actions in the game, either (i) stop the process at x, or (ii) continue the process (at x) in an "infinitesimal sense".

In line with the standard Nash equilibrium formulation we then say that a stopping time $\hat{\tau}$ is a intra-personal equilibrium stopping strategy if

(i) no x-agent wants to stop at x when $\hat{\tau}$ implies no stopping,

(ii) no x-agent wants to continue (at x) in an "infinitesimal sense" when $\hat{\tau}$ implies immediate stopping.

Hence, we have a Nash equilibrium if none of the x-agents wants to deviate from the proposed equilibrium strategy $\hat{\tau}$ with a deviation strategy. In order to make the above motivation precise we thus need to define which strategies are admissible i.e. which strategies are allowed to be considered as possible equilibria or in deviation. The derivation above motivates considering admissible strategies to be given by so called pure Markovian stopping times:

Definition 18. *We say that a stopping time τ is a pure Markovian stopping time if it is a first entry time, i.e. we have*

$$\tau = \inf\{t \geq 0 : X_t \in S\} \tag{3.67}$$

for some measurable set $S \subset E$. We set $\mathcal{T}$ to be the set of all pure Markovian stopping times.

We are now in a position to define an intra-personal Nash equilibrium.

Definition 19. *A stopping time $\hat{\tau} \in \mathcal{T}$ is said to be a (pure Markovian) equilibrium stopping strategy if for all $x \in \mathbb{E}$ we have*

$$J_{\hat{\tau}}(x) \geq G(x), \quad \textit{and} \tag{3.68}$$

$$\liminf_{h \searrow 0} \frac{J_{\hat{\tau}}(x) - J_{\hat{\tau}\circ\theta_h + \tau_h}(x)}{\mathbb{E}_x\left[\tau_h\right]} \geq 0, \tag{3.69}$$

where $\tau_h = \inf\{t \geq 0 : |X_t - X_0| \geq h\}$ is assumed to be a.s. finite.

Let us motivate Definition 19 further. The first condition of the equilibrium, states that using the proposed equilibrium strategy gives a bigger or equal reward compared to immediate stopping for all x. This ensures that condition (i) is satisfied and none of the x-agents would gain by deviating trough stopping. The condition (3.69) is motivated by (ii). In the numerator of (3.69) each agent compares the reward given by the stopping time $\hat{\tau}$ to the one given by the stopping time $\hat{\tau} \circ \theta_h + \tau_h$, which heuristically can be interpreted as continuing until reaching τ_h and then stopping according to $\hat{\tau}$ (cf. Chapter 3.1). Thus, we ensure that it is more beneficial for each agent to stop according to $\hat{\tau}$ then to continue for some time τ_h. By taking the limit $h \searrow 0$ we get $\tau_h \searrow 0$ and by dividing by the term $\mathbb{E}_x\left[\tau_h\right]$, we ensure that the limit can be non-zero. This specifies precisely what we mean by continuing in an "infinitesimal sense" in (ii).

We remark that it is also possible to consider so called mixed Markovian stopping times. This would essentially result in giving the x-agents a third option to act by stopping with an "infinitesimal intensity". Typically, mixed Markovian stopping times are given by the minimum $\tau^{\Psi} \wedge \tau^{S}$ of a pure stopping time τ^{S} and a randomised stopping time of the form

$$\tau^{\Psi} \inf\{t \geq 0 : \Psi_t \geq U\}, \tag{3.70}$$

where Ψ_t is an additive functional of X and $U \sim \exp(1)$ is an independent exponentially distributed random variable. We refer to Borodin and Salminen (2012, Chapter 2) for an exact definition of an additive functional. For an exact definition of the intra-personal equilibrium in the case of mixed strategies we refer to Christensen and Lindensjö (2020) or the first paper of the thesis.

3.4.2 Time-inconsistent control

Consider a control problem where the controlled process X^u is given by the controlled SDE

$$dX_t = \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t \tag{3.71}$$

where μ, σ satisfy the same assumptions as in (3.32). Consider two terminal cost functions $F, G : \mathbb{R}^2 \to R$, we are then interested in the time-inconsistent cost functional

$$J(t, x, u) = \mathbb{E}_{t,x}[F(x, X_T^u)] + G(x, \mathbb{E}_{t,x}[X_T^u]), \tag{3.72}$$

which depends on an admissible control law of the form $u_t = u(t, X_t)$ satisfying properties such that the above is well defined.

Definition 20 (Admissible control law)**.** *Consider $u : [0, T] \times \mathbb{R} \to U \subset \mathbb{R}$. We say that u is an admissible control law if*

(i) For each starting point $(t, x) \in [0, T] \times \mathbb{R}$ the SDE

$$dX_s^u = \mu(t, X_s^u, u(s, X_s))ds + \sigma(s, X_s^u, u(s, X_s))dW_s,\ X_t^u = x,\ s \geq t, \tag{3.73}$$

has a unique strong solution.

(ii) For each starting point $(t, x) \in [0, T] \times \mathbb{R}$, we have that

$$\mathbb{E}_{t,x}[F(x, X_T^u)] < \infty, \quad \mathbb{E}_{t,x}[|X_T^u|] < \infty. \tag{3.74}$$

Since the functions F, G of the cost functional explicitly depend on the starting point x, an optimal stopping rule u^* that would minimize J in classical sense as in (3.36) would typically also depend on the starting value. Hence the Bellman principle of optimality does not hold i.e. the dynamic programming principle is no longer satisfied as in (3.37). Mathematically this means that for general F, G we would typically have that

$$\inf_{u} J(t, x, u) \neq \inf_{u} \mathbb{E}_{t,x}[\operatorname{ess\,inf}_{v} J(s, X_s^u, v)], \tag{3.75}$$

where $s \in (t, T)$ and the infimum is taken over all admissible stopping rules. Thus our minimizing control rule would change if we reassess it at a later time. Similarly to the stopping case, our problem therefore is time-inconsistent and therefore the notion of "optimality" becomes unclear. Following analogues arguments to the stopping case, we thus search for control rules that are given by an intra personal Nash equilibrium.

Definition 21 (Equilibrium definition)**.** *Consider a fixed admissible control law $\hat{u}$. Given an admissible control law u, an initial point $(t, x) \in [0, T) \times \mathbb{R}$ and an $h \in (0, T - t]$ let us define a new control law*

$$u_h(s, y) = \begin{cases} u(s, y), & s \in [t, t+h), \quad y \in \mathbb{R}, \\ \hat{u}(s, y), & s \in [t+h, T], \quad y \in \mathbb{R}. \end{cases} \tag{3.76}$$

We then say that $\hat{u}$ is an equilibrium control law if

$$\limsup_{h \searrow 0} \frac{J(t, x, \hat{u}) - J(x, t, u_h)}{h} \leq 0. \tag{3.77}$$

for all admissible control laws u and initial points $(t, x) \in [0, T) \times \mathbb{R}$.

We remark that the inequality (3.77) is in the other direction as in (3.69) since we are considering a minimization problem. Furthermore let us note that similarly to the time-consistent case, it is possible to characterize an equilibrium control rule as a solution to a so called extended HJB system. We refer to Björk et al. (2021, Chapter 15) for details. It is also possible to generalize the above formulation for singular controls as it is done in Christensen and Lindensjö (2022) or Bodnariu et al. (2025) which is the third paper of the thesis.

Chapter 4

Overview of papers

4.1 Paper I

The first paper of this thesis studies a time inconsistent stopping problem, where the time inconsistency is due to non-exponential discounting. Specifically we consider the minimization problem

$$J_\tau(x) := \mathbb{E}_x\left(\int_0^\tau h(s)f(X_s)ds + h(\tau)g(X_\tau)\right), \tag{4.1}$$

where X is a stochastic process given by the SDE (2.10), f, g are non-negative cost functions and $h : [0, \infty) \to (0, 1]$ is weighted discount function, which means it has the representation

$$h(t) = \int_0^\infty e^{-rt}dF(r), \tag{4.2}$$

where $r > 0$ and F is a distribution function with positive support. This problem has previously been studied before in Tan et al. (2021), where the authors prove that a pure treshold equilibrium is not guaranteed to exist. More specifically, Tan et al. (2021) provides an example where the principle of "smooth fit" fails, in a sense that it is not guaranteed to result in an equilibrium. The contribution of the paper is twofold. Firstly we introduce a new set of mixed strategies, so called local time pushed mixed strategies. Then we provide a suitable intra-personal Nash equilibrium definition for these types of mixed strategies and prove a verification theorem. Secondly, we prove that the example provided by Tan et al. (2021) has a local time pushed mixed equilibrium exactly when the pure threshold equilibrium fails to exist.

The main novelty of the paper is the introduction of the local time pushed mixed stopping times for time-inconsistent stopping problems. These stopping times $\tau^{\Psi,D}$ are given by the minimum $\tau^\Psi \wedge \tau^D$ of a pure (first entry) stopping time τ^D and a mixed stopping time

$$\tau^\Psi = \inf\{t : \Psi_t \geq U\}, \tag{4.3}$$

where $U \sim \exp(1)$ is an independent exponential random variable and the stopping intensity Ψ_t admits the representation:

$$\Psi_t := \sum_{i=1}^{n} \psi_i l_t^{x_i} + \int_0^t \psi(X_s)ds, \tag{4.4}$$

where n, is a fixed number, x_i and $\psi_i \in (0, \infty)$ are fixed constants and ψ is a right continuous function with left hand limits. Here $l_t^{x_i}$ denotes the local time of the process X at the point x_i.

As seen in (4.4), the stopping intensity decomposes into a singular and continuous part. Continuous mixing has been considered in previous papers, such as in Christensen and Lindensjö (2020) and can be interpreted as stopping the process according to a first jump of a Cox process. A singular mixing intensity given by the local time push has not been considered before for time-inconsistent stopping problems. It turns out that allowing this is critical for finding an equilibrium strategy in the example presented.

Future research might include considering different kind of games or optimal stopping problems solved by these stopping strategies, or the development of a general existence result for time-inconsistent stopping problems with non-exponential discounting.

4.2 Paper II

The second paper of this thesis considers a two player controller stopper game with asymmetric information. Particularly, we consider a game where one of the players of the game (in our case the stopper) is not able to observe if the other player is active. This can be interpreted as a game between an owner (stopper) and a manager (controller) of an asset. The manager can either be effective or non effective. An effective manager can exert effort to influence the drift of the asset, while a non-effective manager cannot act at all. The owner receives the returns of the asset at the cost of paying a salary to the manager until the game is stopped. To be more specific, we consider a controlled stochastic process X given by

$$dX_t = \int_0^t (\theta\lambda_s - c)ds + W_t, \tag{4.5}$$

where $c > 0$ is the salary, θ is a Bernoulli random variable indicating if the manager is active, $\lambda_s \in \{\underline{\lambda}, \bar{\lambda}\}$ is a positive control taking 2 values and W_t is a Brownian motion. Both players seek to maximize their respective value functions, with the value function of the owner (stopper) given by

$$\mathcal{J}^1(\tau, \lambda, p) = \mathbb{E}\left[\int_0^\tau e^{-rs}(\theta\lambda_s - c)\,ds\right] \tag{4.6}$$

and the value function of the manager (controller) is the conditional expectation

$$\mathcal{J}^2(\tau, \lambda, p) = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(c - (\lambda_s - \underline{\lambda})^2\right)ds\,\middle|\,\theta = 1\right]. \tag{4.7}$$

The owner cannot observe θ (i.e. if the manager is effective) and thus chooses to stop according to the information given by the process X (i.e. the stopping time is $\{\mathcal{F}_t^X\}$ adapted). It turns out that this stopping is time is chosen w.r.t to the conditional probability $P_t = \mathbb{E}\left[\theta|\mathcal{F}_t^X\right]$, where P_t is computed with help of filtering theory. On the other hand, the manager can observe θ and if $\theta = 1$, can choose to exert extra effort at the continuos cost $(\lambda_s - \underline{\lambda})^2$.

Intuitively an equilibrium to this game would be as follows: Once the process P_t is too small, the owner no longer believes that the manager is effective and decides to stop the game in order to stop paying the salary. On the other hand an effective manager would increase the effort for small P_t in order to keep the game going and continue earning his salary. Therefore, we expect the Nash equilibrium to be of double threshold type i.e. we expect that there exists constants $b_1^*, b_2^* \in (0,1)$ such that

$$\tau_{b_1^*} := \inf\{t \geq 0 : P_t \leq b_1^*\}, \tag{4.8}$$

$$\lambda_{b_2^*}(P_t) := \underline{\lambda} + (\bar{\lambda} - \underline{\lambda})I_{\{P_t < b_2^*\}}, \tag{4.9}$$

is a Nash equilibrium. Indeed, we prove in the paper that such an equilibrium exits (under certain parameter assumptions) and provide equations for determining b_1^*, b_2^*.

A possible extension of this work would be to analyse existence of a Nash equilibrium when allowing the effort rate to take more than two different control values, or to be given by a general (continuos) function.

4.3 Paper III

The third paper of this thesis studies a time inconsistent singular stochastic control problem, where the time inconsistency is due to non-exponential discounting. Specifically we consider the controlled SDE given by the dynamics

$$dX_t^D = \mu(X_t^D)\,dt + \sigma(X_t^D)\,dW_t - dD_t, \quad X_{0-}^D = x \in (l, r), \quad 0 \leq t \leq \tau_l^D. \tag{4.10}$$

Here μ, σ satisfy the usual conditions, $\infty \leq l < r \leq \infty$, τ_l^D is the first hitting time of l of the controlled process X^D (cf. (2.2)) and D is an increasing process which is right continuous with left hand limits and has jumps that do not make X go strictly below l (i.e. analogues assumptions to the ones made in Section 3.2.1). The upper boundary r is assumed to be inaccessible, meaning that it a.s. cannot be reached by the process in finite time i.e. $\mathbb{P}(\tau_r^D = \infty) = 1$. Sufficient as well as necessary conditions for a boundary to be inaccessible can be found in Karatzas and Shreve (1991, Chapter 5.5. C). We are then interested in the time-inconsistent minimization problem

$$J(x; D) := \int_0^\infty \mathbb{E}_x \left[\int_0^{\tau_l^D} e^{-qt} f(X_t^D)\, dt + \int_0^{\tau_l^D} e^{-qt}\, dD_t + e^{-q\tau_l^D} q^{-1} f(l) \right] dF(q), \tag{4.11}$$

where F is a distribution function with positive support and f is a non negative cost function.

In previous works regarding singular (time-inconsistent) stochastic control such as Liang et al. (2024), equilibria are considered to be of Skorokhod reflection type. These strategies are referred in the paper as "strong" threshold strategies. In contrast to these "strong" threshold strategies we introduce a novel class of "mild" threshold control strategies whose corresponding control admits the form

$$D_t := (x - \beta + \delta)\mathbf{1}_{\{x \geq \beta\}} + \int_0^t u_\beta(X_s^D) ds, \quad 0 \leq t \leq \tau_l^D, \tag{4.12}$$

where u_β is picked in such a way such that β becomes inaccessible (in particular we have $\lim_{x \nearrow \beta} u_\beta(x) = \infty$). These strategies act on the process as follows: First if the process starts at $x \in [\beta, \infty)$ there is an initial jump to $\beta - \delta$. Then these type of strategies push the process down with an exploding rate, such that the process will not reach the boundary β. Furthermore, we extend the form in (4.12) to allow for $\delta \to 0$ in a suitable sense, thus allowing the process to start at the inaccessible boundary.

The contribution of the paper is threefold. Firstly we state an appropriate equilibrium definition allowing for strategies incorporating multiple jump/reflection regions as well as an exploding rate, thus incorporate the "mild" and "strong" threshold controls. We then prove a verification theorem for "strong" and "mild" threshold controls providing sufficient as well as necessary conditions for an equilibrium. Lastly, we provide an example where for some parameter values there exists no equilibrium when only considering "strong" threshold controls. However, it is possible to find "mild" threshold equilibria, making these strategies critical for existence of an equilibrium.

Interesting topics for future research, might be to either consider different singular stochastic control problems and games where these type of "mild" threshold strategies appear, or proving a general existence result for time-inconsistent singular control problems with exponential discounting.

4.4 Paper IV

The fourth paper of this thesis studies a dividend problem with random time horizon as well as ruin at zero surplus. The random time horizon is given by an occupation time measuring the time the firm has spent below a certain distress threshold. In the paper we completely characterize the optimal control, leading to different payout structures depending on the distress threshold. In particular we are interested in a stochastic process X, given by a (controlled) Brownian motion with drift

$$X_t^D = \mu t + \sigma W_t - D_t, \ X_0^D = x \geq 0, \tag{4.13}$$

where $\mu, \sigma > 0$ and D is a dividend process satisfying the usual conditions (cf. Section 3.2.1). For any given $y \in [0, \infty)$, we are interested in the optimal dividend problem

$$\begin{aligned} V(x,y) &:= \sup_{D \in \mathcal{E}} J(x;y,D), \quad \text{where} \\ J(x;y,D) &:= \mathbb{E}_x \left[\int_0^{\tau_0^D} e^{-rt} I_{\{w_t^y < e_1\}} dD_t \right], \end{aligned} \tag{4.14}$$

and $\mathcal{E}$ is defined as in (3.50), τ_0^D is the hitting time of 0, e_1 is an exponential random variable with unit mean independent of X, D and w_t^y is an occupation time measuring the time the process spends below y i.e. it is given by

$$\omega_t^y = q \int_0^t I_{\{X_s^D < y\}} ds. \tag{4.15}$$

As seen above, this is an optimal dividend problem, featuring an additional ruin intensity given by the time spent below a certain distress threshold. This can be seen as a more realistic model compared to the classical case, since a prolonged period of low liquidity can cause regulatory intervention or operational collapse. Using the independence of e_1, we are able to reformulate the above problem into an equivalent problem with stochastic (state-dependent) discounting. In particular we have that

$$V(x;y) = \sup_{D \in \mathcal{E}} \mathbb{E}_x \left[\int_0^{\tau_0^D} e^{-rt - w_t^y} dD_t \right].$$

In the paper we characterize the optimal dividend policy for three different cases of y:

(i) If y is large i.e., $y \geq y_u$ for some uniquely determined y_u, we have that the optimal policy is to pay out dividends above a threshold b_{r+q}^* as in the classical case with discounting $r + q$.

(ii) If y is small i.e., $y \leq y_l$ for some uniquely determined $y_l < y_u$, we have that there exists a unique threshold $b^*(y)$ so that we pay out dividends when the process is above $b^*(y)$ (similarly to the classical case).

(iii) If $y \in (y_l, y_u)$, there exists a unique pair $(\underline{b}^*(y), \bar{b}^*(y))$ satisfying $b^*_{r+q} < \underline{b}^*(y) < y < \bar{b}^*(y)$, such that the optimal dividend policy is to pay out when the process is above $\bar{b}^*(y)$ as well as when the process is in the interval $[b^*_{r+q}, \underline{b}^*(y)]$.

A possible extension of this problem for future research might be to consider the uncontrolled process to be given by a solution to a general stochastic differential equation or to allow for an arbitrary state-dependent ruin intensity.

Sammanfattning på svenska

Denna avhandling bidrar till området stokastisk optimering genom att behandla en version av ett optimalt utdelningsproblem, ett stokastiskt differentialspel med ofullständig information, samt spelteoretiska angreppssätt för tidsinkonsistenta stopptids- och kontrollproblem. Avhandlingen baseras på fyra artiklar.

Artikel I behandlar ett spelteoretiskt angreppssätt för ett tidsinkonsistent stopptidsproblem, där tidsinkonsistensen beror på icke-exponentiell diskontering. Vi introducerar en ny klass av blandade stoppstrategier och tillhandahåller en verifikationssats. Vidare betraktar vi ett exempel där det inte finns någon jämvikt om man endast använder rena stopptider. Emellertid kan vi konstruera en jämvikt i klassen av blandade stopptider.

Artikel II behandlar ett stokastiskt spel i kontinuerlig tid mellan en styrande part och en stoppare, med ofullständig information. Stopparen kan ses som ägare till en tillgång och den styrande parten som dess förvaltare som antingen är effektiv eller ineffektiv. Förvaltaren får en lön som betalas av ägaren. En effektiv förvaltare kan välja att anstränga sig till en kostnad för att öka driften hos tillgången, medan en ineffektiv förvaltare inte kan agera. Ägaren kan när som helst välja att stoppa spelet baserat på observationer av tillgångens rörelser. Ägaren kan dock inte direkt observera om förvaltaren är effektiv eller ineffektiv. Vi härleder en Nash-jämvikt för detta spel, given som en tröskellösning beroende på den betingade sannolikheten att förvaltaren är effektiv.

Artikel III behandlar ett tidsinkonsistent singulärt stokastiskt kontrollproblem, där tidsinkonsistensen beror på icke-exponentiell diskontering. Vi introducerar en ny klass av ”milda” tröskelkontroller, vilka ges av en exploderande driftfunktion som genererar en otillgänglig rand för den underliggande diffusionen. Dessa ”milda” kontroller står i kontrast till de ”starka” tröskelkontroller som tidigare har betraktats och som motsvarar Skorokhod-reflektion vid en övre gräns. Vi tillhandahåller ett lämpligt jämviktsvillkor för dessa kontroller samt en verifikationssats. Vidare ger vi ett exempel där ingen ”stark” tröskeljämvikt existerar, men där vi finner en jämvikt med hjälp av ”milda” tröskelkontrollstrategier.

Artikel IV behandlar ett optimalt utdelningsproblem med ruin vid noll överskott samt en ytterligare ruin om överskottet stannar för länge under en viss

krisnivå. Vi löser problemet fullständigt genom att betrakta tre olika fall. Om krisnivån är låg eller hög resulterar den optimala kontrollen i utdelning av vinster när överskottsprocessen når en övre gräns. Om krisnivån antar mellanliggande värden resulterar den optimala kontrollen i utdelning av vinster i två separata intervall.

Sammanlagt bidrar dessa resultat till en vidareutveckling av teorin för stopptids- och kontrollproblem, genom att berika litteraturen med nya problem samt genom att introducera lösningsstrukturer som tidigare inte har betraktats.

Sammanfattningen är baserad på en översättning av ChatGPT (`https://chat.openai.com`).

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Part II

Papers