Int J of Sci and Math Educ (2026) 24:27
https://doi.org/10.1007/s10763-026-10654-2

Students' Experiences of Number Lines in an Algebraic Teaching Tradition: A Study Inspired by the El'konin–Davydov Curriculum

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Received: 12 August 2024 / Accepted: 11 February 2026
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Abstract

This article reports an empirical study investigating how first-grade students in multilingual classes in Sweden experience number lines in an algebraic teaching aligned with the El'konin-Davydov curriculum. A phenomenographic analysis revealed that the students ($n = 150$) experienced number lines in terms of three qualitatively distinct categories: (a) Mathematical properties, (b) Relationships between the properties, and (c) Operations on a number line. These categories involved different types of algebraic thinking, identified in the students' joint analytical work. The higher categories encompass the lower ones, and a greater variety of algebraic thinking was identified in the higher categories. The first, qualitatively lowest, category includes students' experiences with points on a number line and the distances between them. The second category includes experiences about relationships between the properties in category one (the iterated unit). The third and highest category includes aspects of value (that the value is from the starting position to a specific position, distinguishing between positions and values), the direction of the number line (relationships between unknown quantities depending on their locations on the number line), and the relationships between operations (e.g., addition and multiplication). Suppose the number lines had been introduced in a ready-made form consisting of numerical positions; the first and second categories identified in this algebraic teaching might not have been possible. The results indicate that these students might use properties and relationships on a number line to enable their mathematical reasoning.

Keywords Algebraic thinking · El'konin–Davydov curriculum · Number line · Phenomenography

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Published online: 10 March 2026

Introduction

To understand the general structure of numbering, which is at the heart of early mathematics (Adamuz-Povedano et al., 2021; Nunes et al., 2009), primary school children need to develop a deep understanding of the connections between numbers, quantities, and relationships (Carraher et al., 2006; Davydov, 1982; Nunes et al., 2009). However, many children find it challenging to understand relationships, and many teachers struggle to design adequate instructions (Nunes et al., 2009; Woods et al., 2017). Research shows that children's thinking about relationships can be facilitated through tools such as diagrams, graphs, and symbolic representations (Davydov, 1982; Nunes et al., 2009). One such tool is the number line, which has the potential to visualise relationships between numbers (Carraher et al., 2006; Davydov, 1982; Nunes et al., 2009; Saxe et al., 2010) using both known and unknown quantities (Carraher et al., 2006; Davydov, 1982, 2008; Pitta-Pantazi et al., 2025). Despite this potential, number lines in early mathematics education are often provided in a ready-made form, which tends to limit students' engagement in visualising and memorising the number order, rather than reasoning about the general structure of numbers (Davydov, 1982, 1990, 2008; Nunes et al., 2009; Verschaffel et al., 2017). As the level of generalisation that students achieve is influenced by the mediation provided (Adamuz-Povedano et al., 2021; Narváez et al., 2025), young students' ability to discern general arithmetic structures may fail when the tools used are not productive (Davydov, 2008; Narváez et al., 2025). The mediation offered in the present research lessons indicates opportunities for students in multilingual mathematics classrooms to use mathematical properties when reasoning about a number line (Barwell, 2018).

Another issue in early mathematics teaching is that algebra is rarely used to explore structures within arithmetic, even though arithmetic structures are a core strand of algebraic thinking (Kaput, 2008; Mason, 2008; Torres et al., 2024). Research shows that students, without prior knowledge of algebra, can reason about relationships between numbers represented by algebraic symbols on line segments (Carraher et al., 2006; Saxe et al., 2010), a skill also applicable to second-language learners (Eriksson, 2021). Nevertheless, one mathematical area in which number lines have not been used as tools for exploring structure to the same extent as in arithmetic is in algebra (Adamuz-Povedano et al., 2021; Davydov, 2008).

This study, therefore, is based on the idea that early mathematics should bring structures and relationships to the forefront, mediated by algebraic symbols and geometrical figures, as suggested in the algebraic teaching tradition described by the El'konin-Davydov (ED) curriculum (e.g., Davydov, 2008; Dougherty, 2008; Kozulin & Kinard, 2008; Schmittau, 2003, 2005). The ED curriculum has garnered increased research attention over the past decade due to its potential to enhance young students' understanding of arithmetic structures (Coles, 2021; Kieran, 2022; Pitta-Pantazi et al., 2025). However, we do not know enough about how algebraic teaching should be orchestrated in early mathematics teaching to enable understanding of arithmetic relationships (Carraher et al., 2006; Coles, 2021; Dougherty, 2008; Kieran, 2022; Pitta-Pantazi et al., 2025). To contribute knowledge about what algebra and number lines can enable young learners to discern and experience, this study aims to identify various ways of thinking algebraically that are important for teaching when students explore arithmetic structures using number lines. Therefore, a phenomenographic analysis was conducted, revealing students' qualitatively different ways of experiencing number lines, using data from an algebraic teaching approach inspired by the ED curriculum. The research questions for this study are as follows:

1. What qualitatively different ways of experiencing number lines can be identified among young learners participating in the algebraic teaching inspired by the ED curriculum?
2. What can be seen as indicators of algebraic thinking in young students' work with number lines in the algebraic teaching?

Number Lines in Early Mathematics

In elementary and primary school, number sense includes number concepts, the meaning of arithmetic operations, the control of basic arithmetic facts, mental and written arithmetic and arithmetic analyses when solving problems (Adamuz-Povedano et al., 2021; Verschaffel et al., 2017), as well as relational thinking about numbers – all numbers can be explained as relationships with other numbers (Kieran, 2022; Venenciano et al., 2021), altogether also referred to as early algebraic thinking due to algebra as generalised arithmetic (Adamuz-Povedano et al., 2021; Mason, 2008; Radford, 2010b; Saxe et al., 2010). The interest for this article is the beginning of primary school, when children learn to work with countable discrete numbers and represent them on a number line (e.g., Bartolini Bussi & Hua Sun, 2018; Saxe et al., 2010), but they may not recognise crucial details about the numbers that require them to develop functional mental number lines for the future development of number sense (Booth & Siegler, 2006; Nunes et al., 2009). Therefore, it is central in elementary and primary education for students to have systematic support in distinguishing how to construct, use and work with number lines (Nunes et al., 2009). Young students need to experience how numbers are distinguished from each other with iterated equal lengths – thus identifying the unit between the magnitudes, the size of the unit that separates the numbers, and how the magnitude of a number represents the total amount of the iterated units (Booth & Siegler, 2006).

In research on number lines, Bartolini Bussi and Hua Sun (2018) summarise how number sense can be developed when the arithmetical structures were highlighted as balancing ordinal, cardinal and measurement aspects of numbers, connecting the three core concepts of addition, subtraction and numbers, focusing on structural approaches to early number development, emphasising the importance of figural and spatial representations and fostering physical involvement, such as counting using fingers or dancing or jumping on a number line. In this arithmetic tradition, students can comprehend the ordinal relationship of numbers by learning that each number has a specific location on the number line and that subsequent numbers are sequenced in a specific pattern (Booth & Siegler, 2008). Similarly, as students learn that numbers or quantities represent distances from zero, they learn about cardinality. As they learn that each equal-sized interval on a number line represents a specific unit, they can extend their counting skills from counting the number of objects in a set to counting units of length (Bartolini Bussi & Hua Sun, 2018; Booth & Siegler, 2008). They also learn how an arithmetical number line is linear and needs to be understood as containing the same unit between different quantities, unlike a logarithmic number line, in which larger numbers have smaller and smaller units (Booth & Siegler, 2008), which also connects to algebra (Saxe et al., 2010). Research on young students' numerical work on number lines has shown that their spatial capabilities also develop numerical capabilities; for example, higher numbers are often gestured using the right hand and lower numbers using the left hand due to the numbers increasing to the right on a number line (e.g., Lourenco et al., 2018). Young students, as well as adults from non-numeric cultures, can organise numbers of dots along a line without using numerical symbols (Dehaene et al., 2008). However, students who have difficulties with spatial capabilities may struggle to find units between different magnitudes (Booth & Siegler, 2008).

Another way to introduce and work with number lines, which is the focus of this article, is based on the algebraic teaching tradition (e.g., Davydov, 2008). Students are expected to learn numbers, not have numbers as the point of departure; they should not have ready-made number lines to explore numbers, they should explore their construction (Davydov, 2008; Dougherty, 2008; Kozulin & Kinard, 2008; Leontiev, 2005). Instead, in the ED curriculum, it is suggested that the ability to distinguish numbers should begin with more essential preconditions for the numbers, such as measurements and comparisons of quantities (Davydov, 1982, 1990, 2008). In this curriculum, it is suggested that working with number lines should begin with an exploration of their construction using non-numerical symbols (Davydov, 2008; Dougherty, 2008; Leontiev, 2005; Venenciano et al., 2021). This enables an algebraic way of thinking by working with quantities as unknowns grounded in measurements and comparisons, and by connecting the fact that different quantities can be equal, less than, or greater than. In this way, numbers emerge as results of measurement or as relationships between quantities when measurement is performed (Davydov, 1982). Elements essential to explore on a number line are the direction of increasing numbers, the starting point (i.e., zero), and the choice of unit (Davydov, 1982). By using a number line as a tool, including these elements, the structure of numbers can be explored while simultaneously constructing and exploring the number line itself (Carraher et al., 2006; Davydov, 2008; Zuckerman, 2004). By taking measures, it is possible to distinguish units, define values in a specific unit, and see that the formation of numbers is based on studying the relationship between quantities (Davydov, 1982). To distinguish this is in line with the fact that number lines incorporate structures of mathematical concepts, such as relationships between numbers and between operations, which are also included in primarily algebraic thinking (Bartolini Bussi & Hua Sun, 2018; Carraher et al., 2006; Davydov, 2008; Kozulin & Kinard, 2008; Pittalis et al., 2018). Additionally, Kieran (2022) and Venenciano et al. (2021) demonstrated how letter symbolisation is typically understood as algebraic, alongside representations such as manipulatives, graphical representations, and verbal explanations, which support student opportunities for relational thinking of numbers – one aspect of number sense as well as algebra (e.g., Radford, 2014). The following section presents how to connect algebraic thinking and work on number lines to explore the relationships between quantities.

Algebraic Thinking on the Relationships Between the Numbers on a Number Line

The ED curriculum is fundamentally algebraic and grounded in Vygotsky's (1963) idea regarding developing students' thinking with theoretical concepts. Number sense, which aligns with the ED curriculum, is one example of a theoretical concept and, therefore, cannot be developed solely through empirical, everyday concepts (Davydov, 1990, 2008). For young students, understanding numbers should therefore begin with an introduction of ascending structures and relationships among numbers, and continue with concrete numerical examples (Davydov, 1990, 2008; Pitta-Pantazi et al., 2025; Schmittau & Morris, 2004; Venenciano et al., 2021). This process of ascending from abstract to concrete requires specific tools for developmental education, such as algebraic symbols and length models, in collaborative activities (Davydov, 1990, 2008). In the process, students must acquire an understanding of arithmetic structure by identifying concepts to use when analysing relationships between different quantities (Davydov, 1990; Schmittau, 2003). Young students need to, for example, discuss that if $A/B = C$, then $C \cdot B = A$ (provided $B \neq 0$). They can also discuss that if $B > A$, then $C < 1$ for positive integers (Schmittau, 2003). Here, algebraic thinking can be considered a generalisation of arithmetic – arithmetic concerns operations involving numbers, whereas algebra concerns the structure of numbers and relationships between them (e.g., Bourbaki, 1974; Mason, 1996). To explore what algebraic thinking offers the youngest students and what meaning this way of thinking can bring to the development of number sense, an overview of its content, actions, and manifestations is of interest. Because of its general nature, algebra can be described in various ways: by supporting awareness of variables, by rearranging and manipulating expressions, and by expressing generalised arithmetic (Mason, 2008).

One description of algebraic thinking is related to content through the two core aspects suggested by Kaput (2008): (a) systematically symbolising and (b) syntactically guided actions. These two aspects are expressed in three strands that focus on mathematical content (Kaput, 2008). The first strand describes the study of structures, exemplified by algebra as generalised arithmetic; the second describes the study of functions, relations, and joint variations; and the third describes modelling with specific numbers or unknown numbers, modelling with variables, and comparing models (Kaput, 2008).

Another way to describe early algebraic thinking is to focus on different actions (Kieran, 2004, 2018, 2022). As suggested by Kieran (2022), the focus is on analytical, structural, or functional thinking at the earlier generational, transformational, and meta-levels. Analytical thinking involves analysing equalities (Kieran, 2022) and, as suggested by Radford (2014), analysing unknown quantities or variables treated as numerical values. These analyses are not tied to formal alphanumeric signs; they can be expressed in natural languages (verbal or nonverbal), materials or signs (Erbilgni & Gningue, 2023; Kieran, 2022; Radford, 2014; Zazkis & Liljedahl, 2002). Structural thinking can be seen as the exploration of structures by examining relations, properties, and structures within numbers, operations, and expressions, with one example being the relationships between quantities. This exploration can arise from comparisons of quantities derived from measurements notated with algebraic symbols to describe differences, or by using one quantity as a unit to describe another quantity (Bass, 2018; Davydov, 1982). Functional thinking deals with correlating variables (Kieran, 2022). Another way to describe algebraic thinking as actions is to work with (a) relationships between numbers and quantities, not only in calculations; (b) relationships between operations; (c) representations and ways of solving problems; (d) numbers and letters; and (e) the meaning of the equals sign (Kieran, 2004). Blanton et al. (2018) proposed a framework for algebraic thinking, in which generalising, representing, justifying, and reasoning can be actions related to generalised arithmetic, such as concepts associated with equivalence, expressions, equations, and inequalities, as well as functional thinking.

A third way of explaining algebraic thinking, as described by Radford (2010b), is how it can be manifested in both non-symbolic and more traditional symbolic ways of thinking. Radford (2014) specified non-symbolic algebraic thinking as two different actions: factual and contextual. Spoken language and gestures manifest factual algebraic thinking, while contextual algebraic thinking relates to the task's context or the student group.

Of interest for this study on algebraic thinking (see Table 1) is the content of generalised arithmetic (Kaput, 2008) combined with actions focusing on the analysis of relationships between numbers and quantities and between operations and the use of numbers and letters (Blanton et al., 2018; Kieran, 2004, 2022) and, finally, the manifestation of the analyses as factual and contextual (Radford, 2010a, 2010b, 2014). When working with unknown numbers on a number line, as suggested by, for example, Bass (2018), Carraher et al. (2006), Davydov (1982) and Dougherty (2008), an algebraic way of thinking can relate to the structures of numbers and the relationships between them; for example, if $A < B$ and $B < C$, then $A < C$. Such relationships are fundamental to algebra and arithmetic (Davydov, 1982, 1990; Hitt et al., 2016; Leontiev, 2005; Pittalis et al., 2018). Work dealing with general structures in arithmetic constitutes the foundations of algebra (e.g., Bourbaki, 1974; Leontiev, 2005; Pittalis et al., 2018).

Methods

Data

The data used in this study were part of a three-year developmental research project to explore the ED curriculum in Swedish multilingual classrooms. The research lessons were staged once a week during students' regular maths lessons. Over the three years, eight classes of grade 1 students participated. In the first project year,

Table 1 Summary of the different ways of expressing algebraic thinking

	Content (Kaput, 2008)	Actions (Blanton et al., 2018; Kieran, 2004, 2022)	Manifestation (Radford, 2010a, 2010b, 2014)
Algebraic thinking	Systematically symbolising Syntactically guided reasoning Generalised arithmetic Functions, relations and joint variation Modelling with numbers or unknown numbers	Analytical thinking Structural thinking Functional thinking Analyses of relationships between numbers and quantities Relationships between operations Representations and solutions of problems Use of numbers and letters Meaning of the equal sign	Indeterminacy Analyticity Denotation Factual Contextual Symbolic

there were two grade 1 classes; in the second and third years, there were three grade 1 classes each year. The number of classes depended on the participating school's enrollment. A total of 150 students participated in the research project. The students were 6–8 years old, and according to the Swedish school system, this was their first traditional school year in the formal education system. Local screening tests and national assessment tools indicated that the students had varying, relatively low abilities across all areas of mathematics.

Many students who participated in the project were new arrivals in Sweden. At the school, students spoke approximately 20–25 different mother tongues, with the dominant languages being Swedish, Somali, Kurdish, and Arabic. If students had not previously attended school, they started in a lower grade to develop skills in the language of instruction (e.g., Barwell, 2018), which is why students' ages within a given grade varied. All students and parents signed letters of intent to participate in the research lessons, in accordance with the ethical guidelines stipulated by the Swedish Research Council (2017). For more information about the project, see Eriksson (2021). Ordinary class teachers staged the research lessons, which had been designed, reasoned about and revised by a local research team of six teachers and the author of this article, inspired by the ED curriculum. Due to the research design, the teachers considered each task three times over the project years. By doing so, they became pretty familiar with the tasks. The author observed the lessons and documented them using a handheld video camera.

Data consisted of verbatim transcriptions of video recordings from eight Year 1 research lessons, including students' and teachers' actions, with a focus on gestures directed at what was being constructed on the board (e.g., Radford, 2021). The selected episodes serve as typical examples from the eight lessons.

Design of the Research Lessons – Developing and Enhancing the Construction of a Number Line

Among the crucial aspects of the ED curriculum considered in the design of the lessons were students' opportunities to jointly identify a common problem to solve, use tools other than verbal language alone to reason about content, and reason about solutions to the tasks (Davydov, 2008). These aspects were operationalised using tasks from Davydov et al. (2012), why the first semester of Grade 1 focused on content related to equality and inequality, comparisons of quantities, units of measurement, and the number line (Davydov, 1982, 1990, 2008; Davydov et al., 2012; e.g., Eriksson et al., 2024). To this end, the tasks and tools proposed in the ED curriculum consist not only of numerical examples but also of algebraic symbols and graphical representations, such as line segments (Davydov, 1982, 1990, 2008; Davydov et al., 2012). The tasks provided students with the opportunity to jointly reason about different measurement contexts, such as length, area, volume, and number (Davydov, 2008).

One task in the curriculum was analysed in this study (see Fig. below), aimed at the students' reasoning about relationships between quantities and the structure of

Fig. 1 Working with line segments in the ED curriculum (Davydov et al., 2012)

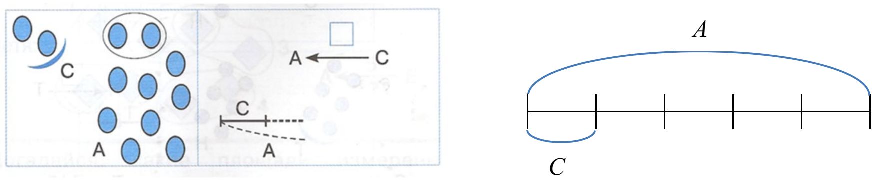


Fig. 2 Task focusing on a unit in the curriculum (Davydov et al., 2012)

number lines. In the project, this task was preceded by work on several tasks that focused on different aspects of number lines within arithmetic structures.

Below, four of these tasks are described; experiences from earlier tasks drove the students' actions in later tasks. At the beginning of the project, one task was to explore two unknown quantities, where a smaller volume *C* and a larger volume *X* were represented using two different length line segments (Fig. 1). The students had to establish that the bottom areas of two cylinders were equal and then realise that they could use only the heights to compare the volumes. Next, the students had to discern that two unknown volumes, *C* and *X*, could be represented as two different line segments and as two marks on the same line segment (Fig. 1).

The work involved exploring the relationship between the two unknown quantities utilising volumes, letter symbols, different length models, and expressions without specific numerical examples. Words related to mathematics, such as large, small, long, short, more, and less, were emphasised, which connected to second language learning (e.g., Barwell, 2018). The task, therefore, encouraged the students to analyse quantities and systematise their results, promoting the development of algebraic thinking (Davydov, 2008; Kieran, 2022). Similar activities were developed to expand this task to explore quantities related to area and countable physical objects (see Eriksson & Sumpter, 2021).

The next task for developing a number line involved how a unit could be used to separate magnitudes, and how this unit could be represented on a line segment and with algebraic symbols. The joint reasoning concerned how many units of *C* could be accommodated within *A* (Fig. 2).

Through this representation, students could analyse a unit as the distance between two magnitudes on the line, which served as a fundamental component of the structure of the number line. Observing that because *C* was the unit of two balls and separated from one (1), the task challenged the students to discern units of different values. The task, therefore, focused on multiplicative structure, as one step on the number line could also be understood as representing 'two'. However, in the

ED curriculum, the structure of multiplication is not explored using a number line; instead, it is explored by identifying an intermediate unit as a measure (e.g., Zuckerman, 2007). In the next task, the unit was equal to one (1). The exploration of the unit as a detail of the structure of a number line exhibits signs of algebraic thinking (Davydov, 1990), which is also comparable to structural and analytical thinking (Kieran, 2022).

Next, line segments were developed with markings for unspecified quantities, which were enhanced to resemble a number line (Fig. 3).

The design choices in this task were intended to help students identify the critical features of a number line (equal spacing, a fixed zero point, and direction). This offered number lines devoid of numbers and, using a special notation, a flag to indicate the line's starting point. Over time, this starting point was changed to represent the zero position. To facilitate a deeper understanding of number lines' structures, various units were employed on the same line (as depicted in Row 2 of Fig. 3), the line was backwards (Row 3), or the starting point – the flag – was placed in different positions (Row 4). Regarding the line in Row 4, students could reason about negative numbers because of the marks on both sides of the flag. To reason about the value, the students placed one finger on the flag and another on the specific marking to visualise that the value is the distance from the starting point to the specific point. In this task (Fig. 3), the students had the opportunity to represent the volume in the flask to the left. Altogether, these number lines were explored when the students engaged in reasoning about the use of the same unit, a designated starting position, the direction of increasing numbers and the difference between a point on a line and a value on a number line, all without reference to specific numbers (Davydov, 2008; Davydov et al., 2012). To continue constructing a number line, the students were given an empty line on the board and asked to turn it into a number line that included as many numbers as possible. They were given a choice of two different units – one longer and one shorter – to develop the distances between the numbers on the number line. Here, it was essential for the students to argue that the shorter unit provided more positions along the line, as the number of markings was inversely related to the unit's size (Davydov et al., 2012).

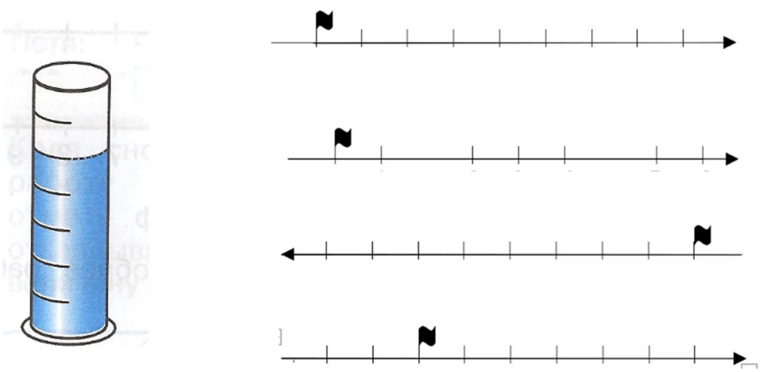


Fig. 3 A task with conceptually challenging number lines in the curriculum (Davydov et al., 2012)

Aspects of the number lines from previous tasks depict the specific task used in this study; see Fig. 4. The task was chosen based on the joint reasoning developed by the students across all eight lessons as they solved it.

In all lessons, this task was staged as a whole-class activity and presented visually on the board. The teachers introduced it by encouraging students to consider what mathematical problems they could address. The excerpts in the findings section are from students' joint reasoning across the eight research lessons. A teacher led the discussions, and the students were engaged in discussing their own and their peers' ways of thinking while sitting at their desks or working on the board. The symbols on the line had no prior meaning for the students other than being symbols for any number.

Method of Analysis

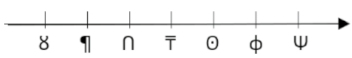
To investigate what algebraic thinking provides young students in terms of opportunities to explore arithmetic relationships, the analysis first focused on qualitatively different experiences with number lines. Then, in the second step of the analysis, these qualitatively different experiences were inductively coded and analysed in relation to algebraic thinking.

The first step of the analysis focused on the students' qualitatively different experiences of number lines, taking phenomenography, as explained by Eriksson (1999) and Marton (2015), as a point of departure. In phenomenography, a distinction is made between first- and second-order perspectives (Eriksson, 1999; Marton, 2015), where the first-order perspective concerns the students' conscious opinions and statements. The second-order perspective, on the other hand, aims to identify students' ways of experiencing, that is, what can be interpreted as the basis for students' statements, as expressed in their overall verbal and non-verbal communication (Eriksson, 1999). Analysing the students' ways of experiencing the number line from a second-order perspective makes it possible to distinguish qualities in the ways of experiencing the phenomenon, and to get an idea of which aspects of the number line the students focus on and what they may disregard. In other words, it becomes possible to see what creates qualitative differences in their experiences (e.g., Eriksson, 1999; Marton, 2015; Pang & Marton, 2003). In the careful and comparative readings of transcripts, the focus was on both the phenomenon students discussed (i.e., the number line) and the actions taken on it (i.e., pointing at the point of departure and the point of magnitude) and an identification of why these actions were taken (often depending on previous work on number lines). Sequences of the students' ways of working with and communicating around the number line reflected different ways of experiencing it. These sequences were identified and colour-coded to compile them into qualitatively distinct categories. During the process of grouping, regrouping, and formulating the outcome space, a total of three categories of ways of experiencing the number line were identified. By identifying

Fig. 4 Task in the lesson for this study (Davydov et al., 2012)

$A = 3K$

$B = 2K$



similarities and differences between the three categories, they could then be organised into a hierarchical outcome space (e.g., Pang & Marton, 2003).

Experiences involving the mathematical properties that must be present on a number line reached the lowest quality, as there were no ideas of relationships between the properties or how these properties could be utilised when working on the line. One example of colour-coding was the students' answer to the teachers' questions about what they could see on the board: a line, some dots, and the distances between them. The second category revealed experiences related to the number line as a means of visualising relationships between different properties. One example concerned the experiences: the unit between the points had to be the same along the whole line for it to constitute a number line. The third category was identified as operations on a number line. In validation of the analyses, in analytical seminars with experienced phenomenographers, three subcategories were identified in this third category: the value from the starting position to the specific position, the direction of a number line and the operations addition and multiplication One example that was colour-coded was students' focus on a common starting point for the number line, interpreted as the experience of the organisation of a number line with the same reference point for values.

In the second step of the analysis, indicators of students' algebraic ways of thinking were identified within the different categories. This analysis was conducted by coding what is said and done, grounding the experiences of the number line according to the descriptions of algebraic thinking presented in Table 1. The codes used are presented in Table 2. The first column presents the algebraic approaches to thinking and the references that support them. The second column identifies the category to which the codes belong, and the third provides examples from the data.

Validation of the analyses was carried out repeatedly, as previously mentioned, in analytical seminars with experienced phenomenographers and researchers in mathematics education. Questions in these seminars concerned whether selected transcripts could be interpreted as describing students' ways of experiencing the phenomenon and their algebraic ways of thinking.

Findings

To answer the two research questions, this section presents three phenomenographic categories, summarised in Table 3.

The categories concern similarities and differences based on how students saw the number line as. The first and second categories are linked to students' experiences of constructing number lines: the mathematical properties necessary for a number line and the relationships between the properties, respectively. The third category is linked to the use of number lines when representing, working with and operating with numbers. The indicators for algebraic thinking are presented at the end of each category. The data chosen to illustrate the categories are examples of joint work from all eight research lessons in Grade 1, when students and teachers worked on the task shown in Fig. 4.

Table 2 Analyses of algebraic thinking using the results in Table 1 as a framework

Codes for Algebraic Thinking	Category	Indicators from the data
Systematically symbolising Kaput (2008) Syntactically guided reasoning Kaput (2008)	1, 2, 3	*We know A, and we know B. [meaning the statements in the task A = 3K; B = 2K]* Interpreted as a systematic symbolising, as the students know the unknown numbers in the expression to represent on the number line.
Generalised arithmetic Kaput (2008) Blanton et al (2018), Kieran (2004, 2022)	1, 2, 3	The number line in the task was used to reason about generalised arithmetic.  *It can be any number, but the distance is K.* This quote is interpreted as students' analyses of symbols for general unknown numbers and their relationships, and as reasoning about the general structure of an arithmetical number line, with the same unit between all positions.
Modelling with numbers or unknown indeterminacy numbers Kaput (2008), Blanton et al (2018), Kieran (2004, 2022), Radford (2010a) Functional thinking, including functions, relations and joint variation Kaput (2008), Blanton et al (2018), Kieran (2004, 2022)	2, 3	*Student 8: We do need to start at the same place and have the same distance.* … *Student 6: We do need to use the same unit [points to the expression and the distances on the line].* Interpreted as modelling with the unit as unknown numbers or a relationship between numbers.
Analytical thinking, including analyses of relationships between numbers and quantities Blanton et al (2018), Kieran (2004, 2022)	2, 3	*It says A and 3K. [Points in the expression.] Then, I think this is K. [The student points to the entire number line and then uses a finger to jump between the symbols* This quote is interpreted as the students' exploration of different representations of the unit, analysing the relationship between different properties
Structural thinking Blanton et al (2018), Kieran (2004, 2022)	3	*… it is what is between that is important. It is the jump from one symbol to another that is important.* Interpreted as the exploration of the structure of magnitudes on the line as values
Relationships between operations Blanton et al (2018), Kieran, (2004, 2022)	3	*Student 10: 3K plus 2K equals 5K.* *Student 8: But 3 times 2 is 6. That is different. Then, three jumps twice.* Interpreted as the students exploring the relationships between addition and multiplication
Representations and solutions of problems Blanton et al (2018), Kieran (2004, 2022)	2, 3	The solution to this task involves the analytical process of identifying relationships and representing expressions on a number line, rather than providing a numerical answer.
Use of numbers and letters Blanton et al (2018), Kieran (2004, 2022) Meaning of the equal sign Blanton et al (2018), Kieran (2004, 2022) Denotation Blanton et al (2018), Kieran (2004, 2022)	1, 2, 3	The students used the numbers and the letters in the task to explore the number line. I think we can make it on the line [points at $A=3K$] because K … the distances… [The student points at the line.]
Factual Radford (2010a, 2010b, 2014)	1, 2, 3	*We do need to start at the same place and have the exact distances.* Interpreted as the need for the point of zero and the unit between the positions.
Contextual Radford (2010a, 2010b, 2014) Symbolic Radford (2010a, 2010b, 2014)	2, 3	The students explored the symbol K as the unit in two different contexts: in the expression and on the number line.

Table 3 Experiences of number lines

1) Mathematical properties
2) Relationships between the properties
3) Operations on a number line
3a) The value is from the starting position to the specific position
3b) The direction of a number line
3c) Addition and multiplication

Category 1: Mathematical Properties

In this category, the students identified a number line as a line possessing specific mathematical properties, which they are familiar with from earlier work: a line, specific positions (including the starting position), the space between these positions, and the direction of the line.

350	Teacher:	What do you see in this task? [Points at the board on which the task in Fig. 4 is shown to the class.]
353	Student 1:	We know A, and we know B
400	Teacher	Yes…
401	Student 2:	There is a line, some points with a distance between them, and an arrow that could describe the direction of the number line. Therefore, it is a number line. [Gets up, goes to the board and points to the different details.]

Here, Student 2 implicitly mentions the properties constituting the structure of a number line (line 401). The student's argument about the 'points' focuses on the positions on the line, while 'a distance between' focuses on the fact that the points were organised with space between them.

Later, Students 3 and 4 argued that one of the positions on the line needed to be the starting position (see lines 501 and 508). In line 438, Student 2 adds that K in the expressions could refer to the distance between the points on the line.

438	Student 2:	I think we can make it on the line [points at $A=3\ K$] because K … the distances… [The student points at the line.]
501	Student 3:	… We need to start here, I think. [The student also stands by the board and points to the first symbol on the line.]
508	Student 4:	But we do need the same starting point [Sits down at his place]

In the previous two excerpts, the students' arguments regarding the properties of a number line, including the line itself, the positions on the line, one of which is a starting position, and the space between the positions, illustrate its general properties.

In one lesson, a student argued that there was no starting point, and another student contradicted that they could choose one.

408	Student 5:	There is no starting point; therefore, this is not a number line, and we cannot use it for anything
413	Student 6:	But if we decide on such a point, we can use it as a reference point on a number line

The properties the students reasoned about were all connected to the foundation of numbers as magnitudes on a line, and to the general aspects of number lines that distinguish them from general lines. The students did not use numerical examples; instead, they used letter symbols as placeholders for unknown or general values.

An algebraic way of thinking is indicated within this category, involving generalised arithmetic. The students' arguments systematically used symbolised values with unknown numbers, allowing them to focus on the general properties of the number line. The general symbols were used to reason about particular properties of this specific number line. This first category concerns the analysis of mathematical properties characterising a number line that do not apply to a general line – properties that the students needed to discern to reason about relationships between numbers when using the number line. Moreover, the mathematical reasoning exhibited signs of algebraic thinking, as evidenced by the use of general symbols when modelling particular positions on the number line.

Category 2: Relationships between the Properties

This category focuses on the students' experiences with the relationships between the properties mentioned in Category 1. These experiences involve the relationship between quantities, including that between distances and quantities on the number line. These relationships constitute a unit, which is the same for an entire number line in arithmetic form.

The first thing the students explored in all the lessons was the unit K in the expression and its relationship to the distances between the positions on the number line. The following excerpt highlights one example of students' joint work on the whiteboard when they explored the relationship between the different representations in the task. The students marked A (from the expression $A=3\ K$) on the number line with unknown quantities. They implicitly reasoned about unit K from the expression as being the same as the distance between the positions, as illustrated by Students 6, 7 and 8 in the following:

627	Student 7:	May I try? It says A and $3K$. Then, I think this is K. [The student points to the entire number line and then uses their finger to jump between the symbols.]
655	Teacher:	Interesting. Now, we are on our way. Do you think we can use this line as a number line? Moreover, do we think we have to start somewhere on the line? Furthermore, what position is on the line?
	Student 8:	We need to start at the same place and have the exact distances
735	Teacher:	Why?
740	Students:	For K
741	Student 6:	We do need to use the same unit [Points to the expression and at the distances on the line.]

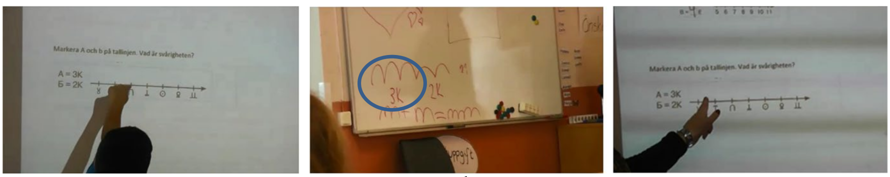
Here, Student 7 (line 627) and Student 6 (line 741) point to *K* in the expression and to the distances on the number line (meaning that the distances represent *K*). The students reasoned about the unit of this number line and labelled it *K,* the same labelling as in the expression. An interpretation is that the students were familiar with identifying units from earlier tasks. Altogether, they analysed and identified the unit in two different contexts: in the expression and on the number line, which was new for them.

This category also includes an exploration of the general structure of a number line, which states that the distances between all positions must be the same and that the unit is iterated; thus, the distances must be represented by a unit. In this task, ¶ = T̄ = T̄⊙ = ⊙ф when = ¶ + 1; T̄ = ¶ + 2; ⊙ = ¶ + 3 (Fig. 5c). This structure was highlighted by the students in the reasoning when they connected the expressions with the number line, as illustrated in the following excerpt:

720	Student 9:	*A* is three. [Points to the expression *A* = 3*K*.]
733	Teacher:	How can we state that?
742	Student 10:	By jumping
743	Teacher:	How? Show us on the board
745	Student 11:	If this is A, we need to jump one, two, three [Points with his finger three jumps on the number line]. We end up here [see Fig. 5a]

750	Student 10:	Mmm…
756	Student 12:	More students want to talk
757	Teacher:	Yes
815	Student 13:	[This student is also by the board]. This is one, two, three, but what is between is important. [Points to the symbols, shown in Fig. 5a, then draws the jumps (3*K*) as arches on a whiteboard beside the number line, circled in Fig. 5b. The student is distinct in the drawing of the arches. Again, he points at the symbols, as shown in Fig. 5c.] It is the jump from one symbol to another that is important
828	Teacher:	Ahh. This is important for everyone to listen to

Here, Students 10 and 13 reason about how the relationships between quantities depend on the distances between positions, with Student 13 explicitly saying, 'But it is what is between that is important.' They state that the distances between positions,



a b c

Fig. 5 Work in a lesson when two students show and reason about the number line and thus the unit, are important for a number line (implicit in the value). Altogether, they present the general expression of a value as a relationship between quantities, depending on the unit, in a general yet contextual manner. They do not use specific numerical examples to model unit K on the number line.

The algebraic thinking in this category was evident in students' analyses of the relationships between the properties identified in Category 1, indicating that Category 1 is encompassed within this second category. The students' joint analytical work involved modelling the unknown unit K, which represents the distance between different positions on the number line, while also exploring the magnitudes of numbers. This analytical work focused on the relationship between the positions on the number line for representing A and B. The relationship was identified as the unit *K*. The students' modelling work with this unit K was staged without using specific numerical examples, signalling an emphasis on algebraic thinking. In this way, the students' algebraic thinking was manifested contextually, as the unit K was addressed in two contexts: as expressions and as distances between positions on the number line. The algebraic work focused on the need for a unit, which is the same for the whole number line, and was also demonstrated in a factual manner using everyday language.

Category 3: Operations on the Number Line

In this category, the students discerned how to represent a quantity on a number line, visualise relationships between numbers, and visualise different counting operations. It consists of three subcategories: the value of a position, the direction of a number line and identifying addition and multiplication. The students could also jointly discern the general relationship between the linear order of numbers using unknown quantities, showing signs of algebraic thinking.

Category 3a: The Value Is From the Starting Position to a Specific Position

This subcategory focuses on the starting position and its relationship to the values of different positions. This relationship concerns how a value is represented on a number line; thus, the distance between the starting position and a specific position. One example was when the students wanted to symbolise *A* on the number line. The teacher asked the students where to start and connected the expression for *A* to the students' discussion:

850	Student 14:	We can mark *A* on the number line
900	Student 15:	It is three. [Makes three arcs in the air with one finger.]
900	Student 15:	It is three. [Makes three arcs in the air with one finger.]
910	Student 16:	We make *K* on the number line
918	Student 17:	We start at the starting point
924	Teacher:	What do we do with the starting position? How many jumps from this position?
930	Students:	Zero. That is the start

939	Teacher:	[The teacher slowly counts and illustrates jumps on the line.] Zero at this starting position. The teacher keeps her left hand in the starting position. Her right hand performs jumps along the number line.] Then one jump, two jumps and three jumps. What is then? What is *A*?
940	Student 5:	The distance is *A*
942	Student 1:	The distance between zero and three is *A*
950	Student 4:	The distance is important

In line 942, the students clarify that the value is the distance from the position of zero to the specific position. The relationship reasoned about is that the distance from the starting position (the position of zero) to a position labelled a is $|a|$; from this, it follows that the distance from the starting position to b is $|b|$. The same detail is shown in line 952 (see Category 3c) when Student 5 jumps along the number line. The student holds one finger in the starting position, and the other hand is positioned to show the value of the numbers; that is, the student implicitly reasoned about the structure of the number line and the relationships between positions and values.

The algebraic thinking in this category consisted of systematically analysing the relationships between the starting point and the different positions on the number line. The analyses concerned the general structure of the relationship between numbers and quantities, namely structural thinking about magnitudes, modelled using symbols for unknown numbers and a letter symbol for the unit.

Category 3b: The Direction of a Number Line

This subcategory concerns the direction from 0 to 1, which specifies the positive orientation of the number line and the direction in which numbers increase. The students reasoned about this as follows:

610	Student 4:	The starting position has to be furthest to the left
615	Student 5:	The direction is there. [The student is at the board and points to the arrow in the task.]
618	Student 4:	That is what I said. So, we need to have the starting position here
620	Student 6:	Greater numbers are in that direction
625	Teacher:	What does that mean?
630	Student 5:	This [points at $\odot$] is less than [ϕ], and this [points at Ψ] is greater than [points at ϕ]; therefore, [points at Ψ] is the greatest

Here, the students identify the direction of this number line and that the values increase to the right. In the context of this number line, it follows, according to Student 5, that $\odot < \phi$, $\Psi > \phi$ and therefore $\Psi > [\phi$ and $\odot]$. Student 5's arguments are implicitly connected to linear order: if there are three points on a line, one point lies in the interval between the other two.

The indicated algebraic thinking involved the students' analyses of the relationships among the unknown numbers, in which they identified which symbols represented larger or smaller values. The analyses focused on relationships between quantities. This way of thinking was manifested factually, modelled on the number line using gestures.

Category 3c: Operations on a Number Line

This category is useful for working with multiple positions on a number line and for understanding relationships between operations. The students reasoned about the structure of addition, suggesting $A+B$. They also reasoned about multiplication and worked with expressions with the unknown unit, as shown in the following excerpt:

947	Student 5:	Go on, two more jumps
950	Student 8:	Why?
952	Student 5:	A is the same as $3K$, and this is $2K$ [meaning $B=2K$]. [The student points at $A=3\ K$. Then, the student points at the number line and jumps to the positions on the line.] A is 1, 2 and 3. [The student has one finger on the left hand pointing at the starting position and jumps with one finger on the right hand.]
957	Student 9:	Go on, two more jumps
958	Teacher:	Why?
965	Student 9:	Because $3+2$ is equal to 5. It says $3K$ and $2K$. [$A=3K$ and $B=2K$.]
966	Student 5:	[The student goes on to jump across two more markings on the number line.]
990	Student 10:	$3K$ plus $2K$ equals $5K$
998	Student 8:	But 3 times 2 is 6. That is different. Then, three jumps twice

Here, the four students expand on the use of the number line in a collective reasoning about adding and multiplying. On lines 947, 952, and 957, the students suggest adding A and B; on line 998, Student 8 suggests multiplying two quantities. Student 8 states that multiplication differs from addition because it involves counting with a unit other than one.

The algebraic thinking in this subcategory includes the students analysing the operations of addition and multiplication in a general manner; multiplication involves counting with a unit other than one. This analysis indicates the student's ability to extend their thinking beyond individual positions on the number line, a sign of generalisation, showing structural thinking about these operations implicitly by using the properties of unit, magnitude, and movement on a number line.

Discussion

Identifying structures and relationships within arithmetic is challenging for students (Nunes et al., 2009; Woods et al., 2017). Therefore, to identify ways of thinking that may be important in early mathematics teaching to help students' exploration of arithmetic structures, this study analysed lessons from a project using the ED curriculum. The ED curriculum suggests that students need tools, models, and joint work to enhance what Vygotsky (1963) describes as theoretical thinking. One example of theoretical thinking is the study of general relationships between numbers, which includes aspects of algebraic thinking important for young students' conceptual understanding of numbers (Adamuz-Povedano et al., 2021; Davydov, 1982; Kaput, 2008; Radford, 2014). According to Davydov (1982, 1990, 2008), tools such as a visual line can mediate students' transition from empirical to theoretical thinking about quantities (see also Radford, 2010a). When explored as a tool, the number line can serve as an emergent conceptual model rather than merely a representational aid (Carraher et al., 2006).

The analytical work in the present study yielded three qualitatively different categories of students' experiences of number lines as: (a) mathematical properties, (b) relationships between the properties, and (c) operations on a number line. These categories are hierarchical in that experiences in higher categories also encompass those in lower ones. The forms of algebraic thinking reflected in the categories included systematic, symbolised, and generalised arithmetic, as described by Kaput (2008), as well as the use of numerical and letter symbols to reason about relationships, as suggested by Kieran (2022). However, the identified algebraic thinking did not involve syntactic, guided reasoning, or functional thinking, as outlined by Kaput (2008) and Kieran (2022), which are often emphasised in studies of early algebraic thinking (Torres et al., 2024). Instead, algebra in this study functioned as a tool to mediate relationships among unknown numbers, enabling generalisation of arithmetic structures (e.g., Narváez et al., 2025). The manifestation of algebraic thinking in the lessons was both factual, relying on everyday words when analysing relationships, and contextual, as the analyses of the unit were conducted in two contexts, in the expressions and on the number line. It was not manifested symbolically (Radford, 2010b, 2014) since formal algebraic operations were not staged. Moreover, in contrast to what is common in early mathematics, the solution to the analysed task was not a numerical answer (e.g., Carraher et al., 2006); instead, it involved an algebraic expression with unknown quantities represented on a number line. In this context, and depending on students' work on previous tasks, the algebraic expression and the number line appeared to enable the exploration of properties essential to understanding relationships among numbers. These properties also serve as gatekeepers to understand operations with numbers (e.g., Adamuz-Povedano et al., 2021; Saxe et al., 2010), including elements of algebraic thinking (e.g., Blanton et al., 2018; Kieran, 2004, 2022). Reflecting on solutions that attend to detailed aspects of numbers–such as the dependence of a number's value and its relation to zero and the need for a common unit for comparison–provides opportunities for more qualified reasoning about numerical relationships and operations.

The findings show that these students used mathematical properties and relationships identified in the lower categories to enhance their reasoning about operations in the higher category. In the highest category, the students' experiences concerned fundamental arithmetical structures, for example, distinguishing between positions and values, that relationships between unknown quantities depend on their locations on the number line, and that addition involves adding different numbers of the same unit while multiplication involves repeated addition, also explained as relationships in a manner of algebraic reasoning (c.f. Adamuz-Povedano et al., 2021). Despite the small sample ($n = 150$ from one school in a single project conducted over three years), these findings indicate that the mathematical properties students discerned might enable more qualified participation in number line work. More qualified and more detailed participation, in turn, indicates learning (Davydov, 1990, 2008; Marton, 2015; Vygotsky, 1963).

When early mathematics revolves predominantly around numerical examples and operations on a ready-made number line, students' experiences of mathematical properties (Category 1) and relationships among these properties (Category 2) are often overlooked (e.g., Carraher et al., 2006; Pitta-Pantazi et al., 2025). To provide the mathematical properties (e.g., the starting position, unit and direction) and to highlight the relationships between the properties (e.g., the value as the distance from the starting point to a specific position and the distance between positions as the unit), the number line and the expressions containing unknown quantities appeared to work as mediating tools for algebraic reasoning (e.g., Bass, 2018; Mason, 2008; Nunes et al., 2009; Radford, 2010a). The findings in Categories 1 and 2 represent preconditions for the number line (e.g., Kieran, 2022; Nunes et al., 2009; Venenciano et al., 2021; Verschaffel et al., 2017), yet these are often taken for granted in early mathematics education (Carraher et al., 2006). Altogether, the students' engagement in the highest Category 3, with reasoning that includes mathematical properties and their relationships, indicates the importance of the two lower categories. Relationships, such as the distinction between points on a line and a value, and understanding value as a relationship from a reference point to another point on the line, were processed from the particular to the general (e.g., Mason, 2008). General structural aspects were modelled (e.g., Kaput, 2008), as seen in the analysis of values and magnitudes in Category 3, also referred to as relational thinking (e.g., Blanton et al., 2018; Venenciano et al., 2021).

It could be argued that, despite the limitations of this study, it contributes to the knowledge of algebraic teaching using the ED curriculum, which has been questioned in research (Coles, 2021; Kieran, 2022; Pitta-Pantazi et al., 2025). The analyses indicate, in line with other studies (e.g., Eriksson & Eriksson, 2021; Eriksson & Sumpter, 2021; Eriksson, 2024), that the multilingual students seemed to be able to reason about relationships jointly, even though both reasoning and relationships are challenging, particularly due to their varying capabilities in the teaching language and due to that they had no support to translation into their mother tongue in the research lessons (e.g., Barwell, 2018).

Acknowledgements This study was funded by the Municipality of Borlänge and Dalarna University in Sweden. Open Access is funded by Stockholm University. Thanks to the teachers and students who took part in the design project and thank you to the researchers who took part in the seminar at Dalarna University and those who have read and discussed the study. Thanks also to Prof Astrid Pettersson, Prof Inger Eriksson and Prof Lovisa Sumpter at Stockholm University for commenting on my work.

Funding Open access funding provided by Stockholm University.

Data Availability Data is stored on separate hard drives, not in any cloud service.

Declarations Prof Lovisa Sumpter who is in the Editorial board for IJSME has earlier been my supervisor. She is not an author of the article but is disqualified (biased) in the publication process.

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